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RICCI CURVATURES ON HERMITIAN MANIFOLDS

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Abstract. In this paper, we introduce the first Aeppli-Chern class for complex manifolds and show that the $(1,1)$- component of the curvature 2-form of the Levi-Civita connection on the anti-canonical line bundle represents this class. We systematically investigate the relationship between a variety of Ricci curvatures on Hermitian manifolds and the background Riemannian manifolds. Moreover, we study non-Kähler Calabi-Yau manifolds by using the first Aeppli-Chern class and the Levi-Civita Ricci-flat metrics. In particular, we construct explicit Levi-Civita Ricci-flat metrics on Hopf manifolds $\mathbb{S}^{2n-1}\times\mathbb{S}^1$. We also construct a smooth family of Gauduchon metrics on a compact Hermitian manifold such that the metrics are in the same first Aeppli-Chern class, and their first Chern-Ricci curvatures are the same and nonnegative, but their Riemannian scalar curvatures are constant and vary smoothly between negative infinity and a positive number. In particular, it shows that Hermitian manifolds with nonnegative first Chern class can admit Hermitian metrics with strictly negative Riemannian scalar curvature.

Contents

1. Introduction

Let (M,h) be a Hermitian manifold and g the background Riemannian metric. It is well-known that, when (M,h) is not Kähler, the complexification of the real curvature tensor R is extremely complicated. Moreover, on the Hermitian holomorphic vector bundle $(T^{1,0}M,h)$, there are two typical connections: the (induced)

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Levi-Civita connection and the Chern connection. The curvature tensors of them are denoted by $\mathfrak{R}$ and Θ respectively. It is known that the complexified Riemannian curvature R, the Hermitian Levi-Civita curvature $\mathfrak{R}$ and the Chern curvature Θ are mutually different. It is also obvious that R is closely related to the Riemannian geometry of M, and Θ can characterize many complex geometric properties of M whereas $\mathfrak{R}$ can be viewed as a bridge between R and Θ, i.e. a bridge between Riemannian geometry and Hermitian geometry.

Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates centered at a point $p \in M$. We can compare the curvature tensors of $\mathfrak{R}$, Θ when restricted on the space $\Gamma(M, \Lambda^{1,1}T^*M \otimes End(T^{1,0}M))$. That is, we can find relations between $\mathfrak{R}_{i\bar{j}k\bar{\ell}}$ and $\Theta_{i\bar{j}k\bar{\ell}}$. We denoted by

$$\mathfrak{R}^{(1)} = \sqrt{-1}\mathfrak{R}^{(1)}_{i\bar{j}} dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}^{(1)}_{i\bar{j}} = h^{k\bar{\ell}}\mathfrak{R}_{i\bar{j}k\bar{\ell}},$$

$$\mathfrak{R}^{(2)} = \sqrt{-1}\mathfrak{R}^{(2)}_{i\bar{j}} dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}^{(2)}_{i\bar{j}} = h^{k\bar{\ell}}\mathfrak{R}_{k\bar{\ell}i\bar{j}},$$

$$\mathfrak{R}^{(3)} = \sqrt{-1}\mathfrak{R}^{(3)}_{i\bar{j}} dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}^{(3)}_{i\bar{j}} = h^{k\bar{\ell}}\mathfrak{R}_{i\bar{\ell}k\bar{j}},$$

and

$$\mathfrak{R}^{(4)} = \sqrt{-1}\mathfrak{R}^{(4)}_{i\bar{j}} dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}^{(4)}_{i\bar{j}} = h^{k\bar{\ell}}\mathfrak{R}_{k\bar{j}i\bar{\ell}}.$$

$\mathfrak{R}^{(1)}$ and $\mathfrak{R}^{(2)}$ are called the *first Levi-Civita Ricci curvature* and the *second Levi-Civita Ricci curvature* of $(T^{1,0}M, h)$ respectively; $\mathfrak{R}^{(3)}$and $\mathfrak{R}^{(4)}$ are the corresponding third and fourth Levi-Civita Ricci curvatures. Similarly, we can define the first Chern-Ricci curvature $\Theta^{(1)}$, the second Chern-Ricci curvature $\Theta^{(2)}$, the third and fourth Chern-Ricci curvatures $\Theta^{(3)}$ and $\Theta^{(4)}$ respectively. As shown in [41], $\mathfrak{R}^{(2)}$ and $\Theta^{(2)}$ are closely related to the geometry of M, for example, we can use them to study the cohomology groups and plurigenera of compact Hermitian manifolds. On the other hand, it is well-known that $\Theta^{(1)}$ represents the first Chern class $c_1(M) \in H^{1,1}_{\bar{\partial}}(M)$. However, in general, the first Levi-Civita Ricci form $\mathfrak{R}^{(1)}$ is not d-closed, and so it can not represent a class in $H^{1,1}_{\bar{\partial}}(M)$.

We introduce two cohomology groups to study the geometry of compact complex (especially, non-Kähler) manifolds, the Bott-Chern cohomology and the Aeppli cohomology:

$$H^{p,q}_{BC}(M) := \frac{\text{Ker} d \cap \Omega^{p,q}(M)}{\text{Im}\partial\bar{\partial} \cap \Omega^{p,q}(M)} \quad \text{and} \quad H^{p,q}_{A}(M) := \frac{\text{Ker}\partial\bar{\partial} \cap \Omega^{p,q}(M)}{\text{Im}\partial \cap \Omega^{p,q}(M) + \text{Im}\bar{\partial} \cap \Omega^{p,q}(M)}.$$

Suppose α is a d-closed (p,q)-form. We denote by $[\alpha]_{BC}$ and $[\alpha]_A$, the corresponding classes in $H^{p,q}_{BC}(M)$ and $H^{p,q}_{A}(M)$ respectively. Let $\text{Pic}(M)$ be the set of holomorphic line bundles over M. As similar as the first Chern class map $c_1 : \text{Pic}(M) \to H^{1,1}_{\bar{\partial}}(M)$, there is a **first Aeppli-Chern class** map

$$c_1^{AC} : \text{Pic}(M) \to H^{1,1}_A(M), \tag{1.1}$$

which can be described as follows.

Definition 1.1. Let $L \to M$ be a holomorphic line bundle over M. The first Aeppli-Chern class is defined as

$$c_1^{AC}(L) = \left[-\sqrt{-1}\partial\bar{\partial}\log h\right]_A \in H^{1,1}_A(M) \tag{1.2}$$

where h is an arbitrary smooth Hermitian metric on L. Note that $-\sqrt{-1}\partial\overline{\partial}\log h$ is the (local) curvature form Θ_h of the Hermitian line bundle (L,h). If we choose a different metric h', then $\Theta_{h'}-\Theta_h=\sqrt{-1}\partial\overline{\partial}\log\left(\frac{h}{h'}\right)$ is globally $\partial\overline{\partial}$-exact. Hence $c_1^{AC}(L)$ is well-defined in $H_A^{1,1}(M)$ and it is independent of the metric h. For a complex manifold M, $c_1^{AC}(M)$ is defined to be $c_1^{AC}(K_M^{-1})$ where K_M^{-1} is the anti-canonical line bundle $\wedge^n T^{1,0}M$.

Note that, for a Hermitian line bundle (L,h), the classes $c_1(L)$, $c_1^{BC}(L)$ and $c_1^{AC}(L)$ have the same $(1,1)$-form representative $\Theta^h=-\sqrt{-1}\partial\overline{\partial}\log h$ (in different classes).

On Kähler manifolds or Hermitian manifolds the first Chern classes and first Bott-Chern classes are well-studied in the literatures by using the Chern connection. In particular, the related Monge-Ampère type equations are extensively investigated since the celebrated work of Yau. However, the geometry of the Levi-Civita connection is not well understood in complex geometry although it has rich real Riemannian geometry structures.

1.1. **The complex geometry of the Levi-Civita connection.** In the first part of this paper, we study (non-Kähler) Hermitian manifolds by using Levi-Civita Ricci forms and Aeppli-Chern class $c_1^{AC}(M)$. At first, we establish the following result which is analogous to the classical result that the (first) Chern-Ricci curvature $\Theta^{(1)}$ represents the first Chern class $c_1(M)$.

Theorem 1.2. *On a compact Hermitian manifold (M,ω), the first Levi-Civita Ricci form $\mathfrak{R}^{(1)}$ represents the first Aeppli-Chern class $c_1^{AC}(M)$ in $H_A^{1,1}(M)$. More precisely,*

$$\mathfrak{R}^{(1)}=\Theta^{(1)}-\frac{1}{2}(\partial\partial^*\omega+\overline{\partial\partial}^*\omega). \tag{1.3}$$

In particular, we obtain

(1) *$\mathfrak{R}^{(1)}=\Theta^{(1)}$ if and only if $d^*\omega=0$, i.e. (M,ω) is a balanced manifold;*
(2) *if $\overline{\partial}\partial^*\omega=0$, then $\mathfrak{R}^{(1)}$ represents the real first Chern class $c_1(M)\in H^2_{dR}(M)$, i.e. $c_1(M)=c_1^{AC}(M)$ in $H^2_{dR}(M)$.*

We also show that, on complex manifolds supporting $\partial\overline{\partial}$-lemma (e.g. manifolds in Fujiki class $\mathscr{C}$ and in particular, Moishezon manifolds), the converse statement of (2) holds.

There is an important class of manifolds, so called Calabi-Yau manifolds which are extensively studied by mathematicians and also physicians. In this paper, a Calabi-Yau manifold is a complex manifold with $c_1(M)=0$ and we will focus on non-Kähler Calabi-Yau manifolds. There are many fundamental results on non-Kähler Hermitian manifolds with vanishing first Bott-Chern classes. They are always characterized by using the first Chern-Ricci curvature $\Theta^{(1)}$ and the related Monge-Ampère type equations (e.g.[19, 26, 52, 53, 54, 55, 56, 8]). For more details on this subject, we refer the reader to the nice survey paper [51].

Next, we make the following observation:

Corollary 1.3. *Let M be a complex manifold. Then*

$$c_1^{BC}(M)=0\Longrightarrow c_1(M)=0\Longrightarrow c_1^{AC}(M)=0.$$

Moreover, on a complex manifold satisfying the $\partial\overline{\partial}$-lemma,

$$c_1^{BC}(M) = 0 \Longleftrightarrow c_1(M) = 0 \Longleftrightarrow c_1^{AC}(M) = 0.$$

That means, it is very natural to study non-Kähler Calabi-Yau manifolds by using the first Aeppli-Chern class c_1^{AC} and the first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$.

Definition 1.4. A Hermitian metric ω on M is called *Levi-Civita Ricci-flat* if

$$\mathfrak{R}^{(1)}(\omega) = 0.$$

It is known that Hopf manifolds $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ are all non-Kähler Calabi-Yau manifolds ($n \geq 2$), i.e. $c_1(M) = 0$. However, there is no Chern Ricci-flat Hermitian metrics on M, i.e. there does **not** exist a Hermitian metric ω such that $\Theta^{(1)}(\omega) = 0$ since $c_1^{BC}(M) \neq 0$ (see Remark 6.1 for more details). On the contrary, we can construct explicit Levi-Civita Ricci-flat metrics on them.

Theorem 1.5. *Let $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ with $n \geq 2$ and ω_0 the canonical metric on M. The perturbed Hermitian metric*

$$\omega = \omega_0 - \frac{4}{n}\mathfrak{R}^{(1)}(\omega_0)$$

is Levi-Civita Ricci-flat, i.e. $\mathfrak{R}^{(1)}(\omega) = 0$.

Theorem 1.5 is an explicit example demonstrating that the zero first Aeppli-Chern class can imply the existence of Levi-Civita Ricci-flat Hermitian metric. This result is also anologous to the following two classical and general results. One is Yau's celebrated solution to Calabi's conjecture

Theorem 1.6 ([63])**.** *Let (M, ω) be a compact Kähler manifold. If the real $(1,1)$ form η represents the first Chern class $c_1(M)$, then there exists a smooth function $\varphi \in C^\infty(M)$ such that the Kähler metric $\widetilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$ has Ricci curvature η, i.e.*

$$Ric(\widetilde{\omega}) = \eta.$$

The other one is Tosatti and Weinkove's Hermitian analogue of Yau's fundamental result:

Theorem 1.7 ([52])**.** *Let (M, ω) be a compact Hermitian manifold. If the real $(1,1)$ form η represents the first Bott-Chern class $c_1^{BC}(M)$, then there exists a smooth function $\varphi \in C^\infty(M)$ such that the Hermitian metric $\widetilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$ has Chern-Ricci curvature η, i.e.*

$$\Theta^{(1)}(\widetilde{\omega}) = \eta.$$

As inspired by Theorem 1.5 and the fundamental Theorem 1.6 and Theorem 1.7, we propose the following problem.

Problem 1.8. Let M be a compact complex manifold. For a fixed Hermitian metric ω_0 on M, and a real $(1,1)$-form η representing $c_1^{AC}(M)$, does there exist a $(0,1)$-form γ such that the Hermitian metric $\omega = \omega_0 + \partial\gamma + \overline{\partial}\overline{\gamma}$ satisfies

$$\mathfrak{R}^{(1)}(\omega) = \eta?$$

In particular, if $c_1^{AC}(M) = 0$ (or $c_1(M) = 0$), does there exist a $(0,1)$-form γ such that the Hermitian metric $\omega = \omega_0 + \partial\gamma + \overline{\partial}\overline{\gamma}$ is Levi-Civita Ricci-flat, i.e. $\mathfrak{R}^{(1)}(\omega) = 0$?

Although the background PDE is not exactly the same as the standard Monge-Ampère equations, we hope similar methods could work.

The first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$ is closely related to possible symplectic structures on Hermitian manifolds. It is easy to see that, on a Hermitian manifold (M,ω), $\mathfrak{R}^{(1)}$ is the $(1,1)$-component of the curvature 2-form of the Levi-Civita connection on K_M^{-1}. Hence, if $\mathfrak{R}^{(1)}$ is strictly positive, it can induce a symplectic structure on M (see Theorem 3.21 and also [37]). Moreover, the symplectic structures thus obtained are not necessarily Kähler.

As applications of Theorem 1.2, we can characterize Hermitian manifolds by using the Levi-Civita Ricci curvature.

Theorem 1.9. *Let (M,h) be a compact Hermitian manifold. If the first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$ is quasi-positive, then the top intersection number $c_1^n(M)>0$. In particular, $H^2_{dR}(M)$, $H^{1,1}_{\overline{\partial}}(M), H^{1,1}_{BC}(M)$ and $H^{1,1}_A(M)$ are all non-zero.*

The Levi-Civita Ricci curvature and the Aeppli-Chern class are also closely related to the algebraic aspects of the anti-canonical line bundle. For example, on a projective manifold M, $c_1(M)$ and $c_1^{AC}(M)$ are "numerically" equivalent, i.e. for any irreducible curve γ in M,

$$c_1^{AC}(M)\cdot\gamma = c_1(M)\cdot\gamma.$$

In particular,

Corollary 1.10.
(1) *if $\mathfrak{R}^{(1)}$ is semi-positive, the anti-canonical line bundle K_M^{-1} is* nef*;*
(2) *if $\mathfrak{R}^{(1)}$ is quasi-positive, then K_M^{-1} is a* big *line bundle.*

It is not hard to see that if the Hermitian manifold (M,g) has positive constant Riemannian sectional curvature, then $\mathfrak{R}^{(1)}$ is positive. On the other hand, since the positivity condition is an open condition, $\mathfrak{R}^{(1)}$ is still positive in a small neighborhood of a positive constant sectional curvature metric. As an application of this observation, one can see the following result of Lebrun which is also observed in [5] and [50].

Corollary 1.11. *On $\mathbb{S}^6$, there is no orthogonal complex structure compatible with metrics in some small neighborhood of the round metric.*

1.2. **Curvature relations on Hermitian manifolds.** In the second part of this paper, we investigate the relations between various Ricci curvatures on Hermitian manifolds. As introduced above, on a Hermitian manifold (M,ω) there are several different types of Ricci curvatures:

(1) the Levi-Civita Ricci curvatures $\mathfrak{R}^{(1)}$, $\mathfrak{R}^{(2)}$, $\mathfrak{R}^{(3)}$, $\mathfrak{R}^{(4)}$;
(2) the Chern Ricci curvatures $\Theta^{(1)}$, $\Theta^{(2)}$, $\Theta^{(3)}$, $\Theta^{(4)}$;
(3) the Hermitian-Ricci curvature $Ric_H=\sqrt{-1}R_{i\overline{j}}dz^i\wedge d\overline{z}^j$ where $R_{i\overline{j}}=h^{k\overline{\ell}}R_{i\overline{j}k\overline{\ell}}$ (which is equal to $R^{(1)}$ and $R^{(2)}$), the third and fourth Hermitian-Ricci curvatures $R^{(3)}$ and $R^{(4)}$;
(4) the $(1,1)$-component of the complexified Riemannian Ricci curvature, $\mathscr{R}ic$.

If (M,ω) is Kähler, all Ricci curvatures are the same, but it is not true on general Hermitian manifolds. We shall explore explicit relations between them by using the Hermitian metric ω and its torsion T. We write them down with a reference curvature, e.g. $\Theta^{(1)}$, to the reader's convenience.

Theorem 1.12. *Let (M,ω) be a compact Hermitian manifold.*

(1) *The Levi-Civita Ricci curvatures are*

$$\mathfrak{R}^{(1)} = \Theta^{(1)} - \frac{1}{2}\left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right);$$

$$\mathfrak{R}^{(2)} = \Theta^{(1)} - \frac{1}{2}\left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right) - \frac{\sqrt{-1}}{4}T\circ\overline{T} + \frac{\sqrt{-1}}{4}T\boxdot\overline{T};$$

$$\mathfrak{R}^{(3)} = \Theta^{(1)} - \frac{1}{2}\left(\sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right) + \left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right)\right) + \frac{\sqrt{-1}}{4}T\boxdot\overline{T} + \frac{T([\partial^*\omega]^{\#})}{4};$$

$$\mathfrak{R}^{(4)} = \Theta^{(1)} - \frac{1}{2}\left(\sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right) + \left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right)\right) + \frac{\sqrt{-1}}{4}T\boxdot\overline{T} + \frac{\overline{T([\partial^*\omega]^{\#})}}{4},$$

where $(\partial^*\omega)^{\#}$ is the dual vector of the $(0,1)$-form $\partial^*\omega$.

