WEAK MIRROR SYMMETRY OF LIE ALGEBRAS

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1. Introduction

It is well known that deformation theory of geometric objects such as complex structures and symplectic structures are dictated by a differential Gerstenhaber algebra (a.k.a. DGA) and the associated cohomology theory [27], [35]. Therefore, DGA plays a key role in mirror symmetry [5] [21]. In developing the algebraic aspects of mirror symmetry, Merkulov proposes the concept of *weak mirror symmetry* [29]. If M is a manifold with a complex structure J and $M^\vee$ is another manifold with a symplectic structure ω, then (M, J) and $(M^\vee, \omega)$ form a weak mirror pair if the associated differential Gerstenhaber algebras $\mathrm{DGA}(M, J)$ and $\mathrm{DGA}(M^\vee, \omega)$ are quasi-isomorphic. The overall goal of this project is to construct all mirror pairs when the manifolds M and $M^\vee$ are solvmanifolds, i.e. homogeneous spaces of simply-connected connected solvable Lie groups, J is an invariant complex structure and ω is an invariant symplectic structure.

In the SYZ-conjecture, one considers the geometry of special Lagrangian fibrations in a Calabi-Yau manifold $L \hookrightarrow M \to B$ with L being a real three-dimensional torus. The mirror image is presumably a (new) Calabi-Yau manifold $M^\vee$ with the special Lagrangian fibrations $L^* \hookrightarrow M^\vee \to B$ where L^* is the dual torus of L [34]. One way to adapt the structure of a Lie group H to resemble this situation is by insisting that the Lie algebra $\mathfrak{h}$ is a semi-direct sum of a subalgebra $\mathfrak{g}$ by an abelian ideal V. The group H is then a semi-direct product, namely the product of the group G corresponding to $\mathfrak{g}$ with V. By restricting our attention to invariant structures on homogenous spaces

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of such groups the geometry of the fibration of H over G becomes encoded in the corresponding objects on the Lie algebra $\mathfrak{h}$ of H. We shall speak somewhat sloppily of $\mathfrak{g}$ as the base and of V as the fiber of the fibration.

Forgetting about the SYZ-conjecture, semi-direct products are still natural objects to study in connection with "weak mirror symmetry". This is so since the direct sum of the bundle of type $(1,0)$ vectors $T^{(1,0)}$ and bundle of $(0,1)$-forms $T^{*(0,1)}$ on a complex manifold carries a natural Lie bracket (Schouten) such that $T^{(1,0)}$ is a sub-algebra and $T^{*(0,1)}$ is an abelian ideal. It is the associated exterior algebra of this semi-direct sum and associated $\bar{\partial}$ complex that eventually controls the deformations of the complex structure.

It is well-known that a symplectic structure defines a flat torsion-free connection on a Lie algebra $\mathfrak{g}$ [14]. As we shall see, a flat torsion-free connection on a Lie algebra $\mathfrak{g}$ also defines a symplectic form ω, not on $\mathfrak{g}$ but on a semi-direct product $\mathfrak{h}^\vee$ of $\mathfrak{g}$ with its dual $\mathfrak{g}^*$, see also [11]. This symplectic form is defined such that it is Lagrangian with respect to both the base $\mathfrak{g}$ and the fiber $V^* = \mathfrak{g}^*$ and so we call the pair $(\mathfrak{h}^\vee, \omega)$ a *Lagrangian semi-direct product*. A torsion-free flat connection on $\mathfrak{g}$ also defines a *totally real semi-direct product* $(\mathfrak{h}, J)$ where J is a complex structure on a semi-direct sum of $\mathfrak{g}$ with itself. Such complex structures are particular cases of *complex product structures*, see [1]. The observation that complex structures and symplectic structures on certain semi-direct products both are related to the notion torsion-free flat connections may be found in [6]. Torsion-free flat connections are also known as affine structures and as such already has widespread application in the study of mirror symmetry, see for instance [2, 9, 23]. Left-invariant torsion-free flat connections on Lie groups are equivalent to (Lie compatible) left-symmetric algebras. Much is known about left-symmetric algebras. In particular existence problems have been examined and it is known that no left-symmetric structure exists on semi-simple algebras. Also, certain nilpotent algebras of dimension greater than nine have no left-symmetric structures, see e.g. [7, 10, 12, 13].

Therefore we confine the scope of our present paper to deal with solvable spaces, i.e. the base $\mathfrak{g}$ and hence the total space $\mathfrak{h}$ are solvable algebras. Recent advance in resolving the Benson-Gordon conjecture means that when we insist that $\mathfrak{h}$ should carry a Kähler structure then $\mathfrak{h}$ is flat and therefore of a very special solvable type [4, 8, 25, 30]. For a nilpotent algebra more is true - it is Kähler only if it is abelian [3, 24]. Therefore an invariant symplectic form ω on a non-abelian nilpotent algebra is of type $(1,1)$ with respect a complex structure J if and only if ω and J determine a non-definite metric. We call such a pair a *pseudo-Kählerian geometry*.

Invariant complex structures, their Dolbeault cohomology and pseudo-Kählerian geometry on nilmanifolds have been a subject of much investigation in recent years, especially when the complex dimension is equal to three, see [16, 17, 18, 19, 20, 32, 33]. In particular, with the recent advance in understanding the cohomology theory on nilmanifolds [32], our computation and results on nilpotent algebras in this paper could be used to provide a full description of the differential Gerstenhaber algebras of any invariant complex structures on nilmanifolds in all dimension.

We have organized ourselves as follows. In the next section, we briefly review the construction of differential Gerstenhaber algebras for complex and symplectic structures, the definition of semi-direct products and establish notations for subsequent computation. In Section 3, we study the complex and symplectic geometry on semi-direct products. We first establish a correspondence between a totally real semi-direct product and flat torsion-free connections on the base in Proposition 3.2. The analogous result for Lagrangian semi-direct products is obtained in Proposition 3.5.

In Section 3.3, we develop the concept of dual semi-direct product and use it as the candidate of "mirror" space. In Section 3.4, we demonstrate a construction of a special Lagrangian structure on the dual semi-direct product whenever a special Lagrangian structure on a semi-direct product is given. This is the result of Proposition 3.10. After a brief revisit to the subject on flat connections in Section 3.5, we prove the first main theorem (Theorem 4.1), which states that the differential Gerstenhaber algebra on a totally real semi-direct product is isomorphic to the differential Gerstenhaber algebra of the Lagrangian dual semi-direct product as constructed in Proposition 3.2.

In Section 4.1, we exhibit with some examples of Kählerian solvable algebras and their mirror partners. In Section 5, we focus on nilpotent algebras of dimension-four and dimension-six. The work on the four-dimensional case is a brief review of past results [31]. Our first step in addressing the issue of finding mirror pairs of special Lagrangian nilpotent algebras in dimension-six begins in Section 5.2. In this section, we determine nilpotent algebras admitting a semi-direct product structures, and then identify their dual semi-direct product space in Table (5.38). In Section 5.3, using results in literature we identify the semi-direct product structures which potentially could admit totally real complex structures or Lagrangian symplectic structures. The result is in Table (5.39). We finish this paper by giving examples of special Lagrangian pseudo-Kähler structures on every algebras in Table (5.39), and identifying their mirror structures. This is the content of Theorem 5.1.

2. Preliminaries

We first recall two well known constructions of differential Gerstenhaber algebras (DGA) [15, 29, 35]. After a motivation due to weak mirror symmetry, we recall the definition of semi-direct product of Lie algebras.

2.1. **DGA of a complex structure.** Suppose J is an integrable complex structure on $\mathfrak{h}$. i.e. J is an endomorphism of $\mathfrak{h}$ such that $J\circ J = -1$ and

$$[x \bullet y] + J[Jx \bullet y] + J[x \bullet Jy] - [Jx \bullet Jy] = 0. \tag{2.1}$$

Then the $\pm i$ eigenspaces $\mathfrak{h}^{(1,0)}$ and $\mathfrak{h}^{(0,1)}$ are complex Lie subalgebras of the complexified algebra $\mathfrak{h}_{\mathbb{C}}$. Let $\mathfrak{f}$ be the exterior algebra generated by $\mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}$, i.e.

$$\mathfrak{f}^n := \wedge^n(\mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}), \quad \text{and} \quad \mathfrak{f} = \oplus_n \mathfrak{f}^n. \tag{2.2}$$

The integrability condition in (2.1) implies that $\mathfrak{f}^1$ is closed under the *Schouten bracket*

$$[x + \alpha \bullet y + \beta] := [x, y] + \iota_x d\beta - \iota_y d\alpha. \tag{2.3}$$

Note that working on a Lie algebra, the Schouten bracket coincides with *Courant bracket* in Lie algebroid theory.

A similar construction holds for the conjugate $\bar{\mathfrak{f}}$, generated by $\mathfrak{h}^{(0,1)} \oplus \mathfrak{h}^{*(1,0)}$.

Let d be the Chevalley-Eilenberg (C-E) differential d for the Lie algebra $\mathfrak{h}, [- \bullet -]$. Then $(\wedge \mathfrak{h}^*, d)$ is a differential graded algebra. Similarly, let $\bar{\partial}$ be the C-E differential for the complex Lie algebra $\bar{\mathfrak{f}}^1$. Note that the natural pairing on $(\mathfrak{h} \oplus \mathfrak{h}^*) \otimes \mathbb{C}$ induces a complex linear isomorphism $(\bar{\mathfrak{f}}^1)^* \cong \mathfrak{f}^1$. Therefore, the C-E differential of the Lie algebra $\bar{\mathfrak{f}}^1$ is a map from $\mathfrak{f}^1$ to $\mathfrak{f}^2$. Denote this operator by $\overline{\partial}$. It turns out that $(\mathfrak{f}, [- \bullet -], \wedge, \overline{\partial})$ form a differential Gerstenhaber algebra which we denote by $\mathrm{DGA}(\mathfrak{h}, J)$. The same construction shows that $(\bar{\mathfrak{f}}, [- \bullet -], \wedge, \partial)$ is a differential Gerstenhaber algebra, conjugate linearly isomorphic to $\mathrm{DGA}(\mathfrak{h}, J)$. The above construction could be carried out similarly on a manifold with a complex structure.

2.2. **DGA of a symplectic structure.** Let $\mathfrak{k}$ be a Lie algebra over $\mathbb{R}$. Suppose that ω is a symplectic form on $\mathfrak{k}$. Then the contraction with ω, $\omega : \mathfrak{k} \to \mathfrak{k}^*$ is a real non-degenerate linear map. Define a bracket $[- \bullet -]_\omega$ on $\mathfrak{k}^*$ by

$$[\alpha \bullet \beta]_\omega := \omega[\omega^{-1}\alpha \bullet \omega^{-1}\beta]. \tag{2.4}$$

It is a tautology that $(\mathfrak{k}^*, [- \bullet -]_\omega)$ becomes a Lie algebra, with the map ω understood as a Lie algebra homomorphism.

