RC-positivity, rational connectedness and Yau's conjecture

Xiaokui Yang

Abstract. In this paper, we give a differential geometric interpretation of Mumford's conjecture on rational connectedness and outline a differential geometric approach. To this end, we introduce a concept of RC-positivity for Hermitian holomorphic vector bundles over compact complex manifolds. We prove that if E is an RC-positive vector bundle over a compact complex manifold X, then any line subbundle of the dual vector bundle E^* can not be pseudo-effective and for any vector bundle A, there exists a positive integer $c_A = c(A, E)$ such that

$$H^0(X, \mathrm{Sym}^{\otimes \ell} E^* \otimes A^{\otimes k}) = 0$$

for $\ell \geq c_A(k+1)$ and $k \geq 0$. Moreover, we obtain that, on a projective manifold X, if the anticanonical bundle $\Lambda^{\dim X} T_X$ is RC-positive, then X is uniruled; if $\Lambda^p T_X$ is RC-positive for every $1 \leq p \leq \dim X$, then X is rationally connected. As applications, we show that if a compact Kähler manifold (X, ω) has positive holomorphic sectional curvature, then $\Lambda^p T_X$ is RC-positive and $H^{p,0}_{\bar\partial}(X) = 0$ for every $1 \leq p \leq \dim X$; in particular, we establish that X is a projective and rationally connected manifold, which confirms a conjecture of Yau.

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1. Introduction

This is a continuation of our previous papers [Yang17a, Yang17b]. In this paper, we give a geometric interpretation of Mumford's conjecture on rational connectedness and propose a differential geometric approach to attack this conjecture. As an application of this approach, we confirm a conjecture of Yau that a compact Kähler manifold with positive holomorphic sectional curvature is a projective and rationally connected manifold. This project is motivated by a number of well-known conjectures proposed by Yau, Mumford, Demailly, Campana, Peternell and etc., and we refer to [Yau74, Yau82, Cam92, KMM92, Kol96, DPS96a, DPS96b, Pet06, HW12, BDPP13, CDP14, CP14, HW15, BC15, Yang16, Cam16, CH17] and the references therein.

A projective manifold X is called *rationally connected* if any two points of X can be connected by some rational curve. It is easy to show that on a rationally connected projective manifold, one has

$$H^0(X,(T_X^*)^{\otimes m})=0, \quad \text{for every} \quad m\geq 1.$$

A well-known conjecture of Mumford says that the converse is also true.

Conjecture 1.1 (Mumford). *Let X be a projective manifold. If*

$$H^0(X,(T_X^*)^{\otimes m})=0, \quad \textit{for every} \quad m\geq 1,$$

then X is rationally connected.

This conjecture holds when $\dim X\leq 3$ ([KMM92]) and not much has been known in higher dimensions, and we refer to [LP17], [CDP14] and [Kol96] for more historical discussions. In [BC15], Brunebarbe and Campana also proposed a stronger conjecture that X is rationally connected if and only if

$$H^0(X,\mathrm{Sym}^{\otimes \ell}\Omega_X^p)=0 \quad \text{for every } \ell>0 \text{ and } 1\leq p\leq \dim X. \tag{1.1}$$

We present a differential geometric interpretation of Mumford's Conjecture 1.1. The starting point is the following differential geometric interpretation on uniruled projective manifold which is obtained in [Yang17b, Corollary 1.9] and [Yang17b, Theorem 4.1],

Theorem 1.2. *Let X be a projective manifold. Then the following are equivalent*

(1) *X is uniruled, i.e. X is covered by rational curves;*
(2) *there exists a Hermitian metric ω on X such that the (Chern) Ricci curvature $\mathrm{Ric}(\omega)$ has at least one positive direction at each point of X;*
(3) *there exists a Hermitian metric ω on X with positive (Chern) scalar curvature.*

By using Theorem 1.2, Mumford's Conjecture 1.1 has an *equivalent* differential geometric interpretation (the equivalence is proved in Proposition 8.5):

Conjecture 1.3. *Let X be a projective manifold. If*

$$H^0(X,(T_X^*)^{\otimes m})=0, \quad \textit{for every} \quad m\geq 1, \tag{1.2}$$

then one (and hence all) of the following holds

(1) *there exists a Hermitian metric* ω *on* X *such that the (Chern) Ricci curvature* $\mathrm{Ric}(\omega)$ *has at least one positive direction at each point of* X*;*
(2) *there exists a Hermitian metric* ω *on* X *with positive (Chern) scalar curvature.*

The key difficulty in solving Conjecture 1.3 arises from how to construct a prescribed Hermitian metric from the given vanishing cohomology (1.2). This is very similar to a vector bundle version of a converse to the Andreotti-Grauert theorem we obtained in [Yang17b, Theorem 1.4].

In order to give geometric interpretations on vanishing conditions in (1.1) and (1.2), we introduce the following concept for Hermitian vector bundles:

Definition 1.4. Let (E, h) be a Hermitian holomorphic vector bundle over a complex manifold X and $R^{(E,h)} \in \Gamma(X, \Lambda^{1,1}T_X^* \otimes \mathrm{End}(E))$ be its Chern curvature tensor. E is called *RC-positive* (resp. *RC-negative*) if for any nonzero local section $a \in \Gamma(X, E)$, there exists some local section $v \in \Gamma(X, T_X)$ such that

$$R^{(E,h)}(v, \overline{v}, a, \overline{a}) > 0. \quad (\quad \text{resp.} \quad < 0)$$

For a line bundle (L, h), it is RC-positive if and only if its Ricci curvature has at least one positive direction at each point of X. This terminology has many nice properties. For instances, quotient bundles of RC-positive bundles are also RC-positive; subbundles of RC-negative bundles are still RC-negative (see Theorem 3.5); the holomorphic tangent bundles of Fano manifolds can admit RC-positive metrics (see Corollary 3.9). However, if a vector bundle E has ample determinant line bundle $\det(E)$, i.e. $c_1(E) > 0$, E is not necessarily RC-positive. The first main result of our paper is

Theorem 1.5. *Let* X *be a* compact complex manifold*. If* E *is an RC-positive vector bundle, then*

(1) *for any line subbundle* F *of* E^**,* F *is not pseudo-effective.*
(2) *for any vector bundle* A*, there exists a positive integer* $c_A = c(A, E)$ *such that*

$$H^0(X, \mathrm{Sym}^{\otimes \ell} E^* \otimes A^{\otimes k}) = 0$$

for $\ell \geq c_A(k + 1)$ *and* $k \geq 0$*.*

It worths to point out that we also obtained in [Yang17b, Corollary 1.6] that

Theorem 1.6. *Let* X *be a projective manifold. The following are equivalent*

(1) L *is an RC-positive line bundle;*
(2) L^* *is not pseudo-effective;*
(3) *for any vector bundle* A*, there exists a positive integer* $c_A = c(A, L)$ *such that*

$$H^0(X, (L^*)^{\otimes \ell} \otimes A^{\otimes k}) = 0$$

for $\ell \geq c_A(k + 1)$ *and* $k \geq 0$*.*

Hence, it is natural to propose the following problem, which is analogous to the converse of the Andreotti-Grauert theorem for line bundles (e.g. [DPS96a, Problem 2.1])

Problem 1.7. Let X be a projective manifold and E be a vector bundle. Suppose that for any vector bundle A, there exists a positive integer $c_A = c(A,E)$ such that

$$H^0(X, \mathrm{Sym}^{\otimes \ell} E^* \otimes A^{\otimes k}) = 0$$

for $\ell \geq c_A(k+1)$ and $k \geq 0$, is $\det(E)$ necessarily RC-positive? Or more generally, is $\Lambda^p E$ RC-positive for every $1 \leq p \leq \mathrm{rank}(E)$?

We expect that Problem 1.7 can shade some lights on solving the differential geometric interpretation (Conjecture 1.3) of Mumford's conjecture.

As a straightforward application of Theorem 1.5 and some well-known criterions for rational connectedness (e.g. [CDP14, Theorem 1.1], [GHS03, Corollary 1.7], [LP17, Proposition 2.1] and [Cam16, Proposition 1.3]), we obtain the second main result of our paper

Theorem 1.8. *Let X be a projective manifold of complex dimension n. Suppose that for every $1 \leq p \leq n$, there exists a smooth Hermitian metric h_p on the vector bundle $\Lambda^p T_X$ such that $(\Lambda^p T_X, h_p)$ is RC-positive, then X is rationally connected.*

Of course, in Theorem 1.8, $\Lambda^p T_X$ can be replaced by $T_X^{\otimes k}$ or $\mathrm{Sym}^{\otimes k} T_X$ ($k \geq 1$). We also propose a converse question to Theorem 1.8, i.e. whether rationally connected manifolds can support RC-positive metrics for every $\Lambda^p T_X$, $1 \leq p \leq \dim X$ (see Question 8.6). It is very easy to verify that if a compact Kähler manifold (X,ω) has positive Ricci curvature, then $(\Lambda^p T_X, \Lambda^p \omega)$ is RC-positive for every $1 \leq p \leq \dim X$ (Corollary 3.9). Hence, by the celebrated Calabi-Yau theorem ([Yau78]), we obtain the classical result of Campana([Cam92]) and Kollár-Miyaoka-Mori ([KMM92]) that Fano manifolds are rationally connected (see also [HW15]). More generally, we obtain

Theorem 1.9. *Let X be a smooth projective manifold. If there exist a Hermitian metric ω on X and a (possibly different) Hermitian metric h on T_X such that*

$$\mathrm{tr}_\omega R^{(T_X,h)} \in \Gamma(X, \mathrm{End}(T_X)) \tag{1.3}$$

is positive definite, then X is rationally connected.