(2) *The Chern-Ricci curvatures are*

$$\Theta^{(2)} = \Theta^{(1)} - \sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right) - \left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right) + \sqrt{-1}T\boxdot\overline{T};$$

$$\Theta^{(3)} = \Theta^{(1)} - \partial\partial^*\omega;$$

$$\Theta^{(4)} = \Theta^{(1)} - \overline{\partial\partial^*}\omega.$$

(3) *The Hermitian-Ricci curvatures are*

$$Ric_H = R^{(1)} = R^{(2)} = \Theta^{(1)} - \frac{1}{2}\left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right) - \frac{\sqrt{-1}}{4}T\circ\overline{T};$$

$$\begin{aligned} R^{(3)} = R^{(4)} \quad &= \quad \Theta^{(1)} - \frac{1}{2}\left(\sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right) + \left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right)\right) \\ &\quad + \frac{\sqrt{-1}}{4}T\boxdot\overline{T} + \frac{\overline{T([\partial^*\omega]^{\#})} + T([\partial^*\omega]^{\#})}{4}. \end{aligned}$$

(4) *The $(1,1)$-component of the complexified Riemannian Ricci curvature is*

$$\begin{aligned} \mathscr{R}ic \quad &= \quad \Theta^{(1)} - \sqrt{-1}(\Lambda\partial\overline{\partial}\omega) - \frac{1}{2}(\partial\partial^*\omega + \overline{\partial\partial^*}\omega) + \frac{\sqrt{-1}}{4}\left(2T\boxdot\overline{T} + T\circ\overline{T}\right) \\ &\quad + \frac{1}{2}\left(T([\partial^*\omega]^{\#}) + \overline{T([\partial^*\omega]^{\#})}\right). \end{aligned}$$

From these curvature relations, one can see clearly the geometry of many Hermitian manifolds with special metrics (e.g. $d^*\omega = 0$, $\partial\overline{\partial}\omega = 0$). In particular, these curvature relations may enlighten the study of various Hermitian Ricci flows (e.g. [47, 48, 49], [41], [26, 53, 54, 57, 27, 28]) by using the well-studied Hamilton's Ricci flow. Of course, it is also natural to define new Hermitian Ricci flows by certain Ricci curvatures with significant geometric meanings, for example, the first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$.

As straightforward consequences, from the Ricci curvature relations, we can also obtain relations between the corresponding scalar curvatures. It is known that the positive scalar curvature can characterize the geometry of the manifolds. In [64], Yau proved that, on a compact Kähler manifold (M,ω), if the total scalar curvature is positive, then all plurigenera $p_m(M)$ vanish, and so the Kodaira dimension of M is $-\infty$. Based on Yau's result, Heier-Wong ([31]) observed that on a projective manifold, if the total scalar curvature is positive, the manifold is uniruled, i.e. it is covered by rational curves. On a compact Hermitian manifold M, Gauduchon showed ([24]) that if the total Chern scalar curvature of a *Gauduchon metric* is positive, then $p_m(M) = 0$ and $\kappa(M) = -\infty$. On the other hand, the Riemannian scalar curvature on Riemannian manifolds is extensively studied. In particular, by Trudinger, Aubin and Schoen's solution to the Yamabe problem, it is well-known that every Riemannian metric is conformal to a metric with constant scalar curvature. To understand the relations between Riemannian geometry and Hermitian geometry, the following relation is of particular interest.

Corollary 1.13. *On a compact Hermitian manifold (M,ω), the Riemannian scalar curvature s and the Chern scalar curvature s_C are related by*

$$s = 2s_C + \left(\langle \partial\partial^*\omega + \overline{\partial\partial}^*\omega, \omega\rangle - 2|\partial^*\omega|^2\right) - \frac{1}{2}|T|^2. \tag{1.4}$$

Moreover, according to the different types of Ricci curvatures, there are different scalar curvatures and the following statements are equivalent:

(1) (M,ω) *is Kähler;*
(2) $\int s\cdot\omega^n = \int 2s_C\cdot\omega^n$*;*
(3) $\int s_C\cdot\omega^n = \int s_R\cdot\omega^n$*;*
(4) $\int s_C\cdot\omega^n = \int s_H\cdot\omega^n$*;*
(5) $\int s_H\cdot\omega^n = \int s_{LC}\cdot\omega^n$.

A similar formulation as (1.4) by using "Lee forms" is also observed by Gauduchon ([25]). See also [58, 59, 3, 11, 12, 1, 33, 23, 4, 42, 34] for some curvature relations on complex surfaces. For more scalar curvature relations, see Corollary 4.2, Remark 4.3, Corollary 4.4 and Corollary 4.5.

1.3. **Special metrics on Hermitian manifolds.** Finally, we study special metrics on Hermitian manifolds. In the following, we give precise examples of Hermitian manifolds on the relations between Ricci curvatures, Chern scalar curvatures and Riemannian scalar curvatures.

Theorem 1.14. *For any $n \geq 2$, there exists an n-dimensional compact complex manifold X with $c_1(X) \geq 0$, such that X admits three different* ***Gauduchon metrics*** *ω_1, ω_2 and ω_3 with the following properties.*

(1) $[\omega_1] = [\omega_2] = [\omega_3] \in H^{1,1}_A(X)$*;*
(2) *they have the same semi-positive Chern-Ricci curvature, i.e.*

$$\Theta^{(1)}(\omega_1) = \Theta^{(1)}(\omega_2) = \Theta^{(1)}(\omega_3) \geq 0;$$

(3) *they have constant positive Chern scalar curvatures.*

Moreover,

(1) ω_1 *has* **positive** *constant Riemannian scalar curvature;*
(2) ω_2 *has* **zero** *Riemannian scalar curvature;*
(3) ω_3 *has* **negative** *constant Riemannian scalar curvature.*

To the best of our knowledge, this is the first example to show that Hermitian manifolds with nonnegative first Chern class can admit Hermitian metrics with strictly negative constant Riemannian scalar curvature.

We observe the following identity on general compact Hermitian manifolds.

Proposition 1.15. *On a compact Hermitian manifold (M,ω), for any $1 \leq k \leq n-1$, we have*

$$\int \sqrt{-1}\partial\omega \wedge \overline{\partial}\omega \cdot \frac{\omega^{n-3}}{(n-3)!} = \|\partial^*\omega\|^2 - \|\partial\omega\|^2, \tag{1.5}$$

and

$$\int \sqrt{-1}\omega^{n-k-1} \wedge \partial\overline{\partial}\omega^k = (n-3)!k(n-k-1)\left(\|\partial\omega\|^2 - \|\partial^*\omega\|^2\right). \tag{1.6}$$

The form

$$\sqrt{-1}\omega^{n-k-1} \wedge \partial\overline{\partial}\omega^k$$

was firstly introduced in [19] by Fu-Wang-Wu to define a generalized Gauduchon's metric. More precisely, a metric ω satisfying $\sqrt{-1}\omega^{n-k-1} \wedge \partial\overline{\partial}\omega^k = 0$ for $1 \leq k \leq n-1$ is called a k-Gauduchon metric. The $(n-1)$-Gauduchon metric is the original Gauduchon metric. It is well-known that, the Hopf manifold $\mathbb{S}^{2n+1} \times S^1$ can not support a metric with $\partial\overline{\partial}\omega = 0$ (SKT) or $d^*\omega = 0$ (balanced metric). However, they showed in [19] that on $\mathbb{S}^5 \times \mathbb{S}^1$, there exists a 1-Gauduchon metric ω, i.e. $\omega \wedge \partial\overline{\partial}\omega = 0$.

A straightforward application of Proposition 1.15 is the following interesting fact:

Corollary 1.16. *If (M,ω) is k-Gauduchon for $1 \leq k \leq n-2$ and also balanced, then (M,ω) is Kähler.*

One can also get the following analogue in the "conformal" setting:

Corollary 1.17. *On a compact complex manifold, the following are equivalent:*

(1) *(M,ω) is conformally Kähler;*
(2) *(M,ω) is conformally k-Gauduchon for $1 \leq k \leq n-2$, and conformally balanced.*

In particular, the following are equivalent:

(3) *(M,ω) is Kähler;*
(4) *(M,ω) is k-Gauduchon for $1 \leq k \leq n-2$, and conformally balanced;*
(5) *(M,ω) is conformally balanced and $\Lambda^2(\partial\overline{\partial}\omega) = 0$.*

We need to point out that the equivalence of (3) and (4) is also proved by Ivanov and Papadopoulos in [34, Theorem 1.3] (see also [13, Proposition 2.4]). However, in [20], Fu-Wang-Wu proved that there exists a non-Kähler 3-fold which can support a 1-Gauduchon metric and a balanced metric simultaneously. By Corollary 1.16, they must be different Hermitian metrics. For more works on Hermitian manifolds with specials metrics, we refer the reader to [1, 2, 4, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 29, 23, 38, 41, 40, 61, 62, 43, 51, 44, 45, 46] and the references therein.

The paper is organized as follows: In Section 2, we introduce several basic terminologies which will be used frequently in the paper. In Section 3, we study the geometry of the Levi-Civita Ricci curvature and prove Theorem 1.2, Corollary 1.3, Theorem 1.9 and Corollary 1.11. In Section 4, we investigate a variety of Ricci curvature and scalar curvature relations over Hermitian manifolds and establish Theorem 1.12 and Corollary 1.13. In Section 5, we study some special metrics on Hermitian manifolds and prove Proposition 1.15, Corollary 1.16 and Corollary 1.17. In Section 6, we construct various precise Hermitian metrics on Hermitian manifolds and prove Theorem 1.5 and Theorem 1.14. In Section 7, we include some straightforward computations on Hermitian manifolds for the reader's convenience.

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2. Background materials

2.1. **Ricci curvature on almost Hermitian manifolds.** Let (M, g, ∇) be a $2n$-dimensional Riemannian manifold with Levi-Civita connection ∇. The tangent bundle of M is denoted by $T_{\mathbb{R}}M$. The curvature tensor of (M, g, ∇) is defined as

$$R(X,Y,Z,W) = g\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\right) \tag{2.1}$$

for any $X, Y, Z, W \in T_{\mathbb{R}}M$. Let $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes \mathbb{C}$ be the complexification of the tangent bundle $T_{\mathbb{R}}M$. We can extend the metric g, the Levi-Civita connection ∇ to $T_{\mathbb{C}}M$ in the $\mathbb{C}$-linear way. For instance, for any $a, b \in \mathbb{C}$ and $X, Y \in T_{\mathbb{C}}M$,

$$g(aX, bY) := ab \cdot g(X, Y). \tag{2.2}$$

Hence for any $a, b, c, d \in \mathbb{C}$ and $X, Y, Z, W \in T_{\mathbb{C}}M$,

$$R(aX, bY, cZ, dW) = abcd \cdot R(X, Y, Z, W). \tag{2.3}$$

Let (M, g, J) be an almost Hermitian manifold, i.e., $J : T_{\mathbb{R}}M \to T_{\mathbb{R}}M$ with $J^2 = -1$, and for any $X, Y \in T_{\mathbb{R}}M$, $g(JX, JY) = g(X, Y)$. We can also extend J to $T_{\mathbb{C}}M$ in the $\mathbb{C}$-linear way. Hence for any $X, Y \in T_{\mathbb{C}}M$, we still have

$$g(JX, JY) = g(X, Y). \tag{2.4}$$

Let $\{x^1, \cdots, x^n, x^{n+1}, \cdots x^{2n}\}$ be the local real coordinates on the almost Hermitian manifold (M, J, g). In order to use Einstein summations, we use the following convention:

$$\{x^i\} \quad \text{for} \quad 1 \leq i \leq n; \qquad \{x^I\} \quad \text{for} \quad n+1 \leq I \leq 2n \quad \text{and} \qquad I = i + n. \tag{2.5}$$

Moreover, we assume,

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^I} \quad \text{and} \quad J\left(\frac{\partial}{\partial x^I}\right) = -\frac{\partial}{\partial x^i}. \tag{2.6}$$

By using real coordinates $\{x^i, x^I\}$, the Riemannian metric is represented by

$$ds_g^2 = g_{i\ell}dx^i \otimes dx^\ell + g_{iL}dx^i \otimes dx^L + g_{I\ell}dx^I \otimes dx^\ell + g_{IJ}dx^I \otimes dx^J,$$

where the metric components $g_{i\ell}, g_{iL}, g_{I\ell}$ and g_{IL} are defined in the obvious way by using $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^I}, \frac{\partial}{\partial x^L}$. By the J-invariant property of the metric g, we have

$$g_{i\ell} = g_{IL}, \quad \text{and} \quad g_{iL} = g_{Li} = -g_{\ell I} = -g_{I\ell}. \tag{2.7}$$

We also use *complex coordinates* $\{z^i, \bar{z}^i\}_{i=1}^n$ on M:

$$z^i := x^i + \sqrt{-1}x^I, \quad \bar{z}^i := x^i - \sqrt{-1}x^I. \tag{2.8}$$

(Note that if the almost complex structure J is integrable, $\{z^i\}_{i=1}^n$ are the local holomorphic coordinates.) We define, for $1 \leq i \leq n$,

$$dz^i := dx^i + \sqrt{-1}dx^I, \quad d\bar{z}^i := dx^i - \sqrt{-1}dx^I \tag{2.9}$$

and

$$\frac{\partial}{\partial z^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial x^I}\right), \quad \frac{\partial}{\partial \overline{z}^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial x^I}\right). \tag{2.10}$$

Therefore,

$$\frac{\partial}{\partial x^i} = \frac{\partial}{\partial z^i} + \frac{\partial}{\partial \overline{z}^i}, \quad \frac{\partial}{\partial x^I} = \sqrt{-1}\left(\frac{\partial}{\partial z^i} - \frac{\partial}{\partial \overline{z}^i}\right). \tag{2.11}$$

By the $\mathbb{C}$-linear extension,

$$J\left(\frac{\partial}{\partial z^i}\right) = \sqrt{-1}\frac{\partial}{\partial z^i}, \quad J\left(\frac{\partial}{\partial \overline{z}^i}\right) = -\sqrt{-1}\frac{\partial}{\partial \overline{z}^i}. \tag{2.12}$$

Let's define a Hermitian form $h : T_{\mathbb{C}}M \times T_{\mathbb{C}}M \to \mathbb{C}$ by

$$h(X,Y) := g(X,Y), \qquad X, Y \in T_{\mathbb{C}}M. \tag{2.13}$$

By J-invariant property of g,

$$h_{ij} := h\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = 0, \quad \text{and} \quad h_{\overline{i}\overline{j}} := h\left(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^j}\right) = 0 \tag{2.14}$$

and

$$h_{i\overline{j}} := h\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}\right) = \frac{1}{2}\left(g_{ij} + \sqrt{-1}g_{iJ}\right). \tag{2.15}$$

It is obvious that $(h_{i\overline{j}})$ is a positive Hermitian matrix. Here, we always use the convention $h_{\overline{j}i} = h_{i\overline{j}}$ since g is symmetric over $T_{\mathbb{C}}M$. The (transposed) inverse matrix of $(h_{i\overline{j}})$ is denoted by $(h^{i\overline{j}})$, i.e. $h^{i\overline{\ell}} \cdot h_{k\overline{\ell}} = \delta^i_k$. One can also show that,

$$h^{i\overline{j}} = 2\left(g^{ij} - \sqrt{-1}g^{iJ}\right) \tag{2.16}$$

From the definition equation (2.13), we see the following well-known relation on $T_{\mathbb{C}}M$:

$$ds_h^2 = \frac{1}{2}ds_g^2 - \frac{\sqrt{-1}}{2}\omega \tag{2.17}$$

where ω is the fundamental 2-form associated to the J-invariant metric g:

$$\omega(X,Y) = g(JX,Y). \tag{2.18}$$

In local complex coordinates,

$$\omega = \sqrt{-1}h_{i\overline{j}}dz^i \wedge d\overline{z}^j. \tag{2.19}$$

In the following, we shall use the components of the complexified curvature tensor R, for example,

$$R_{i\overline{j}k\overline{\ell}} := R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \overline{z}^\ell}\right), \tag{2.20}$$

and in particular we use the following notation for the complexified curvature tensor:

$$R_{ijk\ell} := R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^\ell}\right). \tag{2.21}$$

It is obvious that, the components of ($\mathbb{C}$-linear) complexified curvature tensor have the same properties as the components of the real curvature tensor. We list some properties of $R_{i\overline{j}k\overline{\ell}}$ for examples:

$$R_{i\overline{j}k\ell} = -R_{\overline{j}ik\ell}, \quad R_{i\overline{j}k\overline{\ell}} = R_{k\overline{\ell}i\overline{j}}, \tag{2.22}$$

and in particular, the (first) Bianchi identity holds:

$$R_{i\bar{j}k\bar{\ell}} + R_{ik\bar{\ell}\bar{j}} + R_{i\bar{\ell}\bar{j}k} = 0. \tag{2.23}$$

Definition 2.1. Let $\{e_i\}_{i=1}^{2n}$ be a local orthonormal basis of $(T_{\mathbb{R}}M, g)$, the Riemannian Ricci curvature of (M, g) is

$$Ric(X, Y) := \sum_{i=1}^{2n} R(e_i, X, Y, e_i), \tag{2.24}$$

and the corresponding Riemannian scalar curvature is

$$s = \sum_{j=1}^{2n} Ric(e_i, e_i). \tag{2.25}$$

Lemma 2.2. *On an almost Hermitian manifold (M, h), the Riemannian Ricci curvature of the Riemannian manifold (M, g) satisfies*

$$Ric(X, Y) = h^{i\bar{\ell}}\left[R\left(\frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial \bar{z}^\ell}\right) + R\left(\frac{\partial}{\partial z^i}, Y, X, \frac{\partial}{\partial \bar{z}^\ell}\right)\right] \tag{2.26}$$

for any $X, Y \in T_{\mathbb{R}}M$. The Riemannian scalar curvature is

$$s = 2h^{i\bar{j}}h^{k\bar{\ell}}\left(2R_{i\bar{\ell}k\bar{j}} - R_{i\bar{j}k\bar{\ell}}\right). \tag{2.27}$$

Proof. See Lemma 7.1 in the Appendix. □

In order to formulate the curvature relations more effectively, we introduce new curvature notations as following:

Definition 2.3. The Riemannian Ricci tensor can also be extended to $T_{\mathbb{C}}M$, and we denote the associated $(1,1)$-form component by

$$\mathscr{R}ic = \sqrt{-1}\mathscr{R}_{i\bar{j}}dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathscr{R}_{i\bar{j}} := Ric\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right). \tag{2.28}$$

The *Hermitian-Ricci curvature* of the complexified curvature tensor is

$$Ric_H = \sqrt{-1}R_{i\bar{j}}dz^i \wedge d\bar{z}^j, \quad \text{with} \quad R_{i\bar{j}} := h^{k\bar{\ell}}R_{i\bar{j}k\bar{\ell}}. \tag{2.29}$$

The corresponding *Hermitian scalar curvature* of h is given by

$$s_H = h^{i\bar{j}}R_{i\bar{j}}. \tag{2.30}$$

We also define the *Riemannian type scalar curvature* as

$$s_R = h^{i\bar{\ell}}h^{k\bar{j}}R_{i\bar{j}k\bar{\ell}}. \tag{2.31}$$

It is obvious that $s_H \neq s_R$ in general. Similarly, we can define the third and fourth Hermitian-Ricci curvatures $R^{(3)}$ and $R^{(4)}$ respectively,

$$R^{(3)} = \sqrt{-1}R^{(3)}_{i\bar{j}}dz^i \wedge d\bar{z}^j \quad \text{with} \quad R^{(3)}_{i\bar{j}} = h^{k\bar{\ell}}R_{i\bar{\ell}k\bar{j}},$$

and

$$R^{(4)} = \sqrt{-1}R^{(4)}_{i\bar{j}}dz^i \wedge d\bar{z}^j \quad \text{with} \quad R^{(4)}_{i\bar{j}} = h^{k\bar{\ell}}R_{k\bar{j}i\bar{\ell}}.$$

It is easy to see that $R^{(3)} = R^{(4)}$.

