

U(1)-decomposable self-dual manifolds*

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Abstract: We characterize conformally flat spaces as the only compact self-dual manifolds which are U(1)-equivariantly and conformally decomposable into two complete self-dual Einstein manifolds with common conformal infinity. A geometric characterization of such conformally flat spaces is also given.

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1. Through a deep application of twistor theory, Hitchin discovered a large class of complete self-dual Einstein metrics on non-compact manifolds [9]. All his metrics admit the group $SU(2)$ as group of isometries. Among his examples are families of self-dual Einstein metrics on the unit ball in \mathbb{R}^4 . The conformal boundaries of these metrics are the three-dimensional spheres with left-invariant metrics. This large class of examples prompt the following question. *When can two complete non-compact self-dual Einstein manifolds be glued along a common conformal infinity to produce a compact self-dual manifold?* To address this question, we recall the definition of conformal infinity [14, 5].

Definition 1.1. Suppose that (X, h) is a real analytic oriented four-dimensional Riemannian manifold and (M, g) is a real analytic oriented three-dimensional Riemannian manifold. The conformal manifold $(M, [g])$ is a *conformal infinity* of (X, h) if there is an oriented real analytic conformal manifold $(\hat{X}, [\hat{g}])$ such that

- (1) there is an analytic oriented conformal diffeomorphism from X onto the interior of \hat{X} ;
- (2) there is an analytic oriented conformal diffeomorphism from M onto the boundary of \hat{X} ;
- (3) via the above embeddings, the metric h has a pole along M .

In one of the earliest applications of twistor theory, LeBrun proved that for any oriented three-dimensional analytic conformal manifold $(M, [g])$, there is an oriented four-dimensional Riemannian manifold (X, h) such that $(M, [g])$ is the conformal boundary of (X, h) [14]. Moreover, the metric h is a self-dual Einstein metric with negative scalar curvature. We call the manifold X with the conformal class $[h]$ an *extension* of $(M, [g])$. A metric is *half-hyperbolic* if it is self-dual Einstein with negative scalar curvature.

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The pole of the half-hyperbolic metric h in an extension is of order two along the conformal infinity. And the conformal infinity is totally umbilic in the extension [14]. In general, if (X, h) is an Einstein manifold with a conformal infinity $(M, [g])$, the metric h is complete only if the scalar curvature is negative [5]. A priori, an extension is only one-sided when an orientation on M is chosen [14]. The problem of finding sufficient conditions for M to have two-sided extensions is not yet solved. Neither is the problem of finding sufficient conditions for M to have complete extensions. We focus on two-sided extensions which are not only complete but also compact.

Definition 1.2. ([24]) A compact self-dual manifold (X, g) is *decomposable* along a closed connected oriented hypersurface M if $(X, [g])$ is an extension of $(M, [g|_M])$, $X \setminus M$ is a disjoint union of two smooth manifolds and the half-hyperbolic metric in $[g|_{X \setminus M}]$ is complete. In this case, (X, g) is said to be a *two-sided extension* of $(M, [g|_M])$.

The symmetry group G of a self-dual manifold is the group of orientation preserving conformal transformations. If it acts on the decomposition of a decomposable manifold X equivariantly, the manifold X is said to be G -decomposable.

Regarding Hitchin's half-hyperbolic metrics on the disk, one may attempt gluing such disks equivariantly with respect to $SU(2)$. Due to the presence of a large symmetry group, the possibility is severely limited [23]. The topology also prohibits such operation except when the conformal class on the boundary is the standard one [12, 10]. In this paper, we break the symmetry from $SU(2)$ to $U(1)$.

Theorem 1.3. *If $(X, [g])$ is a compact $U(1)$ -decomposable self-dual manifold along a hypersurface M , then up to a finite covering the triple $(X, [g]; M)$ is conformally equivalent to one of the following:*

- (1) *the sphere S^4 with the standard metric; decomposes along any hyper-sphere;*
- (2) *the product of the metric with curvature 1 on the sphere S^2 and the metric with curvature -1 on a Riemann surface Σ ; the hypersurface M is the product of any hyper-sphere in S^2 with Σ ;*
- (3) *a Hopf manifold H_Υ ; decomposes along $S^1 \times S^2$;*
- (4) *a Fuchsian manifold; decomposes along a three-dimensional Fuchsian manifold defined by the same Fuchsian group.*

Remark 1.4. Let ψ be any real number and γ a real number such that $0 < \gamma < 1$. Define a matrix action on \mathbb{C}^2 by $\Upsilon \cdot (w_1, w_2) = (\gamma e^{i\psi} w_1, \gamma w_2)$. The quotient of $\mathbb{C}^2 \setminus \{0\}$ by the group generated by Υ is a Hopf manifold. It is denoted by H_Υ . It decomposes along the copy of $S^1 \times S^2$ defined by $w_2 = \bar{w}_2$.

Remark 1.5. $SO(2, 1)$ is the group of isometries of the hyperbolic 2-disk \mathcal{H}^2 . A Fuchsian group is an infinite discrete subgroup of $SO(2, 1)$ acting on \mathcal{H}^2 properly discontinuously [13]. It can be treated as a subgroup of $SO(n+1, 1)$ for $n \geq 2$ in an obvious manner. A Fuchsian manifold of dimension n with respect to a Fuchsian group Γ is a compact manifold with fundamental group Γ and developable universal covering.

