

TROPICALIZATION OF THE MODULI SPACE OF STABLE MAPS

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ABSTRACT. Let X be an algebraic variety and let S be a tropical variety associated to X . We study the tropicalization map from the moduli space of stable maps into X to the moduli space of tropical curves in S . We prove that it is a continuous map and that its image is compact and polyhedral. Loosely speaking, when we deform algebraic curves in X , the associated tropical curves in S deform continuously; moreover, the locus of realizable tropical curves inside the space of all tropical curves is compact and polyhedral. Our main tools are Berkovich spaces, formal models, balancing conditions, vanishing cycles and quantifier elimination for rigid subanalytic sets.

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1. INTRODUCTION

Let k be a complete discrete valuation field, X a toric variety over k of dimension d , and C an algebraic curve embedded in X . In tropical geometry (see [56, 43, 31]), one associates to C a piecewise-linear tropical curve C^{trop} embedded in \mathbb{R}^d . In this paper, we study this “tropicalization procedure” in families.

In fact, we do not restrict ourselves to toric target spaces. We work with the framework of global tropicalization using Berkovich spaces (cf. [11, 69, 67, 41]). This is not only more general, but also more natural from our viewpoint.

Roughly speaking, we prove the following results:

- (i) The tropical curve C^{trop} deforms continuously when we deform the algebraic curve C .
- (ii) The locus of realizable tropical curves inside the space of all tropical curves of bounded degree is compact and polyhedral.

More precisely, we fix a k -analytic space X as the target space for curves. We tropicalize X by choosing a strictly semi-stable formal model \mathfrak{X} of X (see Definition 2.3). The associated tropical variety $S_{\mathfrak{X}}$ is homeomorphic to the dual intersection complex of the special fiber \mathfrak{X}_s . We call $S_{\mathfrak{X}}$ the *Clemens polytope*. As in the toric case, analytic curves in X give rise to piecewise-linear tropical curves in $S_{\mathfrak{X}}$.

In order to bound the complexity of the analytic curves in X , we need to bound their degree with respect to a Kähler structure \widehat{L} on X (see Section 5 and [67, 47]). Similarly, we will define a notion of *simple density* ω on $S_{\mathfrak{X}}$ induced by \widehat{L} in order to bound the complexity of the tropical curves in $S_{\mathfrak{X}}$.

We have the following description of the space of tropical curves in $S_{\mathfrak{X}}$ with bounded degree (Theorem 6.1).

Theorem. *Fix two non-negative integers g, n and a positive real number A . Let $M_{g,n}(S_{\mathfrak{X}}, A)$ denote the set of simple n -pointed genus g parametrized tropical curves in $S_{\mathfrak{X}}$ whose degree with respect to ω is bounded by A . Then $M_{g,n}(S_{\mathfrak{X}}, A)$ is naturally a compact topological space with a stratification whose open strata are open convex polyhedrons.*

In view of applications, instead of only considering analytic curves embedded in X , we will consider stable maps into X introduced by Maxim Kontsevich [45]. Let T be a strictly k -analytic space. Assume we have a family over T of n -pointed genus g k -analytic stable maps into X with degree bounded by A . We have the

following set-theoretic tropicalization map

$$\tau_T: T \longrightarrow M_{g,n}(S_{\mathfrak{X}}, A)$$

which sends n -pointed genus g k -analytic stable maps to the associated n -pointed genus g parametrized tropical curves.

Theorem. *The tropicalization map τ_T has the following properties:*

- (i) *It is a continuous map (Theorem 8.1).*
- (ii) *Its image is polyhedral in $M_{g,n}(S_{\mathfrak{X}}, A)$, in the sense that the intersection with every open stratum of $M_{g,n}(S_{\mathfrak{X}}, A)$ is polyhedral (Theorem 9.5).*

It is helpful to reformulate the theorem for the universal family of stable maps. Let $\overline{\mathcal{M}}_{g,n}(X, A)$ denote the moduli stack of n -pointed genus g k -analytic stable maps into X with degree bounded by A . It is a compact k -analytic stack by [67]. The map τ_T above extends to a tropicalization map

$$\tau_M: \overline{\mathcal{M}}_{g,n}(X, A) \longrightarrow M_{g,n}(S_{\mathfrak{X}}, A).$$

The image of the map τ_M consists of so-called *realizable tropical curves*. We denote it by $\overline{M}_{g,n}^{\text{trop}}(X, A)$. It is of much interest in tropical geometry to characterize realizable tropical curves (cf. [56, 59, 64, 58, 66, 25, 60]). We have the following corollary concerning τ_M and the locus of realizable tropical curves.

Corollary. (i) *The tropicalization map τ_M is a continuous map.*
(ii) *The locus $\overline{M}_{g,n}^{\text{trop}}(X, A)$ is compact and polyhedral.*

Discussions and related works. One motivation of our work stems from the speculations by Kontsevich and Soibelman in [46, §3.3] and the works by Gross, Siebert, Hacking and Keel [38, 36, 37]. We will apply our results to the study of non-archimedean enumerative geometry (see [70]). Another motivation is a question asked by Ilia Itenberg during a talk given by the author at Jussieu, Paris.

Moduli spaces of tropical curves in classical contexts were studied by Mikhalkin, Nishinou, Siebert, Gathmann, Markwig, Kerber, Kozlov, Caporaso, Viviani, Brannetti, Melo, Chan, Yu and others in [56, 57, 59, 34, 32, 48, 19, 49, 17, 18, 22, 23, 39, 68].

Very interesting geometry concerning the tropicalization of the moduli space of stable curves is studied in detail by Abramovich, Caporaso and Payne [2]. More generally, we expect to have explicit descriptions for the tropicalization of the

moduli space of stable maps into toric varieties. There are related developments by Cavalieri, Markwig, Ranganathan, Ascher, Molcho, Chen, Satriano and A. Gross [20, 21, 4, 24, 35, 61].

The model-theoretic technique involved in the proof of polyhedrality is inspired by the works of Ducros [29] and Martin [52], and is based on the theory of rigid subanalytic sets developed by Lipshitz and Robinson [50, 51].

We regret a certain asymmetry in our results. On the analytic side, we consider stable maps; while on the tropical side, we consider parametrized tropical curves which are locally embedded. One can define the notion of tropical stable maps and study their moduli space. We conjecture that the tropicalization map from the space of analytic stable maps to the space of tropical stable maps is also continuous and has polyhedral image.

Outline of the paper. In Section 2, we review the basic settings of global tropicalization. Given a k -analytic space X , the tropicalization $S_{\mathfrak{X}}$ of X depends on the choice of a formal model \mathfrak{X} of X . We work with strictly semi-stable formal models for simplicity.

In Section 3, we define the notion of parametrized tropical curve in our context. We explain how analytic curves in X give rise to tropical curves in $S_{\mathfrak{X}}$.

The tropical curves in $S_{\mathfrak{X}}$ that arise from analytic curves satisfy a distinguished geometrical property, called the *balancing condition*. It is a generalization of the classical balancing condition (cf. [55, 59, 64, 5]). The balancing condition in the global setting was first studied in [69] using k -analytic cohomological arguments. In Section 4, we give a different proof which is useful for the purpose of this paper. The main observation is that the tropical weight vectors can be read out directly from certain intersection numbers via the functor of vanishing cycles (Lemma 4.7).

In Section 5, we introduce a combinatorial notion of simple density on $S_{\mathfrak{X}}$. We define the degree of a tropical curve with respect to a simple density. We use it to give a lower bound of the degree of an analytic curve in X with respect to a non-archimedean Kähler structure on X .

In Section 6, we study the space of tropical curves in $S_{\mathfrak{X}}$ with bounded degree. We use some combinatorial arguments from our previous work [68].

In Section 7, we review the moduli stack of k -analytic stable maps constructed in [67].

In Section 8, we prove the continuity of the tropicalization map from the k -analytic moduli space of Section 7 to the tropical moduli space of Section 6. The

proof makes use of the balancing conditions in Section 4 and the formal models of families of k -analytic stable maps developed in [67].

In Section 9, combining the continuity theorem of Section 8 with the quantifier elimination theorem from the model theory of rigid subanalytic sets [50], we prove the polyhedrality of the locus of realizable tropical curves inside the space of all tropical curves.

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2. BASIC SETTINGS OF GLOBAL TROPICALIZATION

In this section, we review the basic settings of global tropicalization. We refer to [67, §2] and [16, §3] for more details (see also [47, 44, 41, 69]).

Let k be a complete discrete valuation field. Denote by k° the ring of integers, $k^{\circ\circ}$ the maximal ideal, and \tilde{k} the residue field.

For $n \geq 0$, $0 \leq d \leq n$ and $a \in k^{\circ\circ} \setminus 0$, put

$$\mathfrak{S}(n, d, a) = \mathrm{Spf} \left(k^\circ \{T_0, \dots, T_n, T_{d+1}^{-1}, \dots, T_n^{-1}\} / (T_0 \cdots T_d - a) \right), \quad (2.1)$$

where Spf denotes the formal spectrum.

Definition 2.1 ([11]). A formal scheme \mathfrak{X} is said to be *finitely presented* over k° if it is a finite union of open affine subschemes of the form

$$\mathrm{Spf} \left(k^\circ \{T_0, \dots, T_n\} / (f_1, \dots, f_m) \right).$$

Let \mathfrak{X} be a formal scheme finitely presented over k° . One can define its generic fiber \mathfrak{X}_η and its special fiber \mathfrak{X}_s following [9]. Its generic fiber \mathfrak{X}_η has the structure of a compact strictly k -analytic space in the sense of Berkovich [7, 8], and its special fiber \mathfrak{X}_s is a scheme of finite type over the residue field \tilde{k} . We denote by $\pi: \mathfrak{X}_\eta \rightarrow \mathfrak{X}_s$ the reduction map from the generic fiber to the special fiber (cf. [9, §1]).

Definition 2.2. Let X be a k -analytic space. A (*finitely presented*) *formal model* of X is a formal scheme \mathfrak{X} finitely presented over k° together with an isomorphism between the generic fiber \mathfrak{X}_η and the k -analytic space X .

Definition 2.3. Let \mathfrak{X} be a formal scheme finitely presented over k° . The formal scheme \mathfrak{X} is said to be *strictly semi-stable* if

- (i) Every point x of \mathfrak{X} has an open affine neighborhood \mathfrak{U} such that the structure morphism $\mathfrak{U} \rightarrow \mathrm{Spf} k^\circ$ factors through an étale morphism $\phi: \mathfrak{U} \rightarrow \mathfrak{S}(n, d, \varpi)$ for some $0 \leq d \leq n$ and a uniformizer ϖ of k .
- (ii) All the intersections of the irreducible components of the special fiber \mathfrak{X}_s are either empty or geometrically irreducible.

Let X be a k -analytic space and let \mathfrak{X} be a strictly semi-stable formal model¹ of X . Let $\{D_i \mid i \in I_{\mathfrak{X}}\}$ denote the set of the irreducible components of the special fiber \mathfrak{X}_s . For every non-empty subset $I \subset I_{\mathfrak{X}}$, put $D_I = \bigcap_{i \in I} D_i$ and

$$J_I = \{j \in I_{\mathfrak{X}} \mid D_{I \cup \{j\}} \neq \emptyset\}. \quad (2.2)$$

Condition (i) of Definition 2.3 implies that all the strata D_I are smooth over the residue field \tilde{k} .

The *Clemens polytope* $S_{\mathfrak{X}}$ is by definition the simplicial subcomplex of the simplex $\Delta^{I_{\mathfrak{X}}}$ such that for every non-empty subset $I \subset I_{\mathfrak{X}}$, the simplex Δ^I is a face of $S_{\mathfrak{X}}$ if and only if the stratum D_I is non-empty. As a special case of [11], one can construct a canonical inclusion map $\theta: S_{\mathfrak{X}} \hookrightarrow \mathfrak{X}_\eta$ and a canonical strong deformation retraction from \mathfrak{X}_η to $S_{\mathfrak{X}}$. For simplicity, we only explain the construction of the retraction map $\tau: \mathfrak{X}_\eta \rightarrow S_{\mathfrak{X}}$, i.e. the final moment of the strong deformation retraction.

