Logarithmic vanishing theorems on compact Kähler manifolds I

Chunle Huang, Kefeng Liu, Xueyuan Wan, and Xiaokui Yang

Abstract. In this paper, we first establish an L^2-type Dolbeault isomorphism for logarithmic differential forms by Hörmander's L^2-estimates. By using this isomorphism and the construction of smooth Hermitian metrics, we obtain a number of new vanishing theorems for sheaves of logarithmic differential forms on compact Kähler manifolds with simple normal crossing divisors, which generalize several classical vanishing theorems, including Norimatsu's vanishing theorem, Gibrau's vanishing theorem, Le Potier's vanishing theorem and a version of the Kawamata-Viehweg vanishing theorem.

1. Introduction

The basic properties of the sheaf of logarithmic differential forms and of the sheaves with logarithmic integrable connections on smooth projective manifolds were developed by Deligne in [7]. Esnault and Viehweg investigated in [10] the relations between logarithmic de Rham complexes and vanishing theorems on complex algebraic manifolds, and showed that many vanishing theorems follow from the degeneration of certain Hodge to de Rham type spectral sequences. For a comprehensive description of the topic, we refer the reader to Esnault and Viehweg's work [11] and also the references therein.

In this paper, we develop an effective analytic method to prove vanishing theorems for the sheaves of logarithmic differential forms on *compact Kähler manifolds*. One of our motivations to develop this method is to prove Fujino's conjecture (e.g. [14, Conjecture 2.4], [15, Problem 1.8],[16, Section 3] and [30, Conjecture 1.2]) on a logarithmic version of the Kollar injectivity theorem. Let X be a compact Kähler manifold of dimension n and $Y = X - D$ where $D = \sum_{i=1}^{s} D_i$ is a simple normal crossing divisor in X. Suppose that E is an Hermitian vector bundle over X. We first describe the key steps and main difficulties in our analytic approach. Let h_Y^E and ω_Y be two smooth metrics on $E|_Y$ and Y respectively, then we need to show

1. K. Liu is supported in part by NSF Grant.

2. X. Wan is partially supported by China Scholarship Council / University of California, Los Angeles Joint PhD. Student.

3. X.Yang is partially supported by China's Recruitment Program of Global Experts and National Center for Mathematics and Interdisciplinary Sciences, Chinese Academy of Sciences.

(a) there is an L^2 fine resolution $(\Omega^{p,\bullet}_{(2)}(X,E,\omega_Y,h^E_Y),\overline{\partial})$ of the sheaf of logarithmic holomorphic differential forms $\Omega^p(\log D)\otimes\mathcal{O}(E)$ whenever the metrics h^E_Y and ω_Y are chosen to be suitable;

(b) the desired curvature conditions for h^E_Y and ω_Y can imply vanishing theorems for $(\Omega^{p,\bullet}_{(2)}(X,E,\omega_Y,h^E_Y),\overline{\partial})$ by using L^2-estimate.

The main difficulties arise from the construction of the Hermitian metric h^E_Y and the Poincaré type metric ω_Y which are suitable for both (a) and (b).

It is well-known that various vanishing theorems are very important in complex analytic geometry and algebraic geometry. For instance, the Akizuki-Kodaira-Nakano vanishing theorem asserts that if L is a positive line bundle over a compact Kähler manifold X, then

$$H^q(X,\Omega^p_X\otimes L)=0\quad\text{for any } p+q\geq\dim X+1.$$

The main purpose of this paper is to investigate logarithmic type Akizuki-Kodaira-Nakano vanishing theorems for the pair (X,D). The first main result of our paper is

Theorem 1.1. *Let X be a compact Kähler manifold of dimension n and $D=\sum_{i=1}^s D_i$ be a simple normal crossing divisor in X. Let N be a line bundle and $\Delta=\sum_{i=1}^s a_iD_i$ be an $\mathbb{R}$-divisor with $a_i\in[0,1]$ such that $N\otimes\mathcal{O}_X([\Delta])$ is a k-positive $\mathbb{R}$-line bundle. Then for any nef line bundle L, we have*

$$H^q(X,\Omega^p_X(\log D)\otimes L\otimes N)=0\quad\textit{for any } p+q\geq n+k+1.$$

As we pointed out before, the key ingredient is the construction of suitable Hermitian metrics. In the analytical setting, the positivity of $\mathbb{R}$-line bundle, which will be defined in Section 2.4, is defined by using positivity of curvature which is very flexible to use, since we can multiply arbitrary real coefficients on the curvature of a line bundle to obtain certain desired curvature property. In particular, the theory of $\mathbb{R}$-divisors (or $\mathbb{R}$-line bundles) in algebraic geometry is not used in this paper, which is also a notable advantage in our analytic approach. On the other hand, the setting in Theorem 1.1 is quite general and it has many straightforward applications in complex analytic geometry and complex algebraic geometry. The first application is the following log type Gibrau's vanishing theorem.

Corollary 1.2. *Let X be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor in X. If L is a nef line bundle and N is a k-positive line bundle over X, then*

$$H^q(X,\Omega^p_X(\log D)\otimes L\otimes N)=0\quad\textit{for any } p+q\geq n+k+1.$$

In particular, we have

Corollary 1.3. *Let X be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor. Suppose that $L \to X$ is an ample line bundle, then*

$$H^q(X, \Omega_X^p(\log D) \otimes L) = 0 \quad \textit{for any } p + q \geq n + 1.$$

This well-known result is proved by Norimatsu ([31]) using analytic methods (see also Deligne-Illusie's proof in [8] by the characteristic p methods). As an analogue to Corollary 1.3, we obtain the following log type Le Potier vanishing theorem for ample vector bundles.

Corollary 1.4. *Let X be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor. Suppose that $E \to X$ is an ample vector bundle of rank r. Then*

$$H^q(X, \Omega_X^p(\log D) \otimes E) = 0 \quad \textit{for any } p + q \geq n + r.$$

As we know that the Kawamata-Viehweg type vanishing theorems have played fundamental roles in algebraic geometry and complex analytic geometry (e.g. [10, 9, 3, 19, 17]). As another application of Theorem 1.1, we get a log type vanishing theorem for k-positive line bundles over compact *Kähler manifolds*, which generalizes a version of the Kawamata-Viehweg vanishing theorem over projective manifolds.

Theorem 1.5. *Let X be a compact Kähler manifold of dimension n and $D = \sum_{i=1}^s D_i$ be a simple normal crossing divisor. Suppose F is a line bundle over X and m is a positive real number such that $mF = L + D'$, where $D' = \sum_{i=1}^s \nu_i D_i$ is an effective normal crossing $\mathbb{R}$-divisor and L is a k-positive $\mathbb{R}$-line bundle. Then*

$$H^q\left(X, \Omega^p(\log D) \otimes F \otimes \mathcal{O}\left(-\sum_{i=1}^s \left(1 + \left[\frac{\nu_i}{m}\right]\right) D_i\right)\right) = 0 \tag{1.1}$$

for $p + q \geq n + k + 1$.

Remark. In particular, if $mF = L + D'$ where L is an ample line bundle and D' is an effective divisor, bypassing Hironaka's desingularization procedure, one obtains the classical Kawamata-Viehweg vanishing from 1.1 by taking $p = n$ and $k = 0$. It is also worth mentioning that, in [21, Theorem 6.1], Luo obtained a version of logarithmic vanishing theorem under the k-ample condition over a smooth projective variety and his proof relies on the hyperplane induction methods on projective manifold (e.g. the existence of very ample divisors). It is apparently different from our unified analytic approaches over *Kähler manifolds*. On the other hand, it is also pointed out in [32, p. 127] that, the k-ampleness is irrelevant to the k-positivity when $1 \leq k \leq \dim X$.