The proof of Theorem 1.3 relies on the existence of a holomorphic contact structure on the twistor space. The evaluation of the contact 1-form on the holomorphic vector field induced by

the symmetry yields effective divisors. The analysis of the algebraic structure of the divisors shows that the conformal structure is conformally flat. Then we apply the theory of developing maps to finish the classification.

One should notice that all conformal classes found in Theorem 1.3 are conformally flat. It remains to be seen if there exists any compact decomposable self-dual manifold which is not conformally flat. This issue is closely related to the problem of finding topological constraints on decomposable manifolds.

2. The twistor space Z of a self-dual manifold is the total space of the sphere bundle of anti-self-dual 2-forms. It is also the projectivization of the bundle of negative spinors. It has a natural integrable complex structure [1, 2]. The horizontal distribution of the fiber bundle Z is a real rank 4 distribution. This is a holomorphic rank 2 distribution if the metric is Einstein [2, 8].

Assume that X is decomposable along M . Let h be a half-hyperbolic metric on $X \setminus M$ and \mathcal{D} the horizontal distribution. We shall extend this distribution across the twistor space over M . The following observation is essentially due to LeBrun [14]. It is also known in other contexts [8].

Proposition 2.1. *Suppose that $(X, [g])$ is a self-dual manifold decomposable along a hypersurface M . Then there is a unique distribution \mathcal{D} defined by the kernel of a twisted holomorphic 1-form θ on the twistor space with the following properties:*

- (1) \mathcal{D} is a holomorphic contact distribution;
- (2) \mathcal{D} fits in the exact sequence $0 \rightarrow \mathcal{D} \rightarrow \mathbf{T}Z \xrightarrow{\theta} K^{-1/2} \rightarrow 0$;
- (3) \mathcal{D} is real in the sense that $\sigma^*\mathcal{D} = \mathcal{D}$, where σ is the real structure;
- (4) the restriction of \mathcal{D} on any real twistor line over $X \setminus M$ is isomorphic to the normal bundle of the real twistor line, i.e., $\mathcal{O}(1) \oplus \mathcal{O}(1)$;
- (5) any real twistor line L over M is Legendrian.

Proof. We review the construction of the distribution \mathcal{D} over $X \setminus M$ as presented in [2]. The complexified tangent bundle of X is the tensor product of bundles of positive and negative spinors Σ^+ and Σ^- . They are $SU(2)$ -bundles if a metric h is chosen in the given conformal class. Let $\{\psi_\alpha : \alpha = 1, 2\}$ and $\{\phi_\alpha : \alpha = 1, 2\}$ be the unitary frames for Σ^+ and Σ^- respectively. With homogeneous coordinates $[\lambda_1, \lambda_2]$ on the fiber of the twistor space, then $\Phi = \sum_{\alpha=1}^2 \lambda_\alpha \phi_\alpha$ is a tautological section. Let $(\omega_{\alpha\beta}^h)$ be the connection matrix of the induced connection on Σ^- with respect to the metric h . Define

$$\theta_\alpha^h = d\lambda_\alpha - \sum_\gamma \omega_{\alpha\gamma}^h \lambda_\gamma, \quad \alpha = 1, 2. \quad (2.2)$$

The horizontal distribution \mathcal{D} over $X \setminus M$ is defined by the kernel of $\theta^h := \lambda_2 \theta_1^h - \lambda_1 \theta_2^h$. This distribution satisfies conditions (1), (2), (3) and (4) in the Proposition.

Choose a point x on M and a small neighborhood U of x in X . There is a function f such that the zero-set of f coincides with $M \cap U$ and it is non-degenerate along M . Moreover, $h = f^{-2}g$ for some self-dual metric g defined in the neighborhood U . With respect to the metric g , we define the 1-forms θ_1^g, θ_2^g as in (2.2). When Φ is the tautological section of the bundle of negative spinors, we define $\sigma_\alpha = \psi_\alpha \otimes \Phi$. As $g = f^2 h$, $\theta_\alpha^h = \theta_\alpha^g + \frac{1}{4} \sum_{\gamma=1}^2 \langle df/f \cdot \phi_\alpha, \psi_\gamma \rangle \sigma_\gamma$, where

‘ \cdot ’ is the Clifford multiplication. Then on $U \setminus M$,

$$\ker(\lambda_2\theta_1^h - \lambda_1\theta_2^h) = \ker \left\{ f(\lambda_2\theta_1^g - \lambda_1\theta_2^g) + \frac{1}{4} \sum_{\gamma=1}^2 \langle df \cdot (\lambda_2\phi_1 - \lambda_1\phi_2), \psi_\gamma \rangle \sigma_\gamma \right\}.$$

It follows that the distribution \mathcal{D} on the twistor space of $U \setminus M$ has a smooth extension across M . Over M , the distribution is the kernel of the 1-form $\sum_{\gamma=1}^2 \langle df \cdot (\lambda_2\phi_1 - \lambda_1\phi_2), \psi_\gamma \rangle \sigma_\gamma$. Since f is non-degenerate along M , the kernel is complex two-dimensional. Since σ_1 and σ_2 are type (1,0) forms on the twistor space, the section θ^h of $\mathbf{T}^*Z \otimes K^{-1/2}$ over the twistor space of $X \setminus M$ has a differentiable extension such that its kernel is two-dimensional. Since θ^h is holomorphic outside the closed set of twistor lines over M , its extension is holomorphic over Z and independent of the choice of the open set U . Since $\theta^h \wedge d\theta^h$ is a non-zero constant on the twistor space over $X \setminus M$, it is constant everywhere. Therefore, the extended distribution is a contact distribution. It follows that \mathcal{D} fits in the exact sequence (2). As holomorphic tangent vectors of twistor lines are vertical and therefore annihilated by the 1-forms σ_1 and σ_2 , real twistor lines over M are Legendrian. \square