In addition, the exterior algebra of the dual $\mathfrak{h}^*$ with the C-E differential d for the Lie algebra $\mathfrak{k}$ is a differential graded Lie algebra. In fact, $(\wedge^\bullet \mathfrak{k}^*, [-\bullet-]_\omega, \wedge, d)$ is a differential Gerstenhaber algebra over $\mathbb{R}$. After complexification we denote this by $\mathrm{DGA}(\mathfrak{k}, \omega)$.

2.3. **Quasi-isomorphisms and isomorphisms.**

Definition 2.1. [29] The Lie algebra $\mathfrak{h}$ with an integrable complex structure J and the Lie algebra $\mathfrak{k}$ with a symplectic structure ω form a weak mirror pair if the differential Gerstenhaber algebras $\mathrm{DGA}(\mathfrak{h}, J)$ and $\mathrm{DGA}(\mathfrak{k}, \omega)$ are quasi-isomorphic.

Suppose that $\phi : \mathrm{DGA}(\mathfrak{k}, \omega) \to \mathrm{DGA}(\mathfrak{h}, J)$ is a quasi-isomorphism. Since the concerned DGAs are exterior algebras generated by finite dimensional Lie algebras, it is natural to examine the property of the Lie algebra homomorphism on the degree-one elements.

$$\phi : \mathfrak{k}^*_{\mathbb{C}} \to \mathfrak{f}^1 = \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}. \tag{2.5}$$

In particular, if the restriction of ϕ to $\mathfrak{k}^*_{\mathbb{C}}$ is an isomorphism, it induces an isomorphism from $\mathrm{DGA}(\mathfrak{h}, J)$ to $\mathrm{DGA}(\mathfrak{k}, \omega)$. It turns out that for a special class of algebras, this is the only situation when quasi-isomorphism occurs.

Proposition 2.2. [15] *Suppose that $\mathfrak{h}$ and $\mathfrak{k}$ are finite dimensional nilpotent Lie algebras of the same dimension, J is an integrable complex structure on $\mathfrak{h}$ and ω is a symplectic form on $\mathfrak{k}$. Then a homomorphism ϕ from* $\mathrm{DGA}(\mathfrak{h}, J)$ *to* $\mathrm{DGA}(\mathfrak{k}, \omega)$ *is a quasi-isomorphism if and only if it is an isomorphism.*

This proposition provides a large class of Lie algebras to work on. So in this paper we focus our attention on a restricted type of weak mirror pairs. Namely, we seek a pair such that the map ϕ in (2.5) is an isomorphism on the degree-one level. Since the Lie algebra $\mathfrak{k}^*$ is tautologically isomorphic to $\mathfrak{k}$ via ω, we concern ourselves with the non-degeneracy of the map

$$\phi \circ \omega : \mathfrak{k}_{\mathbb{C}} \to \mathfrak{k}^*_{\mathbb{C}} \to \mathfrak{f}^1 = \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}. \tag{2.6}$$

When this is an isomorphism one immediately obtains conditions on the structure of $\mathfrak{k}$. The reason for this is that with respect to the

Schouten bracket, $\mathfrak{h}^{(1,0)}$ is a subalgebra of $\mathfrak{f}^1$ and $\mathfrak{h}^{*(0,1)}$ is an abelian ideal. In addition, $\dim \mathfrak{h}^{(1,0)} = \dim \mathfrak{h}^{*(0,1)}$. In other words, $\mathfrak{f}^1$ and $\mathfrak{k}$ are semi-direct products of a very particular form.

2.4. **Semi-direct products.** Let $\mathfrak{g}$ be a Lie algebra, V a vector space and let $\rho\colon \mathfrak{g} \to \mathit{End}(V)$ be a representation. On the vector space $\mathfrak{g} \oplus V$, define

$$[x+u, y+v]_\rho := [x,y] + \rho(x)v - \rho(y)u, \tag{2.7}$$

where x, y are in $\mathfrak{g}$ and u, v are in V. Then this determines a Lie bracket on $\mathfrak{g} \oplus V$. This structure is a particular case of a semi-direct product. As a Lie algebra it is denoted by $\mathfrak{h} = \mathfrak{h}(\mathfrak{g}, \rho) = \mathfrak{g} \ltimes_\rho V$. By construction, V is an abelian ideal and $\mathfrak{g}$ is a complementary subalgebra. One may also consider the semi-direct product as an extension of the algebra $\mathfrak{g}$ by a vector space V.

Note that if H is a simply connected, connected Lie group of $\mathfrak{h}$ and G the connected subgroup of H with $\mathfrak{g} \leqslant \mathfrak{h}$ as above then $H/G \simeq \mathbb{R}^{\dim V}$ as a flat symmetric space.

Conversely, suppose V is an abelian ideal of a Lie algebra $\mathfrak{h}$ and $\mathfrak{g}$ a complementary subalgebra. The adjoint action of $\mathfrak{g}$ on V then gives $\rho\colon \mathfrak{g} \to \mathit{End}(V)$.

In this paper, we are solely interested in the situation when $\dim \mathfrak{g} = \dim V$. In particular, when the vector space V is regarded as the underlying space of the real algebra $\mathfrak{g}$ or its dual $\mathfrak{g}^*$, interesting geometry and other phenomena arise through the representations of $\mathfrak{g}$ as described next.

3. Geometry on semi-direct products

A *left-invariant connection* on a Lie group G is an affine connection ∇ such that $\nabla_{X_g} Y_g = (L_g)_*(\nabla_x y)$, where $X_g = (L_g)_* x,\ Y_g = (L_g)_* y$, i.e. such that covariant differentiation of left-invariant vector fields give rise to left-invariant vector fields. These are in one-to-one correspondence with linear maps $\gamma\colon \mathfrak{g} \to \mathit{End}(\mathfrak{g})$ through $\nabla_x y = \gamma(x) y$. The torsion of

γ is

$$T^\gamma(x,y) := [x,y] - \gamma(x)y + \gamma(y)x$$

and its curvature R^γ

$$R^\gamma(x,y) := \gamma([x,y]) - \gamma(x)\gamma(y) + \gamma(y)\gamma(x).$$

Since all connections considered here are left-invariant, linear maps $\gamma\colon \mathfrak{g} \to End(\mathfrak{g})$ are referred to as connections on $\mathfrak{g}$ and say that γ is *flat* if $R^\gamma = 0$ and *torsion-free* if $T^\gamma = 0$.

3.1. **Totally real semi-direct products.** A complex structure on a real vector space W is an endomorphism J such that $J^2 = -1$. If V is a subspace of W such that $V' = JV$ satisfies $V \oplus V' = W$ then we say that J is *totally real with respect to* V. Given a totally real J any w in W may be written uniquely as $w = x + Jy$ for x, y in V. So $W \cong V \oplus V$ and J may be viewed as the standard complex structure J_0 on $V \oplus V$ given by $(x, y) \mapsto (-y, x)$. Since a basis for W always may be chosen so that $Je_{2i-1} = e_{2i}$ any J is totally real with respect to some V.

If $\mathfrak{g} = (W, [\cdot,\cdot])$ is a Lie algebra we say that J is *integrable* if the Nijenhuis tensor

$$N_J(x,y) := [x,y] - [Jx,Jy] + J([x,Jy] + [Jx,y])$$

is zero for all x and y in W. If J is totally real with respect to V then J is integrable if and only if $N_J(x,y) = 0$ for all $x, y \in V$. This follows by the identity $N_J(x,y) = JN_J(x,Jy)$ valid for all $x, y \in W$.

Definition 3.1. Suppose that $\mathfrak{h}(\mathfrak{g},\rho) = \mathfrak{g} \ltimes_\rho V$ is a semi-direct product Lie algebra. A complex structure on $\mathfrak{g} \ltimes_\rho V$ is totally real if J is totally real with respect to $\mathfrak{g}$ and $J\mathfrak{g} = V$.

Since V is an abelian ideal, $[Jx, Jy] = 0$ for all x, y in $\mathfrak{g}$ and so the Nijenhuis tensor vanishes precisely when

$$[x,y] + J\rho(x)Jy - J\rho(y)Jx = 0. \tag{3.8}$$

for all $x, y \in \mathfrak{g}$. This has the significance that

$$\gamma(x)y := -J\rho(x)Jy \tag{3.9}$$

defines a torsion-free connection on $\mathfrak{g}$. This is flat since

$$\begin{aligned} & \gamma([x,y]) - \gamma(x)\gamma(y) + \gamma(y)\gamma(x) \\ = \quad & -J\rho([x,y])J + J\rho(x)\rho(y)J - J\rho(y)\rho(x)J = 0. \end{aligned}$$

On the other hand, take a flat, torsion-free connection γ on $\mathfrak{g}$. Then the totally real complex structure J on $\mathfrak{h} := \mathfrak{g} \ltimes_\gamma \mathfrak{g}$ defined by $J(x,y) = (-y,x)$ becomes integrable with respect to $[\cdot,\cdot]_\gamma$ by virtue of

$$\begin{aligned} N_J((x,0),(y,0)) &= [(x,0),(y,0)]_\gamma + J[(x,0),(0,y)]_\gamma - J[(y,0),(0,x)]_\gamma \\ &= [(x,0),(y,0)]_\gamma + J((0,\gamma(x)y) - (0,\gamma(y)x)) \\ &= ([x,y] - \gamma(x)y + \gamma(y)x, 0) = (0,0). \end{aligned}$$

This proves our first Proposition, which is at least implicitly contained in [1].

Proposition 3.2. *There is a one-to-one correspondence between flat torsion-free connections on $\mathfrak{g}$ and totally real integrable complex structures on semi-direct products $\mathfrak{g} \ltimes_\rho V$.*

3.2. **Lagrangian semi-direct products.** Let ω be a two-form on a vector space W, i.e. $\omega \in \Lambda^2 W^*$. We may also view ω as a linear map $\omega\colon W \to W^*$ such that $\omega^* = -\omega$ through the identification $(W^*)^* = W$. Let $2m = \dim W$. Then ω is *non-degenerate* if $\omega^m \neq 0$. Equivalently, $\omega\colon W \to W^*$ is invertible.

