MOMENT MAPS, NONLINEAR PDE AND STABILITY IN MIRROR SYMMETRY, I: GEODESICS

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ABSTRACT. In this paper, the first in a series, we study the deformed Hermitian-Yang-Mills (dHYM) equation from the variational point of view as an infinite dimensional GIT problem. The dHYM equation is mirror to the special Lagrangian equation, and our infinite dimensional GIT problem is mirror to Thomas' GIT picture for special Lagrangians. This gives rise to infinite dimensional manifold $\mathcal{H}$ closely related to Solomon's space of positive Lagrangians. In the hypercritical phase case we prove the existence of smooth approximate geodesics, and weak geodesics with $C^{1,\alpha}$ regularity. This is accomplished by proving sharp with respect to scale estimates for the Lagrangian phase operator on collapsing manifolds with boundary. As an application of our techniques we give a simplified proof of Chen's theorem on the existence of $C^{1,\alpha}$ geodesics in the space of Kähler metrics. In two follow up papers, these results will be used to examine algebraic obstructions to the existence of solutions to dHYM [26] and special Lagrangians in Landau-Ginzburg models [27].

1. INTRODUCTION

Mirror symmetry predicts that Calabi-Yau manifolds come in pairs (X, Ω, ω), $(\check{X}, \check{\Omega}, \check{\omega})$ with the property that symplectic geometry on $\check{X}$ is related to complex geometry on X and vice versa. The physical mechanism underlying mirror symmetry is a duality between type IIA string theory compactified on X and type IIB string theory compactified on $\check{X}$. Within this duality there is a correspondence between the D-branes in each theory; we refer the reader to [3] for a nice discussion of D-branes on Calabi-Yau manifolds. In both the type IIA and type IIB theory the physically realistic D-branes are minimizers of some energy functional, and are referred to as BPS, or supersymmetric D-branes.

On the A-model the D-branes, often referred to as A-branes, are known to be Lagrangian submanifolds of $(\check{X}, \check{\Omega}, \check{\omega})$ equipped with flat unitary bundles (as well as certain extended versions of these). On the B-model, the complex geometric side, D-branes, often called B-branes, can be thought of as holomorphic vector bundles, possibly supported on analytic subsets $V \subset X$. In this language Kontsevich's homological mirror symmetry proposal [52]

predicts a correspondence between A-branes on $\check{X}$, and B-branes on X

$$D^b\text{Fuk}(\check{X}) \sim D^b Coh(X).$$

where the left hand side is the derived Fukaya category of $\check{X}$. Mirror symmetry furthermore predicts a duality between the supersymmetric branes on each side of the correspondence. On the symplectic side, the supersymmetric A-branes are known to be special Lagrangian (sLag) submanifolds of $\check{X}$, together flat unitary bundles. On the B-model, however, the supersymmetry constraint is more mysterious. Around 2000, three separate approaches to understanding supersymmetric B-branes were introduced. One approach, by Mariño-Minasian-Moore-Strominger [59], used the Dirac-Born-Infeld+Chern-Simons functional to compute the equations of motion in the case of abelian gauge group. A second approach by Leung-Yau-Zaslow [56] was to use the Strominger-Yau-Zaslow [70] proposal, and a Fourier-Mukai transform, to compute the mirror object to a special Lagrangian in the setting of semi-flat mirror symmetry [54]. Each of these found that the equations of motion corresponded to a holomorphic line bundle $L \to X$ with a hermitian metric h solving the equation

(1.1)
$$\text{Im}(e^{-\sqrt{-1}\hat{\theta}}(\omega + F(h))^n) = 0$$
$$\text{Re}(e^{-\sqrt{-1}\hat{\theta}}(\omega + F(h))^n) > 0$$

where $\hat{\theta}$ is a constant. This equation became known as the *deformed Hermitian-Yang-Mills* (dHYM) equation. A third approach, initiated by Douglas-Fiol-Römelsberger [40] (see also [39]), was inspired by the Donaldson-Uhlenbeck-Yau theorem [36, 76]. They bypassed the equations of motion for BPS B-branes, and instead proposed an algebro-geometric notion called Π-stability. Their proposal can then be summarized as "an object in $D^b Coh(X)$ is a supersymmetric B-brane if it is Π-stable". This idea was taken up by Bridgeland [13] who developed the notion of categorical stability conditions in great generality. Since Bridgeland's pioneering work the subject of stability conditions on categories, and particularly $D^b Coh(X)$ and $D^b\text{Fuk}(\check{X})$, has generated a tremendous amount of interest. However, despite a great deal of progress (see, for example [1, 4, 5, 58] and the references therein), there is no general construction of a Bridgeland stability condition on $D^b Coh(X)$ for X Calabi-Yau of dimension larger than 2 [1].

The goal of this series of papers is to begin to unite these three approaches to supersymmetric B-branes. Indeed, the correspondence between the algebraic notion of supersymmetric A/B- branes, and the geometric notion of supersymmetric A/B-branes has played an important role in the development of mirror symmetry. Even before the introduction of Π-stability, Thomas [72] and Thomas-Yau [73] proposed a notion of stability for Lagrangians and predicted that the Lagrangian L could be deformed by Hamiltonian deformations to a special Lagrangian if and only if L is stable. This

proposal was based in part on a moment map formalism for special Lagrangians discovered by Thomas [72], and in part on the analogy with the Donaldson-Uhlenbeck-Yau theorem, motivated by mirror symmetry. More recently Joyce [51] proposed a very broad update to the Thomas-Yau conjecture in the framework of Bridgeland stability and the mean curvature flow. Broadly, the folklore conjecture is

Conjecture 1.1. *There is a Bridgeland stability condition on $D^b\mathrm{Fuk}(\check{X})$ (resp. $D^bCoh(X)$) so that the isomorphism class of a Lagrangian L (resp. holomorphic vector bundle E) is stable if and only if it contains a special Lagrangian (resp. E admits a metric solving the deformed Hermitian-Yang-Mills equation).*

This conjecture is really two conjectures, the first involving the existence of a Bridgeland stability condition, and the second relating Bridgeland stability and the existence of a solution to a certain nonlinear PDE.

On either side of this conjecture there has been little progress. Haiden-Katzarkov-Kontsevich-Pandit showed that gradient flows of metrics on semistable quiver representations [44], and the Yang-Mills flow on a holomorphic bundle over a Riemann surface [45], give rise to canonically defined filtrations associated with Bridgeland stability conditions, giving evidence for Conjecture 1.1. On the symplectic side Joyce [51] has outlined a program for approaching to Conjecture 1.1, based on understanding the singularity formation and surgery of the Lagrangian mean curvature flow (LMCF). We remark that Neves [61] has shown that finite time singularities of the LMCF are essentially unavoidable, and hence the problem of understanding the long-time behavior of the LMCF is extremely difficult. At the same time, Imagi-Joyce-Oliveira dos Santos [48] have shown how ideas from Floer theory and the Fukaya category can be used to study the singularity formation of the LMCF.

On the holomorphic side, the deformed Hermitian-Yang-Mills equation has recently been studied by Jacob-Yau [50] and the authors and Jacob [23]. In [23] a necessary and sufficient *analytic* condition was given for the existence of solutions to dHYM in the critical phase case. It was observed that these conditions gave rise to algebraic obstructions of "Bridgeland type". However, outside of this result, and for the case of higher rank vector bundles, essentially nothing is known.

This paper is the first in a series taking up the above folklore conjecture, primarily on the *B*-model. In particular, we will develop the algebro-geometric obstruction theory for the deformed Hermitian-Yang-Mills equation on a line bundle in the hypercritical phase case, in this paper and the sequel [26]. We will compare our results with the "expected" Bridgeland stability condition, and, in the third paper in the series [27] we will use mirror symmetry to deduce similar results for Lagrangians in Landau-Ginzburg models mirror to toric Fano varieties.

Our approach to this problem is to study the mirror of an infinite dimensional GIT framework for special Lagrangians due to Thomas [72], and Solomon [68]. To put things in context, let us briefly recall the basic idea of finite dimensional GIT; we refer the reader to [74, 60] for a thorough discussion. Suppose (X, ω) is a projective Kähler manifold acted on by a group G, which is the complexification of compact real Lie group K, acting on (X, ω) by symplectomorphisms. By the Hilbert-Mumford criterion, a point p with finite stabilizer is GIT stable if and only if the orbit of p is closed under all 1-parameter subgroups, which we think of as infinite geodesics in G. The Kempf-Ness theorem makes the the connection with symplectic geometry by associating to G, p a certain K-invariant function f_p, called the Kempf-Ness functional, constructed out of the moment map for the K-action. The Kempf-Ness functional has the following two properties: (i) f_p is convex along all one-parameter subgroups of G/K, and (ii) p is stable if and only if f_p is proper on G/K. Since f_p is convex, the properness can be checked by evaluating the limit slope of f_p along infinite geodesic rays in G/K, a calculation which gives rise to algebraic invariants of the G-action whose sign determine the stability of p. Furthermore, the construction of f_p shows that p is stable if and only if the K-orbit of p contains a zero of the moment map, which is the usual statement of the Kempf-Ness theorem.

On the A-model, [68] Solomon introduces a Riemannian structure, and geodesic equation on the infinite dimensional space of *positive* (or almost calibrated) Lagrangians. These geodesics are the one parameter subgroups in the complexified symplectomorphism group in Thomas' infinite dimensional GIT picture [72, 73]. Solomon also introduces two functionals $\mathcal{C}, \mathcal{J}$, the latter of which is the the Kempf-Ness functional of the GIT problem, and is therefore convex along putative smooth geodesics.

The existence problem for geodesics in the space of positive Lagrangians has recently generated a great deal of interest. Solomon-Yuval [69] demonstrated the existence of smooth geodesics between positive Lagrangians in Milnor fibers. Rubinstein-Solomon [66] studied the existence of geodesics between graphical positive Lagrangians. They prove that if f_0, f_1 are two functions defined on a smoothly bounded domain $D \subset \mathbb{R}^n$ so that

$$x \longmapsto (x, \nabla f_i(x)) \qquad i = 0, 1$$

define positive Lagrangians in $\mathbb{R}^{2n} = \mathbb{C}^n$, then there exists a *continuous function* $F(x, t)$ having $F(x, 0) = f_0$, $F(x, 1) = f_1$ and which solves Solomon's geodesic equation in the weak sense of Harvey-Lawson's Dirichlet duality theory. These weak geodesics were described in terms of envelopes in the hypercritical phase case by Darvas-Rubinstein [30]. The Rubinstein-Solomon approach was extended to compact Riemannian manifolds by Dellatorre [32], and to the deformed Hermitian-Yang-Mills setting, as developed here, by Jacob [49].

In this paper we will develop the mirror of the Thomas-Solomon infinite dimensional GIT picture for the deformed Hermitian-Yang-Mills equation. We describe an infinite dimensional symplectic manifold, admitting an action by a group of symplectomorphisms, together with a space $\mathcal{H} \subset C^\infty(X, \mathbb{R})$ and a Riemannian structure on $\mathcal{H}$, which can be thought of as analogous to G/K in the finite dimensional GIT. We compute the geodesic equation, and introduce a notion of ε-geodesics, which solve an approximate version of the geodesic equation. We also introduce the complexified Calabi-Yau functional (see Definition 2.13), and extract from this functional analogues of the $\mathcal{C}$, and $\mathcal{J}$ functionals, as well as a $\mathbb{C}$-valued functional Z. The $\mathcal{J}$ functional is the Kempf-Ness functional for the GIT problem; it has critical points at solutions of the dHYM equation, and is convex along smooth geodesics. A fundamental issue in the analogy with finite dimensional GIT is that smooth geodesics need not exist. Thus, our main analytic contribution is to prove, in the hypercritical phase case, the existence of weak geodesics connecting points in $\mathcal{H}$, with $C^{1,\alpha}$ regularity. With this much regularity, we can show that the functionals $\mathcal{J}, \mathcal{C}, Z$ are well-defined and we prove that they are convex/concave along these generalized geodesics.

In order to study the existence of regular geodesics we study the manifolds

$$(\mathcal{X}_\varepsilon, \hat{\omega}) = \left(X \times \{t \in \mathbb{C} : \varepsilon e^{-1} < |t| < \varepsilon\}, \hat{\omega} := \pi_X^* \omega + \sqrt{-1} dt \wedge d\bar{t} \right)$$

where $\pi_X : \mathcal{X}_\varepsilon \to X$ is the natural projection. We consider solutions of the specified Lagrangian phase equation

$$(1.2) \qquad \Theta_{\hat{\omega}}(\pi_X^* \alpha + \sqrt{-1} D\overline{D}\varphi) := \sum_{i=0}^{n} \arctan(\mu_i) = h$$

on $\mathcal{X}_\varepsilon$, where $\mu_0, \ldots, \mu_n$ are eigenvalues of $\pi_X^* \alpha + \sqrt{-1} D\overline{D}\varphi$ with respect to $\hat{\omega}$. We prove sharp (with respect to scale) estimates for this equation. As a PDE question, this seems to be of independent interest. Namely, suppose we have a domain of the form $M \times [-\varepsilon, \varepsilon] \subset M \times \mathbb{R}$, where M is a Riemannian manifold, possible with boundary. Suppose φ solves a fully non-linear elliptic equation $F(D^2\varphi) = 0$. By the mean-value theorem, the gradient of $\nabla\varphi$ is of order $\frac{1}{\varepsilon}$ in the thin directions, and by the comparison principle we expect $D^2\varphi \approx \varepsilon^{-2}$ in directions parallel to the $\mathbb{R}$-factor. The question then is whether this lack of regularity in the "thin" directions propagates to directions tangent to M. In the present setting we prove that the lack of regularity *does not* propagate.

Theorem 1.2. *Suppose $\varphi(x, t)$ is a smooth S^1 invariant function on $(\mathcal{X}_\varepsilon, \hat{\omega})$ solving (1.2), with $h : \mathcal{X}_\varepsilon \to ((n-1)\frac{\pi}{2}, (n+1)\frac{\pi}{2})$, and with $\varphi|_{\partial\mathcal{X}_\varepsilon} \in \mathcal{H}$. Then there is a constant C independent of ε so that following estimates hold*

$$\operatorname{osc}_{\mathcal{X}_\varepsilon} \varphi + |\nabla^X \varphi|_{\hat{\omega}} + |\nabla^X \overline{\nabla^X} \varphi|_{\hat{\omega}} \leqslant C$$

$$|\nabla_{\bar{t}} \nabla^X \varphi|_{\hat{\omega}} + |\nabla_t \varphi|_{\hat{\omega}} \leqslant \frac{C}{\varepsilon}$$

$$|\nabla_{\bar{t}}\nabla_t \varphi|_{\hat{\omega}} \leqslant \frac{C}{\varepsilon^2}$$

The proof of these estimates depends on exploiting the geometry of $(\mathcal{X}_\varepsilon, \hat{\omega})$ together with rather subtle concavity properties of the Lagrangian phase operator. Altogether, these estimates imply the existence of smooth ε-geodesics connecting any two potentials in $\mathcal{H}$, and the existence of weak geodesics in $\mathcal{H}$ with $C^{1,\alpha}$ regularity for any $\alpha \in (0,1)$. We show that these weak geodesics have enough regularity to define the complexified Calabi-Yau functional, and prove that $\mathcal{C}$ is affine, $\mathcal{J}$ is convex, and the real and imaginary parts of Z are concave along these weak geodesics. Furthermore, using the existence result for geodesics we show that $\mathcal{H}$ has a well-defined metric structure.

In the sequel to this work [26] we will use ideas from finite dimensional GIT, with geodesics playing the role of one-parameter subgroups. Using algebraic geometry we will construct model infinite rays, analogous to one-parameter subgroups in the space $\mathcal{H}$, and evaluate the limit slope of the Calabi-Yau functional along these model curves in terms of algebraic data. Using the existence of regular geodesics these model curves will give rise to algebro-geometric obstructions to the existence of solutions to dHYM. Furthermore, we will describe how these obstructions fit into a framework which is closely related to Bridgeland stability [13]. In the third paper in this series [27] we will use these techniques to find obstructions to the existence of special Lagrangians in certain Landau-Ginzrang models.

As suggested by finite dimensional GIT, we expect this theory will have applications in proving the existence of solutions to dHYM. In the case of Kähler-Einstein metrics, Berman-Boucksom-Jonsson [7] recently gave a variational proof of the Chen-Donaldson-Sun theorem [19, 20, 21] establishing the existence of Kähler-Einstein metrics on K-stable Fano manifolds. Some of the key ingredients in the approach of [7] are: (i) the existence of $C^{1,\alpha}$ geodesics in the space of Kähler metrics [17]. (ii) the relationship between existence of Kähler-Einstein metrics and the properness of a certain Kempf-Ness functional D (in an appropriate sense) [31], using the metric structure on the space of Kähler metrics. (iii) the extension of D to a function on non-Archimedean metrics, which allows one to study the limit slopes along infinite geodesic rays [7, 11, 12]. One could hope for a similar approach to proving the existence of solutions to dHYM, using the existence of regular geodesics we establish, together with the convexity/concavity properties of the functionals $\mathcal{J}$ and Z. However, for the dHYM equation there are significant new difficulties related to the phase lifting problem, and related Chern number inequalities, which will be discussed in the sequel [26].

The layout of this paper is as follows. In Section 2 we explain the mirror of Thomas' moment map framework and the Thomas-Solomon GIT/variational framework. We compute the geodesic equation, introduce ε-geodesics and study the complexified Calabi-Yau functional. We also outline the approach to existence of regular geodesic. In Section 3 we discuss the properties of

the Lagrangian phase operator, and construct barrier functions which play an important role in later estimates. In Section 4 we prove a priori C^1 estimates for solutions to (1.2). In Section 5 we prove interior C^2 estimates for solutions of (1.2). We also explain how our estimates can be applied to give a streamlined proof of the existence of geodesics in the space of Kähler metrics [17]. In Section 6 we prove boundary C^2 estimates, and combine our work to prove the existence of smooth ε-geodesics, $C^{1,\alpha}$ weak geodesics, and prove various convexity statements for relevant functionals along weak geodesics.

2. The Variational Framework, Geodesics, and Approximate Geodesics

Let (X, ω) be a compact Kähler manifold, and fix a class $[\alpha] \in H^{1,1}(X, \mathbb{R})$. By the $\partial\bar{\partial}$-lemma, the $(1,1)$ forms α' lying in the cohomology class $[\alpha]$ are parametrized by functions $\alpha' = \alpha + \sqrt{-1}\partial\bar{\partial}\varphi$. Consider the integral $\int_X (\omega + \sqrt{-1}\alpha)^n \in \mathbb{C}$. We will always assume that this integral lies in $\mathbb{C}^*$. Then we define a unit complex number $e^{\sqrt{-1}\hat{\theta}}$ depending only on $[\omega], [\alpha]$ by

$$\int_X (\omega + \sqrt{-1}\alpha)^n \in \mathbb{R}_{>0} e^{\sqrt{-1}\hat{\theta}}.$$

Motivated by mirror symmetry we introduce the deformed Hermitian-Yang-Mills (dHYM) equation, which seeks a function $\varphi \in C^\infty(X, \mathbb{R})$ so that $\alpha_\varphi := \alpha + \sqrt{-1}\partial\bar{\partial}\varphi$ satisfies

$$\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_\varphi)^n\right) = 0.$$

We refer the reader to [25] for a brief introduction to the dHYM equation. Note that with our present convention, if $\alpha = c_1(L)$, then we are studying (1.1) on L^{-1}. This convention plays no role, apart from avoiding an abundance of minus signs. To write the dHYM equation more concretely, fix a point $p \in X$, and local holomorphic coordinates $(z_1, \ldots, z_n)$ near p so that $\omega(p)_{\bar{j}i} = \delta_{\bar{j}i}$, and $(\alpha_\varphi)_{\bar{j}i} = \lambda_i \delta_{\bar{j}i}$. Then λ_i are the eigenvalues of the

Hermitian endomorphism $\omega^{-1}\alpha_\varphi$, and

$$\frac{(\omega + \sqrt{-1}\alpha_\varphi)^n}{\omega^n} = \prod_{i=1}^{n}(1 + \sqrt{-1}\lambda_i)$$

$$= r(\alpha_\varphi)e^{\sqrt{-1}\Theta_\omega(\alpha_\varphi)}$$

where

$$(2.1) \qquad r(\alpha_\varphi) = \sqrt{\left(\prod_{i=1}^{n}(1 + \lambda_i^2)\right)}, \qquad \Theta_\omega(\alpha_\varphi) = \sum_{i=1}^{n}\arctan(\lambda_i).$$

The function $r(\alpha_\varphi)$ is called the radius function, while $\Theta_\omega(\alpha_\varphi)$ is called the Lagrangian phase. The deformed Hermitian-Yang-Mills equation seeks α_φ so that

$$\sum_{i=1}^{n}\arctan(\lambda_i) = \hat{\theta} \mod 2\pi.$$

Note that if there is a solution of the deformed Hermitian-Yang-Mills equation then there is a well-defined lift of $\hat{\theta}$ to $\mathbb{R}$. Furthermore, with this formulation it is clear that the dHYM equation is the complex analogue of the special Lagrangian graph equation. The following lemma is due to Jacob-Yau [50].

Lemma 2.1 (Jacob-Yau [50]). *Solutions of the deformed Hermitian-Yang-Mills equation minimize the functional*

$$C^\infty(X, \mathbb{R})\varphi \longrightarrow V(\varphi) := \int_X r(\alpha_\varphi)\omega^n.$$

If φ is a solution of the deformed Hermitian-Yang-Mills equation then

$$0 < V(\varphi) = \left|\int_X (\omega + \sqrt{-1}\alpha)^n\right|.$$

We define the Lagrangian phase operator by

$$(2.2) \qquad \Theta_\omega(\alpha_\varphi) = \sum_{i=1}^{n}\arctan(\lambda_i).$$

In our previous work with Jacob [23] we gave necessary and sufficient analytic conditions for the existence of solutions to the deformed Hermitian-Yang-Mills equation. We proved

Theorem 2.2 (Collins-Jacob-Yau, [23]). *For $\hat{\theta} > (n-2)\frac{\pi}{2}$, there exists a function φ satisfying*

$$\Theta_\omega(\alpha_\varphi) = \hat{\theta}$$

if and only if there exists a function $\underline{\varphi} : X \to \mathbb{R}$ so that $\Theta_\omega(\alpha_{\underline{\varphi}}) > (n-2)\frac{\pi}{2}$, and, for all $1 \leqslant j \leqslant n$

$$(2.3) \qquad \sum_{i \neq j}\arctan(\underline{\lambda}_i) > \hat{\theta} - \frac{\pi}{2}$$

where $\underline{\lambda}_i$ are the eigenvalues of $\omega^{-1}\alpha_\varphi$.

A primary motivation of this series of papers is to understand the implications of existence of solutions to the dHYM equation with the goal of replacing (2.3) with an algebro-geometric condition. In the best case scenario this could give a checkable criterion equivalent to the existence of solutions to dHYM.

Let us briefly discuss the moment map picture, though our primary interest will be the variational framework of the associated GIT problem. Let $L \to (X, \omega)$ be a holomorphic line bundle, and fix a hermitian metric h on L, which induces a unitary structure. We consider the affine space $\mathcal{A}^{1,1}$ of h-unitary connections inducing integrable complex structures on L. Since L is a line bundle, this is just the same as the set of unitary connections $\nabla = d + A$ such that $\bar{\partial}A^{0,1} = 0$. The group $\mathcal{G}$ of gauge transformations acts on $\mathcal{A}^{1,1}$ by in the standard way. If $g = e^\varphi$ is a gauge transformation for some $\varphi : X \to \mathbb{C}$, then

$$A^{0,1} \mapsto A^{0,1} + \bar{\partial}\varphi \qquad A^{1,0} \mapsto A^{1,0} - \partial\overline{\varphi}.$$

We can identify $T_A\mathcal{A}^{1,1} = \mathrm{Ker}\{\bar{\partial} : \bigwedge^{0,1} \to \bigwedge^{0,2}\}$, and hence define a hermitian inner product by

$$T_A\mathcal{A}^{1,1} \ni a, b \longmapsto \langle a, b \rangle_A := -\sqrt{-1} \int_X a \wedge \bar{b} \wedge \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right).$$

In general, this inner product is degenerate, but one can check that it is non-degenerate in an open neighbourhood of a solution to the dHYM equation. The natural complex structure on $T_A\mathcal{A}^{1,1}$ acts by $a \mapsto \sqrt{-1}a$, and we get a symplectic form on $\mathcal{A}^{1,1}$ by taking

$$\Omega_A(a, b) := \mathrm{Im}\left(\langle a, b \rangle_A\right).$$

Let $\mathcal{G}_U$ be the Lie group of unitary gauge transformations of (L, h), and identify $Lie(\mathcal{G}_U) = C^\infty(X, \sqrt{-1}\mathbb{R})$. Let $\sqrt{-1}\varphi \in Lie(\mathcal{G}_U)$. Identify $Lie(\mathcal{G}_U)^*$ with the space of imaginary $2n$-forms on X by the non-degenerate pairing

$$(\sqrt{-1}\varphi, \sqrt{-1}\beta) \longmapsto \int_X \varphi\beta.$$

The infinitesimal action of $\sqrt{-1}\varphi \in Lie(\mathcal{G}_U)$ generates the vector field $\sqrt{-1}\,\bar{\partial}\varphi$ on $\mathcal{A}$. For any $b \in T_A\mathcal{A}^{1,1}$ we consider

$$\langle \sqrt{-1}\,\bar{\partial}\varphi, b \rangle_A = \int_X \bar{\partial}\varphi \wedge \bar{b} \wedge \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right)$$

$$= -\int_X \varphi \cdot \overline{\partial b} \wedge \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right)$$

and so

$$\Omega_A(\sqrt{-1}\,\bar{\partial}\varphi, b) = \frac{\sqrt{-1}}{2} \int_X \varphi \cdot (\overline{\partial b} - \partial b) \wedge \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^{n-1}\right).$$

Now, if we consider $A^{0,1} \mapsto A^{0,1} + tb$, then

$$\frac{d}{dt}\Big|_{t=0} F(A+tb) = -(\overline{\partial}\overline{b} - \partial b)$$

and this form is purely imaginary. Therefore

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\left(\omega - F(A+tb)\right)^n\right)$$

$$= n\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\left(\omega - F(A)\right)^{n-1} \wedge (\overline{\partial}\overline{b} - \partial b)\right)$$

$$= n(\overline{\partial}\overline{b} - \partial b) \wedge \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\left(\omega - F(A)\right)^{n-1}\right).$$

It follows immediately that the moment map for the $\mathcal{G}_U$ action is

$$A \mapsto -\frac{\sqrt{-1}}{2n}\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega - F(A))^n\right),$$

and hence solutions of dHYM correspond exactly to zeroes of the moment map. Despite the fact the the symplectic form is degenerate, we can still hope to use ideas from GIT to study the existence of zeroes of the moment map.

Infinite dimensional GIT frameworks have appeared in several contexts in the study of nonlinear PDE on Kähler manifolds, including the study of the Hermitian-Yang-Mills equation on a compact Kähler manifold [2], certain nonlinear generalizations thereof [55], and the Kähler-Einstein and constant scalar curvature equation [37, 38], among others [35]. More recently, the study of the space of Kähler metrics has drawn a great deal of attention. Since our setting is formally analogous to this latter topic, let us briefly recall this framework. Following Donaldson [35], Mabuchi [57] and Semmes [67], we fix a Kahler class $[\omega]$ on X, and consider

$$\mathcal{H}_{PSH} := \{\varphi \in C^\infty(X, \mathbb{R}) : \omega_\varphi := \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0\}$$

The tangent space to $\mathcal{H}_{PSH}$ at a function φ is $C^\infty(X, \mathbb{R})$, and we can introduce a Riemannian metric by

$$\langle \psi_1, \psi_2 \rangle_\varphi = \int_X \psi_1 \psi_2 \omega_\varphi^n.$$

This makes $\mathcal{H}_{PSH}$ into an infinite dimensional Riemannian manifold. One can then study the geodesic equation on this manifold, which is equivalent to the homogeneous complex Monge-Ampère equation [35, 57, 67]. For any $\varphi_1, \varphi_2 \in \mathcal{H}$ one can find a curve $\varphi(t)$ of potentials for which $\varphi(0) = \varphi_0, \varphi(1) = \varphi_1$, $\varphi(t)$ is $C^{1,1}$ in space and time, satisfies $\omega + \sqrt{-1}\partial\overline{\partial}\varphi(t) \geqslant 0$, and solves the geodesic equation in a weak sense [17] (see also [16, 41, 9, 22, 78] for related work). Furthermore, it is known that $\varphi(t)$ cannot be C^2 in general [28, 29, 53]. Even without better regularity, the existence of weak geodesics plays an important role in linking the existence of solutions to certain nonlinear PDE, including the Kähler-Einstein equation on a Fano manifold [8, 7], and Donaldson [34] and Chen's J-equation [18].

The variational framework studied here is mirror to a variational framework for positive Lagrangians introduced by Solomon [68]. When L is an ample line bundle, this variational structure can be regarded as interpolating between the Riemannian structure for the Hermitian-Yang-Mills equation, and the Donaldson-Mabuchi-Semmes [35, 57, 67] Riemannian structure on the space of Kähler metrics, as will be discussed below.