Suppose that X is G -decomposable. As the half-hyperbolic metric is complete, G is a group of isometries with respect to the metric h except when all connected components of $X \setminus M$ are conformally equivalent to the hyperbolic space \mathcal{H}^4 [11]. In this case, the compact manifold X is the 4-sphere with the Euclidean metric [12]. Therefore, we assume that G is contained in the group of isometries with respect to the half-hyperbolic metric h in at least one of the component of $X \setminus M$. Since the contact distribution \mathcal{D} on the twistor space over $X \setminus M$ is the horizontal distribution with respect to the metric h , the holomorphic transformations induced by the group G on the twistor space are contact transformations over at least an open set. By analyticity, G is a group of contact transformations on Z . Then it is a group of isometries on $X \setminus M$. Assume that $G = \mathrm{U}(1)$. Let V be the *real* holomorphic vector field on the twistor space generated by G . It is an elementary computation to prove the following observation.

Lemma 2.3. *The section $\zeta := \theta(V)$ of the contact line bundle $K^{-1/2}$ is non-trivial, and the divisor of zeroes of ζ is invariant of the one-parameter group of contact transformations.*

Let S be the zero-divisor of the section ζ in the twistor space. The following is a corollary of Lemma 2.3, and [21, Proposition 5.2].

Corollary 2.4. *If X is $\mathrm{U}(1)$ -decomposable, then the conformal class is of non-negative type.*

As the effective divisor S locally defines a complex structure J on the manifold X such that the self-dual metric g is a Kähler metric [19], the following lemma will be very useful.

Lemma 2.5. *Suppose that (g, J) is a Kähler structure on a manifold U and f is a function on U such that $h = f^{-2}g$ is an Einstein metric. Let ∇f be the gradient vector field of the function f with respect to the Kähler metric g . Then $J\nabla f$ is a Killing vector field with respect to both the metric g and the metric h .*

Proof. Let $\{e_1, \dots, e_n\}$ be a local frame on U . Then

$$(\mathcal{L}_{J\nabla f}g)(e_i, e_j) = (\nabla_{e_i}df)(Je_j) + (\nabla_{e_j}df)(Je_i).$$

Since $h = f^{-2}g$ is an Einstein metric, $\text{Ric}_0 f + 2(\nabla df + \frac{1}{4}\Delta f g) = 0$, where Ric_0 is the trace-free part of the Ricci tensor of the metric g [2]. Then

$$\begin{aligned} (\mathcal{L}_{J\nabla f} g)(e_i, e_j) &= -\frac{1}{2}f\{(\text{Ric}_0(e_i, J e_j) + \text{Ric}_0(e_j, J e_i)) \\ &\quad - \frac{1}{4}\Delta f\{g(e_i, J e_j) + g(e_j, J e_i)\}. \end{aligned}$$

Since the metric g is Kähler, for any tangent vectors v and w ,

$$g(v, Jw) = -g(w, Jv) \quad \text{and} \quad \text{Ric}(v, Jw) = -\text{Ric}(w, Jv).$$

It follows that $\mathcal{L}_{J\nabla f} g = 0$. Therefore, $J\nabla f$ is a Killing vector field with respect to the Kähler metric g . Since $df(J\nabla f) = g(\nabla f, J\nabla f) = 0$ and $h = f^{-2}g$, $\mathcal{L}_{J\nabla f} h = 0$. Therefore, $J\nabla f$ is also a Killing vector field with respect to the metric h . \square

On a self-dual manifold, the above theorem has a more precise form.

Proposition 2.6. *Suppose that (X, h) is a half-hyperbolic manifold with Killing vector field V . Let J be the complex structure defined by the twistor section $\theta(V)$. Let f be the function such that $g = f^2 h$ is a Kähler metric with respect to the complex structure of J . Then $V = J\nabla f$.*

Proof. P. Tod discovers that such Einstein metric h is determined by the $\text{SU}(\infty)$ -Toda field equation [26]. He finds local coordinates (x, y, w, t) such that

$$h = \frac{P}{w^2}[e^v(dx^2 + dy^2) + dw^2] + \frac{1}{Pw^2}(dt + \theta)^2,$$

where v and P are functions of (x, y, w) , $\partial/\partial t$ is the Killing vector field V and θ is a 1-form on the (x, y, w) -space satisfying

$$\begin{aligned} v_{xx} + v_{yy} + (e^v)_{ww} &= 0, \\ -2\Lambda P &= 2 - wv_w, \quad \Lambda \text{ is a non-zero constant,} \\ d\theta &= -\frac{\partial P}{\partial x}dy \wedge dw - \frac{\partial P}{\partial y}dw \wedge dx - \frac{\partial(Pe^v)}{\partial w}dx \wedge dy. \end{aligned}$$

Taking $f(x, y, w, t) = w$, one checks that the metric

$$g = f^2 h = P[e^v(dx^2 + dy^2) + dw^2] + \frac{1}{P}(dt + \theta)^2,$$

is a Kähler metric with Kähler form $\Omega = (dt + \theta) \wedge dw + (Pe^v)dx \wedge dy$ [16], and that $J\nabla w = \partial/\partial t = V$, as claimed. \square

The next collection of propositions deal with twistor interpretation of Gauss map of embedded surfaces.

