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Generalised Connected Sums of Quaternionic Manifolds

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Abstract: Using deformations of singular twistor spaces, a generalisation of the connected sum construction appropriate for quaternionic manifolds is introduced. This is used to construct examples of quaternionic manifolds which have no quaternionic symmetries and leads to examples of quaternionic manifolds whose twistor spaces have arbitrary algebraic dimension.

Key words: *Quaternionic manifold, twistor space, deformation theory, algebraic dimension*
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1. Introduction

In Donaldson & Friedman [5] new self-dual structures were constructed on certain connected sums of self-dual manifolds. In particular, they substantially generalised a result of Poon [17] to show that $n\,\mathbb{C}P(2) = \mathbb{C}P(2)\#\cdots\#\,\mathbb{C}P(2)$ has a self-dual structure derived from the self-dual structures on each $\mathbb{C}P(2)$, a result which was also obtained by analytic methods by Floer [6] and dealt with explicitly by LeBrun [13]. The work of Donaldson & Friedman [5] has the appealing feature of giving a complete obstruction theory for this problem. This has recently been complemented by Taubes [21] who shows that if M is any four-manifold then for sufficiently large k, $M\#k\mathbb{C}P(2)$ is self-dual.

In higher dimensions the geometry most closely related to self-duality is quaternionic geometry in the sense of Salamon [19]. Here one studies $4n$-manifolds with structure group $GL(n,\mathbb{H})GL(1,\mathbb{H})$ which admit a compatible torsion-free connection. Such a manifold M is naturally the base space of a non-holomorphic fibration of an S^2-bundle Z over M whose total space admits a natural complex structure. The total space Z is known as the twistor space of M and its complex and real structures together determine the quaternionic structure of the base. The simplest example of a quaternionic manifold is quaternionic projective space $\mathbb{H}P(n)$ which has twistor space $\mathbb{C}P(2n+1)$. Donaldson & Friedman's construction starts with a singular model for the twistor space of the connected sum built from the blow-ups of the twistor spaces of the four-dimensional summands along a twistor line. They then proceed to show that this singular model can be smoothed in the category of twistor spaces. To try and extend this construction to quaternionic manifolds one

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needs a different notion of connected sum. One reason is that the space S^{4n}, which acts as the identity for connected sums, no longer admits a quaternionic structure if $n > 1$ (Gray & Green [8]). Secondly, for this construction the nature of the twistor space is crucial. The four sphere S^4 has twistor space $\mathbb{C}P(3)$, so it is more natural to replace S^{4n} by the model space $\mathbb{H}P(n)$ whose twistor space is $\mathbb{C}P(2n+1)$. This choice is also natural from the point of view of curvature, since $\mathbb{H}P(n)$ is the flat model in the category of quaternionic manifolds.

The usual connected sum construction involves choosing a point in each manifold; a neighbourhood of each point is then removed and the boundaries of the resulting manifolds are identified with opposite orientations. The construction we wish to study is as follows. Let M_1 and M_2 be two quaternionic manifolds of dimension $8n+4$. Suppose each manifold contains an embedded copy of $\mathbb{H}P(n)$ such that the real projectivised normal bundles are isomorphic, then we may remove a disc bundle of each normal bundle and identify the boundaries of the resulting manifolds. This we will call the *generalised connected sum* $M_1 \#_{\mathbb{H}P} M_2$ of M_1 and M_2. Note that this type of construction is well-known to topologists as a connected sum along submanifolds (see for example Kosinski [12]).

The discussion in Section 2 shows that this definition is promising, since the projective space $\mathbb{H}P(2n+1)$ acts as the identity for generalised connected sums. We then go on to discuss the analogue of Donaldson & Friedman's deformation theory. It turns out that the twistor spaces of the embedded $\mathbb{H}P(n)$'s have normal bundle $(2n+2)\mathcal{O}(1) \to \mathbb{C}P(2n+1)$ and so the embedded $\mathbb{C}P(2n+1)$ behaves much like a twistor line in the original theory. The simplest case to deal with is when the group $H^2(Z_i, \Theta_i)$ (where Θ_i is the holomorphic tangent bundle) vanishes. Even when this group does not vanish the construction goes through, but the proof is somewhat harder and is not discussed here, as we do not require it for our examples. The lack of obstructions is in agreement with a result of Griffiths [9] which implies that a neighbourhood of the embedded $\mathbb{C}P(2n+1)$ is biholomorphic to an open set in $\mathbb{C}P(4n+3)$. From our construction we obtain quaternionic structures on k-fold generalised connected sums $M_{(k)} = M_{(1)} \#_{\mathbb{H}P} \cdots \#_{\mathbb{H}P} M_{(1)}$, where $M_{(1)}$ is an $(8n+4)$-dimensional quaternionic analogue of the Hopf surface $S^1 \times S^3$ obtainable as a generalised connected sum of $\mathbb{H}P(2n+1)$ with itself. These structures include the first explicit examples of compact quaternionic manifolds without any quaternionic symmetries. Other such examples are implicit in the work of Joyce [10, 11]. Modifying the basic building block $M_{(1)}$, we show that the twistor space of a compact quaternionic manifold may have arbitrary algebraic dimension. This contrasts strongly with a result of Pontecorvo [16] for the twistor spaces of quaternionic Kähler manifolds.