Let $\mathrm{Div}_0(\mathfrak{X})_{\mathbb{R}}$ denote the vector space of vertical Cartier \mathbb{R} -divisors on \mathfrak{X} . It is of dimension the cardinality of $I_{\mathfrak{X}}$. An effective vertical divisor D on \mathfrak{X} is locally given by a function u up to multiplication by invertible functions. So $\mathrm{val}(u(x))$ defines a continuous function on \mathfrak{X}_η which we denote by φ_D^0 . By linearity, φ_D^0 makes sense for any divisor D in $\mathrm{Div}_0(\mathfrak{X})_{\mathbb{R}}$. Let $\tau: \mathfrak{X}_\eta \rightarrow \mathrm{Div}_0(\mathfrak{X})_{\mathbb{R}}^*$ be the evaluation map defined by $\langle \tau(x), D \rangle = \varphi_D^0(x)$ for any $x \in \mathfrak{X}_\eta$, $D \in \mathrm{Div}_0(\mathfrak{X})_{\mathbb{R}}$. The image of τ can be naturally identified with the Clemens polytope $S_{\mathfrak{X}}$. The identification gives us a canonical embedding

$$S_{\mathfrak{X}} \subset \mathrm{Div}_0(\mathfrak{X})_{\mathbb{R}}^* \simeq \mathbb{R}^{I_{\mathfrak{X}}}. \quad (2.3)$$

We will always regard the Clemens polytope $S_{\mathfrak{X}}$ as embedded in $\mathrm{Div}_0(\mathfrak{X})_{\mathbb{R}}^*$.

¹If the residue field \tilde{k} has characteristic zero and if X is compact quasi-smooth and strictly k -analytic, then strictly semi-stable formal models exist up to passing to a finite extension of k (cf. [65, 44]).

Remark 2.4. Let \mathfrak{S} denote the standard formal scheme $\mathfrak{S}(n, d, a)$ in (2.1). The retraction map $\tau_{\mathfrak{S}}: \mathfrak{S}_{\eta} \rightarrow S_{\mathfrak{S}} \subset \mathbb{R}^{d+1}$ can be written explicitly as follows

$$\begin{aligned} \mathfrak{S}_{\eta} &\longrightarrow \mathbb{R}^{d+1} \\ x &\longmapsto (\text{val } T_0(x), \dots, \text{val } T_d(x)). \end{aligned}$$

The image is the d -dimensional simplex in $\mathbb{R}_{\geq 0}^{d+1}$ given by the equation $\sum x_i = \text{val}(a)$. We remark that in the general case, the retraction map $\tau: \mathfrak{X}_{\eta} \rightarrow S_{\mathfrak{X}} \subset \mathbb{R}^{I_{\mathfrak{X}}}$ is locally of the form above.

The retraction from the k -analytic space X to the Clemens polytope $S_{\mathfrak{X}}$ with respect to the formal model \mathfrak{X} is functorial in the following sense.

Proposition 2.5 (cf. [67, §2]). *Let $f: X \rightarrow Y$ be a morphism of k -analytic spaces. Let \mathfrak{X} and \mathfrak{Y} be strictly semi-stable formal models of X and Y respectively such that the morphism f extends to a morphism $\mathfrak{f}: \mathfrak{X} \rightarrow \mathfrak{Y}$ of formal schemes. Let $\tau_{\mathfrak{X}}: \mathfrak{X}_{\eta} \rightarrow S_{\mathfrak{X}}$ and $\tau_{\mathfrak{Y}}: \mathfrak{Y}_{\eta} \rightarrow S_{\mathfrak{Y}}$ denote the retraction maps. Then there exists a continuous map $S_{\mathfrak{f}}: S_{\mathfrak{X}} \rightarrow S_{\mathfrak{Y}}$, which is affine on every simplicial face of $S_{\mathfrak{X}}$, such that the diagram*

$$\begin{array}{ccc} \mathfrak{X}_{\eta} & \xrightarrow{\mathfrak{f}_{\eta}} & \mathfrak{Y}_{\eta} \\ \downarrow \tau_{\mathfrak{X}} & & \downarrow \tau_{\mathfrak{Y}} \\ S_{\mathfrak{X}} & \xrightarrow{S_{\mathfrak{f}}} & S_{\mathfrak{Y}} \end{array} \quad (2.4)$$

commutes.

3. PARAMETRIZED TROPICAL CURVES AND PARAMETRIZED TROPICALIZATION

In this section, we introduce the notions of parametrized tropical curves, combinatorial types and degenerations of combinatorial types. After that, we explain how analytic curves give rise to tropical curves. We use the settings of Section 2.

Definition 3.1. Let Γ be a finite undirected graph. We denote by $V(\Gamma)$ the set of vertices and by $E(\Gamma)$ the set of edges. For a vertex v of Γ , the degree $\deg(v)$ denotes the number of edges connected to v . For two vertices u, v of Γ , we denote by $E(u, v)$ the set of edges connecting u and v . For a vertex v and an edge e of Γ , we denote $v \in e$ or $e \ni v$ if v is an endpoint of e . A *flag* of Γ is a pair (v, e) consisting of a vertex v and an edge e connected to v .

Definition 3.2. An n -pointed parametrized tropical curve $(\Gamma, (\gamma_i), h)$ in the Clemens polytope $S_{\mathfrak{X}}$ consists of the following data:

- (i) A connected finite graph Γ without self-loops.
- (ii) A continuous map h from the topological realization of Γ to $S_{\mathfrak{X}}$ such that every edge of Γ embeds as an affine segment with rational slope in a face of $S_{\mathfrak{X}}$.
- (iii) Every flag (v, e) of Γ is equipped with a *tropical weight vector* $w_{(v,e)} \in \mathbb{Z}^{I_{\mathfrak{X}}} \setminus 0$, parallel to the direction of $h(e)$ pointing away from $h(v)$. For every edge e of Γ and its two endpoints u and v , we require that $w_{(u,e)} + w_{(v,e)} = 0$.
- (iv) Every vertex v of Γ is equipped with a non-negative integer $g(v)$, called the *genus* of the vertex v .
- (v) A sequence of vertices $\gamma_1, \dots, \gamma_n$ of Γ , called *marked points*. They are not required to be different from each other. For each vertex v of Γ , we denote by $n(v)$ the number of marked points at v .

We denote a 0-pointed parametrized tropical curve simply by (Γ, h) .

Definition 3.3. Let $(\Gamma, (\gamma_i), h)$ be an n -pointed parametrized tropical curve in $S_{\mathfrak{X}}$. We define its *genus* to be the sum

$$b_1(\Gamma) + \sum_{v \in V(\Gamma)} g(v),$$

where $b_1(\cdot)$ denotes the first Betti number. For every vertex v of Γ , we define the *sum of weight vectors* around v to be $\sigma_v := \sum_{e \ni v} w_{(v,e)}$, summing over all edges connected to v . A vertex v of Γ is said to be of *type A* if σ_v is nonzero. Otherwise it is said to be of *type B*. An n -pointed parametrized tropical curve $(\Gamma, (\gamma_i), h)$ is said to be *simple* if every vertex v with $\deg(v) = 2$ and $g(v) = n(v) = 0$ is of type A.

Definition 3.4. Given an n -pointed parametrized tropical curve $(\Gamma, (\gamma_i), h)$ in $S_{\mathfrak{X}}$, one can obtain a unique simple n -pointed parametrized tropical curve as follows: for every vertex v of Γ of type B such that $\deg(v) = 2$ and $g(v) = n(v) = 0$, we remove the vertex v , replace the two edges connected to v by a single edge, and set the tropical weight vectors accordingly. We call this construction *simplification*.

Definition 3.5. An n -pointed *combinatorial type* $(\Gamma, (\gamma_i))$ in $S_{\mathfrak{X}}$ consists of the following data:

- (i) A connected finite graph Γ without self-loops.
- (ii) Every vertex v of Γ is equipped with a non-empty subset $I_v \subset I_{\mathfrak{X}}$.
- (iii) Every flag (v, e) of Γ is equipped with a *tropical weight vector* $w_{(v,e)} \in \mathbb{Z}^{I_{\mathfrak{X}}} \setminus 0$. For every edge e of Γ and its two endpoints u and v , we require that $w_{(u,e)} + w_{(v,e)} = 0$.

- (iv) Every vertex v of Γ is equipped with a non-negative integer $g(v)$, called the *genus* of the vertex v .
- (v) A sequence of vertices $\gamma_1, \dots, \gamma_n$ of Γ , called *marked points*. They are not required to be different from each other. For each vertex v of Γ , we denote by $n(v)$ the number of marked points at v .

Definitions 3.3 and 3.4 carry over to combinatorial types. Given an n -pointed parametrized tropical curve $(\Gamma, (\gamma_i), h)$ in $S_{\mathfrak{X}}$, we can associate to it an n -pointed combinatorial type $(\Gamma, (\gamma_i))$ by letting $I_v \subset I_{\mathfrak{X}}$ be the subset such that the vertex v sits in the relative interior of the face Δ^{I_v} for every vertex v of Γ .

Definition 3.6. An n -pointed combinatorial type in $S_{\mathfrak{X}}$ is said to be *good* if it comes from an n -pointed parametrized tropical curve in $S_{\mathfrak{X}}$.

Definition 3.7. An n -pointed combinatorial type $(\Gamma, (\gamma_i))$ in $S_{\mathfrak{X}}$ is said to be a *degeneration* of an n -pointed combinatorial type $(\Gamma', (\gamma'_i))$ in $S_{\mathfrak{X}}$ if there exists a surjective map $\phi: V(\Gamma') \rightarrow V(\Gamma)$ satisfying the following conditions:

- (i) For any two vertices u', v' of Γ' such that $\phi(u') \neq \phi(v')$, there exists a bijection $\phi: E(u', v') \rightarrow E(\phi(u'), \phi(v'))$ such that for every $e' \in E(u', v')$, we have $w_{(u', e')} = w_{(\phi(u'), \phi(e'))}$ and $w_{(v', e')} = w_{(\phi(v'), \phi(e'))}$. Moreover, we require that every edge e of Γ is of the form $\phi(e')$ for some edge e' of Γ' .
- (ii) For every vertex v' of Γ' , we have $I_{\phi(v')} \subset I_{v'}$.
- (iii) For every vertex v of Γ , let Γ'_v denote the full subgraph of Γ' generated by the preimage $\phi^{-1}(v)$. We require that Γ'_v is connected and that

$$g(v) = b_1(\Gamma'_v) + \sum_{v' \in \phi^{-1}(v)} g(v'),$$

where $b_1(\cdot)$ denotes the first Betti number.

- (iv) We have $\phi(\gamma'_i) = \gamma_i$ for $1 \leq i \leq n$.

Now given a connected compact quasi-smooth strictly k -analytic curve C , n marked points $s_i \in C$ and a morphism $f: C \rightarrow X$, we can obtain a simple n -pointed parametrized tropical curve $(\Gamma, (\gamma_i), h)$ in $S_{\mathfrak{X}}$ by the following steps.

Step 1. Up to passing to a finite separable extension of k , there exists a strictly semi-stable formal model \mathfrak{C} of C and a morphism of formal schemes $\mathfrak{f}: \mathfrak{C} \rightarrow \mathfrak{X}$, such that $\mathfrak{f}_\eta \simeq f$. This is a consequence of [69, Proposition 5.1] using the correspondence between finite sets of type II points in C and semi-stable reductions of C (see [27, 6]).

When C is proper, this is also a special case of [67, Theorem 1.5]. Let $S_{\mathfrak{C}}$ be the Clemens polytope for \mathfrak{C} and $\tau_{\mathfrak{C}}: C \rightarrow S_{\mathfrak{C}}$ the retraction map. By Proposition 2.5, we obtain a map $S_{\mathfrak{f}}: S_{\mathfrak{C}} \rightarrow S_{\mathfrak{x}}$. We put $\Gamma_0 := S_{\mathfrak{C}}$ and $h_0 := S_{\mathfrak{f}}$. For every vertex v of Γ_0 , let \mathfrak{C}_s^v denote the corresponding irreducible component of \mathfrak{C}_s . We set $g(v)$ to be the genus of \mathfrak{C}_s^v . We add to Γ_0 the marked points $\tau_{\mathfrak{C}}(s_i)$, creating new vertices if necessary.