We need to point out that Theorem 1.9 can also be implied by the "Generalized holonomy principle" for *positive curvature* and the main theorem in [CDP14], although the precise result is not stated there. This special case is a refinement under the RC positive curvature condition. Note also that there are non-Kähler manifolds X which can support a Hermitian metric ω on X and a Hermitian metric h on the vector bundle T_X such that

$$\mathrm{tr}_\omega R^{(T_X,h)} \in \Gamma(X, \mathrm{End}(T_X))$$

is positive definite (e.g. Example 7.1).

Let's describe another application of Theorem 1.8 on compact Kähler manifolds with positive holomorphic sectional curvature. S.-T. Yau proposed a well-known conjecture (e.g. [Yau82, Problem 47] and [Wong]) that

Conjecture 1.10 (Yau)**.** *Let X be a compact Kähler manifold. If X has a Kähler metric with positive holomorphic sectional curvature, then X is a projective and rationally connected manifold.*

As applications of Theorem 1.8, we confirm Yau's Conjecture 1.10 in the full generality. More generally, we obtain

Theorem 1.11. *Let (X,ω) be a compact Kähler manifold with positive holomorphic sectional curvature. Then for every $1 \leq p \leq \dim X$, $(\Lambda^p T_X, \Lambda^p \omega)$ is RC-positive and $H^{p,0}_{\bar\partial}(X) = 0$. In particular, X is a projective and rationally connected manifold.*

Remark 1.12. We need to point out that, recently, Heier-Wong confirmed Yau's conjecture in [HW15] in the special case when X is projective. Our method relies on the concept of RC-positivity (e.g. Theorem 3.2 and Theorem 3.5) for Hermitian vector bundles and a minimum principle for Kähler metrics with positive holomorphic sectional curvature (e.g. Lemma 6.1), while their method builds on an average argument and certain integration by parts.

It is clear that, the RC-positivity is defined for all Hermitian metrics. We also obtain

For some related topics on positive holomorphic sectional curvature, we refer to [HW12, HW15, ACH15, Yang16, AHZ16, YZ16, AH17] and the references therein.

Organizations: In Section 2, we recall some background materials on Hermitian manifolds. In Section 3, we introduce the concept of RC-positivity for vector bundles and investigate its geometric properties. In Section 4, we obtain vanishing theorems for RC-positive vector bundles and prove Theorem 1.5. In Section 5, we establish the relation between RC-positive bundles and rational connectedness, and prove Theorem 1.8 and Theorem 1.9. In Section 6, we study positive holomorphic sectional curvature and rational connectedness, and prove Theorem 1.11, Theorem 1.13 and Corollary 1.14. In Section 7, we present several examples to distinguish various curvature conditions. In Section 8, we study the relations between several open conjectures and propose some further questions.

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2. Background materials

Let (E,h) be a Hermitian holomorphic vector bundle over a compact complex manifold X with Chern connection ∇. Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $R^{(E,h)} \in \Gamma(X, \Lambda^{1,1}T_X^* \otimes \mathrm{End}(E))$ has components

$$R_{i\bar{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}} \frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j} \tag{2.1}$$

(Here and henceforth we sometimes adopt the Einstein convention for summation.) We have the trace $\mathrm{tr}R^{(E,h)} \in \Gamma(X, \Lambda^{1,1}T_X^*)$ which has components

$$R^{\alpha}_{i\bar{j}\alpha} = h^{\alpha\overline{\beta}} R_{i\bar{j}\alpha\overline{\beta}} = -\frac{\partial^2 \log\det(h_{\alpha\overline{\beta}})}{\partial z^i \partial \overline{z}^j}.$$

For any Hermitian metric ω_g on X, we can also define $\mathrm{tr}_\omega R^{(E,h)} \in \Gamma(X, \mathrm{End}(E))$ which has components

$$g^{i\bar{j}} R^{\beta}_{i\bar{j}\alpha}.$$

In particular, if (X,ω_g) is a Hermitian manifold, then the Hermitian vector bundle (T_X,g) has Chern curvature components

$$R_{i\bar{j}k\bar{\ell}}=-\frac{\partial^2 g_{k\bar{\ell}}}{\partial z^i\partial\bar{z}^j}+g^{p\bar{q}}\frac{\partial g_{k\bar{q}}}{\partial z^i}\frac{\partial g_{p\bar{\ell}}}{\partial\bar{z}^j}. \tag{2.2}$$

The (first) Chern-Ricci form

$$\mathrm{Ric}(\omega_g)=\mathrm{tr}_g R^{(T_X,g)}\in\Gamma(X,\Lambda^{1,1}T_X^*)$$

of (X,ω_g) has components

$$R_{i\bar{j}}=g^{k\bar{\ell}}R_{i\bar{j}k\bar{\ell}}=-\frac{\partial^2\log\det(g)}{\partial z^i\partial\bar{z}^j}$$

and it is well-known that the Chern-Ricci form represents the first Chern class of the complex manifold X (up to a factor 2π). The second Chern-Ricci tensor

$$\mathrm{Ric}^{(2)}(\omega_g)=\mathrm{tr}_{\omega_g}R^{(T_X,g)}\in\Gamma(X,\mathrm{End}(T_X))$$

of (X,ω_g) has (lowered down) components

$$R^{(2)}_{k\bar{\ell}}=g^{i\bar{j}}R_{i\bar{j}k\bar{\ell}}.$$

If ω_g is not Kähler (i.e. $d\omega_g\neq 0$), $\mathrm{Ric}(\omega_g)$ and $\mathrm{Ric}^{(2)}(\omega_g)$ are not the same. The (Chern) scalar curvature s_g of ω_g is defined as

$$s_g=g^{i\bar{j}}R_{i\bar{j}}.$$

Definition 2.1. A holomorphic vector bundle (E,h) is called *Griffiths positive* if

$$R_{i\bar{j}\alpha\bar{\beta}}v^i\bar{v}^j a^\alpha\bar{a}^\beta>0$$

for any nonzero vectors $v=(v^i)$ and $a=(a^\alpha)$.

A Hermitian manifold (or Kähler) (X,ω) has positive (resp. semi-positive) holomorphic sectional curvature, if for any nonzero vector $\xi=(\xi^1,\cdots,\xi^n)$,

$$R_{i\bar{j}k\bar{\ell}}\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell>0\quad(\text{resp. }\geq 0).$$

3. RC-positive vector bundles on compact complex manifolds

In the section we introduce the concept of RC-positivity for Hermitian vector bundles and investigate its geometric properties.

Definition 3.1. Let L be a holomorphic line bundle over a compact complex manifold X with $\dim_{\mathbb{C}}X=n$. L is called *q-positive*, if there exists a smooth Hermitian metric h on L such that the Chern curvature $R^{(L,h)}=-\sqrt{-1}\partial\bar{\partial}\log h$ has at least $(n-q)$ positive eigenvalues at every point on X.

When $q = n-1$, the concept of $(n-1)$ positivity has very nice geometric interpretations. We established in [Yang17b, Theorem 1.5] the following result which also plays a key role in this paper:

Theorem 3.2. *Let L be a line bundle over a compact complex manifold X with $\dim_{\mathbb{C}} X = n$. The following statements are equivalent:*

(1) *L is $(n-1)$-positive;*
(2) *The dual line bundle L^* is not pseudo-effective.*

In this paper, we extend the concept of $(n-1)$-positivity to vector bundles.

Definition 3.3. A Hermitian holomorphic vector bundle (E,h) over a complex manifold X is called *RC-positive* (resp. *RC-negative*) at point $q \in X$ if for any nonzero $a = (a^1, \cdots, a^r) \in \mathbb{C}^r$, there exists a vector $v = (v^1, \cdots, v^n) \in \mathbb{C}^n$ such that

$$\sum R_{i\bar{j}\alpha\bar{\beta}} v^i \overline{v}^j a^\alpha \overline{a}^\beta > 0 \quad \text{(resp. } < 0) \tag{3.1}$$

at point q. (E,h) is called RC-positive if it is RC-positive at every point of X.

We can also define RC-semi-positivity (resp. RC-semi-negativity) in the same way.

Remark 3.4. From the definition, it is easy to see that,

(1) for any nonzero $u \in \Gamma(X,E)$, as a Hermitian $(1,1)$-form on X, $R^{(E,h)}(u,u) \in \Gamma(X, \Lambda^{1,1}T^*_X)$ has at least one positive eigenvalue, i.e. $R^{(E,h)}(u,u)$ is $(n-1)$-positive;
(2) if a vector bundle (E,h) is Griffiths positive, then (E,h) is RC-positive;
(3) if E is a line bundle, then E is RC-positive if and only if E is $(n-1)$-positive;
(4) if $\dim X = 1$, E is RC-positive if and only if (E,h) is Griffiths positive.