Theorem 1.5 has several variants and the first one is

Corollary 1.6. *Let X be a compact Kähler manifold $D = \sum_{j=1}^s D_j$ be a simple normal crossing divisor of X. Let $[D']$ be a k-positive $\mathbb{R}$-line bundle over X, where*

$D' = \sum_{i=1}^s c_i D_i$ with $c_i > 0$ and $c_i \in \mathbb{R}$. Then

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}_X(-\lceil D' \rceil)) = 0 \text{ for any } p+q < n-k.$$

In particular, when $[D']$ is ample,

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}_X(-\lceil D' \rceil)) = 0, \text{ for } p+q < n. \tag{1.2}$$

Remark. By using Serre duality, one obtains a special case of (1.2)

$$H^q(X, K_X \otimes \mathcal{O}_X(\lceil D' \rceil)) = 0, \text{ for } q > 0.$$

This is proved by Kawamata in [22, Theorem 1] (see also [23, Corollary 1-2-2], [13, Theorem 3.1.7] and [29, Theorem 5.1]).

The second variant is

Corollary 1.7. *Let X be a compact Kähler manifold and $D = \sum_{j=1}^s D_j$ be a simple normal crossing divisor of X. Let $[D']$ be a k-positive $\mathbb{R}$-line bundle over X, where $D' = \sum_{i=1}^s a_i D_i$ with $a_i > 0$ and $a_i \in \mathbb{R}$. If there exists a line bundle L over X and a real number b with $0 < a_j < b$ for all j and $bL = [D']$ as $\mathbb{R}$-line bundles. Then*

$$H^q(X, \Omega^p(\log D) \otimes L^{-1}) = 0$$

for $p+q > n+k$ and $p+q < n-k$.

Note that, Esnault and Viehweg obtained a similar result in [10, Theorem 6.2(a)] for $\mathbb{Q}$-divisors by using the the degeneration of the logarithmic Hodge to de Rham spectral sequence together with the cyclic covering trick over projective manifolds. The third variant is

Corollary 1.8. *Let X be a compact Kähler manifold of dimension n and $D = \sum_{i=1}^s D_i$ be a simple normal crossing divisor in X. Suppose there exist some real constants $a_i \geq 0$ such that $\sum_{i=1}^s a_i D_i$ is a k-positive $\mathbb{R}$-divisor, then for any nef line bundle L, we have*

$$H^q(X, \Omega_X^p(\log D) \otimes L) = 0 \quad \text{for any } p+q \geq n+k+1.$$

Note that Corollary 1.8 generalizes [13, Corollary 3.1.2].
Remark. In a sequel to this paper, we will systematically investigate a number of vanishing theorems in algebraic geometry by using analytic methods introduced in this paper. For instance, we have obtained a version of Theorem 1.1 for k-ample line bundles on algebraic manifolds.

Acknowledgements. The authors would also like to thank Junyan Cao, Yifei Chen, Jean-Pierre Demailly, Osamu Fujino, Shin-ichi Matsumura, Xiaotao Sun, Valentino Tosatti, Jian Xiao and Xiangyu Zhou for their interests and/or discussions.

2. Preliminaries

2.1. **Positivity of vector bundles.** Let E be a holomorphic vector bundle of rank r over a complex manifold M and h be a smooth Hermitian metric on E. There exists a unique connection ∇, called the Chern connection of (E,h), which is compatible with the metric h and complex structure on E. Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on M and $\{e_\alpha\}_{\alpha=1}^r$ be the local holomorphic frames of E. Locally, the curvature tensor of (E,h) takes the form

$$\sqrt{-1}\Theta(E,h)=\sqrt{-1}R_{i\bar{j}\alpha}^{\gamma}dz^i\wedge d\bar{z}^j\otimes e^\alpha\otimes e_\gamma$$

where $R_{i\bar{j}\alpha}^{\gamma}=h^{\gamma\overline{\beta}}R_{i\bar{j}\alpha\overline{\beta}}$ and

$$R_{i\bar{j}\alpha\overline{\beta}}=-\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i\partial\bar{z}^j}+h^{\gamma\overline{\delta}}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial\bar{z}^j}. \tag{2.1}$$

Here and henceforth we adopt the Einstein convention for summation.

Definition 2.1. An Hermitian vector bundle (E,h) is said to be Griffiths-positive, if for any nonzero vectors $u=u^i\frac{\partial}{\partial z^i}$ and $v=v^\alpha e_\alpha$,

$$\sum_{i,j,\alpha,\beta}R_{i\bar{j}\alpha\overline{\beta}}u^i\bar{u}^jv^\alpha\bar{v}^\beta>0.$$

(E,h) is said to be Nakano-positive, if for any nonzero vector $u=u^{i\alpha}\frac{\partial}{\partial z^i}\otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta}R_{i\bar{j}\alpha\overline{\beta}}u^{i\alpha}\bar{u}^{j\beta}>0.$$

(E,h) is said to be dual-Nakano-positive, if for any nonzero vector $u=u^{i\alpha}\frac{\partial}{\partial z^i}\otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta}R_{i\bar{j}\alpha\overline{\beta}}u^{i\beta}\bar{u}^{j\alpha}>0.$$

Definition 2.2 (cf. [32])**.** Let M be a compact complex manifold and $L\to M$ be a holomorphic line bundle over M.

(1) L is called *k-positive* $(0\le k\le n-1)$ if there exists a smooth Hermitian metric h^L on L such that the curvature form $\sqrt{-1}\Theta(L,h^L)=-\sqrt{-1}\partial\overline{\partial}\log h^L$ is semipositive everywhere and has at least $n-k$ positive eigenvalues at every point of M.

(2) L is called *k-ample* $(0\le k\le n-1)$, if L is semi-ample and suppose that L^m is globally generated for some $m>0$, and the maximum dimension of the fiber of the evaluation map $X\to\mathbb{P}\left(H^0(M,L^m)^*\right)$ is $\le k$.

Hence, the concepts of 0-positivity, 0-ampleness and ampleness are the same. However, it is pointed out in [32, p. 127] that, k-ampleness is irrelevant to the metric k-positivity when $k\ge 1$.

2.2. **Simple normal crossing divisors and Poincaré Type Metric.** On a compact Kähler manifold X, a divisor $D = \sum_{i=1}^{s} D_i$ is called a *simple normal crossing divisor* if every irreducible component D_i is smooth and all intersections are transverse. That is, for every $p \in X$, we can choose local coordinates $z_1, \cdots, z_n$ such that $D = (\prod_{i=1}^{k} z_i = 0)$ in a neighborhood of p. The sheaf of germs of differential p-forms on X with at most logarithmic poles along D, denoted $\Omega_X^p(\log D)$ (introduced by Deligne in [6]) is the sheaf whose sections on an open subset V of X are

$$\Gamma(V, \Omega_X^p(\log D)) := \{\alpha \in \Gamma(V, \Omega_X^p \otimes \mathcal{O}_X(D)) \text{ and } d\alpha \in \Gamma(V, \Omega_X^{p+1} \otimes \mathcal{O}_X(D))\}. \tag{2.2}$$

We will consider the complement $Y = X - D$ of a simple normal crossing divisor D in a compact Kähler manifold X. It is well-known that we can choose a local coordinate chart $(W; z_1, ..., z_n)$ of X such that the locus of D is given by $z_1 \cdots z_k = 0$ and $Y \cap W = W_r^* = (\Delta_r^*)^k \times (\Delta_r)^{n-k}$ where Δ_r (resp. Δ_r^*) is the (resp. punctured) open disk of radius r in the complex plane and $r \in (0, \frac{1}{2}]$. Instead of focusing on the compact complex manifold X, we shall give a Kähler metric ω_Y only on the open manifold Y, which enjoys some special asymptotic behaviors along D.

