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Torus Symmetry of Compact Self-Dual Manifolds

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Abstract. We classify compact anti-self-dual Hermitian surfaces and compact four-dimensional conformally flat manifolds for which the group of orientation preserving conformal transformations contains a two-dimensional torus. As a corollary, we derive a topological classification of compact self-dual manifolds for which the group of conformal transformations contains a two-dimensional torus.

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1. Introduction

The symmetry group of a self-dual conformal class of Riemannian metrics $[g]$ on a smooth oriented four-dimensional manifold X is the group of orientation-preserving conformal transformations. We denote this group by $\mathcal{C}^+(X, [g])$ or simply $\mathcal{C}^+$.

There are compact self-dual manifolds with 'large' symmetry groups. For instance, the sphere, the complex projective plane, the Hopf manifold, and the 4-torus, endowed with their standard metrics, are homogeneous. The author has recently proved that if the symmetry group of a compact self-dual manifold X is at least three-dimensional, then X is either the complex projective plane or one of a handful of conformally flat manifolds ([36]). On the other hand, due to LeBrun's hyperbolic Ansatz, there is a large collection of compact self-dual manifolds with one-dimensional symmetry groups ([22]).

In this paper, we study the intermediate case of compact self-dual manifolds with two-dimensional symmetry groups. Our primary aim is to prove the following.

THEOREM A. *Suppose that* (X, g) *is a four-dimensional compact conformally flat manifold. If* $\mathcal{C}^+(X, [g])$ *contains* $\mathrm{U}(1) \times \mathrm{U}(1)$, *then* (X, g) *is either finitely covered by a flat torus or is conformally equivalent to either* (i) *the Euclidean sphere, or* (ii) *the Hopf manifold with a product metric.*

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Recall that a conformal class on a compact manifold is of positive type if and only if it contains a metric with positive constant scalar curvature ([37]). Negative-type conformal classes are similarly defined. All conformal classes described in Theorem A are of positive type; our theorem is in sharp contrast to the existence of large families of compact negative-type conformally flat manifolds with $C^+ = \mathrm{U}(1)$ ([14], [25]). To establish Theorem A, we also prove the next theorem.

THEOREM B. *Let g be a metric and J a complex structure on a compact manifold X so that (X, g, J) is an anti-self-dual Hermitian surface. If* $\dim C^+(X, [g]) \geq 2$, *then the Hermitian structure is, up to a finite covering, conformally equivalent to either* (i) *a complex torus with a flat metric;* (ii) *the product of the Riemann sphere and a Riemann surface of genus at least 2 with metrics of constant curvature or* (iii) *the Hopf manifold with a Vaisman metric.*

Remark. When a Hopf manifold $S^1 \times S^3$ is expressed as the quotient space $\mathrm{C}^2 \backslash \{0\}/\Gamma$, where Γ is generated by the action $(z_1, z_2) \to (\lambda z_1, \lambda z_2)$, $|\lambda| \neq 0, 1$, the Vaisman metric is

$$\frac{4(\mathrm{d}z_1 \mathrm{d}\bar{z}_1 + \mathrm{d}z_2 \mathrm{d}\bar{z}_2)}{z_1 \bar{z}_1 + z_2 \bar{z}_2} \quad ([38]).$$

Theorem B is partly motivated by numerous examples of anti-self-dual Hermitian surfaces with $C^+ = \mathrm{U}(1)$ ([20], [21]). Moreover, there are two different but similar methods to construct self-dual metrics on compact manifolds. One is an application of the Donaldson–Friedman programme [10] in an equivariant deformation setting to construct self-dual metrics with non-trivial group of conformal transformation ([30]). Another is an application of the same programme in a relative deformation setting to construct anti-self-dual Hermitian metrics ([15]). Theorem B demonstrates that these two methods cannot be combined to generate new anti-self-dual Hermitian surfaces with large symmetry groups.

We shall explain that when the symmetry of a compact self-dual manifold is a 2-torus, then the action is fixed-point-free if and only if the conformal class is conformally flat. Then Theorem A yields a topological classification.

THEOREM C. *Let (X, g) be a compact self-dual manifold such that the metric g is not conformally equivalent to a flat metric. Suppose that $C^+ (X, [g])$ contains* $\mathrm{U}(1) \times \mathrm{U}(1)$, *then the manifold X is finitely covered by a connected sum of S^4, $S^2 \times S^2$, $S^1 \times S^3$,* $\mathbf{CP}^2$ *or* $\overline{\mathbf{CP}}^2$ *(i.e.* $\mathbf{CP}^2$ *with an orientation opposite to the standard one).*

Theorem C is a simplified statement of a technical result, Theorem 8.1. These theorems are deduced from Theorem A and Orlik and Raymond's classification of manifolds with a torus action ([26]). Three exotic families of manifolds R, T and L appear in Orlik and Raymond's classification ([27]). Theorem C does not explicitly exclude these manifolds from having self-dual metrics with torus symmetry.

But Chern–Weil theory would imply that if there were self-dual metrics on these manifolds, the metrics are conformally flat, in contradiction to Theorem A.

After reviewing basic material on self-dual manifolds in Section 2, we discuss topological aspects of torus actions in Section 3. Then Sections 4, 5 and 6 are devoted to proving Theorem B and we prove Theorem A in Section 7. Applications of Theorem A are discussed in Section 8 and Theorem C appears as Theorem 8.2.

2. Self-Dual Manifolds and Twistor Spaces

Let W_+ be the self-dual Weyl tensor and W_- the anti-self-dual Weyl tensor of a metric g on an oriented compact four-dimensional manifold X. (X, g) is self-dual if $W_- \equiv 0$, conformally flat if $W_- \equiv 0$ and $W_+ \equiv 0$. The definitions of W_- and W_+ are orientation dependent. Since conformal change of a self-dual metric is self-dual, the symmetry group of a self-dual manifold is the group of orientation-preserving conformal transformations. Compact self-dual manifolds are subjected to a topological constraint. Due to Chern–Weil theory ([3]), the signature $\tau(X)$ of the oriented manifold X is equal to

$$\frac{1}{12\pi^2}\int_X (|W_+|^2 - |W_-|^2)\mathrm{vol}_g.$$

Therefore, $\tau(X)$ is non-negative when X has a self-dual metric g, and equal to zero if and only if the metric g is conformally flat.