Here, we can also define RC-positivity along k linearly independent directions, i.e. for any given nonzero local section $a \in \Gamma(X,E)$, $R^{(E,h)}(\bullet, \bullet, a, \overline{a}) \in \Gamma(X, \Lambda^{1,1}T^*_X)$ is $(n-k)$-positive as a Hermitian $(1,1)$-form on X. It is also a generalization of the Griffiths positivity. This terminology is systematically investigated in the forthcoming paper [Yang].

By using a simple monotonicity formula and Theorem 3.2, we obtain the following properties, which also hold for RC-positivity along k linearly independent directions.

Theorem 3.5. *Let (E,h) be a Hermitian vector bundle over a compact complex manifold X.*

(1) *(E,h) is RC-positive if and only if (E^*,h^*) is RC-negative;*
(2) *If (E,h) is RC-negative, every subbundle S of E is RC-negative;*
(3) *If (E,h) is RC-positive, every quotient bundle Q of E is RC-positive;*
(4) *If (E,h) is RC-positive, every line subbundle L of E^* is not pseudo-effective.*
(5) *If (E,h) is an RC-positive line bundle, then for any pseudo-effective line bundle L, $E \otimes L$ is RC-positive.*

Proof. (1) is obvious. (2) follows from a standard monotonicity formula. Let r be the rank of E and s the rank of S. Without loss of generality, we can assume, at a fixed point $p \in X$, there exists a local holomorphic frame $\{e_1, \cdots, e_r\}$ of E centered at point p such that $\{e_1, \cdots, e_s\}$ is a local holomorphic frame of S. Moreover, we can assume that $h(e_\alpha, e_\beta)(p) = \delta_{\alpha\beta}$, for $1 \leq \alpha, \beta \leq r$. Hence, the curvature tensor of S at point p is

$$R^S_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + \sum_{\gamma=1}^{s} \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \tag{3.2}$$

where $1 \leq \alpha, \beta \leq s$. The curvature tensor of E at point p is

$$R^E_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + \sum_{\gamma=1}^{r} \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \tag{3.3}$$

where $1 \leq \alpha, \beta \leq r$. It is easy to see that

$$R^E|_S - R^S = \sqrt{-1} \sum_{i,j} \sum_{\alpha,\beta=1}^{s} \left(\sum_{\gamma=s+1}^{r} \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \right) dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e^\beta. \tag{3.4}$$

Since R^E is RC-negative, for any (local) nonzero vector $a_E = (a^1, \cdots, a^s, 0, \cdots, 0) \in \mathbb{C}^r$, there exists a vector $v = (v^1, \cdots, v^n)$ such that

$$R^E(v, \bar{v}, a_E, \bar{a}_E) < 0.$$

Let $a = (a^1, \cdots, a^s)$. Then

$$R^S(v, \bar{v}, a, \bar{a}) = R^E(v, \bar{v}, a_E, \bar{a}_E) - \sum_{i,j} \sum_{\alpha,\beta=1}^{s} \left(\sum_{\gamma=s+1}^{r} \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j} \right) v^i \bar{v}^j a^\alpha \bar{a}^\beta < 0. \tag{3.5}$$

The proof of part (3) is similar.

(4). Let L be a line subbundle of E^* where (E, h) is RC-positive. Then by (1) and (2), we know L is RC-negative. Hence L^* is RC-positive and L^* is $(n-1)$-positive. By Theorem 3.2, $L = (L^*)^*$ is not pseudo-effective.

(5). Suppose $E \otimes L$ is not RC-positive, then by Theorem 3.2, $E^* \otimes L^*$ is pseudo-effective and so is $E^* = (E^* \otimes L^*) \otimes L$. This is a contradiction since E is RC-positive. □

Let's recall some basic linear algebra. Let V be a complex vector space and $\dim_{\mathbb{C}} V = r$. Let $A \in \mathrm{End}(V)$. For any $1 \leq p \leq r$, we define $\Lambda^p A \in \mathrm{End}(\Lambda^p V)$ as

$$(\Lambda^p A)(v_1 \wedge \cdots \wedge v_p) = \sum_{i=1}^{p} v_1 \wedge \cdots \wedge A v_i \wedge \cdots \wedge v_p. \tag{3.6}$$

Similarly, for $k \geq 1$ we define $A^{\otimes k} \in \mathrm{End}(V^{\otimes k})$ by

$$\left(A^{\otimes k}\right)(v_1 \otimes \cdots \otimes v_p) = \sum_{i=1}^{p} v_1 \otimes \cdots \otimes Av_i \otimes \cdots \otimes v_p. \tag{3.7}$$

By choosing a basis, it is easy to see that if A is positive definite, then both $\Lambda^p A$ and $A^{\otimes k}$ are positive definite. We have the following important observation:

Theorem 3.6. *Let (X, ω) be a Hermitian manifold and (E, h) be a Hermitian holomorphic vector bundle. If*

$$\mathrm{tr}_\omega R^{(E,h)} \in \Gamma(X, \mathrm{End}(E)) \tag{3.8}$$

is positive definite, then

1. *$(\Lambda^p E, \Lambda^p h)$ is RC-positive for every $1 \leq p \leq \mathrm{rank}(E)$;*
2. *$(E^{\otimes k}, h^{\otimes k})$ is RC-positive for every $k \geq 1$.*

Proof. From the expression of the induced curvature tensor of $(\Lambda^p E, \Lambda^p h)$, it is easy to see that

$$R^{(\Lambda^p E, \Lambda^p h)} = \Lambda^p R^{(E,h)} \in \Gamma(X, \Lambda^{1,1} T_X^* \otimes \mathrm{End}(\Lambda^p E)),$$

and

$$\mathrm{tr}_\omega R^{(\Lambda^p E, \Lambda^p h)} = \mathrm{tr}_\omega \left(\Lambda^p R^{(E,h)}\right) = \Lambda^p \left(\mathrm{tr}_\omega R^{(E,h)}\right) \in \Gamma(X, \mathrm{End}(\Lambda^p E)),$$

is positive definite. Suppose $(\Lambda^p E, \Lambda^p h)$ is not RC-positive, then there exists a point $a \in X$ and a nonzero local section $a \in \Gamma(X, \Lambda^p E)$, such that for any local section $v \in \Gamma(X, T_X)$,

$$R^{(\Lambda^p E, \Lambda^p h)}(v, \overline{v}, a, \overline{a}) \leq 0$$

at point $q \in X$. In particular, we have

$$\left(\mathrm{tr}_\omega R^{(\Lambda^p E, \Lambda^p h)}\right)(a, \overline{a}) \leq 0$$

at point q. This is a contradiction. The proof for part (2) is similar. □

Remark 3.7. It worths to point out that, there exists a vector bundle E such that $\det E$ is ample, or equivalently there exists a Hermitian metric h on E such that

$$R^{\det E} = \mathrm{tr} R^{(E,h)} = -\sqrt{-1}\partial\overline{\partial} \log \det h \in \Gamma(X, \Lambda^{1,1} T_X^*) \tag{3.9}$$

is strictly positive, but E has *no* RC-positive metric (e.g. Example 7.2). Indeed, the curvature traces in (3.8) and (3.9) have totally different geometric meanings.

Corollary 3.8. *Let X be a compact complex manifold. If there exists a* Hermitian metric *ω on X and a (possibly different) Hermitian metric h on T_X such that*

$$\mathrm{tr}_\omega R^{(T_X,h)} > 0 \tag{3.10}$$

then

1. *$(T_X^{\otimes k}, h^{\otimes k})$ is RC-positive for every $k \geq 1$;*

(2) $(\Lambda^p T_X, \Lambda^p h)$ *is RC-positive for every* $1 \leq p \leq \dim X$.

Note that there exist non-Kähler manifolds (Example 7.1) such that they can support a Hermitian metric ω with

$$\mathrm{tr}_\omega R^{(T_X,\omega)} > 0.$$

Corollary 3.9. *Let X be a compact Kähler manifold. If there exists a* Kähler metric *ω such that it has positive Ricci curvature, i.e.*

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial}\log(\omega^n) > 0,$$

then

(1) $(T_X^{\otimes k}, \omega^{\otimes k})$ *is RC-positive for every* $k \geq 1$*;*
(2) $(\Lambda^p T_X, \Lambda^p \omega)$ *is RC-positive for every* $1 \leq p \leq \dim X$.

4. Vanishing theorems for RC-positive vector bundles

In this section, we derive vanishing theorems for RC-positive vector bundles over compact complex manifolds and prove Theorem 1.5.