Corollary 2.4. *On an almost Hermitian manifold* (M,h)*, we have*

$$\mathscr{R}_{i\bar{j}} = 2\left(h^{k\bar{\ell}}R_{k\bar{j}i\bar{\ell}}\right) - R_{i\bar{j}} \tag{2.32}$$

and

$$s = 2h^{i\bar{j}}\mathscr{R}_{i\bar{j}} = 4s_R - 2s_H. \tag{2.33}$$

Proof. (2.32) follows from (2.26) and the Bianchi identity (2.23). (2.33) follows from (2.27). □

2.2. **Curvatures on Hermitian manifolds.** Let (M,h,J) be an almost Hermitian manifold. The Nijenhuis tensor $N_J : \Gamma(M,T_{\mathbb{R}}M)\times\Gamma(M,T_{\mathbb{R}}M)\to\Gamma(M,T_{\mathbb{R}}M)$ is defined as

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]. \tag{2.34}$$

The almost complex structure J is called *integrable* if $N_J\equiv 0$ and then we call (M,g,J) a Hermitian manifold. By Newlander-Nirenberg's theorem, there exists a real coordinate system $\{x^i,x^I\}$ such that $z^i = x^i+\sqrt{-1}x^I$ are local holomorphic coordinates on M. Moreover, we have $T_{\mathbb{C}}M = T^{1,0}M\oplus T^{0,1}M$ where

$$T^{1,0}M = span_{\mathbb{C}}\left\{\frac{\partial}{\partial z^1},\cdots,\frac{\partial}{\partial z^n}\right\} \quad \text{and} \quad T^{0,1}M = span_{\mathbb{C}}\left\{\frac{\partial}{\partial \bar{z}^1},\cdots,\frac{\partial}{\partial \bar{z}^n}\right\}.$$

Let φ be a (p,q)-form on (M,g), and

$$\varphi = \frac{1}{p!q!}\sum_{i_1,\cdots,i_p,j_1,\cdots,j_q}\varphi_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}}dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\overline{z}^{j_1}\wedge\cdots\wedge d\overline{z}^{j_q}, \tag{2.35}$$

where $\varphi_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}}$ is skew symmetric in $i_1,\cdots,i_p$ and also skew symmetric in $j_1,\cdots,j_q$. The local inner product is defined as

$$|\varphi|^2 = \langle\varphi,\varphi\rangle = \frac{1}{p!q!}h^{i_1\overline{\ell_1}}\cdots h^{i_p\overline{\ell_p}}h^{k_1\overline{j_1}}\cdots h^{k_q\overline{j_q}}\varphi_{i_1\cdots i_p\overline{j_1}\cdots\overline{j_q}}\cdot\overline{\varphi_{\ell_1\cdots\ell_p\overline{k_1}\cdots\overline{k_q}}}. \tag{2.36}$$

The norm on $\Omega^{p,q}(M)$ is

$$\|\varphi\|^2 = (\varphi,\varphi) = \int\langle\varphi,\varphi\rangle\frac{\omega^n}{n!}. \tag{2.37}$$

It is well-known that there exists a real isometry $*:\Omega^{p,q}(M)\to\Omega^{n-q,n-p}(M)$ such that

$$(\varphi,\psi) = \int\varphi\wedge *\overline{\psi}, \tag{2.38}$$

for $\varphi,\psi\in\Omega^{p,q}(M)$.

3. Geometry of the Levi-Civita Ricci curvature

3.1. **The Levi-Civita connection and Chern connection on $(T^{1,0}M,h)$.**

3.1.1. *The induced Levi-Civita connection on* $(T^{1,0}M, h)$. Since $T^{1,0}M$ is a subbundle of $T_{\mathbb{C}}M$, there is an induced connection $\widehat{\nabla}$ on $T^{1,0}M$ given by

$$\widehat{\nabla} = \pi \circ \nabla : \Gamma(M, T^{1,0}M) \xrightarrow{\nabla} \Gamma(M, T_{\mathbb{C}}M \otimes T_{\mathbb{C}}M) \xrightarrow{\pi} \Gamma(M, T_{\mathbb{C}}M \otimes T^{1,0}M). \tag{3.1}$$

Moreover, $\widehat{\nabla}$ is a metric connection on the Hermitian holomorphic vector bundle $(T^{1,0}M, h)$ and it is determined by the relations

$$\widehat{\nabla}_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^k} := \Gamma_{ik}^p \frac{\partial}{\partial z^p} \quad \text{and} \quad \widehat{\nabla}_{\frac{\partial}{\partial \bar{z}^j}} \frac{\partial}{\partial z^k} := \Gamma_{\bar{j}k}^p \frac{\partial}{\partial z^p} \tag{3.2}$$

where

$$\Gamma_{ij}^k = \frac{1}{2} h^{k\bar{\ell}} \left(\frac{\partial h_{j\bar{\ell}}}{\partial z^i} + \frac{\partial h_{i\bar{\ell}}}{\partial z^j} \right), \quad \text{and} \quad \Gamma_{\bar{i}j}^k = \frac{1}{2} h^{k\bar{\ell}} \left(\frac{\partial h_{j\bar{\ell}}}{\partial \bar{z}^i} - \frac{\partial h_{j\bar{i}}}{\partial \bar{z}^\ell} \right). \tag{3.3}$$

The curvature tensor $\mathfrak{R} \in \Gamma(M, \Lambda^2 T_{\mathbb{C}}M \otimes T^{*1,0}M \otimes T^{1,0}M)$ of $\widehat{\nabla}$ is given by

$$\mathfrak{R}(X,Y)s = \widehat{\nabla}_X \widehat{\nabla}_Y s - \widehat{\nabla}_Y \widehat{\nabla}_X s - \widehat{\nabla}_{[X,Y]} s \tag{3.4}$$

for any $X, Y \in T_{\mathbb{C}}M$ and $s \in T^{1,0}M$. The curvature tensor $\mathfrak{R}$ has components

$$\mathfrak{R}_{i\bar{j}k}^\ell = -\left(\frac{\partial \Gamma_{ik}^\ell}{\partial \bar{z}^j} - \frac{\partial \Gamma_{\bar{j}k}^\ell}{\partial z^i} + \Gamma_{ik}^s \Gamma_{\bar{j}s}^\ell - \Gamma_{\bar{j}k}^s \Gamma_{si}^\ell \right) \quad \text{and} \quad \mathfrak{R}_{\bar{j}ik}^\ell = -\mathfrak{R}_{i\bar{j}k}^\ell; \tag{3.5}$$

$$\mathfrak{R}_{ijk}^\ell = -\left(\frac{\partial \Gamma_{ik}^\ell}{\partial z^j} - \frac{\partial \Gamma_{jk}^\ell}{\partial z^i} + \Gamma_{ik}^s \Gamma_{sj}^\ell - \Gamma_{jk}^s \Gamma_{si}^\ell \right); \tag{3.6}$$

and

$$\mathfrak{R}_{\bar{i}\bar{j}k}^\ell = -\left(\frac{\partial \Gamma_{\bar{i}k}^\ell}{\partial \bar{z}^j} - \frac{\partial \Gamma_{\bar{j}k}^\ell}{\partial \bar{z}^i} + \Gamma_{\bar{i}k}^s \Gamma_{\bar{j}s}^\ell - \Gamma_{\bar{j}k}^s \Gamma_{\bar{i}s}^\ell \right). \tag{3.7}$$

With respect to the Hermitian metric h on $T^{1,0}M$, we use the convention

$$\mathfrak{R}_{\bullet\bullet k\bar{\ell}} := \sum_{s=1}^n \mathfrak{R}_{\bullet\bullet k}^s h_{s\bar{\ell}}. \tag{3.8}$$

Corollary 3.1 ([41, Proposition 2.1])**.** *We have the following relations:*

$$R_{ijk}^\ell = \mathfrak{R}_{ijk}^\ell, \quad R_{\bar{i}\bar{j}k}^\ell = \mathfrak{R}_{\bar{i}\bar{j}k}^\ell,$$

and

$$R_{i\bar{j}k}^\ell = -\left(\frac{\partial \Gamma_{ik}^\ell}{\partial \bar{z}^j} - \frac{\partial \Gamma_{\bar{j}k}^\ell}{\partial z^i} + \Gamma_{ik}^s \Gamma_{\bar{j}s}^\ell - \Gamma_{\bar{j}k}^s \Gamma_{si}^\ell - \Gamma_{\bar{s}i}^\ell \cdot \overline{\Gamma_{\bar{k}j}^s} \right) = \mathfrak{R}_{i\bar{j}k}^\ell + \Gamma_{\bar{s}i}^\ell \cdot \overline{\Gamma_{\bar{k}j}^s}. \tag{3.9}$$

Next, we define Ricci curvatures and scalar curvatures for $(T^{1,0}M, h, \widehat{\nabla})$.

Definition 3.2. The *first Levi-Civita Ricci curvature* of the Hermitian vector bundle $\left(T^{1,0}M, h, \widehat{\nabla}\right)$ is

$$\mathfrak{R}^{(1)} = \sqrt{-1} \mathfrak{R}_{i\bar{j}}^{(1)} dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}_{i\bar{j}}^{(1)} = h^{k\bar{\ell}} \mathfrak{R}_{i\bar{j}k\bar{\ell}} \tag{3.10}$$

and the *second Levi-Civita Ricci curvature* of it is

$$\mathfrak{R}^{(2)} = \sqrt{-1} \mathfrak{R}_{i\bar{j}}^{(2)} dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}_{i\bar{j}}^{(2)} = h^{k\bar{\ell}} \mathfrak{R}_{k\bar{\ell}i\bar{j}}. \tag{3.11}$$

The *Levi-Civita scalar curvature* of $\widehat{\nabla}$ on $T^{1,0}M$ is denoted by

$$s_{LC} = h^{i\bar{j}}h^{k\bar{\ell}}\mathfrak{R}_{i\bar{j}k\bar{\ell}}. \tag{3.12}$$

Similarly, we can define $\mathfrak{R}^{(3)}$ and $\mathfrak{R}^{(4)}$ as

$$\mathfrak{R}^{(3)} = \sqrt{-1}\mathfrak{R}^{(3)}_{i\bar{j}}dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}^{(3)}_{i\bar{j}} = h^{k\bar{\ell}}\mathfrak{R}_{i\bar{\ell}k\bar{j}},$$

and

$$\mathfrak{R}^{(4)} = \sqrt{-1}\mathfrak{R}^{(4)}_{i\bar{j}}dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}^{(4)}_{i\bar{j}} = h^{k\bar{\ell}}\mathfrak{R}_{k\bar{j}i\bar{\ell}}.$$

3.1.2. *Curvature of the Chern connection on* $(T^{1,0}M, h)$. On the Hermitian holomorphic vector bundle $(T^{1,0}M, h)$, the Chern connection ∇^{Ch} is the unique connection which is compatible with the complex structure and also the Hermitian metric. The curvature tensor of ∇^{Ch} is denoted by Θ and its curvature components are

$$\Theta_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + h^{p\bar{q}}\frac{\partial h_{p\bar{\ell}}}{\partial \bar{z}^j}\frac{\partial h_{k\bar{q}}}{\partial z^i}. \tag{3.13}$$

It is well-known that the *(first) Chern Ricci curvature*

$$\Theta^{(1)} := \sqrt{-1}\Theta^{(1)}_{i\bar{j}}dz^i \wedge d\bar{z}^j \tag{3.14}$$

represents the first Chern class $c_1(M)$ of M where

$$\Theta^{(1)}_{i\bar{j}} = h^{k\bar{\ell}}\Theta_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 \log\det(h_{k\bar{\ell}})}{\partial z^i \partial \bar{z}^j}. \tag{3.15}$$

The *second Chern Ricci curvature* $\Theta^{(2)} = \sqrt{-1}\Theta^{(2)}_{i\bar{j}}dz^i \wedge d\bar{z}^j$ with components

$$\Theta^{(2)}_{i\bar{j}} = h^{k\bar{\ell}}\Theta_{k\bar{\ell}i\bar{j}}. \tag{3.16}$$

The *Chern scalar curvature* s_C of the Chern curvature tensor Θ is defined by

$$s_C = h^{i\bar{j}}h^{k\bar{\ell}}\Theta_{i\bar{j}k\bar{\ell}}. \tag{3.17}$$

Similarly, we can define $\Theta^{(3)}$ and $\Theta^{(4)}$.

The (first) Chern-Ricci curvature $\Theta^{(1)}$ represents the first Chern class $c_1(M)$, but in general, the first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$ does not represent a class in $H^2_{dR}(M)$ or $H^{1,1}_{\bar{\partial}}(M)$, since it is not d-closed. We shall explore more geometric properties of $\mathfrak{R}^{(1)}$ in the following sections.

3.2. **Elementary computations on Hermitian manifolds.** In this subsection, we recall some elementary and well-known computational lemmas on Hermitian manifolds.

Lemma 3.3. *Let* (M, h) *be a compact Hermitian manifold and* $\omega = \sqrt{-1}h_{i\bar{j}}dz^i \wedge d\bar{z}^j$.

$$\partial^*\omega = -\sqrt{-1}\Lambda\left(\bar{\partial}\omega\right) = -2\sqrt{-1}\Gamma^k_{\bar{j}k}d\bar{z}^j \quad \text{and} \quad \bar{\partial}^*\omega = \sqrt{-1}\Lambda\left(\partial\omega\right) = 2\sqrt{-1}\overline{\Gamma^k_{\bar{i}k}}dz^i. \tag{3.18}$$

Proof. By the well-known Bochner formula (e.g. [41]),

$$[\bar{\partial}^*, L] = \sqrt{-1}\left(\partial + \tau\right)$$

where $\tau = [\Lambda, \partial\omega]$, we see $\bar{\partial}^*\omega = \sqrt{-1}\Lambda\left(\partial\omega\right) = 2\sqrt{-1}\overline{\Gamma^k_{\bar{i}k}}dz^i$. □

Lemma 3.4. *Let (M, h, ω) be a Hermitian manifold. For any $p \in M$, there exist local holomorphic "normal coordinates" $\{z^i\}$ centered at p such that*

$$h_{i\bar{j}}(p) = \delta_{ij} \quad \textit{and} \quad \Gamma_{ij}^k(p) = 0. \tag{3.19}$$

In particular, at p, we have

$$\Gamma_{\bar{j}i}^{\bar{k}} = \frac{\partial h_{i\bar{k}}}{\partial \bar{z}^j} = -\frac{\partial h_{i\bar{j}}}{\partial \bar{z}^k}. \tag{3.20}$$

Let T be the torsion tensor of the Hermitian metric ω, i.e.

$$T_{ij}^k = h^{k\bar{\ell}}\left(\frac{\partial h_{j\bar{\ell}}}{\partial z^i} - \frac{\partial h_{i\bar{\ell}}}{\partial z^j}\right). \tag{3.21}$$

In the following, we shall use the conventions:

$$T \boxdot \overline{T} := h^{p\bar{q}} h_{k\bar{\ell}} T_{ip}^k \cdot \overline{T_{jq}^\ell} dz^i \wedge d\bar{z}^j, \quad \text{and} \quad T \circ \overline{T} := h^{p\bar{q}} h^{s\bar{t}} h_{k\bar{j}} h_{i\bar{\ell}} T_{sp}^k \cdot \overline{T_{tq}^\ell} dz^i \wedge d\bar{z}^j. \tag{3.22}$$

It is obvious that the $(1,1)$-forms $T \circ \overline{T}$ and $T \boxdot \overline{T}$ are not the same.

Lemma 3.5. *At a fixed point p with "normal coordinates" (3.19), we have*

$$(T \boxdot \overline{T})_{i\bar{j}} = T_{ip}^k \cdot \overline{T_{jp}^k} = 4 \sum_{p,k} \frac{\partial h_{p\bar{k}}}{\partial z^i} \cdot \frac{\partial h_{k\bar{p}}}{\partial \bar{z}^j} \tag{3.23}$$

and

$$(T \circ \overline{T})_{i\bar{j}} = T_{pq}^j \cdot \overline{T_{pq}^i} = 4 \sum_{p,q} \frac{\partial h_{q\bar{j}}}{\partial z^p} \cdot \frac{\partial h_{i\bar{q}}}{\partial \bar{z}^p}. \tag{3.24}$$

Moreover

$$tr_\omega\left(\sqrt{-1} T \boxdot \overline{T}\right) = tr_\omega\left(\sqrt{-1} T \circ \overline{T}\right) = |T|^2.$$

Lemma 3.6. *At a fixed point p with "normal coordinates" (3.19), we have the $(1,1)$ form*

$$T((\partial^*\omega)^\#) = -4\sqrt{-1} \frac{\partial h_{i\bar{j}}}{\partial z^s} \frac{\partial h_{\ell\bar{\ell}}}{\partial \bar{z}^s} dz^i \wedge d\bar{z}^j, \tag{3.25}$$

where $(\partial^*\omega)^\#$ is the dual vector of the $(0,1)$-form $\partial^*\omega$.

Proof. Since $\partial^*\omega = -2\sqrt{-1}\Gamma_{\bar{s}\ell}^{\ell} d\bar{z}^s$, the corresponding $(1,0)$ type vector field is

$$(\partial^*\omega)^\# = -2\sqrt{-1} h^{i\bar{s}} \Gamma_{\bar{s}\ell}^{\ell} \frac{\partial}{\partial z^i}. \tag{3.26}$$

The $(1,1)$ form

$$T((\partial^*\omega)^\#) = -2\sqrt{-1} h_{k\bar{j}} T_{pi}^k \left(h^{p\bar{s}} \Gamma_{\bar{s}\ell}^{\ell}\right) dz^i \wedge d\bar{z}^j = -4\sqrt{-1} \sum_{\ell,s,i,j} \frac{\partial h_{i\bar{j}}}{\partial z^s} \frac{\partial h_{\ell\bar{\ell}}}{\partial \bar{z}^s} dz^i \wedge d\bar{z}^j. \tag{3.27}$$

□

Lemma 3.7. *At a fixed point p with "normal coordinates" (3.19), we have*

$$\partial\partial^*\omega = \sqrt{-1}(\partial\partial^*\omega)_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

where

$$(\partial\partial^*\omega)_{i\bar{j}} = \sum_q \left(\frac{\partial^2 h_{q\bar{j}}}{\partial z^i \partial \bar{z}^q} - \frac{\partial^2 h_{q\bar{q}}}{\partial z^i \partial \bar{z}^j}\right) + \frac{1}{2}(T \boxdot \overline{T})_{i\bar{j}}. \tag{3.28}$$

Similarly,

$$\overline{\partial\partial}^*\omega=\sqrt{-1}(\overline{\partial\partial}^*\omega)_{i\bar{j}}dz^i\wedge d\bar{z}^j$$

where

$$(\overline{\partial\partial}^*\omega)_{i\bar{j}}=\sum_q\left(\frac{\partial^2 h_{i\bar{q}}}{\partial z^q\partial\bar{z}^j}-\frac{\partial^2 h_{q\bar{q}}}{\partial z^i\partial\bar{z}^j}\right)+\frac{1}{2}(T\boxdot\overline{T})_{i\bar{j}}. \tag{3.29}$$

Proof. Since

$$\partial^*\omega=-2\sqrt{-1}\Gamma_{\bar{s}\ell}^{\ell}d\bar{z}^s=-\sqrt{-1}h^{\ell\bar{q}}\left(\frac{\partial h_{\ell\bar{q}}}{\partial\bar{z}^s}-\frac{\partial h_{\ell\bar{s}}}{\partial\bar{z}^q}\right)d\bar{z}^s,$$

we have

$$\begin{aligned}\partial\partial^*\omega &= -\sqrt{-1}h^{\ell\bar{q}}\left(\frac{\partial^2 h_{\ell\bar{q}}}{\partial z^i\partial\bar{z}^s}-\frac{\partial^2 h_{\ell\bar{s}}}{\partial z^i\partial\bar{z}^q}\right)dz^i\wedge d\bar{z}^s\\
&\quad -\sqrt{-1}\frac{\partial h^{\ell\bar{q}}}{\partial z^i}\left(\frac{\partial h_{\ell\bar{q}}}{\partial\bar{z}^s}-\frac{\partial h_{\ell\bar{s}}}{\partial\bar{z}^q}\right)dz^i\wedge d\bar{z}^s\\
&= \sqrt{-1}\sum_q\left(\frac{\partial^2 h_{q\bar{s}}}{\partial z^i\partial\bar{z}^q}-\frac{\partial^2 h_{q\bar{q}}}{\partial z^i\partial\bar{z}^s}\right)dz^i\wedge d\bar{z}^s\\
&\quad +2\sqrt{-1}\sum_{q,\ell}\frac{\partial h_{q\bar{\ell}}}{\partial z^i}\frac{\partial h_{\ell\bar{q}}}{\partial\bar{z}^s}dz^i\wedge d\bar{z}^s\\
&= \sqrt{-1}\sum_q\left(\frac{\partial^2 h_{q\bar{s}}}{\partial z^i\partial\bar{z}^q}-\frac{\partial^2 h_{q\bar{q}}}{\partial z^i\partial\bar{z}^s}\right)dz^i\wedge d\bar{z}^s+\frac{\sqrt{-1}}{2}T\boxdot\overline{T}.\end{aligned}$$

□

The following lemmas follow from straightforward computations.

