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HERMITIAN MANIFOLDS WITH SEMI-POSITIVE HOLOMORPHIC SECTIONAL CURVATURE

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Abstract. We prove that a compact Hermitian manifold with semi-positive but not identically zero holomorphic sectional curvature has Kodaira dimension $-\infty$. As applications, we show that Kodaira surfaces and hyperelliptic surfaces can not admit Hermitian metrics with semi-positive holomorphic sectional curvature although they have nef tangent bundles.

1. Introduction

In this note, we study compact Hermitian manifolds with semi-positive *holomorphic sectional curvature*. It is well-known that, the holomorphic sectional curvature plays an important role in differential geometry and algebraic geometry, e.g. in establishing the existence and nonexistence of rational curves on projective manifolds. However, the relationships between holomorphic sectional curvature and Ricci curvature, and the algebraic positivity of the (anti-)canonical line bundles, and some birational invariants of the ambient manifolds are still mysterious. In early 1990s, Yau proposed the following question in his "100 open problems in geometry" (e.g. [33, Problem 67] or [24, p.392]):

Question 1.1. If (X, ω) is a compact Kähler manifold with positive holomorphic sectional curvature, is M unirational? Does X have negative Kodaira dimension?

At first, we answer Yau's question partially, but in a more general setting.

Theorem 1.2. *Let (X, ω) be a compact Hermitian manifold with semi-positive holomorphic sectional curvature. If the holomorphic sectional curvature is not identically zero, then X has Kodaira dimension $-\infty$. In particular, if (X, ω) has positive Hermitian holomorphic sectional curvature, then $\kappa(X) = -\infty$.*

A complex manifold X of complex dimension n is called complex parallelizable if there exist n holomorphic vector fields linearly independent everywhere. It is well-known that every complex parallelizable manifold has flat curvature tensor and so identically zero holomorphic sectional curvature. There are many non-Kähler complex parallelizable manifolds of the form G/H, where G is a complex Lie group and H is a discrete co-compact subgroup. On the other hand, a compact complex parallelizable manifold is Kähler if and only if it is a torus ([27, Corollary 2]). Hence, a compact Kähler manifold (X, ω) with identically zero holomorphic sectional curvature must be a torus.

As an application of Theorem 1.2, we obtain new examples of Kähler and non-Kähler manifolds which can not support Hermitian metrics with semi-positive holomorphic sectional curvature.

Corollary 1.3. *Let X be a Kodaira surface or a hyperelliptic surface. Then X has nef tangent bundle, but X does not admit a Hermitian metric with semi-positive holomorphic sectional curvature.*

It is known that Kodaira surfaces and hyperelliptic surfaces are complex manifolds with torsion anti-canonical line bundles. Hence they are all complex Calabi-Yau manifolds. Note also that Kodaira surfaces are all non-Kähler. Here, X is said to be a complex Calabi-Yau manifold if it has vanishing first Chern class, i.e. $c_1(X) = 0$. Moreover,

Corollary 1.4. *Let X be a compact Calabi-Yau manifold. If X admits a Kähler metric with semi-positive holomorphic sectional curvature, then X is a torus.*

Note also that all diagonal Hopf manifolds are non-Kähler Calabi-Yau manifolds with semi-positive holomorphic sectional curvature([23, 31]).

For reader's convenience, we present some recent progress about the relationship between the positivity of holomorphic sectional curvature and the algebraic positivity of the anti-canonical line bundle, which is inspired by the following conjecture of Yau:

Conjecture 1.5. If a compact Kähler manifold (X, ω) has strictly negative holomorphic sectional curvature, then the canonical line bundle K_X is ample.

Bun Wong proved in [29] that if (X, ω) is a compact Kähler surface with negative holomorphic sectional curvature, then the canonical line bundle K_X is ample. Recently, Heier-Lu-Wong showed in [15] that if (X, ω) is a projective threefold with negative holomorphic sectional curvature, then K_X is ample. Moreover, by assuming the still open "abundance conjecture" in algebraic geometry, they also confirmed the conjecture for higher dimensional projective manifolds. On the other hand, Wong-Wu-Yau proved in [30] that if (X, ω) is a compact projective manifold with Picard number 1 and quasi-negative holomorphic sectional curvature, then K_X is ample. As a breakthrough, Wu-Yau [28] confirmed Conjecture 1.5 when X is projective. Building on their ideas, Tosatti and the author proved Conjecture 1.5 in full generality. More precisely, we obtained

