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Potential functions of HKT spaces

Yat Sun Poon[1] **and Andrew Swann**[2]

[1] Department of Mathematics, University of California at Riverside, Riverside, CA 92521, USA
[2] Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark

E-mail: ypoon@math.ucr.edu and swann@imada.sdu.dk

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Abstract
We describe the potential functions for $\mathcal{N} = 4B$ supersymmetric quantum mechanics with $D(2, 1; \alpha)$ symmetry.

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A brief review of HKT theory

Various geometries with torsion arise as the target spaces of two-dimensional sigma models. In the case of $\mathcal{N} = 4B$ rigid supersymmetry, one gets HKT structures: the geometry of a hyperKähler connection with totally skew symmetric torsion [2, 5]. A mathematical background of HKT geometry was reported in [3].

The existence of symmetry under the superconformal group $D(2, 1; \alpha)$ as an extension to $SU(1, 1|1)$ theory is studied in [6]. The superalgebra of $D(2, 1; \alpha)$ is decribed in [6, (3.44)]. Thanks to Michelson and Strominger's work on the local level, we only need to deal with the geometry of a vector field on a HKT-space as given in (8) below. We limit the scope of this paper to such a perspective.

To produce examples of HKT geometry with such symmetry, Michelson and Strominger make use of potential functions. It leads them to ask which HKT structures with symmetry are generated by potential functions. We answer this question in the next few pages. In the context of black-hole moduli, this problem is already tackled by Gutowski and Papadopoulos [4].

An HKT structure consists of the following ingredients: a Riemannian metric g on a smooth manifold M, three integrable complex structures I_r, $r = 1, 2, 3$ on M satisfying the quaternion identities, and a connection ∇ preserving the metric g and all three complex structures. Each complex structure is Hermitian with respect to the metric g. Moreover, the torsion tensor of the connection ∇

$$c(X, Y, Z) := g(X, T^{\nabla}(Y, Z)) \tag{1}$$

is totally skew. Here the torsion is defined as follows.

$$T^{\nabla}(Y, Z) = \nabla_Y Z - \nabla_Z Y - [Y, Z]. \tag{2}$$

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Since this connection ∇ is uniquely determined by the metric g and the complex structures I_r, HKT geometry refers to either the geometry of the connection ∇ or the geometry of the hyper-Hermitian space (g, I_r) [3]. The integrability of I_r implies that

$$T^{\nabla}(I_rX, I_rY) - I_rT^{\nabla}(I_rX, Y) - I_rT^{\nabla}(X, I_rY) - T^{\nabla}(X, Y) = 0 \quad (3)$$

or equivalently the 3-form c is of type (2, 1) + (1, 2) with respect to all I_r:

$$c(X, Y, Z) = c(X, I_rY, I_rZ) + c(I_rX, Y, I_rZ) + c(I_rX, I_rY, Z). \quad (4)$$

The theory of potential functions comes through a description of the Kähler forms [3]. The Kähler forms are defined by $F_r(X, Y) = g(I_rX, Y)$. Define d_r on n-forms by

$$d_rE = (-1)^n I_r dI_r E \quad (5)$$

where $(I_rE)(X_1, \ldots, X_n) := E(-I_rX_1, \ldots, -I_rX_n)$. The above identities on the torsion tensor implies that

$$c = -d_rF_r. \quad (6)$$

This is consistent with the finding in [1, 2.5.2] and corrects a factor in [3, (4)]. By definition [3], a potential for an HKT structure is a function μ such that

$$F_1 = \frac{1}{2}(dd_1 + d_2d_3)\mu, \qquad F_2 = \frac{1}{2}(dd_2 + d_3d_1)\mu, \qquad F_3 = \frac{1}{2}(dd_3 + d_1d_2)\mu. \quad (7)$$

It has been proved [6] that locally a $D(2, 1; \alpha)$ symmetry arises if and only if there is a vector field D obeying the following:

$$\mathcal{L}_D g = 2g, \qquad \mathcal{L}_{(I_rD)} g = 0, \qquad \mathcal{L}_{(I_rD)} I_s = \frac{2}{\alpha + 1}\epsilon^{rst} I_t, \qquad dD^\flat = 0 \quad (8)$$

where $D^\flat$ is the dual one-form of the vector field defined by the metric g. Please note that the sign convention here is different from the one found in [6, (3.48) and (3.50)]. Our convention is determined by the choice that on the flat 4-space when $D = X^a\partial_a$, $[I_1D, I_2D] = -2I_3D$. In particular, the flat space has $D(2, 1; -2)$ symmetry.

More examples of such symmetry on a HKT structure over the hypercomplex space R^{4N} can be generated by a potential function which turns out to be proportional to the norm-square of the vector field D [6, section 3.4]. Conversely [6, Appendix C] shows that any HKT structure compatible with flat Obata connection has a potential. Elementary examples show that potential functions do not always generate a $D(2, 1; \alpha)$ symmetry.