Lemma 3.8. *At a fixed point p with "normal coordinates" (3.19), we have*

$$\mathfrak{R}_{i\bar{j}k\bar{\ell}}=-\frac{1}{2}\left(\frac{\partial^2 h_{i\bar{\ell}}}{\partial z^k\partial\bar{z}^j}+\frac{\partial^2 h_{k\bar{j}}}{\partial z^i\partial\bar{z}^\ell}\right)-\sum_q\frac{\partial h_{q\bar{\ell}}}{\partial z^i}\frac{\partial h_{k\bar{q}}}{\partial\bar{z}^j}, \tag{3.30}$$

$$\begin{aligned}R_{i\bar{j}k\bar{\ell}} &= \mathfrak{R}_{i\bar{j}k\bar{\ell}}-\sum_q\frac{\partial h_{q\bar{j}}}{\partial z^k}\frac{\partial h_{i\bar{q}}}{\partial\bar{z}^\ell}\\
&= -\frac{1}{2}\left(\frac{\partial^2 h_{i\bar{\ell}}}{\partial z^k\partial\bar{z}^j}+\frac{\partial^2 h_{k\bar{j}}}{\partial z^i\partial\bar{z}^\ell}\right)-\sum_q\left(\frac{\partial h_{q\bar{\ell}}}{\partial z^i}\frac{\partial h_{k\bar{q}}}{\partial\bar{z}^j}+\frac{\partial h_{q\bar{j}}}{\partial z^k}\frac{\partial h_{i\bar{q}}}{\partial\bar{z}^\ell}\right)\end{aligned} \tag{3.31}$$

and

$$\Theta_{i\bar{j}k\bar{\ell}}=-\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i\partial\bar{z}^j}+\sum_q\frac{\partial h_{q\bar{\ell}}}{\partial\bar{z}^j}\frac{\partial h_{k\bar{q}}}{\partial z^i}. \tag{3.32}$$

Lemma 3.9. *At a fixed point p with "normal coordinates" (3.19), we have*

$$\mathfrak{R}^{(1)}_{i\bar{j}}=\sum_k\mathfrak{R}_{i\bar{j}k\bar{k}}=-\frac{1}{2}\sum_k\left(\frac{\partial^2 h_{i\bar{k}}}{\partial z^k\partial\bar{z}^j}+\frac{\partial^2 h_{k\bar{j}}}{\partial z^i\partial\bar{z}^k}\right)-\frac{1}{4}(T\boxdot\overline{T})_{i\bar{j}}; \tag{3.33}$$

$$\mathfrak{R}^{(2)}_{i\bar{j}}=\sum_k\mathfrak{R}_{k\bar{k}i\bar{j}}=-\frac{1}{2}\sum_k\left(\frac{\partial^2 h_{i\bar{k}}}{\partial z^k\partial\bar{z}^j}+\frac{\partial^2 h_{k\bar{j}}}{\partial z^i\partial\bar{z}^k}\right)-\frac{1}{4}(T\circ T)_{i\bar{j}}; \tag{3.34}$$

$$\mathfrak{R}^{(3)}_{i\bar j}=\sum_k \mathfrak{R}_{i\bar k k\bar j}=-\frac12\sum_k\left(\frac{\partial^2 h_{k\bar k}}{\partial z^i\partial\bar z^j}+\frac{\partial^2 h_{i\bar j}}{\partial z^k\partial\bar z^k}\right)+\frac14(T(\partial^*\omega)^\#)_{i\bar j};\tag{3.35}$$

$$\mathfrak{R}^{(4)}_{i\bar j}=\sum_k \mathfrak{R}_{k\bar j i\bar k}=-\frac12\sum_k\left(\frac{\partial^2 h_{k\bar k}}{\partial z^i\partial\bar z^j}+\frac{\partial^2 h_{i\bar j}}{\partial z^k\partial\bar z^k}\right)+\frac14\overline{(T(\partial^*\omega)^\#)}_{i\bar j}.\tag{3.36}$$

Lemma 3.10. *At a fixed point p with "normal coordinates" (3.19), we have*

$$\begin{aligned}R^{(1)}_{k\bar\ell}=R^{(2)}_{k\bar\ell}&=-\frac12\sum_s\left(\frac{\partial^2 h_{s\bar\ell}}{\partial z^k\partial\bar z^s}+\frac{\partial^2 h_{k\bar s}}{\partial z^s\partial\bar z^\ell}\right)-\sum_{q,s}\left(\frac{\partial h_{q\bar\ell}}{\partial z^s}\frac{\partial h_{k\bar q}}{\partial\bar z^s}+\frac{\partial h_{k\bar q}}{\partial z^s}\frac{\partial h_{q\bar\ell}}{\partial\bar z^s}\right)\\&=-\frac12\sum_s\left(\frac{\partial^2 h_{s\bar\ell}}{\partial z^k\partial\bar z^s}+\frac{\partial^2 h_{k\bar s}}{\partial z^s\partial\bar z^\ell}\right)-\frac{(T\circ\overline{T})_{k\bar\ell}+(T\boxdot\overline{T})_{k\bar\ell}}{4};\end{aligned}\tag{3.37}$$

$$\begin{aligned}R^{(3)}_{k\bar\ell}=R^{(4)}_{k\bar\ell}&=-\frac12\sum_s\left(\frac{\partial^2 h_{k\bar\ell}}{\partial z^s\partial\bar z^s}+\frac{\partial^2 h_{s\bar s}}{\partial z^k\partial\bar z^\ell}\right)-\sum_{q,s}\left(\frac{\partial h_{q\bar\ell}}{\partial z^k}\frac{\partial h_{s\bar q}}{\partial\bar z^s}+\frac{\partial h_{q\bar s}}{\partial z^s}\frac{\partial h_{k\bar q}}{\partial\bar z^\ell}\right)\\&=-\frac12\sum_s\left(\frac{\partial^2 h_{k\bar\ell}}{\partial z^s\partial\bar z^s}+\frac{\partial^2 h_{s\bar s}}{\partial z^k\partial\bar z^\ell}\right)+\frac14(T(\partial^*\omega)^\#)_{i\bar j}+\frac14\overline{(T(\partial^*\omega)^\#)}_{i\bar j}.\end{aligned}\tag{3.38}$$

Lemma 3.11. *At a fixed point p with "normal coordinates" (3.19), we have*

$$\Theta^{(1)}_{i\bar j}=-\sum_q\frac{\partial^2 h_{q\bar q}}{\partial z^i\partial\bar z^j}+\frac14(T\boxdot\overline{T})_{i\bar j};\tag{3.39}$$

$$\Theta^{(2)}_{i\bar j}=-\sum_q\frac{\partial^2 h_{i\bar j}}{\partial z^q\partial\bar z^q}+\frac14(T\boxdot\overline{T})_{i\bar j};\tag{3.40}$$

$$\Theta^{(3)}_{i\bar j}=-\sum_q\frac{\partial^2 h_{q\bar j}}{\partial z^i\partial\bar z^q}-\frac14(T\boxdot\overline{T})_{i\bar j};\tag{3.41}$$

$$\Theta^{(4)}_{i\bar j}=-\sum_q\frac{\partial^2 h_{i\bar q}}{\partial z^q\partial\bar z^j}-\frac14(T\boxdot\overline{T})_{i\bar j}.\tag{3.42}$$

Lemma 3.12. *At a fixed point p with "normal coordinates" (3.19), the complexified Ricci curvature is*

$$\begin{aligned}\mathscr{R}_{k\bar\ell}&=\frac12\sum_s\left(\frac{\partial^2 h_{s\bar\ell}}{\partial z^k\partial\bar z^s}+\frac{\partial^2 h_{k\bar s}}{\partial z^s\partial\bar z^\ell}\right)-\sum_s\left(\frac{\partial^2 h_{k\bar\ell}}{\partial z^s\partial\bar z^s}+\frac{\partial^2 h_{s\bar s}}{\partial z^k\partial\bar z^\ell}\right)\\&+\sum_{q,s}\left(\frac{\partial h_{q\bar\ell}}{\partial z^s}\frac{\partial h_{k\bar q}}{\partial\bar z^s}+\frac{\partial h_{k\bar q}}{\partial z^s}\frac{\partial h_{q\bar\ell}}{\partial\bar z^s}\right)-2\sum_{q,s}\left(\frac{\partial h_{q\bar\ell}}{\partial z^k}\frac{\partial h_{s\bar q}}{\partial\bar z^s}+\frac{\partial h_{q\bar s}}{\partial z^s}\frac{\partial h_{k\bar q}}{\partial\bar z^\ell}\right)\end{aligned}\tag{3.43}$$

$$\begin{aligned}&=\frac12\sum_s\left(\frac{\partial^2 h_{s\bar\ell}}{\partial z^k\partial\bar z^s}+\frac{\partial^2 h_{k\bar s}}{\partial z^s\partial\bar z^\ell}\right)-\sum_s\left(\frac{\partial^2 h_{k\bar\ell}}{\partial z^s\partial\bar z^s}+\frac{\partial^2 h_{s\bar s}}{\partial z^k\partial\bar z^\ell}\right)\\&+\frac{(T\circ\overline{T})_{k\bar\ell}+(T\boxdot\overline{T})_{k\bar\ell}}{4}+\frac12(T(\partial^*\omega)^\#)_{i\bar j}+\frac12\overline{(T(\partial^*\omega)^\#)}_{i\bar j}.\end{aligned}\tag{3.44}$$

3.3. **Geometry of the first Levi-Civita Ricci curvature.**

Definition 3.13. Let M be a compact complex manifold. A Hermitian metric ω on M is called *balanced*, if $d^*\omega = 0$. ω is called *conformally balanced*, if there exists a smooth function $\varphi : M \to \mathbb{R}$ and a balanced metric ω_B such that $\omega = e^{\varphi}\omega_B$.

Theorem 3.14. *On a compact Hermitian manifold (M,ω), the first Levi-Civita Ricci form $\mathfrak{R}^{(1)}$ represents the first Aeppli-Chern class $c_1^{AC}(M)$ in $H_A^{1,1}(M)$. More precisely,*

$$\mathfrak{R}^{(1)} = \Theta^{(1)} - \frac{1}{2}(\partial\partial^*\omega + \overline{\partial\partial}^*\omega). \tag{3.45}$$

Moreover,

1. $\mathfrak{R}^{(1)}$ *is d-closed if and only if $\partial\overline{\partial\partial}^*\omega = 0$;*
2. *if $\overline{\partial}\partial^*\omega = 0$, then $\mathfrak{R}^{(1)}$ represents the real first Chern class $c_1(M) \in H^2_{dR}(M)$, i.e. $c_1(M) = c_1^{AC}(M)$ in $H^2_{dR}(M)$.;*
3. *if ω is conformally balanced, then $\mathfrak{R}^{(1)}$ represents the first Chern class $c_1(M) \in H^{1,1}_{\overline{\partial}}(M)$ and also the first Bott-Chern class $c_1^{BC}(M) \in H^{1,1}_{BC}(M)$;*
4. $\mathfrak{R}^{(1)} = \Theta^{(1)}$ *if and only if $d^*\omega = 0$, i.e. (M,ω) is a balanced manifold.*

Proof. By formulas (3.39), (3.28), (3.29) and (3.33), we have

$$\begin{aligned}
& \Theta^{(1)}_{i\bar{j}} - \frac{1}{2}(\partial\partial^*\omega + \overline{\partial\partial}^*\omega)_{i\bar{j}} \\
= & -\sum_q \frac{\partial^2 h_{q\bar{q}}}{\partial z^i \partial \bar{z}^j} + \frac{1}{4}(T \boxdot \overline{T})_{i\bar{j}} \\
& -\frac{1}{2}\left(\sum_q \left(\frac{\partial^2 h_{i\bar{q}}}{\partial z^q \partial \bar{z}^j} + \frac{\partial^2 h_{q\bar{j}}}{\partial z^i \partial \bar{z}^q} - 2\frac{\partial^2 h_{q\bar{q}}}{\partial z^i \partial \bar{z}^j}\right) + (T \boxdot \overline{T})_{i\bar{j}}\right) \\
= & -\frac{1}{2}\sum_q \left(\frac{\partial^2 h_{i\bar{q}}}{\partial z^q \partial \bar{z}^j} + \frac{\partial^2 h_{q\bar{j}}}{\partial z^i \partial \bar{z}^q}\right) - \frac{1}{4}(T \boxdot \overline{T})_{i\bar{j}} \\
= & \ \mathfrak{R}^{(1)}_{i\bar{j}}.
\end{aligned}$$

By (3.45), we see $\mathfrak{R}^{(1)}$ represents the first Aeppli-Chern class $c_1^{AC}(M)$, i.e. $[\mathfrak{R}^{(1)}] = [\Theta^{(1)}]$ as classes in $H_A^{1,1}(M)$.

Next we prove the properties of $\mathfrak{R}^{(1)}$.

(1). By (3.45) again, $d\mathfrak{R}^{(1)} = -\frac{1}{2}\left(\overline{\partial}\partial\partial^*\omega + \partial\overline{\partial\partial}^*\omega\right)$. By degree reasons, $d\mathfrak{R}^{(1)} = 0$ if and only if $\partial\overline{\partial\partial}^*\omega = 0$.

(2). If $\overline{\partial}\partial^*\omega = 0$, we have

$$\mathfrak{R}^{(1)} = \Theta^{(1)} - \frac{1}{2}(\partial\partial^*\omega + \overline{\partial\partial}^*\omega) = \Theta^{(1)} - \frac{1}{2}dd^*\omega.$$

Hence $\left[\mathfrak{R}^{(1)}\right] = \left[\Theta^{(1)}\right] \in H^2_{dR}(M)$.

(3). If ω is conformally balanced, there exists a smooth function f and a balanced metric ω_f such that $\omega_f = e^f\omega$. We denote by an extra index f the corresponding quantities with respect to the new metric ω_f. The Christoffel symbols of ω_f are

$$(\Gamma_f)^k_{\bar{i}j} = \frac{1}{2}e^{-f}g^{k\bar{\ell}}\left(\frac{\partial(e^f g_{j\bar{\ell}})}{\partial \bar{z}^i} - \frac{\partial(e^f g_{j\bar{i}})}{\partial \bar{z}^\ell}\right) = \Gamma^k_{\bar{i}j} + \frac{1}{2}\left(\delta_{jk}f_{\bar{i}} - g^{k\bar{\ell}}g_{j\bar{i}}f_{\bar{\ell}}\right).$$

In particular,

$$(\Gamma_f)_{\bar{i}k}^{k} = \Gamma_{\bar{i}k}^{k} + \frac{n-1}{2} f_{\bar{i}}.$$

By (3.18), we obtain

$$\overline{\partial}_f^* \omega_f = \overline{\partial}^* \omega + \sqrt{-1}(n-1)\partial f. \tag{3.46}$$

Therefore,

$$\overline{\partial}\overline{\partial}_f^* \omega_f = \overline{\partial}\overline{\partial}^* \omega - (n-1)\sqrt{-1}\partial\overline{\partial} f \quad \text{and} \quad \partial\overline{\partial}_f^* \omega_f = \partial\overline{\partial}^* \omega. \tag{3.47}$$

Since ω_f is balanced, i.e. $\overline{\partial}_f^* \omega_f = 0$, we obtain

$$\partial\partial^* \omega + \overline{\partial}\overline{\partial}^* \omega = 2(n-1)\sqrt{-1}\partial\overline{\partial} f.$$

Hence, $\mathfrak{R}^{(1)} = \Theta^{(1)} - (n-1)\sqrt{-1}\partial\overline{\partial} f$, i.e. $\left[\mathfrak{R}^{(1)}\right] = \left[\Theta^{(1)}\right] \in H_{BC}^{1,1}(M)$. Hence, $\mathfrak{R}^{(1)}$ represents the first Chern class $c_1(M) \in H_{\overline{\partial}}^{1,1}(M)$ and also the first Bott-Chern class $c_1^{BC}(M) \in H_{BC}^{1,1}(M)$.

(4). $\mathfrak{R}^{(1)} = \Theta^{(1)}$ if and only if $\partial\partial^* \omega + \overline{\partial}\overline{\partial}^* \omega = 0$. By pairing with ω, we see the latter is equivalent to $d^* \omega = 0$. $\square$

Example 3.15. In this example, we shall construct a Hermitian metric with strictly positive $\mathfrak{R}^{(1)}$, but $\Theta^{(1)}$ is not strictly positive. Let M be a Fano manifold with complex dimension $n \geq 2$. By Yau's theorem, there exists a Kähler metric ω on M such that $\mathfrak{R}_\omega^{(1)} = \Theta_\omega^{(1)} > 0$. For any smooth function φ, we define $\omega_t = e^{t\varphi}\omega$. By (3.15), one can see

$$\Theta_{\omega_t}^{(1)} = \Theta_\omega^{(1)} - nt\sqrt{-1}\partial\overline{\partial}\varphi. \tag{3.48}$$

Hence, by (3.45) and (3.47),

$$\mathfrak{R}_{\omega_t}^{(1)} = \mathfrak{R}_\omega^{(1)} - \sqrt{-1}t\partial\overline{\partial}\varphi. \tag{3.49}$$

Let $t_0 = \sup\{t > 0 \mid \Theta_{\omega_t}^{(1)} = \Theta_\omega^{(1)} - nt\sqrt{-1}\partial\overline{\partial}\varphi \geq 0\}$, and $t_1 := \frac{3}{2}t_0$, then

$$\mathfrak{R}_{\omega_{t_1}}^{(1)} = \mathfrak{R}_\omega^{(1)} - t_1\sqrt{-1}\partial\overline{\partial}\varphi > 0 \tag{3.50}$$

but $\Theta_{\omega_{t_1}}^{(1)}$ is not positive definite.

Definition 3.16 ([9])**.** A compact *complex* manifold M is said to satisfy the $\partial\overline{\partial}$-lemma if the following statement holds: if η is d-exact, ∂-closed and $\overline{\partial}$-closed, it must be $\partial\overline{\partial}$-exact. In particular, on such manifolds, for any pure-type form $\psi \in \Omega^{p,q}(M)$, if ψ is $\overline{\partial}$-closed and ∂-exact, then it is $\partial\overline{\partial}$-exact.