Proposition 2.7. *Let (X, h) be a self-dual Einstein manifold with twistor space Z . Suppose that C is a smooth holomorphic curve in Z such that the twistor fibration on C is a diffeomorphism. Suppose that $\text{U}(1)$ is a group of isometries of (X, h) such that its induced action on the twistor space leaves every point of C fixed, then C is Legendrian with respect to the horizontal distribution defined by h .*

Proof. By assumption, $\pi(C)$ is a two-dimensional sub-manifold. It consists of fixed points of the $U(1)$ -action. With respect to the induced metric, $\pi(C)$ is totally geodesic. The Gauss map of $\pi(C)$ is a non-zero section of the bundle \wedge^2 along $\pi(C)$. Taking the normalization of the anti-self-dual part, one obtains a section of the twistor fibration along $\pi(C)$. When $\pi(C)$ is totally geodesic, this section defines a holomorphic horizontal curve C' [4,20]. Let \bar{C}' be its conjugate.

We claim that C coincides with either C' or \bar{C}' . Let x be a point in $\pi(C)$. The isotropy representation of $U(1)$ at x is the direct sum of the isotropy representation of the tangent space of $\pi(C)$ and the action on the normal plane. Since the action is effective, and $\pi(C)$ consists of fixed points, the action on the normal plane is non-trivial. Therefore, the induced action on the three-dimensional space of anti-self-dual 2-forms over x is a non-trivial rotation on a two-dimensional subspace. Then the induced holomorphic $U(1)$ -action on the twistor line L over x has precisely two fixed points. The intersection of L with C' and \bar{C}' are the only fixed points of the $U(1)$ -action on L . Since C intersects L , and the intersection is a fixed point, then $C \cap L$ is either $C' \cap L$ or $\bar{C}' \cap L$. Since this conclusion is true for any point in $\pi(C)$, the proposition follows. \square

Proposition 2.8. *Let $(X, [g])$ be a compact $U(1)$ -decomposable manifold different from S^4 . Let Z be the twistor space with induced contact distribution \mathcal{D} . Suppose that C is a smooth holomorphic curve in Z such that the twistor fibration on C is a diffeomorphism. Suppose that $U(1)$ leaves every point of C fixed, then C is Legendrian.*

Proof. As $\pi(C)$ consists of fixed points, and M is three-dimensional, $\pi(C) \cap M$ consists of at most finitely many real curves. Due to the last proposition, an open subset of C is Legendrian. By closeness of condition, C is Legendrian. \square

Finally let C be an oriented surface embedded in a self-dual manifold with normal bundle \mathbf{N} . We treat \mathbf{N} as a complex line bundle. Let \mathbf{K}^{-1} be the anti-canonical bundle of C . The Chern class of the bundle $\mathbf{K}^{-1}\mathbf{N}$ can be computed by the Gauss–Codazzi equation. One has the following result [4, 20].

Proposition 2.9. *If C is a compact minimal surface isometrically embedded in a self-dual Einstein manifold with negative scalar curvature, then its self-intersection number is subject to the condition that $C \cdot C > \chi(C)$.*

Remark 2.10. When one uses an orientation opposite to the one on X , then $\chi(C) < -C \cdot C$.

3. In this section, we assume that $\zeta = \theta(V)$ does not vanish along any real twistor line. It follows that the zero divisor S is non-singular [21], and the conformal class is of type zero [15].

When S is reducible, it is the disjoint union of a conjugate pair of irreducible nonsingular divisors D and \bar{D} . The restriction of the twistor fibration onto D is an orientation-reversing diffeomorphism [18]. Pulling back the self-dual conformal class from X onto D by the twistor fibration, one has an anti-self-dual Kähler structure on D [19].

If S is irreducible, the restriction of the twistor fibration onto S is an unbranched double covering of X . Pulling back the self-dual conformal structure from X onto S , one has an anti-

self-dual Kähler surface. The contact structure on the twistor space of X is pulled back to a contact structure on the twistor space of S . The lifting of the conformal group to the twistor space defines an inclusion of the symmetry group of X in the group of conformal holomorphic transformations on S . The discussions of the last paragraph on X can now be applied on S . Hence, up to a double covering S is reducible.

As D is invariant of the $U(1)$ -action on the twistor space, and the action on the twistor space is holomorphic, the action on D is holomorphic and conformal. With respect to the Kähler metric on D , the vector field V is a holomorphic Killing vector field on D . It was proved that the surface D is either the flat torus, the conformally-flat Kähler surface $S^2 \times \Sigma$ with $SO(3)$ -symmetry or blow-ups of ruled surfaces of genus at least 2 [17]. The flat torus is a quotient of \mathbb{R}^4 by a lattice group generated by translations in four linearly independent directions. If the torus were decomposable, the contact 1-form on the twistor space is lifted to a contact 1-form θ on the twistor space of the Euclidean space \mathbb{R}^4 . The sphere is a conformal compactification of \mathbb{R}^4 by adding one point ∞ . Then the 1-form θ is uniquely extended to a section of $\mathbf{T}^*\mathbb{CP}^3 \otimes \mathcal{O}(2)$ by Hartog's Theorem. The kernel of θ is two-dimensional except possibly over the twistor line of ∞ . Away from this line, the space of lines over which the splitting type of the kernel of θ differs from $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is an analytic subspace in the space of all lines. By compactness, this analytic subspace has only finitely many components. As the lattice group is generated by four linearly independent vectors, the space of jumping lines cannot be invariant. Then the space of jumping lines consists of infinitely many components. This contradiction shows that X cannot be the flat torus.