Definition 2.3. *Define the space*

$$\mathcal{H} := \{\varphi \in C^\infty(X, \mathbb{R}) : \mathrm{Re}(e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_\varphi)^n) > 0\}$$

A slightly more concrete definition of the space $\mathcal{H}$ in terms of the Lagrangian phase operator (2.2) is

Lemma 2.4. *The space $\mathcal{H}$ can be defined as*

$$\mathcal{H} := \{\varphi \in C^\infty(X, \mathbb{R}) : \Theta_\omega(\alpha_\varphi) \in (\hat{\theta} - \frac{\pi}{2}, \hat{\theta} + \frac{\pi}{2}) \mod 2\pi\}.$$

Remark 2.5. It is clear that if one changes Kahler forms within the class $[\omega]$, then the space $\mathcal{H}$ will change as well. Thus we should really be writing $\mathcal{H}_\omega$ to indicate the dependence on ω, but we will refrain from doing so and hope this causes no confusion.

Recall that the angle $\hat{\theta}$ is a priori only defined modulo 2π.

Lemma 2.6. *Assume that $\mathcal{H} \neq \emptyset$. Then there exists a unique lift of $\hat{\theta}$ to $\mathbb{R}$.*

This lemma is an application of the maximum principle; we refer the reader to [50, 25] for a proof.

It is a simple consequence of the Schur-Horn Theorem [47] and Lemma 3.1 (7) that the space $\mathcal{H}$ is convex when $|\hat{\theta}| \geqslant (n-1)\frac{\pi}{2}$. However, in the lower branches it is not even clear that $\mathcal{H}$ is connected. For our purposes this will not be a significant issue, since $\mathcal{H}$ is embedded in the vector space $C^\infty(X, \mathbb{R})$.

There is a natural Riemannian structure on $\mathcal{H}$ defined in the following way. The tangent space at a point $\varphi \in \mathcal{H}$ is $T_\varphi \mathcal{H} = C^\infty(X, \mathbb{R})$ and we define a non-trivial Riemannian structure by

$$\langle \psi_1, \psi_2 \rangle_\varphi = \int_X \psi_1 \psi_2 \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_\varphi)^n\right).$$

The Riemannian metric gives rise to a notion of geodesics.

Proposition 2.7. *A smooth curve $\varphi(t) \in \mathcal{H}$ with $\varphi(0) = \varphi_0, \varphi(1) = \varphi_1$ is a geodesic if it solves the equation*
(2.4)
$$\ddot{\varphi}\mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_\varphi)^n\right) + n\sqrt{-1}\partial\dot{\varphi}\wedge\overline{\partial}\dot{\varphi}\wedge\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_\varphi)^{n-1}\right) = 0$$

Proof. Let $\varphi(t)$ be a curve in $\mathcal{H}$ with constant speed. Suppose that $\varphi(t, s)$ is surface in $\mathcal{H}$ such that $\varphi(0, s) = \varphi_0$, $\varphi(1, s) = \varphi_1$, and $\varphi(t, 0) = \varphi(t)$.

We may write $\varphi(t,s) = \varphi(t) + s\psi(t,s)$ with $\psi(0,s) = \psi(1,s) = 0$. We will compute the variation of arc-length around $\varphi(t)$. The length is

$$L(s,t) = \int_0^1 dt \sqrt{\int_X (\dot\varphi)^2 \mathrm{Re}\left(e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^n\right)}$$

Taking a derivative gives

$$\frac{d}{ds}\bigg|_{s=0} L = \int_0^1 dt \frac{1}{\|\dot\varphi\|}\int_X \dot\psi\dot\varphi\mathrm{Re}\left(e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^n\right)$$
$$+ \int_0^1 \frac{1}{2\|\dot\varphi\|}\int_X (\dot\varphi)^2 \mathrm{Re}\left(n\sqrt{-1}e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^{n-1}\wedge\sqrt{-1}\partial\bar\partial\psi\right)$$

Integration by parts on the second term yields
(2.5)
$$\frac{d}{ds}\bigg|_{s=0} L = \int_0^1 dt\frac{1}{\|\dot\varphi\|}\int_X \dot\psi\dot\varphi\mathrm{Re}\left(e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^n\right)$$
$$- \int_0^1 dt\frac{n}{\|\dot\varphi\|}\int_X \psi\sqrt{-1}\partial\dot\varphi\wedge\bar\partial\dot\varphi\wedge\mathrm{Im}\left(e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^{n-1}\right)$$
$$- \int_0^1 dt\frac{n}{\|\dot\varphi\|}\int_X \psi\dot\varphi\mathrm{Im}\left(e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^{n-1}\wedge\sqrt{-1}\partial\bar\partial\dot\varphi\right)$$

where we also used that $\sqrt{-1}\partial\bar\partial$ is a real operator, and $\mathrm{Re}(\sqrt{-1}z) = -\mathrm{Im}(z)$. We now integrate by parts in time on the first term.
(2.6)
$$\int_0^1 dt\frac{1}{\|\dot\varphi\|}\int_X \dot\psi\dot\varphi\mathrm{Re}\left(e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^n\right)$$
$$= \int_0^1 dt\frac{d}{dt}\left(\frac{1}{\|\dot\varphi\|}\int_X \psi\dot\varphi\mathrm{Re}\left(e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^n\right)\right)$$
$$- \int_0^1 dt\frac{1}{\|\dot\varphi\|}\int_X \psi\ddot\varphi\mathrm{Re}\left(e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^n\right)$$
$$- \int_0^1 dt\frac{1}{\|\dot\varphi\|}\int_X \psi\dot\varphi\mathrm{Re}\left(n\sqrt{-1}e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^{n-1}\wedge\sqrt{-1}\partial\bar\partial\dot\varphi\right)$$

The last term from (2.6) cancels the last term from (2.5). Furthermore, since $\psi(0,s) = \psi(1,s) = 0$ the first integral vanishes. Thus

$$\frac{d}{ds}\bigg|_{s=0} L = -\int_0^1 dt\frac{1}{\|\dot\varphi\|}\int_X \psi\ddot\varphi\mathrm{Re}\left(e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^n\right)$$
$$- \int_0^1 dt\frac{n}{\|\dot\varphi\|}\int_X \psi\sqrt{-1}\partial\dot\varphi\wedge\bar\partial\dot\varphi\wedge\mathrm{Im}\left(e^{-\sqrt{-1}\hat\theta}(\omega + \sqrt{-1}\alpha_\varphi)^{n-1}\right)$$

which is what we wanted to prove. □

As in the case of geodesics in the space of Kähler metrics, we can reformulate this equation as a degenerate elliptic equation over the product

manifold

$$\mathcal{X} := X \times \mathcal{A} = X \times \{e^{-1} < |t| < 1\}$$

where now t is a coordinate on $\mathbb{C}$. We denote the projections for $\mathcal{X}$ to $X, \mathcal{A}$ by $\pi_X, \pi_\mathcal{A}$ respectively, and let $\sqrt{-1}D\overline{D}$ denote the $\partial\overline{\partial}$ operator on the $n+1$ dimensional manifold $\mathcal{X}$, and $\sqrt{-1}\partial\overline{\partial}$ denote the $\partial\overline{\partial}$ operator on X.

Lemma 2.8. *Suppose $\varphi_0, \varphi_1 \in \mathcal{H}$. A solution $\hat{\varphi}(x,s) \in \mathcal{H}$ of the geodesic equation with $\varphi(x,0) = \varphi_0, \varphi(x,1) = \varphi_1$ is equivalent to a function $\varphi : \mathcal{X} \to \mathbb{R}$ which is S^1 invariant (ie. $\varphi(x,t) = \varphi(x,|t|)$) and solving*

$$(2.7) \qquad \mathrm{Im}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\pi_X^*\omega + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right] = 0$$

$$\mathrm{Re}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\omega + \sqrt{-1}\left(\alpha + \sqrt{-1}\partial\overline{\partial}\varphi\right)\right)^n\right] > 0$$

and with $\varphi(x,1) = \varphi_0(x), \varphi(x,e^{-1}) = \varphi_1$.

Proof. Let $s = -\log|t|$, and set $\varphi(x,t) = \hat{\varphi}(x, -\log|t|)$, and let $\pi = \pi_X$ for simplicity. The second line just expresses that $\varphi(x,t) = \hat{\varphi}(x,s) \in \mathcal{H}$, so it suffices to check (2.7) is equivalent to (2.4). For simplicity denote $\dot{\hat{\varphi}} = \partial_s\hat{\varphi}$, and similarly for higher derivatives. Then

$$\partial_t\varphi = -\frac{1}{t}\dot{\hat{\varphi}}, \quad \partial_{\bar{t}}\varphi = -\frac{1}{\bar{t}}\dot{\hat{\varphi}}, \quad \partial_t\partial_{\bar{t}}\varphi = \frac{1}{|t|^2}\ddot{\hat{\varphi}}$$

Now it is a matter of linear algebra. We expand

$$\left(\pi^*\omega + \sqrt{-1}\left(\pi^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}$$
$$= \sqrt{-1}(n+1)\left(\pi^*\omega + \sqrt{-1}(\pi^*(\alpha_{\hat{\varphi}})\right)^n \wedge \sqrt{-1}\partial_t\overline{\partial}_t\varphi$$
$$- \frac{n(n+1)}{2}(\pi^*\omega + \sqrt{-1}\pi^*(\alpha_{\hat{\varphi}}))^{n-1} \wedge (\partial_t\partial_{\bar{t}}\varphi + \partial_X\partial_{\bar{t}}\varphi + \partial_t\overline{\partial}_X\varphi)^2$$

The third line becomes

$$- n(n+1)\partial_X\partial_{\bar{t}}\varphi \wedge \partial_t\overline{\partial}_X\varphi \wedge (\pi^*\omega + \sqrt{-1}\pi^*(\alpha_{\hat{\varphi}}))^{n-1}$$
$$= n(n+1)\partial_{\bar{t}}\partial_X\varphi \wedge \partial_t\overline{\partial}_X\varphi \wedge (\pi^*\omega + \sqrt{-1}\pi^*(\alpha_{\hat{\varphi}}))^{n-1}$$

After some straightforward algebra we get

$$\frac{(n+1)\sqrt{-1}dt \wedge d\bar{t}}{|t|^2} \wedge \left(\ddot{\hat{\varphi}}\sqrt{-1}\pi^*(\omega + \sqrt{-1}\alpha_{\hat{\varphi}})^n + n\sqrt{-1}\partial\dot{\hat{\varphi}} \wedge \overline{\partial}\dot{\hat{\varphi}} \wedge (\pi^*(\omega + \sqrt{-1}\alpha_{\hat{\varphi}}))^{n-1}\right)$$

Multiplying by $e^{-\sqrt{-1}\hat{\theta}}$ and taking the imaginary part yields equivalence between the geodesic equation and (2.7). $\qquad\square$

Equation (2.7) is a degenerate equation, and hence the existence and regularity of solutions is not guaranteed. In fact, following work of Rubinstein-Solomon [66], Jacob [49] showed that (2.7) is a degenerate elliptic equation which fits the Dirichlet duality theory of Harvey-Lawson [46]. Let us briefly recall how this is done. On $\mathcal{X}$ define metrics

$$(2.8) \qquad\qquad \hat{\omega}_\varepsilon := \pi_X^*\omega + \varepsilon^2\sqrt{-1}dt \wedge d\bar{t}.$$

Then,

Definition 2.9. *The (space-time lifted) degenerate Lagrangian phase operator is defined by*

$$\widetilde{\Theta}_\omega(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi) = \lim_{\varepsilon\to 0}\Theta_{\hat{\omega}_\varepsilon}(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi).$$

As shown in [66], the operator $\widetilde{\Theta}_\omega(\cdot)$ defines a degenerate elliptic operator, and the geodesic equation can be rewritten as

(2.9) $$\widetilde{\Theta}_\omega(\alpha_\varphi) = \hat{\theta}$$

When $|\hat{\theta}| > (n-1)\frac{\pi}{2}$, Jacob [49] proves existence of continuous geodesics building on work of Rubinstein-Solomon [66]. In this context, a continuous geodesic is a viscosity solution of (2.9), in the sense of Harvey-Lawson [46].

Our approach here is different, necessitated by the need for geodesics with better regularity for geometric applications. We will therefore obtain the existence of regular geodesics as limits of smooth solutions to a regularized version of (2.7) which is elliptic.

Definition 2.10. *Suppose $\varphi_0, \varphi_1 \in \mathcal{H}$. An S^1 invariant function $\varphi : \mathcal{X} \to \mathbb{R}$ is said to be an ε-regularized geodesic in the space $\mathcal{H}$ joining φ_0 and φ_1 if φ solves*

(2.10)
$$\mathrm{Im}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\pi^*\omega + \varepsilon^2\sqrt{-1}dt\wedge d\bar{t} + \sqrt{-1}\left(\pi^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right] = 0$$

$$\mathrm{Re}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\omega + \sqrt{-1}\left(\alpha + \sqrt{-1}\partial\overline{\partial}\varphi\right)\right)^n\right] > 0$$

and with $\varphi(x,1) = \varphi_0(x), \varphi(x,e^{-1}) = \varphi_1$. Here, as before, $\sqrt{-1}D\overline{D}$ denotes the $\sqrt{-1}\partial\overline{\partial}$-operator on the $n+1$ dimensional manifold $\mathcal{X}$, while $\sqrt{-1}\partial\overline{\partial}$ denotes the operator on X

Remark 2.11. To streamline some statements, we will view geodesics as ε-regularized geodesics with $\varepsilon = 0$. We will also refer to ε-regularized geodesics as ε-geodesics.

Note that (2.10) is just the deformed Hermitian-Yang-Mills equation on the manifold with boundary, defined with respect to the degenerating metric $\hat{\omega}_\varepsilon$.

Lemma 2.12. *If $\varphi(x,t)$ is an S^1 invariant solution of the deformed Hermitian-Yang-Mills equation on $(\mathcal{X},\hat{\omega}_\varepsilon)$ with $\hat{\theta} > (n-1)\frac{\pi}{2}$, then $\varphi(x,t)$ is an ε-geodesic.*

Proof. This is almost a tautology, except to show that $\varphi(x,t) \in \mathcal{H}$ for all $t \in \mathcal{A}$. That is, we need to show that

$$n\frac{\pi}{2} > \Theta_\omega(\alpha + \sqrt{-1}\partial\overline{\partial}\varphi(x,t)) > \hat{\theta} - \frac{\pi}{2}$$

for all t. The upper bound is trivial while the lower bound follows from the Schur-Horn Theorem [47] together with Lemma 3.1 (7).   □

For our purposes, the most important point of this Riemannian structure is that there exist real valued functions on $\mathcal{X}$ which are convex/concave/linear along geodesics, and which can be used to study the existence problem for the dHYM equation on X.

Definition 2.13. *The complexified Calabi-Yau functional is defined by its differential on $\mathcal{H}$ as follows. Suppose $\varphi \in \mathcal{H}$, and $\psi \in T_\varphi \mathcal{H}$, then*

$$dCY_{\mathbb{C}}(\varphi)(\psi) = \int_X \psi(\omega + \sqrt{-1}\alpha_\varphi)^n.$$

Recall that on the space of Kähler metrics in the class $[\omega]$ the Calabi-Yau functional is defined by its differential at a Kähler potential φ by

$$dCY(\varphi)(\psi) = \int_X \psi \omega_\varphi^n.$$

Hence we can view the functional $CY_{\mathbb{C}}$ as the extension of the Calabi-Yau functional to *complexified* Kähler forms in $H^{1,1}(X, \mathbb{C})$. The next proposition shows that $CY_{\mathbb{C}}$ integrates to a well-defined function.

Proposition 2.14. *The complexified Calabi-Yau functional integrates to a well-defined functional $CY_{\mathbb{C}} : \mathcal{H} \to \mathbb{C}$. If $0 \in \mathcal{H}$, then we can write $CY_{\mathbb{C}}$ explicitly as*

$$CY_{\mathbb{C}}(\varphi) = \frac{1}{n+1} \sum_{j=0}^{n} \int_X \varphi(\omega + \sqrt{-1}\alpha_\varphi)^j \wedge (\omega + \sqrt{-1}\alpha)^{n-j}.$$

Proof. Since the space $\mathcal{H}$ may not be connected, we will instead show that $CY_{\mathbb{C}}$ integrates to a well-defined functional on $C^\infty(X, \mathbb{R})$, which we then restrict to $\mathcal{H}$. Fix a base point in $\mathcal{H}$, which, by changing the background form α we may always take to be $0 \in \mathcal{H}$. Let $\varphi_1 \in C^\infty(X, \mathbb{R})$ be another potential, and suppose that $\varphi(t), \hat{\varphi}(t)$ are two paths in $C^\infty(X, \mathbb{R})$ such that $\varphi(0) = \hat{\varphi}(0) = 0$ and $\varphi(1) = \hat{\varphi}(1) = \varphi_1$. Let $\psi(t) = \hat{\varphi}(t) - \varphi(t)$, and consider $\varphi(t) + s\psi(t)$ for $s \in [0,1]$. We compute
(2.11)
$$\frac{d}{ds} \int_0^1 dt \int_X (\dot{\varphi} + s\dot{\psi})\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_{\varphi+s\psi})^n\right)$$
$$= \int_0^1 dt \int_X \dot{\psi}\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_{\varphi+s\psi})^n\right)$$
$$+ \int_0^1 dt \int_X (\dot{\varphi} + s\dot{\psi})\mathrm{Im}\left(n\sqrt{-1}e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_{\varphi+s\psi})^{n-1} \wedge \sqrt{-1}\partial\bar{\partial}\psi\right)$$

Since $\psi(0) = \psi(1) = 0$, we can integrate by parts in t in the first term on the right hand side of (2.11).

$$\int_0^1 dt \int_X \dot{\psi}\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_{\varphi+s\psi})^n\right)$$
$$= - \int_0^1 dt \int_X \psi\mathrm{Im}\left(n\sqrt{-1}e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_{\varphi+s\psi})^{n-1} \wedge \sqrt{-1}\partial\bar{\partial}(\dot{\varphi} + s\dot{\psi})\right)$$

Finally, since $\sqrt{-1}\partial\overline{\partial}$ is a real operator, integration by parts cancels the second term on the right in (2.11). To obtain the closed form take $\varphi \in \mathcal{H}$, and consider the path $t\varphi$. Then we have

$$CY_{\mathbb{C}}(\varphi)(\psi) = \int_0^1 dt \int_X \varphi(\omega + \sqrt{-1}\alpha_{t\varphi})^n$$

Writing $\omega + \sqrt{-1}\alpha_{t\varphi} = t(\omega + \sqrt{-1}\alpha_{\varphi}) + (1-t)(\omega + \sqrt{-1}\alpha)$ we get

$$CY_{\mathbb{C}}(\varphi)(\psi) = \sum_{j=0}^n \int_0^1 \binom{n}{j} t^j (1-t)^{n-j} dt \int_X \varphi(\omega + \sqrt{-1}\alpha_{\varphi})^j \wedge (\omega + \sqrt{-1}\alpha)^{n-j}$$

and the t integral is easily evaluated by induction to be $\frac{1}{n+1}$ independent of j. $\qquad\square$

Remark 2.15. The above proof shows that $CY_{\mathbb{C}}$ integrates to a well defined functional on $C^\infty(X, \mathbb{R})$, which we can then restrict to $\mathcal{H}$. This avoids obvious technical difficulties in case $\mathcal{H}$ has more than one connected component.

Following Solomon [68] we can extract two particularly useful real valued functions from $CY_{\mathbb{C}}$. Define

$$\mathcal{J}(\varphi) := -\mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}} CY_{\mathbb{C}}\right)$$
$$\mathcal{C}(\varphi) := \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}} CY_{\mathbb{C}}\right).$$

Clearly the $\mathcal{J}$ functional is precisely the Kempf-Ness functional for our infinite dimensional GIT problem. We have the following Corollary of Proposition 2.14.

Corollary 2.16. *Fix a base point $\varphi_0 \in \mathcal{H}$ and let $\varphi(t)$ be a path in $C^\infty(X, \mathbb{R})$ connecting φ_0 to $\varphi_1 \in \mathcal{H}$. Then*

$$\mathcal{J}(\varphi_1) := -\int_0^1 dt \int_X \dot\varphi \mathrm{Im}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_{\varphi(t)})^n\right)$$
$$\mathcal{C}(\varphi_1) := \int_0^1 dt \int_X \dot\varphi \mathrm{Re}\left(e^{-\sqrt{-1}\hat{\theta}}(\omega + \sqrt{-1}\alpha_{\varphi(t)})^n\right)$$

In particular $d\mathcal{J}(\varphi) = 0$ at a point $\varphi \in \mathcal{H}$ if and only if φ solves the deformed Hermitian-Yang-Mills equation.

The next proposition makes the connection with infinite dimensional GIT.

Proposition 2.17. *Let $\varphi(x, s)$ be a curve in $\mathcal{H}$, viewed as an S^1 invariant function on $\mathcal{X}$ by $s = -\log(|t|)$. Then we have the following formula for the second derivatives of $\mathcal{J}$ and $\mathcal{C}$;*

(2.12)
$$\sqrt{-1}\partial_t\partial_{\bar{t}}\mathcal{J}(\varphi(t)) = (\pi_{\mathcal{A}})_* \mathrm{Re}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\pi_X^*\omega + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right].$$

(2.13)
$$\sqrt{-1}\partial_t\partial_{\bar{t}}\mathcal{C}(\varphi(t)) = (\pi_{\mathcal{A}})_* \mathrm{Im}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\pi_X^*\omega + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right].$$

Furthermore, the functional $\mathcal{J}$ is strictly convex along non-trivial, smooth ε-geodesics for all $\varepsilon \geqslant 0$, and $\mathcal{C}$ is affine along ε-geodesics for all $\varepsilon \geqslant 0$.

Proof. Suppose that $\varphi(s)$ is a smooth ε-regularized geodesic in $\mathcal{H}$. We compute

$$
\frac{d^2}{ds^2}CY_{\mathbb{C}}(\varphi(s)) = \int_X \left(\frac{d^2}{ds^2}\varphi\right)\left(\omega + \sqrt{-1}\alpha_{\varphi(s)}\right)^n
$$
$$
+ \int_X \left(\frac{d}{ds}\varphi\right) n\left((\omega + \sqrt{-1}\alpha_{\varphi(s)})^{n-1} \wedge \sqrt{-1}\cdot\sqrt{-1}\partial\bar\partial\left(\frac{d}{ds}\varphi\right)\right)
$$
$$
= \int_X \left(\frac{d^2}{ds^2}\varphi\right)\left(\omega + \sqrt{-1}\alpha_{\varphi(s)}\right)^n
$$
$$
- \sqrt{-1}\int_X n\sqrt{-1}\partial\left(\frac{d}{ds}\varphi\right) \wedge \bar\partial\left(\frac{d}{ds}\varphi\right) \wedge (\omega + \sqrt{-1}\alpha_{\varphi(s)})^{n-1}
$$

From this expression one can easily check directly using local coordinates that $\mathcal{J}$ is convex along geodesics, and $\mathcal{C}$ is affine. However, it is useful to rewrite this expression on $\mathcal{X}$. Let $\pi_X : \mathcal{X} \to X$ be the projection to X, and $\pi_{\mathcal{A}}$ be the projection to annulus. Recall that $s = -\log(|t|)$ where t is a coordinate on $\mathbb{C}$. We can rewrite the above expression using the computation in Lemma 2.8 as

$$
(2.14) \quad \sqrt{-1}\partial_t\partial_{\bar t}CY_{\mathbb{C}} = -\sqrt{-1}(\pi_{\mathcal{A}})_*\left(\pi_X^*\omega + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}
$$

where $(\pi_{\mathcal{A}})_*$ denotes the push forward along the fibration $\mathcal{X} \to \mathcal{A}$ (ie. integration along fibers). Taking the real and imaginary parts of this expression proves (2.12) and (2.13). In order to establish the convexity of $\mathcal{J}$ along ε-regularized geodesics we need to evaluate the sign of

$$
\mathrm{Re}\left[e^{-\sqrt{-1}\hat\theta}\left(\pi_X^*\omega\sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right]
$$
$$
= \mathrm{Re}\left[e^{-\sqrt{-1}\hat\theta}\left(\pi_X^*\omega + \varepsilon^2\sqrt{-1}dt\wedge d\bar t + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right]
$$
$$
- (n+1)\varepsilon^2\sqrt{-1}dt\wedge d\bar t \wedge \pi_X^*\mathrm{Re}\left[e^{-\sqrt{-1}\hat\theta}\left(\omega + \sqrt{-1}\left(\alpha + \sqrt{-1}\partial\bar\partial\varphi\right)\right)^n\right].
$$

In particular, it suffices to show that the fibre integral of the right hand side is positive. By (2.1) we have

$$
\mathrm{Re}\left[e^{-\sqrt{-1}\hat\theta}\left(\pi_X^*\omega + \varepsilon^2\sqrt{-1}dt\wedge d\bar t + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right]
$$
$$
= (n+1)R(\alpha_\varphi)\varepsilon^2\sqrt{-1}dt\wedge d\bar t \wedge \omega^n
$$

where we used that $\Theta_{\hat\omega_\varepsilon}(\alpha_\varphi) = \hat\theta$ since φ is an ε-geodesic. Here $R(\alpha_\varphi)$ denotes the radius function computed on $(\mathcal{X}, \hat\omega_\varepsilon)$. On the other hand, we have

$$
\mathrm{Re}\left[e^{-\sqrt{-1}\hat\theta}\left(\omega + \sqrt{-1}\left(\alpha + \sqrt{-1}\partial\bar\partial\varphi\right)\right)^n\right] = r(\varphi)\cos\left(\theta_\omega(\alpha_\varphi|_X) - \hat\theta\right)\omega^n
$$

where $(\alpha_\varphi)|_X$ denotes the restriction of α_φ to $TX \subset T\mathcal{X}$, and $r(\varphi)$ is the radius function computed on (X, ω). Fix a point $(p_0, t_0) \in \mathcal{X}$, and choose

holomorphic normal coordinates $(w_1, \ldots, w_n)$ for (X, ω) near p_0 so that at (p_0, t_0) we have $\omega_{\bar{j}i} = \delta_{\bar{j}i}$, and $(\alpha + \sqrt{-1}\partial\bar{\partial}\varphi) = \lambda_i \delta_{\bar{j}i}$. Complete this to a set of coordinates on $\mathcal{X}$ by setting $w_0 = t - t_0$. We can then write

$$\alpha + \sqrt{-1}D\overline{D}\varphi = \left(\begin{array}{c|ccc} a_{00} & & \vec{a}_0{}^\dagger & \\ \hline & \lambda_1 & \cdots & 0 \\ \vec{a}_0 & \vdots & \ddots & \vdots \\ & 0 & \cdots & \lambda_n \end{array}\right), \qquad \hat{g}_\varepsilon^{-1} = \left(\begin{array}{c|c} \frac{1}{\varepsilon^2} & 0 \\ \hline 0 & 1_n \end{array}\right)$$

where 1_n is the $n \times n$ identity matrix. By definition $r(\varphi)^2 = \prod_{i=1}^n (1 + \lambda_i^2)$, while

$$1 + \hat{g}_\varepsilon^{-1}\alpha_\varphi \hat{g}_\varepsilon^{-1}\alpha_\varphi = \left(\begin{array}{c|c} 1 + \frac{1}{\varepsilon^4}a_{00}^2 + \frac{1}{\varepsilon^2}|\vec{a}_0|^2 & \frac{a_{00}}{\varepsilon^4}\vec{a}_0{}^\dagger + \frac{1}{\varepsilon^2}\vec{a}_0{}^\dagger D \\ \hline \frac{a_{00}}{\varepsilon^2}\vec{a}_0 + D\vec{a}_0 & 1_n + \frac{1}{\varepsilon^2}\vec{a}_0\vec{a}_0{}^\dagger + D^2 \end{array}\right)$$

The function $R(\varphi)^2$ is the determinant of this matrix. Expanding the determinant along the top row we have

$$R(\varphi)^2 = 1 \cdot \det\left(1_n + \frac{1}{\varepsilon}\vec{a}_0\vec{a}_0{}^\dagger + D^2\right) + \det\left(1_n + \hat{g}_\varepsilon^{-1}\alpha_\varphi \hat{g}_\varepsilon^{-1}\alpha_\varphi\right)$$

where, in the second expression have abusively written

$$1_n = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1_n \end{array}\right).$$

Clearly $1_n + \hat{g}_\varepsilon^{-1}\alpha_\varphi \hat{g}_\varepsilon^{-1}\alpha_\varphi$ is a non-negative definite matrix, and

$$1_n + \frac{1}{\varepsilon}\vec{a}_0\vec{a}_0{}^\dagger + D^2 \geqslant 1_n + D^2.$$

So $R(\varphi) \geqslant \sqrt{\det(1_n + D^2)} = r(\varphi) \geqslant r(\varphi)\cos\left(\Theta_\omega(\alpha_\varphi|_X) - \hat{\theta}\right)$. It follows that

$$(\pi_{\mathcal{A}})_* \mathrm{Re}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\pi_X^*\omega + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right] \geqslant 0$$

in the sense of currents, and so $\mathcal{J}$ is convex along ε-geodesics. Furthermore, if we have equality in the above computation at some point (p_0, t_0) then it is easy to see that we must have $a_{00} = 0 = \vec{a}_0$, and $\theta_\omega(\alpha_\varphi|_X) - \hat{\theta} = 0$. It follows that if $\varepsilon > 0$, and $\mathcal{J}$ is not strictly convex along an ε-geodesics, then $\alpha_\varphi(x, t) = \alpha_\varphi(x, 0)$ is the constant ε-geodesic emanating from a solution of dHYM. Finally, when $\varepsilon = 0$ one can either compute directly, or take a limit as $\varepsilon \to 0$ in the above argument (though this does not give the strict convexity statement).

