Degeneracy of Holomorphic Poisson Spectral Sequence

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Abstract

Through the theory of Lie bi-algebroids and generalized complex structures, one could define a cohomology theory naturally associated to a holomorphic Poisson structure. It is known that it is the hypercohomology of a bi-complex such that one of the two operators is the classical $\overline{\partial}$-operator. Another operator is the adjoint action of the Poisson bivector with respect to the Schouten-Nijenhuis bracket. The hypercohomology is naturally computed by one of the two associated spectral sequences. In a prior publication, the author of this article and his collaborators investigated the degeneracy of this spectral sequence on the second page. In this note, the author investigates the conditions for which this spectral sequence degenerates on the first page. Particular effort is devoted to nilmanifolds with abelian complex structures.

1 Introduction

It is well known that complex structures and symplectic structures are examples of generalized complex structures in the sense of Hitchin [8, 9, 10]. It is now also known that holomorphic Poisson structure plays a fundamental role in generalized geometry [1]. Since a key feature of generalized geometry is to put both complex

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structures and symplectic structures within a single framework, its deformation theory is of great interests [5] [6] [11]. An understanding of cohomology theory of generalized geometry in general and holomorphic Poisson structures in particular becomes necessary. On algebraic surfaces, there has been work done by Xu and collaborators [12] [13]. For nilmanifolds with abelian complex structures, there are recent work done by this author and his collaborators [2] [3] [7]. A common feature of these work is to recognize the cohomology as the hypercohomology of a bi-complex.

In computation of hypercohomology of bi-complex, theoretically one could apply one of the two naturally defined spectral sequences to complete the task. In the case of a holomorphic Poisson structure, the first page of one of the spectral sequences consists of the Dolbeault cohomology of a complex manifold with coefficients in the sheaf of germs of holomorphic polyvector fields. As the Dolbeault cohomology is a well known classical object, it is natural to determine when this spectral sequence degenerates fast. In [2] and [3], we have seen situation when the spectral sequence degenerates on the second page. We push our analysis in the past and investigate the possibility when degeneracy on the first page occurs.

In particular, as a corollary of our Theorem 6 in this article, we could formulate a result as below.

Theorem 1 *Let $M = G/\Gamma$ be a 2-step nilmanifold with abelian complex structure. Let $\mathfrak{c}$ be the center of the Lie algebra $\mathfrak{g}$ of the simply connected Lie group G, and $\mathfrak{t}$ its quotient. Let $\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{1,0}$ be the space of invariant $(1,0)$-vectors. Assume that $\dim \mathfrak{c}^{1,0} = 1$. Let $\overline{\rho}$ span $\mathfrak{c}^{*(0,1)}$. Suppose that Λ_1 is in $\mathfrak{t}^{1,0} \otimes \mathfrak{c}^{1,0}$. Then,*

- *Λ_1 is a holomorphic Poisson structure.*
- *The spectral sequence of its bi-complex degenerates on the second page.*
- *If in addition, $d\rho$ is a non-degenerate $(1,1)$-form, the spectral sequence degenerates on the first page.*

We will develop a proof leading to the above result and provide examples to illustrate whether one could relax the constrains to attain the same result.

2 Complex and Generalized Complex Structures

In this section, we review the basic background materials as seen in [3] to set up notations.

Let M be a smooth manifold. Denote its tangent bundle by TM and the cotangent bundle by T^*M. A generalized complex structure on an even-dimensional manifold M [8] [9] is a subbundle L of the direct sum $\mathcal{T} = (TM \oplus T^*M)_{\mathbb{C}}$ such that

- L and its conjugate bundle $\overline{L}$ are transversal;
- L is maximally isotropic with respect to the natural pairing on $\mathcal{T}$;
- and the space of sections of L is closed with respective to the Courant bracket.