Theorem 1.6 ([26])**.** *Let (X, ω) be a compact Kähler manifold with nonpositive holomorphic sectional curvature. Then the canonical line bundle K_X is nef. Moreover, if (X, ω) has strictly negative holomorphic sectional curvature, then the canonical line bundle K_X is ample.*

For more related discussions on this topic, we refer to [29, 15, 30, 16, 28, 26, 18] and the references therein. One may also wonder whether similar statements hold for compact Kähler manifolds with positive holomorphic sectional curvature. However,

Example 1.7. Let Y be the Hirzebruch surface $Y = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}\right)$ for $k \geq 2$. It is is proved ([19] or [24, p.292]) that Y has a smooth Kähler metric with positive holomorphic sectional curvature. But the anti-canonical line bundle K_Y^{-1} is not ample although K_Y^{-1} is known to be effective. For more details, see Example 3.6.

As an important structure theorem, Heier and Wong proved in [17] that projective manifolds with positive total scalar curvature are uniruled. In partiuclar, projective manifolds with positive holomorphic sectional curvature are uniruled.

2. Preliminaries

Let (E, h) be a Hermitian holomorphic vector bundle over a compact complex manifold X with Chern connection ∇. Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $R^\nabla \in \Gamma(X, \Lambda^2 T^*X \otimes E^* \otimes E)$ has components

$$R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \tag{2.1}$$

(Here and henceforth we sometimes adopt the Einstein convention for summation.) In particular, if (X, ω_g) is a Hermitian manifold, $(T^{1,0}M, \omega_g)$ has Chern curvature components

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{\ell}}}{\partial \bar{z}^j}. \tag{2.2}$$

The (first) Chern-Ricci form $Ric(\omega_g)$ of (X, ω_g) has components

$$R_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 \log \det(g)}{\partial z^i \partial \bar{z}^j}$$

and it is well-known that the Chern-Ricci form represents the first Chern class of the complex manifold X (up to a factor 2π). The Chern scalar curvature s of (X, ω_g) is defined as

$$s = g^{i\bar{j}} R_{i\bar{j}}. \tag{2.3}$$

For a Hermitian manifold (X, ω_g), we define the torsion tensor

$$T_{ij}^k = g^{k\bar{\ell}} \left(\frac{\partial g_{j\bar{\ell}}}{\partial z^i} - \frac{\partial g_{i\bar{\ell}}}{\partial z^j} \right). \tag{2.4}$$

By using elementary Bochner formulas (e.g. [23, Lemma 3.3], or [21, Lemma A.6]), we have

$$\bar{\partial}^* \omega = -\sqrt{-1} T_{ki}^k dz^i. \tag{2.5}$$

Indeed, we have $[\bar{\partial}^*, L] = \sqrt{-1}\,(\partial + \tau)$ with $\tau = [\Lambda, \partial\omega]$. When it acts on constant 1, we obtain

$$\begin{aligned} \bar{\partial}^* \omega = \sqrt{-1}\tau(1) &= \sqrt{-1}\Lambda\,(\partial\omega) \\ &= -\sqrt{-1} g^{k\bar{\ell}} \left(\frac{\partial g_{i\bar{\ell}}}{\partial z^k} - \frac{\partial g_{k\bar{\ell}}}{\partial z^i} \right) dz^i \\ &= -\sqrt{-1} T_{ki}^k dz^i. \end{aligned}$$

Let (X, ω) be a compact Hermitian manifold. (X, ω) has positive (resp. semi-positive) holomorphic sectional curvature, if for any nonzero vector $\xi = (\xi^1, \cdots, \xi^n)$,

$$R_{i\bar{j}k\bar{\ell}} \xi^i \bar{\xi}^j \xi^k \bar{\xi}^\ell > 0 \quad (\text{resp. } \geq 0).$$

3. Hermitian manifolds with semi-positive holomorphic sectional curvature

In this section, we discuss the relationship between the holomorphic sectional curvature and the Kodaira dimension of the ambient manifold. It is also well-known that, on Hermitian manifolds, there are many curvature notations and the curvature relations are more complicated than the relations in the Kähler case because of the non-vanishing of the torsion tensor (e.g.[21, 23]).