Next we show that $\mathcal{C}$ is affine along ε-geodesics. We need to show that

$$0 = (\pi_{\mathcal{A}})_* \mathrm{Im}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\pi_X^*\omega + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right]$$

$$= (\pi_{\mathcal{A}})_* \mathrm{Im}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\pi_X^*\omega + \varepsilon^2\sqrt{-1}dt \wedge d\bar{t} + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right]$$

$$- (n+1)\varepsilon^2\sqrt{-1}dt \wedge d\bar{t} \int_X \mathrm{Im}\left[e^{-\sqrt{-1}\hat{\theta}}\left(\omega + \sqrt{-1}\left(\alpha + \sqrt{-1}\partial\bar{\partial}\varphi\right)\right)^n\right]$$

By the definition of an ε-geodesic the term on the second line is zero, and the term on the third line vanishes by the definition of $\hat{\theta}$. $\qquad\square$

As a consequence of Proposition 2.17 we get a whole S^1 worth of interesting functionals on the space $\mathcal{H}$; namely $e^{\sqrt{-1}\xi}CY_{\mathbb{C}}$. These functionals are either convex or concave along ε-geodesics depending on the choice of ξ. We point out one further functional which will be useful later on.

Definition 2.18. *Suppose that* $\hat{\theta} \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2})$. *We define the Z-functional for* $[\alpha], [\omega]$ *by*

$$(2.15) \qquad\qquad Z(\varphi) = e^{-\sqrt{-1}n\frac{\pi}{2}}CY_{\mathbb{C}}$$

The variation of Z at $\varphi \in \mathcal{H}$ is given by

$$\delta Z(\varphi) = \int_X (\delta\varphi)\mathrm{Re}\left(e^{-\sqrt{-1}\frac{n\pi}{2}}(\omega + \sqrt{-1}\alpha_\varphi)^n\right)$$
$$+ \sqrt{-1}\int_X (\delta\varphi)\mathrm{Im}\left(e^{-\sqrt{-1}\frac{n\pi}{2}}(\omega + \sqrt{-1}\alpha_\varphi)^n\right).$$

If $\varphi \in \mathcal{H}$ then

$$(2.16) \qquad \mathrm{Im}\left(e^{-\sqrt{-1}\frac{n\pi}{2}}(\omega + \sqrt{-1}\alpha_\varphi)^n\right) = r(\varphi)\sin\left(\theta_\omega(\alpha_\varphi) - \frac{n\pi}{2}\right)\omega^n$$

is a negative measure. Furthermore, if φ is a solution of the deformed Hermitian-Yang-Mills equation, then

$$\mathrm{Re}\left(e^{-\sqrt{-1}\frac{n\pi}{2}}(\omega + \sqrt{-1}\alpha_\varphi)^n\right)$$

is a positive measure. Writing Z in terms of $\mathcal{C}, \mathcal{J}$ and applying Proposition 2.17 gives

Corollary 2.19. *The functionals* $\mathrm{Re}(Z), \mathrm{Im}(Z)$ *are concave along smooth* ε-*geodesics.*

From now on we will restrict to the "hypercritical phase" case,

$$\hat{\theta} \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2}).$$

This is used crucially in the analysis in the remainder of the paper. In the sequel [26] we will comment briefly on the new difficulties and phenomena in the case of lower phase. We remark that all our results work just as well under the assumption that $\hat{\theta} \in (-n\frac{\pi}{2}, -(n-1)\frac{\pi}{2})$.

As in classical GIT, the way to link existence of solutions of the dHYM equation with algebraic geometry is via the function $\mathcal{J}$, which can be regarded as a Kempf-Ness type functional. If there is a solution φ_0 of the deformed Hermitian-Yang-Mills equation in $\mathcal{H}$, then for every infinite length (smooth) (ε)-geodesic $\varphi(s)$ emanating from φ_0 we must have

$$0 < \lim_{s\to\infty}\frac{d}{ds}\mathcal{J}(\varphi(s)).$$

In special cases we can evaluate the right hand side as an algebraic invariant, and this gives rise to algebraic obstructions for the existence of solutions to dHYM. The two main difficulties in executing this approach are the lack of smooth geodesics, and the evaluation of the limit slope in terms of algebraic data. In this paper we will essentially completely resolve the first issue when $\hat{\theta} \in ((n-1)\frac{\pi}{2}, n\frac{\pi}{2})$, and in the sequel [26] we will use the existence of regular geodesics and algebro-geometric techniques to evaluate the slope of the functional at infinity in a rather general setting. Furthermore, in the sequel [26] we will explain how the resulting invariants can be interpreted as Bridgeland stability type obstructions. In dimension 3 this will give a relatively complete picture relating the existence of solutions to dHYM and Bridgeland type stability conditions.

Before explaining the plan of attack, let us make a few formal remarks about the Riemannian structure and functionals considered here. Suppose that $[\alpha]$ is a Kähler class, and rescale $\omega \mapsto t\omega$ for $t > 0$. Then we get a family of infinite dimensional Riemannian manifolds $(\mathcal{H}_t, g_t)$, and functionals $\mathcal{J}_t, \mathcal{C}_t$. It is not hard to show that, as $t \to 0$, we have $\hat{\theta}(t) \to n\frac{\pi}{2}$, and so in the "small radius limit" we have

$$\langle \psi_1, \psi_2 \rangle_t \approx \int_X \psi_1 \psi_2 \alpha_\varphi^n + O(t)$$

which is precisely the Donaldson-Mabuchi-Semmes Riemmanian structure on the space of Kähler metrics in the class $[\alpha]$. Similarly we have

$$d(CY_{\mathbb{C}}(t))(\varphi) = (\sqrt{-1})^n \int_X (\delta\varphi)\alpha_\varphi^n + t(\sqrt{-1})^{n-1} \int_X (\delta\varphi) n\alpha_\varphi^{n-1} \wedge \omega + O(t^2)$$

and so $\mathcal{C}_t$ approaches the classical Calabi-Yau functional, while $\mathcal{J}_t$ approaches the J functional of Donaldson [34] and Chen [18].

On the other hand, in the "large radius limit", as $t \to +\infty$ we have that $\hat{\theta} \to 0$, and so

$$\frac{1}{t^n} \langle \psi_1, \psi_2 \rangle_t \approx \int_X \psi_1 \psi_2 \omega^n + O(\frac{1}{t})$$

and so the Riemannian structure converges to the flat metric, while

$$\frac{1}{t^n} d(CY_{\mathbb{C}}(t))(\varphi) = \int_X (\delta\varphi)\alpha_\varphi^n + \frac{1}{t}\sqrt{-1} \int_X (\delta\varphi) n\omega^{n-1} \wedge \alpha_\varphi + O(\frac{1}{t^2}).$$

If $[\alpha] = c_1(L)$, the large radius limit yields the Riemannian metric on the space of hermitian metrics on L, and $\mathcal{J}_t$ converges to the Donaldson functional, which is the Kempf-Ness functional for the infinite dimensional GIT framework related to the Hermitian-Yang-Mills equation (albeit on a line bundle).

The next four sections of the paper will be devoted to proving the existence of smooth ε-geodesics, and $C^{1,\alpha}$ geodesic segments in the space $\mathcal{H}$. Our plan of attack is the following; rescale $\mathbb{C}$ by $t \mapsto \varepsilon t$. The ε-geodesic

equation becomes

$$(2.17) \quad \text{Im} \left[e^{-\sqrt{-1}\hat{\theta}} \left(\pi_X^* \omega + \sqrt{-1} dt \wedge d\bar{t} + \sqrt{-1} \left(\alpha + \sqrt{-1} D\overline{D}\varphi \right) \right)^n \right] = 0$$

$$(2.18) \qquad \text{on } \mathcal{X}_\varepsilon = X \times \mathcal{A}_\varepsilon := X \times \{ \varepsilon e^{-1} \leqslant |t| \leqslant \varepsilon \}$$

$$(2.19) \qquad \text{with } \varphi_0 = \varphi \Big|_{|t|=\varepsilon}, \qquad \varphi_1 = \varphi \Big|_{|t|=e^{-1}\varepsilon}$$

where $\varphi_0, \varphi_1 \in \mathcal{H}$. In particular, rather than work on a fixed manifold with a degenerating metric, we choose to work with a non-degenerate metric at the expense of working on a very "thin" manifold. In order solve the geodesic equation we need to pass to the limit as $\varepsilon \to 0$. This will be possible if we can prove that the solution of (2.17) satisfies estimates of the form

$$|\nabla^X \varphi|_{\hat{\omega}} \leqslant C, \qquad\qquad |\nabla_t \varphi|_{\hat{\omega}} \leqslant \frac{C}{\varepsilon}$$

$$|\nabla^X \overline{\nabla^X} \varphi|_{\hat{\omega}} \leqslant C, \qquad |\nabla_t \overline{\nabla^X} \varphi|_{\hat{\omega}} \leqslant \frac{C}{\varepsilon}, \quad |\nabla_t \nabla_{\bar{t}} \varphi| \leqslant \frac{C}{\varepsilon^2}$$

where $\nabla^X, \overline{\nabla^X}$ denote the covariant derivative along the fibers of $\mathcal{X}_\varepsilon \to \mathcal{A}_\varepsilon$, and C is a uniform constant independent of ε. If this is possible, then the rescaled solutions $\tilde{\varphi} = \varphi(x, \varepsilon t)$ will solve the ε-geodesic equation on $(\mathcal{X}, \hat{\omega}_\varepsilon)$ and be uniformly bounded with respect to the non-degenerate metric $(\mathcal{X}, \hat{\omega})$. We can then pass to the limit to obtain weak solutions of the geodesic equation with $C^{1,\alpha}$ regularity.

Before beginning the proof we need a few easy lemmas regarding the geometry of the manifolds $(\mathcal{X}_\varepsilon, \hat{\omega})$. Throughout the paper we will use the following terminology.

Definition 2.20. *A set of space-time adapted coordinates for $(\mathcal{X}_\varepsilon, \hat{\omega})$, α_φ centered at (p_0, t_0) is the following.*

- *A set of holomorphic normal coordinates $(w_1, \ldots, w_n)$ for (X, ω) centered at p_0 making $\omega_{\bar{j}i} = \delta_{\bar{j}i}$, and $(\alpha_\varphi)_{\bar{j}i} = \lambda_i \delta_{\bar{j}j}$.*
- *The coordinate $w_0 = t - t_0$*

Note that space-time adapted coordinates are, in particular, holomorphic normal coordinates for $(\mathcal{X}_\varepsilon, \hat{\omega})$.

Lemma 2.21. *The manifold $(\mathcal{X}_\varepsilon, \hat{\omega})$ satisfies the following properties*

(1) *The Riemann curvature tensor satisfies $R(\partial_t, \cdot, \cdot, \cdot) = 0$. In particular, in space-time adapted coordinates we have*

$$R_{\bar{j}i\bar{k}p} = 0$$

whenever one of $i, j, k, p = 0$.

(2) *The vector fields $\partial_t, \partial_{\bar{t}}$ are parallel.*

Proof. The proof is trivial. Pick (p_0, t_0) and choose space-time adapted coordinates $(w_0, \ldots, w_n)$ on an open ball B. Both statements follow from the fact that $g_{\bar{0}0} \equiv 1$ on B, $g_{\bar{0}j} \equiv 0$ if $j \neq 0$. $\qquad\square$

3. Analytic Preliminaries

Following the discussion in the previous section, the construction of geo-
desic segments in the space $\mathcal{H}$ will rely on solving the deformed Hermitian-
Yang-Mills equation on thin manifolds with boundary. Let M be a $(n+1) \times$
$(n + 1)$ Hermitian matrix. In order to simplify the notation, and to avoid
confusion with the Lagrangian phase operator on X, we will denote

$$F(M) = \sum_{i=0}^{n} \arctan(\mu_i)$$

where μ_i are the eigenvalues of M. On $\mathcal{X}_\varepsilon$ we will write $\alpha_\varphi = \alpha + \sqrt{-1}D\overline{D}\varphi$.
Over the next four sections we will prove a priori estimates for functions
$\varphi : \mathcal{X}_\varepsilon \to \mathbb{R}$ such that

$$F\left((\hat{\omega})^{-1}(\alpha_\varphi)\right) = h(x, |t|)$$

(3.1)
$$\text{with } \varphi_0 = \varphi\bigg|_{|t|=\varepsilon}, \qquad \varphi_1 = \varphi\bigg|_{|t|=e^{-1}\varepsilon}$$

where $h(x, |t|) : \mathcal{X}_\varepsilon \to \mathbb{R}$ is some given S^1 invariant function, and $\varphi_0, \varphi_1 \in \mathcal{H}$.
We remark that, since the boundary data is clearly S^1 invariant it follows
from the maximum principle that the solution to (3.1) is S^1 invariant as
well. We will impose three extra mild structural conditions that the data
must satisfy:

(C1) There is a constant $\eta_1 > 0$ such that

(3.2)
$$(n-1)\frac{\pi}{2} + \eta_1 \leqslant \Theta_\omega(\alpha_{\varphi_i}) < n\frac{\pi}{2}, \quad i = 0, 1.$$

(C2) There is a constant $\eta_2 > 0$ such that

$$h : \mathcal{X}_\varepsilon \to [(n-1)\frac{\pi}{2} + \eta_1, (n+1)\frac{\pi}{2} - \eta_2]$$

where h is an S^1 invariant function satisfying

(3.3)
$$\Theta_\omega((\alpha_{\varphi_i})) \geqslant h(x, |t|) - \frac{\pi}{2} + \eta_1$$

for $i = 0, 1$.

Condition (C1) is automatic, since the boundary data $\varphi_i \in \mathcal{H}$. Condition
(C2) is also automatically satisfied when $h = \hat{\theta}$, but in order to use the
method of continuity we need to consider the Lagrangian phase equation
with non-constant right hand side. The estimates will exploit several prop-
erties of F, the most basic of which are its first and second derivatives. It is
straightforward to compute that, at a diagonal matrix M, the linearization
DF, and the Hessian D^2F are given by

$$DF(M)(A) = \sum_i \frac{1}{1+\mu_i^2} A_{\bar{i}i}, \quad D^2F(M)(A, A) = \sum_{i,j=0}^{n} \frac{\mu_i + \mu_i}{(1+\mu_i^2)(1+\mu_j^2)}|A_{\bar{j}i}|^2.$$

where A is a Hermitian matrix. The next lemma summarizes the properties of F we will need.

Lemma 3.1. *Suppose $\mu_0 \geqslant \mu_1 \geqslant \cdots \geqslant \mu_n$ are such that $\sum_{i=0}^n \arctan(\mu_i) \geqslant (n-1)\frac{\pi}{2} + \eta_1$ for some $\eta_1 > 0$. The following properties hold,*

(1) *$\mu_0 \geqslant \mu_1 \geqslant \cdots \geqslant \mu_{n-1} > 0$ and $|\mu_n| \leqslant \mu_{n-1}$.*
(2) *If $\mu_n \leqslant 0$, then $\mu_n + \mu_{n-1} \geqslant \tan(\eta)$, and if $\mu_n \geqslant 0$, then $\mu_{n-1} \geqslant \tan(\frac{\eta_1}{2})$.*
(3) *$\sum_{i=0}^n \mu_i > 0$.*
(4) *$\mu_n \geqslant -C(\eta_1)$.*
(5) *If $\sum_{i=0}^{n+1} \arctan(\mu_i) \leqslant (n+1)\frac{\pi}{2} - \eta_2$ then $\mu_n \leqslant C(\eta_2)$.*
(6) *If $\mu_n < 0$, then $\sum_{i=0}^n \frac{1}{\mu_i} < -\tan(\eta)$.*
(7) *For any $\sigma \in [(n-1)\frac{\pi}{2}, (n+1)\frac{\pi}{2})$, the set*

$$\Gamma^\sigma := \{M \in \mathrm{Herm}(n+1) : F(M) \geqslant \sigma\}$$

is convex with boundary a smooth, convex hypersurface.
(8) *There exists a constant A depending on η_1 such that the function*

$$\mu \longmapsto -e^{-A\sum_i \arctan(\mu_i)}$$

is concave on the set $\Gamma^{\sigma+\eta_1}$.

Proof. We refer the reader to [79, 77] for properties (1), (3), (4), (6), (7), while property (8) was observed by the first author with Picard and Wu in [24]. Property (2) is implicit in [79]. First note that

$$(n-1)\frac{\pi}{2} + \arctan(\mu_{n-1}) + \arctan(\mu_n) \geqslant \sum_{i=0}^n \arctan(\mu_i) \geqslant (n-1)\frac{\pi}{2} + \eta_1$$

and hence $\arctan(\mu_{n-1}) + \arctan(\mu_n) \geqslant \eta_1$. Now if $\mu_n < 0$, the sum on the left hand side of the inequality lies in $[\eta_1, \frac{\pi}{2})$, and so by the arctan addition formula we get

$$\arctan\left(\frac{\mu_{n-1} + \mu_n}{1 - \mu_{n-1}\mu_n}\right) \geqslant \eta_1$$

Now $1 - \mu_{n-1}\mu_n > 1$, so we obtain $\mu_{n-1} + \mu_n \geqslant \tan(\eta_1)$. On the other hand, if $\mu_n \geqslant 0$, then we have $2\arctan(\mu_{n-1}) \geqslant \eta_1$, which is the desired estimate. Property (5) is trivial. □

Fix the following notation. Let $\Gamma_{n+1} = \mathbb{R}_{>0}^{n+1}$ be the positive orthant, and let $\Gamma \subset \mathbb{R}^{n+1}$ be the cone over the set $\{\mu \in \mathbb{R}^{n+1} : F(\mu) \geqslant (n-1)\frac{\pi}{2}\}$ and through the origin. By Lemma 3.1 (3) and (6), Γ is an open convex cone contained in

$$\{\mu \in \mathbb{R}^{n+1} : \sum_{i=0}^n \mu_i > 0\}$$

The following definition is due to Székelyhidi [71], building on work of Guan [42].

Definition 3.2. *A smooth function $\varphi : \mathcal{X}_\varepsilon \to \mathbb{R}$ is a $\mathcal{C}$-subsolution of* (3.1) *if, at each point p the following holds: Define $\underline{\mathcal{E}}^i_j = (\hat{\omega})^{i\bar{k}}(\alpha_\varphi)_{\bar{k}j}$. Then the set*

$$\left\{ \mu' \in \Gamma : \sum_{i=0}^{n} \arctan(\mu_i) = h(p), \quad and \ \mu' - \mu(\underline{\mathcal{E}}(p)) \in \Gamma_{n+1} \right\}$$

is bounded. Here $\mu(\underline{\mathcal{E}}(p))$ denotes the eigenvalues of $\underline{\mathcal{E}}(p)$.

The following lemma from [23] gives a simple criterion for a function φ to be a $\mathcal{C}$-subsolution.

Lemma 3.3. *A function φ is a $\mathcal{C}$-subsolution of the equation* (3.1) *if the following holds; at each point $p \in \mathcal{X}_\varepsilon$, if $\mu_0, \ldots, \mu_n$ denote the eigenvalues of $\hat{\omega}^{-1}\alpha_\varphi$, then, for any $j = 0, \ldots, n$ we have*

$$\sum_{\ell \neq j} \arctan(\mu_\ell) > h(p) - \frac{\pi}{2}.$$

Remark 3.4. *Trivially, if φ is a subsolution satisfying $F((\hat{\omega})^{-1}\alpha_\varphi) \geqslant h(x)$, then φ is a $\mathcal{C}$-subsolution.*

The main property of $\mathcal{C}$-subsolutions that we will need is the following

Lemma 3.5 (Székelyhidi [71], Proposition 6). *Let $[a,b] \subset ((n-1)\frac{\pi}{2}, (n+1)\frac{\pi}{2})$, and $\delta, R > 0$. There exists a constant $\kappa_0 > 0$, depending only on a, b, R, δ with the following property. Suppose that $\sigma \in [a,b]$ and A is a Hermitian matrix such that*

$$(\mu(A) - 2\delta I + \Gamma_{n+1}) \cap \partial\Gamma^\sigma \subset B_R(0).$$

Then for any Hermitian matrix M with $\mu(M) \in \partial\Gamma^\sigma$, and $|\mu(M)| > R$ we either have

$$\sum_{p,q} F^{p\bar{q}}(M)\left(A_{\bar{q}p} - M_{\bar{q}p}\right) > \kappa_0 \sum_p F^{p\bar{p}}(M)$$

or $F^{i\bar{i}}(M) > \kappa_0 \sum_p F^{p\bar{p}}(M)$ for all $0 \leqslant i \leqslant n$. Here $F^{p\bar{q}}$ are the coefficients of the linearized operator of F.

Note that if φ is a given subsolution, then it is a $\mathcal{C}$-subsolution, and for each point $p \in \mathcal{X}_\varepsilon$ we can choose $\delta = \delta(p) > 0$, and $R = R(p) > 0$ depending only on $|\sqrt{-1}D\overline{D}\varphi(p)|_{\hat{\omega}}, a, b$ so that we have

$$\left[\mu((\hat{\omega})^{-1}\alpha_\varphi) - 2\delta I + \Gamma_{n+1} \right] \cap \partial\Gamma^\sigma \subset B_R(0)$$

for all $\sigma \in [a,b]$. In particular, the constants $\kappa_0 = \kappa_0(p), R(p)$ depend only on $|\sqrt{-1}D\overline{D}\varphi(p)|_{\hat{\omega}}, a, b$.

In order to solve (3.1) on manifolds with boundary, it is essential to construct a subsolution of the equation with given boundary data. In the case of domains in $\mathbb{C}^n$ this was explained by the first author, Picard and Wu [24], building on work of Guan [43]. We prove the following

Lemma 3.6. *Fix two functions $\varphi_0, \varphi_1 \in \mathcal{H}$, and a function $h : \mathcal{X}_\varepsilon \to \mathbb{R}$. Suppose that this data satisfies structural conditions (C1) and (C2) (see (3.2) and (3.3)). Then there exists a smooth, S^1-invariant function $\underline{\varphi}$ such that $\underline{\alpha} := \pi_X^* \alpha + \sqrt{-1}\partial\bar{\partial}\underline{\varphi}$ satisfies*

- *(i) $F(\underline{\alpha}) \geqslant h(p) + \frac{\eta_1}{2}$ for all $p \in \mathcal{X}_\varepsilon$.*
- *(ii) $\underline{\varphi}|_{\{|t|=\varepsilon\}} = \varphi_0$*
- *(iii) $\underline{\varphi}|_{\{|t|=e^{-1}\varepsilon\}} = \varphi_1$*
- *(iv) On $\partial\mathcal{X}_\varepsilon$ we have*

$$\left|\frac{\partial}{\partial|t|}\underline{\varphi}\right| \leqslant \frac{C}{\varepsilon}, \qquad \left|\frac{\partial^2}{\partial|t|^2}\underline{\varphi}\right| \leqslant \frac{C}{\varepsilon^2}$$

for a constant C depending only on η_1 and $\|\varphi_0 - \varphi_1\|_{L^\infty(X)}$.

Before proving the lemma, we prove a fairly general result which we hope will be of use elsewhere. Let us first recall Demailly's regularized maximum construction [33]. Fix $\theta : \mathbb{R} \to \mathbb{R}$ a smooth, even, positive function with support in $[-1, 1]$ and such that

$$\int_{[-1,1]} \theta(x)dx = 1.$$

For $\delta > 0$, and $(t_0, t_1) \in \mathbb{R}^2$ define

$$M_\delta(t_0, t_1) = \int_{\mathbb{R}^2} \max\{t_0 + h_0, t_1 + h_1\} \frac{1}{\delta}\theta\left(\frac{h_0}{\delta}\right)\frac{1}{\delta}\theta\left(\frac{h_1}{\delta}\right) dh_0 dh_1.$$

The function M_δ has the following properties [33]

- M_δ is non-decreasing in t_i for $i = 0, 1$, smooth and convex.
- $\max\{t_0, t_1\} \leqslant M_\delta(t_0, t_1) \leqslant \max\{t_0 + \delta, t_1 + \delta\}$
- if $t_1 + \delta \leqslant t_0 - \delta$ then $M_\delta(t_0, t_1) = t_0$, and vice versa.
- $M_\delta(t_0 + a, t_1 + a) = M_\delta(t_0, t_1) + a$. In particular

$$\nabla_{(1,1)}M_\delta = \frac{\partial M_\delta}{\partial t_0} + \frac{\partial M_\delta}{\partial t_1} = 1.$$

Let ψ_0, ψ_1 be two C^2 functions. We are going to compute two derivatives of $M(\psi_0, \psi_1)$. We compute

$$\bar{\partial}M_\delta(\psi_0, \psi_1) = \frac{\partial M_\delta}{\partial t_0}(\psi_0, \psi_1)\bar{\partial}\psi_0 + \frac{\partial M_\delta}{\partial t_1}(\psi_0, \psi_1)\bar{\partial}\psi_1$$

and so
(3.4)
$$\partial\bar{\partial}M_\delta(\psi_0, \psi_1) = \frac{\partial M_\delta}{\partial t_0}\partial\bar{\partial}\psi_0 + \frac{\partial M_\delta}{\partial t_1}\partial\bar{\partial}\psi_1 + (\partial\psi_0, \partial\psi_1)D^2 M(\partial\psi_0, \partial\psi_1)^\dagger.$$

We now make two observations. First, by the convexity of M, the second term on the right hand side is non-negative. Secondly, by combining the first and last properties of M_δ the first term on the right hand of (3.4) is a convex combination of $\partial\bar{\partial}\psi_0$, and $\partial\bar{\partial}\psi_1$. We conclude the following very general lemma.

Lemma 3.7. *Suppose $F : \mathrm{Herm}(n) \to \mathbb{R}$ is an elliptic operator such that $\{F \geqslant 0\}$ is convex. Suppose ψ_0, ψ_1 are two functions satisfying $F(\partial\overline{\partial}\psi_i) \geqslant 0$ for $i = 0, 1$. Then*

$$F\left(\partial\overline{\partial}M_\delta(\psi_0, \psi_1)\right) \geqslant 0$$

Proof. By the above computation, using the ellipticity of F we have

$$F\left(\partial\overline{\partial}M_\delta(\psi_0, \psi_1)\right) \geqslant F(t\partial\overline{\partial}\psi_0 + (1-t)\partial\overline{\partial}\psi_1)$$

for some $0 \leqslant t \leqslant 1$. Now $F(\partial\overline{\partial}\psi_i) \geqslant 0$ and $\{F \geqslant 0\}$ convex implies the result. $\qquad\square$

Remark 3.8. The result also holds for real functions, with $\partial\overline{\partial}\psi_i$ replaced by $D^2\psi_i$, provided F is elliptic on the symmetric matrices and $\{F \geqslant 0\}$ is convex.

We can now give the proof of Lemma 3.6.

Proof of Lemma 3.6. Consider the function

$$\psi_0(x, t) = \varphi_0(x) + A_0(|t|^2 - \varepsilon^2) + C_0 \log\left(\frac{|t|^2}{\varepsilon^2}\right)$$

Observe that $\psi_0(x, t) = \varphi_0(x)$ whenever $|t| = \varepsilon$. Furthermore we have

$$\sqrt{-1}D\overline{D}\psi_0 = \sqrt{-1}\partial\overline{\partial}\varphi_0 + A_0\sqrt{-1}dt \wedge d\bar{t}$$

and so

$$F(\alpha_{\psi_0}) = \Theta_\omega(\alpha_{\varphi_0}) + \arctan(A_0)$$

and so if we choose A_0 sufficiently large so that $\arctan(A_0) > \frac{\pi}{2} - \frac{\eta_1}{2}$, then

$$F(\alpha_{\psi_0}) \geqslant h(x, |t|) - \frac{\pi}{2} + \eta_1 + \frac{\pi}{2} - \frac{\eta_1}{2} = h(x, |t|) + \frac{\eta_1}{2}.$$

Similarly define

$$\psi_1(x, t) = \varphi_1(x) + A_1(|t|^2 - e^{-2}\varepsilon^2) - C_1 \log\left(\frac{e^2|t|^2}{\varepsilon^2}\right).$$

Note that $\psi_1(x, t) = \varphi_1(x)$ whenever $|t| = e^{-1}\varepsilon$. A similar computation shows that we can choose A_1 large depending only on η_1 so that $F(\alpha_{\psi_1}) \geqslant h(x, |t|) + \frac{\eta_1}{2}$. Now we have

$$\psi_0(x, t)|_{\{|t| = e^{-1}\varepsilon\}} = \varphi_0(x) - A_0\varepsilon^2(1 - e^{-2}) - 2C_0$$

Choose $C_0 \geqslant \|\varphi_0 - \varphi_1\|_{L^\infty(X)} + 1$, so that

$$\psi_0(x, t)|_{\{|t| = e^{-1}\varepsilon\}} \leqslant \varphi_1(x) - 1 = \psi_1(x)|_{\{|t| = e^{-1}\varepsilon\}} - 1.$$

Similarly

$$\psi_1(x, t)|_{\{|t| = \varepsilon\}} = \varphi_1(x) + A_1\varepsilon^2(1 - e^{-2}) - 2C_1$$

Choosing C_1 large depending on $\|\varphi_0 - \varphi_1\|_{L^\infty(X)}$, and A_1 we can ensure that

$$\psi_1(x, t)|_{\{|t| = \varepsilon\}} \leqslant \varphi_0 - 1 = \psi_0(x)|_{\{|t| = \varepsilon\}} - 1$$

Now set $\underline{\varphi} = M_\delta(\psi_0, \psi_1)$ for $0 < \delta \ll 1$. Since the sublevel sets of the angle operator Γ^σ are convex when $\sigma \geqslant (n-1)\frac{\pi}{2}$, we can apply the above construction using Demailly's regularized maximum to conclude that $\underline{\varphi}$ satisfies (i). Furthermore, $\underline{\varphi}$ satisfies properties $(ii), (iii)$ and the first part of (iv) of the lemma by construction. It remains only to prove the estimates in (iv). For this, note that for $\delta \ll 1$, we have that

$$\underline{\varphi} = \psi_0 \text{ in an open neighbourhood of } |t| = \varepsilon$$
$$\underline{\varphi} = \psi_1 \text{ in an open neighbourhood of } |t| = e^{-1}\varepsilon$$

and so we only need to check the claimed estimates for φ_0, φ_1. The estimates are automatic from the formulas for ψ_i, and the choice of the constants A_i, C_i above. $\qquad\square$

Remark 3.9. A word of caution is in order here. By (3.4) and the construction of $\underline{\varphi}$ it is easy to see that $\sqrt{-1}D\overline{D}\underline{\varphi}$ is *not* bounded above independent of ε on the whole of $\mathcal{X}_\varepsilon$.