It is well-known that all compact Kähler manifolds satisfy the $\partial\overline{\partial}$-lemma. Moreover, if $\mu : \widehat{M} \to M$ is a modification between compact complex manifolds and if the $\partial\overline{\partial}$-lemma holds for $\widehat{M}$, then the $\partial\overline{\partial}$-lemma also holds for M. In particular, Moishezon manifolds and also manifolds in Fujiki class $\mathscr{C}$ satisfy the $\partial\overline{\partial}$-lemma. For more details, we refer to [9, 2, 45, 46] and also the references therein.

In the following, we show on complex manifolds with $\partial\overline{\partial}$-lemma, the converse of (2) and (3) in Theorem 3.14 are also true.

Proposition 3.17. *Let M be a compact complex manifold on which the $\partial\overline{\partial}$-lemma holds. Let ω be a Hermitian metric on M.*

1. *$\mathfrak{R}^{(1)}$ represents the real first Chern class $c_1(M) \in H_{dR}^2(M)$ if and only if $\overline{\partial}\partial^* \omega = 0$;*

(2) $\mathfrak{R}^{(1)}$ *represents the first Chern class* $c_1(M) \in H^{1,1}_{\overline{\partial}}(M)$ *if and only if* ω *is conformally balanced.*

Proof. (1) If $\left[\mathfrak{R}^{(1)}\right] = \left[\Theta^{(1)}\right] \in H^2_{dR}(M)$, by (3.45), $\mathfrak{R}^{(1)} = \Theta^{(1)} - \frac{1}{2}dd^*\omega + \frac{1}{2}\left(\overline{\partial}\partial^*\omega + \partial\overline{\partial}^*\omega\right)$, there exists a 1-form γ such that $d\gamma = \overline{\partial}\partial^*\omega + \partial\overline{\partial}^*\omega$. The compatibility condition $d^2\gamma = 0$ implies $\partial\overline{\partial}\partial^*\omega + \overline{\partial}\partial\overline{\partial}^*\omega = 0$. Hence $\overline{\partial}\partial\overline{\partial}^*\omega = 0$. If we set $\gamma = \gamma^{1,0} + \gamma^{0,1}$, then $\partial\gamma^{1,0} = \partial\overline{\partial}^*\omega$. Therefore, $\partial\gamma^{1,0}$ is both d-closed and ∂-exact. By $\partial\overline{\partial}$-lemma, there exists some η such that $\partial\gamma^{1,0} = \partial\overline{\partial}\eta$. By degree reasons, $\partial\gamma^{1,0} = 0$, that is $\partial\overline{\partial}^*\omega = 0$.
(2). If $\left[\mathfrak{R}^{(1)}\right] = \left[\Theta^{(1)}\right] \in H^{1,1}_{\overline{\partial}}(M)$, there exits $(1,0)$-form τ such that $\overline{\partial}\tau = \partial\partial^*\omega$. So $\overline{\partial}\tau$ is both ∂-closed and $\overline{\partial}$-exact, hence by $\partial\overline{\partial}$-lemma, there exists a smooth function φ such that $\partial\partial^*\omega = \overline{\partial}\tau = \sqrt{-1}\partial\overline{\partial}\varphi$. By [25], there exists a smooth function f such that $\omega_f := e^{\frac{f}{n-1}}\omega$ is a Gauduchon metric, i.e. $\partial\overline{\partial}\omega_f^{n-1} = 0$. We use the index f to denote the operations with respect to the new metric ω_f. For example,

$$\|\partial_f^*\omega_f\|_f^2 = (\partial\partial_f^*\omega_f, \omega_f)_f.$$

We can see $\partial\partial_f^*\omega_f = \partial\partial^*\omega - \sqrt{-1}\partial\overline{\partial}f = \sqrt{-1}\partial\overline{\partial}(\varphi - f)$. Moreover,

$$(\partial\partial_f^*\omega_f, \omega_f)_f = (\sqrt{-1}\partial\overline{\partial}(\varphi - f), \omega_f)_f = \int \sqrt{-1}\partial\overline{\partial}(\varphi - f) \wedge \frac{\omega_f^{n-1}}{(n-1)!} = 0,$$

since ω_f is a Gauduchon metric. Therefore, $\partial_f^*\omega_f = 0$, i.e. ω_f is balanced and so ω is conformally balanced. □

Remark 3.18. By (3.47), a conformally balanced metric ω satisfies $\overline{\partial}\partial^*\omega = 0$. On the other hand, if $H^{0,1}_{\overline{\partial}}(M) = 0$, then $\overline{\partial}\partial^*\omega = 0$ if and only if ω is conformally balanced. On the Hopf manifold $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ with $n \geq 2$, the canonical metric ω_0 satisfies $\overline{\partial}\partial^*\omega_0 = 0$, but it is not conformally balanced.

Corollary 3.19. *Let* M *be a complex manifold. Then*

$$c_1^{BC}(M) = 0 \Longrightarrow c_1(M) = 0 \Longrightarrow c_1^{AC}(M) = 0.$$

Moreover, on a complex manifold satisfying the $\partial\overline{\partial}$*-lemma,*

$$c_1^{BC}(M) = 0 \Longleftrightarrow c_1(M) = 0 \Longleftrightarrow c_1^{AC}(M) = 0.$$

Proof. The first statement is obvious. For the second statement, we only need to show that, on a complex manifold M with $\partial\overline{\partial}$-lemma if $c_1^{AC}(M) = 0$, then $c_1^{BC}(M) = 0$. Indeed, if $c_1^{AC}(M) = 0$, for a Hermitian metric ω on M,

$$\Theta^{(1)} = \partial A + \overline{\partial B}$$

where A and B are $(0,1)$ forms. It is obvious that ∂A is $\overline{\partial}$-closed and ∂-exact, and so by $\partial\overline{\partial}$-lemma, there exists a smooth function f_1 such that $\partial A = \partial\overline{\partial}f_1$. Similarly, there exists smooth function f_2 such that $\partial B = \partial\overline{\partial}f_2$ and so $\Theta^{(1)} = \partial\overline{\partial}(f_1 - \overline{f_2})$. That is $c_1^{BC}(M) = 0$. □

Remark 3.20. It is well-known that $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ $(n \geq 2)$ has $c_1(M) = c_1^{AC}(M) = 0$, but $c_1^{BC}(M) \neq 0$ (e.g. [51, Example 3.3]). It is also interesting to find a complex manifold N with $c_1^{AC}(N) = 0$ but $c_1(N) \neq 0$.

3.4. **Hermitian manifolds with nonnegative $\mathfrak{R}^{(1)}$.** In this subsection, we study the geometry of Hermitian manifolds with nonnegative first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$.

Theorem 3.21. *Let (M,h) be a compact Hermitian manifold. If the first Levi-Civita Ricci curvature $\mathfrak{R}^{(1)}$ is quasi-positive, then the top intersection number $c_1^n(M)>0$. In particular, $H^2_{dR}(M)$, $H^{1,1}_{\bar\partial}(M), H^{1,1}_{BC}(M)$ and $H^{1,1}_A(M)$ are all non-zero.*

Proof. At first, let's recall the general theory for vector bundles. Let ∇^E be a connection on the holomorphic vector bundle E. Let r be the rank of E, then there is a naturally induced connection $\nabla^{\det(E)}$ on the determinant line bundle $\det(E)=\Lambda^r E$,

$$\nabla^{\det(E)}(s_1\wedge\cdots\wedge s_r)=\sum_{i=1}^r s_1\wedge\cdots\wedge\nabla^E s_i\wedge\cdots\wedge s_r. \tag{3.51}$$

The curvature tensor of (E,∇^E) is denoted by $R^E\in\Gamma\left(M,\Lambda^2T^*M\otimes End(E)\right)$ and the curvature tensor of $(\det E,\nabla^{\det(E)})$ is denoted by $R^{\det(E)}\in\Gamma\left(M,\Lambda^2T^*M\right)$. We have the relation that

$$trR^E=R^{\det E}\in\Gamma(M,\Lambda^2T^*M). \tag{3.52}$$

Note that the trace operator is well-defined without using the metric on E. Moreover, $trR^E=R^{\det E}$ is a d-closed 2-form. By Bianchi identity, we know, for any vector bundle (F,∇^F)

$$\nabla^{F\otimes F^*}R^F=0.$$

In particular, if F is a line bundle, $F\otimes F^*=\underline{\mathbb{C}}$ and $\nabla^{F\otimes F^*}=d$. Hence $d\left(R^{\det E}\right)=0$. On the other hand, by Chern-Weil theory (e.g. [65, Theorem 1.9]), $R^{\det E}$ represents the real first Chern class $c_1(E)\in H^2(M,\mathbb{R})$. In fact, let ∇^{Ch} be the Chern connection on the Hermitian holomorphic line bundle $(\det E,h)$, and $\Theta^{\det E}$ be the Chern curvature, then by Chern-Weil theory,

$$R^{\det E}-\Theta^{\det E}=d\beta$$

for some 1-form β. It is well known that the Chern curvature $\Theta^{\det E}$ of the Hermitian line bundle $(\det E,h)$ represents the first Chern class $c_1(E)\in H^{1,1}_{\bar\partial}(M)$.

Now we go back to the setting on the Hermitian manifold (M,ω). Let $E=T^{1,0}M$ with Hermitian metric h induced by ω. With respect to the Levi-Civita connection $\widehat\nabla$ on E, we have a decomposition

$$R^{\det E}=\eta^{2,0}+\eta^{0,2}+\eta^{1,1}.$$

It is obvious that

$$\eta^{1,1}=\sqrt{-1}R^k_{i\bar jk}dz^i\wedge d\bar z^j=\mathfrak{R}^{(1)},\qquad\text{and}\quad \eta^{0,2}=\overline{\eta}^{2,0}. \tag{3.53}$$

It is also easy to see that $\eta^{2,0}=-\partial\bar\partial^*\omega$. Hence

$$\int\left(R^{\det E}\right)^n=\sum_{\ell=0}^{\left[\frac{n}{2}\right]}\binom{n}{2\ell}\binom{2\ell}{\ell}\int(\eta^{2,0}\wedge\overline{\eta}^{2,0})^\ell\wedge\left(\eta^{1,1}\right)^{n-2\ell}. \tag{3.54}$$

It is obvious that, if $\eta^{1,1}$ is quasi-positive,

$$\int(\eta^{2,0}\wedge\overline{\eta}^{2,0})^\ell\wedge\left(\eta^{1,1}\right)^{n-2\ell}\geq 0. \tag{3.55}$$

for $1 \leq \ell \leq \left[\frac{n}{2}\right]$ and $\int \left(\eta^{1,1}\right)^n > 0$. That is

$$\int \left(R^{\det E}\right)^n > 0. \tag{3.56}$$

We obtain

$$\int c_1^n(M) = \int \left(R^{\det E}\right)^n > 0. \tag{3.57}$$

On the other hand, if $H_A^{1,1}(M) = 0$, we obtain $\Theta^{(1)} = \partial B + \overline{\partial} C$ for 1-forms B and C, and so

$$\int_M \left(\Theta^{(1)}\right)^n = \int_M (\partial B + \overline{\partial} C) \wedge \left(\Theta^{(1)}\right)^{n-1} = 0, \tag{3.58}$$

which is a contradiction to $c_1^n(M) > 0$. The non-vanishing of other cohomology groups follows immediately from Corollary 3.19. □

Remark 3.22. (1) Note that, in general,

$$\int_M \left(\Theta^{(1)}\right)^n \neq \int_M \left(\mathfrak{R}^{(1)}\right)^n. \tag{3.59}$$

(2) When M is in the Fujiki class $\mathscr{C}$ and $\mathfrak{R}^{(1)}$ is strictly positive, then M is a Kähler manifold ([7, Theorem 0.2]).

3.5. **Hypothetical complex structures on $\mathbb{S}^6$.** Let (M, h) be a Hermitian manifold with constant Riemannian sectional curvature K, i.e., for any $X, Y, Z, W \in T_{\mathbb{R}} M$,

$$R(X, Y, Z, W) = K \cdot (g(X, W)g(Y, Z) - g(X, Z)g(Y, W)). \tag{3.60}$$

Therefore, by the complexification process,

$$R_{i\overline{j}k\overline{\ell}} = K \cdot h_{i\overline{\ell}} h_{k\overline{j}}, \quad R_{i\overline{j}} = K \cdot h_{i\overline{j}}, \quad \text{and} \quad R_{ijk\overline{\ell}} = R_{\overline{i}\overline{j}k\overline{\ell}} = 0.$$

In particular, $Ric_H = K \cdot \omega$. If $K > 0$, we see from (3.9) that

$$\mathfrak{R}^{(1)} \geq Ric_H = K \cdot \omega_h > 0. \tag{3.61}$$

By Theorem 3.21, $c_1^3(M) > 0$. Now we get Lebrun's result that

Corollary 3.23 ([35])**.** *On $\mathbb{S}^6$, there is no orthogonal complex structure compatible with metrics in some small neighborhood of the round metric.*

4. Curvature relations on Hermitian manifolds

4.1. **Ricci curvature relations.** Let's recall different types of Ricci curvatures on a Hermitian manifold (M, ω):

(1) the Levi-Civita Ricci curvatures $\mathfrak{R}^{(1)}$, $\mathfrak{R}^{(2)}$, $\mathfrak{R}^{(3)}$, $\mathfrak{R}^{(4)}$;
(2) the Chern Ricci curvatures $\Theta^{(1)}$, $\Theta^{(2)}$, $\Theta^{(3)}$, $\Theta^{(4)}$;
(3) the Hermitian-Ricci curvature $Ric_H = \sqrt{-1} R_{i\overline{j}} dz^i \wedge d\overline{z}^j$ where $R_{i\overline{j}} = h^{k\overline{\ell}} R_{i\overline{j}k\overline{\ell}}$ (which is equal to $R^{(1)}$ and $R^{(2)}$), the third and fourth Hermitian-Ricci curvatures $R^{(3)}$ and $R^{(4)}$;
(4) the $(1,1)$-component of the complexified Riemannian Ricci curvature, $\mathscr{R}ic$.

In this subsection, we shall explore explicit relations between them by using ω and its torsion T. We shall prove Theorem 1.12. We also state it as in the following to the reader's convenience.

Theorem 4.1. *Let (M,ω) be a compact Hermitian manifold.*

(1) *The Levi-Civita Ricci curvatures are*

$$\mathfrak{R}^{(1)} = \Theta^{(1)} - \frac{1}{2}\left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right); \tag{4.1}$$

$$\mathfrak{R}^{(2)} = \Theta^{(1)} - \frac{1}{2}\left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right) - \frac{\sqrt{-1}}{4}T\circ\overline{T} + \frac{\sqrt{-1}}{4}T\boxdot\overline{T}; \tag{4.2}$$

$$\mathfrak{R}^{(3)} = \Theta^{(1)} - \frac{1}{2}\left(\sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right) + (\partial\partial^*\omega + \overline{\partial\partial^*}\omega)\right) + \frac{\sqrt{-1}}{4}T\boxdot\overline{T} + \frac{T([\partial^*\omega]^\#)}{4}; \tag{4.3}$$

$$\mathfrak{R}^{(4)} = \Theta^{(1)} - \frac{1}{2}\left(\sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right) + (\partial\partial^*\omega + \overline{\partial\partial^*}\omega)\right) + \frac{\sqrt{-1}}{4}T\boxdot\overline{T} + \frac{\overline{T([\partial^*\omega]^\#)}}{4}, \tag{4.4}$$

where $(\partial^*\omega)^\#$ is the dual vector of the $(0,1)$-form $\partial^*\omega$.

(2) *The Chern-Ricci curvatures are*

$$\Theta^{(2)} = \Theta^{(1)} - \sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right) - (\partial\partial^*\omega + \overline{\partial\partial^*}\omega) + \sqrt{-1}T\boxdot\overline{T}; \tag{4.5}$$

$$\Theta^{(3)} = \Theta^{(1)} - \partial\partial^*\omega; \tag{4.6}$$

$$\Theta^{(4)} = \Theta^{(1)} - \overline{\partial\partial^*}\omega. \tag{4.7}$$

(3) *The Hermitian-Ricci curvatures are*

$$Ric_H = R^{(1)} = R^{(2)} = \Theta^{(1)} - \frac{1}{2}\left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right) - \frac{\sqrt{-1}}{4}T\circ\overline{T}; \tag{4.8}$$

$$\begin{aligned} R^{(3)} = R^{(4)} &= \Theta^{(1)} - \frac{1}{2}\left(\sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right) + (\partial\partial^*\omega + \overline{\partial\partial^*}\omega)\right) \\ &\quad + \frac{\sqrt{-1}}{4}T\boxdot\overline{T} + \frac{\overline{T([\partial^*\omega]^\#)} + T([\partial^*\omega]^\#)}{4}. \end{aligned} \tag{4.9}$$

(4) *The $(1,1)$-component of the Riemannian Ricci curvature is*

$$\begin{aligned} \mathscr{R}ic &= \Theta^{(1)} - \sqrt{-1}(\Lambda\partial\overline{\partial}\omega) - \frac{1}{2}(\partial\partial^*\omega + \overline{\partial\partial^*}\omega) + \frac{\sqrt{-1}}{4}\left(2T\boxdot\overline{T} + T\circ\overline{T}\right) \\ &\quad + \frac{1}{2}\left(T([\partial^*\omega]^\#) + \overline{T([\partial^*\omega]^\#)}\right). \end{aligned} \tag{4.10}$$

Proof. (1). Equation (4.1) is proved in (3.45). By (3.33), (3.34) and (4.1), we see

$$\begin{aligned} \mathfrak{R}^{(2)} &= \mathfrak{R}^{(1)} - \frac{\sqrt{-1}}{4}T\circ\overline{T} + \frac{\sqrt{-1}}{4}T\boxdot\overline{T} \\ &= \Theta^{(1)} - \frac{1}{2}\left(\partial\partial^*\omega + \overline{\partial\partial^*}\omega\right) - \frac{\sqrt{-1}}{4}T\circ\overline{T} + \frac{\sqrt{-1}}{4}T\boxdot\overline{T} \end{aligned}$$

which proves (4.2). A straightforward computation shows

$$\sqrt{-1}\Lambda(\partial\overline{\partial}\omega) = \sqrt{-1}\sum_q\left[\left(\frac{\partial h_{i\overline{j}}}{\partial z^q\partial\overline{z}^q} + \frac{\partial h_{q\overline{q}}}{\partial z^i\partial\overline{z}^j}\right) - \left(\frac{\partial h_{i\overline{q}}}{\partial z^q\partial\overline{z}^j} + \frac{\partial^2 h_{q\overline{j}}}{\partial z^i\partial\overline{z}^q}\right)\right]dz^i\wedge d\overline{z}^j. \tag{4.11}$$

If we write

$$\Theta^{(1)}-\frac{1}{2}\left(\sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right)+\left(\partial\partial^*\omega+\overline{\partial}\overline{\partial}^*\omega\right)\right)+\frac{\sqrt{-1}}{4}T\boxdot\overline{T}:=\sqrt{-1}B_{i\overline{j}}dz^i\wedge d\overline{z}^j,$$

by using formulas (3.39), (4.11), (3.28) and (3.29), we obtain

$$\begin{aligned}
B_{i\overline{j}} &= -\sum_q\frac{\partial^2 h_{q\overline{q}}}{\partial z^i\partial\overline{z}^j}-\frac{1}{2}\sum_q\left[\left(\frac{\partial h_{i\overline{j}}}{\partial z^q\partial\overline{z}^q}+\frac{\partial h_{q\overline{q}}}{\partial z^i\partial\overline{z}^j}\right)-\left(\frac{\partial h_{i\overline{q}}}{\partial z^q\partial\overline{z}^j}+\frac{\partial^2 h_{q\overline{j}}}{\partial z^i\partial\overline{z}^q}\right)\right]\\
&\quad -\frac{1}{2}\sum_q\left(\frac{\partial^2 h_{i\overline{q}}}{\partial z^q\partial\overline{z}^j}+\frac{\partial^2 h_{q\overline{j}}}{\partial z^i\partial\overline{z}^q}-2\frac{\partial^2 h_{q\overline{q}}}{\partial z^i\partial\overline{z}^j}\right)\\
&= -\frac{1}{2}\sum_q\left(\frac{\partial^2 h_{q\overline{q}}}{\partial z^i\partial\overline{z}^j}+\frac{\partial^2 h_{i\overline{j}}}{\partial z^q\partial\overline{z}^q}\right).
\end{aligned}$$

By formula (3.35), we obtain (4.3). The proof of formula (4.4) follows from (3.36), (3.35) and (4.3).