Definition 2.3. We say that the metric ω_Y on Y is of Poincaré type along D, if for each local coordinate chart $(W; z_1, ..., z_n)$ along D the restriction $\omega_Y|_{W^*_{\frac{1}{2}}}$ is equivalent to the usual Poincaré type metric ω_P defined by

$$\omega_P = \sqrt{-1} \sum_{j=1}^{k} \frac{dz_j \wedge d\overline{z}_j}{|z_j|^2 \cdot \log^2 |z_j|^2} + \sqrt{-1} \sum_{j=k+1}^{n} dz_j \wedge d\overline{z}_j. \tag{2.3}$$

Two nonnegative functions or Hermitian metrics f and g defined on $W^*_{\frac{1}{2}}$ are said to be equivalent along D if for any relatively compact subdomain V of W, there is a positive constant C such that $(1/C)g \leq f \leq Cg$ on $V - D$. In this case we shall use the notation $f \sim g$.

As a fundamental result along this line, it is well-known that, see [34, Section 3], there always exists a Kähler metric ω_Y on $Y = X - D$ which is of Poincaré type along D. Furthermore, this metric is complete and of finite volume. Moreover, it has bounded geometry which implies that its curvature tensor and covariant derivatives are bounded. We will use these properties frequently in this paper. The following model example is used in analyzing the integrability of holomorphic sections with respect to the Poincaré type metrics.

Example 2.4. For any positive integer n, the integral

$$\int_0^{\frac{1}{2}} r^{\alpha} (\log r)^n dr$$

is finite if and only if $\alpha > -1$.

2.3. **L^2-Estimates and L^2 Cohomology.** We need the following L^2-estimates, which will be used frequently in this paper.

Lemma 2.5 ([2, 20, 9])**.** *Let (M, ω) be a complete Kähler manifold. Let (E, h^E) be an Hermitian vector bundle over M. Assume that $A = [i\Theta(E, h^E), \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{p,q}T^*M \otimes E$, $q \geq 1$. Then for any form $g \in L^2(X, \Lambda^{p,q}T^*M \otimes E)$ satisfying $\overline{\partial} g = 0$ and $\int_M (A^{-1}g, g)dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^*M \otimes E)$ such that $\overline{\partial} f = g$ and*

$$\int_M |f|^2 dV_\omega \leq \int_M (A^{-1}g, g)dV_\omega.$$

Suppose that ω_Y is a smooth Kähler metric on Y and h_Y^E is a smooth Hermitian metric on $E|_Y$. The sheaf $\Omega^{p,q}_{(2)}(X, E, \omega_Y, h_Y^E)$ (for short $\Omega^{p,q}_{(2)}(X, E)$) over X is defined as follows. On any open subset U of X, the section space $\Gamma(U, \Omega^{p,q}_{(2)}(X, E))$ of $\Omega^{p,q}_{(2)}(X, E)$ over U consists of E-valued (p, q)-forms u with measurable coefficients such that the L^2 norms of both u and $\overline{\partial} u$ are integrable on any compact subset V of U. Here the integrability means that both $|u|^2_{\omega_Y \otimes h_Y^E}$ and $|\overline{\partial}^E u|_{\omega_Y \otimes h_Y^E}$ are integrable on $V - D$. It is well-known that the spaces of global sections of $\Omega^{p,q}_{(2)}(X, E)$ with $\overline{\partial}$ operator form an L^2 Dolbeault complex

$$\Gamma(X, \Omega^{p,0}_{(2)}(X, E)) \to \Gamma(X, \Omega^{p,1}_{(2)}(X, E)) \to \cdots \to \Gamma(X, \Omega^{p,n}_{(2)}(X, E)) \to 0 \tag{2.4}$$

and the associated cohomology groups $H^{p,*}_{(2)}(Y, E)$ are called the L^2 Dolbeault cohomology groups with values in E.

Recall that a sheaf $\mathscr{S}$ over X is called a *fine sheaf* if for any finite open covering $\mathfrak{U} = \{U_j\}$, there is a family of homomorphisms $\{h_j\}$, $h_j : \mathscr{S} \to \mathscr{S}$, such that the support of h_j satisfying that $\mathrm{Supp}(h_j) \subset U_j$ and $\sum_j h_j = \text{identity}$ (cf. [24, Definition 3.13]). For any fine sheaf $\mathscr{S}$, one has $H^q(X, \mathscr{S}) = 0$ for $q \geq 1$ (cf. [24, Theorem 3.9]).

We have already known that if the Kähler metric ω_Y on Y is of Poincaré type along D, then the Kähler metric ω_Y will be complete and with finite volume (cf. [34]). In this case if u is an E-valued (p, q)-form such that u and $\overline{\partial} u$ are L^2 local integrable on U and if f is a smooth function on X, then it is obvious that both fu and $\overline{\partial}(fu)$ will still be L^2 local integrable on U. Thus the sheaf $\Omega^{p,q}_{(2)}(X, E)$ admits a partition of unity and we conclude that $\Omega^{p,q}_{(2)}(X, E)$ is a fine sheaf over X, so we have $H^i(X, \Omega^{p,q}_{(2)}(X, E)) = 0$, for $p, q \geq 0$ and $i \geq 1$.

2.4. **$\mathbb{R}$-divisors and $\mathbb{R}$-linear equivalence.**
For readers' convenience, we explain the notions in Theorem 1.1. Let X be a compact Kähler manifold.

(1). T is called an $\mathbb{R}$-divisor, if it is an element of $\mathrm{Div}_{\mathbb{R}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathrm{Div}(X)$ is the set of divisors in X. Two divisors T_1, T_2 in $\mathrm{Div}_{\mathbb{R}}(X)$ are said to be linearly equivalent, denoted by $T_1 \sim_{\mathbb{R}} T_2$, if their difference $T_1 - T_2$ can be written as a finite sum of principal divisors with real coefficients, i.e.

$$T_1 - T_2 = \sum_{i=1}^{k} r_i(f_i) \tag{2.5}$$

where $r_i \in \mathbb{R}$ and (f_i) is the principal divisor associated to a meromorphic function f_i (cf. [13, 5.2.3]).

(2). An $\mathbb{R}$-line bundle $L = \sum_i a_i L_i$ is a finite sum with some real numbers $a_1, \cdots, a_k$ and certain line bundles $L_1, \cdots, L_k$. Note that we also use "$\otimes$" for operations on line bundles. An $\mathbb{R}$-line bundle $L = \sum_i a_i L_i$ is said to be k-positive if there exist smooth metrics $h_1, \cdots, h_k$ on $L_1, \cdots, L_k$ such that the curvature of the induced metric on L, which is explicitly given by

$$\sqrt{-1}\Theta(L,h) = \sum_{i=1}^{k} a_i \sqrt{-1}\Theta(L_i, h_i)$$

is k-positive. It is easy to see that if there is another expression $L = \sum_{i=1}^{\ell} b_i \widetilde{L}_i$ for the k-positive line bundle L, then by $\partial\overline{\partial}$-lemma on compact Kähler manifolds, we can find smooth metrics on $\widetilde{L}_i$ such that the induced metric on L is also k-positive. The definitions for $\mathbb{Q}$-line bundles and $\mathbb{Q}$-divisors are similar.

Remark 2.6. As we shall see in the proofs of Theorem 1.1 and its applications, the Hermitian metrics on $\mathbb{R}$-line bundles and $\mathbb{R}$-divisors play the key role in the analytic approaches.

3. An L^2-type Dolbeault isomorphism

In this section we will establish an L^2-type Dolbeault isomorphism by using Hörmander's L^2-estimates.

