AN EXTENSION OF A THEOREM OF WU-YAU

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Abstract

We show that a compact Kähler manifold with nonpositive holomorphic sectional curvature has nef canonical bundle. If the holomorphic sectional curvature is negative then it follows that the canonical bundle is ample, confirming a conjecture of Yau. The key ingredient is the recent solution of this conjecture in the projective case by Wu-Yau.

1. Introduction

The holomorphic sectional curvature of a Kähler manifold is by definition equal to the Riemannian sectional curvature of holomorphic planes in the tangent space, and it determines the whole curvature tensor. Despite this simple definition, its properties have remained rather mysterious. In this note, we study its relationship with Ricci curvature, and more precisely how negativity of the holomorphic sectional curvature affects the positivity of the canonical bundle.

Yau's Schwarz Lemma [**18**] implies that if a compact Kähler manifold (X, ω) has nonpositive holomorphic sectional curvature, then X does not contain any rational curve. If X is assumed to be projective, then thanks to Mori's Cone Theorem [**7**] it follows that K_X is nef (which means that $c_1(K_X) = -c_1(X)$ belongs to the closure of the Kähler cone), since if K_X is not nef then X contains a rational curve. We conclude that if X is projective and ω has nonpositive holomorphic sectional curvature then K_X is nef.

Our goal is to extend this result to all Kähler manifolds, not necessarily projective. Note that there are many non-projective Kähler manifolds with nonpositive holomorphic sectional curvature, for example a generic torus.

The Kodaira classification of Kähler surfaces shows that if K_X is not nef then again X contains a rational curve, and we are done. The Cone

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Theorem was very recently extended to Kähler threefolds by Höring-Peternell [**6**], and therefore in this case we can again conclude what we want. In the main result of this note, we prove this in all dimensions (bypassing Mori theory), using very recent ideas of Wu-Yau [**16**]:

Theorem 1.1. *Let (X, ω) be a compact Kähler manifold with nonpositive holomorphic sectional curvature. Then the canonical bundle K_X is nef.*

As a corollary, using the work of Wu-Yau [**16**], we confirm a conjecture of Yau (see e.g. [**16**]), which was very recently settled in the projective case by Wu-Yau [**16**]:

Corollary 1.2. *Let (X, ω) be a compact Kähler manifold with negative holomorphic sectional curvature. Then the canonical bundle K_X is ample.*

Indeed, Theorem 1.1 shows that K_X is nef, and we can then apply [**16**, Theorem 7] and conclude that K_X is ample.

This also gives a proof of another open problem posed by Yau [**14**, p.1313]:

Corollary 1.3. *Let (X, ω) be a compact Kähler manifold with negative holomorphic sectional curvature. Then X is projective.*

Using Yau's Theorem [**19**], we can rephrase Corollary 1.2 by saying that if there exists a Kähler metric with negative holomorphic sectional curvature, then there is a (possibly different) Kähler metric with negative Ricci curvature.

When X is projective, Corollary 1.2 was proved very recently by Wu-Yau [**16**], and we make use of some of their ideas in this note. Earlier work on this conjecture include [**13**] (where it is proved for Kähler surfaces), [**4**] (for projective threefolds), [**15**] (for projective manifolds with Picard number 1), [**5**] (for projective manifolds assuming the abundance conjecture) and finally [**16**] (for all projective manifolds). In this note we observe that a modification of the technique in [**16**], using also an argument by contradiction, can be used to prove Theorem 1.1 as well, and therefore get rid of the projectivity assumption.

Remark 1.4. Corollary 1.2 falls into the circle of ideas around Kobayashi hyperbolicity of complex manifolds, see e.g. [**3**, **8**, **16**] for more on this.

Remark 1.5. As in [**16**, Corollary 4], Theorem 1.1 implies that if (X, ω) be a compact Kähler manifold with nonpositive holomorphic sectional curvature, then the canonical bundle K_Y of every compact complex submanifold Y of X is nef. Similarly, if ω has negative holomorphic sectional curvature, then K_Y is ample.

Remark 1.6. It would be interesting to know whether Corollary 1.2 still valid if we only assume that the holomorphic sectional curvature is nonpositive and strictly negative at one point. This is shown to be true in [**15**] if X is projective with Picard number 1, and in [**5**] for projective surfaces.

