A Hodge-Type Decomposition of Holomorphic Poisson Cohomology on Nilmanifolds

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Abstract

A cohomology theory associated to a holomorphic Poisson structure is the hypercohomology of a bi-complex where one of the two operators is the classical $\overline{\partial}$-operator, while the other operator is the adjoint action of the Poisson bivector with respect to the Schouten-Nijenhuis bracket. The first page of the associated spectral sequence is the Dolbeault cohomology with coefficients in the sheaf of germs of holomorphic polyvector fields. In this note, the authors investigate the conditions for which this spectral sequence degenerates on the first page when the underlying complex manifolds are nilmanifolds with an abelian complex structure. For a particular class of holomorphic Poisson structures, this result leads to a Hodge-type decomposition of the holomorphic Poisson cohomology. We provide examples when the nilmanifolds are 2-step.

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1 Introduction

It is well known that complex structures and symplectic structures are examples of generalized complex structures in the sense of Hitchin [11, 12, 13]. It is now also known that a holomorphic Poisson structure plays a fundamental role in generalized geometry [1]. Since a key feature of generalized geometry is to place both complex structures and symplectic structures within a single framework, its deformation theory is of great interest [8, 9, 14]. An understanding of cohomology theory of generalized geometry in general, and holomorphic Poisson structures in particular, becomes necessary. On algebraic surfaces, there has been work done by Xu and collaborators [16, 17]. Recently, Hong studied holomorphic Poisson cohomology on toric Poisson manifolds [15]. For nilmanifolds with abelian complex structures, there is recent work done by this author and his collaborators [4, 5, 10]. A common feature of these works is to recognize the cohomology as the hypercohomology of a bi-complex.

In computation of hypercohomology of bi-complex, theoretically one could apply one of the two naturally defined spectral sequences to complete the task. In the case of a holomorphic Poisson structure, one of the operators is the classical $\overline{\partial}$-operator. Another operator is the action of a holomorphic bi-vector field Λ on the space of polyvector fields and differential forms via the Schouten bracket. One may choose a filtration so that the first page of the spectral sequence consists of the Dolbeault cohomology of a complex manifold with coefficients in the sheaf of germs of holomorphic polyvector fields, $H^q(M, \Theta^p)$. In this case, the d_1-map on the first page is the holomorphic Poisson structure via the Schouten bracket. Another approach is to apply the holomorphic Poisson structure action first. In such a case, the first page of the spectral sequence consists of a holomorphic version of Lichnerowicz-Poisson cohomology [18, 25].

We focus on the first approach, as Dolbeault cohomology is a well known classical object, and it involves the complex structure only. It is natural to determine when this spectral sequence degenerates quickly. In [4, 5], we have seen situations where the spectral sequence degenerates on the second page. We push our analysis from past work and investigate the possibility when degeneracy on the first page occurs, and establish a direct relation between the holomorphic Poisson cohomology H^n_Λ and the elements in the first page of the spectral sequence. In particular, we could formulate a result as below.

Theorem 1 *Let M be a nilmanifold with an abelian complex structure. Let $H^q(M,\Theta^p)$ be the q^{th} Dolbeault cohomology of the complex manifold with coefficients in the sheaf of germs of holomorphic polyvector fields of degree p. For any invariant holomorphic Poisson structure Λ on M, let H^n_Λ be the n^{th} cohomology of the holomorphic Poisson structure. Then there exists a natural one-to-one linear map*

$$\phi : H^n_\Lambda \to \oplus_{p+q=n} H^q(M,\Theta^p).$$

In particular, $\dim H^n_\Lambda \leq \sum_{p+q=n} \dim H^q(M,\Theta^p)$.

To prove this result, we first review some basic facts on generalized complex structures and the related cohomology theory from the perspective of Lie bi-algebroids, and then address it in the context of the holomorphic Poisson structure Λ. It quickly sets up that the cohomology with respect to Λ is given by the operator $\overline{\partial}_\Lambda = \mathrm{ad}_\Lambda + \overline{\partial}$ where ad_Λ is the action of Λ on the space of sections of the exterior product of the direct sum of $(1,0)$-vectors and $(0,1)$-forms, and $\overline{\partial}$ is the classical operator on complex manifolds. This frames our double complex and its related spectral sequence.

We begin to focus on nilmanifolds in Section 4. A key observation is that the Dolbeault cohomology spaces $H^q(M,\Theta^p)$ are given by invariant elements [4]. It allows us to address the entire computation through that on a finite-dimensional Lie algebra. In Section 4, we refine some details regarding the action of ad_Λ on invariant elements. This sets up the computation in Section 5, and leads to the the proof of Theorem 4, which is also Theorem 1 stated above.

Prior to our investigation on when the map ϕ in Theorem 1 is an isomorphism, in Section 7 we identify the obstruction for degeneracy of the spectral sequence on the first page when the center of the nilpotent algebra covering the manifold M is of real dimension two. Subsequently, we find that it is also sufficient for a particular kind of holomorphic Poisson structure. Details are given in Theorem 5.

Although the degeneracy of a spectral sequence on the first page does not, in general, lead to a decomposition of a hypercohomology in terms of direct sums of entries in the first page of its spectral sequence, our work leading to Theorem 5 could be refined to complete a proof of Theorem 6, which presents a Hodge-type decomposition.

In order to paraphrase Theorem 6 with precision and minimum technicality, let M be as given in the last theorem with an abelian complex structure J. Let $\mathfrak{g}$ be the Lie algebra of the nilpotent group covering M. Let $\mathfrak{c}$ be the center, and $\mathfrak{t}$ be the quotient algebra $\mathfrak{g}/\mathfrak{c}$. Suppose that $\dim_{\mathbb{R}} \mathfrak{c} = 2$. Let ρ be a non-trivial element spanning the space of $(1,0)$-form dual to the center, i.e., $\rho \in \mathfrak{c}^{*(1,0)}$. When the complex structure is abelian, then $d\rho$ is a type $(1,1)$-form on the complexification of $\mathfrak{t}$ [3, 24]. Through contraction, it could be treated as a map from $\mathfrak{t}^{1,0}$ to $\mathfrak{t}^{*(0,1)}$.

Theorem 2 *Suppose that M is a nilmanifold with abelian complex structure and suppose that* $\dim_{\mathbb{R}} \mathfrak{c} = 2$. *If the map $d\rho$ is non-degenerate, there exists a holomorphic Poisson structure* Λ *such that the injective map ϕ in Theorem 1 is an isomorphism.*

In order to better describe the obstruction to degeneracy of the holomorphic Poisson spectral sequence on its first page, in Section 9 we further analyze the necessary condition on this obstruction. The key observations are in Proposition 2 and Proposition 3.

Given Propositions 2 and 3, we examine several series of 2-step nilmanifolds, and present examples for which the obstruction vanishes, as well as examples for which the obstruction is non-trivial so that the map ϕ in Theorem 1 fails to be an isomorphism.

2 Complex and Generalized Complex Structures

In this section, we review the basic background materials as seen in [5] to set up notations.

Let M be a smooth manifold. Denote its tangent bundle by TM and the cotangent bundle by T^*M. A generalized complex structure on an even-dimensional manifold M [11, 12] is a subbundle L of the direct sum $\mathcal{T} = (TM \oplus T^*M)_{\mathbb{C}}$ such that

- L and its conjugate bundle $\overline{L}$ are transversal;

- L is maximally isotropic with respect to the natural pairing on $\mathcal{T}$;
- the space of sections of L is closed with respect to the Courant bracket.

Given a generalized complex structure, the pair of bundles L and $\overline{L}$ makes a (complex) Lie bi-algebroid. The composition of the inclusion of L and $\overline{L}$ in $\mathcal{T}$ with the natural projection onto the summand $TM_{\mathbb{C}}$ becomes the anchor map of these Lie algebroids, which we denote by α. Via the canonical non-degenerate pairing on the bundle $\mathcal{T}$, the bundle $\overline{L}$ is complex linearly identified to the dual of L. Therefore, the Lie algebroid differential of $\overline{L}$ acts on L, which extends to a differential on the exterior algebra of L. For the calculus of Lie bi-algebroids, we follow the conventions in [20]. In particular, for any element Γ in $C^\infty(M, \wedge^k L)$ and elements $a_1, \ldots, a_{k+1}$ in $C^\infty(M, \overline{L})$, the Lie algebroid differential of Γ is defined by the Cartan formula as in exterior differential algebra, namely

$$\begin{aligned}(\overline{\partial}_L \Gamma)(a_1, \ldots, a_{k+1}) = &\sum_{r=1}^{k+1} (-1)^{r+1} \alpha(a_r)(\Gamma(a_1, \ldots, \hat{a}_r, \ldots, a_{k+1})) \\ &+ \sum_{r<s} (-1)^{r+s} \Gamma([\![a_r, a_s]\!], a_1, \ldots, \hat{a}_r, \ldots, \hat{a}_s, \ldots, a_{k+1}). \qquad (1)\end{aligned}$$

The space of sections of the bundle $\overline{L}$ is closed with respect to the Courant bracket if and only if $\overline{\partial}_L \circ \overline{\partial}_L = 0$.

Typical examples of generalized complex structures are classical complex structures and symplectic structures on a manifold [11]. In this section, we focus on the former.

Let $J : TM \to TM$ be an integrable complex structure on the manifold M. The complexified tangent bundle $TM_{\mathbb{C}}$ splits into the direct sum of the bundle of $(1,0)$-vectors $TM^{1,0}$ and the bundle of $(0,1)$-vectors $TM^{0,1}$. Their p^{th} exterior products are respectively denoted by $TM^{p,0}$ and $TM^{0,p}$. Denote their dual bundles by $TM^{*(p,0)}$ and $TM^{*(0,p)}$ respectively.

Define $L = TM^{1,0} \oplus TM^{*(0,1)}$. This defines a generalized complex structure, where its dual is its complex conjugate $\overline{L} = TM^{0,1} \oplus TM^{*(1,0)}$. When one restricts the Courant bracket from the ambient bundle $\mathcal{T} = (TM \oplus T^*M)_{\mathbb{C}}$ to the subbundles L and $\overline{L}$, one then recovers the Schouten-Nijenhuis bracket, often simply known as the Schouten bracket, in classical deformation. The Schouten bracket between

$(1,0)$-vector fields is the Lie bracket of vector fields; the Schouten bracket between a $(1,0)$-vector field and a $(0,1)$-form is related to the Lie derivative of a form by a vector field. The Schouten bracket between two $(0,1)$-forms is equal to zero. These brackets are extended to higher exterior products by observing the rule of exterior multiplication [20].