When the surface D is the blow-up of a ruled surface over a Riemann surface Σ with genus at least 2, we use b to denote the blowing-down map from D to its minimal model Y , and r the projection from Y onto Σ . Y is the projectivization of a rank 2 vector bundle over Σ , and the $U(1)$ -action on D is the lifting of rotations on each fiber [17]. Each irreducible fiber F of the map $r \circ b$ is invariant of the $U(1)$ -action. As the action on F is non-trivial, the vector field V is tangential to F . Since $\theta(V) = 0$ along D , F is Legendrian. The proper transforms of the zero-section C_0 and the infinity section C_∞ of the ruling r are both fixed point sets of the $U(1)$ -action. By Proposition 2.8, C_0 and C_∞ are Legendrian. It follows that the twistor lines through the intersection $n = F \cap C_0$ and $s = F \cap C_\infty$ are not Legendrian. Therefore, when D is identified to X by twistor fibration, n and s are not in $F \cap M$. It follows that $F \cap M$ is a finite union of one-dimensional orbits of the $U(1)$ -action. This intersection is non-empty for otherwise F is holomorphic horizontal with respect to the Einstein distribution of the half-hyperbolic metric. It is a contradiction to Remark 2.10 because the self-intersection number of F is equal to zero, and the complex orientation on D is opposite to the given one on X .

Lemma 3.1. *The manifold M intersects every fiber of the ruling along one circle.*

Proof. Consider the restriction of the function f on an irreducible fiber F . Since f is invariant of the $U(1)$ -action, it descends to the orbit space of the action on $F \setminus \{n, s\}$. The orbit space is an open interval I . If $F \cap M$ consists of at least two circles, then f has two distinct zeroes on the open interval I . Between these two zeroes, there is a point where the derivative of f is equal to zero. Since the vector field V is in the direction of $J\nabla f$, the gradient field ∇f is perpendicular to the orbits of the $U(1)$ -action and is tangent to the fiber F . Therefore, there is a

non-degenerate orbit on F along which $\mathcal{L}_{\nabla f} f = 0$. It is equivalent to $g(V, V) = 0$. This is a contradiction because the fixed point set of the $U(1)$ -action on F is $\{n, s\}$. Therefore, when F is an irreducible fiber, $F \cap M$ consists of only one circle.

Reducible fibers are isolated in the family of all fibers. Let F be a reducible fiber. Let p be a point in $F \cap M$. Let F_0 be an irreducible component of F containing p . Since the singular points of F cannot be in M , the $U(1)$ -orbit of p is a circle. Since M is a compact space and F_0 is a 2-sphere, the intersection $M \cap F_0$ is a finite union of circles. Then the intersection of M and F_0 along the orbit of p is transversal. Via the exponential map at p , one sees that the intersection between M and any irreducible fiber near F is non-empty. And the intersection is along an orbit topologically near the orbit of p . This observation applies to all intersection of M and F . As M intersects any irreducible fiber along exactly one circle, $M \cap F$ consists of exactly one circle even when F is reducible. \square

Lemma 3.1 and Remark 2.10 together imply that when F is reducible, it contains at most one (-1) -curve and no (-2) -curves as its components. It is possible only if F is irreducible. Therefore, X is a minimal surface. Up to a finite covering, D is a product surface. Since every fiber is irreducible, every fiber intersects M along one circle, M is diffeomorphic to $S^1 \times \Sigma$.

We claim that M is isometric to $E \times \Sigma$ where E is a circle perpendicular to the axis of the $U(1)$ -action on S^2 . Let \mathcal{H}^2 be the universal covering of Σ . We use the following coordinates.

$$S^2 \times \mathcal{H}^2 = \{(y_1, y_2, y_3) \times (x, r) : y_1^2 + y_2^2 + y_3^2 = 1, x \in \mathbb{R}, r > 0\}.$$

When V is the rotational field $(y_2 \partial/\partial y_3 - y_3 \partial/\partial y_2)$, then $\nabla f = -JV = -\nabla y_1$. Therefore $f = -y_1 + c$ for some constant c . We conclude that

Proposition 3.2. *If $(X, [g])$ is a type zero $U(1)$ -decomposable manifold, up to a finite covering it is a product $S^2 \times \Sigma$; the conformal class contains the product of the metric with curvature 1 on S^2 and the metric with curvature -1 on Σ ; and the hypersurface M is the product of any hyper-sphere in S^2 with Σ .*

4. When the divisor S contains a real twistor line, the conformal class is of positive type [15].

If the divisor S is a sum $D + \bar{D}$, the divisors D and \bar{D} intersect along a real twistor line L with multiplicity 1. The self-intersection number of L on the surface D is equal to 1. Therefore, D is a blow-up of \mathbb{CP}^2 [18]. Restricting the twistor fibration π to D , we see that X is diffeomorphic to $n\mathbb{CP}^2$. Due to the exact sequence of Proposition 2.1, $c_1(\mathcal{D}) = \frac{1}{2}c_1$ and $c_2(\mathcal{D}) = c_2 - \frac{1}{4}c_1^2$. Let χ and τ be the Euler number and signature of the manifold X respectively, then $c_1^3 = 16(2\chi - 3\tau)$ and $c_1 c_2 = 12(\chi - \tau)$ [7]. By reality of \mathcal{D} , $c_2(\mathcal{D}) \cdot D = c_2(\mathcal{D}) \cdot \bar{D}$. It follows that

$$c_2(\mathcal{D})[D] = \frac{1}{2}c_2(\mathcal{D}) \cdot (D + \bar{D}) = \frac{1}{4}c_1 c_2 - \frac{1}{16}c_1^3 = \chi = 2 + n.$$