The twistor space Z associated to (X, g) is the total space of the bundle of unit anti-self-dual 2-forms. The projection $\pi\colon Z \to X$ is called twistor fibration. When a 2-form Ω on X is identified to an endomorphism J of the tangent space by the identity $\Omega(\cdot,\cdot) = g(\cdot, J\cdot)$, any point z in the twistor space is considered as an almost complex structure on the tangent space $T_{\pi(z)}X$ such that the Riemannian inner product is Hermitian. From this point of view, a section J of the twistor fibration is an almost complex structure on X such that g is Hermitian with respect to J. Moreover, the natural orientation determined by J is opposite to the given one.

There is a tautologically defined almost complex structure on Z such that it is integrable if and only if the metric is self-dual ([1]). And one has the following ([3]).

PROPOSITION. *If $f\colon X \to X$ is an orientation-preserving conformal map, there is a natural lifting $\hat{f}$ of f mapping the twistor space to itself such that*

(1) *$\hat{f}$ is a holomorphic automorphism;*
(2) *$\pi \circ \hat{f} = f \circ \pi$;*
(3) *$\hat{f}$ is real in the sense that $\hat{f} \circ \sigma = \sigma \circ \hat{f}$, where σ is the real structure of the twistor space.*

In other words, the symmetry group of a self-dual manifold acts on the twistor space as a group of *real* holomorphic transformations. For example, when the

Euclidean sphere is considered as the quaternion projective space, the symmetry group is PGL(2, **H**). The twistor space of S^4 is the complex projective space $\mathbf{CP}^3$. Then PGL(2, **H**) acts on $\mathbf{CP}^3$ as a subgroup of PGL(4, **C**) given by

$$\left\{ A \in \mathrm{PGL}(4, \mathbf{C}) \colon A = \begin{pmatrix} \alpha & \beta & \epsilon & \xi \\ -\bar{\beta} & \bar{\alpha} & -\bar{\xi} & \bar{\epsilon} \\ \gamma & \delta & \eta & \theta \\ -\bar{\delta} & \bar{\gamma} & -\bar{\theta} & \bar{\eta} \end{pmatrix} \right\}.$$

3. Topology of Torus Actions

In the seventies, Orlik and Raymond studied the equivariant topology of four-dimensional manifolds with two-dimensional group of diffeomorphisms. Their success is contributed in part by the following theorem ([8]).

THEOREM 3.1. *Let X be a compact four-dimensional oriented manifold. Suppose that its group of orientation-preserving diffeomorphisms contains a two-dimensional torus. Let F be the set of fixed points of the torus action. Then the Euler characteristics of X and F are equal: $\chi(X) = \chi(F)$.*

When the fixed-point-set of a torus action is non-empty, a topological classification was achieved by Pao after the groundwork of Orlik and Raymond. Their results are as follows.

THEOREM 3.2 ([26], [27]). *Let X be a compact oriented four-dimensional manifold with effective* $\mathrm{U}(1) \times \mathrm{U}(1)$ *action. If the action has fixed points, then X is an equivariant connected sum of* S^4, $S^2 \times S^2$, $\mathbf{CP}^2$, $\overline{\mathbf{CP}}^2$, $2(S^1 \times S^3)\#2(S^2 \times S^2)$, *and three families of manifolds R, T, and L.*

THEOREM 3.3 ([28]). *The manifolds R, T, and L are determined as follows:*

(1) *R is homeomorphic to either $(S^2 \times S^2)\#(S^1 \times S^3)$ or $\mathbf{CP}^2\#\overline{\mathbf{CP}}^2\#(S^1 \times S^3)$;*
(2) *T is homeomorphic to either $\mathbf{CP}^2\#\overline{\mathbf{CP}}^2\#\mathbf{CP}^2\#\overline{\mathbf{CP}}^2\#(S^1 \times S^3)$ or $(S^2 \times S^2)\#(S^2 \times S^2)\#(S^1 \times S^3)$;*
(3) *the universal covering space of L is $(S^2 \times S^2)\# \cdots \#(S^2 \times S^2)$.*

Such classification becomes very complicated when the fixed-point-set is empty ([23], [29]). We shall deal with this problem when the torus is a symmetry group of self-dual metrics. Recall that there are two basic theorems concerned with zeros of Killing vector fields.

THEOREM 3.4 ([16]). *Let X be a compact Riemannian manifold and V a Killing vector field. Let $\bigcup_i N_i$ be the decomposition of the zero set of V into its connected components, then $\chi(X) = \Sigma_i \chi(N_i)$.*

THEOREM 3.5 ([4], [16]). *Let X be a compact oriented Riemannian manifold of dimension $2m$. Let V be a Killing vector field. Let $\bigcup_i N_i$ be the decomposition of the zero set of V into its connected components. Let Ω be the curvature form of the bundle of oriented orthonormal frames. Let f be a* SO$(2m)$*-invariant symmetric form of degree m on the Lie algebra $o(2m)$. Then the characteristic number of X defined by f is given by $f(\Omega, \dots, \Omega)[X] = \Sigma_i \mathrm{Res}_f(N_i)$.*

We do not spell out the definition of the residue $\mathrm{Res}_f(N_i)$ but observe that it is contributed by the evaluation of a differential form on the homology class represented by N_i. Important to us is the following.

THEOREM 3.6. *Let (X, g) be a compact self-dual manifold. If $C^+(X, [g])$ contains a two-dimensional torus such that its action is fixed-point-free, then $\chi(X) = 0$, $\tau(X) = 0$ and g is conformally flat.*

Proof. By the Obata theorem ([18]), except when the conformal structure is a Euclidean sphere, the torus is a group of isometries with respect to some metrics in the given conformal class. Theorem 3.4 implies that $\chi(X) = 0$. In Theorem 3.5, for an appropriate f, $f(\Omega, \Omega)$ is the first Pontryagin class. Therefore, one deduces $\tau(X) = 0$. Hence, the metric is conformally flat.