Proposition 4.1. *Let $\mathcal{O}_E(1) \to \mathbb{P}(E^*)$ be the tautological line bundle of $E \to X$. Suppose (E, h^E) is RC-positive, then $\mathcal{O}_E(1)$ is $(\dim X - 1)$-positive over $\mathbb{P}(E^*)$.*

Proof. The proof follows from a standard curvature formula for $\mathcal{O}_E(1)$ induced from (E, h). Suppose $\dim_{\mathbb{C}} X = n$. Let π be the canonical $\mathbb{P}(E^*) \to X$ and $L = \mathcal{O}_E(1)$. Let $(e_1, \cdots, e_r)$ be the local holomorphic frame with respect to a given trivialization on E and the dual frame on E^* is denoted by $(e^1, \cdots, e^r)$. The corresponding holomorphic coordinates on E^* are denoted by $(W_1, \cdots, W_r)$. There is a local section e_{L^*} of L^* defined by

$$e_{L^*} = \sum_{\alpha=1}^{r} W_\alpha e^\alpha \tag{4.1}$$

Its dual section is denoted by e_L. Let h^L the induced quotient metric on L by the morphism $(\pi^* E, \pi^* h^E) \to L$. If $\left(h_{\alpha\overline{\beta}}\right)$ is the matrix representation of h with respect to the basis $\{e_\alpha\}_{\alpha=1}^r$, then h^L can be written as

$$h^L = \frac{1}{h^{L^*}(e_{L^*}, e_{L^*})} = \frac{1}{\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta} \tag{4.2}$$

The curvature of (L, h^L) is

$$R^{h^L} = -\sqrt{-1}\partial\overline{\partial}\log h^L = \sqrt{-1}\partial\overline{\partial}\log\left(\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta\right) \tag{4.3}$$

where ∂ and $\overline{\partial}$ are operators on the total space $\mathbb{P}(E^*)$. We fix a point $p \in \mathbb{P}(E^*)$, then there exist local holomorphic coordinates $(z^1, \cdots, z^n)$ centered at point $s = \pi(p)$ and local holomorphic basis $\{e_1, \cdots, e_r\}$ of E around s such that

$$h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} - R_{i\overline{j}\alpha\overline{\beta}} z^i \overline{z}^j + O(|z|^3) \tag{4.4}$$

Without loss of generality, we assume p is the point $(0, \cdots, 0, [a_1, \cdots, a_r])$ with $a_r = 1$. On the chart $U = \{W_r = 1\}$ of the fiber $\mathbb{P}^{r-1}$, we set $w^A = W_A$ for $A = 1, \cdots, r-1$. By formula (4.3) and (4.4)

$$R^{h^L}(p) = \sqrt{-1}\left(\sum R_{i\overline{j}\alpha\overline{\beta}} \frac{a_\beta \overline{a}_\alpha}{|a|^2} dz^i \wedge d\overline{z}^j + \sum_{A,B=1}^{r-1}\left(1 - \frac{a_B \overline{a}_A}{|a|^2}\right) dw^A \wedge d\overline{w}^B\right) \tag{4.5}$$

where $|a|^2 = \sum_{\alpha=1}^{r} |a_\alpha|^2$. If R^E is RC-positive, then the $(n \times n)$ Hermitian matrix

$$\left[\sum R_{i\overline{j}\alpha\overline{\beta}} \frac{a_\beta \overline{a}_\alpha}{|a|^2}\right]_{i,j=1}^{n} \tag{4.6}$$

has at least one positive eigenvalues. In particular, the curvature R^{h^L} of (L, h^L) has at least r positive eigenvalues over the projective bundle $\mathbb{P}(E^*)$. Since $\dim \mathbb{P}(E^*) = n + r - 1$, we know L is $(n-1)$-positive. □

Theorem 4.2. *Let X be a compact complex manifold of complex dimension n. If E is RC-positive, then for any vector bundle A over $\mathbb{P}(E^*)$, there exists a positive integer $c_A = c(A, E)$ such that*

$$H^{p,q}(\mathbb{P}(E^*), \mathcal{O}_E(\ell) \otimes A^{\otimes k}) = 0 \tag{4.7}$$

when $\ell \geq c_A(k+1)$, $k \geq 0$, $p \geq 0$ and $q > n-1$.

Proof. The proof follows from Proposition 4.1 and an Andreotti-Grauert type vanishing theorem. Let $F = \Omega^p_{\mathbb{P}(E^*)} \otimes K^{-1}_{\mathbb{P}(E^*)}$, then we have

$$H^{p,q}(\mathbb{P}(E^*), \mathcal{O}_E(\ell) \otimes A^{\otimes k}) \cong H^{n+r-1,q}(\mathbb{P}(E^*), \mathcal{O}_E(\ell) \otimes A^{\otimes k} \otimes F). \tag{4.8}$$

By Proposition 4.1, $\mathcal{O}_E(1)$ is $(n-1)$-positive. On the other hand, by [Yang17b, Proposition 2.2], there exists a Hermitian metric h on $\mathcal{O}_E(1)$ and a smooth Hermitian metric ω on $\mathbb{P}(E^*)$ such that $(\mathcal{O}_E(1), h)$ is uniformly $(n-1)$-positive. That is, in local holomorphic coordinates of $\mathbb{P}(E^*)$, at some point $p \in X$,

$$\omega = \sqrt{-1}\sum_i dz^i \wedge d\overline{z}^i, \qquad R^{(\mathcal{O}_E(1),h)} = \sqrt{-1}\sum_i \lambda_i dz^i \wedge d\overline{z}^i$$

where $\lambda_1 \geq \cdots \geq \lambda_{n+r-1}$, $\lambda_r > 0$ and

$$\lambda_r + \cdots + \lambda_{n+r-1} > 0. \tag{4.9}$$

Let $\mathcal{E} = \mathcal{O}_E(\ell) \otimes A^{\otimes k} \otimes F$ and $h^{\mathcal{E}}$ be the induced metric on $\mathcal{E}$ where we fix an arbitrary smooth metric h^A on A. Without loss of generality, we assume A is a line bundle.

By standard Bochner formulas on compact complex manifolds ([Dem, Theorem 1.4 in Chapter VII]), one has

$$\Delta_{\overline{\partial}_{\mathcal{E}}} = \widehat{\Delta}_{\partial_{\mathcal{E}}} + \left[R^{\mathcal{E}}, \Lambda_\omega\right] + T_\omega \tag{4.10}$$

where $\tau = [\Lambda_\omega, \partial\omega]$, $\nabla^{\mathcal{E}} = \partial_{\mathcal{E}} + \overline{\partial}_{\mathcal{E}}$ is the decomposition of the Chern connection, $\widehat{\Delta}_{\partial_{\mathcal{E}}} = (\partial_{\mathcal{E}}+\tau)(\partial_{\mathcal{E}}^*+\tau^*)+(\partial_{\mathcal{E}}^*+\tau^*)(\partial_{\mathcal{E}}+\tau)$, $T_\omega = \left[\Lambda_\omega, \left[\Lambda_\omega, \sqrt{-1}\partial\overline{\partial}\omega\right]\right] - [\partial\omega, (\partial\omega)^*]$. Hence, for any $s \in H^{n+r-1,q}(\mathbb{P}(E^*), \mathcal{E})$, we have

$$\langle\left[R^{\mathcal{E}}, \Lambda_\omega\right] s, s\rangle = \ell\left\langle\left[R^{(\mathcal{O}_E(1),h)}, \Lambda_\omega\right] s, s\right\rangle + k\left\langle\left[R^{(A,h^A)}, \Lambda_\omega\right] s, s\right\rangle + \left\langle\left[R^{(F,h^F)}, \Lambda_\omega\right] s, s\right\rangle.$$

It is obvious that there exist uniform constants $c_0 \in \mathbb{R}$ and $c_1 \in \mathbb{R}$ such that

$$\left\langle\left[R^{(A,h^A)}, \Lambda_\omega\right] s, s\right\rangle \geq c_0|s|^2 \quad \text{and} \quad \langle T_\omega s, s\rangle + \left\langle\left[R^{(F,h^F)}, \Lambda_\omega\right] s, s\right\rangle \geq c_1|s|^2 \tag{4.11}$$

hold on $\mathbb{P}(E^*)$. On the other hand, a straightforward computation shows that when $q > n-1$

$$\ell\left\langle\left[R^{(\mathcal{O}_E(1),h)}, \Lambda_\omega\right] s, s\right\rangle \geq \ell\left(\lambda_r + \cdots + \lambda_{n+r-1}\right)|s|^2.$$

We set

$$\gamma = \inf_{p\in\mathbb{P}(E^*)}\left(\lambda_r + \cdots + \lambda_{n+r-1}\right) > 0$$

and pick a positive number $c_A = c(A, E)$ such that

$$c_A \cdot \gamma + c_0 > 0, \quad c_A \cdot \gamma + c_1 > 1.$$

Hence, for any $s \in H^{n+r-1,q}(\mathbb{P}(E^*), \mathcal{E})$, $q > n-1$ and $\ell \geq c_A(k+1)$, we have

$$\begin{aligned} 0 = \left\langle \Delta_{\overline{\partial}_{\mathcal{E}}} s, s\right\rangle &= \left\langle \widehat{\Delta}_{\partial_{\mathcal{E}}} s, s\right\rangle + \langle\left[R^{\mathcal{E}}, \Lambda_\omega\right] s, s\rangle + \langle T_\omega s, s\rangle \\ &\geq \left\langle \widehat{\Delta}_{\partial_{\mathcal{E}}} s, s\right\rangle + k(c_A\cdot\gamma + c_0)|s|^2 + (c_A\cdot\gamma + c_1)|s|^2 \\ &\geq \left\langle \widehat{\Delta}_{\partial_{\mathcal{E}}} s, s\right\rangle + |s|^2. \end{aligned}$$

An integration over $\mathbb{P}(E^*)$ shows $s = 0$. □

Theorem 4.3. *Let E be an RC-positive vector bundle over a compact complex manifold X. Then for every vector bundle A, there exists a positive integer $c_A = c(A, E)$ such that*

$$H^0(X, \mathrm{Sym}^{\otimes\ell}E^* \otimes A^{\otimes k}) = 0 \tag{4.12}$$

when $\ell \geq c_A(k+1)$ and $k \geq 0$.