With respect to the Lie algebroid $\overline{L}$, we get its differential $\overline{\partial}$ as defined in (1),

$$\overline{\partial} : C^\infty(M, L) \to C^\infty(M, \wedge^2 L). \tag{2}$$

It is extended to a differential of exterior algebras:

$$\overline{\partial} : C^\infty(M, \wedge^p L) \to C^\infty(M, \wedge^{p+1} L). \tag{3}$$

It is an elementary exercise in computation of Lie algebroid differential that when $\overline{\partial}$ is restricted to $(0,1)$-forms, it is the classical $\overline{\partial}$-operator in complex manifold theory; and

$$\overline{\partial} : C^\infty(M, TM^{*(0,1)}) \to C^\infty(M, TM^{*(0,2)})$$

is the $(0,2)$-component of the exterior differential [23]. Similarly, when the Lie algebroid differential is restricted to $(1,0)$-vector fields, then the map becomes

$$\overline{\partial} : C^\infty(M, TM^{1,0}) \to C^\infty(M, TM^{*(0,1)} \otimes TM^{1,0}),$$

the *Cauchy-Riemann operator* as seen in [7].

By virtue of L and $\overline{L}$ being a Lie bi-algebroid pair, the space $C^\infty(M, \wedge^\bullet L)$ together with the Schouten bracket, exterior product and the Lie algebroid differential $\overline{\partial}$ form a differential Gerstenhaber algebra [20, 22]. In particular, if a is a smooth section of $\wedge^{|a|}L$ and b is a smooth section of $\wedge^{|b|}L$, then

$$\begin{aligned} \overline{\partial}[\![a, b]\!] &= [\![\overline{\partial}a, b]\!] + (-1)^{|a|+1}[\![a, \overline{\partial}b]\!]; \quad (4) \\ \overline{\partial}(a \wedge b) &= (\overline{\partial}a) \wedge b + (-1)^{|a|} a \wedge (\overline{\partial}b). \quad (5) \end{aligned}$$

Since $\overline{\partial} \circ \overline{\partial} = 0$, one obtains the Dolbeault cohomology with coefficients in holomorphic polyvector fields. Denoting the sheaf of germs of sections of the p^{th} exterior power of the holomorphic tangent bundle by Θ^p, we have

$$H^\bullet(M, \wedge^\bullet \Theta) \cong \bigoplus_{p,q \geq 0} H^q(M, \Theta^p).$$

In subsequent computations, when $p = 0$, Θ^p represents the structure sheaf $\mathcal{O}$ of the complex manifold M.

Due to the compatibility between $\overline{\partial}$ and the Schouten bracket $[\![-,-]\!]$, and the compatibility between $\overline{\partial}$ and the exterior product $\wedge$ as noted above, the Schouten bracket and exterior product descend to the cohomology space $H^\bullet(M, \Theta^\bullet)$. In other words, the triple $(H^\bullet(M, \Theta^{\bullet,0}), [\![-,-]\!], \wedge)$ forms a Gerstenhaber algebra. When we ignore the exterior product, we call it a Schouten algebra. For example, by center of the Schouten algebra, we mean the collection of elements A in $H^\bullet(M, \Theta^\bullet)$ such that $[\![A, B]\!] = 0$ for all B in this space.

3 Holomorphic Poisson Bi-Complex

A holomorphic Poisson structure on a complex manifold (M, J) is a holomorphic bi-vector field Λ such that $[\![\Lambda, \Lambda]\!] = 0$. The corresponding bundles as a generalized complex structure for Λ are the pair of bundles of graphs $L_{\overline{\Lambda}}$ and $\overline{L}_\Lambda$ where

$$\overline{L}_\Lambda = \{\overline{\ell} + \Lambda(\overline{\ell}) : \overline{\ell} \in \overline{L}\}. \tag{6}$$

While the pair of bundles $L_{\overline{\Lambda}}$ and $\overline{L}_\Lambda$ naturally form a Lie bi-algebroid, so does the pair L and $\overline{L}_\Lambda$ [19]. From this perspective, the Lie algebroid differential of the deformed generalized complex structure $\overline{L}_\Lambda$ acts on the space of sections of the bundle L.

Any smooth section of the bundle L is the sum of a section v of $TM^{1,0}$ and a section $\overline{\omega}$ of $TM^{*(0,1)}$. Given a holomorphic Poisson structure Λ, define ad_Λ by

$$\mathrm{ad}_\Lambda(v + \overline{\omega}) = [\![\Lambda, v + \overline{\omega}]\!]. \tag{7}$$

Proposition 1 [10, Proposition 1] *The action of the Lie algebroid differential of $\overline{L}_\Lambda$ on L is given by*

$$\overline{\partial}_\Lambda = \overline{\partial} + \mathrm{ad}_\Lambda : C^\infty(M, L) \to C^\infty(M, \wedge^2 L). \tag{8}$$

The operator $\overline{\partial}_\Lambda$ extends to act on the exterior algebra of $TM^{1,0} \oplus TM^{*(0,1)} = L$. From now on, for $n \geq 0$ denote

$$K^n = C^\infty(M, \wedge^n L). \tag{9}$$

For $n < 0$, set $K^n = \{0\}$.

Since Λ is a holomorphic Poisson structure, the closure of the space of sections of the corresponding bundles $\overline{L}_\Lambda$ is equivalent to $\overline{\partial}_\Lambda \circ \overline{\partial}_\Lambda = 0$. Therefore, one has a complex with $\overline{\partial}_\Lambda$ being a differential.

Definition 1 *For all $n \geq 0$, the n^{th} holomorphic Poisson cohomology of the holomorphic Poisson structure Λ is the space*

$$H^n_\Lambda(M) := \frac{\text{kernel of } \overline{\partial}_\Lambda : K^n \to K^{n+1}}{\text{image of } \overline{\partial}_\Lambda : K^{n-1} \to K^n}. \tag{10}$$

Given Proposition 1, the identity $\overline{\partial}_\Lambda \circ \overline{\partial}_\Lambda = 0$ is equivalent to a system of three equations,

$$\overline{\partial} \circ \overline{\partial} = 0, \quad \overline{\partial} \circ \mathrm{ad}_\Lambda + \mathrm{ad}_\Lambda \circ \overline{\partial} = 0, \quad \mathrm{ad}_\Lambda \circ \mathrm{ad}_\Lambda = 0. \tag{11}$$

The first identity is equivalent to the complex structure J being integrable; the second identity is equivalent to Λ being holomorphic, and the third is equivalent to Λ being Poisson. Define $A^{p,q} = C^\infty(M, TM^{p,0} \otimes TM^{*(0,q)})$, then

$$\mathrm{ad}_\Lambda : A^{p,q} \to A^{p+1,q}, \quad \overline{\partial} : A^{p,q} \to A^{p,q+1}; \quad \text{and} \quad K^n = \oplus_{p+q=n} A^{p,q}. \tag{12}$$

Therefore, we obtain a bi-complex. We arrange the double indices (p, q) in such a way that p increases horizontally so that ad_Λ maps from left to right, and q increases vertically so that $\overline{\partial}$ maps from bottom to top. As a result, we obtain a *first quadrant* bi-complex.

Definition 2 *Given a holomorphic Poisson structure Λ, its Poisson bi-complex is the triple $\{A^{p,q}, \mathrm{ad}_\Lambda, \overline{\partial}\}$.*

It is now obvious that the (holomorphic) Poisson cohomology $H^\bullet_\Lambda(M)$ theoretically could be computed by each one of the two naturally defined spectral sequences. We choose a filtration given by

$$F^p K^n = \bigoplus_{\substack{p'+q=n \\ p' \geq p}} A^{p',q}.$$

The lowest differential is $\overline{\partial} : A^{p,q} \to A^{p,q+1}$. Therefore, the first sheet of the spectral sequence is the Dolbeault cohomology

$$E_1^{p,q} = H^q(M, \Theta^p). \tag{13}$$

The first page of the spectral sequence is given as below.

$$\begin{array}{ccccccc}
\cdots & \to & \cdots & \to & \cdots & \to \\
H^n(M, \mathcal{O}) & \to & H^n(M, \Theta) & \to & H^n(M, \Theta^2) & \to \\
\cdots & \to & \cdots & \to & \cdots & \to \\
H^1(M, \mathcal{O}) & \to & H^1(M, \Theta) & \to & H^1(M, \Theta^2) & \to \\
H^0(M, \mathcal{O}) & \to & H^0(M, \Theta) & \to & H^0(M, \Theta^2) & \to
\end{array}$$

The differential on this page is

$$d_1^{p,q} = \mathrm{ad}_\Lambda : H^q(M, \Theta^p) \to H^q(M, \Theta^{p+1}). \tag{14}$$

Question 1 *When will the spectral sequence of a Poisson bi-complex degenerate on the first page? In other words, when will* $\mathrm{ad}_\Lambda \equiv 0$ *for all* p, q*?*

It is known in the application of spectral sequences that degeneracy on the first page does not necessarily lead to an isomorphism of the hypercohomology of degree n and a direct sum of the spaces on the first page whose sum of bi-degree is equal to n. A well-known case is on the degeneracy of Frölicher spectral sequence on compact complex surfaces [2]. Therefore, the next question is also relevant.

Question 2 *Is there any holomorphic Poisson structure such that its cohomology has the following decomposition?*

$$H_\Lambda^n \cong \bigoplus_{p+q=n} H^q(M, \Theta^p). \tag{15}$$

In subsequent discussion, we will use $H^{p,q}$ to represent $H^q(M, \Theta^p)$.

4 Nilmanifolds

A compact manifold M is a nilmanifold if there exists a simply-connected nilpotent Lie group G and a lattice subgroup Γ such that M is diffeomorphic to G/Γ. We denote the Lie algebra of the group G by $\mathfrak{g}$, and its center by $\mathfrak{c}$. By definition, $\mathfrak{g}^0 = \mathfrak{g}$, and inductively, for all natural numbers p, $\mathfrak{g}^p = [\![\mathfrak{g}^{p-1}, \mathfrak{g}]\!]$. The algebra $\mathfrak{g}$ is a k-step nilpotent algebra if $\mathfrak{g}^k = \{0\}$ and $\mathfrak{g}^{k-1} \neq \{0\}$. In particular, $\mathfrak{g}^{k-1} \subseteq \mathfrak{c}$. The step of the nilmanifold is the nilpotence of the Lie algebra $\mathfrak{g}$.