Given a generalized complex structure, the pair of bundles L and $\overline{L}$ makes a (complex) Lie bi-algebroid. The composition of the inclusion of L and $\overline{L}$ in $\mathcal{T}$ with the natural projection onto the summand $TM_{\mathbb{C}}$ becomes the anchor map of these Lie algebroids. It is denoted by ϕ. Via the canonical non-degenerate pairing on the bundle $\mathcal{T}$, the bundle $\overline{L}$ is complex linearly identified to the dual of L. Therefore, the Lie algebroid differential of $\overline{L}$ acts on L. It extends to a differential on the exterior algebra of L. For the calculus of Lie bi-algebroids, we follow the conventions in [15]. In particular, for any element Γ in $C^\infty(M, \wedge^k L)$ and elements $a_1, \ldots, a_{k+1}$ in $C^\infty(M, \overline{L})$, the Lie algebroid differential of Γ is defined by the Cartan Formula as in exterior differential algebra, namely

$$\begin{aligned}(\overline{\partial}_L \Gamma)(a_1, \ldots, a_{k+1}) &= \sum_{r=1}^{k+1} (-1)^{r+1} \phi(a_r)(\Gamma(a_1, \ldots, \hat{a}_r, \ldots, a_{k+1})) \\ &+ \sum_{r<s} (-1)^{r+s} \Gamma([\![a_r, a_s]\!], a_1, \ldots, \hat{a}_r, \ldots, \hat{a}_s, \ldots, a_{k+1}). \qquad (1)\end{aligned}$$

The space of sections of the bundle $\overline{L}$ is closed if and only if $\overline{\partial}_L \circ \overline{\partial}_L = 0$.

Typical examples of generalized complex structures are classical complex structures and symplectic structures on a manifold. In this section, we focus on the former.

Let $J : TM \to TM$ be an integrable complex structure on the manifold M. The complexified tangent bundle $TM_{\mathbb{C}}$ splits into the direct sum of bundle of $(1,0)$-vectors $TM^{1,0}$ and bundle of $(0,1)$-vectors $TM^{0,1}$. Their p-th exterior products are respectively denoted by $TM^{p,0}$ and $TM^{0,p}$. Denote their dual bundles by $TM^{*(p,0)}$ and $TM^{*(0,p)}$ respectively.

Define $L = TM^{1,0} \oplus TM^{*(0,1)}$. One gets a generalized complex structure. Its dual is its complex conjugate $\overline{L} = TM^{0,1} \oplus TM^{*(1,0)}$. When one restricts the Courant bracket from the ambient bundle $\mathcal{T} = (TM \oplus T^*M)_{\mathbb{C}}$ to the subbundles L and $\overline{L}$, then one recovers the Schouten-Nijenhuis bracket or simply known as the Schouten bracket in classical deformation. The Schouten bracket between $(1,0)$-vector fields is the Lie bracket of vector fields; the Schouten bracket between a $(1,0)$-vector field and a $(0,1)$-form is the Lie derivative of a form by a vector field. The Schouten bracket between two $(0,1)$-forms is equal to zero. These brackets are extended to higher exterior product by observing the rule of exterior multiplication [15].

With respect to the Lie algebroid $\overline{L}$, we get its differential $\overline{\partial}$ as defined in (1).

$$\overline{\partial} : C^\infty(M, L) \to C^\infty(M, \wedge^2 L). \tag{2}$$

It is extended to a differential of exterior algebras:

$$\overline{\partial} : C^\infty(M, \wedge^p L) \to C^\infty(M, \wedge^{p+1} L). \tag{3}$$

It is an elementary exercise in computation of Lie algebroid differential that when $\overline{\partial}$ is restricted to $(0,1)$-forms, it is the classical $\overline{\partial}$-operation in complex manifold theory; and

$$\overline{\partial} : C^\infty(M, TM^{*(0,1)}) \to C^\infty(M, TM^{*(0,2)})$$

is the $(0,2)$-component of the exterior differential [18]. Similarly, when the Lie algebroid differential is restricted to $(1,0)$-vector fields, then

$$\overline{\partial} : C^\infty(M, TM^{1,0}) \to C^\infty(M, TM^{*(0,1)} \otimes TM^{1,0});$$

and it is the Cauchy-Riemann operator as seen in [4].

By virtue of L and $\overline{L}$ being a pair of Lie bi-algebroid, the space $C^\infty(M, \wedge^\bullet L)$ together with the Schouten bracket, exterior product and the Lie algebroid differential $\overline{\partial}$ form a differential Gerstenhaber algebra [15] [17]. In particular, if a is a smooth section of $\wedge^{|a|}L$ and b is a smooth section of $\wedge^{|b|}L$, then

$$\overline{\partial}[\![a, b]\!] = [\![\overline{\partial} a, b]\!] + (-1)^{|a|+1}[\![a, \overline{\partial} b]\!]; \tag{4}$$

$$\overline{\partial}(a \wedge b) = (\overline{\partial} a) \wedge b + (-1)^{|a|} a \wedge (\overline{\partial} b), \tag{5}$$

Since $\overline{\partial} \circ \overline{\partial} = 0$, one obtains the Dolbeault cohomology with coefficients in holomorphic polyvector fields. Denote the sheaf of germs of sections of the p-th exterior power of the holomorphic tangent bundle by Θ^p, we have

$$H^\bullet(M, \wedge^\bullet \Theta) \cong \oplus_{p,q\geq 0} H^q(M, \Theta^p).$$

In subsequent computation, when $p = 0$, Θ^p means to represent the structure sheaf $\mathcal{O}$ of the complex manifold M.