Theorem 3.1. *Let (X, ω) be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor in X. Let ω_P be a smooth Kähler metric on $Y = X - D$ which is of Poincaré type along D. Then there exists a smooth Hermitian metric h_Y^L on $L|_Y$ such that the sheaf $\Omega^p(\log D) \otimes \mathcal{O}(L)$ over X enjoys a fine resolution given by the L^2 Dolbeault complex $(\Omega_{(2)}^{p,*}(X, L, \omega_P, h_Y^L), \overline{\partial})$, that is, we have an exact sequence of sheaves over X*

$$0 \to \Omega^p(\log D) \otimes \mathcal{O}(L) \to \Omega_{(2)}^{p,*}(X, L, \omega_P, h_Y^L) \tag{3.1}$$

such that $\Omega_{(2)}^{p,q}(X, L, \omega_P, h_Y^L)$ is a fine sheaf for any $0 \le p, q \le n$. In particular,

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}(L)) \cong H_{(2)}^{p,q}(Y, L, \omega_P, h_Y^L) \cong \mathbb{H}_{(2)}^{p,q}(Y, L, \omega_P, h_Y^L). \tag{3.2}$$

Proof. Let h^L be an arbitrary smooth Hermitian metric on L over X. Let σ_i be the defining section of D_i. Fix smooth Hermitian metrics $\|\bullet\|_{D_i}$ on $[D_i]$ such that $\|\sigma_i\|_{D_i} < \frac{1}{2}$. For arbitrarily fixed constants $\tau_i \in (0,1]$, we construct a smooth Hermitian metric $h^L_{\alpha,\tau} := h^L_Y$ on $L|_Y$ as

$$h^L_{\alpha,\tau} = \prod_{i=1}^{s} \|\sigma_i\|^{2\tau_i}_{D_i} (\log^2(\|\sigma_i\|^2_{D_i}))^{\frac{\alpha}{2}} h^L. \tag{3.3}$$

where α is a large positive constant (even integer) to be determined later. It is well known that $\Omega^{p,q}_{(2)}(X, L, \omega_P, h^L_Y)$ are fine sheaves over X since ω_P on $Y = X - D$ is of Poincaré type along D, so we only need to check the exactness of (3.1).

First let us consider the exactness of (3.1) at $q = 0$. Let $(W; z_1, ..., z_n)$ be a local coordinate chart of X along D. Let e be a trivialization section of L on W such that $\frac{1}{2} \leq |e(z)|_{h^L} \leq 1$ over W. Denote

$$\zeta_j = \frac{1}{z_j} dz_j, \text{ for } 1 \leq j \leq t; \text{ and } \zeta_j = dz_j, \text{ for } t+1 \leq j \leq n.$$

Let σ be a holomorphic section of $\Omega^{p,0}_{(2)}(X, L)$ on W. Then we can write

$$\sigma(z) = \sum_{|I|=p} \sigma_I(z) \zeta_{i_1} \wedge \cdots \wedge \zeta_{i_p} \otimes e$$

where $I = (i_1, ..., i_p)$ is a multi-index with $i_1 < \cdots < i_p$ and $\sigma_I(z)$ is a holomorphic function on $W^*_{1/2}$. By definition, we see that σ is L^2 integrable on $W^*_r \triangleq \Delta^{*t}_r \times \Delta^{n-t}_r \subset W^*_{1/2}$ for any $0 < r < 1/2$. Note that the Hermitian metric $h^L_{\alpha,\tau}|_{W^*_{1/2}}$ is equivalent to the following Hermitian metric

$$h^L_\alpha = h^L_\alpha(W^*_{1/2}) = \prod_{i=1}^{t} |z_i|^{2\tau_i} (\log^2 |z_i|^2)^{\frac{\alpha}{2}} h^L. \tag{3.4}$$

If we denote $\{i_1, ..., i_p\} \cap \{1, ..., t\} = \{i_{p1}, ..., i_{pb}\}$, then

$$\|\sigma\|^2_{L^2(W^*_r)} = \sum_{|I|=p} \int_{W^*_r} |e|^2_{h^L} \left(|\sigma_I(z)|^2 \prod_{\nu=1}^{b} \log^2 |z_{i_{p\nu}}|^2 \prod_{i=1}^{t} |z_i|^{2\tau_i} (\log^2 |z_i|^2)^{\frac{\alpha}{2}} \right) \omega^n_P. \tag{3.5}$$

Suppose that the Laurent series representation of $\sigma_I(z)$ on $W^*_{1/2}$ is given by

$$\sigma_I(z) = \sum_{\beta=-\infty}^{\infty} \sigma_{I\beta}(z_{t+1}, ..., z_n) z_1^{\beta_1} \cdots z_t^{\beta_t}, \beta = (\beta_1, ..., \beta_t)$$

where $\sigma_{I\beta}(z_{t+1}, ..., z_n)$ is a holomorphic function on $\Delta^{n-t}_{1/2}$. Then by using polar coordinates and Fubini's theorem (e.g. Example 2.4), we see that σ is L^2 integrable on W^*_r if and only if $\beta_j > -\tau_j$ along D_j. Since $\tau_j \in (0,1]$, we see $\beta_j \geq 0$ and $\sigma_I(z)$ has removable singularity. Hence σ and $\nabla\sigma$ have only logarithmic pole, and σ is a section of $\Omega^p(\log D) \otimes \mathcal{O}(L)$ on W. Conversely, if we choose σ to be a holomorphic section of $\Omega^p(\log D) \otimes \mathcal{O}(L)$ on W, it is easy to check by formula (3.5) that σ is L^2 integrable on W_r^* for any $0 < r < \frac{1}{2}$. Therefore we have proved that (3.1) is exact at $\Omega^{p,0}_{(2)}(X, L)$ for any $\alpha > 0$.

Now we consider the exactness of (3.1) at $q \geq 1$. For any fixed $r \in (0, 1/2)$, we deform the Kähler metric ω_P to be a new Kähler metric $\widetilde{\omega}_P$ on W_r^*, given by

$$\widetilde{\omega}_P = \widetilde{\omega}_P(W_r^*) = \omega_P + \sqrt{-1}\sum_{i=1}^{n} \partial\overline{\partial}\psi_i = \sqrt{-1}\sum_{i=1}^{n} \widetilde{g}_{ii} dz_i \wedge d\overline{z}_i \tag{3.6}$$

where $\psi_i(z) = \frac{1}{r^2 - |z_i|^2}, z \in W_r^*$. Then it is easy to check that $\widetilde{\omega}_P$ is a complete Kähler metric on W_r^*. We define a new Hermitian metric $\widetilde{h}^L_\alpha$ for L on W_r^* as

$$\widetilde{h}^L_\alpha = \widetilde{h}^L_\alpha(W_r^*) = \prod_{i=1}^{t} |z_i|^{2\tau_i} (\log^2 |z_i|^2)^{\frac{\alpha}{2}} \prod_{i=1}^{n} \exp(-2\alpha|z_i|^2 - \alpha\psi_i) h^L. \tag{3.7}$$

Lemma 3.2. *On W_r^* the Chern curvature of $\widetilde{h}^L_\alpha$ satisfies*

$$\sqrt{-1}\Theta(L, \widetilde{h}^L_\alpha) \geq \alpha\widetilde{\omega}_P \tag{3.8}$$

for some large $\alpha > 0$.