A subspace V of W is *isotropic* if $\omega(V,V) = 0$. Equivalently $\omega(V) \subset \mathrm{Ann}(V) \subset W^*$. If $\dim V = \frac{1}{2}W$, then an isotropic V is called a *Lagrangian* subspace. In this case $\omega(V) = \mathrm{Ann}(V)$. A splitting $W = V \oplus V'$ of a vector space W into a direct sum is called Lagrangian with respect to ω if both V and V' are Lagrangian with respect to ω. Two vector spaces W_1 and W_2 with non-degenerate two-form ω_1 and ω_2 are said to be isomorphic if a linear isomorphism $f\colon W_1 \to W_2$ exists such that $f^*\omega_2 = \omega_1$.

Let V be a vector space. Then $V \oplus V^*$ carries a two-form ω given by the canonical pairing $\omega(\alpha, x) = \alpha(x)$ for $\alpha \in V^*$ and $x \in V$ and such that the given splitting is Lagrangian.

Lemma 3.3. *Suppose W is a vector space and ω is a non-degenerate two-form on W. Then W admits a Lagrangian splitting $W = V \oplus V'$ if and only if (W, ω) is isomorphic to $(V \oplus V^*, \langle \cdot, \cdot \rangle)$ (where V may be taken to be $\mathbb{R}^{\frac{1}{2}\dim W}$.*

Proof: Clearly, if $f : W \to V \oplus V^*$ is an isomorphism then the splitting given $U := f^{-1}(V)$, $U' := f^{-1}(V^*)$ is Lagrangian with respect to ω. If, on the other hand $U \oplus U'$ is a Lagrangian splitting of W then $\mathrm{Ann}(V') = \omega(V')$. Furthermore $\mathrm{Ann}(V')$ is canonically isomorph to V^*. Therefore $W \cong V \oplus V^*$ by the map $f(x + x') := x + \omega(x')$. Moreover,

$$\begin{aligned}\langle f(x+x'), f(y+y')\rangle &= \langle x + \omega(x'), y + \omega(y')\rangle \\ &= \omega(x', y) + \omega(x, y') \\ &= \omega(x + x', y + y').\end{aligned}$$

■

Suppose that $\mathfrak{g} = (W, [\cdot, \cdot])$ is a Lie algebra. Then the derivative of a two-form ω with respect to the Chevalley-Eilenberg differential is

$$\begin{aligned}(d\omega)(x, y, z) &= -\left(\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y)\right) \\ &= -\left(\omega([x, y])z - \omega(x)(\mathrm{ad}(y)z) + \omega(y)(\mathrm{ad}(x)z)\right).\end{aligned}$$

So ω is closed if and only if for all x, y:

$$\omega([x, y]) = \mathrm{ad}^*(x)(\omega(y)) - \mathrm{ad}^*(y)(\omega(x)) \tag{3.10}$$

where $(\mathrm{ad}^*(x)\alpha)(y) = -\alpha([x, y])$.

Definition 3.4. Suppose that a Lie algebra $\mathfrak{h}$ is a semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes_\rho V$. It is said to be Lagrangian with respect to a non-degenerate 2-form if the subalgebra $\mathfrak{g}$ and the abelian ideal V are both Lagrangian with respect to ω.

When $\mathfrak{g} \ltimes_\rho V$ is Lagrangian, there is a canonical isomorphism $\omega(V) = \mathrm{Ann}(V) \cong \mathfrak{g}^*$. Similarly, $\omega(\mathfrak{g}) = \mathrm{Ann}(\mathfrak{g}) \cong V^*$. Define

$$\rho^* \colon \mathfrak{g} \to \mathit{End}(V^*) \quad \text{by} \quad (\rho^*(x)\alpha)(u) = -\alpha(\rho(x)u). \tag{3.11}$$

Then

$$(\rho^*(x)\omega(y))(u) = -\omega(y, \rho(x)u) = -\omega(y, [x,u]_\rho) = (\mathrm{ad}^*_\rho(x)\omega(y))(u).$$

Comparing to equation (3.10) it is now clear that ω is closed if and only if

$$\omega([x,y]) = \rho^*(x)(\omega(y)) - \rho^*(y)(\omega(x)) \tag{3.12}$$

for all $x, y \in \mathfrak{g}$. The story now repeats itself. Define

$$\gamma(x)y := \omega^{-1}(\rho^*(x)\omega(y)). \tag{3.13}$$

This defines a flat torsion-free connection on $\mathfrak{g}$ since

$$\begin{aligned} & R^\gamma(x,y)z \\ = \;& \omega^{-1}(\rho^*([x,y])\omega(z)) \\ & -\omega^{-1}(\rho^*(x)\omega(\omega^{-1}(\rho^*(y)\omega(z)))) + \omega^{-1}(\rho^*(y)\omega(\omega^{-1}(\rho^*(x)\omega(z)))) \\ = \;& \omega^{-1}(\rho^*([x,y])\omega(z)) - \omega^{-1}(\rho^*(x)\rho^*(y)\omega(z)) + \omega^{-1}(\rho^*(y)\rho^*(x)\omega(z)) \\ = \;& 0. \end{aligned}$$

Conversely, take a flat torsion-free connection γ on $\mathfrak{g}$. Let ω be the standard skew pairing on $\mathfrak{g} \oplus \mathfrak{g}^*$:

$$\omega(x+u, y+v) = u(y) - v(x).$$

Define the bracket on $\mathfrak{g}\oplus\mathfrak{g}^*$ as the semi-direct product by representation γ^*. Then the semi-direct product is Lagrangian with respect to ω. It follows that if x, y, z are in $\mathfrak{g}$ and u, v, w are in $\mathfrak{g}^*$,

$$d\omega(x,y,z) = d\omega(u,v,w) = d\omega(x,u,v) = 0. \tag{3.14}$$

Moreover,

$$\begin{aligned} (d\omega)(x,y,u) &= -\left(\omega([x,y],u) + \omega(\gamma^*(y)u, x) - \omega(\gamma^*(x)u, y)\right) \\ &= (u([x,y]) - (\gamma^*(y)u)(x) + (\gamma^*(x)u)(y)) \\ &= u([x,y]) + u(\gamma(y)x) - u(\gamma(x)y) \\ &= u([x,y] - \gamma(x)y + \gamma(y)x) = 0. \end{aligned}$$

This gives us the following result (which may also be found in [11]).

Proposition 3.5. *There is a one-to-one correspondence between flat torsion-free connections ρ on Lie algebras $\mathfrak{g}$ and Lagrangian semi-direct products $\mathfrak{g} \ltimes_\rho V$.*

3.3. **From complex structure to two-form, and back.** Proposition 3.2 and Proposition 3.5 of the preceding sections yield a one-to-one correspondence between certain integrable complex structures and certain symplectic forms going via flat, torsion-free connections. In this section, we construct a direct relation between two-forms and complex structures.

Suppose $W = V \oplus JV$, i.e. J is a complex structure on W so that V and JV are totally real. On $W^\vee := V \oplus (JV)^*$, define

$$\omega_J(x+u, y+v) := v(Jx) - u(Jy), \tag{3.15}$$

where x, y are in V and u, v are in $(JV)^*$. Then ω_J is non-degenerate on $W^\vee$ with both V and $(JV)^*$ being Lagrangian.

Conversely suppose ω is a non-degenerate 2-form on $W = V \oplus V'$ with both V and V' being Lagrangian. Write $V' = \omega^{-1}(V^*)$ and set

$$J_\omega(x+u) = -\omega^{-1}(u) + \omega(x) \tag{3.16}$$

for all $x+u$ in $V \oplus V^*$. Clearly, both V and V^* are totally real with respect to J_ω.

When W is a semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes_\rho V$ define the *dual semi-direct product* $\mathfrak{h}^\vee$ by $\mathfrak{h}^\vee := \mathfrak{g} \ltimes_{\rho^*} V^*$ where ρ^* is the dual representation

$$(\rho^*(x)a)(u) := -a(\rho(u)).$$

Suppose J is totally real with respect to the semi-direct product $\mathfrak{h}$. As noted in (3.14), the sole obstruction for the differential of the induced two-form ω_J on $\mathfrak{h}^\vee$ to vanish is due to $(d\omega_J)(x, y, u)$ where x, y are in

$\mathfrak{g}$ and u is in V^*. In the present case,

$$\begin{aligned}(d\omega_J)(x,y,u) &= -\left(\omega_J([x,y],u)+\omega_J([y,u],x)+\omega_J([u,x],y)\right)\\ &= -\left(u(J[x,y])-(\rho^*(y)u)(Jx)+(\rho^*(x)u)(Jy)\right)\\ &= -\left(u(J[x,y])+u(\rho(y)Jx)-u(\rho(x)Jy)\right)\\ &= -u(J[x,y]+\rho(y)Jx-\rho(x)Jy)\\ &= -u(J([x,y]-J\rho(y)Jx+J\rho(x)Jy)).\end{aligned}$$

By (3.8), we have the following.

Lemma 3.6. *Suppose $\mathfrak{h}$ is a semi-direct product $\mathfrak{g}\ltimes_\rho V$ and let J be a totally real complex structure on $\mathfrak{h}$. Then the dual semi-direct product $\mathfrak{h}^\vee$ is Lagrangian with respect to the two-form ω_J. Moreover, ω_J is symplectic if and only if J is integrable.*

Similarly, if ω is a non-degenerate two-form on $\mathfrak{h}$, and the semi-direct product is Lagrangian, then the Nijenhuis tensor of the induced complex structure J_ω is determined by

$$\begin{aligned}N_{J_\omega}(x,y) &= [x,y]-[J_\omega(x),J_\omega(y)]+J_\omega([x,J_\omega(y)]-[y,J_\omega(x)])\\ &= [x,y]-[\omega(x),\omega(y)]-\omega^{-1}([x,\omega(y)]-[y,\omega(x)])\\ &= [x,y]-\omega^{-1}(\rho^*(x)\omega(y)-\rho^*(y)\omega(x)).\end{aligned}$$

By (3.12), we have the following.

Lemma 3.7. *Suppose $\mathfrak{h}$ is a semi-direct product $\mathfrak{g}\ltimes_\rho V$ with a non-degenerate two-form ω. Suppose that the semi-direct product is Lagrangian. Then the complex structure J_ω on $\mathfrak{h}^\vee$ is totally real. Furthermore, J_ω is integrable if and only if ω is symplectic.*

In Lemma 3.6 and Lemma 3.7 (compare [6]), we demonstrate the passages from a totally real complex structure on a semi-direct product to a symplectic structure on the dual semi-direct product, and from a symplectic structure on a Lagrangian semi-direct product to a totally real complex structure on the dual. In the next lemma, we demonstrate that these two processes reverse each other.