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2. The Flat Model

Here we consider forming the generalised connected sum of two quaternionic projective spaces. This is the model for the generalised connected sum construction described in the next section. The twistor space of $\mathbb{H}P(2n+1)$ is $\mathbb{C}P(4n+3)$ with the twistor fibration $\mathbb{C}P(4n+3) \to \mathbb{H}P(2n+1)$ given by $[z_0, z_1, \ldots] \mapsto [z_0 + jz_1, \ldots]$. Choosing a standard copy of $\mathbb{H}P(n)$ in $\mathbb{H}P(2n+1)$ gives a copy of $\mathbb{C}P(2n+1)$ linearly embedded in $\mathbb{C}P(4n+3)$ as the first $(2n+2)$-coordinates. Let $\tilde{Z}$ denote the blow-up of $\mathbb{C}P(4n+3)$ along this $\mathbb{C}P(2n+1)$. This blow-up naturally lives in $\mathbb{C}P(4n+3) \times \mathbb{C}P(2n+1)$ as the variety

$$c_i b_j = c_j b_i,$$

where $[a_0, \ldots, a_{2n+1}, b_0, \ldots, b_{2n+1}] \in \mathbb{C}P(4n+3)$ and $[c_0, \ldots, c_{2n+1}] \in \mathbb{C}P(2n+1)$ are homogeneous coordinates and i, j take all possible values. Now $\mathbb{C}P(4n+3) \times \mathbb{C}P(2n+1)$ may be embedded in $\mathbb{C}P(8(n+1)^2 - 1)$ via the Segré map s

$$s([a_i, b_j], [c_k]) = [a_p c_q, b_r c_s].$$

However under this map the defining equations for $\tilde{Z}$ become linear equations and $s(\tilde{Z})$ lies in a $\mathbb{C}P(6(n+1)^2 + n)$. Note that the image $s(Q)$ of the exceptional divisor $Q = \{ b_i = 0 \} \cong \mathbb{C}P(2n+1) \times \mathbb{C}P(2n+1)$ lies in a $(4(n+1)^2 - 1)$-dimensional linear subspace. We form the singular space with normal crossing $Z = \tilde{Z}_1 \cup_Q \tilde{Z}_2$ by taking two copies of this construction and identifying the exceptional divisors by swapping the two $\mathbb{C}P(2n+1)$-factors. The resulting space embeds in $\mathbb{C}P(8n^2 + 18n + 9)$ which is precisely the target space for the Veronese map of degree two on $\mathbb{C}P(4n+3)$ (see below).

Proposition 2.1. *If $\tilde{Z}_1$ and $\tilde{Z}_2$ are two copies of the blow-up of $\mathbb{C}P(4n+3)$ along a linearly embedded $\mathbb{C}P(2n+1)$, then the resulting singular space with normal crossing $Z = \tilde{Z}_1 \cup_Q \tilde{Z}_2$ may be smoothed to $\mathbb{C}P(4n+3)$.*

Proof. Choose homogeneous coordinates $[u_{ij}, v_{pq}, w_{rs}]$ on $\mathbb{C}P(8n^2+18n+9)$, where the indices run over $\{0, \ldots, 2n+1\}$ but with $p \leq q$ and $r \leq s$. Then the embedding of $\tilde{Z}_1$ is given by

$$u_{ij} = a_i c_j, \qquad v_{pq} = b_p c_q, \qquad w_{rs} = 0$$

and that of $\tilde{Z}_2$ by

$$u_{ij} = a'_j c'_i, \qquad v_{pq} = 0, \qquad w_{rs} = b'_r c'_s.$$

Thus, the singular variety Z is the union of

$$u_{ij} u_{kl} = u_{il} u_{kj}, \qquad u_{ij} v_{pq} = u_{iq} v_{pj}, \qquad v_{pq} v_{rs} = v_{ps} v_{rq}, \qquad w_{rs} = 0$$

and

$$u_{ij} u_{kl} = u_{il} u_{kj}, \qquad v_{rs} = 0, \qquad u_{ij} w_{rs} = u_{is} w_{rj}, \qquad w_{pq} w_{rs} = w_{ps} w_{rq}.$$

Consider the one-parameter family of varieties V_λ defined by

$$\begin{aligned} u_{ij} u_{kl} &= u_{il} u_{kj}, \\ u_{ij} v_{pq} = u_{iq} v_{pj}, &\qquad v_{pq} v_{rs} = v_{ps} v_{rq}, \\ u_{ij} w_{rs} = u_{is} w_{rj}, &\qquad w_{pq} w_{rs} = w_{ps} w_{rq}, \\ v_{pq} w_{rs} &= \lambda u_{pr} u_{qs}. \end{aligned}$$

When $\lambda = 0$ this coincides with the singular variety Z. However, V_1 is precisely the image of the Veronese map $\phi : \mathbb{C}P(4n+3) \to \mathbb{C}P(8n^2+18n+9)$ defined by

$$u_{ij} = d_i e_j, \qquad v_{pq} = d_p d_q, \qquad w_{rs} = e_r e_s,$$

where $[d_0, \dots, d_{2n+1}, e_0, \dots, e_{2n+1}]$ are homogeneous coordinates on $\mathbb{C}P(4n+3)$. □

Corollary 2.2. *The generalised connected sum of two copies of* $\mathbb{H}P(2n+1)$ *is again* $\mathbb{H}P(2n+1)$.