A left-invariant complex structure J on G is said to be *abelian* if on the Lie algebra $\mathfrak{g}$, it satisfies the conditions $J \circ J = -$identity and $[\![JA, JB]\!] = [\![A, B]\!]$ for all A and B in the Lie algebra $\mathfrak{g}$ [3, 24]. If one complexifies the algebra $\mathfrak{g}$ and denotes the $+i$ and $-i$ eigen-spaces of J by $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$, respectively, then the invariant complex structure J is abelian if and only if the complex Lie algebra $\mathfrak{g}^{1,0}$ is abelian.

Denote $\wedge^p \mathfrak{g}^{1,0}$ and $\wedge^q \mathfrak{g}^{*(0,1)}$ respectively by $\mathfrak{g}^{p,0}$ and $\mathfrak{g}^{*(0,q)}$. We will use the notation

$$B^{p,q} = \mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)}.$$

On the nilmanifold M, we consider $\mathfrak{g}^{p,0}$ as invariant $(p,0)$-vector fields and $\mathfrak{g}^{*(0,q)}$ as invariant $(0,q)$-forms. It yields an inclusion map

$$B^{p,q} \hookrightarrow A^{p,q} = C^\infty(M, TM^{p,0} \otimes TM^{*(0,q)}).$$

When the complex structure is also invariant, $\overline{\partial}$ sends $B^{p,q}$ to $B^{p,q+1}$. Given an invariant complex structure and an invariant holomorphic Poisson structure Λ, ad_Λ sends $B^{p,q}$ to $B^{p+1,q}$. Restricting $\overline{\partial}$ to $B^{p,q}$, we then consider the invariant cohomology

$$H^q(\mathfrak{g}^{p,0}) = \frac{\text{kernel of } \overline{\partial} : B^{p,q} \to B^{p,q+1}}{\text{image of } \overline{\partial} : B^{p,q-1} \to B^{p,q}}. \tag{16}$$

The inclusion map yields a homomorphism of cohomology:

$$H^q(\mathfrak{g}^{p,0}) \hookrightarrow H^q(M, \Theta^p).$$

Theorem 3 (See [4, 5]) *On a nilmanifold M with an invariant abelian complex structure, the inclusion $B^{p,q}$ in $A^{p,q} = C^\infty(M, TM^{p,0} \otimes TM^{*(0,q)})$ induces an isomorphism of cohomology spaces. In other words,*

$$H^q(\mathfrak{g}^{p,0}) \cong H^q(M, \Theta^p) = H^{p,q}.$$

Any element A in $B^{p,q}$ acts on $B^{p,q}$ by the Schouten bracket. We denote its action by ad_A, i.e.,

$$\mathrm{ad}_A(B) = [\![A, B]\!].$$

An element A is said to be in the center of the Schouten algebra $\oplus_{p,q} B^{p,q}$ with respect to the Schouten bracket $[\![-,-]\!]$ if and only if $\mathrm{ad}_A \equiv 0$. Similarly, an element A in $H^q(\mathfrak{g}^{p,0})$ is said to be in the center of the Schouten algebra $\oplus_{p,q} H^q(\mathfrak{g}^{p,0})$ if $\mathrm{ad}_A(B)$ is equal to zero on the cohomology level for any B in $\oplus_{p,q} H^q(\mathfrak{g}^{p,0})$.

For each integer $0 \le m \le k+1$, define $\mathfrak{g}_J^m = \mathfrak{g}^m + J\mathfrak{g}^m$. When the complex structure is abelian, it is clear from various definitions that each $\mathfrak{g}_J^m$ is a J-invariant ideal of $\mathfrak{g}$, and we have a filtration of subalgebras:

$$\{0\} = \mathfrak{g}_J^k \subset \mathfrak{g}_J^{k-1} \subseteq \cdots \subseteq \mathfrak{g}_J^{\ell+1} \subseteq \mathfrak{g}_J^{\ell} \subseteq \cdots \subseteq \mathfrak{g}_J^1 \subset \mathfrak{g}_J^0 = \mathfrak{g}.$$

Since the center $\mathfrak{c}$ is J-invariant and it contains $\mathfrak{g}^{k-1}$,

$$\mathfrak{g}_J^{k-1} \subseteq \mathfrak{c}. \tag{17}$$

We complexify the above filtration to get

$$\{0\} = \mathfrak{g}_{J,\mathbb{C}}^k \hookrightarrow \mathfrak{g}_{J,\mathbb{C}}^{k-1} \hookrightarrow \cdots \hookrightarrow \mathfrak{g}_{J,\mathbb{C}}^{\ell+1} \hookrightarrow \mathfrak{g}_{J,\mathbb{C}}^{\ell} \hookrightarrow \cdots \hookrightarrow \mathfrak{g}_{J,\mathbb{C}}^1 \hookrightarrow \mathfrak{g}_{J,\mathbb{C}}^0 = \mathfrak{g}_{\mathbb{C}}. \tag{18}$$

With respect to the eigenspace decomposition for J, there exists a type decomposition for each ℓ

$$\mathfrak{g}_{J,\mathbb{C}}^{\ell} = \mathfrak{g}_J^{\ell,(1,0)} \oplus \mathfrak{g}_J^{\ell,(0,1)}.$$

So, the filtration (18) splits into two. One is for type $(1,0)$-vectors

$$\{0\} \hookrightarrow \mathfrak{g}_J^{k-1,(1,0)} \hookrightarrow \cdots \hookrightarrow \mathfrak{g}_J^{\ell+1,(1,0)} \hookrightarrow \mathfrak{g}_J^{\ell,(1,0)} \hookrightarrow \cdots \hookrightarrow \mathfrak{g}_J^{1,(1,0)} \hookrightarrow \mathfrak{g}_J^{0,(1,0)} = \mathfrak{g}^{(1,0)};$$

Another is for type $(0,1)$-vectors

$$\{0\} \hookrightarrow \mathfrak{g}_J^{k-1,(0,1)} \hookrightarrow \cdots \hookrightarrow \mathfrak{g}_J^{\ell+1,(0,1)} \hookrightarrow \mathfrak{g}_J^{\ell,(0,1)} \hookrightarrow \cdots \hookrightarrow \mathfrak{g}_J^{1,(0,1)} \hookrightarrow \mathfrak{g}_J^{0,(0,1)} = \mathfrak{g}^{(0,1)}.$$

Lemma 1 [4, Lemma 4] *Suppose that the complex structure J is abelian. Then*

- $[\![\mathfrak{g}^{(1,0)}, \mathfrak{g}^{(1,0)}]\!] = 0,$ *and* $[\![\mathfrak{g}^{(0,1)}, \mathfrak{g}^{(0,1)}]\!] = 0.$
- $[\![\mathfrak{g}_J^{a,(1,0)}, \mathfrak{g}_J^{b,(0,1)}]\!] \subseteq \mathfrak{g}_{J,\mathbb{C}}^{1+\max\{a,b\}}.$

In particular, $[\![\mathfrak{g}_J^{a,(1,0)}, \mathfrak{g}^{(0,1)}]\!] \subseteq \mathfrak{g}_{J,\mathbb{C}}^{a+1}$.

For the quotient space, we will adopt the notation

$$\mathfrak{t}_\ell^{(1,0)} = \mathfrak{g}_J^{\ell-1,(1,0)}/\mathfrak{g}_J^{\ell,(1,0)}. \tag{19}$$

Choose a vector space isomorphism so that the short exact sequence of Lie algebras

$$0 \to \mathfrak{g}_J^{\ell,(1,0)} \to \mathfrak{g}_J^{\ell-1,(1,0)} \to \mathfrak{g}_J^{\ell-1,(1,0)}/\mathfrak{g}_J^{\ell,(1,0)} \to 0$$

is turned into a direct sum of vector spaces.

$$\mathfrak{g}_J^{\ell-1,(1,0)} \cong \mathfrak{t}_\ell^{(1,0)} \oplus \mathfrak{g}_J^{\ell,(1,0)}.$$

Inductively,

$$\mathfrak{g}^{(1,0)} = \mathfrak{t}_1^{(1,0)} \oplus \mathfrak{t}_2^{(1,0)} \oplus \cdots \oplus \mathfrak{t}_k^{(1,0)}. \tag{20}$$

Similarly,

$$\mathfrak{g}^{(0,1)} = \mathfrak{t}_1^{(0,1)} \oplus \mathfrak{t}_2^{(0,1)} \oplus \cdots \oplus \mathfrak{t}_k^{(0,1)}.$$

We remark that $\mathfrak{t}_k^{(1,0)} = \mathfrak{g}_J^{k-1,(1,0)}$. Let $\mathfrak{t}_\ell^{*(0,1)}$ be the dual space of $\mathfrak{t}_\ell^{(0,1)}$, and $\mathfrak{t}_\ell^{*(1,0)}$ the dual space of $\mathfrak{t}_\ell^{(1,0)}$.

Lemma 2 [4, Proposition 1] *When the complex structure is abelian and* $\mathfrak{g}$ *is* k*-step nilpotent, then*

1. $\overline{\partial}\mathfrak{g}^{*(0,1)} = \{0\}$.
2. $\overline{\partial}\mathfrak{t}_k^{1,0} = \{0\}$.
3. $\overline{\partial}\mathfrak{t}_{k-1}^{1,0} \subseteq \oplus_{a\le k-1}(\mathfrak{t}_k^{1,0} \otimes \mathfrak{t}_a^{*(0,1)})$.
4. *For all* $1 \le \ell \le k-2$, $\overline{\partial}\mathfrak{t}_\ell^{1,0} \subseteq \bigoplus_{a\le\ell}(\mathfrak{t}_{\ell+1}^{1,0} \otimes \mathfrak{t}_a^{*(0,1)}) \oplus \bigoplus_{a>\ell}\left(\mathfrak{t}_{a+1}^{1,0} \otimes \mathfrak{t}_a^{*(0,1)}\right)$.

As a consequence, all $(0,q)$-forms are $\overline{\partial}$-closed.

Lemma 3 *There is an isomorphism*

$$H^q(\mathfrak{g}^{0,0}) = B^{0,q} = \wedge^q \mathfrak{g}^{*(0,1)}.$$

5 A Bound on Holomorphic Poisson Cohomology

Due to Theorem 3, our investigation on the holomorphic Poisson spectral sequence for any invariant holomorphic Poisson structure Λ is reduced to an analysis on the adjoint action of Λ on the invariant cohomology $H^q(\mathfrak{g}^{p,0})$.