Lemma 3.8. *Let* $\mathfrak{h} = \mathfrak{g} \ltimes_\rho V$ *be a semi-direct product. Let* $\phi\colon \mathfrak{h} \to (\mathfrak{h}^\vee)^\vee$ *be the canonical isomorphism defined by the identification* $(V^*)^* = V$*. If* $\mathfrak{h}$ *is equipped with a totally real complex structure* J *then* $\phi^*(J_{\omega_J}) = J$*. Similarly, if* $\mathfrak{h}$ *is Lagrangian with respect to a symplectic form* ω *then* $\phi^*(\omega_{J_\omega}) = \omega$*.*

Proof: The identification $(V^*)^* = V$ is of course the map $u \mapsto u^{**}$ given by $u^{**}(v^*) := v^*(u)$ for all $u \in V$ and $v^* \in V^*$. Setting $\phi(x + u) := x + u^{**}$ this is an isomorphism of Lie algebras as one may easily check. Suppose J is totally real. Then J_{ω_J} is also totally real, by Lemmas 3.6 and 3.7. Then the second statement follows by checking that $J_{\omega_J} x = \phi(Jx) \in (V^*)^*$. But

$$(J_{\omega_J} x)(u^*) = \omega_J(x)(u^*) = u^*(Jx) = (Jx)^{**}(u^*) = \phi(Jx)(u^*).$$

Similarly, if $\mathfrak{h}$ is Lagrangian with respect to ω then it is also Lagrangian with respect to ω_{J_ω}. Moreover,

$$\omega_{J_\omega}(x, u^{**}) = u^{**}(J_\omega x) = (J_\omega x)(u) = \omega(x, u).$$

This completes the proof. ■

3.4. **Special Lagrangian structures.** A non-degenerate two-form ω and a complex structure J on a vector space W are said to be *compatible* if $\omega(J\xi, J\eta) = \omega(\xi, \eta)$ for all ξ, η in W. In that case $g(\xi, \eta) := \omega(\xi, J\eta)$ is a non-degenerate symmetric two-tensor on W, the induced metric for which J is an orthogonal transformation: $g(J\xi, J\eta) = g(\xi, \eta)$. If g is positive-definite we say (ω, J) is an almost Hermitian pair, otherwise we say (ω, J) is almost pseudo-Hermitian.

Suppose that (ω, J) is an almost pseudo-Hermitian structure on W and a vector subspace V is totally real with respect J. Then V is isotropic with respect to ω if and only if JV is isotropic. If V is a totally real subspace, then the splitting $W = V \oplus JV$ is orthogonal with respect to the induced metric. In addition, it is clear that $g_{|_{JV}}$ is determined by $g_{|_V}$.

Conversely, let J be a complex structure on W and V a totally real subspace. Any inner product g on V could be extended to W by declaring $g(Jx, Jy) = g(x, y)$ for $x, y \in V$ and $g(x, Jy) = 0$. Then (ω, J) is an almost pseudo-Hermitian structure on W.

If W is a Lie algebra $\mathfrak{h} = (W, [\cdot,\cdot])$, we say an almost pseudo-Hermitian pair is pseudo-Kähler if ω is symplectic and J is integrable.

Definition 3.9. Let $\mathfrak{h}$ be a Lie algebra with a semi-direct product structure $\mathfrak{h} = \mathfrak{g} \ltimes_\rho V$. Let (ω, J) be a pseudo-Kähler structure on $\mathfrak{h}$. Then $\mathfrak{h}$ is said to be special Lagrangian if $\mathfrak{g}$ and V are totally real with respect to J and Lagrangian with respect to ω. We then also call (ω, J) a special Lagrangian structure on the semi-direct product $\mathfrak{h}$.

Proposition 3.10. *If (ω, J) is a special Lagrangian structure on a semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes_\rho V$, then (ω_J, J_ω) is a special Lagrangian structure on the dual semi-direct product $\mathfrak{h}^\vee = \mathfrak{g} \ltimes_{\rho^*} V$.*

Proof: In view of Lemma 3.6 and Lemma 3.7, the only issue is to verify that for any $x + u, y + v$ in $\mathfrak{g} \oplus V^*$,

$$\omega_J(J_\omega(x + u), J_\omega(y + v)) = \omega_J(x + u, y + v).$$

Given the compatibility of ω and J, the proof is simply a matter of definitions as given in (3.15) and (3.16). ■

Other than allowing the metric being pseudo-Kähler, Definition 3.9 above is an invariant version of the usual definition of special Lagrangian structures found in literature on mirror symmetry if we extend the metric g and the complex structure J to be left-invariant tensors on the simply connected Lie groups of $\mathfrak{h}$ and $\mathfrak{g}$ (see e.g. [26]). To illustrate this point, note that if $\{e^1, \ldots, e^n\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the (pseudo-)Riemannian metric g, set $u^j = e^j + iJe^j$. Then $\{u^1, \ldots, u^n\}$ is a Hermitian basis of $\mathfrak{h}$. Then the Kähler form ω is

$$\omega = i \sum_{j=1}^{n} u^j \wedge \overline{u}^j = i \sum_j (e^j + iJe^j) \wedge (e^j - iJe^j).$$

The complex volume form is

$$\begin{aligned}\Phi &= u^1 \wedge \cdots \wedge u^n = (e^1 + iJe^1) \wedge \cdots \wedge (e^n + iJe^n) \\ &= e^1 \wedge \cdots \wedge e^n + i^n Je^1 \wedge \cdots \wedge Je^n \\ &\quad + \text{terms mixed with both } e^j \text{ and } Je^k.\end{aligned}$$

When n is odd the real part of Φ restricts to zero on V and the imaginary part restricts to a real volume form. Therefore, the fibers of the quotient map from the Lie group H onto G are special Lagrangian submanifolds.

3.5. **Flat connections and special Lagrangian structures.** Suppose (ω, J) is a special Lagrangian structure on $\mathfrak{h} = \mathfrak{g} \ltimes_\rho V$ and let g be the induced metric. Define $\gamma(x) := -J\rho(x)J$. Then γ is a flat torsion-free connection on $\mathfrak{g}$. Since ω is closed

$$\begin{aligned}(d\omega)(x, y, Jz) &= -(\omega([x, y], Jz) + \omega([y, Jz], x) + \omega([Jz, x], y)) \\ &= -g([x, y], z) + g(\gamma(y)z, x) - g(\gamma(x)z, y) \\ &= -g([x, y] - \gamma^t(y)x + \gamma^t(x)y, z) = 0,\end{aligned}$$

and, since γ is flat,

$$\begin{aligned}&-\gamma^t([x, y]) - \gamma^t(x)\gamma^t(y) + \gamma^t(y)\gamma^t(x) \\ = \;&-(\gamma([x, y] - \gamma(x)\gamma(y) + \gamma(y)\gamma(x))^t = 0.\end{aligned}$$

Therefore, $-\gamma^t$ is another flat torsion-free connection.

On the other hand, suppose that $\mathfrak{g}$ is equipped with a non-degenerate bilinear form g. Let γ be a flat torsion-free connection such that $\gamma' := -\gamma^t$ is also an flat torsion-free connection. Then, as above the complex structure on $\mathfrak{h} := \mathfrak{g} \ltimes_\gamma \mathfrak{g}$ given by $J(x, y) = (-y, x)$ is integrable. We write $x + Jy,\ x, y \in \mathfrak{g}$ for the elements in $\mathfrak{h}$. Define ω on $\mathfrak{h}$ by $\omega(x, y) = \omega(Jx, Jy) = 0$ and $\omega(x, Jy) = g(x, y) = -\omega(y, Jx)$ and set $g(Jx, Jy) = g(x, y)$. Then essentially the same calculation as above shows that $d\omega = 0$ by virtue of γ' being flat and torsion-free.

Remark 3.11. Note that we may equally well choose to work with the integrable complex structure $J'(x, y) = (-y, x)$ on $\mathfrak{h} := \mathfrak{g} \ltimes_{\gamma'} \mathfrak{g}$ and the associated symplectic form ω'. This is of course precisely the "mirror image" of $(\mathfrak{h}, J, \omega)$. This all amounts to

Proposition 3.12. *Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate bilinear form g. Then there is a two-to-one correspondence between special Lagrangian structures on a semi-direct product extending the Lie algebra $\mathfrak{g}$ and flat torsion-free connections γ on $\mathfrak{g}$ such that the dual connection $-\gamma^t$ is also flat and torsion-free.*

4. Canonical isomorphism of DGAs

In this section, we consider the relation between $\mathrm{DGA}(\mathfrak{h}, J)$ and $\mathrm{DGA}(\mathfrak{h}^\vee, \omega_J)$ when $\mathfrak{h}$ is a semi-direct product totally real with respect to a complex structure J.

Let γ be a flat torsion-free connection on a Lie algebra $\mathfrak{g}$. Write V for the associated representation of $\mathfrak{g}$ on itself and consider the usual integrable complex structure J on $\mathfrak{h} = \mathfrak{g} \ltimes_\gamma V$. Then $\mathfrak{k}^1(\mathfrak{h}, J) = \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}$ where $\mathfrak{h}^{(1,0)}$ spanned by $(1-iJ)x$ as x goes through $\mathfrak{g}$ while $\mathfrak{h}^{*(0,1)}$ is generated by $(1-iJ)\alpha$ where α ranges through $V^* \subset \mathfrak{h}^*$. Here J acts on V^* by $(Jv^*)(x+u) = -v^*(Jx+Ju) = -v^*(Jx)$. In particular, $Jv^* \in \mathrm{Ann}(V) \subset \mathfrak{h}^*$.

Now set $\mathfrak{h}^\vee := \mathfrak{g} \ltimes_{\gamma^*} V^*$ and define $\phi\colon \mathfrak{h}^\vee_{\mathbb{C}} \to \mathfrak{k}^1(\mathfrak{h}, J)$ as the tautological map:

$$\phi(x+v^*) := (1-iJ)x + (1-iJ)v^*. \tag{4.17}$$

Recall that the restriction of the Schouten bracket on the space $\mathfrak{k}^1(\mathfrak{h}, J)$ is a Lie bracket.