Step 2. For every flag (v, e) of Γ_0 , we define a weight vector $w_{(v,e)} \in \mathbb{Z}^{I_{\mathfrak{x}}}$ as follows (see also [69, §5]). Let e° denote the relative interior of the edge e . The inverse image $\tau_{\mathfrak{C}}^{-1}(e^\circ)$ is an open annulus in C , which we denote by A . Fix $i \in I_{\mathfrak{x}}$ and let $p_i: \mathbb{R}^{I_{\mathfrak{x}}} \rightarrow \mathbb{R}$ be the projection to the i^{th} coordinate. By Remark 2.4, the map $p_i \circ \tau \circ f|_A$ is given by the valuation of a certain invertible function f_i on A . Let z be a coordinate on the annulus A such that the annulus A is given by $c_1 < |z| < c_2$ and that c_1 corresponds to the vertex v . We write $f_i = \sum_{m \in \mathbb{Z}} f_{i,m} z^m$, where $f_{i,m} \in k$. Since f_i is invertible on A , there exists $m_i \in \mathbb{Z}$ such that $|f_{i,m_i}| r^{m_i} > |f_{i,m}| r^m$ for all $m \neq m_i$ and $c_1 < r < c_2$. We set the i^{th} component of the weight vector $w_{(v,e)}$ to be m_i . By construction, the weight vector $w_{(v,e)}$ is parallel to the direction of $h_0(e)$ pointing away from $h_0(v)$. It is zero if and only if the edge e is mapped to a point by h_0 .

Step 3. For every connected subgraph Γ' of Γ_0 that is mapped to a point by h_0 , we contract it to a vertex v' and set

$$g(v') = b_1(\Gamma') + \sum_{v \in \Gamma'} g(v).$$

We denote the resulting n -pointed parametrized tropical curve by $(\Gamma, (\gamma_i), h)$. After that, we replace $(\Gamma, (\gamma_i), h)$ by its simplification (see Definition 3.4). We note that Step 3 removes the dependence of our construction on the choices of the finite extension of k and the formal model \mathfrak{C} of C .

4. VANISHING CYCLES AND BALANCING CONDITIONS

We use the settings of Section 2.

Let C be a connected proper smooth strictly k -analytic curve and let $f: C \rightarrow X$ be a morphism. Let (Γ, h) be the associated parametrized tropical curve in $S_{\mathfrak{x}}$ constructed in Section 3.

The local shape of the tropical curve (Γ, h) sitting inside $S_{\mathfrak{x}}$ satisfies a distinguished geometrical property, which we refer to as the *balancing condition*. It is

studied in [69, Theorem 1.1]; see also [39, Proposition 1.15] for a related statement. Here we give a different proof using semi-stable reduction of k -analytic curves, which will be useful for later sections.

Let v be a vertex of Γ . Assume that $h(v)$ sits in the relative interior of the face Δ^{I_v} of $S_{\mathfrak{X}}$ corresponding to a subset $I_v \subset I_{\mathfrak{X}}$. Let D_{I_v} denote the closed stratum in the special fiber \mathfrak{X}_s corresponding to the face Δ^{I_v} .

Let $Z_1(D_{I_v})$ denote the group of one-dimensional algebraic cycles in D_{I_v} with integer coefficients. Let $Z_1^+(D_{I_v})$ denote the submonoid consisting of effective cycles, i.e. cycles with non-negative integer coefficients. Let α be the map

$$\begin{aligned} \alpha: Z_1^+(D_{I_v}) &\longrightarrow \mathbb{Z}^{I_{\mathfrak{X}}} \\ Z &\longmapsto \left(Z \cdot \mathcal{O}(D_i)|_{D_{I_v}}, i \in I_{\mathfrak{X}} \right), \end{aligned} \quad (4.1)$$

which sends a one-dimensional cycle Z to its intersection numbers with the restrictions $\mathcal{O}(D_i)|_{D_{I_v}}$ for every $i \in I_{\mathfrak{X}}$.

Theorem 4.1. *The sum σ_v of weight vectors around the vertex v lies in the image of the map α defined in (4.1).*

Now we explain the proof of Theorem 4.1.

Let k^s be a separable closure of k , $\widehat{k^s}$ its completion, and $\widetilde{k^s}$ its residue field. We fix a prime number ℓ different from the characteristic of the residue field \widetilde{k} . Let $R\Psi$ and $R\Phi$ denote the derived functors of nearby cycles and of vanishing cycles respectively (cf. [69, §3] and [9]²). We have the following exact triangle

$$\mathbb{Q}_{\ell, \mathfrak{X}_s} \rightarrow R\Psi\mathbb{Q}_{\ell, \mathfrak{X}_\eta} \rightarrow R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta} \xrightarrow{+1}, \quad (4.2)$$

where $\mathbb{Q}_{\ell, -}$ denotes the constant sheaf with values in \mathbb{Q}_ℓ , and $\mathfrak{X}_{\bar{s}} = \mathfrak{X}_s \times \widetilde{k^s}$.

Let $\bar{D}_{I_v} = D_{I_v} \times \widetilde{k^s}$ and let $j: \bar{D}_{I_v} \hookrightarrow \mathfrak{X}_{\bar{s}}$ denote the closed immersion. We apply j^* to (4.2) and take global sections, we obtain a long exact sequence

$$\cdots \rightarrow R^1\Gamma(j^*R\Psi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}) \xrightarrow{\beta^*} R^1\Gamma(j^*R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}) \xrightarrow{\alpha^*} H_{\text{ét}}^2(\bar{D}_{I_v}, \mathbb{Q}_\ell) \rightarrow \cdots,$$

where α^* is the boundary map.

Put $J = J_{I_v} = \{ j \in I_{\mathfrak{X}} \mid D_{I_v \cup \{j\}} \neq \emptyset \}$. Recall that we have the following calculation of the sheaf of vanishing cycles for a strictly semi-stable formal scheme \mathfrak{X} over k° .

²We use the terminology in [69], which is different from [9, 10].

Theorem 4.2 (cf. [69, Corollary 3.2], [62], [42]). *We have an isomorphism*

$$R^1\Gamma(j^*R\Phi\mathbb{Q}_{\ell,\mathfrak{X}_\eta}) \simeq \text{Coker}(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell^J)(-1),$$

where Δ denotes the diagonal map and the symbol (-1) denotes the Tate twist. Moreover, the boundary map

$$\alpha^*: \text{Coker}(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell^J)(-1) \longrightarrow R^2\Gamma(j^*\mathbb{Q}_{\ell,\mathfrak{X}_s}) \simeq H_{\text{ét}}^2(\bar{D}_{I_v}, \mathbb{Q}_\ell)$$

is induced by the cycle class map in étale cohomology.

Let $\mathfrak{C}, \mathfrak{f}: \mathfrak{C} \rightarrow \mathfrak{X}, (\Gamma_0, h_0)$ be as in Section 3. We observe that in order to prove Theorem 4.1 for the vertex v of (Γ, h) , it suffices to prove that for every vertex v_0 of (Γ_0, h_0) that maps to v , the sum σ_{v_0} of weight vectors around v_0 lies in the image of the map α defined in (4.1).

Let v_0 be a vertex of Γ_0 and let $\mathfrak{C}_s^{v_0}$ denote corresponding irreducible component of the special fiber \mathfrak{C}_s . The assumption that $h_0(v_0)$ lies in the relative interior of the face Δ^{I_v} implies that the image of $\mathfrak{C}_s^{v_0}$ under the map \mathfrak{f}_s is contained in D_{I_v} .

Let $\mathfrak{C}_{\bar{s}} = \mathfrak{C}_s \times \tilde{k}^s, \mathfrak{C}_{\bar{s}}^{v_0} = \mathfrak{C}_s^{v_0} \times \tilde{k}^s$. Denote by $j_C: \mathfrak{C}_{\bar{s}}^{v_0} \rightarrow \mathfrak{C}_{\bar{s}}$ the closed immersion. Put $\mathfrak{f}_{\bar{s}} := \mathfrak{f}_s \times \tilde{k}^s: \mathfrak{C}_{\bar{s}} \rightarrow \mathfrak{X}_{\bar{s}}$. The properness of $\mathfrak{C}_s^{v_0}$ implies an isomorphism $H_{\text{ét}}^2(j_C^*\mathbb{Q}_{\ell,\mathfrak{C}_{\bar{s}}}) \simeq \mathbb{Q}_\ell(-1)$.

Lemma 4.3. *We have the following commutative diagram:*

$$\begin{array}{ccccc} R^1\Gamma(j^*R\Psi\mathbb{Q}_{\ell,\mathfrak{X}_\eta}) & \xrightarrow{\beta^*} & R^1\Gamma(j^*R\Phi\mathbb{Q}_{\ell,\mathfrak{X}_\eta}) & \xrightarrow{\alpha^*} & H_{\text{ét}}^2(j^*\mathbb{Q}_{\ell,\mathfrak{X}_s}) \\ \downarrow \mathfrak{f}_\Psi^* & & \downarrow \mathfrak{f}_\Phi^* & & \downarrow \mathfrak{f}_s^* \\ R^1\Gamma(j_C^*R\Psi\mathbb{Q}_{\ell,\mathfrak{C}_\eta}) & \xrightarrow{\beta_C^*} & R^1\Gamma(j_C^*R\Phi\mathbb{Q}_{\ell,\mathfrak{C}_\eta}) & \xrightarrow{\alpha_C^*} & H_{\text{ét}}^2(j_C^*\mathbb{Q}_{\ell,\mathfrak{C}_s}) \xrightarrow{\simeq} \mathbb{Q}_\ell(-1). \end{array} \quad (4.3)$$

Proof. For any étale sheaf \mathcal{F} on $\mathfrak{C}_{\bar{s}}$, we have the adjunction morphism

$$\mathcal{F} \rightarrow j_{C*}j_C^*\mathcal{F}. \quad (4.4)$$

Applying the derived pushforward functor $R\mathfrak{f}_{\bar{s}*}$ to both sides of (4.4), we obtain a morphism

$$R\mathfrak{f}_{\bar{s}*}\mathcal{F} \rightarrow R\mathfrak{f}_{\bar{s}*}j_{C*}j_C^*\mathcal{F}.$$

Since the image of $\mathfrak{C}_s^{v_0}$ under the map \mathfrak{f}_s is contained in D_{I_v} , the sheaf $R\mathfrak{f}_{\bar{s}*}j_{C*}j_C^*\mathcal{F}$ is supported on $\bar{D}_{I_v} \subset \mathfrak{X}_{\bar{s}}$. Therefore, we obtain a morphism

$$j_*j^*R\mathfrak{f}_{\bar{s}*}\mathcal{F} \rightarrow R\mathfrak{f}_{\bar{s}*}j_{C*}j_C^*\mathcal{F}. \quad (4.5)$$

Moreover, by [9, Corollary 4.5(ii)], we obtain a morphism

$$R\Psi\mathbb{Q}_{\ell, \mathfrak{x}_\eta} \rightarrow R\Psi Rf_*\mathbb{Q}_{\ell, \mathfrak{e}_\eta} \simeq Rf_{\bar{s}*}R\Psi\mathbb{Q}_{\ell, \mathfrak{e}_\eta}.$$

Applying j_*j^* , we obtain a morphism

$$j_*j^*R\Psi\mathbb{Q}_{\ell, \mathfrak{x}_\eta} \rightarrow j_*j^*Rf_{\bar{s}*}R\Psi\mathbb{Q}_{\ell, \mathfrak{e}_\eta}. \quad (4.6)$$

Substituting \mathcal{F} by $R\Psi\mathbb{Q}_{\ell, \mathfrak{e}_\eta}$ in (4.5), we obtain a morphism

$$j_*j^*Rf_{\bar{s}*}R\Psi\mathbb{Q}_{\ell, \mathfrak{e}_\eta} \rightarrow Rf_{\bar{s}*}j_{C*}j_C^*R\Psi\mathbb{Q}_{\ell, \mathfrak{e}_\eta}.$$