Proof. The twistor space $\mathbb{C}P(4n+3)$ of $\mathbb{H}P(2n+1)$ has its real structure given by

$$[z_0, z_1, z_2, z_3, \dots] \mapsto [-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_4, \dots].$$

One may now verify that, for λ real, the real structure induced on V_λ via the above smoothing construction agrees with that induced by the Veronese map ϕ. The proof is completed by applying the

Inverse Twistor Construction. (Pedersen & Poon [14]) *Let* Z *be a complex manifold of complex dimension* $2n+1 \geq 5$ *with a fixed-point-free anti-holomorphic involution* σ. *Then the set* N *of* σ*-invariant rational curves with normal bundle* $2n\mathcal{O}(1)$ *is a quaternionic manifold of real dimension* $4n$.

Indeed one may choose to use this as a definition of a quaternionic manifold.

3. Standard Deformations

Let M_1 and M_2 be two compact quaternionic $(8m+4)$-manifolds, with twistor fibrations $\pi_i : Z_i \to M_i$. Suppose P_1 and P_2 are embedded copies of $\mathbb{H}P(m)$ together with an isomorphism between the sphere bundles of N_1 and N_2, the normal bundles of P_i in M_i. Let $\tilde{Z}_i$ denote the blow-up of Z_i along $\pi_i^{-1}(P_i)$ and let Q_i be the exceptional divisor. If Q_i is isomorphic to $\mathbb{C}P(2m+1) \times \mathbb{C}P(2m+1)$ then we say that the embedded $\mathbb{H}P(m)$'s are *admissible*. In this case, we form the singular space $Z = \tilde{Z}_1 \cup_Q \tilde{Z}_2$, where $Q = Q_1 \cong Q_2$ with the identification swapping the $\mathbb{C}P(2m+1)$-factors. Recall that a *standard deformation* of Z consists of (i) a smooth complex $(n+4m+3)$-dimensional manifold $\mathcal{Z}$ together with a proper holomorphic map $p : \mathcal{Z} \to S$, where S is a neighbourhood of the origin in $\mathbb{C}^n$, (ii) an isomorphism between $p^{-1}(0)$ and Z as complex spaces and (iii) antiholomorphic involutions σ on $\mathcal{Z}$ and S compatible with p and inducing the given real structure on Z. By shrinking S if necessary, we may assume that there are local coordinates $(t_1, \dots, t_n)$ on $\mathbb{C}^n$ such that the singular fibres of p lie over the hypersurface $\{ t_1 = 0 \}$.

Theorem 3.1. *Suppose the projectivised holomorphic normal bundles* $\mathbb{P}(\nu_i)$ *of* $\pi_i^{-1}(P_i)$ *in* Z_i *are trivial and that* $\mathcal{Z} \to S$ *is a standard deformation of the singular space* Z. *Then for sufficiently small* s *in the fixed-point set* S^σ *but not lying in* $\{ t_1 = 0 \}$, *the fibre* $p^{-1}(s)$ *is the twistor space of a quaternionic structure on* $M_1 \#_{\mathbb{H}P} M_2$.

Sketch proof. It is sufficient to consider the case when S is one-dimensional. The real geometry is exactly as in the four-dimensional case (see Donaldson & Friedman [5]) and implies that $\mathcal{Z}$ is fibred by complex lines over a smooth deformation of $G = \tilde{M}_1 \cup_P \tilde{M}_2$, where $\tilde{M}_i$ is the real blow-up of M_i along P_i and P is the real projectivisation $\mathbb{P}_\mathbb{R}(N_1) = \mathbb{P}_\mathbb{R}(N_2)$ of the normal bundle N_i.

Consider a line L which is a fibre of $Z \to G$. If L is not in Q then L is just an ordinary twistor line, so has normal bundle $(4m+2)\mathcal{O}(1)$ in Z. This implies that the normal bundle of L in $\mathcal{Z}$ is $(4m+2)\mathcal{O}(1)\oplus\mathcal{O}$. If L lies in Q, write $\nu_{L,Q}$ for the relative normal bundle given by $0 \to \nu_{L,Q} \to \nu_L \to \nu_Q|_L \to 0$. Then $\nu_{L,Q} = \mathcal{O}(2)\oplus 4m\mathcal{O}(1)$, because a real line lies in a linear $\mathbb{C}\mathrm{P}(1)\times\mathbb{C}\mathrm{P}(1)$ in $\mathbb{C}\mathrm{P}(2m+1)\times\mathbb{C}\mathrm{P}(2m+1)$. Since $\mathbb{P}(\nu_i)$ is trivial, we have $\nu_i \cong (2m+2)\mathcal{O}(k)$ for some k. If L_i is a twistor line of Z_i lying in $V_i = \pi_i^{-1}(P_i)$, then we have the exact sequence

$$0 \longrightarrow \nu_{L_i,V_i} \longrightarrow \nu_{L_i,Z_i} \longrightarrow \nu_i|_{L_i} \longrightarrow 0.$$

In our case this is

$$0 \longrightarrow 2m\mathcal{O}(1) \longrightarrow (4m+2)\mathcal{O}(1) \longrightarrow (2m+2)\mathcal{O}(k) \longrightarrow 0,$$

since the inclusion of L_i in Z_i is just a linear $\mathbb{C}\mathrm{P}(1)$ in $\mathbb{C}\mathrm{P}(2m+1)$. Counting first Chern numbers shows that $k = 1$ and hence $\nu_i = (2m+2)\mathcal{O}(1)$. Thus, in the blow-up $\tilde{Z}_i$ of Z_i along V_i, the exceptional divisor Q_i is isomorphic to $\mathbb{C}\mathrm{P}(2m+1)\times\mathbb{C}\mathrm{P}(2m+1)$ and has normal bundle $\mathcal{O}(1,-1)$. In $\mathcal{Z}$ this gives $\nu_Q = \mathcal{O}(1,-1)\oplus\mathcal{O}(-1,1)$ and $\nu_Q|_L = 2\mathcal{O}$. Thus, $\nu_L = \mathcal{O}(2)\oplus 4m\mathcal{O}(1)\oplus 2\mathcal{O}$ and L has an $(8m+5)$-parameter family of deformations.