Lemma 4.1. *The map $\phi : \mathfrak{h}^\vee_{\mathbb{C}} \to \mathfrak{k}^1(\mathfrak{h}, J)$ is an isomorphism of Lie algebras.*

Proof: This is a straight-forward check. First, if u^*, v^* are in V^*, then $\phi(u^*), \phi(v^*)$ are in $\mathfrak{h}^{*(0,1)}$. Therefore, $[\phi(u^*), \phi(v^*)] = 0 = \phi([u^*, v^*])$. If $x, y \in \mathfrak{g}$, then

$$\begin{aligned}[\phi(x), \phi(y)] &= [(1-iJ)x, (1-iJ)y] = [x,y] - i([x, Jy] + [Jx, y]) \\ &= [x,y] - iJ[x,y] = \phi([x,y])\end{aligned}$$

by integrability of J. Finally, take $x \in \mathfrak{g}$ and $v^* \in V^*$. Then $[\phi(x), \phi(v^*)] \in \mathfrak{h}^{*(0,1)}$. With $y \in \mathfrak{g}$ we get

$$\begin{aligned} & [\phi(x), \phi(v^*)]((1+iJ)y) \\ = \;& [(1-iJ)x, (1-iJ)v^*]((1+iJ)y) \\ = \;& -((1-iJ)v^*)([(1-iJ)x, (1+iJ)y]) \\ = \;& -((1-iJ)v^*)([x,y] + i([x,Jy] - [Jx,y])). \end{aligned}$$

It is apparent that $v^*([x,y]) = 0$. In addition, as $[x, Jy]$ is in V and Jv^* is in $\mathrm{Ann}(V)$, the above is equal to

$$\begin{aligned} = \;& i(Jv^*)([x,y]) - iv^*([x,Jy] - [Jx,y]) \\ = \;& -iv^*(J[x,y] + [x,Jy] - [Jx,y]) = -2iv^*([x,Jy]). \end{aligned}$$

While

$$\begin{aligned} & \phi([x,v^*])((1+iJ)y) = ((1-iJ)[x,v^*])((1+iJ)y) \\ = \;& -i(J[x,v^*])(y) + i[x,v^*](Jy) = 2i[x,v^*](Jy) = -2iv^*([x,Jy]). \end{aligned}$$

■

Recall that the complex structure J on $\mathfrak{h}$ induces a symplectic structure ω_J on $\mathfrak{h}^\vee$. Then the contraction map

$$\omega_J : \mathfrak{h}^\vee \to (\mathfrak{h}^\vee)^*$$

carries the Lie bracket on $\mathfrak{h}^\vee$ to a Lie bracket $[- \bullet -]_{\omega_J}$ on $(\mathfrak{h}^\vee)^*$. Therefore, $\phi \circ \omega_J^{-1}$ is a Lie algebra isomorphism from $(\mathfrak{h}^\vee)^*_{\mathbb{C}}$ to $\mathfrak{f}^1$. It induces an isomorphism from the underlying Gerstenhaber algebra of $\mathrm{DGA}(\mathfrak{h}^\vee, \omega_J)$ to that of $\mathrm{DGA}(\mathfrak{h}, J)$. Next we demonstrate that this map is also an isomorphism of differential graded algebra. i.e.

$$\phi \circ \omega^{-1} \circ d = \bar{\partial} \circ \phi \circ \omega^{-1}. \tag{4.18}$$

We do have an isomorphism at hand. Composing ϕ with complex conjugation on $\mathfrak{h}^\vee_{\mathbb{C}}$ and $\mathfrak{f}^1$ respectively, we get a complex linear map from $\overline{\mathfrak{h}^\vee_{\mathbb{C}}}$ to $\bar{\mathfrak{f}}^1$. But the complexified Lie algebra $\mathfrak{h}^\vee_{\mathbb{C}}$ is isomorphic to $\overline{\mathfrak{h}^\vee_{\mathbb{C}}}$. This yields a Lie algebra isomorphism from $\mathfrak{h}^\vee_{\mathbb{C}}$ to $\bar{\mathfrak{f}}^1$. The dual map induces an isomorphism of the exterior differential algebra generated by the dual vector spaces and the corresponding pair of Chevalley-Eilenberg differentials. This isomorphism *should be* the map given in (4.18). To see that ω plays a proper role, we need more technical details.

As $\phi^* : (\mathfrak{f}^1)^* \to (h^\vee_{\mathbb{C}})^*$, the conjugated map is $\bar{\phi}^* : (\bar{\mathfrak{f}^1})^* \to (\overline{h^\vee_{\mathbb{C}}})^*$. In the next calculation we implicitly identify the isomorphic Lie algebras $(\mathfrak{f}^1)^*$ with $\bar{\mathfrak{f}}^1$ and $\mathfrak{h}^\vee_{\mathbb{C}}$ with its conjugate $\overline{\mathfrak{h}^\vee_{\mathbb{C}}}$. Hence $\bar{\phi}^*$ is identified with the map $\bar{\phi}^* : \mathfrak{f}^1 \to (h^\vee_{\mathbb{C}})^*$. Then $\bar{\phi}^*\phi$ is a map from $\mathfrak{h}^\vee_{\mathbb{C}}$ to $(h^\vee_{\mathbb{C}})^*$. According to [15, Proposition 11], the map $\phi \circ \omega_J^{-1}$ yields an isomorphism of differential graded algebra as in (4.18) if, up to a constant, $\bar{\phi}^*\phi$ is equal to the contract of ω_J. Therefore, we have the following computation.

$$
\begin{aligned}
& (\bar{\phi}^*\phi)(x+u^*)(y+v^*) \\
= \; & (\bar{\phi}^*)((1-iJ)x+(1-iJ)u^*)(y+v^*) \\
= \; & \overline{\phi^*((1+iJ)x+(1+iJ)u^*)}(y+v^*) \\
= \; & \overline{\phi^*((1+iJ)x+(1+iJ)u^*)(y+v^*)} \\
= \; & \overline{((1+iJ)x+(1+iJ)u^*)(\phi(y+v^*))} \\
= \; & \overline{((1+iJ)x+(1+iJ)u^*)((1-iJ)y+(1-iJ)v^*))} \\
= \; & ((1-iJ)x+(1-iJ)u^*)((1+iJ)y+(1+iJ)v^*)) \\
= \; & 2i(u^*(Jy)-v^*(Jx)) = 2i\omega_J(x+u^*, y+v^*).
\end{aligned}
$$

This shows that the isomorphism ϕ defines a DGA structure on the de Rham complex of $\mathfrak{h}^\vee_{\mathbb{C}}$ isomorphic to the one defined by ω_J, since the brackets differ only by multiplication by a constant. In particular, we have

Theorem 4.1. $\mathrm{DGA}(\mathfrak{h}, J)$ *and* $\mathrm{DGA}(\mathfrak{h}^\vee, \omega_J)$ *are isomorphic.*

A similar construction and calculation shows that for a Lagrangian symplectic form ω on $\mathfrak{h} = \mathfrak{g} \ltimes_\rho V$ the associated differential Gerstenhaber algebras $\mathrm{DGA}(\mathfrak{h}, \omega)$ and $\mathrm{DGA}(\mathfrak{h}^\vee, J_\omega)$ are isomorphic.

4.1. **Examples.**

4.1.1. *Kählerian structure on* $\mathbb{R}^n \ltimes \mathbb{R}^n$. Choose $\mathfrak{g} = \mathbb{R}^n$ with trivial Lie bracket. Then a representation of $\mathfrak{g}$ on $V = \mathbb{R}^n$ is given by a linear map $\rho\colon \mathbb{R}^n \to \mathfrak{gl}(n,\mathbb{R})$. Pick a basis $e_1,\dots,e_n$ of $\mathfrak{g}=\mathbb{R}^n$. Represent $\mathfrak{g}$ on $V=\mathbb{R}^n$ with basis $v_1,\dots,v_n$ by declaring $\rho(e_i)$ to be the diagonal matrix with a ρ_i in the i-th place of the diagonal and zero else. Then $\mathfrak{h}=\mathfrak{g}\ltimes_\rho V$ is the Lie algebra given by the structure equations

$$dv^i = -\rho_i e^i \wedge v^i$$

where e^i and v^i are the dual elements. By relabeling the v_i we may suppose $\rho_1,\dots,\rho_p$ to be the non-zero structure constants for a certain $p\le n$. Then

$$\omega_a = \sum_{i=1}^{n} a_i e^i \wedge v^i$$

is symplectic for any n-tuple $a=(a_1,\dots,a_n)\in\mathbb{R}^n$ with $a_i\neq 0$. Moreover, the complex structure J defined by

$$J(e_i)=v_i \quad \text{and} \quad J(v_i)=-e_i$$

is integrable, since $N_J(e_i,e_j)=J\rho(e_i)v_j - J\rho(e_j)v_i = 0$.

It is apparent that ω_a is of type $(1,1)$ with respect to J. In fact,

$$\omega(e_i,Je_i)=\omega(e_i,v_i)=a_i=\omega(v_i,Jv_i),$$

so $\mathfrak{h}$ is Kähler precisely when all a_i are positive (or all negative).

To identify the "mirror image" $\mathfrak{h}^\vee$ of $\mathbb{R}^n\ltimes\mathbb{R}^n$, we first calculate its Lie bracket $[-,-]^\vee$. Since both the base algebra and the ideal are abelian, the only non-trivial brackets are contributed by $e_i\in\mathbb{R}^n$ and $v^j\in(\mathbb{R}^n)^*$. As

$$\begin{aligned} &([e_i,v^j]^\vee)(v_k) = (\rho^*(e_i)v^j)(v_k) = -v^j([e_i,v_k]) = (dv^j)(e_i,v_k)\\ =\ &\rho_j(e^j\wedge v^j)(e_i,v_k) = \rho_j\delta_i^j\delta_k^j. \end{aligned}$$

Therefore,

$$[e_i,v^j]^\vee = 0 \text{ when } i\neq j, \text{ and } [e_i,v^i]^\vee = \rho_i v^i \text{ for each } i.$$

In particular, when $\mathfrak{h}=\mathbb{R}^n\ltimes\mathbb{R}^n$ is given by the trivial representation, which has $\rho_i=0$ for all i, then its corresponding dual direct product $\mathfrak{h}^\vee$ is again the trivial algebra.

In all cases, the symplectic form and complex structure are

$$\omega_J(e_i, v^j) = -v^j(v_i), \quad J_\omega(e_i + v^j) = -\omega^{-1}(v^j) + \omega(e_i) = -\frac{1}{a_j}e_j + a_i v^i.$$

4.1.2. *An example with solvable base.* Take $\mathfrak{g}$ to be the solvable Lie algebra with brackets

$$[e_1, e_3] = -e_5, \qquad [e_1, e_5] = e_3, \qquad [e_3, e_5] = 0.$$

Using $\{e_2, e_4, e_6\}$ as an ordered basis of a vector space V, we represent elements in $End(V)$ by matrices. We choose an inner product on V by declaring this basis orthonormal. Then $\gamma\colon \mathfrak{g} \to End(V)$ defined by

$$\gamma(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \gamma(e_3) = 0 = \gamma(e_5),$$

is a skew-adjoint representation of $\mathfrak{g}$ on V with respect to the standard metric. If we consider V as the underlying vector space of $\mathfrak{g}$, it also defines a torsion-free left-invariant connection on G. Now, the non-zero brackets on $\mathfrak{h} = \mathfrak{g} \ltimes_\gamma V$ is

$$[e_1, e_3] = -e_5, \qquad [e_1, e_4] = -e_6, \qquad [e_1, e_5] = e_3, \qquad [e_1, e_6] = e_4.$$

The reader may now verify that the two-form $\omega = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6$ is symplectic and the complex structure $Je_{2j-1} = e_{2j}$ is integrable. It is apparent that ω is a positive-definite type-(1,1) symplectic form with respect to J. In other words, (ω, J) is a Kähler structure.