4. Anti-self-dual Hermitian Surfaces

A Hermitian structure on a smooth manifold X is a pair of integrable complex structures J and a Riemannian metric g such that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. When X is four-dimensional, (X, g, J) is an *anti-self-dual* Hermitian surface if the metric g is anti-self-dual with respect to the natural orientation of J. Throughout this paper, we choose an orientation on X opposite to the natural orientation of J so that (X, g) is self-dual. With this convention, a self-dual manifold (X, g) admits an anti-self-dual Hermitian structure with the opposite orientation if the complex structure J defines an integrable section of the twistor fibration. $J(X)$ is a holomorphic hypersurface of the twistor space of (X, g). Often, we consider $J(X)$ as an effective divisor. It is denoted by D. $\overline{D}$ denotes the divisor $-J(X)$.

Since D is a section of the twistor fibration, it intersects every fibre of the twistor projection transversally at one point. Therefore, the restriction of the associated bundle $\mathbf{D}\overline{\mathbf{D}}$ of the divisor $D + \overline{D}$ on any fibre of the twistor fibration is a degree 2 holomorphic line bundle. This holomorphic line bundle is also real with respect to the real structure defined by the fibrewise anti-podal map because $\overline{D} = \sigma(D)$ ([1]). Therefore, there is a holomorphic line bundle $\mathbf{F}$ with vanishing first Chern class such that ([33], [34]):

$$\mathbf{D}\overline{\mathbf{D}} \cong \mathbf{K}^{-1/2}\mathbf{F}, \tag{4.1}$$

where $\mathbf{K}^{-1}$ is the anti-canonical bundle of the twistor space.

It is known ([32]) that an anti-self-dual Hermitian structure is conformally equivalent to a Kähler metric if and only if the bundle $\mathbf{F}$ is holomorphically trivial.

It can be seen as follows. As D is a section of the twistor fibration, the fundle **F** is holomorphically trivial on the twistor space if and only if its restriction onto the divisor D is trivial. Since any compact anti-self-dual Hermitian surface is locally conformally Kähler ([6]), there is a locally finite covering of X by simply-connected neighbourhoods $\{U_\alpha: \alpha \in \Lambda\}$ and smooth function f_α on U_α such that $\mathrm{e}^{-f_\alpha}g$ is a Kähler metric on U_α. If ω is the fundamental 2-form of the Hermitian metric g, the locally defined 2-form $\omega_\alpha := \mathrm{e}^{-f_\alpha}\omega$ is closed. It implies that $\mathrm{d}\omega = \mathrm{d}f_\alpha \wedge \omega$. In particular, on any non-empty intersection $U_\alpha \cap U_\beta$, $\mathrm{d}f_\alpha = \mathrm{d}f_\beta$. Then the 1-form $\theta := \mathrm{d}f_\alpha$ is globally defined. The Hermitian metric is conformally equivalent to a Kähler metric if and only if the 1-form θ is exact.

Due to twistor correspondence ([12]), ω defines a holomorphic section of $\mathbf{D}\overline{\mathbf{D}}$ and ω_α defines a holomorphic section of $\mathbf{K}^{-1/2}$ over the open set $\pi^{-1}(U_\alpha)$. Therefore, e^{-f_α} is a locally defined holomorphic section of **F**. It follows that the transition function of the bundle **F** on $\pi^{-1}(U_\alpha \cap U_\beta)$ is the function $\phi_{\alpha\beta} := \mathrm{e}^{f_\beta - f_\alpha}$. Then on $U_\alpha \cap U_\beta$, $\mathrm{d}f_\alpha = \phi_{\alpha\beta}(\mathrm{d}f_\beta)\phi_{\alpha\beta}^{-1} + \mathrm{d}\phi_{\alpha\beta} \cdot \phi_{\alpha\beta}^{-1}$. Therefore, θ is a connection form of **F**. It follows that **F** is trivial if and only if θ is exact. Due to discussions of the last paragraph, (X, g, J) is conformally equivalent to a Kähler metric if and only if the bundle **F** is trivial.

LEMMA 4.2. *If the dimension of the complete linear system containing the divisor D is positive, any two elements in this system are mutually disjoint.*

Proof. Suppose on the contrary that there is a pair of elements in the linear system $|D|$ intersecting at one point z. Within the pencil generated by these two elements, there is an element $\hat{D}$ passing through the conjugate point $\sigma(z)$.

Let L be the real twistor line through z and $\sigma(z)$. Since $D \cdot L$ is equal to 1, $\hat{D}$ is irreducible and non-singular ([35]). As $\hat{D}$ intersects L at z and $\sigma(z)$, it contains L. Due to the adjunction formula ([2]), the self-intersection number of L on the surface $\hat{D}$ is equal to 1. Therefore, the surface $\hat{D}$ is rational ([2]). In particular, $\hat{D}$ is simply-connected. Then the flat bundle **F** in (4.1) is trivial. Therefore the Hermitian structure (X, g, J) is conformally equivalent to a Kähler metric. When the system $|D|$ is as big as a pencil, the algebraic dimension of the twistor space is positive. It follows that the metric g is conformal to a Ricci-flat metric ([34]). Therefore, X is finitely covered by a 4-torus or a K3-surface ([3]).

On the other hand, $\hat{D}$ contains the real twistor line L and intersects all the other real twistor lines transversally at one point. The restriction of the twistor fibration onto the surface $\hat{D}$ is a diffeomorphism from $\hat{D}$ onto $X\#\mathbf{CP}^2$ ([35]). Yet the surface $\hat{D}$ is rational, the diffeomorphism $\hat{D} \cong X\#\mathbf{CP}^2$ is absurd when X is finitely covered by a 4-torus or a K3-surface.