(2). For Chern-Ricci curvature relations, we use similar computations. For example, if we write

$$\Theta^{(1)}-\left(\sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right)+\left(\partial\partial^*\omega+\overline{\partial}\overline{\partial}^*\omega\right)\right)+\sqrt{-1}T\boxdot\overline{T}:=\sqrt{-1}C_{i\overline{j}}dz^i\wedge d\overline{z}^j,$$

by using formulas (3.39), (4.11), (3.28) and (3.29), we obtain

$$\begin{aligned}
C_{i\overline{j}} &= -\sum_q\frac{\partial^2 h_{q\overline{q}}}{\partial z^i\partial\overline{z}^j}-\sum_q\left[\left(\frac{\partial h_{i\overline{j}}}{\partial z^q\partial\overline{z}^q}+\frac{\partial h_{q\overline{q}}}{\partial z^i\partial\overline{z}^j}\right)-\left(\frac{\partial h_{i\overline{q}}}{\partial z^q\partial\overline{z}^j}+\frac{\partial^2 h_{q\overline{j}}}{\partial z^i\partial\overline{z}^q}\right)\right]\\
&\quad -\sum_q\left(\frac{\partial^2 h_{i\overline{q}}}{\partial z^q\partial\overline{z}^j}+\frac{\partial^2 h_{q\overline{j}}}{\partial z^i\partial\overline{z}^q}-2\frac{\partial^2 h_{q\overline{q}}}{\partial z^i\partial\overline{z}^j}\right)+\frac{1}{4}\left(T\boxdot\overline{T}\right)_{i\overline{j}}\\
&= -\sum_q\frac{\partial^2 h_{i\overline{j}}}{\partial z^q\partial\overline{z}^q}+\frac{1}{4}\left(T\boxdot\overline{T}\right)_{i\overline{j}}.
\end{aligned}$$

By formula (3.40), we obtain (4.5). Formula (4.6) follows from (3.39), (3.41) and (3.28). Formula (4.7) follows from (3.39), (3.42) and (3.29).

(3). For Hermitian-Ricci curvatures, by (3.37) and (3.33), we see

$$R^{(1)}=R^{(2)}=\mathfrak{R}^{(1)}-\frac{\sqrt{-1}}{4}T\circ\overline{T}.$$

Therefore, (4.8) follows from (4.1). Similarly, by (3.38), (3.35) and (4.3), we obtain (4.9).

(4). If we write

$$\Theta^{(1)}-\left(\sqrt{-1}\Lambda\left(\partial\overline{\partial}\omega\right)+\frac{1}{2}(\partial\partial^*\omega+\overline{\partial}\overline{\partial}^*\omega)\right):=\sqrt{-1}F_{i\overline{j}}dz^i\wedge d\overline{z}^j,$$

by using formulas (3.39), (4.11), (3.28) and (3.29), we obtain

$$\begin{aligned}
F_{i\bar{j}} &= -\sum_q \frac{\partial^2 h_{q\bar{q}}}{\partial z^i \partial \bar{z}^j} - \sum_q \left[\left(\frac{\partial h_{i\bar{j}}}{\partial z^q \partial \bar{z}^q} + \frac{\partial h_{q\bar{q}}}{\partial z^i \partial \bar{z}^j} \right) - \left(\frac{\partial h_{i\bar{q}}}{\partial z^q \partial \bar{z}^j} + \frac{\partial^2 h_{q\bar{j}}}{\partial z^i \partial \bar{z}^q} \right) \right] \\
&\quad -\frac{1}{2} \sum_q \left(\frac{\partial^2 h_{i\bar{q}}}{\partial z^q \partial \bar{z}^j} + \frac{\partial^2 h_{q\bar{j}}}{\partial z^i \partial \bar{z}^q} - 2\frac{\partial^2 h_{q\bar{q}}}{\partial z^i \partial \bar{z}^j} \right) - \frac{1}{4} \left(T \boxdot \overline{T} \right)_{i\bar{j}} \\
&= \frac{1}{2} \sum_q \left(\frac{\partial^2 h_{q\bar{j}}}{\partial z^i \partial \bar{z}^q} + \frac{\partial^2 h_{i\bar{q}}}{\partial z^q \partial \bar{z}^j} \right) - \sum_q \left(\frac{\partial^2 h_{i\bar{j}}}{\partial z^q \partial \bar{z}^q} + \frac{\partial^2 h_{q\bar{q}}}{\partial z^i \partial \bar{z}^j} \right) - \frac{1}{4} \left(T \boxdot \overline{T} \right)_{i\bar{j}}.
\end{aligned}$$

By (3.44), we obtain (4.10). □

4.2. **Scalar curvature relations.** On a Hermitian manifold (M,ω), we can define five different types of scalar curvatures:

1. s, the scalar curvature of the background Riemannian metric;
2. $s_R = h^{i\bar{\ell}} h^{k\bar{j}} R_{i\bar{j}k\bar{\ell}}$, the Riemannian type scalar curvature;
3. $s_H = h^{i\bar{j}} h^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}$, the scalar curvature of the Hermitian curvature;
4. $s_{LC} = h^{i\bar{j}k\bar{\ell}} \mathfrak{R}_{i\bar{j}k\bar{\ell}}$, the scalar curvature of the Levi-Civita connection;
5. $s_C = h^{i\bar{j}} h^{k\bar{\ell}} \Theta_{i\bar{j}k\bar{\ell}}$, the scalar curvature of the Chern connection.

By Theorem 4.1, we get the corresponding scalar curvature relations:

Corollary 4.2. *Let (M,ω) be a compact Hermitian manifold, then*

$$s = 2s_C + \left(\langle \partial\partial^*\omega + \overline{\partial\partial}^*\omega, \omega\rangle - 2|\partial^*\omega|^2 \right) - \frac{1}{2}|T|^2, \tag{4.12}$$

$$s_{LC} = s_C - \frac{1}{2}\langle \partial\partial^*\omega + \overline{\partial\partial}^*\omega, \omega\rangle = s_C - \langle \partial\partial^*\omega, \omega\rangle, \tag{4.13}$$

$$s_H = s_C - \frac{1}{2}\langle \partial\partial^*\omega + \overline{\partial\partial}^*\omega, \omega\rangle - \frac{|T|^2}{4}, \tag{4.14}$$

and

$$s_R = s_C - \frac{1}{2}|\partial^*\omega|^2 - \frac{1}{4}|T|^2. \tag{4.15}$$

Proof. By (2.33), we know $s = 2h^{i\bar{j}}\mathscr{R}_{i\bar{j}}$. By (4.10),

$$s = 2s_C - \left\langle 2\sqrt{-1}(\Lambda\partial\overline{\partial}\omega), \omega \right\rangle - \left\langle \partial\partial^*\omega + \overline{\partial\partial}^*\omega, \omega \right\rangle + \frac{3|T|^2}{2} - 2|\partial^*\omega|^2,$$

where we use the fact that $tr_\omega T([\partial^*\omega]^\#) = -|\partial^*\omega|^2$(see (3.25)). By (4.5), we have

$$|T|^2 = \langle \sqrt{-1}\Lambda\partial\overline{\partial}\omega, \omega\rangle + \langle \partial\partial^*\omega + \overline{\partial\partial}^*\omega, \omega\rangle, \tag{4.16}$$

we get (4.12). (4.13) and (4.14) follow from (4.1) and (4.8) respectively. For (4.15), it follows from (4.12), (4.13) and the fact that $s = 4s_R - 2s_H$. □

Remark 4.3. Note that we can also define the scalar curvatures as

$$tr_\omega \mathfrak{R}^{(3)}, \quad tr_\omega \Theta^{(3)} \quad \text{and} \quad tr_\omega R^{(3)}$$

The corresponding scalar curvature relations follow immediately from formulas (4.3), (4.6) and (4.9). More precisely, we obtain

$$tr_\omega \mathfrak{R}^{(3)} = tr_\omega \mathfrak{R}^{(4)} = s_C - \frac{1}{4}|T|^2 - \frac{1}{4}|\partial^*\omega|^2; \tag{4.17}$$

$$tr_\omega \Theta^{(3)} = tr_\omega \Theta^{(4)} = s_C - \langle \partial\partial^*\omega, \omega\rangle = s_{LC}; \tag{4.18}$$

and

$$tr_\omega R^{(3)} = tr_\omega R^{(4)} = s_C - \frac{1}{4}|T|^2 - \frac{1}{2}|\partial^*\omega|^2 = s_R. \tag{4.19}$$

Corollary 4.4. *Let (M,ω) be a compact Hermitian manifold. Then the following are equivalent:*

(1) *(M,ω) is Kähler;*
(2) $\int s\cdot\omega^n = \int 2s_C\cdot\omega^n$;
(3) $\int s_C\cdot\omega^n = \int s_R\cdot\omega^n$;
(4) $\int s_C\cdot\omega^n = \int s_H\cdot\omega^n$;
(5) $\int s_H\cdot\omega^n = \int s_{LC}\cdot\omega^n$.

Corollary 4.5. *Let (M,ω) be a compact Hermitian manifold. Then the following are equivalent:*

(1) *(M,ω) is balanced;*
(2) $\int s\cdot\omega^n = \int 2s_R\cdot\omega^n$;
(3) $\int s\cdot\omega^n = \int 2s_H\cdot\omega^n$;
(4) $\int s_C\cdot\omega^n = \int s_{LC}\cdot\omega^n$;
(5) $\int s_R\cdot\omega^n = \int s_H\cdot\omega^n$.

5. Special metrics on Hermitian manifolds

Before discussing special metrics on Hermitian manifolds, we need the following observation which is the integral version of (4.16). We assume $\dim_{\mathbb{C}} M = n \geq 3$.

Proposition 5.1. *On a compact Hermitian manifold (M,ω), for any $1\leq k\leq n-1$, we have*

$$\int \sqrt{-1}\partial\omega\wedge\overline{\partial}\omega\wedge\frac{\omega^{n-3}}{(n-3)!} = \|\partial^*\omega\|^2 - \|\partial\omega\|^2, \tag{5.1}$$

and

$$\int \sqrt{-1}\omega^{n-k-1}\wedge\partial\overline{\partial}\omega^k = (n-3)!k(n-k-1)\left(\|\partial\omega\|^2 - \|\partial^*\omega\|^2\right). \tag{5.2}$$

Proof. At first, it is easy to see

$$\partial\overline{\partial}\omega^k = k\omega^{k-1}\partial\overline{\partial}\omega + k(k-1)\omega^{k-2}\partial\omega\wedge\overline{\partial}\omega. \tag{5.3}$$

On the other hand, the $(1,2)$-form $\alpha := \overline{\partial}\omega - \frac{L\Lambda\overline{\partial}\omega}{n-1}$ is primitive, i.e. $\Lambda\alpha = 0$. Hence, by [60, Proposition 6.29],

$$*(\alpha) = (-1)^{\frac{3(3+1)}{2}}(\sqrt{-1})^{1-2}\frac{L^{n-3}}{(n-3)!}\alpha = -\sqrt{-1}\frac{\omega^{n-3}\wedge\alpha}{(n-3)!}. \tag{5.4}$$

Therefore

$$\left(\partial\omega, \partial\omega - \frac{L\Lambda\partial\omega}{n-1}\right) = (\partial\omega, \overline{\alpha}) = \int \partial\omega \wedge *(\alpha) = -\int \sqrt{-1}\partial\omega\wedge\alpha\wedge\frac{\omega^{n-3}}{(n-3)!}. \tag{5.5}$$

In particular, we have

$$\begin{aligned}
& \int \sqrt{-1}\partial\omega\wedge\overline{\partial}\omega\wedge\frac{\omega^{n-3}}{(n-3)!} \\
= & -\left(\partial\omega, \partial\omega - \frac{L\Lambda\partial\omega}{n-1}\right) + \int\sqrt{-1}\partial\omega\wedge\frac{L\Lambda\overline{\partial}\omega}{n-1}\wedge\frac{\omega^{n-3}}{(n-3)!} \\
= & -\|\partial\omega\|^2 + \frac{\|\Lambda\partial\omega\|^2}{n-1} + \int\sqrt{-1}\partial\omega\wedge\frac{\Lambda\overline{\partial}\omega}{n-1}\wedge\frac{\omega^{n-2}}{(n-3)!}.
\end{aligned}$$

Form (3.18), $\partial^*\omega = -\sqrt{-1}\Lambda\overline{\partial}\omega$, we have

$$\begin{aligned}
& \int\sqrt{-1}\partial\omega\wedge\frac{\Lambda\overline{\partial}\omega}{n-1}\wedge\frac{\omega^{n-2}}{(n-3)!} \\
= & \frac{1}{(n-1)^2}\int\partial^*\omega\wedge\frac{\partial\omega^{n-1}}{(n-3)!} = \frac{n-2}{n-1}\int\partial\partial^*\omega\wedge\frac{\omega^{n-1}}{(n-1)!} \\
= & \frac{n-2}{n-1}(\partial\partial^*\omega, \omega) = \frac{n-2}{n-1}\|\partial^*\omega\|^2.
\end{aligned}$$

Therefore

$$\int\sqrt{-1}\partial\omega\wedge\overline{\partial}\omega\wedge\frac{\omega^{n-3}}{(n-3)!} = \|\partial^*\omega\|^2 - \|\partial\omega\|^2. \tag{5.6}$$

From integration by parts, we also get

$$\int\sqrt{-1}\partial\overline{\partial}\omega\wedge\frac{\omega^{n-2}}{(n-2)!} = \|\partial\omega\|^2 - \|\partial^*\omega\|^2. \tag{5.7}$$

Hence (5.2) follows from (5.3), (5.6) and (5.7). □

Corollary 5.2. *We have the follow relation on a compact Hermitian manifold* (M,ω)

$$\int s_R\cdot\frac{\omega^n}{n!} = \int s_{LC}\cdot\frac{\omega^n}{n!} - \frac{1}{(n-3)!2k(n-k-1)}\int\sqrt{-1}\omega^{n-k-1}\wedge\partial\overline{\partial}\omega^k. \tag{5.8}$$

Proof. It follows from Corollary 4.2 and formula (5.2). □

Fu-Wang-Wu defined in [19] that a Hermitian metric ω satisfying

$$\sqrt{-1}\omega^{n-k-1}\wedge\partial\overline{\partial}\omega^k = 0, \quad 1\leq k\leq n-1 \tag{5.9}$$

is called a k-Gauduchon metric. It is obvious that $(n-1)$-Gauduchon metric, i.e. $\partial\overline{\partial}\omega^{n-1} = 0$ is the original Gauduchon metric. It is well-known that, Hopf manifolds $\mathbb{S}^{2n+1}\times S^1$ can not support Hermitian metrics with $\partial\overline{\partial}\omega = 0$ (SKT) or $d^*\omega = 0$ (balanced). They showed in [19] that on $\mathbb{S}^5\times\mathbb{S}^1$, there exists a 1-Gauduchon metric ω, i.e. $\omega\wedge\partial\overline{\partial}\omega = 0$.

As a straightforward application of Proposition 5.1, we obtain:

Corollary 5.3. *If (M,ω) is k-Gauduchon ($1\leq k\leq n-2$) and also balanced, then (M,ω) is Kähler.*

This is also true in the "conformal" setting:

Corollary 5.4. *On a compact complex manifold, the following are equivalent:*

1. *(M,ω) is conformally Kähler;*
2. *(M,ω) is conformally k-Gauduchon for $1\leq k\leq n-2$, and conformally balanced;*

In particular, the following are also equivalent:

3. *(M,ω) is Kähler;*
4. *(M,ω) is k-Gauduchon for $1\leq k\leq n-2$, and conformally balanced;*
5. *(M,ω) is conformally balanced and $\Lambda^2(\partial\overline{\partial}\omega)=0$.*

Proof. We first show (2) implies (1). Since ω is conformally balanced, $\omega=e^F\omega_B$ for a balanced metric ω_B and a smooth function $F\in C^\infty(M,\mathbb{R})$. By the conformally k-Gauduchon condition, we know there exists $\widetilde{F}\in C^\infty(M,\mathbb{R})$ and a k-Gauduchon metric ω_G such that $\omega=e^{\widetilde{F}}\omega_G$. Let $f=F-\widetilde{F}$, then $\omega_G=e^f\omega_B$. Since ω_G is k-Gauduchon,

$$\left(e^f\omega_B\right)^{n-k-1}\wedge\partial\overline{\partial}\left(e^f\omega_B\right)^k=0,$$

and we obtain

$$\omega_B^{n-k-1}\wedge\partial\overline{\partial}(e^f\omega_B)^k=0. \tag{5.10}$$

Claim: If a balanced metric ω_B satisfies (5.10), then f is a constant and ω_B is Kähler.

Since ω_B is balanced, i.e., $\partial\omega_B^{n-1}=\overline{\partial}\omega_B^{n-1}=0$, we see $\omega_B^{n-k-1}\wedge\partial\omega_B^k=0$. From (5.10), we get

$$e^{kf}\omega_B^{n-k-1}\wedge\partial\overline{\partial}\omega_B^k+\omega_B^{n-1}\wedge\partial\overline{\partial}\left(e^{kf}\right)=0. \tag{5.11}$$

Hence,

$$\int_M e^{kf}\cdot\omega_B^{n-k-1}\wedge\partial\overline{\partial}\omega_B^k=-\int_M\omega_B^{n-1}\wedge\partial\overline{\partial}\left(e^{kf}\right)=0. \tag{5.12}$$

Using integration by parts and the balanced condition $\omega_B^{n-k-1}\wedge\partial\omega_B^k=0$, we obtain

$$\begin{aligned}
0 &= -\sqrt{-1}\int_M e^{kf}\cdot\omega_B^{n-k-1}\wedge\partial\overline{\partial}\omega_B^k\\
&= \sqrt{-1}k(n-k-1)\int_M e^{kf}\omega_B^{n-3}\partial\omega_B\wedge\overline{\partial}\omega_B\\
&= \sqrt{-1}k(n-k-1)(n-3)!\int_M e^{\frac{kf}{2}}\partial\omega_B\wedge\left(e^{\frac{kf}{2}}\overline{\partial}\omega_B\cdot\frac{\omega_B^{n-3}}{(n-3)!}\right)\\
&= -k(n-k-1)(n-3)!\|e^{\frac{kf}{2}}\partial\omega_B\|_B^2,
\end{aligned}$$

and so $\partial\omega_B=0$, i.e. ω_B is Kähler. Note that, in the last step we use the fact that $e^{\frac{kf}{2}}\overline{\partial}\omega_B$ is primitive, i.e.

$$\Lambda\left(e^{\frac{kf}{2}}\overline{\partial}\omega_B\right)=e^{\frac{kf}{2}}\Lambda\overline{\partial}\omega_B=\sqrt{-1}e^{\frac{kf}{2}}\partial^*\omega_B=0,$$

where the norm $\|\bullet\|_B$, ∂^* and the contraction Λ are taken with respect to ω_B. From (5.11), we see that f must be constant if ω_B is Kähler. The proof of the claim is complete. We know ω is conformally Kähler.