Given the bi-degree, we next create a filtration of cohomology. For any natural number n, $B^n = \oplus_{p+q=n} B^{p,q}$. Define

$$F^m B^n = \bigoplus_{\substack{m \le j \le n \\ j+\ell=n}} B^{j,\ell} = B^{n,0} \oplus B^{n-1,1} \oplus \cdots \oplus B^{m+1,n-m-1} \oplus B^{m,n-m}. \tag{21}$$

It follows that $F^0 B^n = B^n$, $F^n B^n = B^{n,0}$. Define $F^m B^n = \{0\}$ when $m \ge n+1$. Therefore, we have a filtration

$$B^n = F^0 B^n \supset F^1 B^n \supset \cdots \supset F^k B^n \supset \cdots \supset F^n B^n \supset F^{n+1} B^n = \{0\}. \tag{22}$$

Define

$$F^m Z^n = \ker \overline{\partial}_\Lambda : F^m B^n \to B^{n+1}, \quad F^m C^n = \text{image } \overline{\partial}_\Lambda : F^m B^{n-1} \to B^n. \tag{23}$$

As $\overline{\partial}_\Lambda$ maps $B^{p,q}$ to $B^{p+1,q} \oplus B^{p,q+1}$, it is apparent that $\overline{\partial}_\Lambda$ maps $F^m B^{n-1}$ to $F^m B^n$. Therefore, $F^m Z^n \supseteq F^m C^n$, and the quotient below is well defined

$$F^m H^n = \frac{F^m Z^n}{F^m C^n}. \tag{24}$$

As $F^{m-1} Z^n \cap F^m C^n = F^{m-1} C^n$, we have an inclusion $F^{m-1} H^n \supset F^m H^n$, and hence a filtration

$$H^n_\Lambda = F^0 H^n \supset F^1 H^n \supset \cdots \supset F^k H^n \supset \cdots \supset F^n H^n \supset F^{n+1} H^n = \{0\}. \tag{25}$$

This induces a vector space isomorphism

$$H^n_\Lambda \cong \bigoplus_{m=0}^{n} \frac{F^m H^n}{F^{m+1} H^n}. \tag{26}$$

Each element in $\frac{F^m H^n}{F^{m+1} H^n}$ is represented by an element α in $F^m B^n$ such that $\overline{\partial}_\Lambda \alpha = 0$. Let us expand α as the sum of elements in $B^{p,q}$. It is given by

$$\alpha = \sum_{\substack{m \le j \le n \\ j+\ell=n}} \alpha^{j,\ell} = \alpha^{n,0} + \alpha^{n-1,1} + \cdots + \alpha^{m+1,n-m-1} + \alpha^{m,n-m}. \tag{27}$$

Therefore,

$$\begin{aligned}
\overline{\partial}_\Lambda \alpha &= \mathrm{ad}_\Lambda \alpha + \overline{\partial}\alpha \\
&= \mathrm{ad}_\Lambda \alpha^{n,0} + \mathrm{ad}_\Lambda \alpha^{n-1,1} + \cdots + \mathrm{ad}_\Lambda \alpha^{m+1,n-m-1} + \mathrm{ad}_\Lambda \alpha^{m,n-m} \\
&\qquad\qquad \overline{\partial}\alpha^{n,0} + \overline{\partial}\alpha^{n-1,1} + \cdots + \overline{\partial}\alpha^{m+1,n-m-1} + \overline{\partial}\alpha^{m,n-m}.
\end{aligned}$$

Since the complex structure is abelian, $\mathrm{ad}_\Lambda \alpha^{n,0} = 0$. By collecting terms according to the bi-degrees in $B^{p,q}$, we obtain

$$\overline{\partial}_\Lambda \alpha = (\mathrm{ad}_\Lambda \alpha^{n-1,1} + \overline{\partial}\alpha^{n,0}) + \cdots + (\mathrm{ad}_\Lambda \alpha^{m,n-m} + \overline{\partial}\alpha^{m+1,n-m-1}) + \overline{\partial}\alpha^{m,n-m}.$$

Therefore, $\overline{\partial}_\Lambda \alpha = 0$ if and only if

$$\mathrm{ad}_\Lambda \alpha^{n-1,1} + \overline{\partial}\alpha^{n,0} = 0, \ldots, \mathrm{ad}_\Lambda \alpha^{m,n-m} + \overline{\partial}\alpha^{m+1,n-m-1} = 0, \quad \text{and} \quad \overline{\partial}\alpha^{m,n-m} = 0.$$

In particular, $\alpha^{m,n-m}$ represents an element in the cohomology space $H^{m,n-m} = H^{n-m}(\mathfrak{g}^{m,0})$.

We use the notation $[-]_\Lambda$ to indicate equivalence classes with respect to the total cohomology operator $\overline{\partial}_\Lambda = \mathrm{ad}_\Lambda + \overline{\partial}$, and use $[-]_{\overline{\partial}}$ to indicate equivalence classes with respect to $\overline{\partial}$. The observation in the last paragraph indicates that for any $n \geq 0$ and $0 \leq m \leq n$, we could define a map

$$\phi_{m,n-m} : F^m H^n \to H^{m,n-m}, \quad \text{where} \quad \phi_{m,n-m}([\alpha]_\Lambda) = [\alpha^{m,n-m}]_{\overline{\partial}}. \tag{28}$$

To check that this map is well defined, consider α and β in $F^m B^n$ such that $[\alpha]_\Lambda = [\beta]_\Lambda$. As their difference is in $F^m C^n$, there exists $\gamma \in F^m B^{n-1}$ such that $\alpha = \beta + \overline{\partial}_\Lambda \gamma$. As

$$\gamma = \sum_{\substack{m \leq j \leq n-1 \\ j+\ell = n-1}} \gamma^{j,\ell} = \gamma^{n-1,0} + \cdots + \gamma^{m+1,n-1-m-1} + \gamma^{m,n-1-m},$$

and $\overline{\partial}_\Lambda \gamma^{j,\ell} = \mathrm{ad}_\Lambda \gamma^{j,\ell} + \overline{\partial}\gamma^{j,\ell}$, the only term of $\overline{\partial}_\Lambda \gamma$ in $B^{m,n-m}$ is given by $\overline{\partial}\gamma^{m,n-1-m}$. It follows that

$$\alpha^{m,n-m} = \beta^{m,n-m} + \overline{\partial}\gamma^{m,n-1-m},$$

and hence $[\alpha^{m,n-m}]_{\overline{\partial}} = [\beta^{m,n-m}]_{\overline{\partial}}$. In other words, the map $\phi_{m,n-m}$ is well defined at cohomology level.

Furthermore, suppose that $[\alpha]_\Lambda$ is in the kernel of the map $\phi_{m,n-m}$. Then there exists $\gamma^{m,n-m-1} \in B^{m,n-m-1}$ such that $\alpha^{m,n-m} = \overline{\partial}\gamma^{m,n-m-1}$, so

$$\alpha = \alpha^{n,0} + \alpha^{n-1,1} + \cdots + \alpha^{m+1,n-m-1} + \overline{\partial}\gamma^{m,n-m-1}$$

and $\overline{\partial}_\Lambda \alpha = 0$. As $[\alpha]_\Lambda = [\alpha - \overline{\partial}_\Lambda \gamma^{m,n-m-1}]_\Lambda$ and

$$\alpha - \overline{\partial}_\Lambda \gamma^{m,n-m-1} = \alpha^{n,0} + \alpha^{n-1,1} + \cdots + \alpha^{m+1,n-m-1} - \mathrm{ad}_\Lambda \gamma^{m,n-m-1}$$

is contained in $F^{m+1}H^n$, the kernel of $\phi_{m,n-m}$ is contained in $F^{m+1}H^n$. On the other hand, by inspecting the bi-degree, $F^{m+1}H^n$ is contained in the kernel of $\phi_{m,n-m}$, so $F^{m+1}H^n$ is equal to the kernel of the map $\phi_{m,n-m}$.

For each n, by taking the direct sum of linear maps $\phi = (\phi_{n,0}, \ldots, \phi_{m,n-m})$, we obtain the next observation.

Theorem 4 *For any invariant holomorphic Poisson structure Λ on a nilmanifold with an abelian complex structure, there exists an injective map ϕ from H^n_Λ into $\oplus_{p+q=n} H^{p,q} = \oplus_{p+q=n} H^q(\mathfrak{g}^{p,0})$. In particular,*

$$\dim H^n_\Lambda \leq \sum_{p+q=n} \dim H^{p,q} = \sum_{p+q=n} \dim H^q(\mathfrak{g}^{p,0}).$$

Now an obvious question is whether the map ϕ is surjective. On a nilmanifold with abelian complex structure, there is an obvious set of cohomology spaces as seen in Lemma 3, namely $H^{0,q} = H^q(\mathfrak{g}^{0,0}) = B^{0,q}$. Suppose that $\alpha^{0,1}$ is in $B^{0,1}$. If the map ϕ is surjective, then there exists $\alpha^{1,0} \in B^{1,0}$ such that $\mathrm{ad}_\Lambda(\alpha^{1,0} + \alpha^{0,1}) = 0$. It means

$$\mathrm{ad}_\Lambda(\alpha^{1,0}) + \left(\overline{\partial}\alpha^{1,0} + \mathrm{ad}_\Lambda(\alpha^{0,1})\right) + \overline{\partial}\alpha^{0,1} = 0.$$

Since the complex structure is abelian, $\mathrm{ad}_\Lambda \alpha^{1,0} = 0$ and $\overline{\partial}\alpha^{0,1} = 0$. It follows that

$$\overline{\partial}\alpha^{1,0} + \mathrm{ad}_\Lambda \alpha^{0,1} = 0.$$

In other words, $\mathrm{ad}_\Lambda \alpha^{0,1}$ is $\overline{\partial}$-exact, a non-trivial criterion for the holomorphic Poisson spectral sequence degenerates on its first page.

6 A Class of Holomorphic Poisson Structures

Since $\overline{\partial}_\Lambda = \mathrm{ad}_\Lambda + \overline{\partial}$, if the action of ad_Λ on $B^{p,q}$ is equal to zero for all p, q, the action of $\overline{\partial}_\Lambda$ on $B^{p,q}$ is equal to the action of $\overline{\partial}$. In such case, ϕ in Theorem 4 is apparently an isomorphism. This would be the case if Λ is a non-trivial element in $\mathfrak{c}^{2,0} = \wedge^2 \mathfrak{c}^{1,0}$.

From now on, we consider the case when $\dim_{\mathbb{C}} \mathfrak{c}^{1,0} = 1$. In terms of the notation in the decomposition (20), let $\mathfrak{t}_k^{1,0} = \mathfrak{c}^{1,0}$. We will use the notation $\mathfrak{t}^{1,0} = \mathfrak{t}_1^{1,0} \oplus \cdots \oplus \mathfrak{t}_{k-1}^{1,0}$.