Remark 1.7. If (X, ω) is a compact Hermitian manifold, then we can still define its holomorphic sectional curvature, using the Chern connection. Is the analog of Theorem 1.1 still true in this more general situation? While the analogous complex Monge-Ampère equation in the Hermitian case is solved in [**2**] (see also [**11**, **12**]), the argument with the Schwarz Lemma uses the Kähler condition for Royden's lemma (which requires the Kähler symmetries of the curvature tensor), and also to make sure that all the Hermitian Ricci curvatures are equal.

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2. Proof of the main theorem

In this section we give the proof of Theorem 1.1, by modifying the method used by Wu-Yau [**16**] to prove Corollary 1.2 in the projective case.

The following is the key lemma, from Wu-Yau [**16**] (cf. [**17**, **15**]), and it is an application of Yau's Schwarz Lemma [**18**] and a trick of Royden [**10**].

Lemma 2.1 ([**16**], Proposition 9)**.** *Let X be a compact Kähler manifold with two Kähler metrics $\omega, \hat{\omega}$, such that ω has holomorphic sectional curvature bounded above by a constant $-\kappa \leqslant 0$, and $\hat{\omega}$ satisfies*

$$\mathrm{Ric}(\hat{\omega}) \geqslant -\lambda\hat{\omega} + \nu\omega, \tag{2.1}$$

for some constants $\lambda, \nu > 0$. Then we have

$$\Delta_{\hat{\omega}} \log \mathrm{tr}_{\hat{\omega}}\omega \geqslant \left(\frac{n+1}{2n}\kappa + \frac{\nu}{n}\right)\mathrm{tr}_{\hat{\omega}}\omega - \lambda.$$

In particular, we have

$$\sup_X \mathrm{tr}_{\hat{\omega}}\omega \leqslant \frac{\lambda}{\frac{n+1}{2n}\kappa + \frac{\nu}{n}}.$$

Proof. Yau's Schwarz Lemma calculation [**18**] gives

$$\Delta_{\hat{\omega}} \log \mathrm{tr}_{\hat{\omega}}\omega \geqslant \frac{1}{\mathrm{tr}_{\hat{\omega}}\omega}\left(\hat{g}^{i\bar{\ell}}\hat{g}^{k\bar{j}}g_{k\bar{\ell}}\hat{R}_{i\bar{j}} - \hat{g}^{i\bar{j}}\hat{g}^{k\bar{\ell}}R_{i\bar{j}k\bar{\ell}}\right).$$

An observation of Royden [**10**, Lemma, p.552] gives

$$\hat{g}^{i\bar{j}}\hat{g}^{k\bar{\ell}}R_{i\bar{j}k\bar{\ell}} \leqslant -\frac{n+1}{2n}\kappa(\mathrm{tr}_{\hat{\omega}}\omega)^2,$$

while (2.1) says that

$$\hat{R}_{i\bar{j}} \geqslant -\lambda \hat{g}_{i\bar{j}} + \nu g_{i\bar{j}},$$

and so

$$\hat{g}^{i\bar{\ell}}\hat{g}^{k\bar{j}}g_{k\bar{\ell}}\hat{R}_{i\bar{j}} \geqslant -\lambda \mathrm{tr}_{\hat{\omega}}\omega + \nu \hat{g}^{i\bar{\ell}}\hat{g}^{k\bar{j}}g_{k\bar{\ell}}g_{i\bar{j}} \geqslant -\lambda \mathrm{tr}_{\hat{\omega}}\omega + \frac{\nu}{n}(\mathrm{tr}_{\hat{\omega}}\omega)^2.$$

q.e.d.