PROPOSITION 4.3. *If G is a 1-parameter group of orientation-preserving conformal transformations of a compact anti-self-dual Hermitian surface, then the group G acts holomorphically except possibly when the Hermitian structure is the Vaisman metric on a Hopf surface.*

Proof. Let D be an effective divisor in the twistor space Z representing the complex structure of an anti-self-dual Hermitian surface. If D is not G-invariant, the system $|D|$ is at least a pencil. Due to Lemma 4.2, $|D|$ is a pencil of mutually disjoint effective divisors. Therefore, the associated map of the bundle $\mathbf{D}$ is a holomorphic fibration from Z onto $\mathbf{CP}^1$. It follows that (X, g, J) is a hyper-Hermitian surface. Among all hyper-Hermitian surfaces ([7]), only the complex tori and Hopf surfaces admit non-trivial 1-parameter group of conformal transformations. Since conformal transformations on flat tori are holomorphic, only Hopf surfaces with the Vaisman metric admit non-holomorphic conformal Killing vector fields.

Remark 4.4. The group of holomorphic transformations on a Hopf surface with Vaisman metric is U(2). While the identity component of the group of conformal transformations is $\mathrm{U}(1) \times \mathrm{SO}(4)$.

5. Non-Kählerian Hermitian Surfaces

In this section, we derive a geometrical classification of non-Kählerian Hermitian surfaces with torus symmetry.

PROPOSITION 5.1. *Suppose that (X, g, J) is an anti-self-dual Hermitian surface. Suppose that g is not conformally Kähler. When $C^+(X, [g])$ contains $\mathrm{U}(1) \times \mathrm{U}(1)$, the Hermitian structure is conformally equivalent to the Vaisman metric on a Hopf surface.*

Proof. In view of Proposition 4.3, we assume that the $\mathrm{U}(1) \times \mathrm{U}(1)$ action is holomorphic. It implies that the holomorphic transformations on the twistor space generated by the torus leave the divisor D invariant. The restriction of the holomorphic transformations from the twistor space onto the hypersurface D are exactly the holomorphic transformations on the given anti-self-dual Hermitian surface.

Let M be the minimal model of D. The torus action on D descends to act on M effectively and holomorphically. D is a class VII surface when (X, g, J) is not conformally Kähler ([6]). As M is minimal, $b_2(M) = \chi(M)$ ([2]).

When $b_2(M) = 0$, Theorem 3.1 implies that the torus action on M is fixed-point-free. Since any exceptional divisor of blowing-up on D is contracted to a fixed point on M, $b_2(M) = 0$ only if the surface D is minimal and $b_2(D) = 0$. Then the torus action on D is free. Since the underlying smooth structure of D is the manifold X, by Theorem 3.6, the signature of X is equal to zero. Then the Hermitian structure is conformally flat. Due to Pontecorvo's theorem ([31]), the only conformally flat Hermitian surface in class VII is the Hopf surface with the Vaisman metric.

Suppose that $b_2(M) \geq 1$. Let V_1, V_2 be linearly independent holomorphic vector fields generated by the torus action on D. $V_1 \wedge V_2$ is a section of the anti-canonical bundle. This section is trivial only if V_1 and V_2 are linearly dependent over the field of meromorphic functions on D. More rigorously, we consider the image sheaf S

of the morphism $\mathcal{O}_D \oplus \mathcal{O}_D \to \mathcal{T}D$ defined by $(f_1, f_2) \to f_1V_1 + f_2V_2$. As $\mathcal{S}$ is a subsheaf of the tangent sheaf $\mathcal{T}D$, it is coherent and torsion-free ([11]). The rank of $\mathcal{S}$ is not equal to zero because $\mathcal{S}$ is the distribution sheaf defined by an effective torus action. When $V_1 \wedge V_2$ vanishes identically, the rank of $\mathcal{S}$ is equal to 1. Then the bi-dual sheaf $\mathcal{S}^{**}$ is locally-free because it is a rank-1 coherent reflexive sheaf ([11]). Via the natural monomorphism of a torsion-free sheaf into its bi-dual sheaf, we consider V_1 and V_2 as holomorphic sections of the line bundle $\mathcal{S}^{**}$. Their ratio is a meromorphic function. This meromorphic function is non-constant because V_1 and V_2 are linearly independent over the field of complex numbers.

A class VII surface has non-constant meromorphic functions only when it is bi-meromorphic to a Hopf surface ([2]). When M is minimal and $b_2(M) \geq 1$, it is not bi-meromorphic to a Hopf surface. Therefore, $V_1 \wedge V_2$ is a non-trivial section of the anti-canonical bundle on D. This section descends to a non-trivial anti-canonical section on M. As the anti-canonical divisor on M is effective, Nakamura shows ([24]) that the manifold M is diffeomorphic to $(S^1 \times S^3)\#n\overline{\mathbf{CP}}^2$, $n \geq 1$. Due to Theorems 3.2 and 3.3, $(S^1 \times S^3)\#n\overline{\mathbf{CP}}^2$ does not admit effective torus action. This contradiction finishes the proof of Proposition 5.1.

6. Anti-Self-Dual Kähler Surfaces

A Kähler surface is anti-self-dual with respect to its natural orientation if and only if the scalar curvature vanishes, i.e. when the metric is scalar-flat ([13]). In this section, we derive a geometrical classification of such surfaces when $\mathcal{C}^+$ contains a two-dimensional torus and then finish the proof of Theorem B.

PROPOSITION 6.1. *Suppose that* (X, g, J) *is an anti-self-dual Hermitian surface. Suppose that* g *is conformally equivalent to a Kähler metric. When* $\mathcal{C}^+(X, [g])$ *contains* $\mathrm{U}(1) \times \mathrm{U}(1)$, *the Kähler structure is finitely covered by a flat torus.*

Proof. After a conformal change, we assume that the metric g is Kähler with respect to the complex structure J. Due to Proposition 4.3, $\mathcal{C}^+$ is a group of holomorphic transformations. As g is Kähler, $\mathcal{C}^+$ is the group of holomorphic isometries.