Lemma 3.2. *There is a neighbourhood of the singular real space G in the real deformation space such that the lines over these points with normal bundle not equal to $(4m+2)\mathcal{O}(1)\oplus\mathcal{O}$ are those lines contained in Q.*

Sketch proof. Again the proof comes down to considering those twistor lines in Q. We get $\gamma_L : H^0(\nu_L) \to H^1(\mathrm{End}(\nu_L)) \cong \mathbb{C}^2$. Note the family of lines in Q has dimension $3+8m$, so it is sufficient to show that γ_L is surjective. In fact we only need to check this in the flat model. This model is the total space of $\mathcal{O}(1,-1)\oplus\mathcal{O}(-1,1)$ and we need to show that it contains a versal deformation of $2\mathcal{O}\oplus 4m\mathcal{O}(1)\oplus\mathcal{O}(2)$. It is sufficient to consider $\mathcal{O}\oplus 4m\mathcal{O}(1)\oplus\mathcal{O}(2)$ in $\mathcal{O}(1,-1)$. But $H^1(\mathrm{End}(\mathcal{O}\oplus 4m\mathcal{O}(1)\oplus\mathcal{O}(2))) = H^1(\mathcal{O}(-2))$ and the local computation is as in four-dimensions. □

The end of the proof of the theorem is exactly as in four-dimensions. □

To apply the deformation theory of Donaldson & Friedman [5] we need the following:

Lemma 3.3. *The following cohomology groups of the exceptional divisor vanish:*
(1) $H^1(Q,\mathcal{O})$;
(2) $H^p(Q_i,\nu_i)$ *for all* p, *where* ν_i *is the normal bundle of* Q_i *in* $\tilde{Z}_i$;
(3) $H^p(Q,\Theta_Q)$ *for* $p=1,2$, *where* Θ_Q *is the holomorphic tangent bundle.*

Proof. Recall the Künneth formula

$$H^i(M_1\times M_2,\mathcal{O}(\pi_1^*E_1\otimes\pi_2^*E_2)) = \sum_{k+l=i} H^k(M,\mathcal{O}(E_1))\otimes H^l(M_2,\mathcal{O}(E_2)),$$

for the tensor product of vector bundles E_i over M_i, where $\pi_i : M_1\times M_2 \to M_i$ are the obvious projections. In our case, $M_1 \cong M_2 \cong \mathbb{C}\mathrm{P}(2m+1)$ and (1) follows from the vanishing of $H^i(\mathbb{C}\mathrm{P}(r),\mathcal{O}(n))$ for $0<i<r$ and $n\in\mathbb{Z}$. For part (2), we have $H^p(Q_i,\nu_i) = H^p(\mathbb{C}\mathrm{P}(2m+1)\times\mathbb{C}\mathrm{P}(2m+1),\mathcal{O}(1,-1))$ which is zero by the above

remarks and the vanishing of $H^0(\mathbb{C}P(r), \mathcal{O}(n))$ and $H^r(\mathbb{C}P(r), \mathcal{O}(-n-r-1))$ for $n < 0$. Part (3) now follows from the Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow (r+1)\mathcal{O}(1) \longrightarrow \Theta_{\mathbb{C}P(r)} \longrightarrow 0.$$

□

Corollary 3.4. (Donaldson & Friedman [5]) *If the groups $H^2(Z_i, \Theta_i)$, for $i = 1, 2$, vanish then there is a complex standard deformation of the singular twistor space Z.*

As in Donaldson & Friedman [5], one now deduces:

Theorem 3.5. *Suppose M_i, $i = 1, \dots, n$, are compact quaternionic $(8m+4)$-manifolds with admissible embedded copies P_i of $\mathbb{H}P(m)$ and isomorphisms between the sphere bundles of the normal bundles N_i of P_i in M_i. Let Z_i denote the twistor space of M_i and Θ_i the holomorphic tangent bundle of Z_i. If all the cohomology groups $H^2(Z_i, \Theta_i)$ vanish then the generalised connected sum $M_1 \#_{\mathbb{H}P} \cdots \#_{\mathbb{H}P} M_n$ admits a quaternionic structure.*