Theorem 3.1. *Let (X,ω) be a compact Hermitian manifold with semi-positive holomorphic sectional curvature. If the holomorphic sectional curvature is not identically zero, then X has Kodaira dimension $\kappa(X)=-\infty$.*

Proof. At a given point $p\in X$, the maximum holomorphic sectional curvature is defined to be

$$H_p := \max_{W\in T_p^{1,0}X, |W|=1} H(W),$$

where $H(W):=R(W,\overline{W},W,\overline{W})$. Since X is of finite dimension, the maximum can be attained. Suppose the holomorphic sectional curvature is not identically zero, i.e. $H_p>0$ for some $p\in X$. For any $q\in X$. We assume $g_{i\overline{j}}(q)=\delta_{ij}$. If $\dim_{\mathbb{C}} X=n$ and $[\xi^1,\cdots,\xi^n]$ are the homogeneous coordinates on $\mathbb{P}^{n-1}$, and ω_{FS} is the Fubini-Study metric of $\mathbb{P}^{n-1}$. At point q, we have the following well-known identity(e.g. [22, Lemma 4.1]):

$$\int_{\mathbb{P}^{n-1}} R_{i\overline{j}k\overline{\ell}}\frac{\xi^i\overline{\xi}^j\xi^k\overline{\xi}^\ell}{|\xi|^4}\omega_{FS}^{n-1} = R_{i\overline{j}k\overline{\ell}}\cdot\frac{\delta_{ij}\delta_{k\ell}+\delta_{i\ell}\delta_{kj}}{n(n+1)} = \frac{s+\widehat{s}}{n(n+1)}. \tag{3.1}$$

where s is the Chern scalar curvature of ω and $\widehat{s}$ is defined as

$$\widehat{s} = g^{i\overline{\ell}}g^{k\overline{j}}R_{i\overline{j}k\overline{\ell}}. \tag{3.2}$$

Hence if (X,ω) has semi-positive holomorphic sectional curvature, then $s+\widehat{s}$ is a non-negative function on X. On the other hand, at point $p\in X$, $s+\widehat{s}$ is strictly positive. Indeed, since $H_p>0$, there exists a nonzero vector $\xi\in T_p^{1,0}X$ such that $H(\xi)=R_{i\overline{j}k\overline{\ell}}\frac{\xi^i\overline{\xi}^j\xi^k\overline{\xi}^\ell}{|\xi|^4}>0$. By (3.1), the integrand is quasi-positive over $\mathbb{P}^{n-1}$, and so $s+\widehat{s}$ is strictly positive at $p\in X$. Note that in general if (X,ω) is not Kähler, s and $\widehat{s}$ are not the same. By [23, Section 4], we have the relation

$$s=\widehat{s}+\langle\overline{\partial}\overline{\partial}^*\omega,\omega\rangle. \tag{3.3}$$

Indeed, we compute

$$\begin{aligned} s-\widehat{s} &= g^{i\overline{j}}g^{k\overline{\ell}}\left(R_{i\overline{j}k\overline{\ell}}-R_{k\overline{j}i\overline{\ell}}\right) = g^{i\overline{j}}g^{k\overline{\ell}}\left(\nabla_{\overline{j}}\left(\frac{\partial g_{i\overline{\ell}}}{\partial z^k}-\frac{\partial g_{k\overline{\ell}}}{\partial z^i}\right)\right)\\ &= g^{i\overline{j}}\nabla_{\overline{j}}T_{ki}^k = g^{i\overline{j}}\frac{\partial T_{ki}^k}{\partial \overline{z}^j} = \langle\overline{\partial}\overline{\partial}^*\omega,\omega\rangle \end{aligned}$$

where we use formula (2.5) in the last identity. Therefore, we have

$$\int_X \widehat{s}\omega^n = \int_X s\omega^n - \int_X |\overline{\partial}^*\omega|^2\omega^n. \tag{3.4}$$

Next we use Gauduchon's conformal method ([12, 13], see also [3, 4]) to find a Hermitian metric $\widetilde{\omega}$ in the conformal class of ω such that $\widetilde{\omega}$ has positive Chern scalar curvature $\widetilde{s}$.