To construct $\mathfrak{h}^\vee$ explicitly, we take the basis $\langle e_1, e_3, e_5\rangle \oplus \langle e^2, e^4, e^6\rangle$ and identify the structure equations.

$$[e_1, e_3] = -e_5, \qquad [e_1, e^6] = -e^4, \qquad [e_1, e_5] = e_3, \qquad [e_1, e^4] = e^6.$$

To construction the corresponding symplectic structure ω_J and J_ω, we apply (3.15) and (3.16) to find that

$$\omega_J = e^1 \wedge e_2 + e^3 \wedge e_4 + e^5 \wedge e_6,$$
$$J_\omega(e_1) = e^2, \quad J_\omega(e_3) = e^4, \quad J_\omega(e_5) = e^6.$$

We note that both $\mathfrak{h}$ and $\mathfrak{h}^\vee$ clearly also may be represented as semi-direct products of $\mathbb{R}$ with $\mathbb{R}^5$ where $\mathbb{R}^5$ is represented on $\mathbb{R}^5$ as a line of transformations skew-symmetric with respect to the standard metric

g. This precisely follows the prescriptions of [30] to make g a flat left-invariant metric on H.

5. Nilpotent algebras of dimension at most six.

In this section, we tackle two problems when the algebra $\mathfrak{h}$ is a nilpotent algebra whose real dimension is at most six.

Problem 1. *Let $\mathfrak{h}$ be a nilpotent algebra. Suppose that it is a semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes V$ and totally real with respect to a complex structure J. Identify the corresponding algebra $\mathfrak{h}^\vee = \mathfrak{g} \ltimes V^*$ and the associated symplectic structure ω_J.*

In view of Lemma 3.8, the above problem is equivalent to finding the associated complex structure on the dual semi-direct product when one is given a semi-direct product which is also Lagrangian with respect to a symplectic structure.

The next problem raises a more restrictive issue.

Problem 2. *Let $\mathfrak{h}$ be a nilpotent algebra. Suppose that it is a semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes V$ and it is special Lagrangian with respect to a pseudo-Kähler structure (J, ω). Identify the corresponding algebra $\mathfrak{h}^\vee = \mathfrak{g} \ltimes V^*$ and the associated pseudo-Kähler structure $(J^\vee, \omega^\vee)$.*

In view of the example in Section 4.1.1, when the algebra $\mathfrak{h}$ is abelian, the dual semi-direct product $\mathfrak{h}^\vee$ is again abelian. The correspondence from (J, ω) to (J_ω, ω_J) is also given. Therefore, we shall exclude this trivial case in subsequent computation although we may include it for the completeness of a statement in a theorem. The first even dimension in which a non-abelian nilpotent algebra occurs is four.

5.1. **Four-dimensional case.** There are two four-dimensional non-trivial nilpotent algebras [22]. Only one of them is a semi-direct product, namely the direct sum of a trivial algebra with a three-dimensional Heisenberg algebra. It happens to be the only one admitting integrable invariant complex structures [33, Proposition 2.3]. Up to equivalence, there exists a basis $\{e_1, e_2, e_3, e_4\}$ on the algebra $\mathfrak{h}$ such that the structure equation is simply $[e_1, e_2] = -e_3$. The corresponding complex structure is

$$J(e_1) = e_2, \quad J(e_3) = e_4, \quad J(e_2) = -e_1, \quad J(e_4) = -e_3. \tag{5.19}$$

It is integrable. A symplectic form is

$$\omega = e^1 \wedge e^4 + e^3 \wedge e^2. \tag{5.20}$$

Consider the subspaces

$$\mathfrak{g} := \langle e_2, e_4 \rangle, \quad V := \langle e_1, e_3 \rangle. \tag{5.21}$$

They determines a semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes_{Ad} V$. It is apparent that this semi-direct product is special Lagrangian the pair (ω, J) above. One may now work through our theory to demonstrate that the mirror image of $(\mathfrak{h}, J, \omega)$ on $(\mathfrak{h}^\vee, J_\omega, \omega_J)$ is isomorphic to $(\mathfrak{h}, J, \omega)$ itself.

5.2. **Algebraic Aspects.** In the next few paragraphs, we identify the six-dimensional nilpotent algebras which is a semi-direct product of a three-dimensional Lie subalgebra $\mathfrak{h}$ and an abelian ideal V by identifying equivalent classes of representation of $\mathfrak{h}$ on V. Once it is done, the construction of the semi-direct product with the dual representation follows naturally. We postpone geometric considerations to the next section.

Since the adjoint action of the nilpotent Lie algebra $\mathfrak{g}$ on the abelian ideal V is a nilpotent representation, by Engel Theorem there exists a basis $\{e_2, e_4, e_6\}$ of V such that the matrix of any $\operatorname{ad} x$, $x \in \mathfrak{g}$, is strictly lower triangular.

In our calculation below, we often express the structure equation on $\mathfrak{h} = \mathfrak{g} \ltimes_{\mathrm{ad}} V$ in terms of the C-E differential on the dual basis $\{e^1, \ldots, e^6\}$. In particular we collect $(de^1, \ldots, de^6)$ in an array. We shall also adopt the shorthand notation that when $de^1 = e^i \wedge e^j + e^\alpha \wedge e^\beta$, then the first entry in this array is $ij + \alpha\beta$ [33]. To name six-dimensional algebras, we use the convention developed in [18].

5.2.1. *Assume that $\mathfrak{g}$ is abelian.* There exists a basis $\{e_1, e_3, e_5\}$ of $\mathfrak{g}$ such that with respect to the ordered basis $\{e_2, e_4, e_6\}$ for V, the adjoint representation of $\mathfrak{g}$ on V is given as below

$$\rho(e_1) = -\begin{pmatrix} 0&0&0\\ a&0&0\\ c&b&0 \end{pmatrix}, \quad \rho(e_3) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ d&e&0 \end{pmatrix}, \quad \rho(e_5) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ f&0&0 \end{pmatrix}. \tag{5.22}$$

Up to equivalence, we have the following

$$\mathfrak{h}_3: \quad \rho(e_1) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&1&0 \end{pmatrix}, \quad \rho(e_3) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{pmatrix}, \quad \rho(e_5) = 0. \tag{5.23}$$

$$\mathfrak{h}_8: \quad \rho(e_1) = 0, \quad \rho(e_3) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{pmatrix}, \quad \rho(e_5) = 0. \tag{5.24}$$

$$\mathfrak{h}_6: \quad \rho(e_1) = -\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}, \quad \rho(e_3) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{pmatrix}, \quad \rho(e_5) = 0. \tag{5.25}$$

$$\mathfrak{h}_{17}: \quad \rho(e_1) = -\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{pmatrix}, \quad \rho(e_3) = 0, \quad \rho(e_5) = 0. \tag{5.26}$$

$$\mathfrak{h}_9: \quad \rho(e_1) = -\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{pmatrix}, \quad \rho(e_3) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{pmatrix}, \quad \rho(e_5) = 0. \tag{5.27}$$

5.2.2. *Assume that $\mathfrak{g}$ is non-abelian.* In this case, $\mathfrak{g}$ is a three-dimensional Heisenberg algebra. Thus there exists a basis $\{e_1, e_3, e_5\}$ of $\mathfrak{g}$ such that $[e_1 \bullet e_3] = -e_5$, and a basis $\{e_2, e_4, e_6\}$ of V such that the adjoint representation of $\mathfrak{g}$ on V is as follows.

$$\rho(e_1) = -\begin{pmatrix} 0&0&0\\ a&0&0\\ c&b&0 \end{pmatrix}, \rho(e_3) = -\begin{pmatrix} 0&0&0\\ d&0&0\\ f&e&0 \end{pmatrix}, \rho(e_5) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ bd-ae&0&0 \end{pmatrix}. \tag{5.28}$$

If $d \neq 0$, by choosing $\{ae_3 - de_1, e_3\}$, we have a new set of $\{e_1, e_3, e_5\}$ such that $d = 0$. If $a \neq 0$, we may consider the new basis $\{e_1, e_3 - \frac{d}{a}e_1, e_5\}$ for $\mathfrak{g}$ and $\{e_2, ae_4 + ce_6, e_6\}$ for V and assume $a = 1$, $d = 0$ and $c = 0$. Then

$$\mathfrak{h} = (0, 0, 0, 12, 13, b14 - f23 + e34 + e25).$$

If $e = 0$, it is further reduced to

$$\mathfrak{h} = (0, 0, 0, a12, 13, b14 - f23).$$

The following becomes easy to verify.

$$\mathfrak{h}_6: \quad \rho(e_1) = -\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}, \quad \rho(e_3) = 0, \quad \rho(e_5) = 0. \tag{5.29}$$

$$\mathfrak{h}_7: \quad \rho(e_1) = -\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}, \quad \rho(e_3) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{pmatrix}, \quad \rho(e_5) = 0. \tag{5.30}$$

$$\mathfrak{h}_{10}: \quad \rho(e_1) = -\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{pmatrix}, \quad \rho(e_3) = 0, \quad \rho(e_5) = 0. \tag{5.31}$$

$$\mathfrak{h}_{11}: \quad \rho(e_1) = -\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&1&0 \end{pmatrix} \quad \rho(e_3) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{pmatrix} \quad \rho(e_5) = 0. \tag{5.32}$$

If $e \neq 0$, consider the new basis

$$\{\ell_1, \ldots, \ell_6\} = \{e^2 e_1 - bee_3 - bfe_5, e_2, ee_3 + fe_5, e^2 e_4, e^3 e_5, e^4 e_6\}.$$

Then the structure equations become $(0, 0, 0, 12, 13, 34 + 25)$. Taking the new dual basis $\{e_2 - e_3, e_1, e_2 + e_3, -e_4 + e_5, e_4 + e_5, 2e_6\}$, we find this algebra isomorphic to $\mathfrak{h}_{19+}$. Note that the algebras $\mathfrak{h}_{19+}$ and $\mathfrak{h}_{19-}$ are isomorphic over $\mathbb{C}$ because one has the map

$$(e_1, e_2, e_3, e_4, e_5, e_6) \mapsto (e_1, e_2, ie_3, e_4, ie_5, e_6).$$

For future reference, we note that the representation of $\mathfrak{g}$ on V is given as below.