Because of this remark it is often more convenient to work with the functions ψ_0, ψ_1 constructed above. In particular, we note the following corollary.

Corollary 3.10. *Fix functions $\varphi_0, \varphi_1 \in \mathcal{H}$, and a function $h : \mathcal{X}_\varepsilon \to \mathbb{R}$. Suppose that this data satisfies structural conditions (C1) and (C2) (see (3.2) and (3.3)). Then there exist smooth, S^1-invariant functions $\hat{\varphi}_i$ for $i = 0, 1$ such that $\hat{\varphi}_i$, and $\hat{\alpha}_i := \pi_X^* \alpha + \sqrt{-1}D\overline{D}\hat{\varphi}_i$ have the following properties;*

 - *(0) $\|\hat{\varphi}_i\|_{L^\infty(\mathcal{X}_\varepsilon)}$ is bounded by a constant depending only on η_1, φ_0, φ_1.*
 - *(i) $\hat{\varphi}_0 = \varphi_0$ on the set $|t| = \varepsilon$, and $\hat{\varphi}_0 < \varphi_1$ on $|t| = e^{-1}\varepsilon$.*
 - *(ii) $\hat{\varphi}_1 = \varphi_1$ on the set $|t| = e^{-1}\varepsilon$, and $\hat{\varphi}_1 < \varphi_0$ on $|t| = \varepsilon$.*
 - *(iii) $F(\hat{\alpha}_i) \geqslant h(x,t) + \frac{\eta_1}{2}$ for $i = 0, 1$*
 - *(iv) For each $i = 0, 1$, $\nabla_t \hat{\alpha}_i = \nabla_{\bar{t}} \hat{\alpha}_i = 0$, and $|\nabla \hat{\alpha}_i|_{\hat{\omega}} \leqslant C$ for a constant C depending only on $\|\varphi_i\|_{C^3(X,\omega)}$.*
 - *(v) For each $i = 0, 1$, $|\sqrt{-1}D\overline{D}\hat{\varphi}_i|_{\hat{\omega}}$ is controlled uniformly in terms of $\eta_1, |\sqrt{-1}\partial\overline{\partial}\varphi_i|_{L^\infty(X,\omega)}$*
 - *(vi) For each $i = 0, 1$, $\sup_{\mathcal{X}_\varepsilon} |\nabla^X \hat{\varphi}_i|_{\hat{g}} = \sup_X |\nabla^X \varphi_i|_g$*
 - *(vii) $(\hat{\alpha}_i)_{\bar{t}X} = (\hat{\alpha}_i)_{\bar{X}t} = 0$.*
 - *(viii) Near $\partial \mathcal{X}_\varepsilon$ we have*

$$\left|\frac{\partial}{\partial|t|}\underline{\varphi}\right| \leqslant \frac{C}{\varepsilon}, \qquad \left|\frac{\partial^2}{\partial|t|^2}\underline{\varphi}\right| \leqslant \frac{C}{\varepsilon^2}$$

for a constant C depending only on η_1 and $\|\varphi_0 - \varphi_1\|_{L^\infty(X)}$.

Proof. Take $\hat{\varphi}_i = \psi_i$ in proof of Lemma 3.6. Then $(0), (i), (ii), (iii), (v), (vi), (viii)$ hold automatically, and we have

$$\hat{\alpha} = \alpha + \sqrt{-1}\partial\overline{\partial}\varphi_0 + A_0\sqrt{-1}dt \wedge d\bar{t}$$

which implies property (vii). Property (iv) follows from the fact that $(\mathcal{X}_\varepsilon, \hat{\omega})$ is the product of (X, ω) with a flat factor. $\qquad\square$

Remark 3.11. The advantage of the subsolutions $\hat{\varphi}_i$ is that the constants R, κ_0 appearing in Lemma 3.5 can be chosen depending only on the data η_1, and the C^2 norm of the boundary data.

We finish with the following simple estimate.

Proposition 3.12. *Suppose* $\varphi : \mathcal{X}_\varepsilon \to \mathbb{R}$ *solves* (3.1)*, and suppose* $\underline{\varphi}$ *is a subsolution of* (3.1)*. Then we have*

$$\underline{\varphi} \leqslant \varphi \leqslant \max\{\|\varphi_0\|_{L^\infty(X)}, \|\varphi_1\|_{L^\infty(X)}\} + C\varepsilon^2$$

for a constant C *depending only on* ω, α.

Proof. The bound $\varphi \geqslant \underline{\varphi}$ follows from the comparison principle. On the other hand, by Lemma 3.1 (3), we have

$$\Delta_{\hat{\omega}}\varphi \geqslant -\mathrm{Tr}_\omega\alpha \geqslant -C.$$

It follows that the function $\varphi + C|t|^2$ is subharmonic and hence achieves its maximum on the boundary. The estimate follows. $\square$

In order to prove the desired estimates, we need to proceed with extreme care, ensuring at every step that constants appearing in the estimate are independent of ε. To ease the presentation we introduce the following terminology.

Definition 3.13. *A constant* C *is uniform if it is independent of* ε*, and is invariant after rescaling* $t \mapsto \varepsilon^{-1}t$.

For example, constants depending only on the boundary data φ_i, (X, ω), the constants η_1, η_2 in (3.2) (3.3), and the norms of spatial derivatives $|(\nabla^X)^k(\overline{\nabla^X})^r h|_{\hat{\omega}}$ are *uniform*. On the other hand, constants depending on $|\nabla^X\nabla_{\bar{t}}h|_{\hat{\omega}}$, and $|\nabla_t\nabla_{\bar{t}}h|_{\hat{\omega}}$ are not uniform, since the norms (measured with respect to $\hat{\omega}$) rescale (unless h is a constant).

4. THE C^1 ESTIMATES

The goal of this section is to prove an a priori gradient estimate for solutions of (3.1). We begin with a uniform spatial gradient estimate. In fact, this estimate can be deduced from the interior spatial C^2 estimate proved in Section 5. However, we include it since it may be independent interest, and we expect it to be applicable to the study of geodesic rays in $\mathcal{H}$.

Proposition 4.1. *Suppose* φ *solves* (3.1) *for boundary data* $\varphi_i \in \mathcal{H}$ *for* $i = 0, 1$*, and with* φ_0, φ_1, h *satisfying* (3.2)*,* (3.3)*. Then there exists a uniform constant* C *so that*

$$|\nabla^X\varphi|_{\hat{g}} \leqslant C$$

Proof. In order to estimate the spatial gradient we adapt ideas of Błocki [9, 10] and Phong-Sturm [63, 64] used to obtain gradient estimates for solutions of the complex Monge-Ampère equation. Before beginning the proof, we choose an appropriate background form α on X_ε. For this section we choose

as our background reference form $\alpha = \pi_X^* \alpha + \sqrt{-1}\partial\overline{\partial}\hat{\varphi}_i$ for $i = 0$ or 1, where $\hat{\varphi}_i$ is one of the subsolutions constructed in Corollary 3.10. To ease notation, we will drop the subscript i in the notation. If we write the solution to (3.1) as $\pi_X^* \alpha + \sqrt{-1}D\overline{D}\varphi$, then $\pi_X^* \alpha + \sqrt{-1}D\overline{D}\varphi = \pi_X^* \alpha + \sqrt{-1}D\overline{D}(\varphi - \hat{\varphi})$. Our goal is to prove

$$|\nabla^X(\varphi - \hat{\varphi})|_{\hat{g}} \leqslant C$$

for a uniform constant C. This implies that $|\nabla^X \varphi|_{\hat{g}} \leqslant C'$ with C' depending in addition on the spatial C^1 norm of $\hat{\varphi}$, which by Corollary 3.10 is uniformly controlled in terms of the boundary data.

With this understood, in order to lighten notation we still write the solution to the equation as $\alpha + \sqrt{-1}D\overline{D}\varphi$, and estimate $|\nabla^X \varphi|_{\hat{g}}$. Following [63, 64, 9, 10] (see also [62]), we apply the maximum principle to the quantity

$$Q := \log\left(|\nabla^X \varphi|^2\right) - \gamma(\varphi)$$

where $\gamma : \mathbb{R} \to \mathbb{R}$ is a function to be determined. Fix a point $(p_0, t_0) \in X_\varepsilon$, and a space-time adapted coordinate system $(w_0, \ldots, w_n)$ centered a (p_0, t_0). We compute

$$\nabla_{\bar{j}}|\nabla^X \varphi|^2 = \sum_{1 \leqslant k,\ell \leqslant n} \hat{g}^{k\bar{\ell}}\left(\nabla_{\bar{j}}\nabla_k \varphi \nabla_{\bar{\ell}}\varphi + \nabla_k \varphi \nabla_{\bar{j}}\nabla_{\bar{\ell}}\varphi\right)$$

$$\nabla_i\nabla_{\bar{j}}|\nabla^X \varphi|^2 = \sum_{1 \leqslant k,\ell \leqslant n} \hat{g}^{k\bar{\ell}}\left(\nabla_i\nabla_{\bar{j}}\nabla_k \varphi \nabla_{\bar{\ell}}\varphi + \nabla_{\bar{j}}\nabla_k \varphi \nabla_i\nabla_{\bar{\ell}}\varphi\right)$$

$$\sum_{1 \leqslant k,\ell \leqslant n} \hat{g}^{k\bar{\ell}}\left(\nabla_i\nabla_k \varphi \nabla_{\bar{j}}\nabla_{\bar{\ell}}\varphi + \nabla_k \varphi \nabla_i\nabla_{\bar{j}}\nabla_{\bar{\ell}}\varphi\right)$$

To deal with the first and last terms, we differentiate the equation to get

$$F^{i\bar{j}}\nabla_k\nabla_i\nabla_{\bar{j}}\varphi = \nabla_k h - \nabla_k \alpha_{\bar{j}i}$$

$$F^{i\bar{j}}\nabla_{\bar{\ell}}\nabla_i\nabla_{\bar{j}}\varphi = \nabla_{\bar{\ell}} h - \nabla_{\bar{\ell}} \alpha_{\bar{j}i}.$$

Commuting derivatives gives

$$\nabla_k\nabla_i\nabla_{\bar{j}}\varphi = \nabla_i\nabla_k\nabla_{\bar{j}}\varphi = \nabla_i\nabla_{\bar{j}}\nabla_k\varphi$$

$$\nabla_{\bar{\ell}}\nabla_i\nabla_{\bar{j}}\varphi = -R_{\bar{\ell}i\bar{j}}{}^{\bar{p}}\nabla_{\bar{p}}\varphi + \nabla_i\nabla_{\bar{j}}\nabla_{\bar{\ell}}\varphi$$

Combining these formulae we get

$$F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}|\nabla^X \varphi|^2 = \sum_{1 \leqslant k,\ell \leqslant n} F^{i\bar{j}}\left(\hat{g}^{k\bar{\ell}}\nabla_{\bar{j}}\nabla_k\varphi\nabla_i\nabla_{\bar{\ell}}\varphi + \nabla_i\nabla_k\varphi\nabla_{\bar{j}}\nabla_{\bar{\ell}}\varphi\right)$$

$$+ 2\mathrm{Re}\left(\sum_{1 \leqslant k,\ell \leqslant n} \hat{g}^{k\bar{\ell}}(\nabla_k h - F^{i\bar{j}}\nabla_k\alpha_{\bar{j}i})\nabla_{\bar{\ell}}\varphi\right)$$

$$+ F^{i\bar{j}}\sum_{\substack{1 \leqslant k,\ell \leqslant n \\ 0 \leqslant p \leqslant n}} \hat{g}^{k\bar{\ell}}R_{\bar{\ell}i\bar{j}}{}^{\bar{p}}\nabla_{\bar{p}}\varphi\nabla_k\varphi$$

By Lemma 2.21 we have that $R_{\bar{\ell}i\bar{j}}{}^{\bar{p}} = 0$ if any of i, j, ℓ, or p are 0, so we can choose a constant C_1, depending only on (X, ω) such that

$$F^{i\bar{j}} \sum_{1 \leqslant k,\ell,p \leqslant n} \hat{g}^{k\bar{\ell}} R_{\bar{\ell}i\bar{j}}{}^{\bar{p}} \nabla_{\bar{p}}\varphi \nabla_k \varphi \geqslant -C_1 |\nabla^X \varphi|^2_{\hat{g}},$$

where we also used that $F^{i\bar{j}} \leqslant \hat{g}^{i\bar{j}}$. Similarly, by Corollary 3.10 we can choose a constant C_2 depending only on the boundary data, and the spatial derivative of h, so that

$$2\mathrm{Re}\left(\sum_{1 \leqslant k,\ell \leqslant n} \hat{g}^{k\bar{\ell}} (\nabla_k h - F^{i\bar{j}} \nabla_k \alpha_{\bar{j}i}) \nabla_{\bar{\ell}}\varphi \right) \geqslant -C_2 |\nabla^X \varphi|_{\hat{g}}.$$

Summarizing we have

$$F^{i\bar{j}} \nabla_i \nabla_{\bar{j}} \log\left(|\nabla^X \varphi|^2_{\hat{g}}\right) \geqslant \frac{1}{|\nabla^X \varphi|^2} \sum_{1 \leqslant k,\ell \leqslant n} F^{i\bar{j}} \hat{g}^{k\bar{\ell}} \left(\nabla_{\bar{j}} \nabla_k \varphi \nabla_i \nabla_{\bar{\ell}}\varphi + \nabla_i \nabla_k \varphi \nabla_{\bar{j}} \nabla_{\bar{\ell}}\varphi \right)$$

$$- \frac{1}{(|\nabla^X \varphi|^2_{\hat{g}})^2} F^{i\bar{j}} \nabla_i |\nabla^X \varphi|^2_{\hat{g}} \nabla_{\bar{j}} |\nabla^X \varphi|^2_{\hat{g}} - C_1 - \frac{C_2}{|\nabla^X \varphi|_{\hat{g}}}$$

for uniform constants C_1, C_2. Abusing notation, define a norm on 1-forms by $\langle \sigma, \sigma' \rangle_F = F^{i\bar{j}} \sigma_i \overline{\sigma'_j}$, and write

$$\sum_{1 \leqslant k,\ell \leqslant n} F^{i\bar{j}} \hat{g}^{k\bar{\ell}} \left(\nabla_{\bar{j}} \nabla_k \varphi \nabla_i \nabla_{\bar{\ell}}\varphi + \nabla_i \nabla_k \varphi \nabla_{\bar{j}} \nabla_{\bar{\ell}}\varphi \right) = |\overline{\nabla}\nabla^X \varphi|^2_{F\otimes\hat{g}} + |\nabla\nabla^X \varphi|^2_{F\otimes\hat{g}}.$$

Define 1-forms

$$S_i := \langle \nabla_i \nabla^X \varphi, \nabla^X \varphi \rangle_{\hat{g}}, \quad T_j := \langle \nabla^X \varphi, \nabla_{\bar{j}} \nabla^X \varphi \rangle_{\hat{g}}$$

so that $\nabla |\nabla^X \varphi|^2_{\hat{g}} = S + T$. By the Cauchy-Schwarz inequality we have

$$|S|^2_F + |T|^2_F \leqslant |\nabla^X \varphi|^2_{\hat{g}} \left(|\overline{\nabla}\nabla^X \varphi|^2_{F\otimes\hat{g}} + |\nabla\nabla^X \varphi|^2_{F\otimes\hat{g}} \right).$$

It follows that

$$|\nabla |\nabla^X \varphi|^2_{\hat{g}}|^2_F - 2\mathrm{Re}\left(\langle \nabla |\nabla^X \varphi|^2_{\hat{g}}, T \rangle_F \right) \leqslant |\nabla^X \varphi|^2_{\hat{g}} \left(|\overline{\nabla}\nabla^X \varphi|^2_{F\otimes\hat{g}} + |\nabla\nabla^X \varphi|^2_{F\otimes\hat{g}} \right).$$

Plugging this into the estimate yields

$$F^{i\bar{j}} \nabla_i \nabla_{\bar{j}} \log\left(|\nabla^X \varphi|^2_{\hat{g}}\right) \geqslant -2\mathrm{Re}\left(\left\langle \frac{\nabla |\nabla^X \varphi|^2_{\hat{g}}}{|\nabla^X \varphi|^2_{\hat{g}}}, \frac{T}{|\nabla^X \varphi|^2_{\hat{g}}} \right\rangle_F \right) - C_1 - \frac{C_2}{|\nabla^X \varphi|_{\hat{g}}}.$$

Let us examine the term T. In our coordinate system we have

$$T_j = \sum_{1 \leqslant k,\ell \leqslant n} \nabla_k \varphi \hat{g}^{k\bar{\ell}} \left((\alpha_\varphi)_{\bar{\ell}j} - \alpha_{\bar{\ell}j} \right).$$

If we let $\mathcal{E}, \mathcal{E}_0$ denote that endomorphisms $\hat{g}^{k\bar{\ell}} (\alpha_\varphi)_{\bar{\ell}j}$, and $\hat{g}^{k\bar{\ell}} \alpha_{\bar{\ell}j}$ respectively, then we can write T invariantly as

$$T = (\mathcal{E} - \mathcal{E}_0) \nabla^X \varphi.$$

Suppose Q achieves an interior maximum at the point (p_0, t_0). At this point we have $\nabla Q = 0$, and $F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}Q \leqslant 0$. Writing out the first of these gives

$$\frac{\nabla|\nabla^X\varphi|^2_{\hat{g}}}{|\nabla^X\varphi|^2_{\hat{g}}} = \gamma'(\varphi)\nabla\varphi$$

and so at (p_0, t_0) we get

$$\mathrm{Re}\left(\left\langle \frac{\nabla|\nabla^X\varphi|^2_{\hat{g}}}{|\nabla^X\varphi|^2_{\hat{g}}}, \frac{T}{|\nabla^X\varphi|^2_{\hat{g}}}\right\rangle_F\right) = \frac{\gamma'(\varphi)}{|\nabla^X\varphi|^2_{\hat{g}}}\mathrm{Re}\left(\langle\nabla\varphi, (\mathcal{E} - \mathcal{E}_0)\nabla^X\varphi\rangle_F\right)$$

For one moment, let us choose another system of holomorphic normal coordinates for $\hat{g}$, $(z_0, \ldots, z_n)$ centered at (p_0, t_0) so that $\hat{g}$ is the identity, and $\mathcal{E}$ is diagonal, with eigenvalues $\mu_0, \ldots, \mu_n$. Write $\nabla^X\varphi = b_i\partial_{z_i}$ for some complex numbers b_i. Then

$$\left|\langle\nabla\varphi, \mathcal{E}\nabla^X\varphi\rangle_F\right| = \left|\sum_i \frac{\mu_i\varphi_i\overline{b_i}}{(1+\mu_i^2)}\right| \leqslant \left(\sum_i \frac{\mu_i^2|b_i|^2}{1+\mu_i^2}\right)^{1/2}\left(\sum_j \frac{|\varphi_j|^2}{1+\mu_j^2}\right)^{1/2}$$

$$\leqslant |\nabla^X\varphi|_{\hat{g}}|\nabla\varphi|_F$$

Similarly, if we write $\mathcal{E}_0\nabla^X\varphi = c_i\partial_{z_i}$, then we have

$$\left|\langle\nabla\varphi, \mathcal{E}_0\nabla^X\varphi\rangle_F\right| = \left|\sum_i \frac{\varphi_i\overline{c_i}}{1+\mu_i^2}\right| \leqslant \left(\sum_i \frac{|c_i|^2}{1+\mu_i^2}\right)^{1/2}\left(\sum_j \frac{|\varphi_j|^2}{1+\mu_j^2}\right)^{1/2}$$

$$\leqslant |\mathcal{E}_0\nabla^X\varphi|_{\hat{g}}|\nabla\varphi|_F$$

By Corollary 3.10 we have $\mathcal{E}_0 TX \subset TX$. Since $\nabla^X\varphi \in TX \subset T\mathcal{X}_\varepsilon$ we get

$$|\mathcal{E}_0\nabla^X\varphi|_{\hat{g}} \leqslant C_3|\nabla^X\varphi|_{\hat{g}}$$

for a uniform constant C_3 depending on the boundary data. Putting everything together gives

(4.1) $\quad 0 \geqslant -C_1 - \dfrac{C_2}{|\nabla^X\varphi|_{\hat{g}}} - C_3\gamma'(\varphi)\dfrac{|\nabla\varphi|_F}{|\nabla^X\varphi|_{\hat{g}}} - \gamma'(\varphi)F^{i\bar{j}}\varphi_{\bar{j}i} - \gamma''(\varphi)|\nabla\varphi|^2_F$

Following Phong-Sturm [63, 64, 62] we choose

$$\gamma(\varphi) = B\varphi - \frac{1}{\varphi + C_4}$$

where $C_4 = -\inf_{\mathcal{X}_\varepsilon}\varphi + 1$, and B is a large positive constant to be determined. Note that

$$B\varphi - 1 \leqslant \gamma(\varphi) \leqslant B\varphi, \quad B \leqslant \gamma'(\varphi) \leqslant B+1, \quad \gamma''(\varphi) = -2\frac{1}{(\varphi + C_4)^3} < 0.$$

We may assume that $|\nabla^X\varphi|_{\hat{g}} > 1$ at the maximum point of Q, for otherwise we're done. We need to consider several cases.

First, suppose that

(4.2) $$\varepsilon_0^2 |\nabla^X \varphi|^2 \geqslant |\nabla \varphi|_F^2$$

for some $0 < \varepsilon_0 < 1$ to be determined. Let R, κ_0 be the constants appearing in Lemma 3.5 for the subsolution $\hat{\varphi}$ and recall that they are uniformly controlled (see Remark 3.11). If every eigenvalue of $\mathcal{E}$ is smaller than R, then we have

$$|\nabla \varphi|_F^2 \geqslant \frac{1}{1+R^2}|\nabla \varphi|_{\hat{g}}^2 \geqslant \frac{1}{1+R^2}|\nabla^X \varphi|_{\hat{g}}^2.$$

If we take ε_0 small so that $100\varepsilon_0^2 \leqslant \frac{1}{1+R^2}$, then under the assumption (4.2) some eigenvalue of $\mathcal{E}$ at (p_0, t_0) must be larger than R. In particular, by Lemma 3.5 we get

$$F^{i\bar{j}}\varphi_{\bar{j}i} = F^{i\bar{j}}(\alpha_\varphi)_{\bar{j}i} - \alpha_{\bar{j}i} \leqslant -\kappa_0 \sum_{p=0}^{n} \frac{1}{1+\mu_p^2} < -\kappa_0 \frac{1}{1+C(\eta)^2}$$

where $C(\eta) := C(\eta_1, \eta_2)$ is the bound for $|\mu_n|$ in Lemma 3.1. Thus, assuming (4.2) implies

$$0 \geqslant -C_1 - C_2 - C_3(B+1)\varepsilon_0 + B\kappa_0 \frac{1}{1+C(\eta)^2} + \frac{2}{\varphi + C_4}|\nabla \varphi|_F^2$$

We now choose B large depending only on C_1, C_2, C_3 so that

$$B\kappa_0 \frac{1}{1+C(\eta)^2} \geqslant C_1 + C_2 + C_3 + 1$$

and choose ε_0 small depending on B so that $\varepsilon_0(B+1) \leqslant 1$. Clearly ε_0, B can be chosen to be uniform constants. With these choices we conclude that Q cannot attain an interior maximum at which (4.2) holds.

We may therefore assume that if Q achieves an interior maximum at (p_0, t_0) then

(4.3) $$\varepsilon_0^2 |\nabla^X \varphi|^2 \leqslant |\nabla \varphi|_F^2$$

at (p_0, t_0). We may also assume that $|\nabla \varphi|_F \geqslant 1$ at (p_0, t_0), for otherwise $|\nabla^X \varphi|^2 \leqslant \varepsilon_0^{-2}$ and we are done. Rearranging (4.1) we get

$$\frac{2}{(1 + \mathrm{osc}_{X_\varepsilon}\varphi)^3}|\nabla \varphi|_F^2 \leqslant C_1 + C_2 + (B+1)(C_3 \frac{|\nabla \varphi|_F}{|\nabla^X \varphi|_{\hat{g}}} + C_5)$$

where we used that

$$F^{i\bar{j}}\varphi_{\bar{j}i} = \sum_i \frac{\mu_i - \alpha_{\bar{i}i}}{1 + \mu_i^2} \leqslant C_5$$

for a uniform constant C_5 by Corollary 3.10. Let us simplify the notation by writing

$$\tilde{\delta} = \frac{2}{(1 + \mathrm{osc}_{\mathcal{X}_\varepsilon}\varphi)^3}, \quad \tilde{B} = (B+1)C_3, \quad \tilde{A} = C_1 + C_2 + (B+1)C_5.$$

so that $\tilde{A}, \tilde{B}, \tilde{\delta}$ are uniform constants and

(4.4)
$$\tilde{\delta}|\nabla\varphi|_F^2 \leqslant \tilde{A} + \tilde{B}\frac{|\nabla\varphi|_F}{|\nabla^X\varphi|_{\hat{g}}}.$$

There are now two cases. First, observe that if $\tilde{\delta} \leqslant \frac{2\tilde{B}}{|\nabla_X\varphi|_{\hat{g}}}$ then we are done. So we may assume that $\tilde{\delta} > \frac{2\tilde{B}}{|\nabla_X\varphi|_{\hat{g}}}$. Upon rearranging (4.4) we obtain

$$\tilde{A} \geqslant \left(\tilde{\delta}|\nabla\varphi|_F - \frac{\tilde{B}}{|\nabla^X\varphi|_{\hat{g}}}\right)|\nabla\varphi|_F \geqslant \frac{\tilde{\delta}}{2}|\nabla\varphi|_F$$

where we used our assumption that $|\nabla\varphi|_F \geqslant 1$. We now use (4.3) to obtain

$$|\nabla^X\varphi|_{\hat{g}} \leqslant \varepsilon_0^{-1}|\nabla\varphi|_F \leqslant \frac{2\tilde{A}}{\varepsilon_0\tilde{\delta}},$$

which is the desired estimate. Since the constants $\varepsilon_0, \tilde{\delta}, \tilde{A}, \tilde{B}$ are uniform, we have shown that, if Q attains an interior maximum it is bounded uniformly. Since φ is bounded uniformly it follows that $|\nabla^X\varphi|_{\hat{g}}^2$ is uniformly bounded. Thus we are finished unless Q attains its maximum on the boundary. But on the boundary Q is clearly bounded from above by a constant depending only on the boundary data. $\qquad\square$

It only remains to estimate the temporal derivative $|\nabla_t\varphi|_{\hat{g}}$. The first step is to reduce the estimate for $|\nabla_t\varphi|_{\hat{g}}$ to a boundary estimate. Write the solution to (3.1) as $\pi^*\alpha + \sqrt{-1}D\overline{D}\varphi$, write $t = u + \sqrt{-1}v$ and compute

$$F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\nabla_u\varphi = F^{i\bar{j}}\nabla_u\nabla_i\nabla_{\bar{j}}\varphi \geqslant -\hat{C}$$

where $\hat{C} = -\sup_{\mathcal{X}_\varepsilon}|\nabla_t h|_{\hat{g}}$. A similar estimate holds for v. In this computation we've used that the curvature vanishes along any ∂_t direction, and that $\nabla_t\pi_X^*\alpha = 0$ by Lemma 2.21. Note that $\hat{C}$ is not a uniform constant. Consider the quantity

$$Q = \partial_u\varphi + A|t|^2 - B(\varphi - \hat{\varphi}),$$

where $\hat{\varphi}$ is either one of the subsolutions constructed in Corollary 3.10. Suppose Q has an interior maximum at (p_0, t_0). Then

$$0 \geqslant -\hat{C} + AF^{\bar{t}t} + BF^{i\bar{j}}((\alpha_{\hat{\varphi}})_{\bar{j}j} - (\alpha_\varphi)_{\bar{j}i})$$

Suppose that $|\mu| > R$ at the point (p_0, t_0). Then by Lemma 3.5 we have

$$0 \geqslant -\hat{C} + B\kappa_0\frac{1}{1 + C(\eta)^2} > 0$$

where $C(\eta) = C(\eta_1, \eta_2)$ is the bound for $|\mu_n|$ (see Lemma 3.1), and we have chosen $B = B'\hat{C}$ for B' large depending only on κ_0, η_1, η_2. If $|\mu| < R$ at the

point (p_0, t_0) then $F^{t\bar{t}} \geqslant \frac{1}{1+R^2}$, and so

$$0 \geqslant -\hat{C} + A\frac{1}{1+R^2} + B\sum_i \frac{(\alpha_{\hat{\varphi}})_{\bar{i}i} - \mu_i}{1+\mu_i^2}$$

$$\geqslant -\hat{C} + A\frac{1}{1+R^2} - B|\alpha_{\hat{\varphi}}|_{\hat{g}} - B(n+1) > 0$$

provided we choose $A = A'\hat{C}$ for A' large depending only on R, n, B' and $\alpha_{\hat{\varphi}}$. In particular, by Corollary 3.10 the constant A' can be chosen uniformly. Similar arguments using $-\partial_u\varphi, \pm\partial_v\varphi$ prove