Combining with (4.6) and taking global sections, we obtain a map

$$f_\Psi^*: R^1\Gamma(j^*R\Psi\mathbb{Q}_{\ell, \mathfrak{x}_\eta}) \rightarrow R^1\Gamma(j_C^*R\Psi\mathbb{Q}_{\ell, \mathfrak{e}_\eta}).$$

Similarly, we have maps

$$f_\Phi^*: R^1\Gamma(j^*R\Phi\mathbb{Q}_{\ell, \mathfrak{x}_\eta}) \rightarrow R^1\Gamma(j_C^*R\Phi\mathbb{Q}_{\ell, \mathfrak{e}_\eta}),$$

and

$$f_s^*: H_{\text{ét}}^2(j^*\mathbb{Q}_{\ell, \mathfrak{x}_s}) \rightarrow H_{\text{ét}}^2(j_C^*\mathbb{Q}_{\ell, \mathfrak{e}_s}).$$

Now the commutativity of (4.3) follows from the functoriality of nearby cycles and vanishing cycles. \square

By Theorem 4.2, we have an isomorphism

$$R^1\Gamma(j^*R\Phi\mathbb{Q}_{\ell, \mathfrak{x}_\eta}) \simeq \text{Coker}(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell^J)(-1). \quad (4.7)$$

For each edge e connected to v_0 , the weight vector $w_{(v_0, e)}$ lives in $\text{Ker}(\mathbb{Z}^J \xrightarrow{\Sigma} \mathbb{Z})$. So it induces a linear map by duality

$$w_{(v_0, e)}^*: R^1\Gamma(j^*R\Phi\mathbb{Q}_{\ell, \mathfrak{x}_\eta}) \rightarrow \mathbb{Q}_\ell(-1).$$

Let $p_c \in \mathfrak{C}_s$ be the point corresponding to the edge e . Let $j_{p_c}: p_c \times \tilde{k}^s \hookrightarrow \mathfrak{C}_s^{v_0}$ denote the inclusion map. By Theorem 4.2 again, we have an isomorphism

$$R^1\Gamma(j_{p_c}^*R\Phi\mathbb{Q}_{\ell, \mathfrak{e}_\eta}) \simeq \text{Coker}(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell)(-1).$$

Let s be the projection map

$$\begin{aligned} R^1\Gamma(j_{p_c}^*R\Phi\mathbb{Q}_{\ell, \mathfrak{e}_\eta}) &\simeq \text{Coker}(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell)(-1) \longrightarrow \mathbb{Q}_\ell(-1) \\ &(x_0, x_1) \longmapsto x_1 - x_0, \end{aligned}$$

where the component x_0 corresponds to the irreducible component $\mathfrak{C}_s^{v_0}$. Let r_C be the restriction map

$$R^1\Gamma(j_C^* R\Phi\mathbb{Q}_{\ell, \mathfrak{C}_\eta}) \rightarrow R^1\Gamma(j_{p_c}^* R\Phi\mathbb{Q}_{\ell, \mathfrak{C}_\eta}).$$

Lemma 4.4. *The composition $s \circ r_C \circ \mathfrak{f}_\Phi^*$ of the following morphisms*

$$R^1\Gamma(j^* R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}) \xrightarrow{\mathfrak{f}_\Phi^*} R^1\Gamma(j_C^* R\Phi\mathbb{Q}_{\ell, \mathfrak{C}_\eta}) \xrightarrow{r_C} R^1\Gamma(j_{p_c}^* R\Phi\mathbb{Q}_{\ell, \mathfrak{C}_\eta}) \xrightarrow{s} \mathbb{Q}_\ell(-1)$$

is equal to the map

$$w_{(v_0, e)}^* : R^1\Gamma(j^* R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}) \rightarrow \mathbb{Q}_\ell(-1).$$

Proof. Let $p = \mathfrak{f}_s(p_c) \in \mathfrak{X}_s$, let $j_p : p \times \tilde{k}^s \hookrightarrow \mathfrak{X}_s$ be the closed immersion, and let

$$r : R^1\Gamma(j^* R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}) \rightarrow R^1\Gamma(j_p^* R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta})$$

be the restriction map. We have a commutative diagram

$$\begin{array}{ccc} R^1\Gamma(j^* R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}) & \xrightarrow{\mathfrak{f}_\Phi^*} & R^1\Gamma(j_C^* R\Phi\mathbb{Q}_{\ell, \mathfrak{C}_\eta}) \\ \downarrow r & & \downarrow r_C \\ R^1\Gamma(j_p^* R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}) & \xrightarrow{\mathfrak{f}_{\Phi, p_c}^*} & R^1\Gamma(j_{p_c}^* R\Phi\mathbb{Q}_{\ell, \mathfrak{C}_\eta}). \end{array} \quad (4.8)$$

It follows from the cohomological interpretation of tropical weight vectors in [69, Lemma 5.8] and [10, Corollary 3.5] that the composition $s \circ \mathfrak{f}_{\Phi, p_c}^* \circ r$ is equal to the map $w_{(v_0, e)}^*$. We conclude our lemma by the commutativity of the diagram in (4.8). \square

Now let

$$\sigma_{v_0} = \sum_{e \ni v_0} w_{(v_0, e)} \in \text{Ker}(\mathbb{Z}^J \xrightarrow{\Sigma} \mathbb{Z}),$$

summing over all edges e of Γ_0 connected to v_0 . Using the isomorphism in (4.7), σ_{v_0} induces a linear map by duality

$$\sigma_{v_0}^* : R^1\Gamma(j^* R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}) \rightarrow \mathbb{Q}_\ell(-1).$$

Lemma 4.5. *The map $\sigma_{v_0}^* : R^1\Gamma(j^* R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}) \rightarrow \mathbb{Q}_\ell(-1)$ is equal to the composition $\alpha_C^* \circ \mathfrak{f}_\Phi^*$ in (4.3).*

Proof. By Theorem 4.2, the map α_C^* in (4.3) is induced by the cycle class map. So α_C^* is the sum of the maps $s \circ r_C$ that we considered in Lemma 4.4 over all edges e connected to v_0 . On the other hand the map

$$\sigma_{v_0}^* : R^1\Gamma(j^* R\Phi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}) \rightarrow \mathbb{Q}_\ell(-1)$$

is the sum of the maps

$$w_{(v_0, e)}^*: R^1\Gamma(j^*R\Phi\mathbb{Q}_{\ell, \mathfrak{x}_\eta}) \rightarrow \mathbb{Q}_\ell(-1)$$

over all edges e connected to v_0 . Therefore Lemma 4.5 follows from Lemma 4.4. \square

Combining Lemma 4.5 and the commutativity of the diagram in (4.3), we obtain the following lemma.

Lemma 4.6. *The map $\sigma_{v_0}^*: R^1\Gamma(j^*R\Phi\mathbb{Q}_{\ell, \mathfrak{x}_\eta}) \rightarrow \mathbb{Q}_\ell(-1)$ is equal to the composition of the following morphisms*

$$R^1\Gamma(j^*R\Phi\mathbb{Q}_{\ell, \mathfrak{x}_\eta}) \xrightarrow{\alpha^*} H_{\text{ét}}^2(j^*\mathbb{Q}_{\ell, \mathfrak{x}_\eta}) \xrightarrow{f_s^*} H_{\text{ét}}^2(\mathfrak{C}_s^{v_0}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathbb{Q}_\ell(-1).$$

Lemma 4.6 can be reformulated as follows using Theorem 4.2 and duality.

Lemma 4.7. *For every $i \in J = J_{I_v}$, the i^{th} component $\sigma_{v_0}^i$ of $\sigma_{v_0} \in \text{Ker}(\mathbb{Z}^J \xrightarrow{\Sigma} \mathbb{Z})$ is equal to the degree of the line bundle $(f_s|_{\mathfrak{C}_s^{v_0}})^*\mathcal{O}(D_i)$ on the curve $\mathfrak{C}_s^{v_0}$. \square*

Now let $Z \in Z_1^+(D_{I_v})$ be the pushforward $f_{s*}([\mathfrak{C}_s^{v_0}])$ of the fundamental class of $\mathfrak{C}_s^{v_0}$. Lemma 4.7 implies that for every $i \in I_{\mathfrak{x}}$, the i^{th} component $\sigma_{v_0}^i$ of σ_{v_0} is equal to the intersection number $Z \cdot \mathcal{O}(D_i)|_{D_{I_v}}$. Therefore, σ_{v_0} lies in the image of the map α in (4.1). Since σ_v equals the sum of σ_{v_0} over all vertices v_0 of Γ_0 that maps to v , we conclude that σ_v lies in the image of the map α , completing the proof of Theorem 4.1.

5. KÄHLER STRUCTURES

As we will study tropical curves and tropicalization of curves in families, we need an extra structure to ensure that the moduli spaces we encounter will be of finite type. In this section, we introduce the notion of simple density and explain its relation with non-archimedean Kähler structures.

We use the settings of Section 2.

Definition 5.1. *A simple density ω on the Clemens polytope $S_{\mathfrak{x}}$ is a collection of numbers $\omega_{I, j} \in (0, +\infty]$ for every face $\Delta^I, I \subset I_{\mathfrak{x}}$ and every vertex $j \in J_I$, such that $\omega_{I, j} \geq \omega_{I', j}$ whenever $I \supset I'$ and $j \in J_I$.*

Definition 5.2. *Let (Γ, h) be a parametrized tropical curve in the Clemens polytope $S_{\mathfrak{x}}$ equipped with a simple density ω . Let v be a vertex of Γ such that $h(v)$ sits in*

the relative interior of a face Δ^{I_v} for some $I_v \subset I_{\mathfrak{X}}$. The *local degree* of (Γ, h) at v with respect to ω is by definition the real number

$$|\sigma_v|_{\omega} := \max_{j \in J_{I_v}} \omega_{I_v, j} \cdot |\sigma_v^j|,$$

where σ_v denotes the sum of weight vectors around v . The *tropical degree* of the parametrized tropical curve (Γ, h) with respect to the simple density ω is the sum of the local degrees over all vertices of Γ . The marked points on a parametrized tropical curve do not contribute to the degree.

Remark 5.3. The definition of tropical degree with respect to a simple density ω carries over to combinatorial types in $S_{\mathfrak{X}}$. The tropical degree of a parametrized tropical curve in $S_{\mathfrak{X}}$ with respect to ω coincides with the tropical degree of its associated combinatorial type with respect to ω .

The notion of simple density is supposed to be an approximation to the non-archimedean Kähler structure introduced in [67]³. We recall that a Kähler structure \widehat{L} on the k -analytic space X with respect to the formal model \mathfrak{X} is a virtual line bundle L on X with respect to \mathfrak{X} equipped with a strictly convex metrization \widehat{L} .

A Kähler structure \widehat{L} on X with respect to \mathfrak{X} induces a simple density ω on the Clemens polytope $S_{\mathfrak{X}}$ in the following way.

For every $i \in I_{\mathfrak{X}}$, let $N^1(D_i)$ denote the numerical classes of divisors in D_i . Put $N^1(D_i)_{\mathbb{R}} = N^1(D_i) \otimes \mathbb{R}$. The curvature of the Kähler structure \widehat{L} is a collection of ample classes $\partial_i \varphi_i \in N^1(D_i)_{\mathbb{R}}$ satisfying the following compatibility condition: for any $I \subset I_{\mathfrak{X}}$, any two vertices $i, j \in I$, we have

$$(\partial_i \varphi_i)|_{D_I} = (\partial_j \varphi_j)|_{D_I}. \quad (5.1)$$

Let Δ^I be a face of $S_{\mathfrak{X}}$ for some $I \subset I_{\mathfrak{X}}$ and let $i \in I$ be a vertex. Let $\partial_I \varphi_i$ be the restriction of $\partial_i \varphi_i$ to the stratum D_I . It is an ample class in $N^1(D_I)_{\mathbb{R}}$. Let j be an element in J_I , and let

$$\omega_{I, j} = \min \{ C \cdot (\partial_I \varphi_i) \mid C \in \overline{\text{NE}}(D_I), |C \cdot \mathcal{O}(D_j)|_{D_I} = 1 \},$$

where $\overline{\text{NE}}(D_I)$ denotes the closure of the cone of effective proper curves in the stratum D_I . Equation (5.1) shows that $\omega_{I, j}$ does not depend on the choice of $i \in I$.