The equivalence of (4) and (3) follows from the proof of the claim in the last paragraph under the condition $\omega = \omega_G$, i.e. $\widetilde{F} = 0$ and $f = F$. Next we show (5) implies (3). In fact, if ω is conformally balanced, i.e. $\omega = e^f\omega_B$ for some balanced metric ω_B and smooth function $f \in C^\infty(M, \mathbb{R})$, the condition $\Lambda^2\partial\overline{\partial}\omega = 0$ implies

$$\Lambda_B^2\partial\overline{\partial}\left(e^f\omega_B\right) = 0$$

where Λ_B is the contraction operator with respect to ω_B. By duality, we have

$$0 = \int_M \partial\overline{\partial}(e^f\omega_B) \wedge \omega_B^{n-2} = \int_M e^f\omega_B \wedge \partial\overline{\partial}\omega_B^{n-2}.$$

As similar as the proof in the last paragraph, we obtain both ω and ω_B are Kähler. □

6. Levi-Civita Ricci-flat and constant negative scalar curvature metrics on Hopf manifolds

In this section, we construct special Hermitian metrics on non-Kähler manifolds related to Hopf manifolds. More precisely,

(1) We construct explicit Levi-Civita Ricci-flat metrics on $\mathbb{S}^{2n-1} \times \mathbb{S}^1$;
(2) We construct a smooth family of explicit Hermitian metrics h_λ with $\lambda \in (-1, +\infty)$ on $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ such that their Riemannian scalar curvature are constants and vary from a positive constant to $-\infty$. In particular, we obtain Hermitian metrics with negative constant Riemannian scalar curvature on Hermitian manifolds with $c_1 \geq 0$;
(3) We construct pluriclosed metrics on the projective bundles over $\mathbb{S}^{2n-1} \times \mathbb{S}^1$.

6.1. **Levi-Civita Ricci-flat metrics on Hopf manifolds.** Let's recall an example in [41, Section 6]. Let $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ be the standard n-dimensional ($n \geq 2$) Hopf manifold. It is diffeomorphic to $\mathbb{C}^n - \{0\}/G$ where G is cyclic group generated by the transformation $z \to \frac{1}{2}z$. It has an induced complex structure from $\mathbb{C}^n - \{0\}$. On M, there is a natural induced metric ω_0 given by

$$\omega_0 = \sqrt{-1}h_{i\overline{j}}dz^i \wedge d\overline{z}^j = \frac{4\delta_{i\overline{j}}}{|z|^2}dz^i \wedge d\overline{z}^j. \tag{6.1}$$

For the reader's convenience, we include some elementary computations:

$$\frac{\partial h_{k\overline{\ell}}}{\partial z^i} = -\frac{4\delta_{k\ell}\overline{z}^i}{|z|^4}, \qquad \frac{\partial h_{k\overline{\ell}}}{\partial \overline{z}^j} = -\frac{4\delta_{k\ell}z^j}{|z|^4}$$

and

$$\frac{\partial^2 h_{k\overline{\ell}}}{\partial z^i \partial \overline{z}^j} = -4\delta_{k\ell}\frac{\delta_{i\overline{j}}|z|^2 - 2\overline{z}^i z^j}{|z|^6}.$$

Similarly, we have

$$\Gamma_{ik}^{\ell} = \frac{1}{2}h^{\ell\overline{q}}\left(\frac{\partial h_{i\overline{q}}}{\partial z^k} + \frac{\partial h_{k\overline{q}}}{\partial z^i}\right) = -\frac{\delta_{i\ell}\overline{z}^k + \delta_{k\ell}\overline{z}^i}{2|z|^2},$$

$$\Gamma_{\overline{j}k}^{\ell} = \frac{1}{2}h^{\ell\overline{q}}\left(\frac{\partial h_{k\overline{q}}}{\partial \overline{z}^j} - \frac{\partial h_{k\overline{j}}}{\partial \overline{z}^q}\right) = \frac{\delta_{jk}z^\ell - \delta_{k\ell}z^j}{2|z|^2},$$

and

$$\frac{\partial \Gamma_{ik}^{\ell}}{\partial \overline{z}^j} = -\frac{\delta_{k\ell}\delta_{ij} + \delta_{i\ell}\delta_{jk}}{2|z|^2} + \frac{\delta_{i\ell} z^j \overline{z}^k + \delta_{k\ell} z^j \overline{z}^i}{2|z|^4},$$

$$\frac{\partial \Gamma_{\overline{j}k}^{\ell}}{\partial z^i} = \frac{\delta_{jk}\delta_{i\ell} - \delta_{k\ell}\delta_{ij}}{2|z|^2} - \frac{(\delta_{jk} z^\ell - \delta_{k\ell} z^j)\overline{z}^i}{2|z|^4}.$$

The curvature components of $\mathfrak{R}$ are

$$\begin{aligned} \mathfrak{R}_{i\overline{j}k\overline{\ell}} &= -h_{p\overline{\ell}}\left(\frac{\partial \Gamma_{ik}^p}{\partial \overline{z}^j} - \frac{\partial \Gamma_{\overline{j}k}^p}{\partial z^i} + \Gamma_{ik}^s \Gamma_{\overline{j}s}^p - \Gamma_{\overline{j}k}^s \Gamma_{si}^p\right) \\ &= \frac{3\delta_{i\ell}\delta_{jk}}{|z|^4} - \frac{2\delta_{i\ell} z^j \overline{z}^k + 2\delta_{jk} z^\ell \overline{z}^i - \delta_{ij}\overline{z}^k z^\ell}{|z|^6}. \end{aligned} \tag{6.2}$$

The complexified curvature components are

$$R_{i\overline{j}k\overline{\ell}} = \mathfrak{R}_{i\overline{j}k\overline{\ell}} + h_{p\overline{\ell}} \Gamma_{\overline{s}i}^p \overline{\Gamma_{\overline{k}j}^s} = \frac{2\delta_{i\ell}\delta_{jk}}{|z|^4} - \frac{\delta_{i\ell} z^j \overline{z}^k + \delta_{jk} z^\ell \overline{z}^i}{|z|^6}. \tag{6.3}$$

The Chern curvature components are

$$\Theta_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 h_{k\overline{\ell}}}{\partial z^i \partial \overline{z}^j} + h^{p\overline{q}} \frac{\partial h_{k\overline{q}}}{\partial z^i} \frac{\partial h_{p\overline{\ell}}}{\partial \overline{z}^j} = \frac{4\delta_{kl}(\delta_{ij}|z|^2 - z^j \overline{z}^i)}{|z|^6}. \tag{6.4}$$

As consequences (see also [41]),

$$\Theta^{(1)} = -\sqrt{-1}\partial\overline{\partial} \log \det(h) = n \cdot \sqrt{-1}\partial\overline{\partial} \log |z|^2, \quad \Theta^{(2)} = \frac{n-1}{4}\omega_h; \tag{6.5}$$

$$\mathfrak{R}^{(1)} = \sqrt{-1}\partial\overline{\partial} \log |z|^2, \qquad \mathfrak{R}^{(2)} = \frac{4-n}{4} \cdot \sqrt{-1}\partial\overline{\partial} \log |z|^2 + \frac{n-1}{16}\omega_h; \tag{6.6}$$

$$Ric_H = \frac{\sqrt{-1}}{2}\partial\overline{\partial} \log |z|^2; \tag{6.7}$$

$$\mathscr{R}ic = \frac{n-1}{2} \cdot \sqrt{-1}\partial\overline{\partial} \log |z|^2 + \frac{n-1}{8}\omega_h \tag{6.8}$$

and

$$s_R = \frac{n^2-n}{8}, \quad s_H = \frac{n-1}{8}, \quad s = \frac{(2n-1)(n-1)}{4}, \quad s_{LC} = \frac{n-1}{4}, \quad s_C = \frac{n(n-1)}{4}. \tag{6.9}$$

Remark 6.1. Let $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ with $n \geq 2$.

(1) Since $\partial \log |z|^2$ is a globally defined $(1,0)$ form on M, one can see from (6.5) that $c_1(M) = 0$, $c_1^{AC}(M) = 0$, but $c_1^{BC}(M) \neq 0$. In particular, the canonical line bundle is topologically trivial but not holomorphically trivial;

(2) Since the Chern scalar curvature $s_C = \frac{n(n-1)}{4}$, one deduces that,

$$H^0(M, mK_M) = 0$$

for any integer $m \geq 1$. In particular, the Kodaira dimension of M is $-\infty$, and so K_M is not a torsion line bundle;

(3) The canonical metric ω_h satisfies $\overline{\partial}\partial^* \omega_h = 0$, but it is well-known that M can not support any balanced metric;

(4) $\partial\overline{\partial}$-lemma does not hold on M;

(5) From the semi-positive $(1,1)$ form $\Theta^{(1)}$, we get $(\Theta^{(1)})^n = 0$ and so the top intersection number $c_1^n(M) = 0$. By Theorem 3.21, M can not admit a Hermitian metric with quasi-positive Hermitian-Ricci curvature. Moreover, the quasi-positive curvature condition in Theorem 3.21 can not be replaced by nonnegative curvature condition.

Next we construct explicit Levi-Civita Ricci-flat Hermitian metrics on all Hopf manifolds and prove Theorem 1.5.

Theorem 6.2. *Let*

$$\widetilde{\omega} = \omega_0 - \frac{4}{n}\mathfrak{R}^{(1)}(\omega_0), \tag{6.10}$$

then the first Levi-Civita Ricci curvature of $\widetilde{\omega}$ *is zero, i.e.*

$$\mathfrak{R}^{(1)}(\widetilde{\omega}) = 0. \tag{6.11}$$

Proof. We consider the perturbed metric

$$\widetilde{\omega} = \omega_0 + 4\lambda\mathfrak{R}^{(1)}(\omega_0), \quad \text{with} \quad \lambda > -1, \tag{6.12}$$

where we know form (6.7) that $\mathfrak{R}^{(1)}(\omega_0) = \sqrt{-1}\partial\overline{\partial}\log|z|^2$. That is

$$\widetilde{h}_{i\overline{j}} = \frac{4}{|z|^2}\left((1+\lambda)\delta_{ij} - \frac{\lambda\overline{z}^i z^j}{|z|^2}\right), \quad \text{and} \quad \widetilde{h}^{i\overline{j}} = \frac{|z|^2}{4}\left(\frac{\delta_{i\overline{j}}}{1+\lambda} + \frac{\lambda z^i\overline{z}^j}{(1+\lambda)|z|^2}\right).$$

Moreover,

$$\frac{\partial\widetilde{h}_{i\overline{j}}}{\partial\overline{z}^\ell} = \frac{8\lambda z^j z^\ell\overline{z}^i}{|z|^6} - \frac{4(1+\lambda)\delta_{ij}z^\ell + 4\lambda\delta_{i\ell}z^j}{|z|^4} \quad \text{and} \quad \frac{\partial\widetilde{h}_{i\overline{j}}}{\partial\overline{z}^\ell} - \frac{\partial\widetilde{h}_{i\overline{\ell}}}{\partial\overline{z}^j} = \frac{4(\delta_{i\ell}z^j - \delta_{ij}z^\ell)}{|z|^4}.$$

The Christoffel symbols of $\widetilde{h}$ are

$$\widetilde{\Gamma}^i_{\overline{j}i} = \frac{1}{2}\widetilde{h}^{i\overline{\ell}}\left(\frac{\partial\widetilde{h}_{i\overline{\ell}}}{\partial\overline{z}^j} - \frac{\partial\widetilde{h}_{i\overline{j}}}{\partial\overline{z}^\ell}\right) = -\frac{(n-1)z^j}{2|z|^2(1+\lambda)},$$

and

$$\partial^*\widetilde{\omega} = -2\sqrt{-1}\widetilde{\Gamma}^i_{\overline{j}i}d\overline{z}^j = \sqrt{-1}\frac{n-1}{1+\lambda}\overline{\partial}\log|z|^2,$$

$$\frac{\partial\partial^*\widetilde{\omega} + \overline{\partial\partial^*}\widetilde{\omega}}{2} = \frac{n-1}{1+\lambda}\cdot\sqrt{-1}\partial\overline{\partial}\log|z|^2,$$

where the adjoint operators $\overline{\partial}^*$ and ∂^* are taken with respect to the new metric $\widetilde{\omega}$. Finally, since $\det(\widetilde{h}_{i\overline{j}}) = (1+\lambda)^{n-1}4^n|z|^{-2n}$, we obtain

$$\Theta^{(1)}(\widetilde{\omega}) = -\sqrt{-1}\partial\overline{\partial}\log\det\widetilde{h}_{i\overline{j}} = n\cdot\sqrt{-1}\partial\overline{\partial}\log|z|^2,$$

and by Theorem 1.2,

$$\mathfrak{R}^{(1)}(\widetilde{\omega}) = \Theta^{(1)}(\widetilde{\omega}) - \frac{\partial\partial^*\widetilde{\omega} + \overline{\partial\partial^*}\widetilde{\omega}}{2} = \left(n - \frac{n-1}{1+\lambda}\right)\cdot\sqrt{-1}\partial\overline{\partial}\log|z|^2.$$

Now it is obvious that when $\lambda = -\frac{1}{n}$, $\mathfrak{R}^{(1)}(\widetilde{\omega}) = 0$. $\square$

Remark 6.3. By using the same ideas and also the Ricci curvature relations in Section 4, one can construct various "Einstein metrics" by using different Ricci curvatures introduced in the previous sections.

6.2. **Hermitian metrics of constant negative scalar curvatures on Hopf manifolds.** Using the same setting as in Theorem 6.2, we let

$$\widetilde{\omega} = \omega_0 + 4\lambda \mathfrak{R}^{(1)}(\omega_0), \quad \text{with} \quad \lambda > -1. \tag{6.13}$$

Hence, the Chern scalar curvature of $\widetilde{\omega}$ is

$$\widetilde{s}_C = \langle \Theta^{(1)}(\widetilde{\omega}), \widetilde{\omega}\rangle = \langle n \cdot \sqrt{-1}\partial\overline{\partial}\log|z|^2, \widetilde{\omega}\rangle = \frac{n(n-1)}{4(1+\lambda)}. \tag{6.14}$$

On the other hand, as computed in Theorem 6.2.

$$\langle \partial\partial^*\widetilde{\omega} + \overline{\partial\partial^*}\widetilde{\omega}, \widetilde{\omega}\rangle = 2\left\langle \frac{n-1}{1+\lambda} \cdot \sqrt{-1}\partial\overline{\partial}\log|z|^2, \widetilde{\omega}\right\rangle = \frac{(n-1)^2}{2(1+\lambda)^2}. \tag{6.15}$$

Similarly,

$$|\partial^*\widetilde{\omega}|^2 = \frac{(n-1)^2}{4(1+\lambda)^2}, \tag{6.16}$$

where the norms are taken with respect to the new metric $\widetilde{\omega}$. Moreover, for the torsion $\widetilde{T}$ of $\widetilde{\omega}$, it is

$$\widetilde{T}^m_{\ell j} = \widetilde{h}^{m\overline{s}}\left(\frac{\partial \widetilde{h}_{j\overline{s}}}{\partial z^\ell} - \frac{\partial \widetilde{h}_{\ell\overline{s}}}{\partial z^j}\right) = \frac{\delta_{m\ell}\overline{z}^j - \delta_{mj}\overline{z}^\ell}{(1+\lambda)|z|^2}$$

and so

$$|\widetilde{T}|^2 = \widetilde{h}_{m\overline{n}} \cdot \widetilde{h}^{j\overline{i}} \cdot \widetilde{h}^{\ell\overline{k}} \cdot \widetilde{T}^m_{\ell j} \cdot \overline{\widetilde{T}^n_{ki}} = \frac{n-1}{2(1+\lambda)^2}. \tag{6.17}$$

Finally, by the scalar curvature relation formula (4.12), we see the Riemannian scalar curvature $\widetilde{s}$ of $\widetilde{\omega}$ is

$$\begin{aligned} \widetilde{s} &= 2\widetilde{s}_C + \left(\langle \partial\partial^*\widetilde{\omega} + \overline{\partial\partial^*}\widetilde{\omega}, \widetilde{\omega}\rangle - 2|\partial^*\widetilde{\omega}|^2\right) - \frac{1}{2}|\widetilde{T}|^2 \\ &= 2\widetilde{s}_C - \frac{1}{2}|\widetilde{T}|^2 \\ &= \frac{n(n-1)}{2(1+\lambda)^2}\left[\lambda - \frac{1-2n}{2n}\right]. \end{aligned} \tag{6.18}$$

It is easy to see that when $\lambda \in (-1, \infty)$,

$$\widetilde{s} \in \left(-\infty, \frac{n^2(n-1)}{4}\right]. \tag{6.19}$$

More precisely, when $\lambda = \frac{1-n}{n}$, $\widetilde{s} = \frac{n^2(n-1)}{4}$. Hence, for the smooth family of Hermitian metrics $\widetilde{\omega} = \omega_0 + 4\lambda\mathfrak{R}^{(1)}(\omega_h)$, with $\lambda > -1$,

(1) $\lambda > \frac{1-2n}{2n}$, $\widetilde{\omega}$ has positive constant Riemannian scalar curvature $\widetilde{s}$;
(2) when $\lambda = \frac{1-2n}{2n}$, $\widetilde{\omega}$ has constant zero Riemannian scalar curvature $\widetilde{s}$;
(3) when $-1 < \lambda < \frac{1-2n}{2n}$, $\widetilde{\omega}$ has negative constant Riemannian scalar curvature $\widetilde{s}$. Note also that when $\lambda \to -1$, $\widetilde{s} \to -\infty$.