Potential functions and special homotheties

We relax the condition for $D(2, 1; \alpha)$ symmetry as follows: on an HKT space, a vector field is called a special homothety if there are distinct non-zero constants a and b such that

$$\mathcal{L}_D g = ag \quad (9)$$

$$\mathcal{L}_{(I_rD)} g = 0 \quad (10)$$

$$\mathcal{L}_{(I_rD)} I_s = b\epsilon^{rst} I_t. \quad (11)$$

We shall see that the dual 1-form $D^\flat$ is necessarily closed so that the conditions found in (8) are all fulfilled by a special homothety with appropriate choice of constants. More importantly, the function

$$\mu := \frac{2}{a(a - b)} g(D, D) \quad (12)$$

is an HKT potential. In particular, any HKT space with $D(2, 1; \alpha)$ symmetry is locally generated by a potential function. Although this result does not require b to be non-zero, in an upcoming report, we shall investigate additional geometric structures associated to $D(2, 1; \alpha)$ symmetry when $b \neq 0$. Besides, this requirement is also consistent with definitions in the superalgebra [6, footnote 8].

To verify our claims above, note that condition (9) implies the following:

$$\begin{aligned} ag(Y, Z) &= g(\nabla_Y D, Z) + g(Y, \nabla_Z D) + c(Z, D, Y) + c(Y, D, Z) \\ &= g(\nabla_Y D, Z) + g(Y, \nabla_Z D). \end{aligned}$$

As a consequence the symmetric part of ∇D is $\frac{1}{2}aI_0$ where I_0 is the identity. Let β be the skew-symmetric part of ∇D. We have

$$\nabla D = \tfrac{1}{2}aI_0 + \beta. \tag{13}$$

Since the connection preserves the complex structures, the above identity is equivalent to

$$g(\nabla_Y(I_r D), Z) = \tfrac{1}{2}aF_r(Y, Z) - \beta(Y, I_r Z). \tag{14}$$

On the other hand, condition (10) implies that

$$g(\nabla_Y(I^r D), Z) + g(\nabla_Z(I^r D), Y) = (\mathcal{L}_{(I_r D)}g)(Y, Z) = 0. \tag{15}$$

Therefore, $\beta(Y, I_r Z) + \beta(Z, I_r Y) = 0$. Then

$$\beta(Y, I_1 Z) = \beta(I_1 Y, Z) = \beta(I_2 I_3 Y, Z) = \beta(I_3 Y, I_2 Z) = \beta(Y, I_3 I_2 Z) = -\beta(Y, I_1 Z). \tag{16}$$

Therefore, $\beta = 0$. This is equivalent to

$$\nabla D = \tfrac{1}{2}aI_0 \quad \text{and} \quad \nabla(I_r D) = \tfrac{1}{2}aI_r. \tag{17}$$

In particular, since ∇ is a metric connection,

$$d(g(D, D)) = 2g(\nabla D, D) = ag(\cdot, D) = aD^\flat. \tag{18}$$

Therefore, $D^\flat$ is d-closed as claimed.

More generally, if λ is a constant and $\mu = \lambda g(D, D)$, then $d\mu = a\lambda D^\flat$, and hence $I_r d\mu = a\lambda(I_r D)^\flat$. This gives

$$\begin{aligned} dI_r d\mu(Y, Z) &= a\lambda(d(I_r D)^\flat)(Y, Z) \\ &= a\lambda(g(\nabla_Y(I_r D), Z) - g(\nabla_Z(I_r D), Y) + g(I_r D, T^\nabla(Y, Z))) \\ &= a\lambda(aF_r(Y, Z) + c(I_r D, Y, Z)). \end{aligned}$$

In other words,

$$dd_r\mu = dI_r d\mu = a^2\lambda F_r + a\lambda\iota_{(I_r D)}c. \tag{19}$$

Further, let I_r, I_s, I_t be a positive permutation of I_1, I_2, I_3.

$$\begin{aligned} d_s d_t\mu(Y, Z) &= -I_s dI_s I_t d\mu(Y, Z) = -I_s dI_r d\mu(Y, Z) = -(dI_r d\mu)(I_s Y, I_s Z) \\ &= -a^2\lambda F_r(I_s Y, I_s Z) - a\lambda c(I_r D, I_s Y, I_s Z) \\ &= -a^2\lambda g(I_r I_s Y, I_s Z) - a\lambda c(I_r D, I_s Y, I_s Z) \\ &= a^2\lambda g(I_s I_r Y, I_s Z) - a\lambda c(I_r D, I_s Y, I_s Z) \\ &= a^2\lambda F_r(Y, Z) - a\lambda c(I_r D, I_s Y, I_s Z) \end{aligned}$$

Thus

$$\tfrac{1}{2}(dd_r + d_s d_t)\mu = a^2\lambda F_r + \tfrac{1}{2}a\lambda\left(\iota_{(I_r D)}c - I_s\iota_{(I_r D)}c\right). \tag{20}$$

By (10) and (11), we have

$$\begin{aligned} bF_r &= \mathcal{L}_{(I_s D)}F_t = d\iota_{(I_s D)}F_t + \iota_{(I_s D)}dF_t = -d(I_r D)^\flat - I_t\iota_{(I_r D)}d_t F_t \\ &= -\frac{1}{a\lambda}dI_r d\mu + I_t\iota_{(I_r D)}c = -aF_r - \iota_{(I_r D)}c + I_s\iota_{(I_r D)}c \end{aligned}$$

using $I_r \iota_{(I_r D)} c = \iota_{(I_r D)} c$ from equation (19). Thus

$$\tfrac{1}{2}(dd_r + d_s d_t)\mu = \tfrac{1}{2}(a-b)a\lambda F_r.$$

Taking

$$\lambda = \frac{2}{a(a-b)} \tag{21}$$

gives the claimed result.

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