$$\mathfrak{h}_{19}: \quad \rho(e_1) = -\begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&b&0 \end{pmatrix} \quad \rho(e_3) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ f&e&0 \end{pmatrix} \quad \rho(e_5) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ -e&0&0 \end{pmatrix}. \tag{5.33}$$

The last cases are due to $a = d = 0$. If it is not already equivalent to a previous case, they are equivalent to one of the following.

$$\mathfrak{h}_4: \quad \rho(e_1) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&1&0 \end{pmatrix}, \quad \rho(e_3) = -\begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{pmatrix}, \quad \rho(e_5) = 0. \tag{5.34}$$

$$\mathfrak{h}_8: \quad \rho(e_1) = 0, \quad \rho(e_3) = 0, \quad \rho(e_5) = 0. \tag{5.35}$$

5.2.3. *The dual semi-direct products.* Next, we go on to identify the Lie algebra structure for $\mathfrak{g} \ltimes V^*$ for each representation above. Recall that $\mathrm{ad} = \rho$ has matrix presentations as given in (5.22) and (5.28) depending on whether the algebra $\mathfrak{g}$ is abelian or not. The ordered base for V is given by $\{e_2, e_4, e_6\}$. To express the dual representation ρ^*, we shall do it in terms of the ordered base $\{e^6, e^4, e^2\}$. It amounts to taking the negative of the "transpose with respect to the *opposite* diagonal". Explicitly, the representation corresponding to (5.22) and (5.28) are respectively,

(5.36)
$$\rho^*(e_1) = -\begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & a & 0 \end{pmatrix}, \quad \rho^*(e_3) = -\begin{pmatrix} 0 & 0 & 0 \\ e & 0 & 0 \\ d & 0 & 0 \end{pmatrix}, \quad \rho^*(e_5) = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ f & 0 & 0 \end{pmatrix}.$$

(5.37)
$$\rho^*(e_1) = -\begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & a & 0 \end{pmatrix}, \quad \rho^*(e_3) = -\begin{pmatrix} 0 & 0 & 0 \\ e & 0 & 0 \\ f & d & 0 \end{pmatrix}, \quad \rho^*(e_5) = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ bd-ae & 0 & 0 \end{pmatrix}.$$

It is now a straight-forward exercise to find the next proposition. For instance, to find $\mathfrak{h}^\vee$ when $\mathfrak{h}$ is $\mathfrak{h}_3$ as the semi-direct product of an abelian algebra with an abelian ideal, we consider the representation (5.23). By (5.36), the corresponding dual representation is

$$\rho^*(e_1) = -\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho^*(e_3) = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho^*(e_5) = 0.$$

By (5.25) we find that $\mathfrak{h}_3^\vee$ is isomorphic to $\mathfrak{h}_6$ as a semi-direct product of an abelian algebra with an abelian ideal. However, if we consider $\mathfrak{h}_6$ as the semi-direct product of a Heisenberg algebra and an abelian ideal, we consider the dual representation of (5.29). Using (5.37) and re-ordering basis on V^*, we find that the algebra $\mathfrak{h}_6^\vee$ is isomorphic to $\mathfrak{h}_6$. This example reminds us that semi-direct product structure on a given Lie algebra is not unique, and hence its dual semi-direct product structure would change accordingly. Otherwise, elementary consideration such as above provides all necessary information to complete the next proposition.

Proposition 5.1. *Suppose that $\mathfrak{h}$ is a six-dimensional nilpotent Lie algebra given as a semi-direct product $\mathfrak{g} \ltimes_\rho V$. Then it is one of the* *algebras given in the left-most column of* Table (5.38). *Its dual semi-direct product* $\mathfrak{h}^\vee := \mathfrak{g} \ltimes_{\rho^*} V^*$ *is given in the same table as checked.*

(5.38)

$\mathfrak{h} \backslash \mathfrak{h}^\vee$	$\mathfrak{h}_3$	$\mathfrak{h}_6$	$\mathfrak{h}_8$	$\mathfrak{h}_9$	$\mathfrak{h}_{17}$	$\mathfrak{h}_4$	$\mathfrak{h}_6$	$\mathfrak{h}_7$	$\mathfrak{h}_8$	$\mathfrak{h}_{10}$	$\mathfrak{h}_{11}$	$\mathfrak{h}_{19}$
$\mathfrak{h}_3$		✓										
$\mathfrak{h}_6$	✓											
$\mathfrak{h}_8$			✓									
$\mathfrak{h}_9$				✓								
$\mathfrak{h}_{17}$					✓							
$\mathfrak{h}_4$								✓				
$\mathfrak{h}_6$							✓					
$\mathfrak{h}_7$						✓						
$\mathfrak{h}_8$									✓			
$\mathfrak{h}_{10}$										✓		
$\mathfrak{h}_{11}$											✓	
$\mathfrak{h}_{19}$												✓

In Table (5.38), the upper left corner is due to the correspondence between semi-direct products of a three-dimensional abelian algebra with an abelian ideal. The lower right corner is due to semi-direct products of a three-dimensional Heisenberg algebra with an abelian ideal.

5.3. **Geometric Aspects.** Recall that if $\mathfrak{h} = \mathfrak{g} \ltimes V$ admits a totally real integrable complex structure, then $\mathfrak{h}^\vee$ is Lagrangian with respect to a symplectic structure ω_J. As $\mathfrak{h}_6^\vee = \mathfrak{h}_3$ when the base is abelian and $\mathfrak{h}_3$ does not admit invariant symplectic structure, $\mathfrak{h}_6$ as the semi-direct product of an abelian ideal with an abelian subalgebra would not admit totally real integrable complex structure. For the same reason, $\mathfrak{h}_{19}$ does not admit compatible complex structure. On the other hand, $\mathfrak{h}_{17}$ simply would not admit any complex structure [33].

When $\mathfrak{h}_8$ is the semi-direct product of the Heisenberg algebra with an abelian ideal as given in (5.35), the integrability of a compatible integrable complex structure as given in (3.8) implies that the algebra

$\mathfrak{g}$ is abelian. This contradiction implies that when $\mathfrak{h}_8$ admits a semi-direct product structure with a compatible complex structure, then it is the semi-direct product of an abelian subalgebra and an abelian ideal.

Therefore, the potential identification from $(\mathfrak{h}, J)$ to $(\mathfrak{h}^\vee, \omega)$ is reduced to the next table.

(5.39)

$(\mathfrak{h}, J) \setminus (\mathfrak{h}^\vee, \omega)$	$\mathfrak{h}_6$	$\mathfrak{h}_8$	$\mathfrak{h}_9$	$\mathfrak{h}_4$	$\mathfrak{h}_6$	$\mathfrak{h}_7$	$\mathfrak{h}_{10}$	$\mathfrak{h}_{11}$
$\mathfrak{h}_3$	✓							
$\mathfrak{h}_8$		✓						
$\mathfrak{h}_9$			✓					
$\mathfrak{h}_4$						✓		
$\mathfrak{h}_6$					✓			
$\mathfrak{h}_7$				✓				
$\mathfrak{h}_{10}$							✓	
$\mathfrak{h}_{11}$								✓

5.3.1. *Totally real semi-direct products.* Among all the algebras identified in Table (5.39) above, the representation of $\mathfrak{g}$ on V has the form

$$\rho(e_1) = -\begin{pmatrix} 0 & 0 & 0 \\ A & 0 & 0 \\ 0 & B & 0 \end{pmatrix}, \quad \rho(e_3) = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix}, \quad \rho(e_5) = 0. \tag{5.40}$$

where A, B, C are respectively zero or one. In addition, $[e_1, e_3] = -De_5$ where D is equal to zero or one, depending on whether $\mathfrak{g}$ is abelian or not. In particular, the potentially non-trivial structure equations are

$$[e_1, e_2] = -Ae_4, \quad [e_1, e_4] = -Be_6, \quad [e_3, e_2] = -Ce_6, \quad [e_1, e_3] = -De_5.$$

If J is a complex structure such that the semi-direct product is totally real, then there is a 3×3-matrix (a_{ij}) such that $Je_{2i-1} = \sum_j a_{ij} e_{2j}$. In addition, it is integrable if and only if $N_J(e_1, e_3) = N_J(e_1, e_5) = N_j(e_3, e_5) = 0$. Given these constraints, one could apply elementary method to identify the set of complex structures for each set of parameters (A, B, C, D) corresponding to an algebra in the left-most column of Table (5.39). We leave it as exercise. Instead we focus on special phenomena.

The first special case is when the algebra is $\mathfrak{h}_3$ as it does not admit invariant symplectic structure. However, due to a classification of complex structure [33, Proposition 3.4], up to equivalence, there is a unique complex structure on $\mathfrak{h}_3$. With respect to our notations here, it is given by $A = 0, B = 1, C = 1, D = 0$. Making use of the dual representation ordered as in (5.36), and with respect to the bases $\{e_1, e_3, e_5\}$ on $\mathfrak{g}$ and $\{e^6, e^4, e^2\}$ on V^*, the structure equation on $\mathfrak{h}_3^\vee = \mathfrak{h}_6$ is

$$[e_1, e^6] = -e^4, \quad [e_3, e^6] = -e^2.$$

Then the 2-form on $\omega_J = e^1 \wedge e_2 + e^3 \wedge e_4 + e^5 \wedge e_6$ is a symplectic form on $\mathfrak{h}_6$.

The second special case is concerned with $\mathfrak{h}_6$. This algebra as a semidirect product is given by $A = D = 1$ and $B = C = 0$. In dual form, the structure equations in the present coordinates are

$$de^4 = e^{12} \quad \text{and} \quad de^5 = e^{13}.$$

It follows that the constraints for J to be integrable are

$$a_{31} = 0, \quad a_{32} = a_{21}, \quad a_{33} = 0. \tag{5.41}$$

Therefore,

$$Je_5 = a_{32}e_4, \quad \text{or} \quad a_{32}Je_4 = -e_5. \tag{5.42}$$

In particular, $a_{21} = a_{32} \neq 0$.