Lemma 4 *Suppose that $V \in \mathfrak{t}_k^{1,0}$ and $T \in \mathfrak{t}_{k-1}^{1,0}$. If $\dim_{\mathbb{C}} \mathfrak{c} = 1$, then $\Lambda = V \wedge T$ is a holomorphic Poisson structure.*

Proof: By item 2 of Lemma 2, $\overline{\partial}(V \wedge T) = (\overline{\partial} V) \wedge T - V \wedge (\overline{\partial} T) = -V \wedge (\overline{\partial} T)$. By item 3 of the same lemma $V \wedge (\overline{\partial} T)$ is an element in $\oplus_{a \leq k-1} (\mathfrak{t}_k^{2,0} \otimes \mathfrak{t}_a^{*(0,1)})$. Since $\mathfrak{t}_k^{1,0} = \mathfrak{c}^{1,0}$ when $\dim \mathfrak{c}^{1,0} = 1$, $\mathfrak{t}_k^{2,0} = \{0\}$. Therefore, $V \wedge (\overline{\partial} T)$ vanishes.

Since the complex structure is abelian $[\![V \wedge T, V \wedge T]\!] = 0$, $\Lambda = V \wedge T$ is an invariant holomorphic Poisson structure. ∎

Next, we refine an observation in [6, 24].

Lemma 5 *The Schouten bracket between elements in $\mathfrak{t}_a^{1,0}$ and $\mathfrak{t}_h^{*(0,1)}$ are given below.*

- $[\![\mathfrak{t}_a^{1,0}, \mathfrak{t}_h^{*(0,1)}]\!] = \{0\}$, *if $h \leq a$.*
- $[\![\mathfrak{t}_a^{1,0}, \mathfrak{t}_h^{*(0,1)}]\!] \subset \oplus_{b<h} \mathfrak{t}_b^{*(0,1)}$, *if $h = a+1$.*
- $[\![\mathfrak{t}_a^{1,0}, \mathfrak{t}_h^{*(0,1)}]\!] \subset \mathfrak{t}_{h-1}^{*(0,1)}$, *if $h \geq a+2$.*

Proof: Suppose that $V_a \in \mathfrak{t}_a^{1,0}$, $\overline{V}_b \in \mathfrak{t}_b^{0,1}$ and $\overline{\omega}^h \in \mathfrak{t}_h^{*(0,1)}$, then

$$[\![V_a, \overline{\omega}^h]\!](\overline{V}_b) = (\iota_{V_a} d\overline{\omega}^h)\overline{V}_b = -\overline{\omega}^h([\![V_a, \overline{V}_b]\!]).$$

By Lemma 1, $[\![V_a, \overline{V}_b]\!]$ is contained in $\mathfrak{t}_{\max\{a,b\}+1}^{1,0}$. Therefore, the evaluation $\overline{\omega}^h([\![V_a, \overline{V}_b]\!])$ is non-zero only if $h = \max\{a, b\} + 1$. The first bullet point is now evident. Furthermore, when $h = a+1$, then the evaluation may be non-zero only if $b \leq a$, i.e, $b < h$. It yields the second observation. When $h \geq a+2$, the evaluation may be non-zero only if $\max\{a, b\} + 1 = b + 1$ and hence $b = h - 1$. It yields the last observation of this lemma. ∎

Lemma 6 [4, Corollary 1] *Suppose that $\mathfrak{g}$ is k-step nilpotent, then*

1. *For all m, $[\![\mathfrak{c}^{1,0}, \mathfrak{g}^{*(0,1)}]\!] = \{0\}$.*
2. $[\![\mathfrak{t}^{1,0}_{k-1}, \mathfrak{t}^{*(0,1)}]\!] = \{0\}$.
3. $[\![\mathfrak{t}^{1,0}_{k-1}, \mathfrak{c}^{*(0,1)}]\!] \subseteq \mathfrak{t}^{*(0,1)}$.
4. *Let $V \in \mathfrak{t}^{1,0}_k$ and $T \in \mathfrak{t}^{1,0}_{k-1}$, and $\Lambda = V \wedge T$, then $\mathrm{ad}_\Lambda(\mathfrak{t}^{*(0,1)}) = \{0\}$; and $\mathrm{ad}_\Lambda(\mathfrak{c}^{*(0,1)}) \subseteq \mathfrak{c}^{1,0} \otimes \mathfrak{t}^{*(0,1)}$.*

Proof: With $\mathfrak{c}^{1,0} = \mathfrak{t}^{1,0}_k$ and $\mathfrak{t}^{*(0,1)} = \bigoplus_{1\le\ell\le k-1} \mathfrak{t}^{*(0,1)}_\ell$, the first three items are the direct consequence of Lemma 5. To address the last item, we note that the first item implies that for all ℓ,

$$\mathrm{ad}_\Lambda(\mathfrak{t}^{*(0,1)}_\ell) = V \wedge \mathrm{ad}_T(\mathfrak{t}^{*(0,1)}_\ell).$$

By the first item, the only non-trivial case is when $\ell = k$. The result then follows from the second item. ■

7 Obstruction for Degeneracy

Suppose that $\Lambda = V\wedge T$ as given in the last section, so it is an invariant holomorphic Poisson structure. We will denote the dual element of V by ρ, which then spans $\mathfrak{t}^{*(1,0)}_k = \mathfrak{c}^{*(1,0)}$.

Prior to our investigation on the map ϕ in Theorem 4, we explore the necessary condition for the spectral sequence of the bi-complex of Λ to degenerate on the first page.

Consider the action of ad_Λ on the 0^{th} column on the first page of the spectral sequence.

$$\mathrm{ad}_\Lambda : H^q(\mathfrak{g}^{0,0}) \to H^q(\mathfrak{g}^{1,0}).$$

By Lemma 3, $H^q(\mathfrak{g}^{0,0}) = B^{0,q} = \wedge^q(\oplus_{1\le\ell\le k}\mathfrak{t}^{*(0,1)}_\ell)$. By Lemma 5, the action of ad_Λ on $B^{0,q}$ is equal to zero, except possibly when an element is of the form $\overline{\rho}\wedge\overline{\Omega}$ where $\overline{\Omega}$ is in $\wedge^{q-1}(\oplus_{1\le\ell\le k-1}\mathfrak{t}^{*(0,1)}_\ell)$.

In particular, $\overline{\rho}$ represents an element in $H^1(\mathfrak{g}^{0,0})$. By Lemma 6,

$$\mathrm{ad}_\Lambda \overline{\rho} = [\![V \wedge T, \overline{\rho}]\!] = V \wedge [\![T, \overline{\rho}]\!] \in \mathfrak{c}^{1,0} \otimes \mathfrak{t}^{*(0,1)}$$

It is $\overline{\partial}$-exact only if there exists X in $\mathfrak{t}^{1,0}$ such that

$$V \wedge [\![T, \overline{\rho}]\!] = [\![\Lambda, \overline{\rho}]\!] = \overline{\partial} X. \tag{29}$$

To summarize our computation so far, we note the following.

Lemma 7 *If the holomorphic Poisson spectral sequence for $\Lambda = V \wedge T$ degenerates on the first page, there exists a vector X in $\mathfrak{t}^{1,0}$ such that $\mathrm{ad}_\Lambda \overline{\rho} = \overline{\partial} X$.*

We will examine the conditions under which such a vector X exists. We demonstrate next that its existence is the only obstruction for the holomorphic Poisson spectral sequence to degenerate on its first page.

Given the dimension constraint on $\mathfrak{c}^{1,0}$, Lemma 6 shows that the action of ad_Λ is possibly non-trivial only when it acts on components of type

$$\Upsilon \in \mathfrak{g}^{p,0} \otimes \mathfrak{t}^{*(0,q-1)} \otimes \mathfrak{c}^{*(0,1)}.$$

Given such a Υ, there exist finitely many $\overline{\Omega}_i$ in $\mathfrak{t}^{*(0,q-1)}$ and the same number of Θ_i in $\mathfrak{g}^{p,0}$ such that

$$\Upsilon = \overline{\rho} \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i).$$

Therefore, $\mathrm{ad}_\Lambda \Upsilon$ is equal to

$$[\![\Lambda, \overline{\rho} \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i)]\!] = V \wedge [\![T, \overline{\rho} \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i)]\!] - T \wedge [\![V, \overline{\rho} \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i)]\!].$$

By Lemma 2 and Lemma 6, the last term on the right hand side is identically zero. By Lemma 6 again, the first term on the right hand side is equal to

$$V \wedge [\![T, \overline{\rho}]\!] \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i) = [\![V \wedge T, \overline{\rho}]\!] \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i)$$

If there exists X in $\mathfrak{t}^{1,0}$ such that $[\![V \wedge T, \overline{\rho}]\!] = [\![\Lambda, \overline{\rho}]\!] = \overline{\partial} X$,

$$[\![\Lambda, \Upsilon]\!] = [\![V \wedge T, \overline{\rho}]\!] \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i) = \overline{\partial} X \wedge \left(\sum_i (\Theta_i \wedge \overline{\Omega}_i) \right). \tag{30}$$

Given that $\overline{\rho}$ is $\overline{\partial}$-closed and every element in $B^{0,q-1}$ is $\overline{\partial}$-closed,

$$\overline{\partial}\Upsilon = \overline{\partial}(\overline{\rho} \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i)) = -\overline{\rho} \wedge \overline{\partial}(\sum_i (\Theta_i \wedge \overline{\Omega}_i)) = -\overline{\rho} \wedge (\sum_i (\overline{\partial}\Theta_i) \wedge \overline{\Omega}_i) \quad (31)$$

By Lemma 2, $\overline{\partial}\Theta_i$ does not have any $\overline{\rho}$ component. Therefore, $\overline{\partial}\Upsilon = 0$ if and only if $\overline{\partial}(\sum_i (\Theta_i \wedge \overline{\Omega}_i)) = 0$. It follows that Identity (30) is further transformed to

$$[\![\Lambda, \Upsilon]\!] = \overline{\partial}(X \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i)). \quad (32)$$

Lemma 8 *When* $\mathrm{ad}_\Lambda \overline{\rho} = \overline{\partial} X$, ad_Λ *is a zero map on* $H^{p,q}$ *for all* p, q. *Moreover, if* $\Upsilon = \overline{\rho} \wedge \sum_j (\Theta_j \wedge \overline{\Omega}_j)$ *is* $\overline{\partial}$*-closed, then* $\mathrm{ad}_\Lambda(\Upsilon)$ *is* $\overline{\partial}$*-exact. To be precise,*

$$\mathrm{ad}_\Lambda(\Upsilon) = \overline{\partial}(X \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i)).$$

As a consequence, the necessary condition for the spectral sequence of the bi-complex of Λ to degenerate on the first page is also sufficient.