Lemma 4.2. *There exists a uniform constant C so that*

$$|\nabla_t\varphi|_{\hat{g}} \leqslant C(1 + \sup_{X_\varepsilon}|\nabla_t h|_{\hat{g}}) + \sup_{\partial X_\varepsilon}|\nabla_t\varphi|_{\hat{g}}$$

It remains only to estimate $|\nabla_t\varphi|$ on the boundary. Since φ is S^1 invariant, it suffices to consider the boundary derivative with respect to the coordinate $r = |t|$. Consider the boundary $|t| = \varepsilon$. Then $\varphi \geqslant \hat{\varphi}_0$ in $\mathcal{X}_\varepsilon$, and $\varphi = \hat{\varphi}_0$ on $\partial\mathcal{X}_\varepsilon$ we have

$$\frac{\partial}{\partial r}\Big|_{r=\varepsilon}\varphi \leqslant \frac{\partial}{\partial r}\Big|_{r=\varepsilon}\hat{\varphi}_0 \leqslant \frac{C}{\varepsilon}$$

by Corollary 3.10. Similarly,

$$\frac{\partial}{\partial r}\Big|_{r=e^{-1}\varepsilon}\varphi \geqslant \frac{\partial}{\partial r}\Big|_{r=e^{-1}\varepsilon}\hat{\varphi}_1 \geqslant -\frac{C}{\varepsilon}.$$

For the remaining estimates we construct barriers from above. To estimate near $\{r = \varepsilon\}$ consider

$$\psi_0 = \varphi_0 - A_0(|t|^2 - \varepsilon^2) - C_0\log\left(\frac{|t|^2}{\varepsilon^2}\right).$$

Clearly $\psi_0 = \varphi_0 = \varphi$ on $\{r = \varepsilon\}$. Furthermore we have

$$\Delta_{\hat{\omega}}\psi_0 = \Delta_\omega\varphi_0 - A_0$$

and so we can choose A_0 large depending only on $\|\varphi_0\|_{C^2(X,\omega)}$ so that $\Delta_{\hat{\omega}}\psi_0 \leqslant 0$. We next choose C_0 large depending only on $A_0, \|\varphi_0 - \varphi_1\|_{L^\infty(X)}$ so that on $\{r = e^{-1}\varepsilon\}$ we have

$$\psi_0\Big|_{|t|=e^{-1}\varepsilon} = \varphi_0 + A_0(1 - e^{-2})\varepsilon^2 + 2C_0 \geqslant \varphi_1.$$

By Lemma 3.1 we have

$$\Delta_{\hat{\omega}}\psi_0 \leqslant 0 < \Delta_{\hat{\omega}}\varphi$$

and $\psi_0 \geqslant \varphi$ on the boundary, with equality when $|t| = \varepsilon$. By the maximum principle we obtain

$$\frac{\partial}{\partial r}\Big|_{r=\varepsilon}\varphi \geqslant \frac{\partial}{\partial r}\Big|_{r=\varepsilon}\psi_0 \geqslant -\frac{C}{\varepsilon}$$

for a uniform constant C. Similar estimates work near $\{r = e^{-1}\varepsilon\}$ to prove estimates near $\{r = e^{-1}\varepsilon\}$. We have therefore proved

Theorem 4.3. *Suppose φ solves* (3.1) *for boundary data $\varphi_i \in \mathcal{H}$ for $i = 0, 1$, and with φ_0, φ_1, h satisfying* (3.2), (3.3). *Then there exists a uniform constant C so that*

$$|\nabla^X \varphi|_{\hat{g}} \leqslant C, \qquad |\nabla_t \varphi|_{\hat{g}} \leqslant C \left(1 + \sup_{X_\varepsilon} |\nabla_t h|_{\hat{g}} \right) + \frac{1}{\varepsilon} \right)$$

5. INTERIOR C^2 ESTIMATES

This section comprises the heart of the analysis towards proving the existence of geodesics in the space $\mathcal{H}$. The goal is to prove the following theorem

Theorem 5.1. *Suppose φ solves* (3.1) *for boundary data $\varphi_i \in \mathcal{H}$ for $i = 0, 1$, and with φ_0, φ_1, h satisfying* (3.2), (3.3). *Then there exists a uniform constant C so that*

$$(5.1) \qquad\qquad |\nabla^X \overline{\nabla^X} \varphi|_{\hat{\omega}} \leqslant C$$

$$(5.2) \qquad |\nabla^X \nabla_{\bar{t}} \varphi| \leqslant C \left(1 + \|h_{\bar{t}t}\|_{L^\infty(\mathcal{X}_\varepsilon)} + \|h_t\|_{L^\infty(\mathcal{X}_\varepsilon)}^2 + \sup_{\partial \mathcal{X}_\varepsilon} |\varphi_{\bar{t}t}|_{\hat{g}} \right)^{\frac{1}{2}}$$

$$(5.3) \qquad |\nabla_t \nabla_{\bar{t}} \varphi|_{\hat{g}} \leqslant C \left(1 + \|h_{\bar{t}t}\|_{L^\infty(\mathcal{X}_\varepsilon)} + \|h_t\|_{L^\infty(\mathcal{X}_\varepsilon)}^2 + \sup_{\partial \mathcal{X}_\varepsilon} |\varphi_{\bar{t}t}|_{\hat{g}} \right)$$

The uniform estimate for the spatial C^2 norm, (5.1), is the most difficult of the three estimates. Let us briefly recall what is known in this direction. In joint work with Jacob [23] the authors proved a C^2 estimate for solutions of $F = h$, when $h : X \to ((n-1)\frac{\pi}{2}, (n+1)\frac{\pi}{2})$ on compact manifolds without boundary, provided a subsolution exists. In the current setting the same estimate works to prove an interior C^2 estimate of the form

$$|\nabla\overline{\nabla}\varphi|_{\hat{g}} \leqslant C(1 + \sup_{\mathcal{X}_\varepsilon} |\nabla\varphi|_{\hat{g}}^2) + \sup_{\partial\mathcal{X}_\varepsilon} |\nabla\overline{\nabla}\varphi|_{\hat{g}}$$

Ignoring the troublesome boundary term, Theorem 4.3 gives the bound $|\nabla\varphi|_{\hat{g}}^2 \leqslant \frac{1}{\varepsilon^2}$, and this bound is saturated if $\varphi_0 \neq \varphi_1$. Nevertheless, the above estimate would be good enough to prove the existence of solutions to (3.1) on $\mathcal{X}_\varepsilon$, but not good enough to deduce regularity of the rescaled solutions as $\varepsilon \to 0$.

In order to prove a uniform estimate for the spatial C^2 norm, we will apply the maximum principle to a quantity involving the largest *spatial* eigenvalue. That is, for each point $(p, t) \in \mathcal{X}_\varepsilon$ define $\lambda_1(p, t)$ to be the largest eigenvalue of $(\alpha + \sqrt{-1}D\overline{D}\varphi)|_X$ with respect to ω. We will bound this quantity from above. As a first step, we need to compute two derivatives of λ_1.

Proposition 5.2. *Suppose $\mathcal{E}$ is a smooth section of $\mathrm{End}(T\mathcal{X}_\varepsilon)$ which is hermitian with respect to $\hat{g}$. Define $\chi = \mathcal{E}|_{TX} = \pi\mathcal{E}\pi$, where π is the orthogonal projection to $TX \subset T\mathcal{X}_\varepsilon$ defined by $\hat{g}$. Suppose $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ are the eigenvalues of χ at $(p,t) \in X_\varepsilon$, and let $e_1, \ldots, e_n$ be the corresponding unit length, orthonormal eigenvectors. Let (z_i) be holomorphic coordinates on $\mathcal{X}_\varepsilon$ centered at (p_0, t_0). Then at (p_0, t_0) we have*

$$\nabla_i \lambda_\alpha = \langle (\nabla_i \mathcal{E})e_\alpha, e_\alpha \rangle, \qquad \nabla_{\bar{i}}\lambda_\alpha \langle (\nabla_{\bar{i}}\mathcal{E})e_\alpha, e_\alpha \rangle$$

and,

$$\nabla_i \nabla_{\bar{j}} \lambda_\alpha = \langle (\nabla_i \nabla_{\bar{j}} \mathcal{E})e_\alpha, e_\alpha \rangle$$
$$+ \sum_{\beta \neq \alpha} \frac{\langle (\nabla_i \mathcal{E})e_\alpha, e_\beta \rangle \overline{\langle (\nabla_j \mathcal{E})e_\alpha, e_\beta \rangle}}{\lambda_\alpha - \lambda_\beta} + \frac{\langle (\nabla_{\bar{j}}\mathcal{E})e_\alpha, e_\beta \rangle \overline{\langle (\nabla_{\bar{i}}\mathcal{E})e_\alpha, e_\beta \rangle}}{\lambda_\alpha - \lambda_\beta}$$

where we view e_α as vectors in $T\mathcal{X}_\varepsilon$ by the inclusion $TX \hookrightarrow T\mathcal{X}_\varepsilon$.

Proof. First note that, by assumption the eigenvalues λ_α are smooth functions near (p_0, t_0), and we can find smooth spatial vector fields denoted e_α which are sections of $TX \subset T\mathcal{X}_\varepsilon$, so that

$$\chi e_\alpha(p, t) = \lambda_\alpha(p, t)e_\alpha, \qquad \|e_\alpha(p, t)\|_{\hat{g}} = 1$$

and clearly we have $e_\alpha(p_0, t_0) = e_\alpha$. Now, since $e_\alpha = \pi e_\alpha$, we can write

$$(5.4) \qquad \mathcal{E}e_\alpha = \chi e_\alpha + (1 - \pi)\mathcal{E}e_\alpha$$

and so $\lambda_\alpha = \langle \mathcal{E}e_\alpha, e_\alpha \rangle$. We now differentiate this equation to get

$$\nabla_i \lambda_\alpha = \langle (\nabla_i \mathcal{E})e_\alpha, e_\alpha \rangle + \langle \mathcal{E}(\nabla_i e_\alpha), e_\alpha \rangle + \langle \mathcal{E}e_\alpha, \nabla_{\bar{i}}e_\alpha \rangle$$
$$= \langle (\nabla_i \mathcal{E})e_\alpha, e_\alpha \rangle + \langle (\nabla_i e_\alpha), \mathcal{E}e_\alpha \rangle + \lambda_\alpha \langle e_\alpha, \nabla_{\bar{i}}e_\alpha \rangle$$
$$= \langle (\nabla_i \mathcal{E})e_\alpha, e_\alpha \rangle + 2\lambda_\alpha \langle (\nabla_i e_\alpha), e_\alpha \rangle$$
$$= \langle (\nabla_i \mathcal{E})e_\alpha, e_\alpha \rangle$$

where we used that $\mathcal{E}$ is hermitian, and $\|e_\alpha\| = 1$, so that $\langle \nabla_i e_\alpha, e_\alpha \rangle = 0$. Similarly we have

$$\nabla_{\bar{i}}\lambda_\alpha = \langle (\nabla_{\bar{i}}\mathcal{E})e_\alpha, e_\alpha \rangle.$$

Before proceeding, let us remark that since $\mathcal{E}$ is hermitian, for any vectors $V, W \in T\mathcal{X}_\varepsilon$ there holds

$$(5.5) \qquad \langle (\nabla_i \mathcal{E})V, W \rangle = \langle V, (\nabla_{\bar{i}}\mathcal{E})W \rangle$$

and similarly for barred indices. Next we compute

$$(5.6) \quad \nabla_i \nabla_{\bar{j}} \lambda_\alpha = \langle (\nabla_i \nabla_{\bar{j}} \mathcal{E})e_\alpha, e_\alpha \rangle + \langle (\nabla_{\bar{j}}\mathcal{E})(\nabla_i e_\alpha), e_\alpha \rangle + \langle (\nabla_{\bar{j}}\mathcal{E})e_\alpha, \nabla_{\bar{i}}e_\alpha \rangle.$$

At the point (p_0, t_0) we claim that

$$\nabla_i e_\alpha = \sum_{\beta \neq \alpha} a_{i\beta}^\alpha e_\beta, \qquad \nabla_{\bar{i}}e_\alpha = \sum_{\beta \neq \alpha} a_{\bar{i}\beta}^\alpha e_\beta.$$

Let us explain how to see this for the first expression, with the second expression being treated in the same way. Let ∂_t denote the vector field generated by the time direction. Since $e_1, \ldots, e_n$ span the orthogonal complement of

span$\{\partial_t\} \subset T\mathcal{X}_\varepsilon$, it suffices to show that $\langle \nabla_i e_\alpha, \partial_t \rangle = 0$. To do this we differentiate the equation $\langle e_\alpha, \partial_t \rangle = 0$ to get

$$0 = \langle \nabla_i e_\alpha, \partial_t \rangle + \langle e_\alpha, \nabla_{\bar{i}} \partial_t \rangle$$

and use that by Lemma 2.21, ∂_t is parallel. In fact, in this case one can use that ∂_t is a holomorphic vector field, but this does not work to prove the analogous claim for $\nabla_{\bar{i}} e_\alpha$. A similar computation shows that $\langle \nabla_{\bar{i}} e_\alpha, \partial_t \rangle = 0$. Finally, the fact that e_i does not appear in the sum follows from $\|e_i\| = 1$. Now we can solve for the $a_{i\beta}^\alpha$. We differentiate (5.4) to get

$$(\nabla_i \mathcal{E}) e_\alpha + \mathcal{E}(\nabla_i e_\alpha) = \nabla_i \lambda_\alpha e_\alpha + \lambda_\alpha \nabla_i e_\alpha + \nabla_i((1 - \pi)\mathcal{E} e_\alpha).$$

Taking the inner product with e_β for $\beta \neq \alpha$, and using that $\mathcal{E}$ is hermitian gives

$$\langle (\nabla_i \mathcal{E}) e_\alpha, e_\beta \rangle + \lambda_\beta a_{i\beta}^\alpha = \lambda_\alpha \langle \nabla_i e_\alpha, e_\beta \rangle + \langle \nabla_i((1 - \pi)\mathcal{E} e_\alpha), e_\beta \rangle$$
$$= \lambda_\alpha a_{i\beta}^\alpha + \langle ((1 - \pi)\mathcal{E} e_\alpha), \nabla_{\bar{i}} e_\beta \rangle$$
$$= \lambda_\alpha a_{i\beta}^\alpha$$

where in the last two lines we used that $e_\beta, \nabla_{\bar{i}} e_\beta$ are orthogonal to span$\{\partial_t\} \ni (1 - \pi)\mathcal{E} e_\alpha$. So

$$(\lambda_\alpha - \lambda_\beta) a_{i\beta}^\alpha = \langle (\nabla_i \mathcal{E}) e_\alpha, e_\beta \rangle.$$

Similarly, we have

$$(\lambda_\alpha - \lambda_\beta) a_{\bar{i}\beta}^\alpha = \langle (\nabla_{\bar{i}} \mathcal{E}) e_\alpha, e_\beta \rangle.$$

Plugging this into (5.6), using (5.5) and doing some algebra we obtain

$$\nabla_i \nabla_{\bar{j}} \lambda_\alpha = \langle (\nabla_i \nabla_{\bar{j}} \mathcal{E}) e_\alpha, e_\alpha \rangle$$
$$+ \sum_{\beta \neq \alpha} \frac{\langle (\nabla_i \mathcal{E}) e_\alpha, e_\beta \rangle \overline{\langle (\nabla_j \mathcal{E}) e_\alpha, e_\beta \rangle}}{\lambda_\alpha - \lambda_\beta} + \frac{\langle (\nabla_{\bar{j}} \mathcal{E}) e_\alpha, e_\beta \rangle \overline{\langle (\nabla_{\bar{i}} \mathcal{E}) e_\alpha, e_\beta \rangle}}{\lambda_\alpha - \lambda_\beta}$$

By noting that λ_1 is a smooth function provided $\lambda_1 > \lambda_2$ we have

$$\nabla_i \nabla_{\bar{i}} \lambda_1 = \langle (\nabla_i \nabla_{\bar{i}} \mathcal{E}) e_1, e_1 \rangle + \sum_{\beta \neq 1} \frac{|\langle (\nabla_i \mathcal{E}) e_1, e_\beta \rangle|^2 + |\langle (\nabla_{\bar{i}} \mathcal{E}) e_1, e_\beta \rangle|^2}{\lambda_1 - \lambda_\beta}.$$

$\square$

Next we compute the linearized operator applied to the largest spatial eigenvalue λ_1 of $\mathcal{E} := \hat{\omega}^{-1}(\pi^* \alpha + \sqrt{-1} D \overline{D} \varphi)$. First, we have to perturb this endomorphism to ensure that the largest spatial eigenvalue is smooth. Fix a point $(p_0, t_0) \in \mathcal{X}_\varepsilon$, and choose holomorphic normal coordinates $(z_0, \ldots, z_n)$ for $\hat{\omega}$ centered at (p_0, t_0) so that $\mathcal{E}(p_0, t_0)$ is diagonal with eigenvalues $\mu_0 \geqslant \mu_1 \geqslant \cdots \geqslant \mu_n$, and we may assume that $\mu_0 > \mu_n$. Consider a matrix $B = (B_j^i) = B_{ii} \delta_j^i$ defined near (p_0, t_0) with the property that $B_{00} = 0 = B_{nn} < B_{n-1n-1} < \cdots < B_{11}$. Let e_β be an $\hat{\omega}$ orthonormal frame of eigenvectors for $\mathcal{E}|_{TX}$ at (p_0, t_0), with corresponding eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, and assume that $\lambda_1 \gg \lambda_n$. As usual, we regard the e_i as vectors in $(T\mathcal{X}_\varepsilon)_{p_0, t_0}$. Furthermore, we make the following stipulation; if $\lambda_1 = \mu_0$, then e_1 is in the

span of the vectors with eigenvalue μ_0. After possibly rotating our original coordinate system, we choose $e_1 = \partial_{z_0}$. We choose the B_{ii} in the following way. We require that

$$Be_1 = 0.$$

If $e_1 = \partial_{z_0}$ this is redundant, otherwise we have gained a single linear equation, and hence we can choose $B \in \mathbb{R}^{n+1}_{\geq 0}$ in the complement of at most 3 hyperplanes. On the orthogonal complement of $\text{span}\{\partial_{z_0}, \partial_{z_n}, e_1\}$ we clearly have that B is positive definite. Extending B to be constant in our local coordinate patch, we can view it as a local holomorphic section of $\text{End}(T\mathcal{X}_\varepsilon)$. Consider the endomorphism $\tilde{\mathcal{E}} := \mathcal{E} - B$. Clearly the eigenvalues of $\tilde{\mathcal{E}}$ are less or equal the eigenvalues of $\mathcal{E}$. Let $\tilde{\mu}_0 \geqslant \tilde{\mu}_1 \geqslant \cdots \geqslant \tilde{\mu}_n$ denote the eigenvalues of $\tilde{\mathcal{E}}$ at (p_0, t_0), and $\tilde{\lambda}_1 \geqslant \tilde{\lambda}_2 \geqslant \cdots \geqslant \tilde{\lambda}_n$ denote the eigenvalues of $\tilde{\mathcal{E}}|_{TX}$. Then we have

$$\tilde{\mu}_n = \mu_n, \quad \tilde{\mu}_0 = \mu_0, \quad \tilde{\lambda}_1 = \lambda_1 \qquad \text{at } (p_0, t_0).$$

Furthermore, we have $\tilde{\lambda}_\beta < \lambda_\beta$ for all $\beta \neq 1, n$. Since $\lambda_n \ll \lambda_1$, by assumption, this implies that $\tilde{\lambda}_1 > \tilde{\lambda}_2 \geqslant \cdots \geqslant \tilde{\lambda}_n$. Thus $\tilde{\lambda}_2 < \tilde{\lambda}_1 \leqslant \lambda_1$ near (p_0, t_0), with $\tilde{\lambda}_1 = \lambda_1$ at (p_0, t_0). This is the desired perturbation. We now compute
(5.7)
$$F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\tilde{\lambda}_1 = \sum_{i=0}^{n} \frac{1}{1+\mu_i^2}\nabla_i\nabla_{\bar{i}}\tilde{\lambda}_1$$
$$= \sum_i \frac{1}{1+\mu_i^2}\langle(\nabla_i\nabla_{\bar{i}}\tilde{\mathcal{E}})e_1, e_1\rangle + \sum_{i=0}^{n}\sum_{\beta\neq 1} \frac{|\langle(\nabla_i\tilde{\mathcal{E}})e_1, e_\beta\rangle|^2 + |\langle(\nabla_{\bar{i}}\tilde{\mathcal{E}})e_1, e_\beta\rangle|^2}{(\tilde{\lambda}_1 - \tilde{\lambda}_\beta)(1+\mu_i^2)}.$$

Now at (p_0, t_0)

$$\nabla_i\tilde{\mathcal{E}}^k_p = \nabla_i(\alpha_\varphi)_{\bar{k}p},$$

$$\nabla_i\nabla_{\bar{j}}\tilde{\mathcal{E}}^k_p = \nabla_i\nabla_{\bar{j}}\left((\hat{g})^{k\bar{\ell}}(\alpha_\varphi)_{\bar{\ell}p} - B^k_p\right) = (\hat{g})^{k\bar{\ell}}\nabla_i\nabla_{\bar{j}}(\alpha_\varphi)_{\bar{\ell}p},$$

since B is a local holomorphic section. At (p_0, t_0) write $e_\beta = V^j_\beta \partial_{z_j}$, for complex numbers V^j_β, and $1 \leqslant \beta \leqslant n$. For simplicity we denote

$$V_\beta = V^j_\beta \partial_{z_j}$$

regarded as vector fields defined in an open neighbourhood of (p_0, t_0). Then at (p_0, t_0) we have

(5.8) $$\sum_i \frac{1}{1+\mu_i^2}\langle(\nabla_i\nabla_{\bar{i}}\tilde{\mathcal{E}})e_1, e_1\rangle = V^p_1\overline{V^k_1}\sum_i \frac{1}{1+\mu_i^2}\nabla_i\nabla_{\bar{i}}(\alpha_\varphi)_{\bar{k}p}.$$

Using that V_1 has constant coefficients, and α_φ is closed we compute

$$
\begin{aligned}
\nabla_{\bar{V}_1}\nabla_{V_1}(\alpha_\varphi)_{\bar{j}i} &= \overline{V_1^k}V_1^p\nabla_{\bar{k}}\nabla_p(\alpha_\varphi)_{\bar{j}i} \\
&= \overline{V_1^k}V_1^p\nabla_{\bar{k}}\nabla_i(\alpha_\varphi)_{\bar{j}p} \\
&= \overline{V_1^k}V_1^p\left(\nabla_i\nabla_{\bar{k}}(\alpha_\varphi)_{\bar{j}p} + R_{\bar{k}i}{}^s{}_p(\alpha_\varphi)_{\bar{j}s} - R_{\bar{k}i\bar{j}}{}^{\bar{s}}(\alpha_\varphi)_{\bar{s}p}\right) \\
&= \overline{V_1^k}V_1^p\left(\nabla_i\nabla_{\bar{j}}(\alpha_\varphi)_{\bar{k}p} + R_{\bar{k}i}{}^s{}_p(\alpha_\varphi)_{\bar{j}s} - R_{\bar{k}i\bar{j}}{}^{\bar{s}}(\alpha_\varphi)_{\bar{s}p}\right) \\
&= \sum_{k,p=0}^n \overline{V_1^k}V_1^p\left(\nabla_i\nabla_{\bar{j}}(\alpha_\varphi)_{\bar{k}p} + R_{\bar{k}i}{}^j{}_p\mu_j - R_{\bar{k}i\bar{j}}{}^{\bar{p}}\mu_p\right).
\end{aligned}
$$

(5.9)

We combine equations (5.8) (5.9) to obtain at (p_0, t_0)

(5.10)
$$
\sum_i \frac{1}{1+\mu_i^2}\langle(\nabla_i\nabla_{\bar{i}}\tilde{\mathcal{E}})e_1, e_1\rangle = \sum_{i=0}^n \frac{1}{1+\mu_i^2}\nabla_{\bar{V}_1}\nabla_{V_1}(\alpha_\varphi)_{\bar{i}i}
$$
$$
- \sum_{i,k,p=0}^n \frac{1}{1+\mu_i^2}R_{\bar{k}i}{}^i{}_p\mu_i V^p\overline{V^k} + \sum_{i,k,p}\frac{1}{1+\mu_i^2}R_{\bar{k}i\bar{i}}{}^{\bar{p}}\mu_p V^p\overline{V^k}.
$$

Differentiate the equation $F(\mathcal{E}) = h$ in the V_1 direction to get

$$
\nabla_{V_1}h = F^{i\bar{j}}\nabla_{V_1}(\alpha_\varphi)_{\bar{j}i} = \sum_{i=0}^n \frac{1}{1+\mu_i^2}\nabla_{V_1}(\alpha_\varphi)_{\bar{i}i},
$$

(5.11)
$$
\nabla_{\bar{V}_1}\nabla_{V_1}h = F^{i\bar{j}}\nabla_{\bar{V}_1}\nabla_{V_1}(\alpha_\varphi)_{\bar{j}i} + F^{i\bar{j},m\bar{\ell}}\nabla_{V_1}(\alpha_\varphi)_{\bar{j}i}\nabla_{\bar{V}_1}(\alpha_\varphi)_{\bar{\ell}m}
$$
$$
= \sum_{i=0}^n \frac{1}{1+\mu_i^2}\nabla_{\bar{V}_1}\nabla_{V_1}(\alpha_\varphi)_{\bar{i}i} - \sum_{i,j=0}^n \frac{\mu_i+\mu_j}{(1+\mu_i^2)(1+\mu_j^2)}|\nabla_{V_1}(\alpha_\varphi)_{\bar{j}i}|^2.
$$

Before substituting (5.10) into (5.11), we note the following easy, but useful lemma.

Lemma 5.3. *In the above notation, at (p_0, t_0) we have*

$$
\lambda_1 = \sum_{i=0}^n |V_1^i|^2\mu_i, \qquad \sum_{i=0}^n |V_1^i|^2 = 1
$$

In particular, λ_1 is a convex combination of the μ_i, $\lambda_1 \leqslant \mu_0$, with equality if and only if V_1 is in the span of the eigenvectors with eigenvalue μ_0.

Proof. The proof is just a consequence of writing the equations $\lambda_1 = \langle\mathcal{E}e_1, e_1\rangle$, and $\|e_1\|^2 = 1$ in our local coordinate system, and using the definition of V_1. $\square$

We note the following simple corollary

Corollary 5.4. *For all $j \neq n$ we have*

$$
|V_1^j|^2\mu_j \leqslant \lambda_1 + \max\{0, -\mu_n\} \leqslant \lambda_1 + C(\eta_1)
$$

Proof. Just observe that, by Lemma 3.1 the $i = n$ term is the only term is the only possible negative contribution to the sum

$$\lambda_1 = \sum_{i=0}^{n} |V_1^i|^2 \mu_i.$$

Since $\sum_{i=0}^{n} |V_1^i|^2 = 1$ we can rearrange, and apply Lemma 3.1 to conclude. $\qquad\square$

Let $C_1 > 0$ be a two-sided bound for the sectional curvature of $\hat{\omega}$, which by Lemma 2.21 depends only on a bound for the sectional curvature of (X, ω), and is therefore a uniform constant. Then we have
(5.12)

$$\sum_{k,p} R_{\bar{k}i\bar{i}}{}^p \mu_p V^p \overline{V^k} \geqslant -C_1 \sum_{p=0}^{n} |\mu_p| |V^p|^2 = -C_1 \sum_{p=0}^{n} \mu_p |V_1^p|^2 - C_1(|\mu_n| - \mu_n)|V_1^n|^2$$

$$= -C_1 \lambda_1 - C_2$$

where we used Lemma 5.3 and that $\mu_n > -C(\eta_1)$ by Lemma 3.1. For the remaining curvature term we use that $|\frac{\mu_i}{1+\mu_i^2}| \leqslant 1$ together with $\sum_i |V_1^i|^2 = 1$ to get

$$(5.13) \qquad \sum_{i,k,p=0}^{n} \frac{1}{1 + \mu_i^2} R_{\bar{k}i}{}^i{}_p \mu_i V^p \overline{V^k} \leqslant C_1(n+1).$$

Returning to equation (5.10), substituting the bounds (5.12),(5.13), and equation (5.11) we obtain

$$\sum_i \frac{1}{1 + \mu_i^2} \langle (\nabla_i \nabla_{\bar{i}} \tilde{\mathcal{E}}) e_1, e_1 \rangle \geqslant \sum_{i,j=0}^{n} \frac{\mu_i + \mu_j}{(1 + \mu_i^2)(1 + \mu_j^2)} |\nabla_{V_1} (\alpha_\varphi)_{\bar{j}i}|^2 - C\lambda_1 - C$$

for a constant C depending only on n, η, and bound for $\nabla_{\bar{V}_1} \nabla_{V_1} h$ and (X, ω). Note that, since V_1 is spatial, the constant C is uniform. Finally, by noting that $\lambda_1 = \tilde{\lambda}_1 > \tilde{\lambda}_\beta$ for $\beta \neq 1$ we arrive at

Proposition 5.5. *In the above notation, at the point (p_0, t_0) we have*

$$F^{i\bar{j}} \nabla_i \nabla_{\bar{j}} \log(\tilde{\lambda}_1) \geqslant \frac{1}{\lambda_1} \sum_{i,j=0}^{n} \frac{\mu_i + \mu_j}{(1 + \mu_i^2)(1 + \mu_j^2)} |\nabla_{V_1} (\alpha_\varphi)_{\bar{j}i}|^2$$

(5.14)

$$- \frac{1}{\lambda_1^2} \sum_{i=0}^{n} \frac{1}{1 + \mu_i^2} |\nabla_i (\alpha_\varphi)_{\bar{V}_1 V_1}|^2 - C$$

for a uniform constant C.