Let ω be a symplectic structure on $\mathfrak{h}_6$ such that $\mathfrak{g}$ and V are both Lagrangian. If $b_{ij} := \omega(e_i, e_j)$, then by using that ω is closed we obtain that $b_{54} = b_{56} = 0$ (recall that $e_5 \in [\mathfrak{h}_6, \mathfrak{h}_6]$ and e_4, e_6 belong to the center) and $b_{52} = b_{43}$. Let us now assume that ω and J are compatible. Then $\omega(Je_4, Je_3) = \omega(e_4, e_3)$. It is equivalent to

$$\omega(a_{32}Je_3, Je_3) = a_{32}\omega(e_4, e_3) = a_{32}b_{43}.$$

By (5.42) above,

$$\begin{aligned} & a_{32}b_{43} = -\omega(e_5, Je_3) \\ = \;& -\omega(e_5, a_{21}e_2 + a_{22}e_4 + a_{23}e_6) = -a_{21}b_{52} - a_{22}b_{54} - a_{23}b_{56} \\ = \;& -a_{21}b_{52} = -a_{21}b_{43} = -a_{32}b_{43}. \end{aligned}$$

Since $a_{32} \neq 0$, it is possible only when $b_{43} = 0$. As $b_{52} = b_{43} = 0$ and $b_{54} = b_{56} = 0$, ω would have been degenerate. It should that h_6 does not admit any special Lagrangian structure with respect to any semi-direct product decomposition.

5.3.2. *Family of special Lagrangian algebras.* In this paragraph, we establish the existence of special Lagrangian structures on the algebras $\mathfrak{h}_4, \mathfrak{h}_7, \mathfrak{h}_9, \mathfrak{h}_{10}$ and $\mathfrak{h}_{11}$. As it turns out, they could be considered as a family of special Lagrangian structures with "jumping" algebraic Lie structures, and hence jumping complex and symplectic structures.

We fix a basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ of a real vector space $\mathfrak{h}$ and consider also fixed structures J and ω defined by

$$Je_{2j-1} = e_{2j}, \qquad Je_{2j} = -e_{2j-1}, \qquad \omega = e^{16} - e^{25} + e^{34}. \tag{5.43}$$

The 2-form ω is type (1,1) with respect to J, and the non-degenerate symmetric bilinear form $g(-,-) := \omega(-, J-)$ has signature $(4, 2)$. If $\mathfrak{g} = \langle e_1, e_3, e_5\rangle$ and $V = \langle e_2, e_4, e_6\rangle$ then $\mathfrak{g}$ and V are totally real with respect to J and maximally isotropic with respect to ω.

Let (a, b, c, d) be real numbers. For each member of the family of Lie brackets

$$(0, 0, 0, a12, b13, c14 + d23),$$

the corresponding Lie algebra $\mathfrak{h}$ is the semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes V$, and the ideal V is abelian. The constraint on $\mathfrak{h}_{a,b}$ so that ω is closed is equivalent to $a + b + d = 0$. To find the constraints on $\mathfrak{h}_{a,b}$ so that J is integral, we choose a basis for the $(1, 0)$-forms with $\omega^j = e^{2j-1} + ie^{2j}$, $1 \leq j \leq 3$. As $d\omega^1 = 0$ and $d\omega^2$ is type $(1, 1)$, the sole constraint is due to $\omega^1 \wedge \omega^2 \wedge d\omega^3 = 0$. Given the structure equations, the integrability of J is equivalent to $b - c - d = 0$. It then follows that for any $a, b \in \mathbb{R}$,

$$\Big(\mathfrak{h}_{a,b} = (0, 0, 0, a12, b13, (a + 2b)14 - (a + b)23), J, \omega\Big), \tag{5.44}$$

is a family of special Lagrangian pseudo-Kähler structures on nilpotent Lie algebras. Since a non-zero scalar multiple of a Lie bracket gives rise to just a homothetic change in the metric, we will restrict ourselves to the curve $\{\mathfrak{h}_{a,b} : a^2 + b^2 = 1,\ b \geq 0\}$. The following isomorphisms can be checked by using the new basis on the right:

$$\mathfrak{h}_{\pm 1,0} \simeq \mathfrak{h}_9, \quad \{e_2, e_1, e_5, e_3, -e_4, -e_6\};$$

$$\mathfrak{h}_{0,1} \simeq \mathfrak{h}_4, \quad \{e_1, e_3, e_2, \tfrac{1}{2}e_4, e_5, e_6\};$$

$$\mathfrak{h}_{-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}} \simeq \mathfrak{h}_{10}, \quad \{e_1, e_2, e_3, -\tfrac{1}{\sqrt{2}}e_4, \tfrac{1}{\sqrt{2}}e_5, -\tfrac{1}{\sqrt{2}}e_6\};$$

$$\mathfrak{h}_{-\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}} \simeq \mathfrak{h}_7, \quad \{e_1, e_2, e_3, -\tfrac{2}{\sqrt{5}}e_4, \tfrac{1}{\sqrt{5}}e_5, \tfrac{1}{\sqrt{5}}e_6\};$$

$\mathfrak{h}_{a,b} \simeq \mathfrak{h}_{11}$ if $a, b, a + 2b, a + b \neq 0$,

$$\{re_1, \tfrac{1}{r}e_2, \tfrac{1}{r}e_3, ae_4, be_5, ra(a + 2b)e_6\},$$

where $r = -\left(\frac{a+b}{a(a+2b)}\right)^{1/3}$.

Isomorphism classes of such structures translate themselves in this context as the orbits of the natural action of the group

$$U(2, 1) = \{\varphi \in GL_6(\mathbb{R}) : \varphi J \varphi^{-1} = J,\ \omega(\varphi\cdot, \varphi\cdot) = \omega\},$$

on the space of Lie brackets [28]. Let $\mathfrak{h}_{a,b}$, $\mathfrak{h}_{a',b'}$ be two points in the curve isomorphic as Lie algebras to $\mathfrak{h}_{11}$, and assume there exists $\varphi \in U(2, 1)$ such that

$$\varphi.[\cdot, \cdot]_{a,b} := \varphi[\varphi^{-1}\cdot, \varphi^{-1}\cdot]_{a,b} = [\cdot, \cdot]_{a',b'}.$$

By using that φ must leave invariant the subspaces (which coincide for both Lie algebras)

$$[\mathfrak{h}, \mathfrak{h}] = \langle e_4, e_5, e_6\rangle_{\mathbb{R}}, \qquad [\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]] = \langle e_6\rangle_{\mathbb{R}},$$

$$\mathfrak{z} = \langle e_5, e_6\rangle_{\mathbb{R}} \text{ (center)}, \quad \{x \in \mathfrak{h} : [x, \mathfrak{h}] \subset \mathfrak{z}\} = \langle e_3, e_4\rangle_{\mathbb{R}}$$

as well as their orthogonal complements relative to both ω and the metric $g = \omega(\cdot, J\cdot)$, one can easily show that the matrix of φ with respect to the basis $\{e_1, e_2, ..., e_6\}$ is necessarily diagonal, and so it has the form

$$\varphi = \begin{bmatrix} s & & & & & \\ & s & & & & \\ & & \pm 1 & & & \\ & & & \pm 1 & & \\ & & & & 1/s & \\ & & & & & 1/s \end{bmatrix}, \qquad s \neq 0.$$

It follows that $a' = \pm\frac{1}{s^2}a$, $b' = \pm\frac{1}{s^2}b$ and hence $(a', b') = \pm(a, b)$.

We summarize the results obtained above in the following proposition.

Proposition 5.2. *With complex structure J and symplectic structure ω given in (5.43), with $t \in [0, \pi]$ the function*

$$\Big((0,0,0,(\cos t)12,(\sin t)13,(\cos t + 2\sin t)14 - (\cos t + \sin t)23), J, \omega\Big)$$

determines a closed curve in the moduli space of isomorphism classes of special Lagrangian pseudo-Kähler structures on the space of six-dimensional nilpotent Lie algebras. It contains exactly one structure on each one of $\mathfrak{h}_4$ ($t = \pi/2$), $\mathfrak{h}_7$ ($t = \arctan(-\frac{1}{2})$), $\mathfrak{h}_9$ ($t = 0, \pi$) $\mathfrak{h}_{10}$ ($t = 3\pi/4$), and the remaining is a continuous family on $\mathfrak{h}_{11}$.

Theorem 5.1. *Suppose that a six-dimensional nilpotent algebra admits a special Lagrangian semi-direct product structure, then the algebra is isomorphic to one of the following: $\mathfrak{h}_1, \mathfrak{h}_4, \mathfrak{h}_7, \mathfrak{h}_8, \mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{11}$. Their weak mirror images are respectively given below.*

$(\mathfrak{h}, J, \omega)$	$\mathfrak{h}_1$	$\mathfrak{h}_4$	$\mathfrak{h}_7$	$\mathfrak{h}_8$	$\mathfrak{h}_9$	$\mathfrak{h}_{10}$	$\mathfrak{h}_{11}$
$(\mathfrak{h}^\vee, \omega_J, J_\omega)$	$\mathfrak{h}_1$	$\mathfrak{h}_7$	$\mathfrak{h}_4$	$\mathfrak{h}_8$	$\mathfrak{h}_9$	$\mathfrak{h}_{10}$	$\mathfrak{h}_{11}$

In addition,

- *All special Lagrangian semi-direct product structure (J, ω) on $\mathfrak{h}_1, \mathfrak{h}_8$, $\mathfrak{h}_9$ and $\mathfrak{h}_{10}$ are self-mirror. It means that there is an isomorphism $\mathfrak{h}_\ell \cong \mathfrak{h}_\ell^\vee$ and there are quasi-isomorphisms.*

$$\mathrm{DGA}(\mathfrak{h}_\ell^\vee, \omega_J) \approx \mathrm{DGA}(\mathfrak{h}_\ell, \omega) \approx \mathrm{DGA}(\mathfrak{h}_\ell^\vee, J_\omega) \approx \mathrm{DGA}(\mathfrak{h}_\ell, J).$$

- *The algebras $\mathfrak{h}_7$ and $\mathfrak{h}_4$ form a mirror pair of special Lagrangian semi-direct products.*
- *The mirror of a special Lagrangian semi-direct structure on $\mathfrak{h}_{11}$ is again a special Lagrangian semi-direct structure on $\mathfrak{h}_{11}$, but the two pseudo-Kähler structures are not equivalent.*

Proof: The statements on $\mathfrak{h}_1$ are trivial.

As noted in Section 5.3.1, the only algebras admitting special Lagrangian structures are given in Table (5.39). $\mathfrak{h}_3$ is eliminated because it does not admit invariant symplectic structure.

The construction in this section and the paragraph concerned with a construction on $\mathfrak{h}_6$ at the end of Section 5.3.1 demonstrate the "existence" of special Lagrangian structures on the named algebras.

The isomorphisms between DGA$(\mathfrak{h}, J)$ and DGA$(\mathfrak{h}^\vee, \omega_J)$ are given by Theorem 4.1 and the identification of algebraic mirrors $\mathfrak{h}^\vee$ as given in Table (5.39). The validity of the claim on $\mathfrak{h}_{11}$ is due to an analysis in [15, Section 5.5]. ■

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