4. Examples

We start by forming the self-sum of $\mathbb{H}P(2n+1)$. Let $\mathbb{H}P(n)_0 = \{[a_0, \dots, a_n, 0, \dots, 0]\}$ and $\mathbb{H}P(n)_1 = \{[0, \dots, 0, b_0, \dots, b_n]\}$ be the two embedded $\mathbb{H}P(n)$'s. The subgroup of $Sp(2n+2)$ preserving these two sets is $Sp(n+1) \times Sp(n+1)$ which we will refer to as $Sp(n+1)_0 \times Sp(n+1)_1$ in order to distinguish the factors. The sphere bundle $S\nu_{\mathbb{R}}$ of the real normal bundle of $\mathbb{H}P(n)_0$ in $\mathbb{H}P(2n+1)$ may be identified with

$$S_r = \left\{ [a_0, \dots, a_n, b_0, \dots, b_n] : |a_0|^2 + \dots + |a_n|^2 = 1, |b_0|^2 + \dots + |b_n|^2 = r^2 \right\},$$

for any $r > 0$. Fixing a point x of $\mathbb{H}P(n)_0$, we see that the action of $Sp(n+1)_1$ on the fibre $S_x\nu_{\mathbb{R}}$ is just the standard action of $Sp(n+1)$ on the unit sphere in $\mathbb{H}^{n+1}$. In particular this action is transitive. Calculating the stabiliser of a point and noting that $\bigcup_{r>0} S_r = \mathbb{H}P(2n+1) \setminus (\mathbb{H}P(n)_0 \cup \mathbb{H}P(n)_1)$ gives

Proposition 4.1. *The self-sum of $\mathbb{H}P(2n+1)$ over $\mathbb{H}P(n)$ is topologically*

$$M_{(1)} = S_r \times S^1 = \frac{Sp(n+1)\, Sp(n+1)}{Sp(n) \Delta\, Sp(1)\, Sp(n)} \times S^1,$$

where $\Delta\, Sp(1)$ denotes the subgroup of $Sp(n+1) \times Sp(n+1)$ consisting of $\operatorname{diag}(1, \dots, 1, q) \times \operatorname{diag}(q, 1, \dots, 1)$, for $q \in Sp(1)$.

This construction leaves many embedded $\mathbb{H}P(n)$'s unchanged, for example $\mathbb{H}P(n)$'s given by equations $\lambda a_i + \mu b_i = 0$, where $\lambda\mu \neq 0$. So we may use the generalised connected sum construction to obtain a series of manifolds

$$M_{(k)} = M_{(1)} \#_{\mathbb{H}P} \cdots \#_{\mathbb{H}P} M_{(1)} = k \#_{\mathbb{H}P} M_{(1)}.$$

Theorem 4.2. *The manifolds $M_{(k)}$ are compact manifolds with $b_1(M_{(k)}) = k$ and admit flat quaternionic structures.*

If k is greater than or equal to 3 and the connected sums are performed with respect to generic totally geodesic disjoint copies of $\mathbb{H}P(n)$, then the resulting manifolds have no connected group of quaternionic symmetries.

On the other hand the maximal group of effective quaternionic symmetries of $M_{(k)}$ *for the quaternionic structures arising from this construction has dimension*

$$\begin{cases} 8(n+1)^2-1, & \text{if } k=1,\\ 4(n+1)^2-1, & \text{if } k>1,\end{cases}$$

and these dimensions may be realised.

Proof. To show $M_{(k)}$ is quaternionic it is sufficient to show that $M_{(1)}$ admits a flat quaternionic structure whose twistor space $Z(1)$ satisfies $H^2(\Theta_{Z(1)})=0$. Let $[c,d]$ be homogeneous coordinates on $\mathbb{CP}(4n+3)$ such that $\mathbb{CP}_0=\{d=0\}\cong\mathbb{CP}(2n+1)$ and $\mathbb{CP}_1=\{c=0\}\cong\mathbb{CP}(2n+1)$ are the twistor spaces of $\mathbb{HP}(n)_0$ and $\mathbb{HP}(n)_1$, respectively. Then $Z(1)$ may be described as the quotient of $\mathbb{CP}(4n+3)\setminus(\mathbb{CP}_0\cup\mathbb{CP}_1)$ by the action $[c,d]\mapsto[c,2d]$. This map preserves the twistor fibration $[c_0,c_1,\ldots,d_0,d_1,\ldots]\mapsto[c_0+jc_1,\ldots,d_0+jd_1,\ldots]$ and induces an action on $\mathbb{HP}(2n+1)\setminus(\mathbb{HP}(n)_0\cup\mathbb{HP}(n)_1)$ such that the quotient by this action is diffeomorphic to $M_{(1)}$. The action on $\mathbb{CP}(4n+3)$ is holomorphic and preserves the real structure, so $Z(1)$ is the twistor space of a quaternionic manifold. It is flat as it is covered by the standard twistor space structure on an open subset of $\mathbb{CP}(4n+3)$ (see Salamon [19], for the definition of flat). Clearly $Z(1)$ has a holomorphic projection p to $\mathbb{CP}_0\times\mathbb{CP}_1$ and we have an exact sequence

$$0\longrightarrow\mathcal{O}_{Z(1)}\longrightarrow\Theta_{Z(1)}\longrightarrow p^*\Theta_{\mathbb{CP}_0\times\mathbb{CP}_1}\longrightarrow 0.$$

When $n=0$, this is precisely the situation described by Pontecorvo [15] and his calculations are easily adapted to show

$$h^i(\Theta_{Z(1)})=\begin{cases} 8(n+1)^2-1, & \text{for } i=0,1,\\ 0, & \text{for } i\geq 2.\end{cases}$$

In particular, $H^2(\Theta_{Z(1)})=0$, as required. The calculation also gives

$$H^0(\Theta_{Z(1)})=\{\,(A,B)\in M_{2n+2}(\mathbb{C})\times M_{2n+2}(\mathbb{C}):\operatorname{Tr}A=-\operatorname{Tr}B\,\}.$$

Since $\mathbb{HP}(n)$ and S^{4n+3} have no homology in dimensions 1 and 2, the assertion on b_1 is clear.