Proof. It is easy to show that

$$\sqrt{-1}\partial\overline{\partial}\log|z_i|^2 = 0 \quad \text{and} \quad -\sqrt{-1}\partial\overline{\partial}\log(\log(|z_j|^2))^2 = \frac{2\sqrt{-1}dz_j \wedge d\overline{z}_j}{|z_j|^2(\log(|z_j|^2))^2}. \tag{3.9}$$

The curvature of $(L, \widetilde{h}^L_{\alpha,\tau})$ is given by

$$\begin{aligned}
\sqrt{-1}\Theta(L, \widetilde{h}^L_\alpha) = & -\sqrt{-1}\sum_{i=1}^{t} \tau_i \partial\overline{\partial}\log|z_i|^2 - \sqrt{-1}\frac{\alpha}{2}\sum_{i=1}^{t}\partial\overline{\partial}\log(\log^2|z_i|^2) \\
& + 2\sqrt{-1}\alpha\sum_{i=1}^{n}\partial\overline{\partial}|z_i|^2 + \sqrt{-1}\alpha\sum_{i=1}^{n}\partial\overline{\partial}\psi_i + \sqrt{-1}\Theta(h^L) \\
\geq & -\sqrt{-1}\frac{\alpha}{2}\sum_{i=1}^{t}\partial\overline{\partial}\log(\log^2|z_i|^2) + \sqrt{-1}\alpha\sum_{i=1}^{n}\partial\overline{\partial}|z_i|^2 + \sqrt{-1}\alpha\sum_{i=1}^{n}\partial\overline{\partial}\psi_i \\
\geq & \ \alpha\widetilde{\omega}_P,
\end{aligned}$$

if we choose α large enough so that $\sqrt{-1}\alpha\sum_{i=1}^{n}\partial\overline{\partial}|z_i|^2 + \sqrt{-1}\Theta(h^L) \geq 0$ on W_r^*. ☐

Lemma 3.3. *On the chart W_r^*, the vector bundle $V := \Omega^p_Y \otimes K_Y^{-1} \otimes L|_Y$ with the induced metric h^V by $\widetilde{\omega}_P$ and $\widetilde{h}^L_\alpha$ is Nakano positive when α is large enough. Moreover, for any $u \in \Gamma(W^*_{\frac{r}{2}}, \Lambda^{n,q}T^*Y \otimes V)$ we have*

$$\langle\left[\sqrt{-1}\Theta(V, h^V), \Lambda_{\widetilde{\omega}_P}\right] u, u\rangle \geq C|u|^2 \tag{3.10}$$

where C is a positive constant independent of u.

Proof. Note that the metric $\widetilde{\omega}_P$ on the holomorphic tangent bundle TY is of the splitting form, i.e.

$$\widetilde{\omega}_P = \sum_{i=1}^{n} \omega_i(z_i), \tag{3.11}$$

and that the metric $\omega_i(z_i)$ depends only on the variable z_i. Hence, by using curvature formula (2.1), in local computations, we can treat $(TY, \widetilde{\omega}_P)$ as a direct sum of line bundles $\oplus_{i=1}^{n}(F_i, \omega_i)$. It is easy to check that the curvature of (F_i, ω_i)

$$|\sqrt{-1}\partial\overline{\partial}\log\omega_i| \leq C\widetilde{\omega}_P \tag{3.12}$$

for some positive constant C independent of α. Hence, in local computations, the curvature of $V = \Omega_Y^p \otimes K_Y^{-1} \otimes L|_Y$ is the curvature of a direct sum of line bundles $L|_Y \otimes F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1}$. Therefore, by using the curvature estimate (3.8), when $\alpha > (n-p+1)C$, the curvature of each summand $L \otimes F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1}$ is strictly positive. That means V is Nakano positive. The inequality (3.10) follows from a straightforward calculation. □

Lemma 3.4. *The sequence of (3.1) is exact at $q \geq 1$. That is, on a small local chart $W^*_{\frac{r}{2}} = (\Delta^*_{\frac{r}{2}})^k \times (\Delta_{\frac{r}{2}})^{n-k}$, for any $\overline{\partial}$-closed L-valued (p,q) form η on $W^*_{\frac{r}{2}}$, if it is L^2-integrable with respect to (ω_P, h^L_α), then there exists an L-valued $(p,q-1)$ form f on $W^*_{\frac{r}{2}}$ such that f is L^2-integrable with respect to (ω_P, h^L_α) and $\overline{\partial} f = \eta$.*

Proof. For simplicity, we write $W = W^*_{\frac{r}{2}}$. Suppose $\eta \in \Gamma(W, \Lambda^{p,q}T^*Y \otimes L)$ is $\overline{\partial}_L$ closed and L^2-integrable with respect to (ω_P, h^L_α). Note that $V = \Omega^p_Y \otimes K_Y^{-1} \otimes L|_Y$, we have

$$\Gamma(W, \Lambda^{n,q}T^*Y \otimes V) \cong \Gamma(W, \Lambda^{p,q}T^*Y \otimes L|_Y). \tag{3.13}$$

Since $\widetilde{h}^L_{\alpha,\tau} \sim h^L_{\alpha,\tau}$, $\omega_P \sim \widetilde{\omega}_P$ on W, by Lemma 3.3 and Lemma 2.5, there exists

$$f \in \Gamma(W, \Lambda^{n,q-1}T^*Y \otimes V) \cong \Gamma(W, \Lambda^{p,q-1}T^*Y \otimes L)$$

such that $\overline{\partial} f = \eta$ on $W^*_{\frac{r}{2}}$, and f is L^2- integrable with respect to $(\widetilde{\omega}_P, \widetilde{h}^L_\alpha)$. By restricting to $W = W^*_{\frac{r}{2}}$, we have that f is also L^2-integrable over W with respect to (ω_P, h^L_α). □

Given the exact sequence in (3.1), the isomorphisms in (3.2) are clear. The proof of Theorem 3.1 is completed. □

Remark 3.5. (1) The isomorphism (3.2) holds up to equivalence of metrics. More precisely, if $\widetilde{\omega}_P \sim \omega_P$ and $\widetilde{h}^L_Y \sim h^L_Y$, then

$$\mathbb{H}^{p,q}_{(2)}(Y, L, \omega_P, h^L_Y) \cong \mathbb{H}^{p,q}_{(2)}(Y, L, \widetilde{\omega}_P, \widetilde{h}^L_Y).$$

(2) From the proof of Theorem 3.1, it is easy to see that the isomorphism in Theorem 3.1 also works for vector bundles.

4. Logarithmic vanishing theorems

In this section, we prove Theorem 1.1 and several applications described in the first section.

Theorem 4.1 (=Theorem 1.1)**.** *Let X be a compact Kähler manifold of dimension n and $D = \sum_{i=1}^{s} D_i$ be a simple normal crossing divisor in X. Let N be a line bundle and $\Delta = \sum_{i=1}^{s} a_i D_i$ be an $\mathbb{R}$-divisor with $a_i \in [0,1]$ such that $N \otimes \mathcal{O}_X([\Delta])$ is a k-positive $\mathbb{R}$-line bundle. Then for any nef line bundle L, we have*

$$H^q(X, \Omega_X^p(\log D) \otimes L \otimes N) = 0 \quad \text{for any } p+q \geq n+k+1.$$

Proof. Let ω_0 be a fixed Kähler metric on X. Let $F = N \otimes \mathcal{O}_X([D])$. Since F is a k-positive $\mathbb{R}$-line bundle, there exist smooth metrics h^N and $h^{[D_i]}$ on F and $[D_i]$ respectively, such that the curvature form of the induced metric h^F on F

$$\sqrt{-1}\Theta(F, h^F) = \sqrt{-1}\Theta(N, h^N) + \sqrt{-1}\sum_{i=1}^{s} a_i \Theta([D_i], h^{[D_i]}) \tag{4.1}$$

is semipositive and has at least $n-k$ positive eigenvalues at each point of X.