Proof. By the classical Le Potier isomorphism over compact complex manifolds (e.g. [SS85, Theorem 5.16]), we have

$$H^q\left(\mathbb{P}(E^*), \mathcal{O}_E(\ell)\otimes\Omega^p_{\mathbb{P}(E^*)}\otimes(\pi^*A^*)^{\otimes k}\right) = H^q\left(X, \mathrm{Sym}^{\otimes\ell}E\otimes\Omega^p_X\otimes(A^*)^{\otimes k}\right)$$

where $\pi : \mathbb{P}(E^*) \to X$ is the projection. By Theorem 4.2, if we take $p = q = n$,

$$H^{n,n}\left(\mathbb{P}(E^*), \mathcal{O}_E(\ell) \otimes (\pi^* A^*)^{\otimes k}\right) = H^{n,n}\left(X, \mathrm{Sym}^{\otimes \ell} E \otimes (A^*)^{\otimes k}\right) = 0$$

when $\ell \geq c_A(k+1)$ and $k \geq 0$. By the Serre duality, we obtain (4.12). $\square$

The proof of Theorem 1.5. If (E,h) is RC-positive, by Theorem 3.5, any line subbundle F of (E^*, h^*) can not be pseudo-effective. The second part of Theorem 1.5 is exactly Theorem 4.3. $\square$

Corollary 4.4. *Let X be a compact complex manifold and E be a vector bundle. If there exist a Hermitian metric ω on X and a Hermitian metric h on E such that*

$$\mathrm{tr}_\omega R^{(E,h)} \in \Gamma(X, \mathrm{End}(E))$$

is quasi-positive definite, then

$$H^0(X, (\Lambda^p E^*)^{\otimes \ell}) = 0 \quad \textit{for every} \quad \ell > 0 \quad \textit{and} \quad 1 \leq p \leq \mathrm{rank}(E). \tag{4.13}$$

Proof. It is well-known that (e.g. [LY12, Theorem 5.2] or [Gau77]) if $\mathrm{tr}_\omega R^{(E,h)} \in \Gamma(X, \mathrm{End}(E))$ is quasi-positive definite, then $H^0(X, E^*) = 0$. On the other hand, according to the proof of Theorem 3.6, we obtain that $\mathrm{tr}_\omega R^{(E^{\otimes \ell}, h^{\otimes \ell})}$ and $\mathrm{tr}_\omega R^{(\Lambda^p E, \Lambda^p h)}$ are also quasi-positive definite. Hence we obtain (4.13). $\square$

5. RC positivity and rational connectedness

In this section, we prove Theorem 1.8 and Theorem 1.9. The proofs rely on a classical criterion for rational connectedness proved in [CDP14, Theorem 1.1], see also [GHS03, Corollary 1.7], [LP17, Proposition 2.1] and [Cam16, Proposition 1.3]. By using this criterion, we are ready to prove rational connectedness under various curvature conditions.

The proof of Theorem 1.8. It follows from part (4) of Theorem 3.5 and [CDP14, Theorem 1.1]. For readers' convenience, we give a proof here following [CDP14, Theorem 1.1]. Indeed, if $(\Lambda^p T_X, h_p)$ is RC-positive for every $1 \leq p \leq \dim X$, then by part (4) of Theorem 3.5, line subbundles of Ω_X^p can not be pseudo-effective. In particular, when $p = n$, K_X is not pseudo-effective. Thanks to [BDPP13], X is uniruled. Let $\pi : X \dashrightarrow Z$ be the associated MRC fibration of X. After possibly resolving the singularities of π and Z, we may assume that π is a proper morphism and Z is smooth. By a result of Graber, Harris and Starr [GHS03, Corollary 1.4], it follows that the target of the MRC fibration is either a point or a positive dimensional variety which is not unruled. Suppose X is not rationally connected, then $\dim Z \geq 1$. Hence Z is not uniruled, by [BDPP13] again, K_Z is pseudo-effective. Since $K_Z =$

$\Omega_Z^{\dim Z} \subset \Omega_X^{\dim Z}$ is pseudo-effective, we get a contradiction. Hence X is rationally connected. (One can also use the vanishing theorem in Theorem 1.5 to deduce the rational connectedness of X.) $\square$

The proof of Theorem 1.9. It follows Theorem 3.6 and Theorem 1.8. If the trace $\mathrm{tr}_\omega R^{(T_X,h)}$ is positive definite, by Theorem 3.6, $(\Lambda^p T_X, \Lambda^p h)$ is RC-positive for every $1 \leq p \leq \dim X$. Hence, by Theorem 1.8, X is rationally connected. $\square$

As an application of Theorem 1.9 and the classical result of Uhlenbeck-Yau and Donaldson ([UY86, Don87]), one has

Corollary 5.1. *Let X be a projective manifold. If there exists a polarization $[H]$ such that $(T_X, [H])$ is poly-stable with positive slope. Then X is rationally connected.*

6. Positive holomorphic sectional curvature and rational connectedness

In this section, we describe more applications of Theorem 1.8 on RC-positive vector bundles and prove Theorem 1.11, Theorem 1.13 and Corollary 1.14.

6.1. **Compact Kähler manifolds with positive holomorphic sectional curvature.**

We begin with an algebraic curvature relation on a compact Kähler manifold (X, ω). By the Kähler symmetry, we have

$$R_{i\bar{j}k\bar{\ell}} = R_{k\bar{\ell}i\bar{j}} = R_{k\bar{j}i\bar{\ell}}.$$

At a given point $q \in X$, the minimum holomorphic sectional curvature is defined as

$$\min_{W \in T_qX, |W|=1} H(W),$$

where $H(W) := R(W, \overline{W}, W, \overline{W})$. Since X is of finite dimension, the minimum can be attained. The following result is essentially obtained in [Yang17, Lemma 4.1] (see also some variants in [Gol98, p. 312], [Bre10, p. 136] and [BKT13, Lemma 1.4]). For the sake of completeness, we include a proof here.

Lemma 6.1. *Let (X, ω) be a compact Kähler manifold and q be an arbitrary point on X. Let $e_1 \in T_qX$ be a unit vector which minimizes the holomorphic sectional curvature H of ω at point q, then*

$$2R(e_1, \overline{e}_1, W, \overline{W}) \geq \left(1 + |\langle W, e_1\rangle|^2\right) R(e_1, \overline{e}_1, e_1, \overline{e}_1) \tag{6.1}$$

for every unit vector $W \in T_qX$.

Proof. Let $e_2 \in T_qX$ be any unit vector orthogonal to e_1. Let

$$f_1(\theta) = H(\cos(\theta)e_1 + \sin(\theta)e_2), \quad \theta \in \mathbb{R}.$$

Then we have the expansion

$$\begin{aligned} f_1(\theta) &= \cos^4(\theta) R_{1\bar{1}1\bar{1}} + \sin^4(\theta) R_{2\bar{2}2\bar{2}} \\ &\quad + 2\sin(\theta)\cos^3(\theta)\left[R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}}\right] + 2\cos(\theta)\sin^3(\theta)\left[R_{1\bar{2}2\bar{2}} + R_{2\bar{1}2\bar{2}}\right] \\ &\quad + \sin^2(\theta)\cos^2(\theta)[4R_{1\bar{1}2\bar{2}} + R_{1\bar{2}1\bar{2}} + R_{2\bar{1}2\bar{1}}]. \end{aligned}$$

Since $f_1(\theta) \geq R_{1\bar{1}1\bar{1}}$ for all $\theta \in \mathbb{R}$ and $f_1(0) = R_{1\bar{1}1\bar{1}}$, we have

$$f_1'(0) = 0 \quad \text{and} \quad f_1''(0) \geq 0.$$

By a straightforward computation, we obtain

(6.2)

$$f_1'(0) = 2(R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}}) = 0, \quad f_1''(0) = 2\left(4R_{1\bar{1}2\bar{2}} + R_{1\bar{2}1\bar{2}} + R_{2\bar{1}2\bar{1}}\right) - 4R_{1\bar{1}1\bar{1}} \geq 0.$$

Similarly, if we set $f_2(\theta) = H(\cos(\theta)e_1 + \sqrt{-1}\sin(\theta)e_2)$, then

$$\begin{aligned} f_2(\theta) &= \cos^4(\theta) R_{1\bar{1}1\bar{1}} + \sin^4(\theta) R_{2\bar{2}2\bar{2}} \\ &\quad + 2\sqrt{-1}\sin(\theta)\cos^3(\theta)\left[-R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}}\right] + 2\sqrt{-1}\cos(\theta)\sin^3(\theta)\left[-R_{1\bar{2}2\bar{2}} + R_{2\bar{1}2\bar{2}}\right] \\ &\quad + \sin^2(\theta)\cos^2(\theta)[4R_{1\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}]. \end{aligned}$$