By the Inverse Twistor Construction (Pedersen & Poon [14]), the complexification of the group of diffeomorphisms preserving the quaternionic structure is isomorphic to the group of holomorphic transformations of the twistor space. In the connected sum construction, we have a singular space $Z=\tilde{Z}_1\cup_Q\tilde{Z}_2$. Let τ^0 be the sheaf of holomorphic automorphisms of Z. By semi-continuity, $h^0(\tau^0)$ is an upper-bound for the dimension of the space of holomorphic automorphisms of any small deformation of Z. From the short exact sequence

$$0\longrightarrow\tau^0\longrightarrow q_*(\Theta_{\tilde{Z}_1,Q_1}\oplus\Theta_{\tilde{Z}_2,Q_2})\longrightarrow\Theta_Q\longrightarrow 0,$$

where the normalisation $q:(\tilde{Z}_1,Q_1)\amalg(\tilde{Z}_2,Q_2)\to(Z,Q)$ is just the obvious map, we have an exact sequence

$$0\longrightarrow H^0(\tau^0)\longrightarrow H^0(\Theta_{Z_1,V_1})\oplus H^0(\Theta_{Z_2,V_2})\longrightarrow H^0(\Theta_Q)\longrightarrow H^1(\tau^0)\longrightarrow\cdots,$$

since $H^0(\Theta_{\tilde{Z}_i,Q_i})\cong H^0(\Theta_{Z_i,V_i})$.

We consider the group $H^0(\Theta_{Z_i,V_i})$ in the case that $Z_i = Z(1)$. The space $\mathbb{C}P' \subset \mathbb{C}P(4n+3)$ over P_i is a $\mathbb{C}P(2n+1)$ given by equations $C'c + D'd = 0$, for some matrices C' and D' in $M_{n+1}(\mathbb{H}) \subset M_{2n+2}(\mathbb{C})$. The condition that $\mathbb{C}P'$ does not meet $\mathbb{C}P_0$ and $\mathbb{C}P_1$ is that C' and D' be invertible. Write $D = C'^{-1}D'$ and $\mathbb{C}P_D = \mathbb{C}P' = \{c + Dd = 0\}$. The group $H^0(\Theta_{Z(1),\mathbb{C}P_D})$ consists of elements of $H^0(\Theta_{Z(1)})$ which preserve $\mathbb{C}P_D$, thus

$$\begin{aligned} H^0(\Theta_{Z(1),\mathbb{C}P_D}) &= \left\{ (DBD^{-1}, B) \in M_{2n+2}(\mathbb{C}) \times M_{2n+2}(\mathbb{C}) : \operatorname{Tr} B = 0 \right\} \\ &\cong \mathfrak{sl}(2n+2, \mathbb{C}). \end{aligned}$$

This also happens to describe the induced action on the exceptional divisor $Q_D = \mathbb{P}(\nu) \times \mathbb{C}P_D$ of the blow-up of $Z(1)$ at $\mathbb{C}P_D$. Note that two such subspaces $\mathbb{C}P_D$ and $\mathbb{C}P_{D'}$ do not intersect if and only if $\det(D - D') \neq 0$.

We first investigate the possible symmetries of $M_{(2)}$. Let Z be the singular space $\tilde{Z}(1) \cup_Q \tilde{Z}(1)$, where the first $Z(1)$ is blown-up along $\mathbb{C}P_D$ and the second $Z(1)$ is blown-up along $\mathbb{C}P_{D'}$. The space $H^0(\tau^0)$ consists of elements of $H^0(\Theta_{\tilde{Z}(1),Q_D}) \oplus H^0(\Theta_{\tilde{Z}(1),Q_{D'}})$ which agree on Q, that is

$$\begin{aligned} &\left\{ (DAD^{-1}, A) \oplus (D'BD'^{-1}, B) : DAD^{-1} = B,\ A = D'BD'^{-1},\ \operatorname{Tr} A = \operatorname{Tr} B = 0 \right\} \\ &\cong \left\{ A \in \mathfrak{sl}(2n+2, \mathbb{C}) : A = (D'D)A(D'D)^{-1} \right\}. \end{aligned}$$

From Steinberg's results on the adjoint action (see Carter [4]), one sees that this last set has dimension between $4(n+1)^2 - 1$ and $2n + 1$. The upper-bound is achieved when $D'D$ is a multiple of the identity; the lower-bound occurs when $D'D \in GL(n+1, \mathbb{H}) \subset GL(2n+2, \mathbb{C})$ is a regular element, for example when $D'D$ is diagonalisable with all eigenvalues distinct. Using the explicit smoothing arising from Section 2, one may check that the condition $\operatorname{Tr} B = 0$ ensures that for small deformations $h^0(\Theta_{Z(2)}) = h^0(\tau^0)$. Taking D and D' to be diagonal matrices our assertions about the maximal group of symmetries are now immediate.