By the definition of D , the restriction of V is a section of the bundle \mathcal{D} on the surface D . While D is invariant of the $U(1)$ -action, V is a section of the tangent bundle of D . Since $\chi(D) = \chi(\mathbb{CP}^2 \# n\bar{\mathbb{CP}}^2) = 3 + n$, and it is not equal to $c_2(\mathcal{D})[D]$, the zero set of V on D contains one-dimensional component. Therefore, when D is blown down to \mathbb{CP}^2 , in appropriate homogeneous

coordinates, the U(1)-action on \mathbb{CP}^2 is

$$[z_0, z_1, z_2] \rightarrow [e^{ip\phi} z_0, z_1, z_2] \quad (4.1)$$

for some real number p . Since L is the intersection of two invariant divisors, it is invariant of the action. Let $x = \pi(L)$, then x is a fixed point on X .

Lemma 4.2. *The U(1)-action on the twistor line L is trivial.*

Proof. As L is invariant, it is the zero set of one of the three coordinate functions. Assume that L is defined by $z_2 = 0$. Let L_0 be the line defined by $z_0 = 0$, and L_1 the line defined by $z_1 = 0$. Let p be the intersection point of L and L_1 , and q the intersection point of L and L_0 . As all points of blowing-up are away from the real twistor line L , we ignore blowing-ups when we study geometry in a neighborhood of the twistor line L . Since the action on L is non-trivial, and the vector field V is contact to \mathcal{D} along S , L is Legendrian. It follows that the fixed point x is contained in M . As M is invariant, the connected component of fixed point set containing x in M is a one-dimensional space F . As the action on L_0 is trivial, $\pi(L_0)$ is a component of the fixed point set of the U(1)-action on X . This is a two-dimensional sphere containing x . The set $\pi(L_0)$ must contain an open subset of F . For otherwise, the isotropy representation of U(1) at x is trivial, and then the induced U(1)-action on the twistor line would have been trivial. Let y be a fixed point contained in F , and near x such that the real twistor line L_y over y intersects L_0 at a point z near q . As y is contained in M , L_y is Legendrian. Let L_{pz} be the line on D joining z and p . As the U(1)-action leaves L_{pz} invariant, and the action on it is non-trivial, L_{pz} is Legendrian. Finally, $\pi(L_0)$ is a set of fixed points. By Proposition 2.8, L_0 is Legendrian. Then the three curves L_0 , L_y and L_{pz} are all Legendrian. But these curves intersect at q such that their tangents at q span the tangent space of the twistor space at q . This is impossible because the contact distribution is two-dimensional. Therefore the U(1)-action on the real twistor line L is trivial. \square

Proposition 4.3. *Suppose that the zero divisor of $\theta(V)$ is reducible, then the decomposable manifold is the round sphere.*

Proof. If X is not the sphere, the surface D is the blow-up of \mathbb{CP}^2 at least once. In the coordinates given in (4.1), let $p = [1, 0, 0]$. The fixed point set of the U(1)-action on \mathbb{CP}^2 consists of the point p and the line L defined by $z_0 = 0$. Since the line L is a real twistor line, none of the points of blowing-up is on L . Since the blowing-ups preserve the U(1)-action, the first point of blowing-up is the point p . Let D_0 be the blow-up of \mathbb{CP}^2 at p . Let b_0 be the blowing-down map from D_0 onto \mathbb{CP}^2 . Let E be the exceptional divisor of p on D_0 . Let F be the divisor class on D_0 representing the lines on \mathbb{CP}^2 through the point p . Any subsequent blow-ups are on the curve E . Let b be the blowing-down map from D onto D_0 . One considers D_0 as a ruled surface with F as fiber class, and E the infinity section.

By (4.1), $\pi(L) = x$ is an isolated fixed point of the U(1)-action on X . Since M is three-dimensional, its fixed point set consists of one-dimensional components. Therefore, x is not contained in M . Hence, L is not Legendrian. After an orientation change, $X \setminus \{x\}$ is the blow-up of \mathbb{C}^2 . The conformal class of $[g]$ contains a Kähler metric g on $X \setminus \{x\}$. There exists a function

f on $X \setminus \{x\}$ such that $h = f^{-2}g$ is an Einstein metric on $X \setminus \{x \cup M\}$. By Lemma 2.6, $V = J\nabla f$ is a Killing vector field with respect to g on $X \setminus \{x\}$.

By the proof of Lemma 3.1, every element in the complete linear system of the proper transform of $|F|$ is irreducible. It follows that $D = D_0$. Therefore, the self-dual manifold is diffeomorphic to \mathbb{CP}^2 with positive scalar curvature. Due to [21], the conformal class contains the Fubini–Study metric. Since the Fubini–Study metric is an Einstein metric with positive scalar curvature, and it is not conformally flat. Brinkman’s result [3] implies that it is not decomposable. The proof of Proposition 4.3 is completed. \square