In order to apply the maximum principle, we need a lower bound for the quantity appearing on the right hand side of (5.14). Note that $\nabla_i (\alpha_\varphi)_{V_1 \bar{V}_1} =$

$\nabla_{V_1}(\alpha_\varphi)_{\bar{V}_1 i}$ since α_φ is closed and V_1 has constant coefficients. Thus, all the negative terms appearing in

$$(5.15) \qquad -\sum_{i=0}^{n} \frac{1}{1+\mu_i^2} |\nabla_i(\alpha_\varphi)\bar{V}_1 V_1|^2$$

have partners appearing in the sum

$$(5.16) \qquad \frac{1}{\lambda_1} \sum_{i,j=0}^{n} \frac{\mu_i + \mu_j}{(1+\mu_i^2)(1+\mu_j^2)} |\nabla_{V_1}(\alpha_\varphi)_{\bar{j}i}|^2.$$

There are three significant difficulties in attaining the estimate we want. The first is that the $i = j = n$ term appear in (5.16) will be non-positive when $\mu_n \leqslant 0$, and even when $\mu_n > 0$, the contribution can be arbitrarily small. Of course, by Lemma 3.1, we know that $-e^{-AF}$ is concave. However, this only implies that the bad term $i = j = n$ term appearing in (5.16) can be controlled at the expense of *all* the terms with $i = j \neq n$ appearing in (5.16). Unfortunately, terms like $|\nabla_{V_1}\alpha_{i\bar{i}}|^2$ evidently appear in $|\nabla_i\alpha_{V_1\bar{V}_1}|^2$. Thus, invoking concavity of $-e^{-AF}$ leaves us with no way to control (5.15). This is in stark contrast to the case of concave elliptic operators [71, 42], where the terms in (5.16) for $i \neq j$ are not used.

The second significant difficulty occurs when trying to invoke that the gradient vanishes at the maximum point. More precisely, a natural approach to controlling (5.15), is to apply the fact that the gradient of our test function vanishes at the maximum. Our test function will be of the form

$$\tilde{Q} := \log(\tilde{\lambda}_1) + H(\varphi)$$

for a specially chosen function H. Thus, at a maximum we will have

$$\frac{\nabla_i(\alpha_\varphi)_{\bar{V}_1 V_1}}{\lambda_1} = -H'(\varphi)\nabla_i\varphi.$$

Such an argument is essentially doomed, since we do not control $|\nabla_i\varphi|^2_{\hat{g}}$ uniformly. In particular, the vector ∂_{z_i} could have a component pointing in the time direction, and by Theorem 4.3, such a component would contribute a term of order ε^{-2} to the estimate. Again, this is in contrast to the case of concave elliptic operators on compact manifolds [71, 42], where this argument is used repeatedly to obtain a C^2 estimate [71, 42].

To see the final significant problem, note that (5.15) contains a term like $\frac{1}{(1+\mu_0^2)\lambda_1^2}|\nabla_{V_1}\alpha_{\bar{0}0}|^2$, and the corresponding term in (5.16) is $\frac{2\mu_0}{(1+\mu_0^2)^2\lambda_1}|\nabla_{V_1}\alpha_{\bar{0}0}|^2$. Since λ_1 is the largest *spatial* eigenvalue, it can (and indeed must) happen that $\mu_0 \gg \lambda_1$, so that

$$\frac{2\mu_0}{(1+\mu_0^2)^2\lambda_1} \ll \frac{1}{(1+\mu_0^2)\lambda_1^2}.$$

Again, when studying concave elliptic operators on compact manifolds [71, 42] we have the $\mu_0 = \lambda_1$, and the two terms above are easily comparable.

There are two key points in the arguments that follow. The first is to split the eigenvalues μ_i according to whether they are large, or small, compared to λ_1. Roughly speaking, if μ_i is small relative to λ_1 we expect that ∂_{z_i} can contain at most a very small component pointing in the temporal direction. On the other hand, if μ_i is very large relative to λ_1 the we must make efficient use of the coefficients V_1^i that appear in (5.15), together with the following trivial observation from Lemma 5.3: if $\mu_i \gg \lambda_1$, then $|V_1^i|^2 \ll 1$.

Before beginning the estimate, let us fix a suitable background form, and some notation. Let $\hat{\varphi}$ be one of the subsolutions constructed in Corollary 3.10, and write

$$\hat{\alpha} = \pi_X^* \alpha + \sqrt{-1} D \overline{D} \hat{\varphi}$$

Then we can write $\alpha_\varphi = \hat{\alpha} + \sqrt{-1} D \overline{D} (\varphi - \hat{\varphi})$. To simplify the notation, let us denote $\alpha = \alpha_\varphi$.

As a first step, we are going to address the first issue outlined above by estimating the term $\nabla_{V_1} \alpha_{\bar{n}n}$, assuming $\mu_n < 0$. We first observe that $\alpha_{\bar{n}n} = \mu_n$ at (p_0, t_0), and so by differentiating the equation we have

$$\frac{1}{1+\mu_n^2} \nabla_{V_1} \alpha_{\bar{n}n} = \nabla_{V_1} h - \sum_{k<n} \frac{1}{1+\mu_k^2} \nabla_{V_1} \alpha_{\bar{k}k}.$$

By Cauchy-Schwarz we have

$$(5.17) \qquad \frac{|\nabla_{V_1} \alpha_{\bar{n}n}|^2}{(1+\mu_n^2)^2} \leqslant (1+\delta_1^{-1})|\nabla_{V_1} h|^2 + (1+\delta_1)\left| \sum_{k<n} \frac{1}{1+\mu_k^2} \nabla_{V_1} \alpha_{\bar{k}k} \right|^2$$

for a constant δ_1 to be determined. Fix a constant $\frac{1}{2} > \delta_0 > 0$. This constants will be determined in the course of the proof. We say that μ_β is big if $\mu_\beta \geqslant \delta_0 \lambda_1$, and small if $\mu_\beta < \delta_0 \lambda_1$. Define

$$B = \{k \in \{0, 1, \dots, n\} : \mu_k \geqslant \delta_0 \lambda_1\}, \qquad S = \{k \in \{0, 1, \dots, n\} : \mu_k < \delta_0 \lambda_1\}.$$

Clearly $\mu_n \in S$ if $\mu_n < 0$ and also if λ_1 is sufficiently large, depending on δ_0. We write

$$\left| \sum_{k<n} \frac{1}{1+\mu_k^2} \nabla_{V_1} \alpha_{\bar{k}k} \right|^2 = \left| \sum_{k\in B} \frac{\nabla_{V_1} \alpha_{\bar{k}k}}{(1+\mu_k^2)} \right|^2 + \left| \sum_{k\in S, \, k<n} \frac{\nabla_{V_1} \alpha_{\bar{k}k}}{(1+\mu_k^2)} \right|^2$$

$$+ 2\mathrm{Re}\left(\sum_{k\in B, \ell\in S, \, \ell<n} \frac{\nabla_{V_1} \alpha_{\bar{k}k} \overline{\nabla_{V_1} \alpha_{\bar{\ell}\ell}}}{(1+\mu_k^2)(1+\mu_\ell^2)} \right).$$

Each sum needs to be estimated differently. Starting with the last sum over S, using Cauchy-Schwarz we estimate

$$\left| \sum_{k\in S, \, k<n} \frac{\nabla_{V_1} \alpha_{\bar{k}k}}{(1+\mu_k^2)} \right|^2 \leqslant \left(\sum_{k\in S, k<n} \frac{\mu_k |\nabla_{V_1} \alpha_{\bar{k}k}|^2}{(1+\mu_k^2)^2} \right) \cdot \left(\sum_{j\in S, j<n} \frac{1}{\mu_j} \right)$$

where we used that $\mu_j > 0$ for $j \neq n$ by Lemma 3.1. Since $\sum_{i=0}^{n} \arctan(\mu_i) > (n-1)\frac{\pi}{2} + \eta_1$, and $\mu_n < 0$ we have

$$(5.18) \qquad \sum_{j \in S, j < n} \frac{1}{\mu_j} + \frac{1}{\mu_n} < -\tan(\eta_1)$$

by Lemma 3.1, together with $\mu_\ell > 0$ if $\ell \in B$. This estimate is clearly not accurate if $|\mu_n|$ is small. In that case we instead have the estimate

$$(5.19) \qquad \sum_{j \in S, j < n} \frac{1}{\mu_j} + \frac{1}{\mu_n} < \frac{n+1}{\tan(\eta_1)} + \frac{1}{\mu_n}$$

where we used that $\mu_i > \tan(\eta_1)$ for all $i < n$ by Lemma 3.1. Estimate (5.19) is more precise when $|\mu_n| < \frac{\tan(\eta_1)}{(n+1)+(\tan(\eta_1))^2}$, and less precise otherwise. By considering each case separately it follows that

$$\mu_n \left(\sum_{j \in S, j < n} \frac{1}{\mu_j} \right) \geqslant -\frac{(n+1)}{(n+1)+(\tan(\eta_1))^2}.$$

The final estimate we obtain is
(5.20)
$$\mu_n \left| \sum_{k \in S,\, k < n} \frac{\nabla_{V_1} \alpha_{\bar{k}k}}{(1+\mu_k^2)} \right|^2 \geqslant \left(-\frac{(n+1)}{(n+1)+(\tan(\eta_1))^2} \right) \cdot \left(\sum_{k \in S,\, k < n} \frac{\mu_k |\nabla_{V_1} \alpha_{\bar{k}k}|^2}{(1+\mu_k^2)^2} \right).$$

Next we estimate the cross terms.

$$\left| \mathrm{Re} \left(\sum_{k \in B, \ell \in S, \ell < n} \frac{\nabla_{V_1} \alpha_{\bar{k}k} \overline{\nabla_{V_1} \alpha_{\bar{\ell}\ell}}}{(1+\mu_k^2)(1+\mu_\ell^2)} \right) \right| \leqslant \sum_{k \in B, \ell \in S, \ell < n} \frac{|\nabla_{V_1} \alpha_{\bar{k}k}||\nabla_{V_1} \alpha_{\bar{\ell}\ell}|}{(1+\mu_k^2)(1+\mu_\ell^2)}$$

$$\leqslant \sum_{k \in B, \ell \in S, \ell < n} \frac{(n+1)|\nabla_{V_1} \alpha_{\bar{k}k}|^2}{\varepsilon_1 \mu_\ell (1+\mu_k^2)^2} + \sum_{k \in B, \ell \in S, \ell < n} \frac{\varepsilon_1 \mu_\ell |\nabla_{V_1} \alpha_{\bar{\ell}\ell}|^2}{(n+1)(1+\mu_\ell^2)^2}$$

$$\leqslant \frac{(n+1)^2}{\varepsilon_1 \tan(\eta_1)} \sum_{k \in B} \frac{|\nabla_{V_1} \alpha_{\bar{k}k}|^2}{(1+\mu_k^2)^2} + \varepsilon_1 \sum_{\ell \in S, \ell < n} \frac{\mu_\ell |\nabla_{V_1} \alpha_{\bar{\ell}\ell}|^2}{(1+\mu_\ell^2)^2},$$

where $\varepsilon_1 > 0$ is a constant to be determined, and we have again used the lower bound $\mu_\ell > \tan(\eta_1) > 0$ for $\ell < n$. Finally, we estimate the big terms using the Cauchy-Schwarz inequality

$$\left| \sum_{k \in B} \frac{\nabla_{V_1} \alpha_{\bar{k}k}}{(1+\mu_k^2)} \right|^2 \leqslant (n+1) \sum_{k \in B} \frac{|\nabla_{V_1} \alpha_{\bar{k}k}|^2}{(1+\mu_k^2)^2}.$$

We now combine these estimates to estimate (5.16) from below when $\mu_n < 0$. We have

$$\sum_{i,j=0}^{n} \frac{\mu_i + \mu_j}{(1+\mu_i^2)(1+\mu_j^2)}|\nabla_{V_1}\alpha_{\bar{j}i}|^2 \geqslant \sum_{i,j=0,i\neq j}^{n-1} \frac{\mu_i + \mu_j}{(1+\mu_i^2)(1+\mu_j^2)}|\nabla_{V_1}\alpha_{\bar{j}i}|^2$$

$$+ \left(1 + (1+\delta_1)\left(\frac{-(n+1)}{(n+1)+(\tan(\eta_1))^2} + 2\mu_n\varepsilon_1\right)\right)\sum_{\ell\in S,\ell<n}\frac{2\mu_\ell}{(1+\mu_\ell^2)^2}|\nabla_{V_1}\alpha_{\bar{\ell}\ell}|^2$$

$$+ \sum_{k\in B}\frac{2|\nabla_{V_1}\alpha_{\bar{k}k}|^2}{(1+\mu_k^2)^2}\left(\mu_k + (1+\delta_1)\mu_n\left(\frac{2(n+1)^2}{\varepsilon_1\tan(\eta_1)} + (n+1)\right)\right)$$

$$+ 2(1+\delta_1^{-1})\mu_n|\nabla_{V_1}h|^2.$$

For ease of notation set

$$\kappa_1 = \frac{(\tan(\eta_1))^2}{2[(n+1)+(\tan(\eta_1))^2]}.$$

Examining the coefficient in front of the small terms yields

$$1 + (1+\delta_1)\left(\frac{-(n+1)}{(n+1)+(\tan(\eta_1))^2} + 2\mu_n\varepsilon_1\right) = 2\kappa_1 - \delta_1(1-2\kappa_1) + 2(1+\delta_1)\mu_n\varepsilon_1.$$

Take $\delta_1 = \kappa_1$, then

$$1 + (1+\delta_1)\left(\frac{-(n+1)}{(n+1)+(\tan(\eta_1))^2} + 2\mu_n\varepsilon_1\right) = \kappa_1 + 2\kappa_1^2 + 2(1+\kappa_1)\mu_n\varepsilon_1$$

$$\geqslant \kappa_1$$

provided we take $\varepsilon_1 = \frac{\kappa_1^2}{(1+\kappa_1)C(\eta_1)}$, where $\mu_n > -C(\eta_1)$ is a lower bound for μ_n. Since $\kappa_1, \delta_1, \varepsilon_1$ are universal we have

$$\mu_k + (1+\delta_1)\mu_n\left(\frac{2(n+1)^2}{\varepsilon_1\tan(\eta_1)} + (n+1)\right) \geqslant \mu_k - 2C_1,$$

$$2(1+\delta_1^{-1})\mu_n|\nabla_{V_1}h|^2 \geqslant -C_2$$

for universal constants C_1, C_2. We have proved

Proposition 5.6. *We have the following lower bound for* (5.16).

$$\sum_{i,j=0}^{n} \frac{\mu_i + \mu_j}{(1+\mu_i^2)(1+\mu_j^2)}|\nabla_{V_1}\alpha_{\bar{j}i}|^2 \geqslant -C_2 + \sum_{\substack{0\leqslant i,j\leqslant n \\ i\neq j}} \frac{\mu_i + \mu_j}{(1+\mu_i^2)(1+\mu_j^2)}|\nabla_{V_1}\alpha_{\bar{j}i}|^2$$

$$+ \kappa_1 \sum_{\ell\in S,\ell<n}\frac{2\mu_\ell}{(1+\mu_\ell^2)^2}|\nabla_{V_1}\alpha_{\bar{\ell}\ell}|^2$$

$$+ \sum_{k\in B}\frac{(2\mu_k - C_1)|\nabla_{V_1}\alpha_{\bar{k}k}|^2}{(1+\mu_k^2)^2}$$

where $C_1 = C_2 = 0$ and $\kappa_1 = 1$ if $\mu_n \geqslant 0$, and if $\mu_n < 0$, then

$$\kappa_1 = \frac{(\tan(\eta_1))^2}{2[(n+1) + (\tan(\eta_1))^2]}$$

and $C_1, C_2 \geqslant 0$ are uniform constants.

We now turn our focus to estimating the gradient term (5.15). Fix $0 \leqslant i \leqslant n$ and write

$$
\begin{aligned}
|\nabla_i \alpha_{\bar{V}_1 V_1}|^2 &= \left| \sum_{j=0}^n V_1^j \nabla_{V_1} \alpha_{\bar{j}i} \right|^2 \\
&\leqslant \sum_{j,\ell \in B} |V_1^j||V_1^\ell||\nabla_{V_1} \alpha_{\bar{j}i}||\nabla_{V_1} \alpha_{\bar{\ell}i}| \\
&\quad + 2 \sum_{j \in B, \ell \in S} |V_1^j||V_1^\ell||\nabla_{V_1} \alpha_{\bar{j}i}||\nabla_{V_1} \alpha_{\bar{\ell}i}| \\
&\quad + \sum_{j,\ell \in S} |V_1^j||V_1^\ell||\nabla_{V_1} \alpha_{\bar{j}i}||\nabla_{V_1} \alpha_{\bar{\ell}i}|
\end{aligned}
$$

For the big terms, we use Cauchy-Schwarz in the following way
(5.21)

$$
\begin{aligned}
\sum_{j,\ell \in B} |V_1^j||V_1^\ell||\nabla_{V_1} \alpha_{\bar{j}i}||\nabla_{V_1} \alpha_{\bar{\ell}i}| &\leqslant \left(\sum_{j,\ell \in B} |V_1^\ell|^2 \frac{\mu_\ell}{\mu_j} |\nabla_{V_1} \alpha_{\bar{j}i}|^2 \right)^{1/2} \left(\sum_{j,\ell \in B} |V_1^j|^2 \frac{\mu_j}{\mu_\ell} |\nabla_{V_1} \alpha_{\bar{\ell}i}|^2 \right)^{1/2} \\
&= \sum_{j,\ell \in B} |V_1^\ell|^2 \frac{\mu_\ell}{\mu_j} |\nabla_{V_1} \alpha_{\bar{j}i}|^2 \\
&= \left(\lambda_1 - \sum_{\ell \in S} |V_1^\ell|^2 \mu_\ell \right) \sum_{j \in B} \frac{1}{\mu_j} |\nabla_{V_1} \alpha_{\bar{j}i}|^2
\end{aligned}
$$

where in the last line we used Lemma 5.3. For the small terms we apply Cauchy-Schwarz to get
(5.22)

$$
\begin{aligned}
\sum_{j,\ell \in S} |V_1^j||V_1^\ell||\nabla_{V_1} \alpha_{\bar{j}i}||\nabla_{V_1} \alpha_{\bar{\ell}i}| &\leqslant \left(\sum_{j,\ell \in S} |V_1^\ell|^2 |\nabla_{V_1} \alpha_{\bar{j}i}|^2 \right)^{1/2} \left(\sum_{j,\ell \in S} |V_1^j|^2 |\nabla_{V_1} \alpha_{\bar{\ell}i}|^2 \right)^{1/2} \\
&= \sum_{j,\ell \in S} |V_1^\ell|^2 |\nabla_{V_1} \alpha_{\bar{j}i}|^2 \leqslant \sum_{j \in S} |\nabla_{V_1} \alpha_{\bar{j}i}|^2
\end{aligned}
$$

where in the last line we used Lemma 5.3 again. Finally we estimate the cross terms.
(5.23)
$$2 \sum_{j \in B, \ell \in S} |V_1^j||V_1^\ell||\nabla_{V_1} \alpha_{\bar{j}i}||\nabla_{V_1} \alpha_{\bar{\ell}i}| \leqslant \varepsilon_0 \sum_{j \in B, \ell \in S} |V_1^j|^2 |V_1^\ell|^2 |\nabla_{V_1} \alpha_{\bar{j}i}|^2 + \frac{1}{\varepsilon_0} \sum_{j \in B, \ell \in S} |\nabla_{V_1} \alpha_{\bar{\ell}i}|^2$$
$$\leqslant \varepsilon_0 \sum_{j \in B} |V_1^j|^2 |\nabla_{V_1} \alpha_{\bar{j}i}|^2 + \frac{n+1}{\varepsilon_0} \sum_{\ell \in S} |\nabla_{V_1} \alpha_{\bar{\ell}i}|^2$$
$$\leqslant \varepsilon_0 \left(\lambda_1 - \sum_{\ell \in S} |V_1^\ell|^2 \mu_\ell \right) \sum_{j \in B} \frac{1}{\mu_j} |\nabla_{V_1} \alpha_{\bar{j}i}|^2$$
$$+ \frac{n+1}{\varepsilon_0} \sum_{\ell \in S} |\nabla_{V_1} \alpha_{\bar{\ell}i}|^2$$

for a constant $\varepsilon_0 > 0$ to be determined. In the final inequality we used the following estimate. If $j \in B$, we have $\mu_j > 0$, so Lemma 5.3 shows

$$|V_1^j|^2 \mu_j \leqslant \sum_{k \in B} |V_1^k|^2 \mu_k = \lambda_1 - \sum_{\ell \in S} |V_1^\ell|^2 \mu_\ell.$$

Combining estimates (5.21) (5.22) and (5.23) we arrive at

$$|\nabla_i \alpha_{\bar{V}_1 V_1}|^2 \leqslant (1 + \varepsilon_0) \left(\lambda_1 - \sum_{\ell \in S} |V_1^\ell|^2 \mu_\ell \right) \sum_{j \in B} \frac{1}{\mu_j} |\nabla_{V_1} \alpha_{\bar{j}i}|^2$$
$$+ (1 + \frac{n+1}{\varepsilon_0}) \sum_{j \in S} |\nabla_{V_1} \alpha_{\bar{j}i}|^2.$$

Every term appearing on the right hand side of this estimate has a partner appearing on the right hand side of the estimate in Proposition 5.6 *except* the term $i = j = n$. For this term we return to (5.17) and apply Cauchy-Schwarz to get

$$|\nabla_{V_1} \alpha_{\bar{n}n}|^2 \leqslant 2|\nabla_{V_1} h|^2 + 2n \sum_{\ell < n} \frac{|\nabla_{V_1} \alpha_{\bar{\ell}\ell}|^2}{(1 + \mu_\ell^2)^2}.$$

Summarizing we have proved

Proposition 5.7. *For any constant $\varepsilon_0 \in (0, 1)$ we have the following estimates.*

(1) *For $0 \leqslant i < n$*

$$|\nabla_i \alpha_{\bar{V}_1 V_1}|^2 \leqslant (1 + \varepsilon_0) \left(\lambda_1 - \sum_{\ell \in S} |V_1^\ell|^2 \mu_\ell \right) \sum_{j \in B} \frac{1}{\mu_j} |\nabla_{V_1} \alpha_{\bar{j}i}|^2$$
$$+ (1 + \frac{n+1}{\varepsilon_0}) \sum_{j \in S} |\nabla_{V_1} \alpha_{\bar{j}i}|^2.$$

(2) *For $i = n$ we have*

$$|\nabla_n \alpha_{\bar{V}_1 V_1}|^2 \leqslant (1 + \varepsilon_0) \left(\lambda_1 - \sum_{\ell \in S} |V_1^\ell|^2 \mu_\ell \right) \sum_{j \in B} \frac{1}{\mu_j} |\nabla_{V_1} \alpha_{\bar{j}n}|^2$$

$$+ (1 + \frac{n+1}{\varepsilon_0}) \sum_{j \in S, j < n} |\nabla_{V_1} \alpha_{\bar{j}n}|^2$$

$$+ 2(1 + \frac{n+1}{\varepsilon_0}) C_3 + 2n(1 + \frac{n+1}{\varepsilon_1}) \sum_{\ell < n} \frac{|\nabla_{V_1} \alpha_{\bar{\ell}\ell}|^2}{(1 + \mu_\ell^2)^2}.$$

for a uniform constant C_3.

We are now ready to combine Propositions 5.6 and 5.7 to obtain the key estimate towards the interior C^2 estimate. Recall that we need to estimate the following quantity from below;

$$(5.24) \qquad \frac{1}{\lambda_1} \sum_{i,j=0}^{n} \frac{\mu_i + \mu_j}{(1 + \mu_i^2)(1 + \mu_j^2)} |\nabla_{V_1} \alpha_{\bar{j}i}|^2 - \frac{1}{\lambda_1^2} \sum_{i=0}^{n} \frac{1}{1 + \mu_i^2} |\nabla_i \alpha_{\bar{V}_1 V_1}|^2.$$

In order to do this, we will estimate a related quantity. Let us define

$$\Gamma := \sum_{\ell \in S} |V_1^\ell|^2 \mu_\ell$$

and note that, by definition of S we have $\Gamma \leqslant \delta_0 \lambda_1 \leqslant \frac{1}{2}\lambda_1$. Consider

$$\Upsilon := \frac{(1 - c_0 \lambda_1^{-1})}{\lambda_1 (\lambda_1 - \Gamma)(1 + \varepsilon_0)} \sum_{i=0}^{n} \frac{1}{1 + \mu_i^2} |\nabla_i \alpha_{\bar{V}_1 V_1}|^2$$

Where c_0 is a positive constant to be determined, and we assume that

$$(5.25) \qquad\qquad\qquad\qquad \lambda_1 > c_0.$$

From now on, in order to ensure that our constants can be chosen consistently, we will note each constraint as a separate equation. By Proposition 5.7 we have

$$\Upsilon \leqslant \frac{(1 - c_0 \lambda_1^{-1})}{\lambda_1} \sum_{\substack{0 \leqslant i \leqslant n \\ j \in B}} \frac{|\nabla_{V_1} \alpha_{\bar{j}i}|^2}{\mu_j (1 + \mu_i^2)}$$

$$+ \frac{2(n+1)(1 - c_0 \lambda_1^{-1})}{\lambda_1 (\lambda_1 - \Gamma)(1 + \varepsilon_0)\varepsilon_0} \sum_{\substack{0 \leqslant i \leqslant n, j \in S \\ (i,j) \neq (n,n)}} \frac{|\nabla_{V_1} \alpha_{\bar{j}i}|^2}{1 + \mu_i^2}$$

$$+ \frac{4n(n+1)(1 - c_0 \lambda_1^{-1})}{\lambda_1 (1 + \mu_n^2)(\lambda_1 - \Gamma)(1 + \varepsilon_0)\varepsilon_0} \sum_{\ell < n} \frac{|\nabla_{V_1} \alpha_{\bar{\ell}\ell}|^2}{(1 + \mu_\ell^2)^2}$$

$$+ \frac{4(n+1)(1 - c_0 \lambda_1^{-1})}{\lambda_1 (1 + \mu_n^2)(\lambda_1 - \Gamma)(1 + \varepsilon_0)\varepsilon_0} C_3$$

We are going to estimate

$$\frac{1}{\lambda_1} \sum_{i,j=0}^{n} \frac{\mu_i + \mu_j}{(1+\mu_i^2)(1+\mu_j^2)} |\nabla_{V_1} \alpha_{\bar{j}i}|^2 - \Upsilon$$

from below, by making use of Proposition 5.6, and comparing the coefficients of $|\nabla_{V_1} \alpha_{\bar{j}i}|^2$ term by term. There are four cases depending on (i,j).

Case 1: $0 \leqslant i \leqslant n, j \in B$, and $i \neq j$.
Case 2: $i = j \in B$.
Case 3: $0 \leqslant i \leqslant n, j \in S$ and $i \neq j$.
Case 4: $0 \leqslant i < n$, and $i = j \in S$.

Case 1. We we have to compare $\frac{\mu_i + \mu_j}{\lambda_1(1+\mu_i^2)(1+\mu_j^2)}$ and $\frac{(1-c_0\lambda_1^{-1})}{\lambda_1(1+\mu_i^2)\mu_j}$. Ignoring the common factors it suffices to estimate

$$\frac{\mu_i + \mu_j}{(1+\mu_j^2)} - \frac{(1 - c_0\lambda_1^{-1})}{\mu_j} = \frac{c_0\lambda_1^{-1}\mu_j^2 + \mu_i\mu_j + c_0\lambda_1^{-1} - 1}{\mu_j(1+\mu_j^2)}.$$

We now use that $j \in B$, so $\mu_j\lambda_1^{-1} > \delta_0$. Since $\mu_i \geqslant \mu_n \geqslant -C(\eta_1)$ we get the estimate

$$\frac{c_0\lambda_1^{-1}\mu_j^2 + \mu_i\mu_j + c_0\lambda_1^{-1} - 1}{\mu_j(1+\mu_j^2)} \geqslant \frac{(c_0\delta_0 - C(\eta_1))\mu_j - 1}{\mu_j(1+\mu_j^2)} > 0$$

as long as we have

(5.26) $$c_0 > \delta_0^{-1}(C(\eta_1) + 1), \quad \text{and} \quad \lambda_1 > \delta_0^{-1}$$

where the latter condition guarantees $\mu_j > 1$ since $j \in B$.