We now show that there is a configuration for which $Z(3)$ has no holomorphic symmetries. Write $Z = \tilde{Z}(1) \cup_Q \tilde{Z}(1) \cup_{Q'} \tilde{Z}(1)$, where the first and last $Z(1)$'s are blown-up along $\mathbb{C}P_{\mathrm{Id}}$ and the middle $Z(1)$ is blown-up along $\mathbb{C}P_D$ and $\mathbb{C}P_{D'}$. Let D be a diagonal matrix $\operatorname{diag}(i\lambda_0, \ldots, i\lambda_n, -i\lambda_0, \ldots, -i\lambda_n)$ with $\lambda_0 > \lambda_1 > \ldots > \lambda_n > 0$. Write $D' = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$, where $\alpha \in GL(n+1, \mathbb{R}) \subset GL(n+1, \mathbb{C})$ is a diagonalisable matrix with distinct eigenvalues such that none of the eigenvectors coincide with any of the standard basis vectors. Then D and D' are regular elements and $D - D'$ is invertible. Thus $\mathbb{C}P_D \cap \mathbb{C}P_{D'} = \emptyset$. Also, the set of $\operatorname{Ad} D$-invariant elements in $\mathfrak{sl}(2n+2, \mathbb{C})$ is precisely the set of diagonal matrices. However, the condition on α ensures that none of these are $\operatorname{Ad} D'$-invariant, so the singular space Z has no holomorphic symmetries. Semi-continuity now implies that small deformations also have no symmetries, so this applies to some of the $Z(3)$'s as required. □

Remarks. (1) Note that $M_{(k)}$ always admits an effective topological action of $PSL(n+1, \mathbb{H})$.

(2) The symmetry group of $Z(1)$ is $SL(2n+2,\mathbb{C}) \times SL(2n+2,\mathbb{C}) \times \mathbb{C}^*$ and the corresponding connected group of quaternionic symmetries of $M_{(1)}$ is (finitely covered by) $SL(n+1,\mathbb{H}) \times SL(n+1,\mathbb{H}) \times S^1$.

(3) If M is a quaternionic Kähler manifold of positive scalar curvature, then an easy application of the Akizuki-Nakano vanishing theorem (Akizuki & Nakano [1]) shows that the twistor space Z of M automatically satisfies $H^2(Z,\Theta) = 0$ (cf. Salamon [18]). There are two natural examples of quaternionic Kähler manifolds with an embedded projective space of the right dimension. The first is the complex Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{2n+3})$ of two planes in $\mathbb{C}^{2n+3}$; the embedded $\mathbb{HP}(n)$ is the set of quaternionic planes in $\mathbb{H}^{n+1} \subset \mathbb{H}^{n+1} \oplus \mathbb{C} \subset \mathbb{C}^{2n+3}$. The second is the Grassmannian $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^7)$ of oriented four-planes in $\mathbb{R}^7$; here $\mathbb{HP}(1)$ is embedded by identifying it with $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^5)$. However, in neither case is the normal bundle of the twistor space of the embedded $\mathbb{HP}(n)$ of the correct type to apply our construction. It may well be that our constraint that $\mathbb{HP}(n)$ be admissible implies that $M = \mathbb{HP}(2n+1)$. Indeed, if $\mathbb{HP}(n)$ is admissible a result of Griffiths [9] implies that the embedded $\mathbb{CP}(2n+1)$ has an open neighbourhood equivalent to the usual linear embedding of $\mathbb{CP}(2n+1)$ in $\mathbb{CP}(4n+3)$. However, this result says nothing about the real structure and there is a result of Burstall [3] which states that even as complex contact manifolds the twistor spaces of the known (that is symmetric) compact quaternionic Kähler manifolds in any fixed dimension are birationally equivalent.

5. Algebraic Dimension

The algebraic dimension of a compact complex manifold X is defined to be the transcendence degree over $\mathbb{C}$ of the space of meromorphic functions on X (see for example Ueno [22]). In this section we calculate the algebraic dimension of various twistor spaces generalising $Z(1)$. First note that in the construction of $M_{(1)}$ it was not necessary to take the embedded quaternionic projective spaces to be of the same dimension. In fact our earlier analysis goes through for disjoint totally geodesic copies of $\mathbb{HP}(a)$ and $\mathbb{HP}(b)$ in $\mathbb{HP}(a+b+1)$. Blowing-up the embedded twistor spaces results in isomorphic exceptional divisors and the resulting normal crossing may be deformed as before. We thus obtain quaternionic structures on

$$M_{(a,b)} = \frac{Sp(a+1)\,Sp(b+1)}{Sp(a)\Delta\,Sp(1)\,Sp(b)} \times S^1.$$

Note that $M_{(a,b)}$ may be realised as a discrete quotient of the quaternionic join (Swann [20]) of $\mathbb{HP}(a)$ and $\mathbb{HP}(b)$.

A twistor space $Z(a,b)$ for a quaternionic structure on $M_{(a,b)}$ may be obtained exactly in the same way that $Z(1)$ was constructed above, that is as a discrete quotient of $\mathbb{CP}(2a+2b+3)\setminus(\mathbb{CP}(2a+1)\cup\mathbb{CP}(2b+1))$ by the action $[c,d] \mapsto [c,2d]$. In particular, the next result covers the twistor space $Z(1)$ as the special case $a = b = n$.

Proposition 5.1. *The twistor space $Z(a,b)$ has algebraic dimension $2a+2b+2$. In particular, it is not Moishezon and is not the twistor space of a quaternionic Kähler structure.*

Proof. We have a holomorphic projection $Z(a,b) \to \mathbb{CP}(2a+1)\times\mathbb{CP}(2b+1)$, so the algebraic dimension $\mathrm{a}(Z(a,b))$ of $Z(a,b)$ is at least that of $\mathbb{CP}(2a+1)\times\mathbb{CP}(2b+1)$, that is $\mathbf{a}(Z(a,b)) \geq 2a+2b+2$. On the other hand $Z(a,b)$ is easily seen to contain a three-dimensional holomorphic submanifold isomorphic to the twistor space Z of a Hopf surface $S^1 \times S^3$ considered by Pontecorvo [15]. Pontecorvo shows that Z has algebraic dimension 2, and since this has co-dimension $2a+2b$, Ueno [22] implies $2 \geq \mathbf{a}(Z(a,b)) - (2a+2b)$. Thus $\mathbf{a}(Z(a,b)) = 2a+2b+2$.