A minimal scalar-flat Kähler surface is isometrically covered by a flat torus, a K3-surface with the Calabi–Yau metric, or a conformally flat Kähler metric on a ruled surface of genus at least 2([6]). The K3-surface is excluded from our investigation because it does not admit non-trivial holomorphic vector fields. If (X, g) is a conformally flat Kähler metric on a ruled surface, its universal covering space is $S^2 \times \mathcal{H}$ with metrics of constant curvature $+1$ on S^2 and -1 on the upper-half-plane $\mathcal{H}$. In this case, $\mathcal{C}^+$ does not contain $\mathrm{U}(1) \times \mathrm{U}(1)$. The flat torus remains the last possibility when the complex structure is minimal.

If X is not minimal, it is the blow-up of a ruled surface ([6]). Since the torus is a group of holomorphic transformations, it leaves exceptional divisors of blowing-up invariant. The torus action on X descends to act on a minimal model M and leaves all points of blowing-up fixed. As the torus action on the M has fixed points, Theorems 3.2 and 3.3 show that the minimal model is rational.

Let V_1 and V_2 be independent holomorphic vector fields generated by the torus action. Due to the vanishing of the scalar curvature, the canonical bundle does not admit non-trivial sections ([39]). Therefore, $V_1 \wedge V_2$ vanishes identically. As in the proof of Proposition 5.1, we consider the distribution sheaf $\mathcal{S}$ of the vector fields V_1 and V_2. As $V_1 \wedge V_2$ vanishes identically, the rank of $\mathcal{S}$ is equal to 1. V_1 and V_2 are holomorphic sections of $\mathcal{S}^{**}$. Let C_1 and C_2 be their divisor of zeros. This is a pair of linearly equivalent and geometrically distinct effective divisors.

As the minimal model is a rational surface, X is blown down to a Hirzebruch surface S_n with a certain degree n, where $n \geq 0$. Let f be a blowing-down map. As C_1 is linearly equivalent to C_2, $f(C_1)$ and $f(C_2)$ are linearly equivalent. The torus action on X descends to acts on S_n and leaves $f(C_1)$ and $f(C_2)$ invariant. We consider Hirzebruch surfaces as projectivization of rank-2 vector bundles over $\mathbf{CP}^1$. Then the invariant divisors of holomorphic torus actions on S_n are composed of one pair of fibres of the projection from S_n onto $\mathbf{CP}^1$, the infinity section and a zero section. Except when $n = 0$, the zero section and the infinity section are not linearly equivalent. Therefore, $f(C_1)$ and $f(C_2)$ are linearly equivalent to a fibre of the projection. When $n = 0$, $f(C_1)$ and $f(C_2)$ are linearly equivalent to a fibre if we choose a projection from S_0 onto $\mathbf{CP}^1$ carefully. In particular, $f(C_1)$ and $f(C_2)$ do not intersect. Hence, neither $f(C_1)$ nor $f(C_2)$ passes through any point of blowing-up. Let q be a point of blowing-up. As the torus action has two fixed points on $f(C_1)$ and two fixed points on $f(C_2)$, if X is the blow-up of S_n, then the torus action has at least five fixed points on S_n. As the Euler characteristic of a Hirzebruch surface is equal to 4, effective torus actions have exactly four fixed points. Therefore, no torus acts effectively and conformally on compact non-minimal scalar-flat Kähler surfaces. Combining this observation with those in the second paragraph, the proof of Proposition 6.1 is completed.

Remark 6.2. As an application of classification of self-dual manifolds with semi-free U(1)-symmetry, LeBrun classifies compact anti-self-dual Kähler surfaces with non-trivial holomorphic vector fields. Proposition 6.1 also follows from his works ([22]).

THEOREM B. *Let g be a metric and J a complex structure on a compact manifold X so that (X, g, J) is an anti-self-dual Hermitian surface. If* $\dim \mathcal{C}^+(X, [g]) \geq 2$, *then the Hermitian structure is, up to a finite covering, conformally equivalent to either* (i) *a complex torus with a flat metric;* (ii) *the product of the Riemann sphere and a Riemann surface of genus at least 2 with metrics of constant curvature; or* (iii) *the Hopf manifold with a Vaisman metric.*

Proof. If $\mathcal{C}^+(X, [g])$ contains a two-dimensional torus, we apply Propositions 5.1 and 6.1. If the maximal torus in $\mathcal{C}^+(X, [g])$ is one-dimensional, then $\mathcal{C}^+$ is non-abelian and hence at least three-dimensional. Then, we apply Theorems 1.2 and 1.3 of [36].

7. Conformally Flat Manifolds

There are conformally flat metrics on connected sums of the Hopf manifold $n(S^1 \times S^3)$ such that $\mathcal{C}^+$ is equal to SO(3). These SO(3)-symmetric conformal classes have non-trivial deformations ([30]). There are also families of U(1)-symmetric conformally flat metrics on $n(S^1 \times S^3)$ such that the scalar curvature of these metrics varies from positive to negative ([14], [25]). Such interesting examples are in sharp contrast to the theorem below.

THEOREM A. *Suppose that (X, g) is a four-dimensional compact conformally flat manifold. If $\mathcal{C}^+(X, [g])$ contains* $\mathrm{U}(1) \times \mathrm{U}(1)$, *then (X, g) is finitely covered by a flat torus or is conformally equivalent to either the Euclidean sphere, or the Hopf manifold with a product metric.*

Proof. Let T be a two-dimensional torus contained in $\mathcal{C}^+$. Assume that the conformal structure is not a round sphere. Due to the Obata theorem ([18]), the torus is a group of isometries with respect to some metrics in the given conformal class. Let g be such a metric.

(i) In terms of orbit structure of a torus action, there are three different cases. The first is when the torus action is non-singular in the sense that every orbit of the torus action is two-dimensional ([23]). The second is when the action is singular but fixed-point-free. The last case is when the torus action has fixed points.