Let $\{\lambda_{\omega_0}^j(h^F)\}_{j=1}^n$ be the eigenvalues of $\sqrt{-1}\Theta(F, h^F)$ with respect to ω_0 such that $\lambda_{\omega_0}^j(h^F) \leq \lambda_{\omega_0}^{j+1}(h^F)$ for all j. Thus for any $j \geq k+1$ we have

$$\lambda_{\omega_0}^j(h^F) \geq \lambda_{\omega_0}^{k+1}(h^F) \geq \min_{x\in X}\left(\lambda_{\omega_0}^{k+1}(h^F)(x)\right) =: c_0 > 0.$$

We set $\delta = \frac{c_0}{32n^2}$. Without loss of generality, we assume $\delta \in (0,1)$. Since L is nef, there exists a smooth metric h_δ^L on L such that

$$\sqrt{-1}\Theta(L, h_\delta^L) = -\sqrt{-1}\partial\overline{\partial}\log h_\delta^L > -\delta\omega_0. \tag{4.2}$$

Let σ_i be the defining section of D_i. Fix smooth metrics $h_{D_i} := \|\cdot\|_{D_i}^2$ on line bundles $[D_i]$, such that $\|\sigma_i\|_{D_i} < \frac{1}{2}$. Write the curvature form of $[D_i]$ as $c_1(D_i) = \sqrt{-1}\Theta([D_i], h_{D_i})$. We define $h^\Delta := \prod_{i=1}^{s} h_{D_i}^{a_i}$, then the curvature form of (Δ, h^Δ) is

$$-\sqrt{-1}\partial\overline{\partial}\log h^\Delta = -\sqrt{-1}\partial\overline{\partial}\log\prod_{i=1}^{s} h_{D_i}^{a_i}. \tag{4.3}$$

For simplicity, we set

$$\mathscr{F} := L \otimes N = L \otimes F \otimes \mathcal{O}_X(-[\Delta]). \tag{4.4}$$

The induced metric on $\mathscr{F}$ is defined by

$$h_{\alpha,\epsilon,\tau}^{\mathscr{F}} = h_\delta^L \cdot h^F \cdot (h^\Delta)^{-1} \cdot \prod_{i=1}^{s} \|\sigma\|_{D_i}^{2\tau_i} \left(\log^2(\epsilon\|\sigma_i\|_{D_i}^2)\right)^{\frac{\alpha}{2}}.$$

Here the constant $\alpha > 0$ is chosen to be large enough and the constants $\tau_i, \epsilon \in (0,1]$ are to be determined later. Note that the smooth metric $h^F \cdot (h^\Delta)^{-1}$ on $N = F \otimes \mathcal{O}_X(-[\Delta])$

is the same as h^N up to a globally defined function over X. A straightforward computation shows that

$$\begin{aligned}
&\sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) \\
&= \sqrt{-1}\Theta(F,h^F) + \sqrt{-1}\Theta(L,h^L_\delta) + \sum_{i=1}^{s}(\tau_i - a_i)c_1(D_i) \\
&\quad + \sum_{i=1}^{s}\frac{\alpha c_1(D_i)}{\log(\epsilon\|\sigma_i\|^2_{D_i})} + \sqrt{-1}\sum_{i=1}^{s}\frac{\alpha\partial\log\|\sigma_i\|^2_{D_i}\wedge\overline{\partial}\log\|\sigma_i\|^2_{D_i}}{(\log(\epsilon\|\sigma_i\|^2_{D_i}))^2}.
\end{aligned} \tag{4.5}$$

Since $a_i \in [0,1]$, for a fixed large α, we can choose $\tau_1, \cdots, \tau_s \in (0,1]$ and ϵ such that $\tau_i - a_i$, ϵ are small enough and

$$-\frac{\delta}{2}\omega_0 \leq \sqrt{-1}\sum_{i=1}^{s}(\tau_i - a_i)c_1(D_i) \leq \frac{\delta}{2}\omega_0, \quad -\frac{\delta}{2}\omega_0 \leq \sum_{i=1}^{s}\frac{\alpha c_1(D_i)}{\log(\epsilon\|\sigma_i\|^2_{D_i})} \leq \frac{\delta}{2}\omega_0. \tag{4.6}$$

Note that the constants τ_i and ϵ are thus fixed, and the choice of ϵ depends on α. We set

$$\omega_Y = \sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + 2(4n+1)\delta\omega_0. \tag{4.7}$$

It is easy to check that ω_Y is a Poincaré type Kähler metric on Y. By (4.1), (4.2), (4.5) and (4.6), one has on Y

$$\sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) \geq \sqrt{-1}\Theta(F,h^F) - 2\delta\omega_0. \tag{4.8}$$

Since $\sqrt{-1}\Theta(F,h^F)$ is a semipositive (1,1) form, we see that on Y

$$\omega_Y = \sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + 2(4n+1)\delta\omega_0 \geq 8n\delta\omega_0. \tag{4.9}$$

This implies that

$$\sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) = \omega_Y - 2(4n+1)\delta\omega_0 \geq -\frac{1}{4n}\omega_Y.$$

By exactly the same argument as in the proof of Theorem 3.1 (see also Remark 3.5), when α is large enough, we obtain

$$H^q\left(X, \Omega^p_X(\log D)\otimes\mathscr{F}\right) \cong H^{p,q}_{(2)}\left(Y, \mathscr{F}, \omega_Y, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right). \tag{4.10}$$

Next, we prove the vanishing of the L^2 cohomology groups by using Lemma 2.5. On a local chart of Y, we may assume that $\omega_0 = \sqrt{-1}\sum_{i=1}^{n}\eta_i\wedge\overline{\eta}_i$ and

$$\sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) = \sqrt{-1}\sum_{i=1}^{n}\lambda^i_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)\eta_i\wedge\overline{\eta}_i.$$

Then

$$\begin{aligned}\sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) &= \sqrt{-1}\sum_{i=1}^{n} \lambda^i_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)\eta_i \wedge \overline{\eta}_i \\ &= \sqrt{-1}\sum_{i=1}^{n} \frac{\lambda^i_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)}{\lambda^i_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + 2(4n+1)\delta}\eta'_i \wedge \overline{\eta}'_i \\ &= \sqrt{-1}\sum_{i=1}^{n} \frac{16n^2\lambda^i_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)}{16n^2\lambda^i_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + (4n+1)c_0}\eta'_i \wedge \overline{\eta}'_i\end{aligned}$$

where

$$\eta'_i = \eta_i \cdot \sqrt{\lambda^i_{\omega_0}(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}) + 2(4n+1)\delta}.$$

Note that $\omega_Y = \sqrt{-1}\sum_{i=1}^n \eta'_i \wedge \overline{\eta}'_i$, and so the eigenvalues of $\sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)$ with respect to ω_Y are

$$\gamma_i := \frac{16n^2\lambda^i_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)}{16n^2\lambda^i_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + (4n+1)c_0} < 1.$$

Thus $\gamma_j \in [-\frac{1}{4n}, 1)$. On the other hand, by (4.8) one has

$$\lambda^j_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) \geq \lambda^j_{\omega_0}(h^F) - 2\delta.$$

Hence for any $j \geq k+1$, we have

$$\lambda^j_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) \geq \min_{x\in X}\left(\lambda^{k+1}_{\omega_0}(h^F)(x)\right) - 2\delta = c_0 - 2\delta = \left(1 - \frac{1}{16n^2}\right)c_0 > 0.$$

It also implies that for $j \geq k+1$,

$$\begin{aligned}\gamma_j &= \frac{16n^2\lambda^i_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right)}{16n^2\lambda^i_{\omega_0}\left(h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right) + (4n+1)c_0} \\ &\geq \frac{16n^2(1-\frac{1}{16n^2})c_0}{16n^2(1-\frac{1}{16n^2})c_0 + (4n+1)c_0} = 1 - \frac{1}{4n}.\end{aligned}$$

For any section $u \in \Gamma(Y, \Lambda^{p,q}TY \otimes \mathscr{F})$, we obtain

$$\begin{aligned}\left\langle\left[\sqrt{-1}\Theta\left(\mathscr{F}, h^{\mathscr{F}}_{\alpha,\epsilon,\tau}\right), \Lambda_{\omega_Y}\right]u, u\right\rangle &\geq \left(\sum_{i=1}^{q}\gamma_i - \sum_{j=p+1}^{n}\gamma_j\right)|u|^2 \\ &\geq \left((q-k)\left(1-\frac{1}{4n}\right) - \frac{k}{4n} - (n-p)\right)|u|^2 \\ &= \left((q+p-n-k) - \frac{q-k}{4n} - \frac{k}{4n}\right)|u|^2 \\ &\geq \frac{1}{2}|u|^2.\end{aligned}$$

Thus, Theorem 4.1 follows from (4.10) and Lemma 2.5. □

As applications of Theorem 4.1, we obtain

Corollary 4.2. *Let X be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor. Suppose that N is a k-positive line bundle and L is a nef line bundle, then*

$$H^q(X, \Omega_X^p(\log D) \otimes N \otimes L) = 0 \quad \text{for any } p + q \geq n + k + 1.$$

In particular, one can deduce the following well-known result.