From $f_2'(0) = 0$ and $f_2''(0) \geq 0$, one can see

$$-R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}} = 0, \qquad 2\left(4R_{1\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}\right) - 4R_{1\bar{1}1\bar{1}} \geq 0. \tag{6.3}$$

Hence, from (6.2) and (6.3), we obtain

$$R_{1\bar{1}1\bar{2}} = R_{1\bar{1}2\bar{1}} = 0, \quad \text{and} \quad 2R_{1\bar{1}2\bar{2}} \geq R_{1\bar{1}1\bar{1}}. \tag{6.4}$$

For an arbitrary unit vector $W \in T_qX$, if W is parallel to e_1, i.e. $W = \lambda e_1$ with $|\lambda| = 1$,

$$2R(e_1, \bar{e}_1, W, \overline{W}) = 2R(e_1, \bar{e}_1, e_1, \bar{e}_1).$$

Suppose W is not parallel to e_1. Let e_2 be the unit vector

$$e_2 = \frac{W - \langle W, e_1\rangle e_1}{|W - \langle W, e_1\rangle e_1|}.$$

Then e_2 is a unit vector orthogonal to e_1 and

$$W = ae_1 + be_2$$

where $a = \langle W, e_1\rangle$, $b = |W - \langle W, e_1\rangle e_1|$ and $|a|^2 + |b|^2 = 1$. Hence

$$2R(e_1, \bar{e}_1, W, \overline{W}) = 2|a|^2 R_{1\bar{1}1\bar{1}} + 2|b|^2 R_{1\bar{1}2\bar{2}},$$

since we have $R_{1\bar{1}1\bar{2}} = R_{1\bar{1}2\bar{1}} = 0$ by formula (6.4). By formula (6.4) again,

$$2R(e_1, \bar{e}_1, W, \overline{W}) \geq (2|a|^2 + |b|^2) R_{1\bar{1}1\bar{1}} = (1 + |a|^2) R_{1\bar{1}1\bar{1}}$$

which completes the proof of Lemma 6.1. □

Remark 6.2. The Kähler condition is substantially used in the proof of Lemma 6.1.

Now we are ready to prove Theorem 1.11, that is

Theorem 6.3. *Let (X,ω) be a compact Kähler manifold with positive holomorphic sectional curvature. Then for every $1 \leq p \leq \dim X$, $(\Lambda^p T_X, \Lambda^p \omega)$ is RC-positive and $H^{p,0}_{\overline{\partial}}(X) = 0$. In particular, X is a projective and rationally connected manifold.*

Proof. Suppose $(\Lambda^p T_X, \Lambda^p \omega)$ is not RC-positive. By Definition 3.3, there exist a point $q \in X$ and a nonzero vector $a \in \Gamma(X, E)$ where $E = \Lambda^p T_X$ such that the Hermitian $(1,1)$-form

$$R^E(\bullet, \bullet, a, \overline{a}) \in \Gamma(X, \Lambda^{1,1} T^*_X) \tag{6.5}$$

is semi-negative at point q. We choose $e_1 \in \Gamma(X, T_X)$ at point q such that

$$R(e_1, \overline{e}_1, e_1, \overline{e}_1) = H(e_1) = \min_{W \in T_q X, |W|=1} H(W) > 0.$$

Hence, by Lemma 6.1,

$$2R(e_1, \overline{e}_1, W, \overline{W}) \geq \left(1 + |\langle W, e_1 \rangle|^2\right) R(e_1, \overline{e}_1, e_1, \overline{e}_1) > 0 \tag{6.6}$$

for every unit vector $W \in T_q X$. In particular,

$$R(e_1, \overline{e}_1, \bullet, \bullet) \in \Gamma(X, \mathrm{End}(T_X))$$

is positive definite at point q. Hence,

$$R^E(e_1, \overline{e}_1, \bullet, \bullet) \in \Gamma(X, \mathrm{End}(\Lambda^p T_X)) \tag{6.7}$$

is also positive definite at point q. Therefore,

$$R^E(e_1, \overline{e}_1, a, \overline{a}) > 0$$

which is a contradiction to (6.5). Hence, we deduce that $(\Lambda^p T_X, \Lambda^p \omega)$ is RC-positive.

For any $s \in H^0(X, \Omega^p_X) \cong H^{p,0}_{\overline{\partial}}(X)$, we have

$$\sqrt{-1}\partial\overline{\partial}|s|^2_g = \langle \nabla s, \nabla s \rangle_g + R^E(\bullet, \bullet, s, \overline{s}).$$

Note that $E = \Lambda^p T_X$. Suppose $|s|^2_g$ attains its maximum at some point q. Let $e_1 \in T_q X$ be the unit vector which minimizes the holomorphic sectional curvature. Hence, at point q, we have

$$R^E(e_1, \overline{e}_1, s, \overline{s}) \leq 0.$$

However, by Lemma 6.1, we know $R^E(e_1, \overline{e}_1, \bullet, \bullet)$ is positive definite at point q. Therefore, we must have $s(q) = 0$. That means $s = 0$ and $H^{p,0}_{\overline{\partial}}(X) = 0$. In particular, we have $H^{2,0}_{\overline{\partial}}(X) = H^{0,2}_{\overline{\partial}}(X) = 0$. Hence, X is projective. By Theorem 1.8, X is rationally connected. $\square$

6.2. **Compact complex manifolds with non-negative holomorphic sectional curvature.** In this subsection, we investigate Hermitian metrics with non-negative holomorphic sectional curvature.

Proposition 6.4. *Let (X,ω) be a compact Hermitian manifold with semi-positive holomorphic sectional curvature. If the holomorphic sectional curvature is not identically zero, then*

(1) *there exists a Gauduchon metric ω_G on X such that*

$$\int_X \mathrm{Ric}(\omega_G)\wedge \omega_G^{n-1}>0;$$

(2) *K_X is not pseudo-effective;*

(3) *there exists a Hermitian metric h on K_X^{-1} such that (K_X^{-1},h) is RC-positive.*

Moreover, if in addition, X is projective, then X is uniruled.

Remark 6.5. Proposition 6.4 is a straightforward application of Theorem [Yang16, Theorem 1.2], [Yang17b, Theorem 4.1] and the classical result of [BDPP13]. To demonstrate the essential difficulty in proving Conjecture 1.15 for higher dimensional compact complex manifolds and also the significant difference from the Kähler case, we include a detailed proof for Proposition 6.4.

Proof. By [Yang17b, Theorem 4.1], we know (1), (2) and (3) are mutually equivalent. Hence, we only need to prove one of them, for instance (1). We follow the steps in [Yang16, Theorem 4.1] for readers' convenience. At a given point $p\in X$, the maximum holomorphic sectional curvature is defined to be

$$H_p:=\max_{W\in T_pX,|W|=1}H(W),$$

where $H(W):=R(W,\overline{W},W,\overline{W})$. Suppose the holomorphic sectional curvature is not identically zero, i.e. $H_p>0$ for some $p\in X$. For any $q\in X$. We assume $g_{i\bar j}(q)=\delta_{ij}$. If $\dim_{\mathbb{C}}X=n$ and $[\xi^1,\cdots,\xi^n]$ are the homogeneous coordinates on $\mathbb{P}^{n-1}$, and ω_{FS} is the Fubini-Study metric of $\mathbb{P}^{n-1}$. At point q, we have the following well-known identity:

$$\int_{\mathbb{P}^{n-1}}R_{i\bar jk\bar\ell}\frac{\xi^i\overline{\xi}^j\xi^k\overline{\xi}^\ell}{|\xi|^4}\omega_{FS}^{n-1}=R_{i\bar jk\bar\ell}\cdot\frac{\delta_{ij}\delta_{k\ell}+\delta_{i\ell}\delta_{kj}}{n(n+1)}=\frac{s+\widehat{s}}{n(n+1)}. \tag{6.8}$$

where s is the Chern scalar curvature of ω and $\widehat{s}$ is defined as

$$\widehat{s}=g^{i\bar\ell}g^{k\bar j}R_{i\bar jk\bar\ell}. \tag{6.9}$$

Hence if (X,ω) has semi-positive holomorphic sectional curvature, then $s+\widehat{s}$ is a non-negative function on X. On the other hand, at point $p\in X$, $s+\widehat{s}$ is strictly positive since $H_p>0$. By (6.8), the integrand is quasi-positive over $\mathbb{P}^{n-1}$, and so $s+\widehat{s}$ is strictly positive at $p\in X$. By [LY17, Section 4], we have the relation

$$s=\widehat{s}+\langle\overline{\partial\partial^*\omega},\omega\rangle. \tag{6.10}$$

Therefore, we have

$$\int_X \widehat{s}\omega^n = \int_X s\omega^n - \int_X |\overline{\partial}^*\omega|^2\omega^n. \tag{6.11}$$