Case 2 In this case we need to estimate

(5.27) $$\frac{(2\mu_j - C_1)}{1 + \mu_j^2} - \frac{(1 - c_0\lambda_1^{-1})}{\mu_j} - \frac{4n(n+1)(1 - c_0\lambda_1^{-1})}{(1+\mu_n^2)(1+\mu_j^2)(\lambda_1 - \Gamma)(1+\varepsilon_0)\varepsilon_0}$$

Observe that, from the definition of Γ we have

$$\lambda_1 - \Gamma = \lambda_1 - \sum_{\ell \in S} |V_1^\ell|^2 \mu_\ell \geqslant (1 - \delta_0)\lambda_1 \geqslant \frac{1}{2}\lambda_1$$

Therefore we can estimate (5.27) from below by

$$\frac{1}{\mu_j(1+\mu_j^2)}\left(\mu_j^2 - C_1\mu_j - 1 - 8n(n+1)\frac{\mu_j}{\lambda_1\varepsilon_0}\right) > 0$$

provided $\mu_j > C_1 + 1 + \frac{8n(n+1)}{\lambda_1\varepsilon_0}$. Since $j \in B$ this is guaranteed as soon as

(5.28) $$\lambda_1 \geqslant \delta_0^{-1}\left(C_1 + 1 + \frac{8n(n+1)}{\lambda_1\varepsilon_0}\right)$$

Case 3 We have to estimate

$$\frac{\mu_i + \mu_j}{(1 + \mu_j^2)} - \frac{2(n+1)(1 - c_0\lambda_1^{-1})}{(\lambda_1 - \Gamma)(1 + \varepsilon_0)\varepsilon_0} \geqslant \frac{\mu_i + \mu_j}{(1 + \mu_j^2)} - \frac{4(n+1)}{\lambda_1\varepsilon_0}$$

We now consider three subcases separately, according to whether μ_j is relatively large or relatively small compared to μ_n, or $j = n$.

Case 3a First suppose $j < n$, and that $\mu_i + \mu_j \geqslant \frac{3}{4}\mu_j$ (which is always the case if $\mu_n \geqslant 0$). Then we have

$$\mu_i + \mu_j - \frac{4(n+1)(1 + \mu_j^2)}{\lambda_1\varepsilon_0} \geqslant \frac{3\mu_j}{4} - \frac{4(n+1)}{\varepsilon_0}\frac{(1 + \mu_j^2)}{\lambda_1}$$

$$\geqslant \frac{3\mu_j}{4} - \frac{4(n+1)}{\varepsilon_0\lambda_1} - \frac{4(n+1)\delta_0\mu_j}{\varepsilon_1}$$

where we used that $\frac{1+\mu_j^2}{\lambda_1} < \frac{1}{\lambda_1} + \delta_0\mu_j$ by the definition of S. Using that $j \neq n$, if we choose

(5.29) $$\delta_0 < \frac{\varepsilon_0}{16(n+1)}$$

we get

$$\mu_i + \mu_j - \frac{4(n+1)(1 + \mu_j^2)}{\lambda_1\varepsilon_1} \geqslant \frac{1}{2}\mu_j - \frac{4(n+1)}{\varepsilon_0\lambda_1}$$

$$\geqslant \frac{\tan(\eta_1)}{4} - \frac{4(n+1)}{\varepsilon_0\lambda_1} > 0$$

provided

(5.30) $$\lambda_1 > \frac{16(n+1)}{\varepsilon_0\tan(\eta_1)}.$$

Case 3b In this case we still assume $j < n$, but that $\mu_i + \mu_j \leqslant \frac{3}{4}\mu_j$. Then necessarily $i = n$, $\mu_n < 0$, and we get that $\mu_j \leqslant 4C(\eta_1)$ for a constant depending only on η_1. Since $\mu_j + \mu_n > \tan(\eta_1)$ by Lemma 3.1 we conclude

(5.31) $$\frac{\mu_i + \mu_j}{(1 + \mu_j^2)} - \frac{4(n+1)}{\lambda_1\varepsilon_0} \geqslant \frac{\tan(\eta_1)}{1 + 16C(\eta_1)^2} - \frac{4(n+1)}{\lambda_1\varepsilon_0} > 0$$

provided

(5.32) $$\lambda_1 > \frac{4(n+1)(1 + 16C(\eta_1)^2)}{\varepsilon_0\tan(\eta_1)}$$

Case 3c If $j = n$, and $0 \leqslant i < n$, then (5.31) also holds, and the same estimate works.

Case 4 Finally, we consider the fourth case. We need to estimate from below the quantity

$$\kappa_1 \frac{2\mu_j}{(1+\mu_j^2)} - \frac{4(n+1)(1-c_0\lambda_1^{-1})}{\lambda_1(1+\varepsilon_0)\varepsilon_0} - \frac{8n(n+1)(1-c_0\lambda_1^{-1})}{(1+\mu_n^2)\lambda_1(1+\varepsilon_0)(1+\mu_j^2)\varepsilon_0}$$

$$\geqslant \kappa_1 \frac{2\mu_j}{(1+\mu_j^2)} - \frac{c_1}{\lambda_1\varepsilon_0} - \frac{c_1}{\lambda_1(1+\mu_j^2)\varepsilon_0}$$

where $c_1 > 0$ depends only on n, η_1, η_2, and where κ_1 is the constant appearing in Proposition 5.6. We therefore need to determine the sign of

$$2\kappa_1\varepsilon_0\lambda_1\mu_j - c_1(1+\mu_j^2) - c_1 = \mu_j(2\kappa_1\varepsilon_0\lambda_1 - c_1\mu_j) - 2c_1.$$

Since $j \in S$ we have $\mu_j \leqslant \delta_0\lambda_1$ and so

$$\mu_j(2\kappa_1\varepsilon_0\lambda_1 - c_1\mu_j) - 2c_1 \geqslant \mu_j\lambda_1(2\kappa_1\varepsilon_0 - c_1\delta_0) - 2c_1 > 0$$

provided

(5.33) $$\delta_0 < \frac{\kappa_1\varepsilon_0}{c_1} \qquad \lambda_1 > \frac{4c_1}{\tan(\eta_1)\kappa_1\varepsilon_0}$$

We can now choose the constants $\delta_0, \varepsilon_0, c_0$ consistently. From (5.29) and (5.33) we see that it suffices to choose $\delta_0 < \max\{c\kappa_1\varepsilon_0, \frac{1}{2}\}$ for $0 < c < 1$ a positive, uniform constant depending only on n, η_1, η_2. Recall that κ_1 is the uniform constant appearing in Proposition 5.6. From (5.26) we choose $c_0 = K_0\varepsilon_0^{-1}$ for a positive uniform constant K_0 depending only on n, η_1. Finally, from (5.28) we conclude

Proposition 5.8. *For every constant $\varepsilon_0 > 0$, there exists a uniform constant C_4 depending only on n, η_1, η_2 so that if $\lambda_1 \geqslant C_4\varepsilon_0^{-2}$ we have*

$$\frac{1}{\lambda_1}\sum_{i,j=0}^{n} \frac{\mu_i + \mu_j}{(1+\mu_i^2)(1+\mu_j^2)}|\nabla_{V_1}\alpha_{\bar{j}i}|^2 - \frac{1}{\lambda_1^2}\sum_{i=0}^{n} \frac{1}{1+\mu_i^2}|\nabla_i\alpha_{\bar{V_1}V_1}|^2$$

$$\geqslant -\frac{2\varepsilon_0}{\lambda_1^2}\sum_{i=0}^{n} \frac{1}{1+\mu_i^2}|\nabla_i\alpha_{\bar{V_1}V_1}|^2 - \frac{C_2}{\lambda_1} - \frac{C_5}{\varepsilon_0\lambda_1^2}$$

For uniform constants C_2, C_5.

Proof. By combining Propositions 5.6 and 5.7 we have shown that

$$\frac{1}{\lambda_1}\sum_{i,j=0}^{n} \frac{\mu_i + \mu_j}{(1+\mu_i^2)(1+\mu_j^2)}|\nabla_{V_1}\alpha_{\bar{j}i}|^2 - \Upsilon \geqslant -\frac{C_2}{\lambda_1} - C(n,\eta_1,\eta_2)\frac{(1-c_0\lambda_1^{-1})}{\varepsilon_1\lambda_1^2}C_3.$$

We only need to compare Υ with the negative term containing $|\nabla_i\alpha_{\bar{V_1}V_1}|^2$. Choose $C_4 = 100K_0$, where K_0 is the constant defining c_0 above. If $\lambda_1 > C_4\varepsilon_0^{-2}$, then $c_0\lambda_1^{-1} < \varepsilon_0$ and we have

$$\frac{(1-c_0\lambda_1^{-1})}{(\lambda_1 - \Gamma)(1+\varepsilon_0)} - \frac{1}{\lambda_1} \geqslant \frac{1-\varepsilon_0}{\lambda_1(1+\varepsilon_0)} - \frac{1}{\lambda_1}$$

$$\geqslant \frac{-2\varepsilon_0}{\lambda_1}.$$

The result follows. □

We can now prove the interior, spatial C^2 estimate. Recall that we have written $\alpha = \alpha_\varphi = \hat\alpha + \sqrt{-1}D\overline{D}\varphi$. Normalize φ so that $\inf_{X_\varepsilon} \varphi = 0$. Define $H : [0, \sup_{X_\varepsilon} \varphi]$ by

$$H(t) = -2At + \frac{A\tau}{2}t^2$$

for constants $A \gg 0, \tau > 0$ to be determined. By Proposition 3.12, and Corollary 3.10 we have a uniform bound for $\|\varphi\|_{L^\infty}$, and so we can choose $\tau < \min\{\frac{1}{2\|\varphi\|_{L^\infty}}, 1\}$ so that

$$-2A \leqslant H' \leqslant -A \qquad H'' = A\tau > 0.$$

We apply the maximum principle to the test function

$$Q := \log(\lambda_1) + H(\varphi).$$

If this quantity achieves its maximum on ∂X_ε, then it is uniformly bounded in terms of the boundary data, and we obtain

$$\lambda_1 \leqslant Ce^{-A\operatorname{osc}_{X_\varepsilon}\varphi},$$

which is the desired estimate. Otherwise, suppose Q attains an interior maximum at (p_0, t_0). Fix holomorphic normal coordinates $(z_0, \ldots, z_n)$ for $\hat\omega$ in a neighborhood of (p_0, t_0) so that α is diagonal at (p_0, t_0) with eigenvalues $\mu_0 \geqslant \mu_1 \geqslant \cdots \geqslant \mu_n$. We perform the perturbation described at the beginning of the section, and consider the quantity

$$\tilde{Q} = \log(\tilde\lambda_1) + H(\varphi).$$

By construction we have $\tilde{Q} \leqslant Q$ near (p_0, t_0), and $\tilde{Q}(p_0, t_0) = Q(p_0, t_0)$. Thus $\tilde{Q}$ achieves an interior maximum at (p_0, t_0). Applying the linearized operator at (p_0, t_0) we have

$$0 \geqslant F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\tilde{Q}.$$

The linearized operator applied to the $H(\varphi)$ term gives

$$F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}H(\varphi) = H'(\varphi)\sum_i \frac{1}{1+\mu_i^2}\nabla_i\nabla_{\bar{i}}\varphi + H''(\varphi)\sum_i \frac{1}{1+\mu_i^2}|\nabla_i\varphi|^2$$

$$= -H'(\varphi)\sum_i \frac{\hat\alpha_{\bar{i}i} - \mu_i}{1+\mu_i^2} + A\tau\sum_i \frac{1}{1+\mu_i^2}|\nabla_i\varphi|^2$$

Thus, at (p_0, t_0) we have

$$0 \geqslant F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\tilde{Q} \geqslant \frac{1}{\lambda_1}\sum_{i,j=0}^{n}\frac{\mu_i + \mu_j}{(1+\mu_i^2)(1+\mu_j^2)}|\nabla_{V_1}\alpha_{\bar{j}i}|^2$$

$$-\frac{1}{\lambda_1^2}\sum_{i=0}^{n}\frac{1}{1+\mu_i^2}|\nabla_i\alpha_{\bar{V}_1V_1}|^2 - C$$

$$-H'(\varphi)\sum_i\frac{\hat{\alpha}_{\bar{i}i} - \mu_i}{1+\mu_i^2} + A\tau\sum_i\frac{1}{1+\mu_i^2}|\nabla_i\varphi|^2$$

Combining Proposition 5.5 and Proposition 5.8 there is a uniform constant C_4 so that if $\lambda_1 \geqslant C_4\varepsilon_0^{-2}$, then we have

$$0 \geqslant -\frac{2\varepsilon_0}{\lambda_1^2}\sum_{i=0}^{n}\frac{1}{1+\mu_i^2}|\nabla_i\alpha_{\bar{V}_1V_1}|^2 - C - \frac{C_2}{\lambda_1} - \frac{C_5}{\varepsilon_0\lambda_1^2}$$

$$-H'(\varphi)\sum_i\frac{\hat{\alpha}_{\bar{i}i} - \mu_i}{1+\mu_i^2} + A\tau\sum_i\frac{1}{1+\mu_i^2}|\nabla_i\varphi|^2$$

for uniform constants C, C_2, C_5. At the maximum of $\tilde{Q}$ we have $\nabla_i\tilde{Q} = 0$, which gives

$$\frac{\nabla_i\alpha_{\bar{V}_1V_1}}{\lambda_1} = -H'\nabla_i\varphi$$

So

$$\frac{1}{\lambda_1^2}\sum_{i=0}^{n}\frac{1}{1+\mu_i^2}|\nabla_i\alpha_{\bar{V}_1V_1}|^2 = (H')^2\sum_i\frac{1}{1+\mu_i^2}|\nabla_i\varphi|^2$$

Choose $\varepsilon_0 = \max\{\frac{\tau}{10A}, 1\}$, then $2\varepsilon_0(H')^2 \leqslant 8\varepsilon_0A^2 < \tau A$ and so we get

$$0 \geqslant -C - \frac{C_2}{\lambda_1} - \frac{10AC_5}{\tau\lambda_1^2}$$

$$+ A\sum_i\frac{\hat{\alpha}_{\bar{i}i} - \mu_i}{1+\mu_i^2}$$

We may assume that $\lambda_1 > R$, where R is the uniform constant appearing in Lemma 3.5 for the subsolution $\hat{\alpha}$ (see Remark 3.11). Since $\mu_0 \geqslant \lambda_1$ we can apply Lemma 3.5. Since $|\mu_n|$ is bounded by Lemma 3.1 we can always assume we are in the first case of Lemma 3.5 and so we get

$$\sum_i\frac{\hat{\alpha}_{\bar{i}i} - \mu_i}{1+\mu_i^2} > \kappa_0\sum_{i=1}^{n}\frac{1}{1+\mu_i^2} > \tilde{\kappa}_0$$

for $\tilde{\kappa}_0$ a uniform constant. Finally, choose A so that $A\tilde{\kappa}_0 = C + C_2 + 1$, which is a uniform constant. Then we obtain

$$0 \geqslant 1 - \frac{10(C + C_2 + 1)}{\tau\tilde{\kappa}_0\lambda_1^2}.$$

In particular, $\lambda_1 < C_6$ for a uniform constant C_6. This implies

$$\lambda_1 \leqslant C_6 e^{A(\varphi - \inf_{\mathcal{X}_\varepsilon} \varphi)}$$

finishing the proof of (5.1).

We next move on to estimating the size of $\alpha_{\bar{t}t}$. This is easily done by the maximum principle. Recall that by Lemma 3.1 there is a constant A depending only on η_1 so that $G = -e^{-AF}$ is concave. Consider the quantity

$$Q = \alpha_{\bar{t}t} - C\varphi,$$

and recall that $\alpha = \hat{\alpha} + \sqrt{-1}D\overline{D}\varphi$. Differentiating the equation $G = -e^{-Ah}$ twice, using that G is concave, α is closed, and the curvature vanishes in any temporal direction yields

$$G^{i\bar{j}}\nabla_i\nabla_{\bar{j}}(\alpha_\varphi)_{\bar{t}t} \geqslant A(\nabla_t\nabla_{\bar{t}}h + A|\nabla_t h|^2)e^{-Ah}$$

Choose $C = \frac{A\|h_{\bar{t}t}\|_{L^\infty(X_\varepsilon)} + A^2\|h_t\|^2_{L^\infty(X_\varepsilon)} + 1}{\tilde{\kappa}_0}$ where $\tilde{\kappa}_0 = \frac{\kappa_0}{1+C(\eta)^2}$, and κ_0 is the uniform constant appearing in Lemma 3.5 for $\hat{\alpha}$, while $C(\eta)$ is a bound for $|\mu_n|$ (see Lemma 3.1). Suppose that Q achieves an interior maximum on $\mathcal{X}_\varepsilon$ at the point (p_0, t_0). Choose holomorphic normal coordinates near (p_0, t_0) such that α is diagonal with eigenvalues $\mu_0, \ldots, \mu_n$. We get

$$0 \geqslant A(\nabla_t\nabla_{\bar{t}}h + A|\nabla_t h|^2)e^{-Ah} + C\sum_i \frac{\hat{\alpha}_{\bar{i}i} - \mu_i}{1 + \mu_i^2}.$$

If $\alpha_{\bar{t}t} > R$, then by Lemma 3.5 and our choice of C we obtain

$$0 \geqslant A(\nabla_t\nabla_{\bar{t}}h + A|\nabla_t h|^2)e^{-Ah} + A(\|h_{\bar{t}t}\|_{L^\infty(X_\varepsilon)} + A\|h_t\|^2_{L^\infty(X_\varepsilon)} + 1)$$

a contradiction. Thus, $(\alpha_\varphi)_{\bar{t}t} \leqslant R$, and so

$$Q \leqslant Q(p_0, t_0) \leqslant R + \frac{A\|h_{\bar{t}t}\|_{L^\infty(X_\varepsilon)} + A\|h_t\|^2_{L^\infty(X_\varepsilon)} + 1}{\tilde{\kappa}_0}\|\varphi\|_{L^\infty(X_\varepsilon)}$$

unless Q attains its maximum on the boundary. Summarizing we have

Proposition 5.9. *There is a uniform constant C so that*

$$|(\alpha_\varphi)_{\bar{t}t}|_{\hat{g}} \leqslant C\left(1 + \|h_{\bar{t}t}\|_{L^\infty(\mathcal{X}_\varepsilon)} + \|h_t\|^2_{L^\infty(\mathcal{X}_\varepsilon)}\right) + \sup_{\partial\mathcal{X}_\varepsilon}|(\alpha_\varphi)_{\bar{t}t}|_{\hat{g}}$$

To estimate the off diagonal terms we use that $\alpha_\varphi \geqslant -C(\eta_1)\hat{\omega}$. Pick any point $(p_0, t_0) \in \mathcal{X}_\varepsilon$, and choose space-time adapted coordinates so that $(\alpha_\varphi)|_{TX}$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$. At p_0 we have

$$\alpha_\varphi + (C(\eta_1)+1)\hat{\omega} = \begin{pmatrix} (\alpha_\varphi)_{\bar{t}t} + C(\eta_1) + 1 & (\alpha_\varphi)_{\bar{1}t} & \cdots & (\alpha_\varphi)_{\bar{n}t} \\ (\alpha_\varphi)_{\bar{t}1} & \lambda_1 + C(\eta_1) + 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha_\varphi)_{\bar{n}t} & 0 & \cdots & \lambda_n + C(\eta_1) + 1 \end{pmatrix}$$

Let D be the $n \times n$ matrix with $D_{\bar{j}i} = (\lambda_1 + C(\eta) + 1)\delta_{\bar{j}i}$. Since $\lambda_i + C(\eta_1) + 1 > 1$ for all i we can compute the determinant as

$$0 \leqslant \det(\alpha_\varphi + (C(\eta_1) + 1)\hat{\omega}) = \det(D)\left((\alpha_\varphi)_{\bar{t}t} + C(\eta_1) + 1 - \sum_i \frac{|(\alpha_\varphi)_{\bar{t}i}|^2}{\lambda_i + C(\eta_1) + 1}\right).$$

Now since $1 \leqslant \lambda_i + C(\eta_1) + 1 \leqslant C$ for a uniform constant C by the spatial C^2 estimate we obtain

$$|(\alpha_\varphi)_{\bar{t}i}| \leqslant C \left(|(\alpha_\varphi)_{\bar{t}t}| + C(\eta_1) + 1\right)^{1/2}$$

which finishes the proof of Theorem 5.1.

Remark 5.10. Note that the spatial C^2 estimate is independent of the estimate for the spatial gradient. In particular, Proposition 4.1 can be obtained directly from the C^2 estimate by applying the elliptic theory along the fibers of $\mathcal{X}_\varepsilon \to \mathcal{A}_\varepsilon$. Nevertheless, we have decided to include the estimate as it may have applications to the existence of geodesic rays in $\mathcal{H}$; see for example [62, 63, 64].

6. Boundary C^2 estimates and existence of solutions to the degenerate Lagrangian phase equation

It remains only to prove the boundary estimates. The boundary estimates are based on estimates of Guan [42, 43] which are in turn inspired by estimates of Trudinger [75]. Similar ideas were used by the first author, Picard and Wu to solve the Dirichlet problem for the Lagrangian phase operator [24]. In fact, the proof here is much simpler than the boundary estimates for the Lagrangian phase operator established in [24] due to the special structure of the boundary.

Without loss of generality, we work near $\{|t| = \varepsilon\}$. Consider the function

$$v = (\varphi - \hat{\varphi}_0) + c_0(\varepsilon - |t|) - N(\varepsilon - |t|)^2$$

for constants $N, c_0 > 0$ to be determined. For simplicity, let us write $\hat{\varphi}_0 = \hat{\varphi}$. Our goal is to choose uniform constants c_0, N so that $F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}v < -\varepsilon_0$ near $|t| = \varepsilon$ for a uniform constant ε_0, and so that $v > 0$ on a neighboourhood of $\{r = \varepsilon\}$. We compute at a point (p_0, t_0) in coordinates where $\hat{\omega}$ is the identity and α_φ is diagonal.

$$F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}v = \sum_i \frac{\mu_i - \hat{\alpha}_{\bar{i}i}}{1 + \mu_i^2} - c_0 F^{t\bar{t}}\frac{1}{4|t|} - \frac{N}{2}F^{t\bar{t}} + N\frac{(\varepsilon - |t|)}{2|t|}F^{t\bar{t}}.$$

Suppose $|\mu| \leqslant R$, where R is the constant in Lemma 3.5 for $\hat{\varphi}_0$. Then we have $1 \geqslant F^{t\bar{t}} > (1 + R^2)^{-1}$. Also, $\hat{\alpha}_{\bar{i}i} > -C(\eta_1)$. Combining these estimates gives

$$(6.1) \quad F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}v \leqslant (n+1) + C(\eta_1) - c_0\frac{F^{t\bar{t}}}{4|t|} - \frac{N}{2(1 + R^2)} + N\frac{(\varepsilon - |t|)}{2|t|}.$$

If instead $|\mu| > R$ the we apply Lemma 3.5 and conclude

$$(6.2) \qquad F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}v < -\kappa_0 - c_0\frac{F^{t\bar{t}}}{4|t|} - \frac{N}{2}F^{t\bar{t}} + N\frac{(\varepsilon - |t|)}{2|t|},$$

for a uniform constant κ_0. We consider v on the domain

$$\Omega_\delta := X \times \{(1-\delta)\varepsilon \leqslant |t| \leqslant \varepsilon\}$$

where $0 < \delta < 1 - e^{-1}$ is to be determined. On Ω_δ we have

$$0 \leqslant \frac{(\varepsilon - |t|)}{|t|} \leqslant \frac{\delta}{1-\delta} < 1.$$

Choose N so that

$$\frac{N}{2(1+R^2)} = n + 2 + C(\eta_1),$$

and note that N is uniform. Then when $|\mu| < R$ we have

$$F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}v \leqslant -1 + N\frac{\delta}{(1-\delta)} < -\frac{1}{2}$$

provided $\delta < \frac{1}{2N+1}$. When $|\mu| > R$ we have

$$F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}v \leqslant -\kappa_0 + N\frac{\delta}{(1-\delta)} < -\frac{\kappa_0}{2}$$

provided $\delta < \frac{\kappa_0}{2N+\kappa_0}$. We choose $\delta = \min\{\frac{1}{2N+1}, \frac{\kappa_0}{2N+\kappa_0}\}$, which is a uniform constant. It remains only to determine c_0. On the $|t| = \varepsilon$ component of $\partial\Omega_\delta$ we have $v = 0$, while on the $|t| = (1-\delta)\varepsilon$ component we have

$$v \geqslant \delta\varepsilon - N(\delta\varepsilon)^2 \geqslant \delta\varepsilon\left(c_0 - \frac{1}{2}\varepsilon\right) > 0$$

provided we take $c_0 = 1$. Summarizing we have

Lemma 6.1. *There exist uniform constants δ, N so that the function*

$$v := (\varphi - \hat{\varphi}_0) + (\varepsilon - |t|) - N(\varepsilon - |t|)^2$$

satisfies

$$F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}v \leqslant -\frac{\kappa_0}{2}, \qquad v \geqslant 0$$

on $\Omega_\delta := X \times \{(1-\delta)\varepsilon \leqslant |t| \leqslant \varepsilon\}$.

We are going to estimate the tangent-normal derivatives of φ near the boundary $\{|t| = \varepsilon\}$. Fix a point $(p_0, t_0) \in \partial X_\varepsilon$, and space-time adapted coordinates $(w_0, \ldots, w_n) = (w_0, w')$. For $1 \leqslant \ell \leqslant n$ we compute

$$F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\nabla_\ell(\varphi - \hat{\varphi}) = \nabla_\ell h - \sum_{1 \leqslant i,j \leqslant n} F^{i\bar{j}}\nabla_\ell\hat{\varphi}$$

where in the last line we used that $\nabla_t\nabla_\ell\hat{\varphi} = \nabla_{\bar{t}}\nabla_\ell\hat{\varphi} = 0$ (see Corollary 3.10). Next we compute

$$F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\nabla_{\bar{\ell}}(\varphi - \hat{\varphi}) = \nabla_{\bar{\ell}}h - F^{i\bar{j}}\sum_{1 \leqslant k \leqslant n} R_{\bar{\ell}i\bar{j}}{}^{\bar{k}}\nabla_{\bar{k}}\varphi - \sum_{1 \leqslant i,j \leqslant n} F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\nabla_{\bar{\ell}}\hat{\varphi}$$

where we used that $R_{\bar{\ell}i\bar{j}k} = 0$ if $k = 0$. Write $w_\ell = x_\ell + \sqrt{-1}y_\ell$. Then combining the spatial gradient bound from Theorem 4.3 with Corollary 3.10 we have

$$-C \leqslant F^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\nabla_{\partial_{x_\ell}}(\varphi - \hat{\varphi}) \leqslant C$$

for a uniform constant C. Clearly the same estimate holds for ∂_{y_ℓ}. We now consider the quantity

$$Q_\pm := Av \pm \partial_{x_\ell}(\varphi - \hat{\varphi}) + B\sum_{i=i}^n |w_i|^2 - \frac{D}{2}\log\left(\frac{|t|^2}{\varepsilon^2}\right)$$

for positive constants A, B, D to be determined, which is defined on the set

$$\hat{\Omega}_\delta := \left\{\sum_{i=1}^n |w_i|^2 < (\delta')^2\right\} \times \{(1-\delta)\varepsilon \leqslant |t| \leqslant \varepsilon\}$$

where δ' is a constant depending only on (X, ω) (namely, the size of the coordinate chart on which holomorphic normal coordinates are defined). On $|t| = \varepsilon$ we have $Q_\pm \geqslant 0$ with equality at $w' = 0$. On $w' = \delta'$ we have

$$Q_\pm \geqslant -\sup_{\mathcal{X}_\varepsilon}|\nabla^X\varphi|_{\hat{g}} + B(\delta')^2.$$

By Proposition 4.1 we can therefore choose a uniform constant B large enough so that $Q_\pm \geqslant 0$ on $|w'| = \delta'$. Finally, when $|t| = (1-\delta)\varepsilon$ we have

$$Q_\pm \geqslant -\sup_{X_\varepsilon}|\nabla^X\varphi|_{\hat{g}} + \frac{D}{2}\log\left(\frac{1}{(1-\delta)^2}\right)$$

Since $\delta > 0$ is uniform, by Proposition 4.1 we can choose D large and uniform so that $Q_\pm \geqslant 0$ on $|t| = (1-\delta)\varepsilon$. Finally, we compute

$$F^{\bar{i}j} \leqslant -A\frac{\kappa_0}{2} + \sup_{\mathcal{X}_\varepsilon}|\nabla^X\varphi|_{\hat{g}} + B\sum_{1\leqslant i\leqslant n}F^{i\bar{i}} \leqslant -A\frac{\kappa_0}{2} + \sup_{X_\varepsilon}|\nabla^X\varphi|_{\hat{g}} + Bn$$

Therefore, another application of Proposition 4.1 shows that we can choose A uniform, sufficiently large so that $F^{i\bar{j}}Q_\pm \leqslant 0$. Since $Q_\pm \geqslant 0$ on $\partial\hat{\Omega}_\delta$ we have $Q_\pm \geqslant 0 = Q_\pm(p_0, t_0)$. We conclude that $\partial_r Q_\pm \leqslant 0$. Now at (p_0, t_0) we have $\partial_r\partial_{x_\ell}\hat{\varphi} = 0$ and $\partial_r v = \partial_r\varphi - \partial_r\hat{\varphi} - 1$. Thus by Theorem 4.3 we obtain

$$|\partial_r\partial_{x_\ell}\varphi| \leqslant \frac{C}{\varepsilon}$$

Repeating the argument with y_ℓ yields

Proposition 6.2. *There is a uniform constant C so that*

$$\sup_{\partial X_\varepsilon}\left(|\nabla_t\overline{\nabla^X}\varphi|_{\hat{g}} + |\nabla_{\bar{t}}\nabla^X\varphi|_{\hat{g}}\right) \leqslant \frac{C}{\varepsilon}$$

Finally, we estimate $\nabla_t\nabla_{\bar{t}}\varphi$ on the boundary. In fact the estimate we need follows from a lemma of Caffarelli-Nirenberg-Spruck [15], which we now recall

Lemma 6.3. *Consider the $n \times n$ hermitian matrix*

$$
M := \begin{pmatrix}
a & b_1 & b_2 & \cdots & b_n \\
\overline{b_1} & d_1 & 0 & \cdots & 0 \\
\overline{b_2} & 0 & d_2 & \cdots & 0 \\
\vdots & \vdots & \cdots & \ddots & \vdots \\
\overline{b_n} & 0 & 0 & \cdots & d_n
\end{pmatrix}
$$

where we assume $a \geqslant 1$. Let $\mu_0, \mu_1, \ldots, \mu_n$ be the eigenvalues of M. For all $0 < \varepsilon_0 \ll 1$, there exists $\delta(\varepsilon_0) > 0$ depending only on $d_1, \ldots, d_n, \varepsilon$ such that, if $\sum_{i=1}^{n} \frac{|b_i|^2}{a} < \delta(\varepsilon_0)$ then

$$
|\mu_i - d_i| < \varepsilon_0.
$$

for $i = 1, \ldots, n$. Furthermore, we have

$$
\mu_0 = a \left(1 + O\left(\sum_{i=1}^{n} \frac{|b_i|^2}{a} \right) \right)
$$

with implied constants depending only on $d_1, \ldots, d_n$.