Due to the compatibility between $\overline{\partial}$ and the Schouten bracket $[\![-,-]\!]$ and the compatibility between $\overline{\partial}$ and the exterior product $\wedge$ as noted above, the Schouten bracket and exterior product descend to the cohomology space $H^\bullet(M, \Theta^\bullet)$. In other words, the triple $(H^\bullet(M, \wedge^\bullet TM^{1,0}), [\![-,-]\!], \wedge)$ forms a Gerstenhaber algebra. When we ignore the exterior product, we call it a Schouten algebra. For example, by center of the Schouten algebra, we mean the collection of elements A in $H^\bullet(M, \Theta^\bullet)$ such that $[\![A, B]\!] = 0$ for all B in this space.

3 Holomorphic Poisson Bi-Complex

A holomorphic Poisson structure on a complex manifold (M, J) is a holomorphic bi-vector field Λ such that $[\![\Lambda, \Lambda]\!] = 0$. The corresponding bundles as generalized complex structure for Λ are the pair of bundles of graphes $L_{\overline{\Lambda}}$ and $\overline{L}_\Lambda$ where

$$\overline{L}_\Lambda = \{\overline{\ell} + \Lambda(\overline{\ell}) : \overline{\ell} \in \overline{L}\}. \tag{6}$$

While the pair of bundles $L_{\overline{\Lambda}}$ and $\overline{L}_\Lambda$ form naturally a Lie bi-algebroid, so does the pair L and $\overline{L}_\Lambda$ [14]. From this perspective, the Lie algebroid differential of the deformed generalized complex structure $\overline{L}_\Lambda$ acts on the space of sections of the bundle L.

Any smooth section of the bundle L is the sum of a section v of $T^{1,0}M$ and a section $\overline{\omega}$ of $T^{*(0,1)}M$. Given a holomorphic Poisson structure Λ, define ad_Λ by

$$\mathrm{ad}_\Lambda(v + \overline{\omega}) = [\![\Lambda, v + \overline{\omega}]\!]. \tag{7}$$

Proposition 1 See (Proposition 1) in [7]. *The action of the Lie algebroid differential of $\overline{L}_\Lambda$ on L is given by*

$$\overline{\partial}_\Lambda = \overline{\partial} + \mathrm{ad}_\Lambda : C^\infty(M, L) \to C^\infty(M, \wedge^2 L). \tag{8}$$

The operator $\overline{\partial}_\Lambda$ extends to act on the exterior algebra of $T^{1,0}M \oplus T^{*(0,1)}M = L$. From now on, for $n \geq 0$ denote

$$K^n = C^\infty(M, \wedge^n L). \tag{9}$$

For $n < 0$, set $K^n = \{0\}$.

Since Λ is a holomorphic Poisson structure, the closure of the space of section of the corresponding bundles $\overline{L}_\Lambda$ is equivalent to $\overline{\partial}_\Lambda \circ \overline{\partial}_\Lambda = 0$. Therefore, one has a complex with $\overline{\partial}_\Lambda$ being a differential.

Definition 1 *For all $n \geq 0$, the n-th holomorphic Poisson cohomology of the holomorphic Poisson structure Λ is the space*

$$H^n_\Lambda(M) := \frac{\text{kernel of } \ \overline{\partial}_\Lambda : K^n \to K^{n+1}}{\text{image of } \ \overline{\partial}_\Lambda : K^{n-1} \to K^n}. \tag{10}$$

Given Proposition 1, the identity $\overline{\partial}_\Lambda \circ \overline{\partial}_\Lambda = 0$ is equivalent to a system of three.

$$\overline{\partial} \circ \overline{\partial} = 0, \quad \overline{\partial} \circ \mathrm{ad}_\Lambda + \mathrm{ad}_\Lambda \circ \overline{\partial} = 0, \quad \mathrm{ad}_\Lambda \circ \mathrm{ad}_\Lambda = 0. \tag{11}$$