Let $\omega_G = f_0^{\frac{1}{n-1}}\omega$ be a Gauduchon metric (i.e. $\partial\overline{\partial}\omega_G^{n-1} = 0$) in the conformal class of ω for some positive weight function $f_0 \in C^\infty(X)$. Let $s_G, \widehat{s}_G$ be the corresponding scalar curvatures with respect to the Gauduchon metric ω_G. Then we have

$$\begin{aligned} \int_X s_G\omega_G^n &= -n\int_X \sqrt{-1}\partial\overline{\partial}\log\det(\omega_G)\wedge\omega_G^{n-1} \\ &= -n\int_X f_0\sqrt{-1}\partial\overline{\partial}\log\det(\omega)\wedge\omega^{n-1} \\ &= \int_X f_0 s\omega^n. \end{aligned} \tag{6.12}$$

By using a similar equation as (6.11) for $s_G, \widehat{s}_G$ and ω_G, we obtain

$$\int_X \widehat{s}_G\omega_G^n = \int_X f_0\widehat{s}\omega^n. \tag{6.13}$$

Therefore, if $s + \widehat{s}$ is quasi-positive, we obtain

$$\begin{aligned} \int_X s_G\omega_G^n &= \frac{\int_X(s_G+\widehat{s}_G)\omega_G^n}{2} + \frac{\int_X(s_G-\widehat{s}_G)\omega_G^n}{2} \\ &= \frac{\int_X(s_G+\widehat{s}_G)\omega_G^n}{2} + \frac{\|\overline{\partial}_G^*\omega_G\|^2}{2} = \frac{\int_X f_0(s+\widehat{s})\omega^n}{2} + \frac{\|\overline{\partial}_G^*\omega_G\|^2}{2} > 0 \end{aligned} \tag{6.14}$$

where the third equation follows from (6.12) and (6.13). □

The proof of Theorem 1.13. From the definition of positive holomorphic holomorphic sectional curvature and RC-positivity, it is easy to see that if a Hermitian metric ω has positive holomorphic sectional curvature, then (T_X, ω) is RC-positive. On the other hand, by Proposition 6.4, if ω has positive holomorphic sectional curvature, then there exists a (possibly different) Hermitian metric $\widetilde{h}$ on K_X^{-1} such that $(K_X^{-1}, \widetilde{h})$ is RC-positive. □

The proof of Corollary 1.14. We first show X is indeed projective. Suppose $\sigma \in H^0(X, K_X)$ is not zero, then K_X is $\mathbb{Q}$-effective. However, by Proposition 6.4, K_X is not pseudo-effective. This is a contradiction. Hence we deduce $H^{2,0}(X) = H^{0,2}(X) = H^0(X, K_X) = 0$. We know X is projective. Now Corollary 1.14 follows from Theorem 1.8 and Theorem 1.13. □

Let $f : X \to Y$ be a smooth submersion between projective manifolds. It is well-known that if X is rationally connected, then Y is rationally connected. As analogous to this result, we have:

Proposition 6.6. *Let $f : X \to Y$ be a smooth submersion between compact complex manifolds. Suppose X admits a Hermitian metric h such that $(\Lambda^p T_X, h)$ is RC-positive for some $1 \leq p \leq \dim Y$, then $\Lambda^p T_Y$ admits an RC-positive Hermitian metric.*

Proof. It follows from part (3) of Theorem 3.5. □

Similarly, we have

Corollary 6.7. *Let $f : X \to Y$ be a smooth submersion between compact complex manifolds. Suppose X admits a Hermitian metric h with positive holomorphic sectional curvature, then Y has a Hermitian metric with positive holomoprhic sectional curvature. In particular, if in addition, Y is projective, then Y is uniruled.*

Proof. It follows from part (3) of Theorem 3.5, formula (3.5) and Theorem 1.13. □

7. Examples

In this section, we present a few examples to make a distinction between various terminologies.

Example 7.1. Let X be a compact complex manifold. If there exist a Hermitian metric ω on X and a Hermitian metric h on the vector bundle T_X such that

$$\mathrm{tr}_\omega R^{(T_X,h)} \in \Gamma(X, \mathrm{End}(T_X))$$

is positive definite, then $\det T_X$ is not necessarily positive, i.e. X is not necessarily Fano.

Let's recall an example in [LY17, Section 6]. Let $M = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ be the standard n-dimensional ($n \geq 2$) Hopf manifold. It is diffeomorphic to $\mathbb{C}^n - \{0\}/G$ where G is cyclic group generated by the transformation $z \to \frac{1}{2}z$. It has an induced complex structure from $\mathbb{C}^n - \{0\}$. On M, there is a natural induced metric ω given by

$$\omega = \sqrt{-1} h_{i\bar{j}} dz^i \wedge d\bar{z}^j = \sqrt{-1} \frac{4\delta_{i\bar{j}}}{|z|^2} dz^i \wedge d\bar{z}^j. \tag{7.1}$$

The curvature components are

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 h_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + h^{p\bar{q}} \frac{\partial h_{k\bar{q}}}{\partial z^i} \frac{\partial h_{p\bar{\ell}}}{\partial \bar{z}^j} = \frac{4\delta_{kl}(\delta_{ij}|z|^2 - z^j \bar{z}^i)}{|z|^6}. \tag{7.2}$$

Hence

$$\mathrm{tr}_\omega R^{(T_X,\omega)} = \left[\frac{(n-1)\delta_{k\ell}}{|z|^2} \right]_{k,\ell} > 0.$$

Example 7.2. Let E be a vector bundle such that $\det E$ is ample. In general, E is not necessarily RC-positive.

Let L be an ample line bundle over a compact complex manifold X and $E = L^{\otimes 2} \oplus L^{-1}$. Then $c_1(E) = c_1(L) > 0$. However, E does not admit a Hermitian metric h such that (E, h) is RC-positive. Indeed, it follows from part (1) of Theorem 1.5. We can also write down an explicit toy model. Let h be a positive metric on L, i.e. $R = -\sqrt{-1}\partial\overline{\partial} \log \det h > 0$. Hence,

$$R^E = 2R \otimes \mathrm{Id}_2 - R \otimes \mathrm{Id}_1$$

and we have

$$R^E(v, \overline{v}, a, \overline{a}) = R(v, \overline{v})(2|a^2|^2 - |a^1|^2)$$

where $a = (a_1, a_2)$. Hence $(E, h^2 \otimes h^{-1})$ can not be RC-positive.

Similarly, by a conformal perturbation construction, we know there exists a *Hermitian non-Kähler metric* ω on a projective manifold X, such that

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial} \log(\omega^n) > 0,$$

but (T_X, ω) is *not* RC-positive. Note that on such a manifold, there does exist a Kähler metric ω_0 such that (T_X, ω_0) is RC-positive, thanks to the Calabi-Yau theorem.

Example 7.3. Let $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1})$ be a Hirzebruch surface. As well-known, it is a rationally connected projective manifold and admits Kähler metrics with positive holomorphic sectional curvature. We refer to [Yau74, Hit75, HW12, HW15, ACH15, Yang16, AHZ16, YZ16, AH17] for detailed studies. We show by the following example that the canonical bundle of a rational connected manifold (resp. a projective manifold with positive holomorphic sectional curvature) is not necessarily nef.

Proposition 7.4. *For any $k \geq 4$, the anti-canonical bundle K_Y^{-1} of the Hirzebruch surface $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1})$ is effective but not nef.*

Proof. We show K_Y^{-1} is not nef. Since $Y \cong \mathbb{P}(E^*)$ with $E = \mathcal{O}_{\mathbb{P}^1}(k-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, by using the projection formula (e.g.[Laz04b, p.89]), we have

$$K_Y = \mathcal{O}_Y(-2) \otimes \pi^*(K_{\mathbb{P}^1} \otimes \det(E)) = \mathcal{O}_Y(-2) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(k-4)). \tag{7.3}$$

In particular,

$$\mathcal{O}_Y(2) = K_Y^{-1} \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(k-4)). \tag{7.4}$$

Suppose K_Y^{-1} is nef, then $\mathcal{O}_Y(1)$ is also nef since $k \geq 4$. Therefore $E = \mathcal{O}_{\mathbb{P}^1}(k-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ is a nef vector bundle. This is a contradiction.

We prove K_Y^{-1} is effective. Since $Y \cong \mathbb{P}(E^*)$ with $E = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}$. We have

$$K_Y^{-1} = \mathcal{O}_Y(2) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(2-k)). \tag{7.5}$$

By the direct image formula (e.g. [Laz04b, p.90]), we get

$$\begin{aligned} H^0(Y, K_Y^{-1}) &= H^0(Y, \mathcal{O}_Y(2)\otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(2-k))) \\ &\cong H^0(\mathbb{P}^1, \pi_*(\mathcal{O}_Y(2)\otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(2-k)))) \\ &= H^0(\mathbb{P}^1, \mathrm{Sym}^{\otimes 2}E\otimes \mathcal{O}_{\mathbb{P}^1}(2-k)) \neq 0, \end{aligned}$$

since we have

$$\mathrm{Sym}^{\otimes 2}E = \mathcal{O}_{\mathbb{P}^1}(2k)\oplus \mathcal{O}_{\mathbb{P}^1}(k)\oplus \mathcal{O}_{\mathbb{P}^1}.$$

Hence, K_Y^{-1} is effective. □

8. Some related questions and discussions

In this section, we gather several conjectures in complex algebraic geometry and give their differential geometric interpretations.