The ampleness of the class $\partial_I \varphi_i$ in $N^1(D_I)_{\mathbb{R}}$ implies that $\omega_{I, j}$ is a positive real number. The fact that $\omega_{I, j} \geq \omega_{I', j}$ for any $I \supset I'$ follows from the inclusion

³It is based on the work of Kontsevich and Tschinkel [47], see also [16].

$\overline{\text{NE}}(D_I) \subset \overline{\text{NE}}(D_{I'})$. So the collection

$$\omega = \{ \omega_{I,j} \mid I \subset I_{\mathfrak{x}}, j \in J_I \}$$

is a simple density on the Clemens polytope $S_{\mathfrak{x}}$ (Definition 5.1).

Definition 5.4. The simple density ω on $S_{\mathfrak{x}}$ constructed above is called the simple density induced by the Kähler structure \widehat{L} on X .

Now let C be a connected proper smooth k -analytic curve and $f: C \rightarrow X$ a morphism. The *degree* of the morphism f with respect to the Kähler structure \widehat{L} is by definition the degree of the virtual line bundle f^*L on the curve C .

Let (Γ, h) denote the associated parametrized tropical curve in $S_{\mathfrak{x}}$. We can relate the degree of the morphism f to the tropical degree (Definition 5.2) of $(\Gamma, (\gamma_i), h)$ as follows.

Proposition 5.5. *The degree of the morphism $f: C \rightarrow X$ with respect to the Kähler structure \widehat{L} is greater than or equal to the tropical degree of the associated parametrized tropical curve (Γ, h) with respect to the simple density ω on the Clemens polytope $S_{\mathfrak{x}}$ induced by \widehat{L} .*

Proof. Let $\mathfrak{C}, \mathfrak{f}: \mathfrak{C} \rightarrow \mathfrak{X}, (\Gamma_0, h_0)$ be as in Section 3.

Let v_0 be a vertex of Γ_0 and let $\mathfrak{C}_s^{v_0}$ denote the corresponding irreducible component of \mathfrak{C}_s . Assume that $h_0(v_0)$ sits in the relative interior of a face Δ^{I_v} for some subset $I_v \subset I_{\mathfrak{x}}$. Let σ_{v_0} be the sum of weight vectors around v_0 .

Let $i \in I_v$. Let us call the degree of the line bundle $(\mathfrak{f}_s|_{\mathfrak{C}_s^{v_0}})^*(\partial_i \varphi_i)$ on the curve $\mathfrak{C}_s^{v_0}$ the *local degree* of $\mathfrak{f}: \mathfrak{C} \rightarrow \mathfrak{X}$ at the irreducible component $\mathfrak{C}_s^{v_0}$. By the construction of the simple density ω and Lemma 4.7, the local degree of $\mathfrak{f}: \mathfrak{C} \rightarrow \mathfrak{X}$ at the irreducible component $\mathfrak{C}_s^{v_0}$ is at least $\omega_{I_v, j} \cdot |\sigma_{v_0}^j|$ for any $j \in J_{I_v}$. Taking maximum over $j \in J_{I_v}$, we deduce that the local degree of $\mathfrak{f}: \mathfrak{C} \rightarrow \mathfrak{X}$ at the irreducible component $\mathfrak{C}_s^{v_0}$ is at least $|\sigma_{v_0}|_{\omega}$ (see Definition 5.2).

Moreover, we observe that the tropical degree of (Γ_0, h_0) is greater than or equal to the tropical degree of (Γ, h) . Now the proposition follows from the fact that the degree of a virtual line bundle on a k -analytic curve equals the degree of its curvature ([67, Proposition 5.7]). \square

Remark 5.6. We do not need the following fact in this paper, but it is worth pointing out. We use the setting in the proof of Proposition 5.5 and the terminology from [67]. The strictly convex metrization \widehat{L} determines a germ of a strictly convex

simple function φ_i at the vertex i up to addition by linear functions. By the definition of the derivative $\partial_i \varphi_i$ and by Lemma 4.7, we have

$$\deg \left((\mathbf{f}_s|_{\mathbf{e}_s^{v_0}})^* (\partial_i \varphi_i) \right) = \sum_{j \in J_{I_v}} \varphi_i(j) \cdot \deg \left((\mathbf{f}_s|_{\mathbf{e}_s^{v_0}})^* \mathcal{O}(D_j) \right) = \sum_{j \in J_{I_v}} \varphi_i(j) \cdot \sigma_{v_0}^j.$$

As a result, the degree of the morphism $f: C \rightarrow X$ with respect to the Kähler structure \widehat{L} can be read out from the corresponding tropical curve.

6. THE MODULI SPACE OF TROPICAL CURVES

In this section, we study the structure of the space of tropical curves with bounded degree. The goal is to prove the following theorem.

Theorem 6.1. *Let $S_{\mathbf{x}}$ be a Clemens polytope equipped with a simple density ω . Fix two non-negative integers g, n and a positive real number A . Denote by $M_{g,n}(S_{\mathbf{x}}, A)$ the set of simple n -pointed genus g parametrized tropical curves in $S_{\mathbf{x}}$ whose tropical degree with respect to ω is bounded by A . Then $M_{g,n}(S_{\mathbf{x}}, A)$ is naturally a compact topological space with a stratification whose open strata are open convex polyhedrons. The strata are in one-to-one correspondence with good simple n -pointed genus g combinatorial types in $S_{\mathbf{x}}$ whose tropical degree is bounded by A .*

Definition 6.2. For any $w \in \mathbb{Z}^{I_{\mathbf{x}}}$, we define its *norm* $|w| := \sqrt{\sum (w^i)^2}$.

Lemma 6.3. *There exists an integer N , such that for any n -pointed parametrized tropical curve $(\Gamma, (\gamma_i), h) \in M(S_{\mathbf{x}}, A)$, the number of type A vertices of Γ and the norms of the sums σ_v of tropical weight vectors around all type A vertices are bounded by N .*

Proof. It follows from Definitions 3.3 and 5.2. □

Lemma 6.4. *There exists an integer N , such that for any n -pointed parametrized tropical curve $(\Gamma, (\gamma_i), h) \in M_{g,n}(S_{\mathbf{x}}, A)$, the number of vertices of Γ is bounded by N .*

Proof. Let $(\Gamma, (\gamma_i), h) \in M_{g,n}(S_{\mathbf{x}}, A)$. By computing the Euler characteristic of Γ , we have

$$\#V(\Gamma) - \#E(\Gamma) = 1 - b_1(\Gamma) \geq 1 - g. \quad (6.1)$$

Since $(\Gamma, (\gamma_i), h)$ is simple, by Lemma 6.3, the number of vertices of Γ of degree less than 3 can be bounded uniformly by an integer N_0 . Therefore, we have

$$\#E(\Gamma) \geq \frac{3}{2}(\#V(\Gamma) - N_0).$$

Combining with (6.1), we deduce that

$$\#V(\Gamma) \leq 3N_0 + 2g - 2.$$

□

Lemma 6.5. *There exists an integer N , such that for any n -pointed parametrized tropical curve $(\Gamma, (\gamma_i), h) \in M_{g,n}(S_{\mathfrak{X}}, A)$, the number of edges of Γ is bounded by N .*

Proof. It follows from Lemma 6.4 and the inequality (6.1). □

Definition 6.6. Let $(\Gamma, (\gamma_i), h)$ be an n -pointed parametrized tropical curve in $S_{\mathfrak{X}}$. Let (v_0, e_0) be a flag of Γ . Let $i \in I_{\mathfrak{X}}$. A *path* P starting from the flag (v_0, e_0) with direction i is a sequence of flags $((v_0, e_0), (v_1, e_1), \dots, (v_{l_P}, e_{l_P}))$ satisfying the following conditions:

- (i) The vertices v_j, v_{j+1} are the two endpoints of e_j for $0 \leq j < l_P$;
- (ii) We have $w_{(v_j, e_j)}^i > 0$ for $0 \leq j \leq l_P$;
- (iii) Let v'_{l_P} denote the endpoint of e_{l_P} different from v_{l_P} , then v'_{l_P} is a vertex of type A. We say that the path P ends at the vertex v'_{l_P} .

Lemma 6.7. *There exists an integer N , such that for any n -pointed parametrized tropical curve $(\Gamma, (\gamma_i), h) \in M_{g,n}(S_{\mathfrak{X}}, A)$, any flag (v_0, e_0) of Γ , the norm of the weight vector $w_{(v_0, e_0)}$ is bounded by N .*

Proof. We fix $i \in I_{\mathfrak{X}}$. Let $(\Gamma, (\gamma_i), h) \in M_{g,n}(S_{\mathfrak{X}}, A)$ and let (v_0, e_0) be a flag of Γ . Let $m := w_{(v_0, e_0)}^i$ be the i^{th} component of the weight vector $w_{(v_0, e_0)}$. We assume that $m \geq 0$, otherwise we replace v_0 by the other endpoint of e_0 . We will show that m can be bounded independently of $(\Gamma, (\gamma_i), h)$.

We use the ideas from [68, §3]. We claim that there exists a collection of m paths P_1, \dots, P_m starting from the flag (v_0, e_0) with direction i such that for any flag (v, e) of Γ , the number of times that (v, e) occurs in P_1, \dots, P_m is bounded by $|w_{(v, e)}^i|$. Such a collection of m paths can be constructed as follows.

We assign to each edge e of Γ an integer $c^i(e)$ called the *capacity* (in the i^{th} direction). For each edge e of Γ , we set initially $c^i(e) = |w_{(v, e)}^i|$, where v is an endpoint of e . By definition, for every vertex v of type B , we have

$$\sum_{e \ni v, w_{(v, e)}^i < 0} c^i(e) = \sum_{e \ni v, w_{(v, e)}^i > 0} c^i(e), \quad (6.2)$$

where the left sum is over every edge e connected to v with $w_{(v, e)}^i < 0$, and the right sum is over every edge e connected to v with $w_{(v, e)}^i > 0$.

To construct the path P_1 , we start with the flag (v_0, e_0) , and we decrease the capacity $c^i(e_0)$ by 1. Suppose we have constructed a sequence of flags $(v_0, e_0), \dots, (v_j, e_j)$ satisfying Definition 6.6 Conditions (i)-(ii). Let v'_j denote the endpoint of the edge e_j different from v_j . We choose (v_{j+1}, e_{j+1}) to be a flag of Γ such that

- (i) $v_{j+1} = v'_j$,
- (ii) $w^i_{(v_{j+1}, e_{j+1})} > 0$,
- (iii) $c^i(e_{j+1}) > 0$.

If such a flag does not exist, we stop. We note that in this case v'_j is necessarily a vertex of type A because of Eq. (6.2). If such a flag exists, we let (v_{j+1}, e_{j+1}) be the next flag in the path P_1 and we decrease the capacity $c^i(e_{j+1})$ by 1. We iterate until we stop. As a result we obtain the path P_1 . Then we construct the path P_2 by the same procedure as the construction of P_1 but with respect to the decreased capacities $c^i(e)$. Since the path P_1 ends at a vertex of type A, we note that after the construction of P_1 , the decreased capacities $c^i(e)$ still satisfy Eq. (6.2) for every vertex of type B except possibly for the vertex v_0 . Therefore, the path P_2 must also end at a vertex of type A. We iterate this procedure and we obtain a collection of m paths P_1, \dots, P_m .

By construction, for any vertex v of Γ of type A, the number of paths ending on v is bounded by $|\sigma_v^i|$. In other words, except for at most $|\sigma_v^i|$ paths, the other paths starting from (v_0, e_0) with direction i that reach the vertex v must continue going. Therefore, by Lemma 6.3, the number m can be bounded independently of $(\Gamma, (\gamma_i), h)$. \square

Remark 6.8. Proposition 5.5 combined with Lemmas 6.4, 6.5 and 6.7 controls the complexity of the tropical curves obtained from analytic curves in X with bounded degrees. More precisely, given a positive real number A , there exists an integer N , such that for any connected proper smooth k -analytic curve C , any morphism $f : C \rightarrow X$ with degree bounded by A , the number of vertices, the number of edges and the norms of the weight vectors of the edges of the associated parametrized tropical curve in $S_{\mathbb{X}}$ are all bounded by N . We remark that this finiteness property can also be obtained as a consequence of Lemma 4.7 and the boundedness of the moduli stack of formal stable maps ([67, Corollary 9.5]).