Let $\omega_G = f_0^{\frac{1}{n-1}}\omega$ be a Gauduchon metric (i.e. $\partial\overline{\partial}\omega_G^{n-1} = 0$) in the conformal class of ω for some strictly positive weight function $f_0 \in C^\infty(X)$ ([12, 13]). Let $s_G, \widehat{s}_G$ be the corresponding scalar curvatures with respect to the Gauduchon metric ω_G. Then we have

$$\begin{aligned}
\int_X s_G\omega_G^n &= -n\int_X \sqrt{-1}\partial\overline{\partial}\log\det(\omega_G)\wedge\omega_G^{n-1} \\
&= -n\int_X \left(\sqrt{-1}\partial\overline{\partial}\log\det(\omega) + \frac{n}{n-1}\sqrt{-1}\partial\overline{\partial}\log f_0\right)\wedge\omega_G^{n-1} \\
&= -n\int_X \sqrt{-1}\partial\overline{\partial}\log\det(\omega)\wedge\omega_G^{n-1} \\
&= -n\int_X f_0\sqrt{-1}\partial\overline{\partial}\log\det(\omega)\wedge\omega^{n-1} \\
&= \int_X f_0 s\omega^n, \qquad (3.5)
\end{aligned}$$

where we use the Stokes' theorem and the fact that ω_G is Gauduchon in the third identity. Similarly, by using the proof of formula (2.5), we have the relation

$$\begin{aligned}
\overline{\partial}_G^*\omega_G &= \sqrt{-1}\Lambda_G(\partial\omega_G) \\
&= \sqrt{-1}\Lambda(\partial\omega) - \sqrt{-1}f_0^{-\frac{1}{n-1}}\left(\frac{\partial f_0^{\frac{1}{n-1}}}{\partial z^k} - n\frac{\partial f_0^{\frac{1}{n-1}}}{\partial z^k}\right)dz^k \\
&= \overline{\partial}^*\omega + \sqrt{-1}\partial\log f_0.
\end{aligned}$$

Since ω_G is Gauduchon, we obtain

$$\begin{aligned}
\int_X \langle\overline{\partial\partial}_G^*\omega_G, \omega_G\rangle\omega_G^n &= n\int_X \overline{\partial\partial}_G^*\omega_G\wedge\omega_G^{n-1} = n\int_X \overline{\partial\partial}^*\omega\wedge\omega_G^{n-1} \\
&= \int_X f_0\langle\overline{\partial\partial}^*\omega, \omega\rangle\omega^n. \qquad (3.6)
\end{aligned}$$

By using a similar equation as (3.4) for $s_G, \widehat{s}_G$ and ω_G, we obtain

$$\begin{aligned}
\int_X \widehat{s}_G\omega_G^n &= \int_X s_G\omega_G^n - \int_X \langle\overline{\partial\partial}_G^*\omega_G, \omega_G\rangle\omega_G^n \\
&= \int_X f_0 s\omega^n - \int_X f_0\langle\overline{\partial\partial}^*\omega, \omega\rangle\omega^n = \int_X f_0\widehat{s}\omega^n. \qquad (3.7)
\end{aligned}$$

where we use equations (3.5), (3.6) in the second identity, and (3.3) in the third identity. Therefore, if $s + \widehat{s}$ is quasi-positive, we obtain

$$\begin{aligned}
\int_X s_G\omega_G^n &= \frac{\int_X(s_G + \widehat{s}_G)\omega_G^n}{2} + \frac{\int_X(s_G - \widehat{s}_G)\omega_G^n}{2} \\
&= \frac{\int_X(s_G + \widehat{s}_G)\omega_G^n}{2} + \frac{\|\overline{\partial}_G^*\omega_G\|^2}{2} = \frac{\int_X f_0(s+\widehat{s})\omega^n}{2} + \frac{\|\overline{\partial}_G^*\omega_G\|^2}{2} > 0 \qquad (3.8)
\end{aligned}$$

where the third equation follows from (3.5) and (3.7).

Next, there exists a Hermitian metric h on K_X^{-1} which is conformal to $\det(\omega_G)$ on K_X^{-1} such that the scalar curvature s_h of (K_X^{-1}, h) with respect to ω_G is a constant, and more precisely we have

$$s_h = -\mathrm{tr}_{\omega_G}\sqrt{-1}\partial\overline{\partial}\log h = \frac{\int_X s_G\omega_G^n}{\int_X \omega_G^n}.$$

Indeed, let $f \in C^\infty(X)$ be a strictly positive function satisfying

$$s_G - \mathrm{tr}_{\omega_G}\sqrt{-1}\partial\overline{\partial} f = \frac{\int_X s_G\omega_G^n}{\int_X \omega_G^n} \tag{3.9}$$

then $h = f\det(\omega_G)$ is the metric we need. Note that the existence of solutions to (3.9) is well-known by Hopf's lemma.