The Moishezon condition says that $\mathbf{a}(X) = \dim X$, so this is clearly not satisfied here. The assertion about quaternionic Kähler metrics follows from Pontecorvo [16], where it is shown that the twistor space of a quaternionic Kähler manifold is either Moishezon or has algebraic dimension 0 or 1. □

More generally one may construct compact quaternionic manifolds as quotients of open sets in $\mathbb{HP}(n)$ by free, properly-discontinuous, co-compact actions of discrete subgroups of $GL(n+1,\mathbb{H})$. For example, consider the action on $\mathbb{HP}(n)$ given by $[q_0, q_1, \dots] \mapsto [\lambda_0' q_0, \lambda_1' q_1, \dots]$. If we assume, for simplicity, that the λ_i' are non-zero complex numbers, then the induced action on the twistor space $\mathbb{CP}(2n+1)$ is given by $[z_0, z_1, z_2, z_3, \dots] \mapsto [\lambda_0' z_0, \bar{\lambda}_0' z_1, \lambda_1' z_2, \bar{\lambda}_1' z_3, \dots]$.

Lemma 5.2. *Consider the action of $\mathbb{Z}$ on $\mathbb{CP}(n)$, $n \geq 2$, generated by*

$$[z_0, z_1, \dots] \mapsto [\lambda_0 z_0, \lambda_1 z_1, \dots],$$

where $\lambda_0, \dots, \lambda_n$ are non-zero complex numbers satisfying

$$1 \leq |\lambda_0| = \dots = |\lambda_i| < |\lambda_{i+1}| = \dots = |\lambda_n|, \tag{5.1}$$

for some i, $0 \leq i < n$. This action is free, properly-discontinuous and co-compact on an open set of $\mathbb{CP}(n)$ and has a fundamental domain U lying between the boundaries

$$\begin{aligned} |z_0|^2 + \dots + |z_i|^2 &= |z_{i+1}|^2 + \dots + |z_n|^2, \\ \frac{|z_0|^2}{|\lambda_0|^2} + \dots + \frac{|z_i|^2}{|\lambda_i|^2} &= \frac{|z_{i+1}|^2}{|\lambda_{i+1}|^2} + \dots + \frac{|z_n|^2}{|\lambda_n|^2}. \end{aligned}$$

(As usual, only include one boundary component in the fundamental domain.) Let X be the quotient manifold. Then the field of meromorphic functions on X is generated by (the push-forwards of) the functions

$$z_0^{a_0} z_1^{a_1} \dots z_n^{a_n},$$

where $a = (a_0, \dots, a_n)$ runs over n-tuples satisfying

$$\sum_{i=0}^{n} a_i = 0 \qquad \text{and} \qquad \lambda_0^{a_0} \lambda_1^{a_1} \dots \lambda_n^{a_n} = 1. \tag{5.2}$$

Proof. First note that the definition of the fundamental domain is consistent with restriction to a coordinate linear subspace of $\mathbb{CP}(n)$ for which the restricted action still satisfies (5.1).

Let $S \subset \{0, \dots, n\}$ be a maximal subset such that the $\{\lambda_i : i \in S\}$ are independent in the sense that they satisfy no non-trivial relation (5.2). Reorder the indices so that $S = \{0, \dots, r\}$. Then X has algebraic dimension at least $n-r$ and contains a copy of the corresponding quotient X_r from $\mathbb{CP}(r)$. The lemma will follow by showing that the algebraic dimension of X is precisely $n-r$.

If $r = 1$, then by the proof of the previous proposition the corresponding quotient from $\mathbb{C}P(2)$ has algebraic dimension 1 and hence X has algebraic dimension $n - 1 = n - r$.

If $r > 1$, let i_r denote the integer i appearing in (5.1) for the induced action on $\mathbb{C}P(r)$. By the Levi Extension Theorem (see Barth et al. [2]), the lift of any meromorphic function on X_r to $\mathbb{C}P(r) \setminus (\mathbb{C}P(i_r) \cup \mathbb{C}P(r - i_r))$ extends to an invariant meromorphic function on $\mathbb{C}^r = \{z_0 \neq 0\}$. Examining the Laurent expansion of such a function about the origin shows that it must be identically zero if no non-trivial relation (5.2) is satisfied. Thus X_r has algebraic dimension zero and again X has algebraic dimension $n - r$. □

Now taking appropriate actions induced from quaternionic space as above and combining the above lemma with the result of Pontecorvo [16] for the algebraic dimension of the twistor space of a quaternionic Kähler manifold quoted above gives:

Proposition 5.3. *Given $n \geq 0$ and a such that $0 \leq a \leq 2n + 1$, there exists a real $4n$-dimensional compact quaternionic manifold whose complex $(2n + 1)$-dimensional twistor space has algebraic dimension a.*

Note that in the case $n = 1$ this result is already known and the special case of the twistor space of a Hopf surface is dealt with by Gauduchon [7].

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