Proposition 6.9. *Let $\alpha := (\Gamma, (\gamma_i))$ be a combinatorial type occurring in $M_{g,n}(S_{\mathbb{X}}, A)$. Let M_α denote the subset of $M_{g,n}(S_{\mathbb{X}}, A)$ with combinatorial type α . Then M_α is naturally an open convex polyhedron in a real affine space, i.e. a subset of a real*

affine space given by the conjunction of finitely many linear strict inequalities. Moreover, let \overline{M}_α denote its closure in the real affine space. We have a natural map

$$\iota_\alpha: \overline{M}_\alpha \longrightarrow M_{g,n}(S_{\mathfrak{X}}, A).$$

The image of ι_α coincides with the union

$$\bigcup_{\alpha' \preceq \alpha} M_{\alpha'}$$

in $M_{g,n}(S_{\mathfrak{X}}, A)$, where the union is over every combinatorial type α' which is the simplification of a degeneration of α .

Proof. Our proposition is related to [55, Proposition 2.23], [63, Lemma 2.2] and [33, Propositions 3.9, 3.12].

An n -pointed parametrized tropical curve $(\Gamma, (\gamma_i), h)$ in $S_{\mathfrak{X}}$ with the given combinatorial type $(\Gamma, (\gamma_i))$ is determined by the position $h(v)$ of every vertex v of Γ . Since Γ is connected and the tropical weight vectors are given by the combinatorial type, if we fix a ‘‘root vertex’’ v_0 of Γ , the set M_α is naturally a subset of the real affine space $\mathbb{R}^{I_{\mathfrak{X}}} \times \mathbb{R}^{E(\Gamma)}$ whose coordinates are

- the position $h(v_0) \in S_{\mathfrak{X}} \subset \mathbb{R}^{I_{\mathfrak{X}}}$ of the fixed root vertex v_0 ,
- the lengths of the image $h(e) \subset S_{\mathfrak{X}}$ for every edge e of Γ .

It is cut out by finitely many linear equations and linear strict inequalities corresponding to the following conditions:

- (i) For every vertex v of Γ , the image $h(v)$ lies in the relative interior of the face Δ^{I_v} of $S_{\mathfrak{X}}$.
- (ii) The lengths of the images $h(e)$ are positive.
- (iii) For every cycle in the graph Γ , the image of this cycle closes up in $S_{\mathfrak{X}}$.

Note that a different choice of the root vertex v_0 corresponds to an affine automorphism. So we have proved the first part of our proposition. To describe the closure \overline{M}_α , it suffices to change the Conditions (i)-(ii) above to the following:

- (i') For every vertex v of Γ , the image $h(v)$ lies in the face Δ^{I_v} of $S_{\mathfrak{X}}$.
- (ii') The lengths of the images $h(e)$ are non-negative.

So strict inequalities change to non-strict inequalities and we obtain all possible degenerations of the combinatorial type $(\Gamma, (\gamma_i))$. \square

Proof of Theorem 6.1. Lemmas 6.4, 6.5 and 6.7 imply that there are only finitely many combinatorial types occurring in the set $M_{g,n}(S_{\mathfrak{X}}, A)$. Now we can construct

$M_{g,n}(S_{\mathfrak{X}}, A)$ by attaching the closed polyhedrons of the form \overline{M}_α in Proposition 6.9 according to the partial order given by degenerations of combinatorial types. We conclude our proof of the theorem by Proposition 6.9. \square

7. THE MODULI STACK OF NON-ARCHIMEDEAN ANALYTIC STABLE MAPS

In this section, we consider families of k -analytic curves in our k -analytic space X . In order to compactify the universal family of such curves, we introduce the analog of Kontsevich's stable map in non-archimedean analytic geometry.

Definition 7.1 (cf. [45, §1.1], [3, §2]). Let T be a k -analytic space. An n -pointed genus g k -analytic stable map $(C \rightarrow T, (s_i), f)$ into X over T consists of a morphism $C \rightarrow T$, a morphism $f: C \rightarrow X$ and n morphisms $s_i: T \rightarrow C$ such that

- (i) The morphism $C \rightarrow T$ is a proper flat⁴ family of curves;
- (ii) The geometric fibers of $C \rightarrow T$ are reduced with at worst double points as singularities, and are of arithmetic genus g ;
- (iii) The n morphisms $s_i: T \rightarrow C$ are disjoint sections of $C \rightarrow T$ which land in the smooth locus of $C \rightarrow T$;
- (iv) (Stability condition) For any geometric point t in T , the automorphism group of the fiber $(C_t \rightarrow \{t\}, (s_i(t)), f_t: C_t \rightarrow X)$ is a finite k -analytic group⁵.

Let \widehat{L} be a Kähler structure on X with respect to a strictly semi-stable formal model \mathfrak{X} of X . Fix a positive real number A . Let $\overline{\mathcal{M}}_{g,n}(X, A)$ denote the moduli stack of n -pointed genus g k -analytic stable maps into X whose degree with respect to \widehat{L} is bounded by A .

The main result of [67] is the following theorem.

Theorem 7.2 (Non-archimedean Gromov compactness). *The stack $\overline{\mathcal{M}}_{g,n}(X, A)$ is a compact k -analytic stack. If we assume moreover that the k -analytic space X is proper and that the residue field \tilde{k} has characteristic zero, then $\overline{\mathcal{M}}_{g,n}(X, A)$ is a proper k -analytic stack.*

Roughly, the theorem above means that the abstract moduli stack $\overline{\mathcal{M}}_{g,n}(X, A)$ locally looks like k -analytic spaces. We will study the tropicalization of the k -analytic moduli stack $\overline{\mathcal{M}}_{g,n}(X, A)$ in Sections 8 and 9.

⁴We refer to [28, 1, 14, 15] for the notion of flatness in non-archimedean analytic geometry.

⁵It is a finite constant group when the field k has characteristic zero.

8. CONTINUITY OF TROPICALIZATION

Every k -analytic stable map into the k -analytic space X gives rise to a parametrized tropical curve in the Clemens polytope $S_{\mathfrak{X}}$. So we obtain a map from the moduli space of k -analytic stable maps to the moduli space of tropical curves. We call this map the tropicalization map of the moduli space of k -analytic stable maps. In Section 8, we prove the continuity of this tropicalization map using the balancing conditions in Section 4 and the formal models of families of stable maps developed in [67]. In Section 9, we prove that the image of this tropicalization map is compact and polyhedral using the continuity theorem together with the quantifier elimination theorem from the model theory of rigid subanalytic sets.

Let X be a k -analytic space, \mathfrak{X} a strictly semi-stable formal model of X , and \widehat{L} a Kähler structure on X with respect to \mathfrak{X} . Let ω be the simple density on the Clemens polytope $S_{\mathfrak{X}}$ induced by the Kähler structure \widehat{L} (Definition 5.4).

Fix two non-negative integers g, n and a positive real number A . Let $M_{g,n}(S_{\mathfrak{X}}, A)$ denote the space of simple n -pointed genus g parametrized tropical curves in $S_{\mathfrak{X}}$ whose degree with respect to ω is bounded by A (see Theorem 6.1).

Now let T be a strictly k -analytic space and let $(C \rightarrow T, (s_i), f)$ be an n -pointed genus g k -analytic stable map into X over T . Assume that the degree of every geometric fiber $(C_t \rightarrow \{t\}, (s_i(t)), f_t: C_t \rightarrow X)$ with respect to \widehat{L} is bounded by A . By Proposition 5.5, we obtain a set-theoretic tropicalization map τ_T from T to $M_{g,n}(S_{\mathfrak{X}}, A)$.

Theorem 8.1. *The map $\tau_T: T \rightarrow M_{g,n}(S_{\mathfrak{X}}, A)$ is continuous.*

We begin the proof by introducing some new notions.

Definition 8.2. Let ν be a positive integer. A ν -semi-stable subdivision $\overline{S_{\mathfrak{X}}}$ of the Clemens polytope $S_{\mathfrak{X}}$ is a finite rational simplicial subdivision of $S_{\mathfrak{X}}$ such that the vertices of $\overline{S_{\mathfrak{X}}}$ lie in the lattice $\frac{1}{\nu}\mathbb{Z}^{I_{\mathfrak{X}}} \subset \mathbb{R}^{I_{\mathfrak{X}}}$ and that every d -dimensional simplex of $\overline{S_{\mathfrak{X}}}$ has volume $\sqrt{d+1}/(\nu^d d!)$.

Let $\overline{S_{\mathfrak{X}}}$ be a ν -semi-stable subdivision of $S_{\mathfrak{X}}$. Let $\overline{I_{\mathfrak{X}}}$ denote the set of vertices of $\overline{S_{\mathfrak{X}}}$. Let $\Delta^{\overline{I_{\mathfrak{X}}}}$ denote the simplex in $\mathbb{R}_{\geq 0}^{\overline{I_{\mathfrak{X}}}}$ given by the equation $\sum x_i = 1/\nu$. We have natural embeddings $\overline{S_{\mathfrak{X}}} \subset \Delta^{\overline{I_{\mathfrak{X}}}} \subset \mathbb{R}^{\overline{I_{\mathfrak{X}}}}$. So we can consider parametrized tropical curves and combinatorial types in $\overline{S_{\mathfrak{X}}}$ as in Definitions 3.2 and 3.5. The map $\overline{S_{\mathfrak{X}}} \rightarrow S_{\mathfrak{X}}$ induces a linear map of vector spaces $\mathbb{R}^{\overline{I_{\mathfrak{X}}}} \rightarrow \mathbb{R}^{I_{\mathfrak{X}}}$ and a morphism

of lattices $\mathbb{Z}^{\overline{I_{\mathfrak{x}}}} \rightarrow \mathbb{Z}^{I_{\mathfrak{x}}}$. In this way, parametrized tropical curves in $\overline{S_{\mathfrak{x}}}$ project to parametrized tropical curves in $S_{\mathfrak{x}}$.

Definition 8.3. A *subdivision datum* $\mathcal{D} = (\overline{S_{\mathfrak{x}}}, (\Gamma, (\gamma_i)))$ consists of

- (i) a ν -semi-stable subdivision $\overline{S_{\mathfrak{x}}}$ of the Clemens polytope $S_{\mathfrak{x}}$ for a positive integer ν , and
- (ii) an n -pointed combinatorial type $(\Gamma, (\gamma_i))$ in $\overline{S_{\mathfrak{x}}}$.

Given a subdivision datum $\mathcal{D} = (\overline{S_{\mathfrak{x}}}, (\Gamma, (\gamma_i)))$, let $M_{g,n}(\overline{S_{\mathfrak{x}}}, A)$ be the space of simple n -pointed genus g parametrized tropical curves in $\overline{S_{\mathfrak{x}}}$ that project to parametrized tropical curves in $S_{\mathfrak{x}}$ with tropical degree bounded by A . Via projection followed by simplification, we obtain a natural bijection $M_{g,n}(\overline{S_{\mathfrak{x}}}, A) \xrightarrow{\sim} M_{g,n}(S_{\mathfrak{x}}, A)$.

The space $M_{g,n}(\overline{S_{\mathfrak{x}}}, A)$ has a natural stratification whose open strata corresponds to combinatorial types in $\overline{S_{\mathfrak{x}}}$. So it is a subdivision of the stratification of $M_{g,n}(S_{\mathfrak{x}}, A)$.

Let $\Xi(\mathcal{D})$ be the finite set of n -pointed combinatorial types $(\Gamma', (\gamma'_i))$ in $\overline{S_{\mathfrak{x}}}$ such that

- (i) $(\Gamma', (\gamma'_i))$ occurs in the space $M_{g,n}(\overline{S_{\mathfrak{x}}}, A)$,
- (ii) $(\Gamma, (\gamma_i))$ is the simplification of a degeneration of $(\Gamma', (\gamma'_i))$.