The first identity is equivalent to the complex structure J being integrable; the second identity is equivalent to Λ being holomorphic, and the third is equivalent to Λ being Poisson. Define $A^{p,q} = C^\infty(M, TM^{p,0} \otimes TM^{*(0,q)})$, then

$$\mathrm{ad}_\Lambda : A^{p,q} \to A^{p+1,q}, \quad \overline{\partial} : A^{p,q} \to A^{p,q+1}; \quad \text{and} \quad K^n = \oplus_{p+q=n} A^{p,q}. \tag{12}$$

Therefore, we obtain a bi-complex. We arrange the double indices (p, q) in such a way that p increases horizontally so that ad_Λ maps from left to right, and q increases vertically so that $\overline{\partial}$ maps from bottom to top.

Definition 2 *Given a holomorphic Poisson structure Λ, its Poisson bi-complex is the triple $\{A^{p,q}, \mathrm{ad}_\Lambda, \overline{\partial}\}$.*

It is now obvious that the (holomorphic) Poisson cohomology $H^\bullet_\Lambda(M)$ theoretically could be computed by each one of the two naturally defined spectral sequences. We choose a filtration given by

$$F^p K^n = \oplus_{p'+q=n, p' \geq p} A^{p',q}.$$

The lowest differential is $\overline{\partial} : A^{p,q} \to A^{p,q+1}$. Therefore, the first sheet of the spectral sequence is the Dolbeault cohomology

$$E_1^{p,q} = H^q(M, \Theta^p). \tag{13}$$

The first page of the spectral sequence is given as below.

$$\begin{array}{ccccccc}
\cdots & \to & \cdots & \to & \cdots & \to \\
H^n(M, \mathcal{O}) & \to & H^n(M, \Theta) & \to & H^n(M, \Theta^2) & \to \\
\cdots & \to & \cdots & \to & \cdots & \to \\
H^1(M, \mathcal{O}) & \to & H^1(M, \Theta) & \to & H^1(M, \Theta^2) & \to \\
H^0(M, \mathcal{O}) & \to & H^0(M, \Theta) & \to & H^0(M, \Theta^2) & \to
\end{array}$$

The differential on this page is

$$d_1^{p,q} = \mathrm{ad}_\Lambda : H^q(M, \Theta^p) \to H^q(M, \Theta^{p+1}). \tag{14}$$

Question 1 *When will the spectral sequence of a Poisson bi-complex degenerates on the first page? In other words, when will* $\mathrm{ad}_\Lambda \equiv 0$ *for all* p, q*?*

Prior to us investigating non-trivial sufficient conditions for spectral sequence to degenerate on the first page, we consider necessary conditions. For instance, we should derive the necessary conditions for the d_1-map on the first row and the first column vanishes.

Given the definition of d_1 in (14), the following observation regarding the first row of the first page in the spectral sequence is trivial.

Proposition 2 *Suppose that the bi-complex of a holomorphic Poisson structure* Λ *degenerates on the first page, then for any holomorphic polyvector fields* Υ*, i.e. elements in* $\oplus_{p\geq 0} H^0(M, \Theta^p)$*,* $[\![\Lambda, \Upsilon]\!] = 0$*.*

Note that the restriction of the Schouten bracket from the full cohomology space $\oplus_{p,q} H^q(M, \Theta^p)$ to the subspace of holomorphic polyvector fields $\oplus_{p\geq 0} H^0(M, \Theta^p)$ turns the latter into a subalgebra. The above proposition means that Λ is in the center of the Schouten algebra $\oplus_{p\geq 0} H^0(M, \Theta^p)$.

The observation above immediately demonstrates that the spectral sequence of well known holomorphic Poisson structures will not degenerate on its first page. Complex projective spaces are the cases at hand. On the other hand, if the space of holomorphic polyvector fields is abelian with respect to the Schouten bracket, then the obstruction from the first row to degenerate vanishes.

Next, we consider the d_1-map on the first column.

$$d_1^{0,q} = \mathrm{ad}_\Lambda : H^q(M, \mathcal{O}) \to H^q(M, \Theta). \tag{15}$$

Suppose $\overline{\omega}$ represents a class in $H^q(M, \mathcal{O})$, then $\mathrm{ad}_\Lambda(\overline{\omega})$ represents the zero class in $H^q(M, \Theta)$ if it is $\overline{\partial}$-exact. Therefore, we have the following observation.