Finally, we deduce that the conformal metric

$$\widetilde{\omega} := f^{\frac{1}{n}} f_0^{\frac{1}{n-1}}\omega = f^{\frac{1}{n}}\omega_G$$

is a Hermitian metric with positive Chern scalar curvature. Indeed, the Chern scalar curvature $\widetilde{s}$ is,

$$\begin{aligned} \widetilde{s} &= -\mathrm{tr}_{\widetilde{\omega}}\sqrt{-1}\partial\overline{\partial}\log\det(\widetilde{\omega}^n) \\ &= -\mathrm{tr}_{\widetilde{\omega}}\sqrt{-1}\partial\overline{\partial}\log h \\ &= -f^{-\frac{1}{n}}\mathrm{tr}_{\omega_G}\sqrt{-1}\partial\overline{\partial}\log h \\ &= f^{-\frac{1}{n}}\frac{\int_X s_G\omega_G^n}{\int_X \omega_G^n} > 0. \end{aligned}$$

Hence, if $\sigma \in H^0(X, mK_X)$ for some positive integer m, by the standard Bochner formula with respect to the metric $\widetilde{\omega}$, one has

$$tr_{\widetilde{\omega}}\sqrt{-1}\partial\overline{\partial}|\sigma|^2_{\widetilde{\omega}} = |\nabla'\sigma|^2_{\widetilde{\omega}} + m\widetilde{s}\cdot|\sigma|^2_{\widetilde{\omega}} \tag{3.10}$$

where $|\bullet|_{\widetilde{\omega}}$ is the pointwise norm on mK_X induced by $\widetilde{\omega}$ and ∇' is the $(1,0)$ component of the Chern connection on mK_X. Since $\widetilde{s}$ is strictly positive, by maximum principle we have $|\sigma|^2_{\widetilde{\omega}} = 0$, i.e. $\sigma = 0$. Now we deduce the Kodaira dimension of X is $-\infty$. □

It is easy to see that, on a Kähler manifold (X,ω), if the total scalar curvature $\int_X s\omega^n$ is positive, then $\kappa(X) = -\infty$ (e.g. [32, Theorem 1] or [17, Theorem 1.1]). However, in general, it is not true for non-Kähler metrics which can be seen from the following example.

Example 3.2. Let $(\mathbb{T}^2,\omega)$ be a torus with the flat metric. For any non-constant real smooth function $f \in C^\infty(\mathbb{T}^2)$, the Hermitian metric $\omega_f = e^f\omega$ has strictly positive total Chern scalar curvature and $\kappa(\mathbb{T}^2) = 0$. Indeed, $\det(\omega_f) = e^{2f}\det(\omega)$ and

$$Ric(\omega_f) = -\sqrt{-1}\partial\overline{\partial}\log\det(\omega_f) = Ric(\omega) - 2\sqrt{-1}\partial\overline{\partial} f = -2\sqrt{-1}\partial\overline{\partial} f.$$

The total scalar curvature of ω_f is given by

$$\begin{aligned} \int s_f\cdot\omega_f^2 &= \int tr_{\omega_f}Ric(\omega_f)\cdot\omega_f^2 \\ &= 2\int Ric(\omega_f)\wedge\omega_f = -4\int\sqrt{-1}\partial\overline{\partial} f\wedge e^f\omega \\ &= 4\int\left(\sqrt{-1}\partial f\wedge\overline{\partial} f\right)e^f\omega \\ &= 4\|\partial f\|^2_{\omega_f} > 0 \end{aligned}$$

since f is not a constant function, where we use the Stokes' theorem in the fourth identity.

Note that, a special case of Theorem 3.1 is proved in [4] that when X is a surface or a threefold and (X,ω) has strictly positive holomorphic sectional curvature, then X has Kodaira dimension $-\infty$.

As an application of Theorem 3.1, we have

Corollary 3.3. *Let X be a Kodaira surface or a hyperelliptic surface. Then X has nef tangent bundle, but X does not admit a Hermitian metric with semi-positive holomorphic* sectional *curvature.*