For each n -pointed combinatorial type $(\Gamma', (\gamma'_i)) \in \Xi(\mathcal{D})$, let $\Delta_{(\Gamma', (\gamma'_i))}^{\circ}$ denote the open stratum of $M_{g,n}(\overline{S_{\mathfrak{x}}}, A)$ corresponding to $(\Gamma', (\gamma'_i))$. Put

$$U(\mathcal{D}) := \bigcup_{(\Gamma', (\gamma'_i)) \in \Xi(\mathcal{D})} \Delta_{(\Gamma', (\gamma'_i))}^{\circ}.$$

By the construction of the topology on the space $M_{g,n}(S_{\mathfrak{x}}, A)$ in the proof of Theorem 6.1, we see that $U(\mathcal{D})$ is an open subset of $M_{g,n}(S_{\mathfrak{x}}, A)$. Moreover, by the construction of the polyhedral structure on every stratum of $M_{g,n}(S_{\mathfrak{x}}, A)$ in the proof of Proposition 6.9, we observe that the subdivision $M_{g,n}(\overline{S_{\mathfrak{x}}}, A)$ of $M_{g,n}(S_{\mathfrak{x}}, A)$ can be as fine as possible when we refine the subdivision $\overline{S_{\mathfrak{x}}}$ of $S_{\mathfrak{x}}$. So we have proved the following lemma.

Lemma 8.4. *The subsets $U(\mathcal{D})$ for all subdivision data $\mathcal{D} = (\overline{S_{\mathfrak{x}}}, (\Gamma, (\gamma_i)))$ form a base for the topology on $M_{g,n}(S_{\mathfrak{x}}, A)$.*

The next lemma relates degenerations of combinatorial types with formal models.

Lemma 8.5. *Let \mathfrak{X} be a formal scheme of finite presentation over k° . Let $(\mathfrak{C} \rightarrow \mathfrak{X}, (\mathfrak{s}_i), \mathfrak{f})$ be an n -pointed genus g formal stable map into \mathfrak{X} over \mathfrak{X} (cf. [67, §8]). Let $\pi_{\mathfrak{X}}: \mathfrak{X}_\eta \rightarrow \mathfrak{X}_s$ denote the reduction map. Let \bar{b}, \bar{b}' be two points in \mathfrak{X}_s such that \bar{b}' is a specialization of \bar{b} . Let $(C_b, (s_i), f_b)$ (resp. $(C_{b'}, (s'_i), f_{b'})$) be an analytic stable map into X corresponding to a point in $\pi_{\mathfrak{X}}^{-1}(\bar{b})$ (resp. $\pi_{\mathfrak{X}}^{-1}(\bar{b}')$). Let $(\Gamma_b, (\gamma_i), h_b)$ and $(\Gamma_{b'}, (\gamma'_i), h_{b'})$ be the associated parametrized tropical curves in $S_{\mathfrak{X}}$ respectively. Let $(\Gamma_b, (\gamma_i))$ and $(\Gamma_{b'}, (\gamma'_i))$ be the associated combinatorial types in $S_{\mathfrak{X}}$ respectively. Then $(\Gamma_b, (\gamma_i))$ is a degeneration of $(\Gamma_{b'}, (\gamma'_i))$.*

Proof. Let $(C_{\bar{b}}, (\bar{s}_i), f_{\bar{b}})$ (resp. $(C_{\bar{b}'}, (\bar{s}'_i), f_{\bar{b}'})$) be the algebraic stable map into \mathfrak{X}_s corresponding to the point \bar{b} (resp. \bar{b}'). Then the nodal curve $C_{\bar{b}'}$ is a degeneration of the nodal curve $C_{\bar{b}}$. In other words, $C_{\bar{b}}$ is a smoothing of $C_{\bar{b}'}$. So we obtain a surjective map from the set of the irreducible components of $C_{\bar{b}'}$ to the set of the irreducible components of $C_{\bar{b}}$. This induces a surjective map $\phi: V(\Gamma_{b'}) \rightarrow V(\Gamma_b)$ satisfying Definition 3.7 Conditions (iii) and (iv). It also induces bijections $\phi: E(u', v') \rightarrow E(\phi(u'), \phi(v'))$ for every pair of vertices u', v' of $\Gamma_{b'}$ such that $\phi(u') \neq \phi(v')$. Since any node of $C_{\bar{b}}$ remains a node of $C_{\bar{b}'}$, any edge of Γ_b is of the form $\phi(e')$ for some edge e' of $\Gamma_{b'}$. Therefore, Definition 3.7 Condition (i) is satisfied except possibly for the equalities of tropical weight vectors.

The assumption that \bar{b}' is a specialization of \bar{b} implies that if an irreducible component $C_{\bar{b}}^v$ of $C_{\bar{b}}$ maps to a closed stratum D_I of \mathfrak{X}_s , then the irreducible components of $C_{\bar{b}'}$ corresponding to the degeneration of $C_{\bar{b}}^v$ map to D_I as well. Therefore, Definition 3.7 Condition (ii) is satisfied.

Now let \mathfrak{X}_s^b be the Zariski closure of the point \bar{b} . Let $\mathfrak{C}_s^b := \mathfrak{C}_s \times_{\mathfrak{X}_s} \mathfrak{X}_s^b$. Let σ_0 be a node of $C_{\bar{b}}$. The node σ_0 remains a node for the family of nodal curves \mathfrak{C}_s^b over \mathfrak{X}_s^b . So we obtain a section $\sigma: \mathfrak{X}_s^b \rightarrow \mathfrak{C}_s^b$. Put $\tilde{\mathfrak{X}}_s^b := \mathfrak{X}_s^b \times \tilde{k}^s$, $\tilde{\mathfrak{C}}_s^b := \mathfrak{C}_s^b \times \tilde{k}^s$ and $\tilde{\sigma} := \sigma \times \tilde{k}^s: \tilde{\mathfrak{X}}_s^b \rightarrow \tilde{\mathfrak{C}}_s^b$.

Assume that $f_s: \mathfrak{C}_s \rightarrow \mathfrak{X}_s$ maps the section σ into a closed stratum D_I of \mathfrak{X}_s . Let $\bar{D}_I := D_I \times \tilde{k}^s$ and let $j: \bar{D}_I \hookrightarrow \mathfrak{X}_s$ denote the closed immersion as in Section 4. Since $\mathfrak{C} \rightarrow \mathfrak{X}$ is a family of nodal curves, the sheaf $\bar{\sigma}^* R^1 \Phi_{\mathbb{Q}_\ell, \mathfrak{C}_\eta}$ is locally constant. Let Λ be the constant sheaf on \mathfrak{X}_s^b associated to the \mathbb{Q}_ℓ -vector space $j^* R^1 \Phi_{\mathbb{Q}_\ell, \mathfrak{X}_\eta}$. The morphism $f: \mathfrak{C} \rightarrow \mathfrak{X}$ of formal schemes induces a morphism $\lambda: \Lambda \rightarrow \bar{\sigma}^* R^1 \Phi_{\mathbb{Q}_\ell, \mathfrak{C}_\eta}$ of locally constant sheaves on \mathfrak{X}_s^b . Assume that $\sigma(\bar{b}')$ corresponds to an edge $e' \in E(u', v')$ for two vertices u', v' of $\Gamma_{b'}$. Then $\sigma(\bar{b})$ corresponds to the edge $\phi(e')$ of Γ_b . By Lemma 4.4, the weight vector $w_{(u', e')}$ (resp. $w_{(\phi(u'), \phi(e'))}$) can be computed by the stalk of the morphism λ at a geometric point over \bar{b}' (resp. \bar{b}).

Since the morphism λ is locally constant and the base \mathfrak{X}_s^b is connected, we deduce that $w_{(u',e')} = w_{(\phi(u'),\phi(e'))}$. In other words, the equalities of weight vectors in Definition 3.7 Condition (i) hold. So we have proved our lemma. \square

Remark 8.6. The part concerning vanishing cycles in the proof of Lemma 8.5 means intuitively that the winding numbers do not change under deformations.

Proof of Theorem 8.1. In order to prove the continuity of the map $\tau_T: T \rightarrow M_{g,n}(S_{\bar{\mathfrak{X}}}, A)$, it suffices to show that for any point $b \in T$ and any subdivision datum $\mathcal{D} = (\overline{S_{\bar{\mathfrak{X}}}}, (\Gamma, (\gamma_i)))$ such that the associated open subset $U(\mathcal{D})$ contains the point $\tau_T(b) \in M_{g,n}(S_{\bar{\mathfrak{X}}}, A)$, the inverse image $\tau_T^{-1}(U(\mathcal{D}))$ is a neighborhood of b in T .

We fix b and \mathcal{D} as above. Up to passing to a finite extension of the ground field k , we can find a strictly semi-stable formal model $\bar{\mathfrak{X}}$ for the k -analytic space X such that the associated Clemens polytope $S_{\bar{\mathfrak{X}}}$ is isomorphic to $\overline{S_{\bar{\mathfrak{X}}}}$ (cf. [44]). Using [67, Theorem 1.5], replacing T by a quasi-étale covering if necessary, one can find a formal model $\mathfrak{t}: \mathfrak{X} \rightarrow \overline{\mathcal{M}}_{g,n}(\bar{\mathfrak{X}}, A)$ for the morphism $t: T \rightarrow \overline{\mathcal{M}}_{g,n}(X, A)$, i.e. an n -pointed genus g formal stable map $(\mathfrak{C} \rightarrow \mathfrak{X}, (\mathfrak{s}_i), \mathfrak{f})$ into $\bar{\mathfrak{X}}$ over \mathfrak{X} which gives back the k -analytic stable map $(C \rightarrow T, (s_i), f)$ into X over T when passing to generic fibers. The finite ground field extension and the quasi-étale covering above are allowed in virtue of the descent of open immersions (cf. [26, Theorem 4.2.7]).

Let $\pi_{\bar{\mathfrak{X}}}: T \simeq \mathfrak{X}_{\eta} \rightarrow \mathfrak{X}_s$ denote the reduction map. Let $\bar{b} := \pi_{\bar{\mathfrak{X}}}(b)$ and let \mathfrak{X}_s^b be the Zariski closure of the point \bar{b} . By Lemma 8.5, for any point $\bar{b}' \in \mathfrak{X}_s^b$ and any analytic stable map corresponding to a point in $\pi_{\bar{\mathfrak{X}}}^{-1}(\mathfrak{X}_s^b)$, its associated tropical curve belongs to the subset $U(\mathcal{D})$. In other words, we have

$$\pi_{\bar{\mathfrak{X}}}^{-1}(\mathfrak{X}_s^b) \subset \tau_T^{-1}(U(\mathcal{D})).$$

By the anti-continuity of the reduction map, the set $\pi_{\bar{\mathfrak{X}}}^{-1}(\mathfrak{X}_s^b)$ is open in T for the Berkovich topology. Since $b \in \pi_{\bar{\mathfrak{X}}}^{-1}(\mathfrak{X}_s^b)$ by construction, we have proved that $\tau_T^{-1}(U(\mathcal{D}))$ is a neighborhood of b in T . Using Lemma 8.4, we conclude that the map $\tau_T: T \rightarrow M_{g,n}(S_{\bar{\mathfrak{X}}}, A)$ is continuous. \square

Corollary 8.7. *The tropicalization map $\tau_M: \overline{\mathcal{M}}_{g,n}(X, A) \rightarrow M_{g,n}(S_{\bar{\mathfrak{X}}}, A)$ is a continuous map.*

Corollary 8.8. *Let $\overline{M}_{g,n}^{\text{trop}}(X, A)$ denote the image of the tropicalization map τ_M . Then $\overline{M}_{g,n}^{\text{trop}}(X, A)$ is a compact subset of $M_{g,n}(S_{\bar{\mathfrak{X}}}, A)$.*

Proof. It follows from Theorem 7.2 and Corollary 8.7. \square

9. POLYHEDRALITY VIA QUANTIFIER ELIMINATION

In Corollary 8.8, we defined $\overline{M}_{g,n}^{\text{trop}}(X, A)$ to be the subset of $M_{g,n}(S_{\mathfrak{x}}, A)$ consisting of tropical curves in $S_{\mathfrak{x}}$ that arise from n -pointed genus g stable maps into X with degree bounded by A . In this section, we will show the polyhedral nature of this subset.