When S is irreducible, it is the blow-up of a ruled surface Q at conjugate pair of points [18]. Let b be the blowing down map, and r the fibration of Q over a Riemann surface Σ . We use F to denote the class of fibers of the map $r \circ b$. The twistor lines contained in S are irreducible non-singular fibers of this map [18]. As $c_1(S) = \frac{1}{2}c_1$ and $c_2(S) = c_2 - \frac{1}{4}c_1^2$, $8(1 - g) - k = c_1^2(S)[S] = 2(2\chi - 3\tau)$, and $4(1 - g) + k = c_2(S)[S] = 2\chi$, where g is the genus of Σ , and k is the number of blowing-ups. Since the conformal class is of positive type, there are no non-trivial anti-self-dual harmonic 2-forms [6]. Therefore, $b_2(X) = \tau$ and

$$k = 2\tau = 2b_2(X), \quad g = b_1(X). \quad (4.4)$$

Let X_S be the collection of real twistor lines contained in S . This is a real analytic subspace of X . Let S_0 be S with all the twistor lines in it removed. Since any real twistor line contained in S is an irreducible non-singular fiber of the map $r \circ b$, S_0 contains all the reducible fibers. The twistor fibration is an unbranched double covering from S_0 to $X_0 := X \setminus X_S$. Let M_0 be the intersection $M \cap S_0$. The restriction of the self-dual conformal class on X_0 is pulled back to S_0 to define an anti-self-dual Kähler structure. Let g be a Kähler metric on S_0 and f a function such that $h = f^{-2}g$ is a half-hyperbolic metric. By Proposition 2.6, $V = J\nabla f$. We apply the proof of Lemma 3.1 on S_0 to conclude that when F is a fiber of the map $r \circ b$, $F \cap M$ consists of only one circle, and that S is a minimal surface. By (4.4), the conformal structure on X is conformally flat.

When $g = 0$, the conformal class is the the round sphere [21]. Any totally umbilic hypersurface is a hyper-sphere. When $g \geq 1$, let Γ be the fundamental group of X . Let \tilde{X} be the universal covering of X . Denote the twistor space of the induced conformal class on \tilde{X} by \tilde{Z} . \tilde{Z} is the universal covering of Z . Note that Γ acts on \tilde{X} as a group of conformal transformations. Since X is orientable, the action of Γ preserves orientation on \tilde{X} .

Since the conformal structure is of positive type, the development map is a conformal embedding from \tilde{X} into S^4 [25]. The $U(1)$ -action is lifted to a one-dimensional group G of conformal transformations on \tilde{X} . Via the development map, it is uniquely extended to a subgroup of $SO(5, 1)$. Similarly, Γ is a subgroup of $SO(5, 1)$. We consider S^4 as the space of projective quaternions \mathbb{HP}^1 with homogeneous coordinates $[q_0, q_1]$. Its twistor space is the complex projective space \mathbb{CP}^3 . With homogeneous coordinates, the twistor projection is $[z_0, z_1, z_2, z_3] \mapsto [z_0 + z_1j, z_2 + z_3j]$.

When G is non-compact, the flow of \tilde{X} is either the entire sphere, \mathbb{R}^4 or $\mathbb{R}^4 \setminus \{0\}$ [13]. The first case is possible only if $X = \tilde{X} = S^4$. If $\tilde{X} = \mathbb{R}^4$, the group G has at most one fixed point on \tilde{X} . The fixed point set of the G -action on \tilde{X} covers the fixed point set of the $U(1)$ -action on X . As Γ is not finite, the fixed point set in \mathbb{R}^4 is empty. Since the $U(1)$ -action is

fixed-point-free, then $g = 1$ and $b_1 = 1$. Up to a finite covering, the fundamental group Γ is an infinite discrete Abelian group with one generator. In particular, it is contained in a non-compact one-dimensional Abelian group $\hat{\Gamma}$. As $\hat{\Gamma}$ acts on \mathbb{R}^4 without fixed points, it is a group generated by Euclidean motions on \mathbb{R}^4 [13]. Then the quotient space $X = \mathbb{R}^4/\Gamma$ would have been a compact space with flat metric. This is a contradiction as X is of positive type.

If $\tilde{X} = \mathbb{R}^4 \setminus \{0\}$, \tilde{M} is a totally umbilic hypersurface in $\mathbb{R}^4 \setminus \{0\}$. It is either a hyper-sphere or hyper-plane. Since \tilde{M} is invariant of the flow of G , it is contained in a hyper-plane through the origin. Up to coordinate changes, \tilde{M} is defined by $x_4 = 0$. As G leaves the origin of \mathbb{R}^4 fixed, it is contained in $\text{CO}^+(4)$. To leave \tilde{M} invariant, G is a subgroup of $\mathbb{R}^+ \times \text{SO}(2)$. Since the G -action on $\mathbb{R}^4 \setminus \{0\}$ is fixed-point-free, $g = b_1 = 1$. Up to a finite covering, the fundamental group Γ is freely generated by one element ρ . As ρ leaves $\{0, \infty\}$ on \mathbb{R}^4 invariant, ρ^2 leaves the origin of \mathbb{R}^4 fixed. Therefore, it is contained in $\text{CO}^+(4)$. Since ρ^2 leaves \tilde{M} invariant and commutes with G , ρ^4 is contained in the same subgroup $\mathbb{R}^+ \times \text{SO}(2)$. It is represented by

$$\gamma \begin{pmatrix} \cos \psi & \sin \psi & 0 & 0 \\ -\sin \psi & \cos \psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for some real numbers γ and ψ . Take ρ^{-4} if necessary, we assume that $0 < \gamma < 1$. Therefore, up to a finite covering, X is the Hopf manifold H_γ . The decomposition is along a copy of $S^1 \times S^2$ defined by $x_4 = 0$.