Proof. We give the proof, since the statement in [15] is not exactly what we need. Just as in [15], the eigenvalues are given by the zeroes of

$$
\det \begin{pmatrix}
1 - \frac{\mu}{a} & \frac{b_1}{a} & \frac{b_2}{a} & \cdots & \frac{b_n}{a} \\
\overline{b_1} & d_1 - \mu & 0 & \cdots & 0 \\
\overline{b_2} & 0 & d_2 - \mu & \cdots & 0 \\
\vdots & \vdots & \cdots & \ddots & \vdots \\
\overline{b_n} & 0 & 0 & \cdots & d_n - \mu
\end{pmatrix} = 0.
$$

Expanding the determinant gives

$$
0 = (1 - \frac{\mu}{a}) \left(\prod_{i=1}^{n} (d_i - \mu) \right) + \sum_{j=1}^{n} (-1)^j \frac{|b_j|^2}{a} \prod_{i \neq j} (d_i - \mu).
$$

Introduce parameters $t_0 = \frac{1}{a}$ and $t_i = \frac{|b_i|^2}{a}$, and write this equation as

$$
P(t_0, t_1 \ldots, t_n, \lambda) = 0
$$

for a polynomial P. When $(t_1, \ldots, t_n) = 0$ we have that $\mu = d_i$ is a zero. Since the roots of $P(t_0, \ldots, t_n)$ depend continuously on the coefficients we get that, for all $\varepsilon_0 > 0$, there exists $\delta > 0$ depending only on $d_1, \ldots, d_n, \varepsilon_0$ such that if $|t| < \delta$, then

$$
|\mu_i - d_i| < \varepsilon_0.
$$

For the last eigenvalue we set $\mu = a\gamma$ and consider

$$
\det \begin{pmatrix}
1 - \gamma & \frac{b_1}{a} & \frac{b_2}{a} & \cdots & \frac{b_n}{a} \\
\frac{\overline{b_1}}{a} & \frac{d_1}{a} - \gamma & 0 & \cdots & 0 \\
\frac{\overline{b_2}}{a} & 0 & \frac{d_2}{a} - \gamma & \cdots & 0 \\
\vdots & \vdots & \cdots & \ddots & \vdots \\
\frac{\overline{b_n}}{a} & 0 & 0 & \cdots & \frac{d_n}{a} - \gamma
\end{pmatrix} = 0
$$

which we write as

$$
0 = (1 - \gamma) \left(\prod_{i=1}^{n} (t_0 d_1 - \gamma) \right) + \sum_{j=1}^{n} (-1)^j t_0 t_j \prod_{i \neq j} (t_0 d_i - \gamma).
$$

When $t = 0, \gamma = 1$ is a simple zero, and hence the implicit function theorem gives that for $|t| < \delta$, there is a constant C depending on $d_1, \ldots, d_n$ such that

$$
|1 - \gamma(t)| \leqslant C|t|
$$

whence

$$
\mu_0(t) = a(1 + O(|t|))
$$

with constants depending only on $d_1, \ldots, d_n$. $\qquad \square$

This lemma, together with the boundary tangent-normal estimates in Proposition 6.2 immediately implies the normal-normal estimate. Suppose there is a point $(p_0, t_0) \in \partial X_\varepsilon$ where $(\alpha_\varphi)_{\bar{t}t} \geqslant \frac{K}{\varepsilon^2}$ for some constant K to be determined (note that the lower bound is automatic from Lemma 3.1). Fix space-time adapted coordinates at (p_0, t_0), and let μ_i $0 \leqslant i \leqslant n$ be the eigenvalues of α_φ. By the tangent-normal estimates we have

$$
\sum_i \frac{|(\alpha_\varphi)_{\bar{t}i}|^2}{(\alpha_\varphi)_{\bar{t}t}} \leqslant \frac{C}{K}
$$

where C is a uniform constant. Fixing $\varepsilon_0 > 0$, if C/K is sufficiently small depending only on the bound for the spatial C^2 norm and ε_0, then we have

$$
|\mu_0 - \alpha_{\bar{t}t}| < \varepsilon_0, \qquad |\alpha_{\bar{i}i} - \mu_i| \leqslant \varepsilon_0.
$$

It follows that

$$
\left| \sum_{i=0}^{n} \arctan(\mu_i) - \left(\sum_{i=1}^{n} \arctan((\alpha_\varphi)_{\bar{i}i}) + \arctan((\alpha_\varphi)_{\bar{t}t}) \right) \right| \leqslant (n+1)\varepsilon_0,
$$

since the derivative of arctan is bounded by 1. On the boundary we have $\varphi = \underline{\varphi}$ and $\underline{\varphi}$ is a subsolution, satisfying

$$
\sum_{i=0}^{n} \arctan(\underline{\mu}_i) \geqslant h + \frac{\eta_1}{2}
$$

where $\underline{\mu}_i$ are the eigenvalues of $\underline{\alpha} := \pi_X^* \alpha + \sqrt{-1}D\overline{D}\underline{\varphi}$. By the Schur-Horn theorem [47] and the convexity of the super-level sets $\{F \geqslant \sigma\}$ for $\sigma \geqslant (n-1)\frac{\pi}{2}$ we have

$$\sum_{i=1}^{n} \arctan(\underline{\alpha}_{\bar{i}i}) \geqslant \sum_{i=0}^{n} \arctan(\underline{\mu}_i) - \arctan(\underline{\alpha}_{\bar{t}t}).$$

So, we have

$$h(x) + \frac{\eta_1}{2} \leqslant \sum_{i=0}^{n} \arctan(\underline{\mu}_i) \leqslant \sum_{i=1}^{n} \arctan(\underline{\alpha}_{\bar{i}i}) + \arctan(\underline{\alpha}_{\bar{t}t})$$

$$\leqslant (n+1)\varepsilon_0 + \sum_{i=0}^{n} \arctan(\mu_i) - \arctan((\alpha_\varphi)_{\bar{t}t}) + \arctan(\underline{\alpha}_{\bar{t}t})$$

$$= (n+1)\varepsilon_0 + h(x) - \arctan((\alpha_\varphi)_{\bar{t}t}) + \arctan(\underline{\alpha}_{\bar{t}t}).$$

This implies

$$\frac{\eta_1}{2} + \arctan((\alpha_\varphi)_{\bar{t}t}) \leqslant (n+1)\varepsilon_0 + \arctan(\underline{\alpha}_{\bar{t}t}).$$

By Lemma 3.6 we have

$$\underline{\alpha}_{\bar{t}t} \leqslant \frac{C_1}{\varepsilon^2}$$

for a uniform constant C_1. Choose ε_0 sufficiently small so that $(n+1)\varepsilon_0 \leqslant \frac{\eta_1}{2}$, we conclude that $K \leqslant C_1$. Thus we conclude

Proposition 6.4. *There exists a uniform constant C so that*

$$\sup_{\partial \mathcal{X}_\varepsilon} |\nabla_t \nabla_{\bar{t}} \varphi| \leqslant \frac{C}{\varepsilon^2}$$

Combining this with Theorem 5.1, and Theorem 4.3 we conclude

Theorem 6.5. *Suppose $\varphi(x,t)$ is a smooth S^1 invariant function on $(\mathcal{X}_\varepsilon, \hat{\omega})$ with $\alpha_\varphi := \alpha + \sqrt{-1}D\overline{D}\varphi(x,t)$ solving the Lagrangian phase equation*

$$F(\hat{\omega}^{-1}\alpha_\varphi) = h(x, |t|).$$

with $\varphi(x,\varepsilon) = \varphi_0$ and $\varphi(x, \varepsilon e^{-1}) = \varphi_1$, and $\varphi_i \in \mathcal{H}$. Suppose in addition that φ_0, φ_1, h satisfy the structural conditions $(C1), (C2)$ with constants η_1, η_2. The following estimates hold

$$\mathrm{osc}_{\mathcal{X}_\varepsilon} \varphi + |\nabla^X \varphi|_{\hat{\omega}} + |\nabla^X \overline{\nabla^X} \varphi| \leqslant C$$

$$|\nabla_t \varphi|_{\hat{\omega}} \leqslant C \left(1 + \sup_{\mathcal{X}_\varepsilon} |\nabla_t h|_{\hat{\omega}} + \frac{1}{\varepsilon} \right)$$

$$|\nabla_{\bar{t}} \nabla^X \varphi| \leqslant C \left(1 + \frac{1}{\varepsilon} + \sqrt{\sup_{\mathcal{X}_\varepsilon} |\nabla_t \nabla_{\bar{t}} h|_{\hat{g}}} + \sqrt{\sup_{\mathcal{X}_\varepsilon} |\nabla_t h|_{\hat{g}}^2} \right)$$

$$|\nabla_t \nabla_{\bar{t}} \varphi| \leqslant C \left(1 + \frac{1}{\varepsilon^2} + \sup_{\mathcal{X}_\varepsilon} |\nabla_t \nabla_{\bar{t}} h|_{\hat{g}} + \sup_{\mathcal{X}_\varepsilon} |\nabla_t h|_{\hat{g}}^2 \right)$$

where C is a uniform constant depending only on $\varphi_0, \varphi_1, \nabla^X \overline{\nabla^X} h, (X, \omega)$ and the structural constants η_1, η_2.

First we apply these estimates to solve the Dirichlet problem for the Lagrangian phase operator on $\mathcal{X}_\varepsilon$.

Theorem 6.6. *Suppose $\varphi_0, \varphi_1 \in \mathcal{H}$, and $h(x, |t|) : \mathcal{X}_\varepsilon \to \mathbb{R}$ is a smooth function satisfying structural constraints $(C1), (C2)$. Then, on $\mathcal{X}_\varepsilon$, there exists a smooth, S^1 invariant solution of the equation*

$$F(\hat{\omega}^{-1} \alpha_\varphi) = h(x, |t|).$$

with boundary values φ_0, φ_1. In particular, for any $\varphi_0, \varphi_1 \in \mathcal{H}$ there is a unique, smooth ε geodesic joining φ_0, φ_1.

Proof. The corollary follows easily from Theorem 6.5. Let $\underline{h}(x) = F(\alpha_{\underline{\varphi}})$ where $\underline{\varphi}$ is the function constructed in Lemma 3.6 and consider the equation

$$F(\hat{\omega}^{-1} \alpha_{\varphi_u}) = (1 - u)\underline{h} + uh$$

where $u \in [0, 1]$, and φ_u has boundary values φ_0, φ_1. Note that structural constraints $(C1), (C2)$ hold uniform for the functions $(1 - u)\underline{h} + uh$. Let I be the set of $u \in [0, 1]$ for which this equation admits a solution. By the implicit function theorem I is open. Suppose $I \ni u_i \to u_*$ Combining the estimates in Theorem 6.5 with the Evans-Krylov theorem, and arguing as in [24] we conclude that I is closed. $\square$

Since we have obtained estimates that scale appropriately we can pass to the limit as $\varepsilon \to 0$ to get weak solutions to the space-time lifted degenerate Lagrangian phase equation, see Definition 2.9. Before explaining how this is done, let us explain how to make sense of the limiting equation weakly. First note that if $\varphi : \mathcal{X} \to \mathbb{R}$ is bounded and satisfies $\sqrt{-1}D\overline{D}\varphi \geqslant -C\hat{\omega}$, then we can define

$$(\pi_X^* \alpha + \sqrt{-1}D\overline{D}\varphi)^k$$

as a (k, k) current for all k using the Bedford-Taylor theory [6]. In particular, for such functions φ

$$(\pi_X^* \omega + \sqrt{-1}(\pi_X^* \alpha + \sqrt{-1}D\overline{D}\varphi))^{n+1}$$

defines a complex measure on $\mathcal{X}$ and hence the equation

$$(6.3) \qquad \mathrm{Im}\left(e^{-\sqrt{-1}h(x,|t|)}(\pi_X^* \omega + \sqrt{-1}(\pi_X^* \alpha + \sqrt{-1}D\overline{D}\varphi))^{n+1}\right) = 0$$

can be interpreted as an equality of measures. With this in mind we have

Theorem 6.7. *Suppose $\varphi_0, \varphi_1 \in \mathcal{H}$, and $h(x, |t|) : \mathcal{X} \to \mathbb{R}$ is a smooth function satisfying structural constraints $(C1), (C2)$. Then there exists an S^1 invariant function φ so that $\varphi \in C^{1,\alpha}$, for all $\alpha \in (0, 1)$, $\sqrt{-1}D\overline{D}\varphi \in L^\infty(\mathcal{X}_\varepsilon, \hat{\omega})$, and φ solves*

$$(6.4) \qquad \mathrm{Im}\left(e^{-\sqrt{-1}h(x,|t|)}(\omega + \sqrt{-1}(\alpha + \sqrt{-1}D\overline{D}\varphi))^{n+1}\right) = 0.$$

pointwise a.e., and in the sense of pluripotential theory, with boundary values φ_0, φ_1. *Equivalently,* φ *solves*

$$\widetilde{\Theta}_\omega(\hat{\omega}^{-1}\alpha_\varphi) = h(x, |t|)$$

in the sense of Harvey-Lawson's Dirichlet Duality. In particular, for any $\varphi_0, \varphi_1 \in \mathcal{H}$ *there is a unique weak geodesic joining* φ_0, φ_1.

Proof. Given $h(x, |t|) : \mathcal{X} \to \mathbb{R}$, we consider

$$h_\varepsilon(x, t) = h\left(x, \frac{t}{\varepsilon}\right) : \mathcal{X}_\varepsilon \to \mathbb{R}.$$

Clearly we have

$$\nabla^X \nabla_{\bar{t}} h_\varepsilon = \frac{1}{\varepsilon} \nabla^X \nabla_{\bar{t}} h \qquad \nabla_t \nabla_{\bar{t}} h_\varepsilon = \frac{1}{\varepsilon^2} \nabla_t \nabla_{\bar{t}} h.$$

By Theorem 6.6 we have functions $\psi_\varepsilon : \mathcal{X}_\varepsilon \to \mathbb{R}$ solving

$$F(\hat{\omega}^{-1}\alpha_{\psi_\varepsilon}) = h_\varepsilon.$$

Let $\varphi_\varepsilon = \psi_\varepsilon(x, \varepsilon t) : \mathcal{X} \to \mathbb{R}$. Then φ_ε solve the Lagrangian phase equation $F((\hat{\omega}_\varepsilon)^{-1}\alpha_{\varphi_\varepsilon}) = h$, or equivalently

$$\mathrm{Im}\left(e^{-\sqrt{-1}h}\left(\pi_X^*\omega + \varepsilon^1\sqrt{-1}dt \wedge d\bar{t} + \sqrt{-1}\left(\pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi\right)\right)^{n+1}\right) = 0$$

on $\mathcal{X}$. By Theorem 6.5, φ_ε satisfies

$$\|\varphi_\varepsilon\|_{L^\infty} + \sup_{\mathcal{X}}|\nabla\varphi_\varepsilon|_{\hat{\omega}} + \|\sqrt{-1}D\overline{D}\varphi_\varepsilon\|_{L^\infty(\mathcal{X},\hat{\omega})} \leqslant C$$

for a uniform constant C independent of ε. We can therefore take a limit as $\varepsilon \to 0$ and get $\varphi_\varepsilon \to \varphi$ where the convergence is uniform in $C^{1,\alpha}$. Clearly

$$C\hat{\omega} \geqslant \pi_X^*\alpha + \sqrt{-1}D\overline{D}\varphi \geqslant -C\hat{\omega}$$

and so by the continuity of the Monge-Ampère operator along uniformly convergent sequences [6] we conclude that φ is a weak solution of (6.4). Next we argue that φ is also a solution of the space-time lifted degenerate Lagrangian phase equation in the sense of Harvey-Lawson. We refer the reader to the work of Rubinstein-Solomon [66] for the construction of the degenerate Lagrangian phase operator. The two key properties we need are

$$F((\hat{\omega}_\varepsilon)^{-1}\alpha_\psi) \to \widetilde{\Theta}(\alpha_\psi) \quad \text{as } \varepsilon \to 0$$

by [66, Theorem A.3], and that $\widetilde{\Theta}(\cdot)$ is upper-semi continuous on the space of hermitian matrices [66]. By the Harvey-Lawson theory, and [66, Theorem 5.1] we need to show that if u is a C^2 function defined on a ball $B \subset \mathcal{X}$, with $u \geqslant \varphi$, and $u(p) = \varphi(p)$, then

$$(6.5) \qquad \widetilde{\Theta}(\alpha + \sqrt{-1}D\overline{D}u)(p) \geqslant h(p)$$

and similarly, that if u touches φ from below, then

$$(6.6) \qquad \widetilde{\Theta}\left(-\left(\alpha + \sqrt{-1}D\overline{D}u\right)\right)(p) \geqslant -h(p)$$

Our proof of this result is based on the proof of the analogous elementary result in the viscosity theory [14, Proposition 2.9], with the added complication that the background metric used to construct the elliptic operator is not constant.

Everything is local, so we may assume that we're working in $B_1 \subset \mathbb{C}^n$, and that $p = 0$. Let B_r denote a ball of radius r centered at 0. Suppose $u \geqslant \varphi$ on B_1. Fix $\delta > 0$. Since $\varphi_\varepsilon \to \varphi$ uniformly on B_1, for any $\eta > 0$ we can choose $\varepsilon < \varepsilon_0(\eta, \delta)$ sufficiently small so that

$$u_\eta := u + \eta |z|^2 \geqslant \varphi_\varepsilon \quad \text{on } \partial B_\delta.$$

Since $u(p) = \varphi(p)$, it follows that for ε sufficiently small depending on η, δ, $u_\eta - \varphi_\varepsilon$ has an interior minimum at some point $p_{(\varepsilon, \eta)} \in B_\delta(p)$. For now, let us suppress the dependence on η, and write $p_{(\varepsilon, \eta)} = p_\varepsilon$. At this point we have

(6.7)
$$\alpha_u(p_\varepsilon) + \eta \sum_{i=0}^n \sqrt{-1} dz_i \wedge d\bar{z}_i \geqslant \alpha_{\varphi_\varepsilon}(p_\varepsilon)$$

Up to taking a subsequence we can assume that $p_\varepsilon \to p_* \in \overline{B_\delta(p)} \subset B_{2\delta}(p)$ as $\varepsilon \to 0$. For ε sufficiently small we have

$$\alpha_u(p_*) + 2\eta \sum_{i=0}^n \sqrt{-1} dz_i \wedge d\bar{z}_i \geqslant \alpha_u(p_\varepsilon) + \eta \sum_{i=0}^n \sqrt{-1} dz_i \wedge d\bar{z}_i$$

as $(1, 1)$ forms on $\mathbb{C}^n$. Now, since $\hat{\omega}_\varepsilon = \pi_X^* \omega + \varepsilon^2 \sqrt{-1} dt \wedge d\bar{t}$ is a product, there is a constant C independent of ε so that

$$-C|x-y| \leqslant \hat{\omega}_\varepsilon(x)^{-1} - \hat{\omega}_\varepsilon(y)^{-1} \leqslant C|x-y|, \qquad \hat{\omega}_\varepsilon^{-1}(x) > C^{-1} \sum_{i=0}^n \sqrt{-1} dz_i \wedge d\bar{z}_i$$

for any points $x, y \in B_1 \subset \mathbb{C}^{n+1}$, where $\hat{\omega}_\varepsilon$ is regarded as a Kähler metric on $\mathbb{C}^{n+1}$. Thus, for $\varepsilon \ll \eta$ sufficiently small we have

$$\hat{\omega}_\varepsilon(p_*)^{-1} \left(\alpha_u(p_*) + \left(4\eta \sum_{i=0}^n \sqrt{-1} dz_i \wedge d\bar{z}_i \right) \right)$$

$$\geqslant \hat{\omega}_\varepsilon(p_\varepsilon)^{-1} \left(\alpha_u(p_*) + 2\eta \sum_{i=0}^n \sqrt{-1} dz_i \wedge d\bar{z}_i \right)$$

$$\geqslant \hat{\omega}_\varepsilon(p_\varepsilon)^{-1} \left(\alpha_u(p_\varepsilon) + \eta \sum_{i=0}^n \sqrt{-1} dz_i \wedge d\bar{z}_i \right)$$

Combining this inequality with (6.7) and applying the elliptic operator F gives

$$F \left((\hat{\omega}_\varepsilon)^{-1} \left(\alpha_u(p_*) + 4\eta \sum_{i=0}^n \sqrt{-1} dz_i \wedge d\bar{z}_i \right) \right) \geqslant F(\hat{\omega}_\varepsilon^{-1} \alpha_{\varphi_\varepsilon})(p_\varepsilon) = h(p_\varepsilon),$$

for all ε sufficiently small. We now take a limit as $\varepsilon \to 0$ to get

$$(6.8) \qquad \widetilde{\Theta}\left(\left(\alpha_u(p_*) + 4\eta \sum_{i=0}^{n} \sqrt{-1}dz_i \wedge d\bar{z}_i\right)\right) \geqslant h(p_*)$$

We now reinstate the dependence on η, and write $p_* = p_\eta$. Since (6.8) holds for all $\eta > 0$, we can take a limit as $\eta \to 0$. Up to taking a subsequence we can assume $p_\eta \to p_\infty \in \overline{B_\delta(p)}$. By the upper-semi continuity of $\widetilde{\Theta}$ we get

$$\widetilde{\Theta}\left(\alpha_u(p_\infty)\right) \geqslant h(p_\infty).$$

This holds for all $\delta > 0$, and so we may finally take a limit as $\delta \to 0$, applying the upper semi-continuity again to conclude

$$\widetilde{\Theta}\left(\alpha_u(p)\right) \geqslant h(p).$$

The same argument works to prove (6.6). $\qquad\square$

The next corollary is essential for infinite dimensional GIT.

Corollary 6.8. *Let $\varphi_0, \varphi_1 \in \mathcal{H}$, and let $\varphi(x, s)$ be a weak geodesic with $\varphi(x, 0) = \varphi_0$, $\varphi(x, 1) = \varphi_1$. Then the functional $CY_\mathbb{C}$ is well-defined along the curve $\varphi(x, s)$. Furthermore $\mathcal{C}$ is affine, $\mathcal{J}$ is convex, and $\mathrm{Re}(Z), \mathrm{Im}(Z)$ are concave.*

Proof. That $CY_\mathbb{C}$ is well defined follows from the Bedford-Taylor theory [6], together with the bounds $\|\varphi(x, s)\|_{L^\infty(X)} \leqslant C$, and $-C\omega \leqslant \sqrt{-1}\partial\bar{\partial}\varphi$, as discussed above. Next, let $\varphi_\varepsilon(x, s)$ be ε-geodesics joining φ_0, φ_1. Since $\varphi_\varepsilon \to \varphi$ the Bedford-Taylor theory [6] implies that

$$CY_\mathbb{C}(\varphi_\varepsilon) \to CY_\mathbb{C}(\varphi)$$

as $\varepsilon \to 0$. The properties of $\mathcal{C}, \mathcal{J}, \mathrm{Re}(Z), \mathrm{Im}(Z)$ along $\varphi(x, s)$ follow from the corresponding properties along $\varphi_\varepsilon(x, s)$; see Proposition 2.17, and Corollary 2.19 $\qquad\square$

6.1. **Applications to Homogeneous Monge-Ampère.** Our techniques can be used to give a simplified proof of the existence of geodesics in the space of Kähler metrics [17], in particular avoiding Błocki's gradient estimate [9, 10]. We briefly describe how this is done.

Let (X, ω) be a Kähler manifold, and

$$\mathcal{H} = \{\varphi \in C^\infty(X, \mathbb{R}) : \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$$

be the space of Kähler metrics. A geodesic in this space with respect to the Donaldson-Mabuchi-Semmes metric is equivalent to a solution of the homogeneous complex Monge-Ampère equation on $\mathcal{X} = X \times \mathcal{A}$. That is, a solution of

$$(\pi_X^*\omega + \sqrt{-1}D\overline{D}\varphi)^{n+1} = 0 \quad \text{on } \mathcal{X}$$

with boundary values $\varphi_0, \varphi_1 \in \mathcal{H}$. As above, we approximate this equation by the degenerating Monge-Ampère equations

$$(\pi_X^*\omega + \varepsilon^2\sqrt{-1}dt \wedge d\bar{t} + \sqrt{-1}D\overline{D}\varphi)^{n+1} = (n+1)\varepsilon^2\sqrt{-1}dt \wedge d\bar{t} \wedge \pi_X^*\omega^n.$$

Rescaling $t \mapsto \varepsilon t$, we can view this as the non-degenerate Monge-Ampère equation

$$(\hat{\omega} + \sqrt{-1}D\overline{D}\varphi)^{n+1} = \hat{\omega}^{n+1}$$

on $\mathcal{X}_{\varepsilon}$ with boundary values $\varphi_0, \varphi_1 \in \mathcal{H}$. Here, as before

$$\hat{\omega} = \pi_X^* \omega + \sqrt{-1}dt \wedge d\bar{t}$$

is a product metric on $\mathcal{X}_{\varepsilon}$. The C^0 estimate follows from the maximum principle, comparing with sub and supersolutions as in Section 3. For the spatial C^2 estimate, we argue as in Section 5, applying the maximum principle to λ_1, the largest eigenvalue of

$$\left(\hat{\omega} + \sqrt{-1}D\overline{D}\varphi\right)\big|_{TX}$$

measured with respect to $\hat{\omega}$. Let $F(M) = \log(\det(M))$. Fix a point (p_0, t_0) and local holomorphic normal coordinates $(z_0, \ldots, z_n)$ so that $\hat{\omega}_{\bar{j}i} = \delta_{\bar{j}i}$ and $\omega_\varphi := (\hat{\omega} + \sqrt{-1}D\overline{D}\varphi)$ is diagonal with entries $\mu_0 \leqslant \cdots \leqslant \mu_n$. Let $V_1 \in T_{p_0}X$ be the unit spatial eigenvector achieving λ_1. Let $F^{i\bar{j}}$ be the linearized operator of F. Then following the computation in Section 5 we compute

$$F^{i\bar{j}} \nabla_i \nabla_{\bar{j}} \log(\lambda_1) \geqslant -C + \frac{1}{\lambda_1} \sum_{i,j=0}^{n} \frac{1}{\mu_i \mu_j} |\nabla_{V_1}(\omega_\varphi)_{\bar{j}i}|^2 - \frac{1}{\lambda_1^2} \sum_{i=0}^{n} \frac{1}{\mu_i} |\nabla_i(\omega_\varphi)_{\bar{V}_1 V_1}|^2.$$

for a uniform constant C. Using Cauchy-Schwarz we now estimate

$$\frac{1}{\mu_i} |\nabla_i(\omega_\varphi)_{\bar{V}_1 V_1}|^2 = \left| \sum_j \overline{V_1^j} \nabla_{V_1}(\omega_\varphi)_{\bar{j}i} \right|^2$$

$$= \sum_{0 \leqslant j, \ell \leqslant n} V_1^\ell \overline{V_1^j} \nabla_{V_1}(\omega_\varphi)_{\bar{j}i} \overline{\nabla_{V_1}(\omega_\varphi)_{\bar{\ell}i}}$$

$$\leqslant \left(\sum_{0 \leqslant j, \ell \leqslant n} |V_1^\ell|^2 \frac{\mu_\ell}{\mu_j} |\nabla_{V_1}(\omega_\varphi)_{\bar{j}i}|^2 \right)$$

$$= \lambda_1 \sum_{0 \leqslant j \leqslant n} \frac{1}{\mu_j} |\nabla_{V_1}(\omega_\varphi)_{\bar{j}i}|^2.$$

Thus we have that $F^{i\bar{j}} \nabla_i \nabla_{\bar{j}} \log(\lambda_1) \geqslant -C$, and arguing as in the second author's proof of the Calabi conjecture [78] we get

$$\text{Tr}_{\hat{\omega}} \left(\hat{\omega} + \sqrt{-1}D\overline{D}\varphi\right)\big|_{TX} \leqslant C e^{-C(\varphi - \inf_{\mathcal{X}} \varphi)}.$$

In other words, for every $t \in \mathcal{A}$ we have an L^∞ bound for $\text{Tr}_{\hat{\omega}}\sqrt{-1}\partial\overline{\partial}\varphi(t)$ on X. By the elliptic theory applied on (X, ω) we conclude that $|\nabla_X \varphi(t)|_\omega$ is uniformly bounded. The estimate for $\nabla_t \varphi$ is easily obtained from the maximum principle, as in Section 4. The remainder of the argument is the same, applying the boundary estimates for complex Monge-Ampère [17, 41].

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