Theorem 5 *Let* $M = G/\Gamma$ *be a k-step nilmanifold with an abelian complex structure. Let* $\mathfrak{c}$ *be the center of the Lie algebra* $\mathfrak{g}$ *of the simply connected Lie group* G. *Let* $\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{1,0}$ *be the space of invariant* $(1,0)$*-vectors. Assume that* $\dim \mathfrak{c}^{1,0} = 1$. *Suppose that* $\Lambda = V \wedge T$ *where* V *is in* $\mathfrak{c}^{1,0}$ *and* T *is in* $\mathfrak{t}^{1,0}_{k-1}$. *Let* $\overline{\rho}$ *span* $\mathfrak{c}^{*(0,1)}$. *The spectral sequence of the bi-complex of* Λ *degenerates on the first page if and only if* $\mathrm{ad}_\Lambda \overline{\rho}$ *is* $\overline{\partial}$*-exact.*

This theorem answers Question 1 for nilmanifolds with an abelian complex structure.

8 Hodge-Type Decomposition

Given the discussion on degeneracy on first page, the answer to Question 2 becomes useful. We now examine when the map $\phi_{m,n-m}$ is surjective. Suppose that $\alpha^{m,n-m}$ represents a class in $H^{m,n-m}$. In particular, it is in

$$B^{m,n-m} = \mathfrak{g}^{m,0} \otimes \wedge^{n-m}(\mathfrak{t}^{*(0,1)} \oplus \mathfrak{c}^{*(0,1)}) = \mathfrak{g}^{m,0} \otimes (\mathfrak{t}^{*(0,n-m)} \oplus \mathfrak{c}^{*(0,1)} \otimes \mathfrak{t}^{*(0,n-m-1)}).$$

There exists finitely many Θ_i and Π_j in $\mathfrak{g}^{m,0}$, $\overline{\Omega}_i$ in $\mathfrak{t}^{*(0,n-m-1)}$ and $\overline{\Gamma}_j$ in $\mathfrak{t}^{*(0,n-m)}$ such that

$$\alpha^{m,n-m} = \sum_j \Pi_j \wedge \overline{\Gamma}_j + \overline{\rho} \wedge \left(\sum_i \Theta_i \wedge \overline{\Omega}_i \right). \tag{33}$$

By Lemma 2,

$$\begin{aligned} \overline{\partial}\alpha^{m,n-m} &= \overline{\partial}\left(\sum_j \Pi_j \wedge \overline{\Gamma}_j \right) - \overline{\rho} \wedge \overline{\partial}\left(\sum_i \Theta_i \wedge \overline{\Omega}_i \right) \\ &= \sum_j (\overline{\partial}\Pi_j) \wedge \overline{\Gamma}_j - \overline{\rho} \wedge \left(\sum_i (\overline{\partial}\Theta_i) \wedge \overline{\Omega}_i \right). \end{aligned}$$

By the same lemma,

$$\begin{aligned} \overline{\rho} \wedge \left(\sum_i (\overline{\partial}\Theta_i) \wedge \overline{\Omega}_i \right) &\in \mathfrak{g}^{m,0} \otimes \mathfrak{c}^{*(0,1)} \otimes \mathfrak{t}^{*(0,n-m)} \\ (\overline{\partial}\Pi_i) \wedge \overline{\Gamma}_j &\in \mathfrak{g}^{m,0} \otimes \mathfrak{t}^{*(0,n-m+1)}. \end{aligned}$$

As they are in different components, when $\overline{\partial}\alpha^{m,n-m} = 0$, each of them is equal to zero.

$$\overline{\partial}(\sum_j \Pi_j \wedge \overline{\Gamma}_j) = 0, \quad \overline{\rho} \wedge \left(\sum_i (\overline{\partial}\Theta_i) \wedge \overline{\Omega}_i \right) = 0. \tag{34}$$

Therefore, to prove that the map $\phi_{m,n-m}$ is surjective, it suffices to examine two cases independently; that is, either when

$$\alpha^{m,n-m} = \sum_j \Pi_j \wedge \overline{\Gamma}_j \in \mathfrak{g}^{m,0} \otimes \mathfrak{t}^{*(0,n-m)},$$

or when

$$\alpha^{m,n-m} = \overline{\rho} \wedge \left(\sum_i \Theta_i \wedge \overline{\Omega}_i \right) \in \mathfrak{g}^{m,0} \otimes \mathfrak{c}^{*(0,1)} \otimes \mathfrak{t}^{*(0,n-m-1)}.$$

In the former case, Lemma 6 implies $\mathrm{ad}_\Lambda \alpha^{m,n-m} = 0$, and hence $\overline{\partial}_\Lambda \alpha^{m,n-m} = 0$. Therefore, $\alpha^{m,n-m}$ represents a class in $F^m H^n$ such that $\phi_{m,n-m}([\alpha^{m,n-m}]_\Lambda) = [\alpha^{m,n-m}]_{\overline{\partial}}$.

In the latter case, by Identity (34), $\overline{\partial}\alpha^{m,n-m} = 0$ implies that $\sum_i \Theta_i \wedge \overline{\Omega}_i$ is $\overline{\partial}$-closed. Hence,

$$\begin{aligned}
\mathrm{ad}_\Lambda \alpha^{m,n-m} &= \mathrm{ad}_\Lambda \left(\overline{\rho} \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i) \right) = (\mathrm{ad}_\Lambda \overline{\rho}) \wedge \left(\sum_i (\Theta_i \wedge \overline{\Omega}_i) \right) \\
&= (\overline{\partial} X) \wedge \left(\sum_i (\Theta_i \wedge \overline{\Omega}_i) \right) = \overline{\partial} \left(X \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i) \right).
\end{aligned}$$

Define

$$\alpha^{m+1,n-(m+1)} = -X \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i).$$

Since it is in $\mathfrak{g}^{m+1,0} \otimes \mathfrak{t}^{*(0,n-m-1)}$, Lemma 6 implies that $\mathrm{ad}_\Lambda(\alpha^{m+1,n-(m+1)}) = 0$. Therefore,

$$\begin{aligned}
& \overline{\partial}_\Lambda (\alpha^{m+1,n-(m+1)} + \alpha^{m,n-m}) \\
= \ & \mathrm{ad}_\Lambda(\alpha^{m+1,n-(m+1)}) + \overline{\partial}(\alpha^{m+1,n-(m+1)}) + \mathrm{ad}_\Lambda(\alpha^{m,n-m}) + \overline{\partial}(\mathrm{ad}_\Lambda(\alpha^{m,n-m})) \\
= \ & -\overline{\partial} \left(X \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i) \right) + \overline{\partial} \left(X \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i) \right) = 0.
\end{aligned}$$

Finally,

$$\alpha^{m+1,n-(m+1)} + \alpha^{m,n-m} = -X \wedge \sum_i (\Theta_i \wedge \overline{\Omega}_i) + \overline{\rho} \wedge \left(\sum_i \Theta_i \wedge \overline{\Omega}_i \right)$$

represents a class in $F^m H^n$ such that $\phi_{m,n-m}([\alpha^{m+1,n-(m+1)} + \alpha^{m,n-m}]_\Lambda) = [\alpha^{m,n-m}]_{\overline{\partial}}$, and the map $\phi_{m,n-m}$ is surjective.

Theorem 6 *Let $M = G/\Gamma$ be a k-step nilmanifold with an invariant abelian complex structure. Let $\mathfrak{g}$ be the Lie algebra of the covering group, $\mathfrak{c}$ the center. Let V be in $\mathfrak{c}^{1,0}$ and ρ be the dual of V. Let T be in $\mathfrak{t}^{1,0}_{k-1}$. If $\mathrm{ad}_\Lambda(\overline{\rho})$ is $\overline{\partial}$-exact, the holomorphic Poisson cohomology of $\Lambda = V \wedge T$ has a Hodge-type decomposition:*

$$H^n_\Lambda \cong \oplus_{p+q=n} H^q(M, \Theta^p).$$

9 $\overline{\partial}$-Exactness of $\mathrm{ad}_\Lambda \overline{\rho}$

In this section, we explore when there exists X such that

$$[\![V \wedge T, \overline{\rho}]\!] = \mathrm{ad}_\Lambda \overline{\rho} = \overline{\partial} X. \tag{35}$$

On the left of the equality above,

$$[\![V \wedge T, \overline{\rho}]\!] = V \wedge [\![T, \overline{\rho}]\!] = V \wedge \iota_T d\overline{\rho}. \tag{36}$$

To compute $\overline{\partial} X$, we applies the Cartan formula to evaluate $\overline{\partial} X$ on a generic element $\overline{Y}$ in $\mathfrak{t}^{0,1}$ and on ρ.

$$\begin{aligned}
\overline{\partial} X(\rho, \overline{Y}) &= -X([\![\rho, \overline{Y}]\!]) = X([\![\overline{Y}, \rho]\!]) = X(\iota_{\overline{Y}} d\rho) \\
&= d\rho(\overline{Y}, X) = -d\rho(X, \overline{Y}) = -(\iota_X d\rho)(\overline{Y}).
\end{aligned}$$

Therefore, $\overline{\partial} X = -V \wedge \iota_X d\rho$. Now comparing (35) with (36), we obtain a rather simple identity below.

$$\iota_T d\overline{\rho} = -\iota_X d\rho. \tag{37}$$

Since the complex structure J is abelian, $d\rho$ is a type $(1,1)$-form. So is $d\overline{\rho}$. We could treat their contractions with elements in $\mathfrak{t}^{1,0}$ as linear maps.

$$d\rho, \quad d\overline{\rho} \quad : \mathfrak{t}^{1,0} \to \mathfrak{t}^{*(0,1)}. \tag{38}$$

Suppose that $d\overline{\rho}$ has a non-trivial kernel and $\iota_T d\overline{\rho} = 0$. In such case, $\mathrm{ad}_{V \wedge T} \overline{\rho} = 0$. However, since $d\overline{\omega}^j = 0$ for all j, $\mathrm{ad}_{V \wedge T} \overline{\omega}^j = 0$. As the complex structure is abelian, the adjoint action of $V \wedge T$ on $\mathfrak{g}^{1,0}$ is identically zero. Therefore, $\mathrm{ad}_{V \wedge T}$ is identically zero on $B^{p,q}$ for all $p, q \geq 0$. In such a case, the action of ad_Λ is trivial. Hence as an action on $B^{p,q}$, $\overline{\partial}_\Lambda = \overline{\partial}$. The Poisson cohomology is simply the direct sum of Dolbeault cohomology,

$$H^n_\Lambda(M) \cong \bigoplus_{p+q=n} H^q(\mathfrak{g}^{p,0}) = \bigoplus_{p+q=n} H^q(M, \Theta^p).$$

On the other hand, $d\rho(\mathfrak{t}^{1,0})$ is a proper subspace of $\mathfrak{t}^{*(0,1)}$ if $d\rho$ degenerates. If T is not in the kernel of $d\overline{\rho}$ and if $\iota_T d\overline{\rho}$ is in the complement of $d\rho(\mathfrak{t}^{1,0})$, Equation (37) does not have a solution. In the next section, we will present an example to demonstrate that such situation does occur. We summarize our discussion when $d\rho$ degenerates as below.