Proof. It is well-known that the holomorphic tangent bundles of Kodaira surfaces or hyperelliptic surfaces are nef (e.g. [10] or [31]). On the other hand, if X is either a Kodaira surface or a hyperelliptic surface, then X has torsion canonical line bundle, i.e. $K_X^{\otimes m} = \mathcal{O}_X$ for some positive integer m ([6, p.244]). In particular, we have $\kappa(X) = 0$. Suppose X has a Hermitian metric ω with semi-positive holomorphic sectional curvature, by Theorem 3.1, (X,ω) has constant zero holomorphic sectional curvature. Then (X,ω) is a Kähler surface [5, Theorem 1]. Since all Kodaira surfaces are non-Kähler, we deduce that Kodaira surfaces can not admit Hermitian metrics with semi-positive holomorphic sectional curvature. Suppose (X,ω) is a hyperelliptic surface with constant zero holomorphic sectional curvature. So ω is a Kähler metric with constant zero holomorphic sectional curvature, and we deduce (X,ω) is flat since the curvature tensor is determined by the holomorphic sectional curvature. Indeed, for any $Y,Z\in T_p^{1,0}X$, expand

$$R(Y+\lambda Z,\overline{Y+\lambda Z},Y+\lambda Z,\overline{Y+\lambda Z})\equiv 0$$

into powers of λ and $\overline{\lambda}$. Using the Kähler symmetry, the $|\lambda|^2$ term gives $R(Y,\overline{Y},Z,\overline{Z}) = 0$. Now if we expand

$$R(Y+\lambda Z,\overline{Y+\lambda Z},A+\mu B,\overline{A+\mu B})\equiv 0$$

into powers of $\lambda,\overline{\lambda},\mu,\overline{\mu}$, the $\overline{\lambda\mu}$ term gives $R(Y,\overline{Z},A,\overline{B}) = 0$ for any $(1,0)$-vectors $Y,Z,A,B\in T_p^{1,0}X$. Since (X,ω) is flat, X is a complex parallelizable manifold (e.g. [11, Proposition 2.4] and [2]). However, it is proved in [27, Corollary 2] that a complex parallelizable manifold is Kähler if and only if it is a torus. This is a contradiction. $\square$

Let X be a complex manifold. X is said to be a complex Calabi-Yau manifold if $c_1(X) = 0$.

Corollary 3.4. *Let X be a compact Calabi-Yau manifold. If X admits a Kähler metric with semi-positive holomorphic sectional curvature, then X is a torus.*

Proof. Let X be a compact Kähler Calabi-Yau manifold, then it is well-known that (e.g. [25, Theorem 1.5]), K_X is a holomorphic torsion, i.e. there exists a positive integer m such that $K_X^{\otimes m} = \mathcal{O}_X$. In particular, $\kappa(X) = 0$. Suppose X has a smooth Kähler metric ω with semi-positive holomorphic sectional curvature, then by Theorem 3.1, X has constant zero holomorphic sectional curvature. As shown in Corollary 3.3, (X,ω) is flat and so it is a complex parallelizable manifold, i.e. X is a torus. $\square$

Remark 3.5. As shown in [31], the Hopf surface $H_{a,b}$ (and every diagonal Hopf manifold [23]) has a Hermitian metric with semi-positive holomorphic bisectional curvature. Since $b_2(H_{a,b}) = b_2(\mathbb{S}^1 \times \mathbb{S}^3) = 0$, we see $c_1(H_{a,b}) = 0$ and so $H_{a,b}$ is a non-Kähler Calabi-Yau manifold with semi-positive holomorphic sectional curvature.

Finally, we want to use the following well-known example to demonstrate that the positivity of the holomorphic sectional curvature can not imply the ampleness of the anti-canonical line bundle although the negativity of the holomorphic sectional curvature does imply the ampleness of the canonical line bundle (e.g. [26] or Theorem 1.6).

Example 3.6. Let Y be the Hirzebruch surface $Y := \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}\right)$ for $k \geq 2$ which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. It is known ([19] or [24, p.292], or [1]) that Y has a smooth Kähler metric with positive holomorphic sectional curvature. Next, we show K_Y^{-1} is not ample although K_Y^{-1} is known to be effective. Let $E := \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}$, $Y = \mathbb{P}(E^*)$ and $\mathcal{O}_Y(1)$ be the tautological line bundle of Y. The following adjunction formula is well known (e.g.[20, p.89])

$$K_Y = \mathcal{O}_Y(-2) \otimes \pi^*(K_{\mathbb{P}^1} \otimes \det E) \tag{3.11}$$

where π is the projection $Y = \mathbb{P}(E^*) \to \mathbb{P}^1$. In particular, we have

$$\mathcal{O}_Y(2) = K_Y^{-1} \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(k-2)). \tag{3.12}$$

Suppose K_Y^{-1} is ample, then $\mathcal{O}_Y(1)$ is aslo ample since $k \geq 2$. Therefore, by definition ([14]), $E = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}$ is an ample vector bundle. This is a contradiction.

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