When G is compact, it is a copy of $\text{SO}(2)$ in $\text{PGL}(2, \mathbb{H})$. The G -action on S^4 has at least two fixed points. Taking isotropy representation at one of the fixed points o , we embed G in $\text{SO}(4)$. In $\text{PGL}(2, \mathbb{H})$, it is represented by the matrix

$$\text{diag}(e^{ip_1\phi}, e^{ip_2\phi}). \quad (4.5)$$

The corresponding action on S^4 is $[1, q] \mapsto [1, e^{-ip_1\phi} q e^{ip_2\phi}]$, and the fixed point o is $[1, 0]$.

The fixed point set of the $\text{U}(1)$ -action is two-dimensional if and only if $p_1 = \pm p_2$. In such cases, we assume that $p_1 = p_2$. Then the fixed point set is the two-dimensional sphere E defined by $x_3 = x_4 = 0$ in \mathbb{R}^4 when $q = x_1 + x_2i + x_3j + x_4k$. As Γ commutes with G , it is contained in $\text{PGL}(2, \mathbb{C}) = \text{SO}(3, 1)$ the group of conformal transformations of the 2-sphere E .

Consider the conformal model of the 4-sphere as a real quadric in \mathbb{RP}^5 defined by $t_1^2 + t_2^2 + t_3^2 + t_4^2 = 2t_0t_5$ [11]. Let $\text{SO}(2)$ be the group of rotations on the (t_3, t_4) -coordinates. Then the Γ -action is trivial on t_3 and t_4 coordinates. When X is decomposable, there is an invariant totally umbilic hypersurface. Via the development map, one finds a hyper-sphere invariant of the $\text{SO}(2)$ -action and the Γ -action. As it is invariant of the rotation group, the linear equation defining the hyper-sphere is in (t_0, t_1, t_2, t_5) -coordinates. Up to a Möbius transformation, we assume that the hyper-plane is defined $t_2 = 0$. As Γ preserves this hyper-plane, it is contained in $\text{SO}(2, 1)$, acting on the (t_0, t_1, t_5) -coordinates. In other words, X is a Fuchsian manifold.

Consider the surface S in the twistor space again. If $g \geq 2$, the zero and infinity sections are contained in the fixed point set of the holomorphic $\text{U}(1)$ -action on the twistor space. Via the twistor projection, the fixed point set of the $\text{U}(1)$ -action on X is at least two-dimensional. Results in the last paragraph shows that such X is a Fuchsian manifold. Therefore, the last case to be investigated is when $g = 1$ with $p_1 \neq \pm p_2$. Since Γ commutes with G , Γ is a subgroup of

$\mathrm{PGL}(1, \mathbb{H}) = \mathrm{Sp}(1)$ when $p_1 = 0$ or $p_2 = 0$. As $\mathrm{Sp}(1)$ is compact, Γ is finite. This is impossible when $g = b_1 = 1$. Hence $p_1 \neq 0$ and $p_2 \neq 0$. It follows that elements in Γ are represented by diagonal matrices of non-zero complex numbers

$$\mathrm{diag}(r_1 e^{it_1}, r_2 e^{it_2}). \quad (4.6)$$

Since any holomorphic vector field on an open set of \mathbb{CP}^3 is uniquely extended to a global holomorphic vector field, one deduces from the Euler sequence that the 1-form θ on the twistor space \tilde{Z} extends to a section $\hat{\theta}$ of $T^*\mathbb{CP}^3 \otimes \mathcal{O}(2)$. The space of global sections of this bundle is six-dimensional. It is spanned by

$$\begin{aligned} \theta_1 &= z_0 dz_1 - z_1 dz_0, & \theta_2 &= z_2 dz_3 - z_3 dz_2, & \theta_3 &= z_0 dz_2 - z_2 dz_0, \\ \theta_4 &= z_3 dz_1 - z_1 dz_3, & \theta_5 &= z_0 dz_3 - z_3 dz_0, & \theta_6 &= z_1 dz_2 - z_2 dz_1. \end{aligned}$$

With respect to the usual real structure [2], $\{\theta_1, \theta_2, \theta_3 - \theta_4, i(\theta_3 + \theta_4), \theta_5 - \theta_6, i(\theta_5 + \theta_6)\}$ is a real basis. When $p_1 \neq \pm p_2$, the induced actions of (4.5) are non-trivial rotations of the planes spanned by $\theta_3 - \theta_4$, $i(\theta_3 + \theta_4)$ and $\theta_5 - \theta_6$, $i(\theta_5 + \theta_6)$ respectively. As $\hat{\theta}$ is an eigenvector of all these rotations, $\hat{\theta} = a_1 \theta_1 + a_2 \theta_2$ for some real numbers a_1 and a_2 . Then $\hat{\theta} \wedge d\hat{\theta} = 2a_1 a_2 (\theta_1 \wedge dz_2 \wedge dz_3 + \theta_2 \wedge dz_0 \wedge dz_1)$. On the other hand, the induced action of (4.6) with respect to this basis is the identity on θ_1 and multiplication by $(r_2/r_1)^2$ on θ_2 . As $\hat{\theta}$ is an eigenvector of this action, and $(r_2/r_1)^2 \neq 1$ when the group Γ is non-compact, $a_1 a_2 = 0$. It means that $\hat{\theta}$ cannot be the extension of a contact form. Therefore, this case cannot occur.

We conclude that when the divisor S is irreducible, the decomposable manifold X is described by (1), (2) and (4) of Theorem 1.3. Together with Proposition 3.2 and Proposition 4.3, the proof of Theorem 1.3 is now completed.

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