Proposition 2 *If $d\overline{\rho}$ degenerates and T is in its kernel, then $\overline{\partial}_\Lambda = \overline{\partial}$ for $\Lambda = V \wedge T$. In such a case, the holomorphic Poisson cohomology has a Hodge-type composition.*

If $d\overline{\rho}$ is non-degenerate, then so is $d\rho$. Therefore, for any T the equation (37) always has a unique solution.

Proposition 3 *Let $M = G/\Gamma$ be a k-step nilmanifold with an abelian complex structure. Let $\mathfrak{c}$ be the center of the Lie algebra $\mathfrak{g}$ of the simply connected Lie group G. Assume that $\dim \mathfrak{c}^{1,0} = 1$. Let V span $\mathfrak{c}^{1,0}$ and ρ span the dual space. If $d\rho$ is non-degenerate, then for any T in $\mathfrak{t}^{1,0}_{k-1}$ and $\Lambda = V \wedge T$, $\mathrm{ad}_\Lambda(\overline{\rho})$ is $\overline{\partial}$-exact. In particular, the holomorphic Poisson cohomology for such Λ has a Hodge-type decomposition.*

10 Examples

We now focus on 2-step nilmanifolds. Let $\mathfrak{t} = \mathfrak{g}/\mathfrak{c}$. Below are some facts shown in Sections 2 and 3 of [21]. Since $\mathfrak{g}$ is 2-step nilpotent, $\mathfrak{t}$ is abelian. As a vector space, $\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{1,0}$, and $\mathfrak{g}^{*(1,0)} = \mathfrak{t}^{*(1,0)} \oplus \mathfrak{c}^{*(1,0)}$. The only non-trivial Lie brackets in $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ are of the form $[\mathfrak{t}^{1,0}, \mathfrak{t}^{0,1}] \subset \mathfrak{c}^{1,0} \oplus \mathfrak{c}^{0,1}$. We assume that $\dim_{\mathbb{C}} \mathfrak{c}^{1,0} = 1$.

Explicitly, there exists a real basis $\{X_k, JX_k : 1 \leq k \leq n\}$ for $\mathfrak{t}$ and $\{Z, JZ\}$ a real basis for $\mathfrak{c}$. The corresponding complex bases for $\mathfrak{t}^{1,0}$ and $\mathfrak{c}^{1,0}$ are respectively composed of the following elements:

$$T_k = \frac{1}{2}(X_k - iJX_k) \quad \text{and} \quad V = \frac{1}{2}(Z - iJZ). \tag{39}$$

The structure equations of $\mathfrak{g}$ are determined by

$$[\![\overline{T}_k, T_j]\!] = E_{kj}V - \overline{E}_{jk}\overline{V} \tag{40}$$

for some constants E_{kj}. Let $\{\omega^k : 1 \leq k \leq n\}$ be the dual basis for $\mathfrak{t}^{*(1,0)}$, and let $\{\rho\}$ be the dual basis for $\mathfrak{c}^{*(1,0)}$. The dual structure equations for (40) are

$$d\rho = \sum_{i,j} E_{ji}\omega^i \wedge \overline{\omega}^j \quad \text{and} \quad d\omega^k = 0. \tag{41}$$

Equivalently,

$$d\overline{\rho} = -\sum_{i,j} \overline{E}_{ji}\omega^j \wedge \overline{\omega}^i \quad \text{and} \quad d\overline{\omega}^k = 0. \tag{42}$$

It follows that

$$[\![T_j, \overline{\rho}]\!] = \mathcal{L}_{T_j}\overline{\rho} = \iota_{T_j} d\overline{\rho} = -\sum_i \overline{E}_{ji}\overline{\omega}^i. \tag{43}$$

By Cartan formula (1),

$$\overline{\partial} T_j = \sum_k E_{kj}\overline{\omega}^k \wedge V = \left(\sum_k E_{kj}\overline{\omega}^k\right) \wedge V. \tag{44}$$

Example 1. Consider a one-dimensional central extension of the Heisenberg algebra $\mathfrak{h}_{2n+1}$ of real dimension $2n+1$. Let $\{X_j, Y_j, Z, A : 1 \leq j \leq n\}$ be basis with structure equations

$$[\![X_j, Y_j]\!] = -[\![Y_j, X_j]\!] = Z, \quad \text{for all} \quad 1 \leq j \leq n. \tag{45}$$

The real center $\mathfrak{c}$ is spanned by Z and A. Define an almost complex structure by

$$JX_j = Y_j, \quad JY_j = -X_j, \quad JZ = A, \quad JA = -Z.$$

It is an abelian complex structure. Let $V = \frac{1}{2}(Z - iA)$ and $T_j = \frac{1}{2}(X_j - iY_j)$, then the complex structure equations become

$$[\![\overline{T}_j, T_j]\!] = -\frac{i}{2}(V + \overline{V}).$$

Therefore, $E_{jj} = -\frac{i}{2} = -\overline{E}_{jj}$. Hence $d\rho$ is non-degenerate and serves as an example for Theorem 5 and Proposition 3.

Example 2. On the direct sum of two Heisenberg algebras $\mathfrak{h}_{2m+1} \oplus \mathfrak{h}_{2n+1}$, choose a basis $\{X_j, Y_j, Z, A_k, B_k, C; 1 \leq j \leq m, 1 \leq k \leq n\}$ so that the non-zero structure equations are given by

$$[\![X_j, Y_j]\!] = -[\![Y_j, X_j]\!] = Z, \quad [\![A_k, B_k]\!] = -[\![B_k, A_k]\!] = C. \tag{46}$$

Define an almost complex structure J by

$$JX_j = Y_j, \quad JA_k = B_k, \quad JZ = C.$$

In fact, this defines an abelian complex structure. Let

$$V = \frac{1}{2}(Z - iC), \quad S_j = \frac{1}{2}(X_j - iY_j), \quad T_k = \frac{1}{2}(A_k - iB_k),$$

so $\mathfrak{c}^{1,0}$ is spanned by V. It follows that the non-zero complex structure equations are given as below

$$[\![\overline{S}_j, S_j]\!] = -\frac{i}{2}(V+\overline{V}), \quad [\![\overline{T}_k, T_k]\!] = \frac{1}{2}(V-\overline{V}).$$

We then have the structure constants

$$E_{jj} = -\frac{i}{2} = -\overline{E}_{jj}, \quad \text{and} \quad E_{kk} = \frac{1}{2} = \overline{E}_{kk}$$

for all $1 \le j \le m$ and $1 \le k \le n$. It is now obvious that $d\rho$ is non-degenerate and serves as example for Theorem 5 and Proposition 3.

Example 3. Consider a real vector space P_{4n+2} spanned by

$$\{X_{4k+1}, X_{4k+2}, X_{4k+3}, X_{4k+4}, Z_1, Z_2; 0 \le k \le n-1\}.$$

Define a Lie bracket by

$$[\![X_{4k+1}, X_{4k+2}]\!] = -\frac{1}{2}Z_1, \quad [\![X_{4k+1}, X_{4k+4}]\!] = -\frac{1}{2}Z_2, \quad [\![X_{4k+2}, X_{4k+3}]\!] = -\frac{1}{2}Z_2.$$

Define an abelian complex structure J on P_{4n+2} by

$$JX_{4k+1} = X_{4k+2}, \quad JX_{4k+3} = -X_{4k+4}, \quad JZ_1 = -Z_2,$$

and define

$$V = \frac{1}{2}(Z_1 + iZ_2), \quad T_{2k+1} = \frac{1}{2}(X_{4k+1} - iX_{4k+2}), \quad T_{2k+2} = \frac{1}{2}(X_{4k+3} + iX_{4k+4}).$$

The non-zero structure equations in terms of complex vectors are

$$[\![\overline{T}_{2k+1}, T_{2k+1}]\!] = \frac{i}{4}(V+\overline{V}); \quad [\![\overline{T}_{2k+1}, T_{2k+2}]\!] = -\frac{1}{4}(V-\overline{V}).$$

Hence

$$E_{2k+1,2k+1} = \frac{i}{4}, \quad E_{2k+1,2k+2} = -\frac{1}{4}, \quad E_{2k+2,2k+1} = -\frac{1}{4}.$$

It follows that $d\rho$ is non-degenerate, and hence (37) is solvable, providing another example for Theorem 5 and Proposition 3.

Example 4. Consider a real vector space W_{4n+6} spanned by

$$\{X_{4k+1}, X_{4k+2}, X_{4k+3}, X_{4k+4}, Z_1, Z_2; 0 \leq k \leq n\}.$$

Define a Lie bracket by

$$\begin{aligned} [\![X_{4k+1}, X_{4k+3}]\!] &= -\frac{1}{2}Z_1, \qquad & [\![X_{4k+1}, X_{4k+4}]\!] &= -\frac{1}{2}Z_2, \\ [\![X_{4k+2}, X_{4k+3}]\!] &= -\frac{1}{2}Z_2, \qquad & [\![X_{4k+2}, X_{4k+3}]\!] &= \frac{1}{2}Z_1. \end{aligned}$$

One can define an abelian complex structure J by

$$JX_{4k+1} = X_{4k+2}, \quad JX_{4k+3} = -X_{4k+4}, \quad JZ_1 = -Z_2.$$

For $0 \leq k \leq n$, define

$$V = \frac{1}{2}(Z_1 + iZ_2), \quad T_{2k+1} = \frac{1}{2}(X_{4k+1} - iX_{4k+2}), \quad T_{2k+2} = \frac{1}{2}(X_{4k+3} + iX_{4k+4}).$$

It is now a straightforward computation to show that the non-zero structure equations are

$$[\![\overline{T}_{2k+1}, T_{2k+2}]\!] = -\frac{1}{2}V, \quad [\![\overline{T}_{2k+2}, T_{2k+1}]\!] - = \frac{1}{2}\overline{V}. \tag{47}$$

Except when

$$E_{2k+1,2k+2} = -\frac{1}{2}, \quad \text{for all} \quad 0 \leq k \leq n,$$

all other structure constants are equal to zero. In particular,

$$d\rho = -\frac{1}{2}\sum_{k=0}^{n} \omega^{2k+2} \wedge \overline{\omega}^{2k+1}, \quad \text{and} \quad d\overline{\rho} = \frac{1}{2}\sum_{k=0}^{n} \omega^{2k+1} \wedge \overline{\omega}^{2k+2}. \tag{48}$$

Moveover,

$$\overline{\partial} T_{2k+1} = 0, \quad \overline{\partial} T_{2k+2} = -\frac{1}{2}\overline{\omega}^{2k+1} \wedge V. \tag{49}$$

Treating $d\rho$ and $d\overline{\rho}$ as maps from $\mathfrak{t}^{1,0}$ to $\mathfrak{t}^{0,1}$, their image spaces are transversal. Therefore, given $T \in \mathfrak{t}^{1,0}$ such that there exists $X \in \mathfrak{t}^{1,0}$ with $\iota_T d\overline{\rho} = -\iota_X d\rho$ only if $\iota_T d\overline{\rho} = 0$. It is possible only when T is the complex linear span of $\{T_2, \ldots, T_{2n+2}\}$. In this case, it simply means that $[\![T, \overline{\rho}]\!] = 0$, and hence $\mathrm{ad}_\Lambda \overline{\rho} = 0$. This example illustrates the situation in Proposition 2 as well as the non-trivial conditions in Theorem 5.