Corollary 4.3. *Let X be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor. Suppose that $L \to X$ is an ample line bundle, then*

$$H^q(X, \Omega_X^p(\log D) \otimes L) = 0 \quad \text{for any } p + q \geq n + 1.$$

As an analogue to Corollary 4.3, we obtain the following log type Le Potier vanishing theorem for ample vector bundles.

Corollary 4.4. *Let X be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor. Suppose that $E \to X$ is an ample vector bundle of rank r. Then*

$$H^q(X, \Omega_X^p(\log D) \otimes E) = 0 \quad \text{for any } p + q \geq n + r.$$

Proof. Let $\pi : \mathbb{P}(E^*) \to X$ be the projective bundle of E and $\mathcal{O}_E(1)$ be the tautological line bundle. By using the Le Potier isomorphism (e.g. [32, Theorem 5.16]), we have

$$H^q(X, \Omega_X^p(\log D) \otimes E) \cong H^q(\mathbb{P}(E^*), \Omega_{\mathbb{P}(E^*)}^p(\log \pi^* D) \otimes \mathcal{O}_E(1)). \tag{4.11}$$

On the other hand, it is easy to see that $\pi^{-1}D$ is also a simple normal crossing divisor. Hence, Corollary 4.4 follows from Corollary 4.2. □

By using the same strategy as in the proof of Theorem 4.1, we also obtain several log type Nakano vanishing theorems for vector bundles on X. For instance,

Proposition 4.5. *Let E be a vector bundle of rank r and L be a line bundle on X.*

(1) *If E is Nakano positive (resp. Nakano semi-positive) and L is nef (resp. ample), then for any $q \geq 1$*
$$H^q(X, \Omega_X^n(\log D) \otimes E \otimes L) = 0.$$
(2) *If E is dual-Nakano positive (resp. dual-Nakano semi-positive) and L is nef (resp. ample), then for any $p \geq 1$*
$$H^n(X, \Omega_X^p(\log D) \otimes E \otimes L) = 0.$$
(3) *If E is globally generated and L is ample, then for any $p \geq 1$*
$$H^n(X, \Omega_X^p(\log D) \otimes E \otimes L) = 0.$$

Indeed, the vector bundle $E \otimes L$ in Proposition 4.5 is either Nakano positive or dual Nakano positive (e.g. [28]). Hence, the proof is very similar to (but simpler than) that in Theorem 4.1.

5. Applications

In this section, we present several straightforward applications of Theorem 1.1 over compact Kähler manifolds, which are also closely related to a number of classical vanishing theorems in algebraic geometry.

Theorem 5.1. *Let X be a compact Kähler manifold of dimension n and $D = \sum_{i=1}^s D_i$ be a simple normal crossing divisor. Suppose F is a line bundle over X and m is a positive real number such that $mF = L + D'$, where $D' = \sum_{i=1}^s \nu_i D_i$ is an effective normal crossing $\mathbb{R}$-divisor and L is a k-positive $\mathbb{R}$-line bundle. Then*

$$H^q\left(X, \Omega^p(\log D) \otimes F \otimes \mathcal{O}_X\left(-\sum_{i=1}^s \left(1 + \left[\frac{\nu_i}{m}\right]\right) D_i\right)\right) = 0 \tag{5.1}$$

for $p + q \geq n + k + 1$.

Proof. Let

$$N = F \otimes \mathcal{O}_X\left(-\sum_{i=1}^s \left(1 + \left[\frac{\nu_i}{m}\right]\right) D_i\right)$$

and

$$\Delta = \sum_i \left(1 + \left[\frac{\nu_i}{m}\right] - \frac{\nu_i}{m}\right) D_i.$$

We have that

$$N \otimes \mathcal{O}_X([\Delta]) = \frac{1}{m} L, \tag{5.2}$$

which is a k-positive $\mathbb{R}$-line bundle. Hence we can apply Theorem 4.1 to complete the proof. □

Corollary 5.2. *Let X be a compact Kähler manifold $D = \sum_{j=1}^s D_j$ be a simple normal crossing divisor of X. Let $[D']$ be a k-positive $\mathbb{R}$-line bundle over X, where $D' = \sum_{i=1}^s c_i D_i$ with $c_i > 0$ and $c_i \in \mathbb{R}$. Then*

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}_X(-\lceil D' \rceil)) = 0 \text{ for any } p + q < n - k.$$

In particular, when $[D']$ is ample,

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}_X(-\lceil D' \rceil)) = 0, \text{ for } p + q < n. \tag{5.3}$$

Proof. Let

$$N = \mathcal{O}_X(-D) \otimes \lceil D' \rceil, \quad \text{and} \quad \Delta = \sum_i (1 + c_i - \lceil c_i \rceil) D_i.$$

It is easy to see that

$$N \otimes \mathcal{O}_X([\Delta]) = [D'] \tag{5.4}$$

which is a k-positive $\mathbb{R}$-line bundle. By using Theorem 4.1, one has

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}_X(-D) \otimes \lceil D' \rceil) = 0 \tag{5.5}$$

for any $p+q \geq n+k+1$. By Serre duality and the isomorphism

$$(\Omega_X^p(\log D))^* \cong \Omega_X^{n-p}(\log D) \otimes \mathcal{O}_X(-K_X - D), \tag{5.6}$$

we see that (5.5) is equivalent to

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}_X(-\lceil D' \rceil)) = 0$$

for any $p+q < n-k$. The proof is complete. □

Corollary 5.3. *Let X be a compact Kähler manifold and $D = \sum_{j=1}^s D_j$ be a simple normal crossing divisor of X. Let $[D']$ be a k-positive $\mathbb{R}$-line bundle over X, where $D' = \sum_{i=1}^s a_i D_i$ with $a_i > 0$ and $a_i \in \mathbb{R}$. If there exists a line bundle L over X and a real number b with $0 < a_j < b$ for all j, and $bL = [D']$ as $\mathbb{R}$-line bundles, then*

$$H^q(X, \Omega^p(\log D) \otimes L^{-1}) = 0$$

for $p+q > n+k$ and $p+q < n-k$.