(ii) Suppose that the torus action is non-singular. In terms of Conner and Raymond's Reduction Theorem ([9]), there are two possibilities according to whether the action is homological injective or not. When the torus action is homological injective, the Reduction Theorem asserts that there is a finite subgroup Δ in T such that the manifold X is equivariant to $T \times_\Delta \Sigma$, where Σ is a two-dimensional manifold. Let T_1 and T_2 be Abelian subgroups such that $T = T_1 \times T_2$. Denote $T_2 \times \Sigma$ by Y. Then we have a T-equivariant covering map $\psi: T_1 \times Y \to X$.

When the torus action is not homological injective, the Reduction Theorem asserts that there is a splitting $T = T_1 \times T_2$, where T_1 and T_2 are one-dimensional subgroups, and a finite subgroup $\Delta \subset T_1$ so that X is Δ-fold equivariantly covered by the product of T_1 and a compact three-dimensional Seifert manifold Y. Again, we have a T-equivariant covering map $\psi: T_1 \times Y \to X$.

In both cases, the metric on X is pulled back to the product manifold $T_1 \times Y$. The action of T_1 is free of any points with non-trivial isotropy. As the covering map ψ is T-equivariant, the group T is contained in the group of isometries of the pull-back metric $\tilde{g} := \psi^* g$. Let ω be the 1-form obtained by contracting the metric $\tilde{g}$ with the Killing vector field generated by the action of T_1. When $T_1 \times Y$ is considered as a principal T_1-bundle over Y, ω is a connection form. Using this connection, one defines a metric h on Y such that the projection ϕ from $T_1 \times Y$ onto Y is a Riemannian submersion. As the metric $\tilde{g}$ on each fibre of this projection has T_1 as group of isometries, the fibres are Euclidean circles. Therefore, the metric on $T_1 \times Y$ is $\tilde{g} = r(y)\,\mathrm{d}\theta^2 + \phi^* h$, where $r(y)$ is a function of Y and $\mathrm{d}\theta^2$ is the standard metric on the unit circle. As the metric g is comformally flat, so is

$[1/r(y)]\tilde{g}$. Since $[1/r(y)]\tilde{g}$ is a product metric, the Riemannian manifold (Y, h) is a three-dimensional space form ([18]). While (Y, h) has a circle of conformal symmetries, it is either covered by the flat Euclidean space or the Euclidean 3-sphere. Therefore, the manifold X is finitely covered by a flat torus or the Hopf manifold with a product metric.

(iii) From this paragraph to paragraph (vi), we assume that the torus action is fixed-point-free but it has one-dimensional orbits.

Let $\tilde{X}$ be the universal covering of X, π the covering map. Let $\delta: \tilde{X} \to S^4$ be a developing map of the conformally flat metric $\pi^* g$. Since δ is conformal, it is lifted to a holomorphic map $\tilde{\delta}$ from the twistor space $\tilde{Z}$ over $\tilde{X}$ into the twistor space over S^4. We have a commutative diagram.

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{\delta}} & \mathbf{CP}^3 \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\delta} & S^4. \end{array}$$

The torus action on (X, g) is lifted to a two-dimensional abelian group of symmetries $\tilde{T}$ on $(\tilde{X}, \pi^* g)$. Let $\tilde{X}_i$ be the lifting of the Killing vector field X_i. Via the developing map, $\tilde{X}_1$ and $\tilde{X}_2$ are uniquely extended to conformal Killing vector fields on the sphere ([18]). Their extensions are denoted by the same symbols. This pair of conformal Killing vector fields is generated by a two-dimensional group of conformal symmetries $\hat{T}$ of the sphere. Since the torus action on X is fixed-point-free, so is the action on $\tilde{X}$. Therefore, $\delta(\tilde{X})$ is a $\hat{T}$-invariant subset of S^4 and is disjoint from the fixed points of the $\hat{T}$-action.

The fundamental group $\pi_1(X)$ acts on $\tilde{X}$ by deck transformations. It is a subgroup of isometries of $(\tilde{X}, \pi^* g)$ and its action commutes with the $\tilde{T}$-action. The development map δ induces a homomorphism ρ from $\pi_1(X)$ into PGL(2, **H**). As $\pi_1(X)$ commutes with $\tilde{T}, \rho(\pi_1(X))$ leaves both $\delta(\tilde{X})$ and the fixed-point-set of $\hat{T}$ on S^4 invariant.

(iv) Since the torus action on X has a one-dimensional orbit, so is the $\tilde{T}$-action on $\tilde{X}$. As the developing map is a local diffeomorphism, the extended action on S^4 also has one-dimensional orbit. Let G be a one-dimensional subgroup of T such that it acts trivially on a one-dimensional submanifold of X. Then the fixed-point-set of its counterpart $\hat{G}$ in $\hat{T}$ is at least one-dimensional. As the fixed-point-set of any non-compact one-dimensional group in the group of conformal symmetries of the Euclidean sphere is at most a set of two isolated points ([18]), $\hat{G}$ is a compact subgroup of SO(5).

Let R be another one-dimensional subgroup of T such that G and R together generate T. Consider its counterpart $\hat{R}$ in $\hat{T}$. If $\hat{R}$ is compact, then $\hat{T}$ is compact. In this case, the $\hat{T}$-action on the sphere has two fixed points. If $\hat{R}$ is non-compact, the fixed-point-set of $\hat{R}$ consists of at most two points. As the fixed points of $\hat{R}$ are isolated, they are also fixed by the action of $\hat{G}$ because $\hat{G}$ is a connected group and its action commutes with the action of $\hat{R}$. As a conclusion, the fixed-point-set of $\hat{T}$ consists of at most two points.