It is also apparent that if we choose $\Lambda = V \wedge T_{2k+1}$, then

$$\mathrm{ad}_\Lambda(\overline{\rho}) = V \wedge \iota_{T_{2k+1}} d\overline{\rho} = -\frac{1}{2} V \wedge \overline{\omega}^{2k+2},$$

which is not $\overline{\partial}$-exact. Therefore, the map ϕ in Theorem 4 fails to be surjective and the holomorphic Poisson spectral sequence fails to degenerate on its first page.

In this example, if we vary T from $\{T_2, \ldots, T_{2n+2}\}$ to $\{T_1, \ldots, T_{2n+1}\}$ and define $\Lambda = V \wedge T$, the dimension of the cohomology $H^\bullet_\Lambda$ jumps down.

Consider $\Lambda = V \wedge T$ where T is in the linear span of $\{T_2, \ldots, T_{2n+2}\}$. The map ϕ is then an identity map. In particular,

$$H^1_\Lambda = H^0(\mathfrak{g}^{1,0}) \oplus H^1(\mathfrak{g}^{0,0}). \tag{50}$$

From the structural equations (49),

$$H^0(\mathfrak{g}^{1,0}) = \{V, T_1, \ldots, T_{2n+1}\}, \quad H^1(\mathfrak{g}^{0,0}) = \mathfrak{g}^{*(0,1)}. \tag{51}$$

Therefore,

$$\dim H^1_\Lambda = \dim H^0(\mathfrak{g}^{1,0}) + \dim H^1(\mathfrak{g}^{0,0}) = (n+2) + (2n+3) = 3n+5. \tag{52}$$

A Further Observation. We can consider the virtual parameter space of generalized complex deformations, utilizing the previous example. For the second hypercohomology, we have the Hodge-like decomposition

$$H^2_\Lambda = H^0(\mathfrak{g}^{2,0}) \oplus H^1(\mathfrak{g}^{1,0}) \oplus H^2(\mathfrak{g}^{0,0}). \tag{53}$$

We could consider $H^0(\mathfrak{g}^{2,0}) = H^0(M, \Theta^{2,0})$ as deformations of holomorphic Poisson structures by varying Λ in the space of holomorphic bivector fields without changing the complex structure. Moreover, we could consider $H^1(\mathfrak{g}^{1,0}) = H^1(M, \Theta^{1,0})$ to be the virtual parameter space of classical deformation of complex structure. It remains to examine the nature of $H^2(\mathfrak{g}^{0,0}) = H^2(M, \mathcal{O})$.

By Lemma 3

$$H^2(\mathfrak{g}^{0,0}) = \mathfrak{g}^{*(0,2)} = \mathfrak{c}^{*(0,1)} \otimes \mathfrak{t}^{*(0,1)} \oplus \mathfrak{t}^{*0,2}. \tag{54}$$

Note that every element in $\mathfrak{t}^{*0,2}$ is not only $\overline{\partial}$-closed but also d-closed. Therefore, the deformations that they generate are B-field transformations of Λ as a generalized complex structure. In terms of generalized complex geometry, they are equivalent to the given holomorphic Poisson structure $\Lambda = V \wedge T$ [11]. However, elements in $\mathfrak{c}^{*(0,1)} \otimes \mathfrak{t}^{*(0,1)}$ are $\overline{\partial}$-closed but not d-closed. For instance, consider $\overline{\Omega} = \overline{\rho} \wedge \overline{\omega}^{2h+1}$ where ω^{2h+1} is dual to T_{2h+1}. As it satisfies both $\overline{\partial}_\Lambda \overline{\Omega} = 0$ and $[\![\overline{\Omega}, \overline{\Omega}]\!] = 0$, it defines an integrable generalized deformation of Λ [11]. The corresponding Lie algebroid differential after deformation is given by

$$\overline{\delta} = \overline{\partial}_\Lambda + [\![\overline{\Omega}, -]\!]. \tag{55}$$

As both $\Lambda = V \wedge T$ and $\overline{\omega}^{2h+1}$ are in the center of Schouten algebra $\oplus_{p,q} B^{p,q}$,

$$\overline{\delta} = \overline{\partial} - \overline{\omega}^{2h+1} \wedge [\![\overline{\rho}, -]\!]. \tag{56}$$

Therefore, every element in $\mathfrak{g}^{*(0,1)}$ is in the kernel of $\overline{\delta}$. Next we examine the action of $\overline{\delta}$ on $\mathfrak{g}^{1,0}$. Apparently, $\overline{\delta} V = 0$. For each k, according to (49)

$$\begin{aligned} \overline{\delta} T_{2k+1} &= \overline{\partial} T_{2k+1} - \overline{\omega}^1 \wedge [\![\overline{\rho}, T_{2k+1}]\!] = \overline{\omega}^{2h+1} \wedge [\![T_{2k+1}, \overline{\rho}]\!] = \frac{1}{2} \overline{\omega}^{2h+1} \wedge \overline{\omega}^{2k+2} \\ \overline{\delta} T_{2k+2} &= \overline{\partial} T_{2k+2} - \overline{\omega}^{2h+1} \wedge [\![\overline{\rho}, T_{2k+2}]\!] = \overline{\partial} T_{2k+2} = -\frac{1}{2} \overline{\omega}^{2k+1} \wedge V. \end{aligned}$$

In particular, on the first cohomology level

$$\ker \overline{\delta} = \mathfrak{c}^{1,0} \oplus \mathfrak{g}^{*(0,1)}. \tag{57}$$

In view of the result in (52), the first cohomology changes after deformation. The class $\overline{\Omega} = \overline{\rho} \wedge \overline{\omega}^{2h+1}$ defines a non-trivial generalized complex deformation of $\Lambda = V \wedge T$.

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References

[1] M. Bailey, *Local classification of generalize complex structures*, J. Differential Geom. **95** (2013), 1–37.

[2] W. Barth, C. Peters & A. Van de Ven, *Compact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag (1984) Berlin.

[3] M. L. Barberis, I. G. Dotti Miatello & R. J. Matello, On certain locally homogeneous Clifford manifolds, Ann. Glob. Anal. Geom. **13** (1995) 289–301.

[4] Z. Chen, A. Fino & Y. S. Poon, *Holomorphic Poisson structures and its cohomology on nilmanifolds*, Differential Geom. Appl., **44** (2016), 144–160.

[5] Z. Chen, D. Grandini & Y. S. Poon, *Cohomology of holomorphic Poisson structures*, Complex Manifolds, **2** (2015), 34–52.

[6] S. Console & A. Fino, *Dolbeault cohomology of compact nilmanifolds*, Transform. Groups, **6** (2001), 111–124.

[7] P. Gauduchon, *Hermitian connections and Dirac operators*, Bollettino U.M.I. **11B** (1997), 257–288.

[8] R. Goto, *Deformations of generalized complex and generalized Kähler structures*, J. Differential Geom. **84** (2010), 525–560.

[9] R. Goto, *Unobstucted deformations of generalized complex structures induced by C^∞ logarithmic symplectic structures and logarithmic Poisson structures*, in *Geometry and Topology of manifolds*, Springer Proc. Math. Stat., **154** (2016), 159–183. Preprint in arXiv:1501.03398v1.

[10] D. Grandini, Y. S. Poon & B. Rolle, *Differential Gerstenhaber algebras of generalized complex structures*, Asian J. Math. **18** (2014), 191–218.

[11] M. Gualtieri, *Generalized complex geometry*, Ann. of Math. **174** (2011), 75–123.

[12] N. J. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math. **54** (2003), 281–308.

[13] N. J. Hitchin, *Instantons, Poisson structures, and generalized Kähler geometry*, Commun. Math. Phys. **265** (2006), 131–164.

[14] N. J. Hitchin, *Deformations of holomorphic Poisson manifolds*, Mosc. Math. J. **669** (2012), 567–591.

[15] W. Hong, *Poisson cohomology of holomorphic toric Poisson manifolds*, preprint (2016) arXiv:1611.08485.

[16] W. Hong & P. Xu, *Poisson cohomology of Del Pezzo surfaces*, J. Algebra **336** (2011), 378–390.

[17] C. Laurent-Gengoux, M. Stiéson & P. Xu, *Holomorphic Poisson manifolds and holomorphic Lie algebroids*, Int. Math. Res. Not. IMRN, (2008). doi: 10.1093/imrn/rnn088

[18] A. Lichnerowicz, *Les variétés de Poisson et leurs algébres de Lie associées*, J. Differential Geom. **12** (1977), 253–300.

[19] Z. J. Liu, A. Weinstein & P. Xu, *Manin triples for Lie bialgebroids*, J. Differential Geom. **45** (1997), 547–574.

[20] K. C. H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, London Math. Soc. Lecture Notes Series **213**, Cambridge U Press, 2005.

[21] C. Maclaughlin, H. Pedersen, Y. S. Poon & S. Salamon, *Deformation of 2-step nilmanifolds with abelian complex structures*, J. London Math. Soc. **73** (2006) 173–193.

[22] Y. S. Poon, *Extended deformation of Kodaira surfaces*, J. reine angew. Math. **590** (2006), 45–65.

[23] B. Rolle, *Construction of weak mirrir pairs by deformations*, Ph.D. Thesis, University of California at Riverside. (2011).

[24] S. M. Salamon, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), 311–333.

[25] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progress in Mathematics **118** (1994) Birhäuser. ISBN 3-7643-5016-4.

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