8.1. Mumford's conjecture and uniruledness conjecture.

In [Yang17b, Theorem 4.1] and [Yang17b, Corollary 1.6], we proved that

Theorem 8.1. *On a compact complex manifold X, the following are equivalent:*

1. *the dual line bundle L^* is not pseudo-effective;*
2. *L is RC-positive;*
3. *there exists a Hermitian metric h on L and a Hermitian metric ω on X such that the scalar curvature $\mathrm{tr}_\omega(-\sqrt{-1}\partial\overline{\partial}\log h) > 0$.*

Moreover, if X is projective, then they are also equivalent to

4. *for any ample line bundle A, there exists a positive integer c_A such that*

$$H^0(X, K_X^{\otimes \ell}\otimes A^{\otimes k}) = 0$$

for $\ell \geq c_A(k+1)$ and $k\geq 0$.

The classical result of [BDPP13] says that a projective manifold is uniruled if and only if the canonical bundle K_X is not pseudo-effective. Hence, one can formulate the uniruledness conjecture as

Conjecture 8.2. *Let X is a projective manifold. Then $\kappa(X) = -\infty$ is equivalent to one (and hence all) of the following*

1. *X is uniruled.*
2. *K_X is not pseudo-effective;*
3. *K_X^{-1} is RC-positive, i.e. there exists a Hermitian metric ω on X such that the Ricci curvature $\mathrm{Ric}(\omega)$ has at least one positive direction at each point;*
4. *there exists a Hermitian metric ω on X with positive (Chern) scalar curvature;*

(5) *for any ample line bundle A, there exists a positive integer c_A such that*
$$H^0(X, K_X^{\otimes \ell} \otimes A^{\otimes k}) = 0$$
for $\ell \geq c_A(k+1)$ and $k \geq 0$.

Conjecture 8.3 (Mumford). *Let X be a projective manifold. If*
$$H^0(X, (T_X^*)^{\otimes m}) = 0, \quad \textit{for every} \quad m \geq 1,$$
then X is rationally connected.

It is well-known that the uniruledness conjecture can imply Conjecture 8.3 (e.g. [GHS03, Corollary 1.7], see also Proposition 8.5).

Conjecture 8.4. *Let X be a projective manifold. If*
$$H^0(X, (T_X^*)^{\otimes m}) = 0, \quad \textit{for every} \quad m \geq 1,$$
then one (and hence all) of the following holds

(1) *X is uniruled;*
(2) *K_X is not pseudo-effective;*
(3) *there exists a Hermitian metric ω on X with positive (Chern) scalar curvature;*
(4) *K_X^{-1} is RC-positive, i.e. there exists a Hermitian metric ω on X such that the Ricci curvature $\mathrm{Ric}(\omega)$ has at least one positive direction at each point;*
(5) *for any ample line bundle A, there exists a positive integer c_A such that*
$$H^0(X, K_X^{\otimes \ell} \otimes A^{\otimes k}) = 0$$
for $\ell \geq c_A(k+1)$ and $k \geq 0$.

Proposition 8.5. *We have the following relations*
$$\textit{Conjecture } 8.2 \Longrightarrow \textit{Conjecture } 8.3 \Longleftrightarrow \textit{Conjecture } 8.4. \tag{8.1}$$

Proof. Conjecture 8.2 $\Longrightarrow$ Conjecture 8.4. Since $K_X = \det T_X^*$, it is well-known that for any positive integer ℓ, $K_X^{\otimes \ell}$ is a subbundle of $(T_X^*)^{\otimes m}$ for some large m. Hence $H^0(X, (T_X^*)^{\otimes m}) = 0$ for every $m \geq 1$ can imply $H^0(X, K_X^{\otimes \ell}) = 0$, i.e. $\kappa(X) = -\infty$. By assuming Conjecture 8.2, we obtain Conjecture 8.4.

Conjecture 8.3 $\Longrightarrow$ Conjecture 8.4. It follows from Theorem 1.2 and the fact that rational connected manifolds are uniruled.

Conjecture 8.4 $\Longrightarrow$ Conjecture 8.3. The proof follows from Theorem 8.1, Theorem 1.6 and a well-known argument in algebraic geometry (e.g. [CDP14, Theorem 1.1], [GHS03, Corollary 1.7], [LP17, Proposition 2.1]), which is also very similar to that of Theorem 1.8. Suppose
$$H^0(X, (T_X^*)^{\otimes m}) = 0, \quad \text{for every} \quad m \geq 1.$$
By assuming Conjecture 8.4, we know K_X is not pseudo-effective. Hence X is uniruled, thanks to the classical result of [BDPP13]. Let $\pi : X \dashrightarrow Z$ be the associated

MRC fibration of X. After possibly resolving the singularities of π and Z, we may assume that π is a proper morphism and Z is smooth. By [GHS03, Corollary 1.4], it follows that the target of the MRC fibration is either a point or a positive dimensional variety which is not unruled. Suppose X is not rationally connected, then $\dim Z \geq 1$. Hence Z is not uniruled, and by [BDPP13] again, K_Z is pseudo-effective. By assuming Conjecture 8.4, we obtain

$$H^0(Z, (T_Z^*)^{\otimes m_0}) \neq 0$$

for some positive integer m_0. We obtain $H^0(X, (T_X^*)^{\otimes m_0}) \neq 0$ since $(T_Z^*)^{\otimes m_0} \subset (T_X^*)^{\otimes m_0}$. This is a contradiction. $\square$

If X is rationally connected, K_X is not pseudo-effective. Hence there exists a Hermitian metric h on $K_X^{-1} = \Lambda^{\dim X} T_X$ such that (K_X^{-1}, h) is RC-positive. We propose a generalization of this fact:

Question 8.6. Let X be a rationally connected projective manifold. Do there exist smooth Hermitian metrics h_p on vector bundles $\Lambda^p T_X$ $(1 \leq p \leq \dim X)$ such that $(\Lambda^p T_X, h_p)$ are all RC-positive? Do there exist smooth Hermitian metrics g_p on vector bundles $T_X^{\otimes p}$ $(p \geq 1)$ such that $(T_X^{\otimes p}, g_p)$ are all RC-positive?

A natural generalization of Conjecture 8.4 is

Question 8.7. Let X be a projective manifold. Suppose

$$H^0(X, (T_X^*)^{\otimes m}) = 0, \quad \text{for every} \quad m \geq 1.$$

We can ask the same question as in Question 8.6.

8.2. **A partial converse to the Andreotti-Grauert theorem: the vector bundle version.** We propose several questions on vector bundles as a converse to Theorem 1.5:

Question 8.8. Let X be a projective manifold and E be a vector bundle. Suppose for every vector bundle A, there exists a positive integer $c_A = c(A, E)$ such that

$$H^0(X, \mathrm{Sym}^{\otimes \ell} E^* \otimes A^{\otimes k}) = 0 \tag{8.2}$$

for $\ell \geq c_A(k+1)$ and $k \geq 0$. Does E^* admit a pseudo-effective line subbundle? Do there exist smooth Hermitian metrics h_p on vector bundles $\Lambda^p E$ $(1 \leq p \leq \mathrm{rank}(E))$ such that $(\Lambda^p E, h_p)$ are all RC-positive? Do there exist smooth Hermitian metrics g_p on vector bundles $E^{\otimes p}$ (resp. $\mathrm{Sym}^{\otimes p} E$) $(p \geq 1)$ such that $(E^{\otimes p}, h_p)$ (resp. $(\mathrm{Sym}^{\otimes p} E, h_p)$) are all RC-positive?

Question 8.9. Consider the similar question as in Question 8.8 if we replace the vanishing condition (8.2) by

$$H^0(X, (E^*)^{\otimes \ell} \otimes A^{\otimes k}) = 0.$$

8.3. **Existence of RC-positive metrics on vector bundles.** In this subsection, we propose several questions on the existence of RC-positive metrics. The celebrated Kodaira embedding theorem establishes that a line bundle is ample if and only if carries a smooth metric with positive curvature. The analogous correspondence for vector bundles is proposed by P. Griffiths ([Gri69]):

Conjecture 8.10 (Griffiths)**.** *If E is an ample vector bundle over a compact complex manifold X, then E admits a Griffiths positive metric.*

When $\dim X = 1$, this conjecture is proved in [CF90]. The following conjecture can be implied by Griffiths' Conjecture 8.10:

Conjecture 8.11. *If E is an ample vector bundle over a projective manifold X, then*

(1) *$(E^{\otimes k}, h^{\otimes k})$ is RC-positive for every $k \geq 1$;*
(2) *$(\Lambda^p E, \Lambda^p h)$ is RC-positive for every $1 \leq p \leq \mathrm{rank}(E)$.*

As a converse to Proposition 4.1, we also propose the following

Question 8.12. Let E be a vector bundle over a compact complex manifold X. Suppose the tautological line bundle $\mathcal{O}_E(1)$ is $(\dim X - 1)$-positive over $\mathbb{P}(E^*)$. Is the vector bundle E necessarily RC-positive?

When $\dim X = 1$, Question 8.12 has an affirmative answer, thanks to [CF90].

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Address of Xiaokui Yang: Morningside Center of Mathematics, Institute of Mathematics, Hua Loo-Keng Center of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China.

E-mail address: xkyang@amss.ac.cn