(v) Suppose that the fixed-point-set of $\hat{T}$ consists of only one point, ∞. Then $\hat{R}$ is non-compact. The vector field generated by $\hat{R}$ is essential as defined by Obata ([18]). Taking stereographical projection from ∞, we obtain $\hat{R}$ as a group of Euclidean isometries. Let U be any small open subset in S^4 on which δ^{-1} is defined. The flow of U under $\hat{R}$ is either S^4 or S^4 with one point removed. As $\delta(\tilde{X})$ is $\hat{T}$-invariant, $\delta(\tilde{X}) = S^4\backslash\infty$. As in the proof of the Obata theorem in [18], one can show that δ is an embedding. To be precise, we use the orbit structure of the group $\hat{R}$ to construct a conformal map ϕ from $\delta(\tilde{X})$ into $\tilde{X}$ as follows. Given $\hat{x}$ in $\delta(\tilde{X})$, there is a point $\hat{y}$ in $\delta(U)$ and an element $\hat{R}_t$ in the group $\hat{R}$ such that $\hat{R}_t(\hat{y}) = \hat{x}$. Then $\phi(\hat{x}) := R_t \circ \delta^{-1}_{|\delta(U)} \circ \hat{R}_{-t}(\hat{x})$. ϕ is surjective. Otherwise, taking the developing map in a small neighbourhood of a boundary point q of $\phi(\delta(\tilde{X}))$, we can extend ϕ across q. This is a contradiction. Since the map $\delta \circ \phi$ is an everywhere defined conformal map of $\mathbf{R}^4$, it is a diffeomorphism. As ϕ is surjective, δ is injective. Therefore, δ is an embedding and the universal covering of X is the Euclidean flat space $\mathbf{R}^4$. Moreover, the homomorphism ρ is an isomorphism and the conformal structure on X is Kleinian. In particular, $\pi_1(X)$ acts on $\mathbf{R}^4$ freely and properly discontinuously. It is possible only when $\pi_1(X)$ is a subgroup of Euclidean isometries. Then $\mathbf{R}^4/\pi_1(X)$ is a flat Riemannian manifold. Without finite torsion, it must be a flat torus.

(vi) Suppose that the fixed-point-set of $\hat{T}$ consists of two points, ∞ and 0. Taking the stereographic projection from ∞, we obtain $\delta(\tilde{X})$ as a $\hat{T}$-invariant subset of $\mathbf{R}^4\backslash 0$.

As $\hat{G}$ is a compact subgroup of SO(5) leaving ∞ and 0 fixed, it is a subgroup of SO(4). As the fixed-point-set of $\hat{G}$ is at least one-dimensional, $\hat{G}$ consists of rotations on a two-dimensional plane, leaving the orthogonal plane fixed. Considering PGL(2, **H**) as a subgroup of PGL(4, **C**), $\hat{G}$ is generated by the diagonal matrix $\mathrm{diag}(e^{ip\theta}, e^{-ip\theta}, e^{ip\theta}, e^{-ip\theta})$, where p is a non-zero real number.

As $\rho(\pi_1(X))$ commutes with $\hat{T}$, elements in $\rho(\pi_1(X))$ either leave both ∞ and 0 invariant or send these two points to each other. Hence, elements in $\rho(\pi_1(X))$ are represented by the following two types of matrices in PGL(4, **C**):

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \bar{\alpha} & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \bar{\beta} \end{pmatrix} \begin{pmatrix} 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \bar{\eta} \\ \psi & 0 & 0 & 0 \\ 0 & \bar{\psi} & 0 & 0 \end{pmatrix}.$$

It follows that the divisor $S := \{\mathbf{z}: z_0 z_2 = 0\}$ in $\mathbf{CP}^3$ is invariant of the action of $\rho(\pi_1(X))$. Let D be the component $\{\mathbf{z}: z_0 = 0\}$. If every element in

$\rho(\pi_1(X))$ is a diagonal matrix, $\hat{\delta}^{-1}(D)$ is invariant of the deck transformations and descends into the twistor space over X. Denote this divisor by J. If some of the elements in $\rho(\pi_1(X))$ is not a diagonal matrix, $\hat{\delta}^{-1}(S)$ is invariant of the deck transformations and descends onto the twistor space over X. Since some of the deck transformations are bi-holomorphisms between the divisor D and the divisor $\{\mathbf{z}: z_2 = 0\}$, J is isomorphic to $\pi(\hat{\delta}^{-1}(S))$. In both cases, we obtain the divisor J in the twistor space.

The only real twistor line contained in D is the line $\{\mathbf{z}: z_0 = 0,\ z_1 = 0\}$. As this line is over ∞ and ∞ is not contained in $\delta(\tilde{X})$, $\hat{\delta}^{-1}(D)$ intersects every real twistor line in $\tilde{Z}$ transversally at one point. Therefore, J defines an integrable complex structure such that (X, g, J) is a conformally flat Hermitian surface. Due to Theorem B, (X, g) is finitely covered by either a flat torus or a Hopf manifold with a product metric.

(vii) Finally, we assume that the torus symmetry has a fixed point. Denote one of the fixed points by $\imath$. Denote one of the preimages of $\imath$ in the universal covering space $\tilde{X}$ by ∞. Then we define the developing map by starting from a small neighbourhood U of ∞. The image of ∞ in S^4 is denoted by the same symbol.

Suppose that $\hat{T}$ is non-compact. Let $\hat{R}$ be a non-compact one-dimensional subgroup. The flow of U under the group $\hat{R}$ is either the sphere S^4 or the sphere with one point removed $S^4\backslash 0$ ([18]). As $\delta(\tilde{X})$ is $\hat{T}$-invariant, it is either S^4 or $S^4\backslash 0$. In both cases, we use the orbit structure of the group $\hat{R}$ to construct backwards a conformal map ϕ of $\delta(\tilde{X})$ into $\tilde{X}$ as in paragraph (v). If $\delta(\tilde{X}) = S^4$, $\phi(S^4)$ is an open and closed set of $\tilde{X}$. Therefore, $\tilde{X} = \phi(S^4)$ is compact, and δ is a diffeomorphism. If $\delta(\tilde{X}) = S^4\backslash 0$, ϕ is still surjective and δ is injective as seen in (v). Therefore, X is conformally covered by S^4 or $S^4\backslash 0$.

Since the action of $\hat{T}$ on S^4 or $S^4\backslash 0$ has at most two fixed points, and these fixed points are the preimages of the fixed point $\imath$ on X, the degree of the covering map is at most 2. By the compactness of X, the universal covering is compact. Therefore, $\tilde{X} = S^4$. It follows that $\tilde{X} = X$. Due to Kuiper's theorem ([17]), the conformal structure of (X, g) is equivalent to a Euclidean sphere.