Proof. Let b' be a real number such that $\max_j a_j < b' < b$ and set

$$N = L^{-1}, \quad \Delta = \frac{D'}{b'} = \sum_{j=1}^s \frac{a_j}{b'} D_j.$$

Let

$$F = L^{-1} \otimes \mathcal{O}_X([D]) = L^{-1} + \frac{D'}{b'} = \frac{b-b'}{bb'} D'.$$

It is easy to see that F is a k-positive $\mathbb{R}$-line bundle and the coefficients of Δ are in $[0,1]$. By Theorem 4.1, we obtain

$$H^q(X, \Omega^p(\log D) \otimes L^{-1}) = 0 \quad \text{ for } p+q > n+k.$$

On the other hand, we can set

$$N = L \otimes \mathcal{O}_X(-D), \quad \Delta = \sum_{j=1}^s \left(1 - \frac{a_j}{2b}\right) D_j, \quad \text{and} \quad F = N \otimes \mathcal{O}_X([D]) = \frac{D'}{2b}.$$

It is easy to see that F is a k-positive $\mathbb{R}$-line bundle and the coefficients of Δ are in $[0,1]$. By Theorem 4.1 again, we get

$$H^q(X, \Omega^p(\log D) \otimes L \otimes \mathcal{O}_X(-D)) = 0, \quad \text{ for } p+q > n+k.$$

By Serre duality and the isomorphism (5.6), we have

$$H^q(X, \Omega^p(\log D) \otimes L^{-1}) = 0$$

for any $p+q < n-k$. □

Corollary 5.4. *Let X be a compact Kähler manifold of dimension n and $D = \sum_{i=1}^{s} D_i$ be a simple normal crossing divisor in X. Suppose there exist some real constants $a_i \geq 0$ such that $\sum_{i=1}^{s} a_i D_i$ is a k-positive $\mathbb{R}$-divisor, then for any nef line bundle L, we have*

$$H^q(X, \Omega_X^p(\log D) \otimes L) = 0 \quad \text{for any } p + q \geq n + k + 1.$$

Proof. We can set $N = \mathcal{O}_X$ and $\Delta = \frac{1}{1+\sum_{i=1}^{s} a_i} \sum_{i=1}^{s} a_i D_i$. Then $N \otimes \mathcal{O}([\Delta]) = \mathcal{O}([\Delta])$ is a k-positive $\mathbb{R}$-line bundle. □

References

[1] Ambro, F. An injectivity theorem, Compositio Math. **150** (2014) 999–1023.

[2] Andreotti, A., Vesentini, E. Carleman estimates for the Laplace-Beltrami equation on complex manifolds. Publications Mathmatiques de l'IHS, 1965, 25: 81-130.

[3] Cao, J.-Y. Numerical dimension and a Kawamata-Viehweg-Nadel-type vanishing theorem on compact Kähler manifolds. Compos. Math. 150 (2014), no. 11, 1869–1902.

[4] Campana, F.; Paun, M. Orbifold generic semi-positivity: an application to families of canonically polarized manifolds. Ann.Inst. Fourier. **65** (2015), 835–861.

[5] Campana, F.; Paun, M. Foliations with positive slopes and birational stability of orbifold cotangent bundles. arXiv:1508.02456.

[6] Deligne, P. Théorème de Lefschetz et critères de dégénérescence de suites spectrales, Publ. Math. Inst. Hautes Études Sci. **35** (1969), 107–126.

[7] Deligne, P. Equations differentielles à points singuliers réguliers. Springer Lect. Notes Math. **163** (1970).

[8] Deligne, Pierre; Illusie, Luc. Relèvements modulo p^2 et décomposition du complexe de de Rham. Invent. Math. **89** (1987), no. 2, 247–270.

[9] Demailly, J.-P. *Analytic Methods in Algebraic Geometry*, Higher Education Press, Surveys of Modern Mathematics, Vol. 1, 2010

[10] Esnault, H.; Viehweg, E. Logarithmic de Rham complexes and vanishing theorems. Invent. Math., **86**(1986), 161–194.

[11] Esnault, H.; Viehweg, E. Lectures on vanishing theorems, DMV Seminar, Band 20, Birkhauser Verlag (1992).

[12] Fujiki, A. An L^2 Dolbeault lemma and its applications. Publ. RIMS, Kyoto Univ. **28** (1992) 845–884.

[13] Fujino, O. Foundation of the minimal model program. Book online `https://www.math.kyoto-u.ac.jp/~fujino/foundation1.pdf`.

[14] Fujino, O. Enoki's injectivity theorem. `https://www.math.kyoto-u.ac.jp/~fujino/enoki-inj.pdf`.

[15] Fujino, O. Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. **47** (2011), 727–789.

[16] Fujino, O. A transcendental approach to Kollár's injectivity theorem II, J. Reine Angew Math. **681** (2013), 149–174.

[17] Fujino, O.; Matsumura, S. Injectivity theorem for pseudo-effective line bundles and its applications.arXiv:1605.02284

[18] Goodman, J. Affine open subsets of algebraic varieties and ample divisors, Ann. of Math., **89**(1969), 160–183.

[19] Guan, Q.-A; Zhou, X.-Y. A proof of Demailly's strong openness conjecture. Ann. of Math. (2) **182** (2015), no. 2, 605–616.

[20] Hörmander, L. L^2 estimates and existence theorems for the $\overline{\partial}$ operator. Acta Mathematica, 1965, 113(1): 89-152.

[21] Luo, H.-Z. Stability of algebraic manifolds. Massachusetts Institute of Technology, 1998.

[22] Kawamata, Y. On the cohomology of $\mathbb{Q}$-divisors. Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), 34–35.

[23] Kawamata, Y.; Matsuda, K.; Matsuki, K. Introduction to the minimal model program, Advanced Studies in Pure Math 10, 283-360, North-Holland, Amsterdam (1987).

[24] Kodaira, K. Complex Manifolds and Deformation of Complex structures, Springer (1986).

[25] Kollár, J. Higher direct images of dualizing sheaves. I, Ann. of Math. (2) **124** (1986) 171–202.

[26] Pardon, W.; Stern, M. L^2-cohomology of complex projective varieties, J. Amer. Math. Soc. **4** (1991), no.3 603–621.

[27] Liu, K.-F.; Sun, X.-F.; Yang, X.-K. Positivity and vanishing theorems for ample vector bundles. J. Algebraic Geom. **22** (2013), 303–331.

[28] Liu, K.-F.; Yang, X.-K. Effective vanishing theorems for ample and globally generated vector bundles. Comm. Anal. Geom. **23** (2015), no.4, 797–818.

[29] Matsuki, K.; Olsson, M. Kawamata-Viehweg vanishing as Kodaira vanishing for stacks. Math. Res. Lett. **12** (2005), 207–217.

[30] Matsumura, S. A transcendental approach to injectivity theorem for log canonical pairs. arXiv:1607.07213.

[31] Norimatsu, Y. Kodaira vanishing theorem and Chern classes for ∂-manifolds. Proceedings of the Japan Academy, Series A, Mathematical Sciences, **54**(1978), 107–108.

[32] Shiffman, B.; Sommese, A.J. *Vanishing theorems on complex manifolds*. Progress in Mathematics, vol. 56. Birkhauser Boston Inc, Boston (1985)

[33] Wells, R O. Comparison of de Rham and Dolbeault cohomology for proper surjective mappings, Pacific. J. Math.,**53**(1974), no. 1, 281–300.

[34] Zucker, S. Hodge theory with degenerating coefficients: L^2 cohomology in the Poincaré metric. Ann. of Math., **109**(1979), 415–476.

Chunle Huang, Center of Mathematical Science, Zhejiang University, Hangzhou, 310027, CHINA.
E-mail address: `chunle@zju.edu.cn`

Kefeng Liu, Department of Mathematics, Capital Normal University, Beijing, 100048, China

Department of Mathematics, University of California at Los Angeles, California 90095
E-mail address: `liu@math.ucla.edu`

Xueyuan Wan, Chern Institute of Mathematics & LPMC, Nankai University, 300071
E-mail address: `xywan@mail.nankai.edu.cn`

Xiaokui Yang, Morningside Center of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China

Hua Loo-Keng Key Laboratory of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing,100190, China
E-mail address: `xkyang@amss.ac.cn`