We now prove Theorem 1.1 by contradiction. If K_X is not nef, then there exists $\varepsilon_0 > 0$ such that the class $\varepsilon_0[\omega] - c_1(X)$ is nef but not Kähler. Then for every $\varepsilon > 0$ the class $(\varepsilon + \varepsilon_0)[\omega] - c_1(X)$ is Kähler, and so we can find a smooth function φ_ε such that

$$(\varepsilon + \varepsilon_0)\omega - \mathrm{Ric}(\omega) + \sqrt{-1}\partial\overline{\partial}\varphi_\varepsilon > 0.$$

By a theorem of Yau [**19**] and Aubin [**1**], we can find a smooth function ψ_ε such that

$$(\varepsilon + \varepsilon_0)\omega - \mathrm{Ric}(\omega) + \sqrt{-1}\partial\overline{\partial}(\varphi_\varepsilon + \psi_\varepsilon) > 0,$$

and

$$((\varepsilon + \varepsilon_0)\omega - \mathrm{Ric}(\omega) + \sqrt{-1}\partial\overline{\partial}(\varphi_\varepsilon + \psi_\varepsilon))^n = e^{\varphi_\varepsilon + \psi_\varepsilon}\omega^n.$$

We let $u_\varepsilon = \varphi_\varepsilon + \psi_\varepsilon$ and $\omega_\varepsilon = (\varepsilon + \varepsilon_0)\omega - \mathrm{Ric}(\omega) + \sqrt{-1}\partial\overline{\partial}(\varphi_\varepsilon + \psi_\varepsilon)$, so that we can write

$$\omega_\varepsilon^n = e^{u_\varepsilon}\omega^n. \tag{2.2}$$

Differentiating this, we see that

$$\mathrm{Ric}(\omega_\varepsilon) = \mathrm{Ric}(\omega) - \sqrt{-1}\partial\overline{\partial}u_\varepsilon = -\omega_\varepsilon + (\varepsilon + \varepsilon_0)\omega. \tag{2.3}$$

We may therefore apply Lemma 2.1, with $\kappa = 0, \lambda = 1, \nu = \varepsilon + \varepsilon_0$, and obtain

$$\sup_X \mathrm{tr}_{\omega_\varepsilon}\omega \leqslant \frac{n}{\varepsilon + \varepsilon_0}, \tag{2.4}$$

which is bounded uniformly independent of ε (as ε approaches zero). Furthermore, at any point $x \in X$ where u_ε achieves its maximum, we have that $((\varepsilon + \varepsilon_0)\omega - \mathrm{Ric}(\omega))(x) > 0$ and

$$e^{\sup_X u_\varepsilon} = e^{u_\varepsilon(x)} \leqslant \frac{((\varepsilon + \varepsilon_0)\omega - \mathrm{Ric}(\omega))^n}{\omega^n}(x) \leqslant C,$$

independent of ε small. This proves a uniform upper bound for u_ε, and hence we obtain

$$\sup_X \frac{\omega_\varepsilon^n}{\omega^n} \leqslant C. \tag{2.5}$$

Combining (2.4), (2.5) and the elementary inequality

$$\mathrm{tr}_\omega \omega_\varepsilon \leqslant \frac{1}{(n-1)!}(\mathrm{tr}_{\omega_\varepsilon}\omega)^{n-1}\frac{\omega_\varepsilon^n}{\omega^n},$$

we conclude that

$$\sup_X \mathrm{tr}_\omega \omega_\varepsilon \leqslant C, \tag{2.6}$$

and (2.4) and (2.6) together give

$$C^{-1}\omega \leqslant \omega_\varepsilon \leqslant C\omega. \tag{2.7}$$

Using (2.2) this implies that $\inf_X u_\varepsilon \geqslant -C$ as well. We claim that the following higher order estimates hold

$$\|\omega_\varepsilon\|_{C^k(X,\omega)} \leqslant C_k, \tag{2.8}$$

where C_k is independent of ε, for all $k \geqslant 0$. These are essentially standard (following the work of Yau [**19**]), but since the precise setting is not readily found in the literature (we have no control on our reference metrics $(\varepsilon+\varepsilon_0)\omega - \mathrm{Ric}(\omega) + \sqrt{-1}\partial\overline{\partial}\varphi_\varepsilon$ as $\varepsilon \to 0$), we provide a proof below.

Assuming first that we have (2.8), we can immediately conclude the proof of Theorem 1.1. Indeed, using (2.8) together with (2.7), with the Ascoli-Arzelà theorem and a diagonal argument, we obtain that there exists a sequence $\varepsilon_i \to 0$ such that ω_{ε_i} converge smoothly to a Kähler metric ω_0 which satisfies

$$[\omega_0] = \varepsilon_0[\omega] - c_1(X),$$

which is a contradiction to the fact that this class is not Kähler.