The proof uses quantifier elimination from model theory together with the continuity theorem of Section 8. The model-theoretic approach here is inspired by Antoine Ducros' work [29]. The model theory of algebraically closed valued fields is used by Ducros in [29] after algebraization of k -analytic situations. However, since we will deal not only with one single k -analytic curve but also with families of k -analytic curves, it is not clear how to apply the standard algebraization techniques. Therefore we resort to the model theory of rigid subanalytic sets developed by Leonard Lipshitz and Zachary Robinson [50, 51].

Let us take a quick review following [50, 51, 52, 54]. We restrict to the case where the ground field k is of discrete valuation because the theory in the general case is more involved.

The language \mathcal{L}_{an}^D of the theory of rigid subanalytic sets consists of three sorts: \mathcal{O} , \mathfrak{m} and Γ_0 . There are binary function symbols $+$, $-$, \cdot on \mathcal{O} and \mathfrak{m} , a relation symbol $<$ and a constant 0 on Γ_0 . Moreover, there are functions symbols $D_0: \mathcal{O}^2 \rightarrow \mathcal{O}$, $D_1: \mathcal{O}^2 \rightarrow \mathfrak{m}$, $|\cdot|: \mathcal{O} \rightarrow \Gamma_0$, and for each $f \in k^\circ \langle T_1, \dots, T_m \rangle \llbracket S_1, \dots, S_n \rrbracket$, there is a function symbol $f: \mathcal{O}^m \times \mathfrak{m}^n \rightarrow \mathcal{O}$. If in addition f is in the ideal $(\varpi, S_1, \dots, S_n)$, where ϖ denotes a uniformizer of k , there is a function symbol $f: \mathcal{O}^m \times \mathfrak{m}^n \rightarrow \mathfrak{m}$. A term in the language \mathcal{L}_{an}^D of the form $f: \mathcal{O}^m \times \mathfrak{m}^n \rightarrow \mathcal{O}$ will be called a *D-function* for simplicity.

Given a non-archimedean field extension $k \subset K$, one can associate to K a standard \mathcal{L}_{an}^D -structure. We note that $\mathcal{O}, \mathfrak{m}, \Gamma_0$ are interpreted as $K^\circ, K^\circ, |K|$ respectively, and D_0, D_1 are interpreted as

$$D_0(x, y) = \begin{cases} x/y & \text{if } |x| \leq |y| \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$D_1(x, y) = \begin{cases} x/y & \text{if } |x| < |y| \\ 0 & \text{otherwise.} \end{cases}$$

The other symbols have obvious interpretations.

Let \mathbb{D} (resp. \mathbb{D}°) denote the closed (resp. open) unit disc over k , considered as a k -analytic space. Set $\Gamma := \sqrt{|k^*|}$.

Definition 9.1 (cf. [52]). A subset S of $\mathbb{D}^m \times (\mathbb{D}^\circ)^n \times \Gamma^l$ is called *subanalytic* if it is a boolean combination of subsets of the form

$$\{(x, y, \gamma) \in \mathbb{D}^m \times (\mathbb{D}^\circ)^n \times \Gamma^l \mid |f(x, y)|\gamma^u c \leq |g(x, y)|\},$$

where f, g are D -functions⁶, $u \in \mathbb{Z}^l$ and $c \in \Gamma$.

Remark 9.2. Subanalytic sets are exactly those definable by first order formulas without quantifiers in the language \mathcal{L}_{an}^D .

The following theorem has many variants in the literature [13, 12, 30, 40, 29]. The theory of rigid subanalytic sets provides us another proof via the quantifier elimination theorem of Lipshitz [50, Theorem 3.8.2].

Theorem 9.3 (cf. [53]). *Let $Y \subset \mathbb{D}^m$ be a closed immersion and let $S \subset Y \subset \mathbb{D}^m$ be a subanalytic set as in Definition 9.1. Let h_1, \dots, h_l be l D -functions on \mathbb{D}^m . Then $(\text{val } h_1, \dots, \text{val } h_l)(S) \cap \mathbb{R}^l$ is a finite polyhedral complex in \mathbb{R}^l of dimension less than or equal to the dimension of Y , where we put $\text{val } 0 = +\infty$.*

We generalize Theorem 9.3 to the relative case. Let $Y \subset \mathbb{D}^{m'}$ and $T \subset \mathbb{D}^m$ be closed immersions. Let $p_0: \mathbb{D}^{m'} \rightarrow \mathbb{D}^m$ be a morphism which restricts to a morphism $p: Y \rightarrow T$ of k -affinoid spaces. Let f be a morphism from Y to the analytification $(\mathbb{G}_{m,k}^n)^{an}$ of the algebraic torus $\mathbb{G}_{m,k}^n$ over k . Let G be the definable set

$$G = \left\{ (t, \gamma) \in T \times \Gamma^l \mid \exists y \in Y \left((p(y) = t) \wedge (|f|(y) = \gamma) \right) \right\}.$$

Using quantifier elimination for rigid subanalytic sets, there exists a positive integer N , D -functions $q_1, \dots, q_N, q'_1, \dots, q'_N$ on \mathbb{D}^m , $u_1, \dots, u_N \in \mathbb{Z}^l$ and $c_1, \dots, c_N \in \Gamma$ such that the set G can be written as a boolean combination of the subsets

$$\{(t, \gamma) \in T \times \Gamma^l \mid |q_i(t)|\gamma_i^{u_i} c_i \leq |q'_i(t)|\}, \text{ for } i = 1, \dots, N.$$

By Theorem 9.3, the intersection

$$(\text{val } q_1, \dots, \text{val } q_N, \text{val } q'_1, \dots, \text{val } q'_N)(T) \cap \mathbb{R}^{2N}$$

⁶To evaluate the norm of a D -function on a point of the k -analytic space $\mathbb{D}^m \times (\mathbb{D}^\circ)^n$, it suffices to pass to a ground field extension making the point rational.

is a finite polyhedral complex in \mathbb{R}^{2N} of dimension less than or equal to the dimension of T .

Let $M_{g,n}(S_{\mathfrak{X}}, A)_u$ denote the quotient of $M_{g,n}(S_{\mathfrak{X}}, A)$ by the following equivalence relation: two parametrized tropical curves $(\Gamma, (\gamma_i), h)$ and $(\Gamma', (\gamma'_i), h')$ in $M_{g,n}(S_{\mathfrak{X}}, A)$ are in the same equivalence class if $h(\Gamma) = h'(\Gamma')$ as subsets of $S_{\mathfrak{X}}$ and $h(\gamma_i) = h'(\gamma'_i)$ for $1 \leq i \leq n$. As the space $M_{g,n}(S_{\mathfrak{X}}, A)$, the space $M_{g,n}(S_{\mathfrak{X}}, A)_u$ is also naturally a compact topological space with a stratification whose open strata are open convex polyhedrons.

Proposition 9.4. *We use the settings in Section 8. Let $\tau_T: T \rightarrow M_{g,n}(S_{\mathfrak{X}}, A)$ denote the set-theoretically defined tropicalization map, and let u denote the quotient map from $M_{g,n}(S_{\mathfrak{X}}, A)$ to $M_{g,n}(S_{\mathfrak{X}}, A)_u$. Then the image of T under the composite map $u \circ \tau_T$ is polyhedral in $M_{g,n}(S_{\mathfrak{X}}, A)_u$, in the sense that its intersection with every open stratum of $M_{g,n}(S_{\mathfrak{X}}, A)_u$ is polyhedral.*

Proof. We can assume that T is a k -affinoid space. Choose a finite covering of the formal model \mathfrak{X} by affine open subschemes of the form \mathfrak{U} as in Definition 2.3. Let \mathfrak{U} be an element in the covering and let $\{Y_j\}$ be a finite affinoid covering of the inverse image $f^{-1}(\mathfrak{U}_\eta)$. Using the explicit description of the map $\tau: X \rightarrow S_{\mathfrak{X}}$ in Remark 2.4, we apply the model-theoretic arguments above to the morphisms $Y_j \rightarrow T$, and apply Theorem 9.3 to the tropicalization of the marked points s_i . We deduce that the image of T under the composite map $u \circ \tau_T$ is polyhedral in $M_{g,n}(S_{\mathfrak{X}}, A)_u$. \square

Theorem 9.5. *We use the setting of Proposition 9.4. The image of T under the map $\tau_T: T \rightarrow M_{g,n}(S_{\mathfrak{X}}, A)$ is polyhedral in $M_{g,n}(S_{\mathfrak{X}}, A)$, in the sense that its intersection with every open stratum of $M_{g,n}(S_{\mathfrak{X}}, A)$ is polyhedral.*

Proof. We can assume that T is a k -affinoid space. Let n' be an integer greater than or equal to n . Let $M_{g,n'}^v(S_{\mathfrak{X}}, A)$ denote the subset of $M_{g,n'}(S_{\mathfrak{X}}, A)$ consisting of parametrized tropical curves $(\Gamma, (\gamma_i), h)$ such that $n(v) \geq 1$ for every vertex v of Γ . Let $M_{g,n'}^v(S_{\mathfrak{X}}, A)_u$ denote the quotient of $M_{g,n'}^v(S_{\mathfrak{X}}, A)$ by the equivalence relation introduced before Proposition 9.4. We note that both spaces $M_{g,n'}^v(S_{\mathfrak{X}}, A)$ and $M_{g,n'}^v(S_{\mathfrak{X}}, A)_u$ are naturally compact topological spaces with a stratification whose open strata are open convex polyhedrons. Moreover, the fibers of the quotient map $u: M_{g,n'}^v(S_{\mathfrak{X}}, A) \rightarrow M_{g,n'}^v(S_{\mathfrak{X}}, A)_u$ are finite.

Up to passing to a finite quasi-étale covering of T , we can add enough new marked points so that the tropicalization map

$$\tau'_T: T \longrightarrow M_{g,n'}(S_{\mathfrak{x}}, A)$$

factorizes through the inclusion $M_{g,n'}^v(S_{\mathfrak{x}}, A) \subset M_{g,n'}(S_{\mathfrak{x}}, A)$ for some $n' \geq n$. In order to show that the image $\tau_T(T)$ is polyhedral in $M_{g,n}(S_{\mathfrak{x}}, A)$, it suffices to show that the image $\tau'_T(T)$ is polyhedral in $M_{g,n'}^v(S_{\mathfrak{x}}, A)$.

Let O be an open stratum of the space $M_{g,n'}^v(S_{\mathfrak{x}}, A)_u$. Let O_1, \dots, O_m denote the connected components of the inverse image $u^{-1}(O)$ in $M_{g,n'}^v(S_{\mathfrak{x}}, A)$. We will show that the intersection $\tau'_T(T) \cap O_i$ is polyhedral for all $i = 1, \dots, m$. Let $T_O \subset T$ denote the inverse image $(u \circ \tau'_T)^{-1}(O)$. By Theorem 8.1, the restriction $(\tau'_T)|_{T_O}: T_O \rightarrow u^{-1}(O)$ is a continuous map. Therefore, for any $i = 1, \dots, m$, the inverse $T_{O_i} := ((\tau'_T)|_{T_O})^{-1}(O_i)$ is a union of connected components of T_O . The model-theoretic arguments above imply that the map $u \circ \tau'_T: T \rightarrow M_{g,n}(S_{\mathfrak{x}}, A)_u$ is given by the norms of a finite collection of D -functions $\{q_j\}$. Therefore, the subspace $T_O \subset T$ is a subanalytic set. Since the subspace T_{O_i} is a union of connected components of T_O , it is also subanalytic (cf. [54, §3]). Restricting the D -functions $\{q_j\}$ to T_{O_i} and applying Theorem 9.3, we conclude that the image $\tau'_T(T_{O_i}) = \tau'_T(T) \cap O_i$ is polyhedral, completing the proof. \square

Corollary 9.6. *Let $\overline{M}_{g,n}^{\text{trop}}(X, A)$ denote the image of the tropicalization map*

$$\tau_M: \overline{M}_{g,n}(X, A) \rightarrow M_{g,n}(S_{\mathfrak{x}}, A).$$

Then $\overline{M}_{g,n}^{\text{trop}}(X, A)$ is a compact and polyhedral in $M_{g,n}(S_{\mathfrak{x}}, A)$.

Proof. The corollary follows from Theorems 7.2, 8.1 and 9.5. \square

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