When $\hat{T}$ is compact, it has two fixed points on the sphere. As these points are account for all preimages of a fixed point of the torus action on X, the degree of the covering map from $\tilde{X}$ onto X is at most 2. Therefore, $S^4 = \tilde{X} = X$. Due to Kuiper's theorem, the conformal structure is the one of the Euclidean sphere.

The proof of Theorem A is completed.

8. Corollaries

An immediate corollary to Theorem A is a topological classification of compact self-dual manifolds with torus symmetry.

THEOREM 8.1. *Let* (X, g) *be a compact self-dual manifold such that* $C^+(X, [g])$ *contains* $\mathrm{U}(1) \times \mathrm{U}(1)$. *Then* X *is either* (i) *finitely covered by the 4-torus,* (ii) *the*

Hopf manifold $S^1 \times S^3$, *or* (iii) *an equivariant connected sum of* S^4, $S^2 \times S^2$, $\mathbf{CP}^2$, $\overline{\mathbf{CP}}^2$, $2(S^1 \times S^3)\#2(S^2 \times S^2)$, *and the three families of manifolds* R, T, *and* L.

Proof. If the signature of X is equal to zero, the metric is conformally flat. Then we apply Theorem A. If the signature of X is not equal to zero, the torus action has fixed points. Then we apply Theorem 3.2.

A simplified version of Theorem 8.1 can be stated as follows.

THEOREM 8.2. *Let* (X, g) *be a compact self-dual manifold such that the metric* g *is not conformally equivalent to a flat metric. Suppose that* $C^+(X, [g])$ *contains* $\mathrm{U}(1) \times \mathrm{U}(1)$, *then the manifold* X *is finitely covered by a connected sum of* S^4, $S^2 \times S^2$, $\mathbf{CP}^2$, $\overline{\mathbf{CP}}^2$, *and* $S^1 \times S^3$.

Proof. After Theorem A, we only need investigate the case when the torus action has fixed points. The manifold L appears in an equivariant decomposition of the manifold X only when one excises a neighbourhood of a point with non-trivial finite isotropy from X ([27]). After taking a finite covering, we avoid this procedure. Then the proof of this theorem is concluded by observing that the manifolds R and T are connected sums of $S^2 \times S^2$, $\mathbf{CP}^2$, $\overline{\mathbf{CP}}^2$, and $S^1 \times S^3$.

We are also interested in knowing if the type of self-dual conformal classes is related to the existence of torus symmetry. The next theorem shows that non-negative type self-dual conformal classes are relatively rare.

THEOREM 8.3. *Let* (X, g) *be a compact self-dual manifold with non-negative scalar curvature. Suppose that* $C^+(X, [g])$ *contains* $\mathrm{U}(1)\times\mathrm{U}(1)$. *Then the manifold* X *is either* (i) *the sphere,* (ii) *a connected sum of the complex projective plane,* (iii) *finitely covered by the 4-torus or* (iv) *the Hopf manifold.*

Proof. Assuming that the manifold is not described by Theorem A, then the topology of X is determined by Theorems 3.2 and 3.3. When the scalar curvature is non-negative, a Bochner type argument shows that either (X, g) is finitely covered by a scalar-flat Kähler surface with opposite natural orientation or the intersection form of X is positive-definite ([5], [19]).

When (X, g) is finitely covered by a compact scalar-flat Kähler surface, Proposition 6.1 shows that (X, g) is finitely covered by the 4-torus. To conclude the proof of this theorem, observe that on the list of Theorem 8.2, only connected sums of the complex projective plane have positive-definite intersection form.

When we apply an equivariant version of the Donaldson–Friedman programme to construct self-dual metrics with symmetry on connected sums of compact self-dual manifolds ([30]), we need building blocks. They are compact self-dual manifolds admitting symmetries *with fixed points*. For this reason, it is useful to know which manifolds on the lists of Theorems 3.2 and 3.3 admit self-dual metrics. After Orlik and Raymond, manifolds on these lists are called *elementary*.

THEOREM 8.4. *Suppose that* (X, g) *is a compact self-dual manifold and* $\mathcal{C}^+(X, [g])$ *contains* $\mathrm{U}(1) \times \mathrm{U}(1)$. *If the manifold is elementary with respect to a given torus action, then* (X, g) *is conformally equivalent to either the Euclidean sphere or the complex projective plane with a Fubini–Study metric.*

Proof. Among all elementary manifolds, only $\mathbf{CP}^2$ has positive signature. Due to Theorem A and Theorem 8.1, the sphere is the only elementary manifold with vanishing signature admitting a conformally flat structure. However, the only conformally flat structure on a sphere is the standard one ([17]).

To determine self-dual structures on $\mathbf{CP}^2$, observe that any torus action on $\mathbf{CP}^2$ contains a semi-free $\mathrm{U}(1)$-action. To be precise, any effective torus action on $\mathbf{CP}^2$ is equivariant to the following torus action. Parametrize the torus by $0 \leq \theta < 2\pi$, $0 \leq \phi < 2\pi$. The action is given by

$$(\theta, \phi) \cdot [z_0, z_1, z_2] \rightarrow [\mathrm{e}^{im\theta} z_0, z_1, \mathrm{e}^{in\phi} z_2],$$

where $[z_0, z_1, z_2]$ is a homogeneous coordinate on the complex projective plane, and (m, n) is a pair of coprime integers. Then the action of the subgroup $\{t \mapsto (t, kt), 0 \leq t < 2\pi\}$, where $k \neq 0$, 1 is a real number, is semi-free.

It is proved in [22] that when a compact self-dual manifold with strictly positive definite intersection form has semi-free $\mathrm{U}(1)$-symmetry, then it is obtained by a hyperbolic Ansatz over the hyperbolic 3-space. This construction on complex projective plane yields the conformal class containing the Fubini–Study metric.

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