For the reader's convenience, we now give the proof of the higher-order estimates (2.8), following [**19**]. Let

$$S = |\nabla^\omega \omega_\varepsilon|^2_{\omega_\varepsilon},$$

where ∇^ω is the covariant derivative of ω. It is also equal to $S = |T|^2_{\omega_\varepsilon}$, where T is the tensor which is given by the difference of the Christoffel symbols of ω_ε and ω. Yau's C^3 calculation gives (cfr. [**9**, (2.44)])

$$\begin{aligned}\Delta_{\omega_\varepsilon} S &= |\nabla T|^2_{\omega_\varepsilon} + |\overline{\nabla} T|^2_{\omega_\varepsilon} - 2\mathrm{Re}\big(g_\varepsilon^{i\overline{p}} g_\varepsilon^{j\overline{q}} (g_\varepsilon)_{k\overline{\ell}} \overline{T^\ell_{pq}} (\nabla_i R^k_j - \nabla^{\overline{\ell}} R(\omega)_{i\overline{\ell}j}{}^k)\big) \\ &\quad + T^k_{ij}\overline{T^\ell_{pq}}\big(R^{i\overline{p}} g_\varepsilon^{j\overline{q}} (g_\varepsilon)_{k\overline{\ell}} + g_\varepsilon^{i\overline{p}} R^{j\overline{q}} (g_\varepsilon)_{k\overline{\ell}} - g_\varepsilon^{i\overline{p}} g_\varepsilon^{j\overline{q}} R_{k\overline{\ell}}\big),\end{aligned}$$

where ∇ denotes the covariant derivative of ω_ε, $R_{k\overline{\ell}}$ its Ricci curvature, $R(\omega)$ is the curvature tensor of ω, and we raise indices using ω_ε. Using (2.3) together with (2.7) we obtain

$$\Delta_{\omega_\varepsilon} S \geqslant -C_0 S - C.$$

But we also have

$$\begin{aligned}\Delta_{\omega_\varepsilon} \mathrm{tr}_\omega \omega_\varepsilon &= \Delta_\omega u_\varepsilon - R_\omega + g^{i\overline{\ell}}\, g_\varepsilon^{p\overline{j}} g_\varepsilon^{k\overline{q}} \nabla^\omega_i (g_\varepsilon)_{k\overline{j}} \nabla^\omega_{\overline{\ell}} (g_\varepsilon)_{p\overline{q}} + g^{i\overline{\ell}}\, g_\varepsilon^{p\overline{j}} (g_\varepsilon)_{k\overline{\ell}} R(\omega)_{p\overline{j}i}{}^k \\ &\geqslant C_1^{-1} S - C,\end{aligned}$$

where R_ω is the scalar curvature of ω, and so the maximum principle applied to $S + C_1(C_0+1)\mathrm{tr}_\omega\omega_\varepsilon$ gives $\sup_X S \leqslant C$, independent of ε

small. This implies that $\|\omega_\varepsilon\|_{C^1(X,\omega)} \leqslant C$, and then a standard bootstrap argument gives all the higher order estimates (2.8).

References

[1] T. Aubin *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) **102** (1978), no. 1, 63–95, MR0494932, Zbl 0374.53022.

[2] P. Cherrier *Équations de Monge-Ampère sur les variétés Hermitiennes compactes*, Bull. Sc. Math (2) **111** (1987), 343–385, MR0921559, Zbl 0629.58028.

[3] S. Diverio *Rational curves on Calabi-Yau threefolds and a conjecture of Oguiso*, Oberwolfach Rep. **9** (2012), no. 3, 2612–2615, MR3156735.

[4] G. Heier, S.Y.S. Lu, B. Wong *On the canonical line bundle and negative holomorphic sectional curvature*, Math. Res. Lett. **17** (2010), no. 6, 1101–1110, MR2729634, Zbl 1233.14005.

[5] G. Heier, S.Y.S. Lu, B. Wong *Kähler manifolds of semi-negative holomorphic sectional curvature*, J. Differential Geom. (in press)

[6] A. Höring, T. Peternell *Minimal models for Kähler threefolds*, Invent. Math. **203**

[18] S.-T. Yau *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), 197–204, MR0486659, Zbl 0424.53040.

[19] S.-T. Yau *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), no.3, 339–411, MR0480350, Zbl 0369.53059.

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