Theorem 6.4. *For any $n \geq 2$, there exists an n-dimensional compact complex manifold X with $c_1(X) \geq 0$, such that X admits three different **Gauduchon metrics** ω_1, ω_2 and ω_3 with the following properties.*

(1) $[\omega_1] = [\omega_2] = [\omega_3] \in H^{1,1}_A(X)$;

(2) *they have the same semi-positive Chern-Ricci curvature, i.e.*

$$\Theta^{(1)}(\omega_1) = \Theta^{(1)}(\omega_2) = \Theta^{(1)}(\omega_3) \geq 0;$$

(3) *they have constant positive Chern scalar curvatures.*

Moreover,

(1) ω_1 *has* **positive** *constant Riemannian scalar curvature;*
(2) ω_2 *has* **zero** *Riemannian scalar curvature;*
(3) ω_3 *has* **negative** *constant Riemannian scalar curvature.*

Proof. Let $Y = \mathbb{S}^3 \times \mathbb{S}^1$ and ω_0 the canonical metric. It is easy to see that $\partial\overline{\partial}\omega_0 = 0$, i.e. ω_0 is a Gauduchon metric. Indeed,

$$\partial\overline{\partial}\omega_0 = \partial\left(-\frac{4z^\ell}{|z|^4}d\overline{z}^\ell \wedge dz^i \wedge d\overline{z}^i\right) = \left(-\frac{4\delta_{k\ell}}{|z|^4} + \frac{8\overline{z}^k z^\ell}{|z|^6}\right)dz^k \wedge d\overline{z}^\ell \wedge dz^i \wedge d\overline{z}^i = 0$$

since the complex dimension of Y is 2. As in the previous paragraph, let

$$\omega_\lambda := \omega_0 + 4\lambda\mathfrak{R}^{(1)}(\omega_0) = \omega_0 + 4\sqrt{-1}\lambda\partial\overline{\partial}\log|z|^2, \quad \text{with} \quad \lambda > -1. \tag{6.20}$$

Hence $[\omega_\lambda] = [\omega_0] \in H^{1,1}_A(Y)$ for any $\lambda > -1$, and ω_λ are all Gauduchon metrics. On the other hand, ω_λ has first Chern-Ricci curvature

$$\Theta^{(1)}(\omega_\lambda) = 2\sqrt{-1}\partial\overline{\partial}\log|z|^2 \geq 0.$$

The Chern scalar curvature are

$$s_C(\omega_\lambda) = \frac{1}{2(1+\lambda)} > 0 \tag{6.21}$$

since $\lambda > -1$. Since $n = 2$, by formula (6.18), the Riemannian scalar curvature of ω_λ is

$$s(\omega_\lambda) = \frac{1}{(1+\lambda)^2}\left[\lambda + \frac{3}{4}\right]. \tag{6.22}$$

Therefore

(1) $\lambda > -\frac{3}{4}$, ω_λ has positive constant Riemannian scalar curvature;
(2) when $\lambda = -\frac{3}{4}$, ω_λ has constant zero Riemannian scalar curvature;
(3) when $-1 < \lambda < -\frac{3}{4}$, ω_λ has negative constant Riemannian scalar curvature.

Therefore, in dimension 2, Theorem 6.4 is proved.

For higher dimensional cases, let (Z, ω_Z) be any compact Kähler manifold with zero (Chern) scalar curvature. For example we can take (Z, ω_Z) as the standard $(n-2)$-torus $(\mathbb{T}^{n-2}, \omega_{\mathbb{T}})$. Hence the background Riemannian metric has zero Riemannian scalar curvature. Let X be the product manifold $Y \times Z$, and ω_X the product metric of ω_Z and ω_λ. It is obvious that (X, ω_X) is a Gauduchon metric with semi-positive Chern-Ricci curvature and positive constant Chern scalar curvature. Moreover, the Riemannian scalar curvature of ω_X equals the Riemannian scalar curvature of ω_λ thanks to the product structure. $\square$

6.3. **Pluriclosed metrics on the projective bundles over Hopf manifolds.** Let $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ with $n \geq 2$, and $E = T^{1,0}M$. Suppose $X = \mathbb{P}(E^*)$ and $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$ is the tautological line bundle of the fiber bundle $\pi : X \to M$. By the adjunction formula,

$$K_X = L^{-n} \otimes \pi^* \left(K_M \otimes \det E\right) \tag{6.23}$$

we see

$$K_X = L^{-n}. \tag{6.24}$$

It is obvious, when restricted to the fiber $X_s := \mathbb{P}(E_s^*) \cong \mathbb{P}^{n-1}$, $K_X|_{X_s} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-n)$. Hence, K_X is not topologically trivial and moreover, $c_1(X) \geq 0$. However, by a straightforward calculation we see $c_1^{2n-1}(X) = 0$ where $\dim_{\mathbb{C}} X = 2n-1$. In fact, as described in [39], there is a natural Hermitian metric g on L induced by the Hermitian metric ω_h on $E = T^{1,0}M$. Since the Chern curvature tensor of $(E = T^{1,0}M, h)$ is Griffiths-semi-positive (see formula (6.4), or [41, Proposition 6.1]), the curvature $(1,1)$-form Θ_g of (L, g) is semi-positive and strictly positive on each fiber. We can see that $\left(\Theta_{\omega_h}\right)^{2n-1} = 0$ and so $c_1^{2n-1}(X) = 0$. It is easy to see that the Chern scalar curvature of X is strictly positive, i.e.

$$n \cdot tr_g \Theta_g > 0.$$

Hence, $H^0(X, mK_X) = 0$ for any integer $m > 0$. Now we summarize the discussion as follows.

Proposition 6.5. *Let $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ with $n \geq 2$, and $E = T^{1,0}M$. Suppose $X = \mathbb{P}(E^*)$.*

(1) *K_X^{-1} admits a Hermitian metric with semi-positive Chern curvature;*
(2) *K_X^{-1} is not topologically trivial, i.e. $c_1(X) \neq 0$. Moreover, $c_1(X) \geq 0$, but $c_1^{2n-1}(X) = 0$;*
(3) *$H^0(X, mK_X) = 0$ for any integer $m > 0$.*

Next we consider a more general setting. Suppose $n \geq 2$ and $k \geq 1$. Let $M_n = \mathbb{S}^{2n-1} \times \mathbb{S}^1$. $E_k = \underbrace{T^{1,0}M_n \oplus \cdots \oplus T^{1,0}M_n}_{k \text{ copies}}$ and $X_{n,k} = \mathbb{P}(E_k^*) \to M_n$. Since $\pi : X_{n,k} \to M_n$ is a proper holomorphic submersion and M_n can not admit any balanced metric, by [43, Proposition 1.9], $X_{n,k}$ can not support balanced metrics. In particular, $X_{n,k}$ is not in the Fujiki class $\mathscr{C}$. On the other hand, we consider a special case $\pi : X_{2,k} \to M_2$. It is obvious that E_k has an induced Hermitian metric with Griffiths-semi-positive curvature. Let F be the tautological line bundle of $X_{2,k} \to M_2$. The induced Hermitian metric on F has semi-positive curvature tensor Θ_F which is also strictly positive on each fiber. Now we can construct a family of Hermitian metrics ω with $\partial\overline{\partial}\omega = 0$ on $X_{2,k}$. Let ω_0 be the canonical metric (6.1) on $M_2 = \mathbb{S}^3 \times \mathbb{S}^1$, and it is obvious that $\partial\overline{\partial}\omega_0 = 0$. Then for any $\lambda > 0$,

$$\omega := \pi^*(\omega_0) + \lambda \Theta_F \tag{6.25}$$

is a Hermitian metric on $X_{2,k}$. Moreover, it satisfies $\partial\overline{\partial}\omega = 0$.

Proposition 6.6. *Suppose $n \geq 2$ and $k \geq 1$. Let $M_n = \mathbb{S}^{2n-1} \times \mathbb{S}^1$. $E_k = \underbrace{T^{1,0}M_n \oplus \cdots \oplus T^{1,0}M_n}_{k \text{ copies}}$ and $X_{n,k} = \mathbb{P}(E_k^*)$. Then*

(1) *$X_{n,k}$ can not support any balanced metric;*

(2) *$X_{2,k}$ admits a Hermitian metric ω with $\partial\overline{\partial}\omega = 0$.*

7. Appendix: the Riemannian Ricci curvature and $*$-Ricci curvature

In this appendix, we provide more details on the complexification of Riemannian Ricci curvatures on almost Hermitian manifolds.

7.1. The Riemannian Ricci curvature.

Lemma 7.1. *On an almost Hermitian manifold (M, h), the Riemannian Ricci curvature of the background Riemannian manifold (M, g) satisfies*

$$Ric(X,Y) = h^{i\overline{\ell}}\left[R\left(\frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial \overline{z}^\ell}\right) + R\left(\frac{\partial}{\partial z^i}, Y, X, \frac{\partial}{\partial \overline{z}^\ell}\right)\right] \tag{7.1}$$

for any $X, Y \in T_{\mathbb{R}}M$. The Riemannian scalar curvature of (M, g) is

$$s = 2h^{i\overline{j}}h^{k\overline{\ell}}\left(2R_{i\overline{\ell}k\overline{j}} - R_{i\overline{j}k\overline{\ell}}\right). \tag{7.2}$$

Proof. For any $X, Y \in T_{\mathbb{R}}M$, by using real coordinates $\{x^i, x^I\}$ and the relations (2.7), (2.11), (2.15) and (2.16), one can see:

$$\begin{aligned}
& Ric(X,Y) \\
= \; & g^{i\ell}R\left(\frac{\partial}{\partial x^i}, X, Y, \frac{\partial}{\partial x^\ell}\right) + g^{iL}R\left(\frac{\partial}{\partial x^i}, X, Y, \frac{\partial}{\partial x^L}\right) \\
& + g^{I\ell}R\left(\frac{\partial}{\partial x^I}, X, Y, \frac{\partial}{\partial x^\ell}\right) + g^{IL}R\left(\frac{\partial}{\partial x^I}, X, Y, \frac{\partial}{\partial x^L}\right) \\
= \; & g^{i\ell}R\left(\frac{\partial}{\partial z^i} + \frac{\partial}{\partial \overline{z}^i}, X, Y, \frac{\partial}{\partial z^\ell} + \frac{\partial}{\partial \overline{z}^\ell}\right) + \sqrt{-1}g^{iL}R\left(\frac{\partial}{\partial z^i} + \frac{\partial}{\partial \overline{z}^i}, X, Y, \frac{\partial}{\partial z^\ell} - \frac{\partial}{\partial \overline{z}^\ell}\right) \\
& + \sqrt{-1}g^{I\ell}R\left(\frac{\partial}{\partial z^i} - \frac{\partial}{\partial \overline{z}^i}, X, Y, \frac{\partial}{\partial z^\ell} + \frac{\partial}{\partial \overline{z}^\ell}\right) - g^{IL}R\left(\frac{\partial}{\partial z^i} - \frac{\partial}{\partial \overline{z}^i}, X, Y, \frac{\partial}{\partial z^\ell} - \frac{\partial}{\partial \overline{z}^\ell}\right) \\
= \; & 2g^{i\ell}\left[R\left(\frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial \overline{z}^\ell}\right) + R\left(\frac{\partial}{\partial \overline{z}^i}, X, Y, \frac{\partial}{\partial z^\ell}\right)\right] \\
& + 2\sqrt{-1}g^{iL}\left[-R\left(\frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial \overline{z}^\ell}\right) + R\left(\frac{\partial}{\partial \overline{z}^i}, X, Y, \frac{\partial}{\partial z^\ell}\right)\right] \\
= \; & h^{i\overline{\ell}}R\left(\frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial \overline{z}^\ell}\right) + h^{\ell\overline{i}}R\left(\frac{\partial}{\partial \overline{z}^i}, X, Y, \frac{\partial}{\partial z^\ell}\right) \\
= \; & h^{i\overline{\ell}}\left[R\left(\frac{\partial}{\partial z^i}, X, Y, \frac{\partial}{\partial \overline{z}^\ell}\right) + R\left(\frac{\partial}{\partial z^i}, Y, X, \frac{\partial}{\partial \overline{z}^\ell}\right)\right].
\end{aligned}$$

By using the symmetry $R(X, Y, Z, W) = R(Y, X, W, Z)$ for $X, Y, Z, W \in T_{\mathbb{C}}M$, we get the following formula for the Riemannian scalar curvature,

$$\begin{aligned}
s &= h^{i\overline{j}}h^{k\overline{\ell}}\left(R_{ik\overline{\ell}\overline{j}} + R_{i\overline{\ell}k\overline{j}} + R_{\overline{j}k\overline{\ell}i} + R_{\overline{j}\overline{\ell}ki}\right) \\
&= 2h^{i\overline{j}}h^{k\overline{\ell}}\left(R_{ik\overline{\ell}\overline{j}} + R_{i\overline{\ell}k\overline{j}}\right) \\
&= 2h^{i\overline{j}}h^{k\overline{\ell}}\left(2R_{i\overline{\ell}k\overline{j}} - R_{i\overline{j}k\overline{\ell}}\right).
\end{aligned}$$

The proof of Lemma 7.1 is complete. $\square$

7.2. $*$-Ricci curvature and $*$-scalar curvature.

Definition 7.2. Let $\{e_i\}_{i=1}^{2n}$ be an orthonormal basis of $(T_{\mathbb{R}}M, g)$, the (real) $*$-Ricci curvature of (M, g) is defined to be (e.g. [58])

$$Ric^*(X,Y) := \sum_{i=1}^{2n} R(e_i, X, JY, Je_i), \tag{7.3}$$

for any $X, Y \in T_{\mathbb{R}}M$. The $*$-scalar curvature (with respect to the Riemannian metric) is defined to be

$$s^* = \sum_{j=1}^{2n} Ric^*(e_i, e_i). \tag{7.4}$$

It is easy to see that, for any $X, Y \in T_{\mathbb{R}}M$.

$$Ric^*(X,Y) = Ric^*(JY, JX). \tag{7.5}$$

Lemma 7.3. *We have the following formula for $*$-Ricci curvature,*

$$\begin{aligned} Ric^*(X,Y) &= \sqrt{-1}h^{\ell\bar{i}} R\left(\frac{\partial}{\partial \bar{z}^i}, X, JY, \frac{\partial}{\partial z^\ell}\right) - \sqrt{-1}h^{i\bar{\ell}} R\left(\frac{\partial}{\partial z^i}, X, JY, \frac{\partial}{\partial \bar{z}^\ell}\right) && (7.6) \\ &= \sqrt{-1}h^{k\bar{\ell}} R\left(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^\ell}, X, JY\right). && (7.7) \end{aligned}$$

The $*$-scalar curvature is

$$s^* = 2h^{i\bar{j}}h^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} = 2s_H. \tag{7.8}$$

Proof. As similar as the computations in Lemma 7.1, we can write down the $*$-Ricci curvature in real coordinates $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^I}\}$ and show:

$$Ric^*(X,Y) = \sqrt{-1}h^{\ell\bar{i}} R\left(\frac{\partial}{\partial \bar{z}^i}, X, JY, \frac{\partial}{\partial z^\ell}\right) - \sqrt{-1}h^{i\bar{\ell}} R\left(\frac{\partial}{\partial z^i}, X, JY, \frac{\partial}{\partial \bar{z}^\ell}\right).$$

On the other hand,

$$\begin{aligned} & Ric^*(X,Y) + Ric^*(JY, JX) \\ = \; & \sqrt{-1}h^{\ell\bar{i}} R\left(\frac{\partial}{\partial \bar{z}^i}, X, JY, \frac{\partial}{\partial z^\ell}\right) - \sqrt{-1}h^{i\bar{\ell}} R\left(\frac{\partial}{\partial z^i}, X, JY, \frac{\partial}{\partial \bar{z}^\ell}\right) \\ & -\sqrt{-1}h^{\ell\bar{i}} R\left(\frac{\partial}{\partial \bar{z}^i}, JY, X, \frac{\partial}{\partial z^\ell}\right) + \sqrt{-1}h^{i\bar{\ell}} R\left(\frac{\partial}{\partial z^i}, JY, X, \frac{\partial}{\partial \bar{z}^\ell}\right) \\ = \; & 2\sqrt{-1}h^{k\bar{\ell}} R\left(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^\ell}, X, JY\right), \end{aligned}$$

where the last step follows by Bianchi identity. Therefore (7.7) follows by the relation (7.5). For the scalar curvature s^*, by definition, it is

$$\begin{aligned} s^* = \; & g^{jk} Ric^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) + g^{jK} Ric^*\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^K}\right) \\ & + g^{Jk} Ric^*\left(\frac{\partial}{\partial x^J}, \frac{\partial}{\partial x^k}\right) + g^{JK} Ric^*\left(\frac{\partial}{\partial x^J}, \frac{\partial}{\partial x^K}\right). \end{aligned}$$

By using the symmetry $R(X,Y,Z,W)=R(Y,X,W,Z)$ and (7.6), we see

$$\begin{aligned} s^* &= (\sqrt{-1}h^{\ell\bar{i}})\left[\sqrt{-1}h^{k\bar{j}}R\left(\frac{\partial}{\partial\bar{z}^i},\frac{\partial}{\partial\bar{z}^j},\frac{\partial}{\partial z^k},\frac{\partial}{\partial z^\ell}\right)-\sqrt{-1}h^{j\bar{k}}R\left(\frac{\partial}{\partial\bar{z}^i},\frac{\partial}{\partial z^j},\frac{\partial}{\partial\bar{z}^k},\frac{\partial}{\partial z^\ell}\right)\right] \\ &- (\sqrt{-1}h^{i\bar{\ell}})\left[\sqrt{-1}h^{k\bar{j}}R\left(\frac{\partial}{\partial z^i},\frac{\partial}{\partial\bar{z}^j},\frac{\partial}{\partial z^k},\frac{\partial}{\partial\bar{z}^\ell}\right)-\sqrt{-1}h^{j\bar{k}}R\left(\frac{\partial}{\partial z^i},\frac{\partial}{\partial z^j},\frac{\partial}{\partial\bar{z}^k},\frac{\partial}{\partial\bar{z}^\ell}\right)\right] \\ &= -h^{i\bar{j}}h^{k\bar{\ell}}R_{\bar{j}\bar{\ell}ki}+h^{i\bar{j}}h^{k\bar{\ell}}R_{\bar{j}k\bar{\ell}i}+h^{i\bar{j}}h^{k\bar{\ell}}R_{i\bar{\ell}k\bar{j}}-h^{i\bar{j}}h^{k\bar{\ell}}R_{ik\bar{\ell}\bar{j}} \\ &= 2h^{i\bar{j}}h^{k\bar{\ell}}R_{i\bar{j}k\bar{\ell}}=2s_H, \end{aligned}$$

where the last step follows by Bianchi identity. □

Remark 7.4. It is easy to see that $*$-Ricci curvature is neither symmetric nor skew-symmetric. For example, by using the Hermitian Ricci tensor and (7.7), we see the following submatrix of the real matrix representation of $*$-Ricci curvature:

$$\begin{aligned} Ric^*\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) &= \sqrt{-1}h^{k\bar{\ell}}R\left(\frac{\partial}{\partial z^k},\frac{\partial}{\partial\bar{z}^\ell},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^J}\right) \\ &= -h^{k\bar{\ell}}R\left(\frac{\partial}{\partial z^k},\frac{\partial}{\partial\bar{z}^\ell},\frac{\partial}{\partial z^i}+\frac{\partial}{\partial\bar{z}^i},\frac{\partial}{\partial z^j}-\frac{\partial}{\partial\bar{z}^j}\right) \\ &= \left(R_{\bar{i}\bar{j}}-R_{ij}\right)+(R_{i\bar{j}}-R_{\bar{i}j}) \\ &= \left(R_{\bar{i}\bar{j}}-R_{ij}\right)+(R_{i\bar{j}}+R_{j\bar{i}}). \end{aligned} \tag{7.9}$$

The first part in (7.9) is skew symmetric whereas the second part is symmetric. Hence, as a real $(0,2)$ tensor, it is impossible to define the positivity or negativity for the $*$-Ricci curvature. That is $Ric^*(X,X)>0$ for all nonzero vector $X\in T_{\mathbb{R}}M$ can not happen on any almost Hermitian manifold. To make the $*$-Ricci tensor a symmetric tensor, an extra condition as

$$R(X,Y,Z,W)=R(X,Y,JZ,JW) \tag{7.10}$$

is sufficient (see, e.g. [30, 58]).

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