Proposition 3 *Suppose that the bi-complex of a holomorphic Poisson structure* Λ *degenerates on the first page, then for any* $\overline{\partial}$*-closed* $(0, q)$*-form* $\overline{\omega}$*, there exists* $\Gamma \in C^\infty(M, T^{1,0}M \otimes T^{*(0,q-1)})$ *such that*

$$\mathrm{ad}_\Lambda(\overline{\omega}) = \overline{\partial}(\Gamma). \tag{16}$$

Given the obvious necessary conditions by observing the d_1-map on the first row and the first column on the first page, we now could post a question to guide our investigation.

Question 2 *Suppose that* Λ *is in the center of the Schouten algebra of the holomorphic polyvector fields* $\oplus_{p\geq 0} H^0(M, \Theta^p)$ *and that for any* $\overline{\partial}$*-closed* $(0, q)$*-form* $\overline{\omega}$*,* $\mathrm{ad}_\Lambda(\overline{\omega})$ *is* $\overline{\partial}$*-exact. Is it necessarily that* $d_1^{p,q} \equiv 0$ *for all* p, q*?*

In the rest of this paper, we find answers to these questions on a class of nilmanifolds.

4 2-step Nilmanifolds

A compact manifold M is a nilmanifold if there exists a simply-connected nilpotent Lie group G and a lattice subgroup Γ such that M is diffeomorphic to G/Γ. We denote the Lie algebra of the group G by $\mathfrak{g}$ and its center by $\mathfrak{c}$. The step of the nilmanifold is the nilpotence of the Lie algebra $\mathfrak{g}$. A left-invariant complex structure

J on G is said to be abelian if on the Lie algebra $\mathfrak{g}$, it satisfies the conditions $J \circ J = -$identity and $[\![JA, JB]\!] = [\![A, B]\!]$ for all A and B in the Lie algebra $\mathfrak{g}$. If one complexifies the algebra $\mathfrak{g}$ and denotes the $+i$ and $-i$ eigen-spaces of J respectively by $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$, then the invariant complex structure J is abelian if and only if the complex Lie algebra $\mathfrak{g}^{1,0}$ is abelian.

Denote $\wedge^k \mathfrak{g}^{1,0}$ and $\wedge^k \mathfrak{g}^{*(0,1)}$ respectively by $\mathfrak{g}^{k,0}$ and $\mathfrak{g}^{*(0,k)}$. We will use the following notation.

$$B^{p,q} = \mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)}.$$

Assume that the Lie algebra $\mathfrak{g}$ is 2-step nilpotent, i.e. $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{c}$. In such a case, we call the manifold $M = G/\Gamma$ a 2-step nilmanifold [16].

On the nilmanifold M, we consider $\mathfrak{g}^{k,0}$ as invariant $(k, 0)$-vector fields and $\mathfrak{g}^{*(0,k)}$ as invariant $(0, k)$-forms. It yields an inclusion map

$$B^{p,q} \hookrightarrow A^{p,q} = C^\infty(M, T^{p,0}M \otimes T^{*(0,q)}M).$$

When the complex structure is also invariant, $\overline{\partial}$ sends $B^{p,q}$ to $B^{p,q+1}$. Given an invariant complex structure and an invariant holomorphic Poisson structure Λ, ad_Λ sends $B^{p,q}$ to $B^{p+1,q}$. Restricting $\overline{\partial}$ to $B^{p,q}$, we then consider the invariant cohomology.

$$H^q(\mathfrak{g}^{p,0}) = \frac{\text{kernel of } \overline{\partial} : B^{p,q} \to B^{p,q+1}}{\text{image of } \overline{\partial} : B^{p,q-1} \to B^{p,q}}. \tag{17}$$

The inclusion map yields a homomorphism of cohomology:

$$H^q(\mathfrak{g}^{p,0}) \hookrightarrow H^q(M, \Theta^p).$$

Theorem 2 (See [2] and [3]) *On a 2-step nilmanifold M with an invariant abelian complex structure, the inclusion $B^{p,q}$ in $A^{p,q} = C^\infty(M, T^{p,0}M \otimes T^{*(0,q)}M)$ induces an isomorphism of cohomology. In other words,*

$$H^q(\mathfrak{g}^{p,0}) \cong H^q(M, \Theta^p).$$

Given any element A in $B^{p,q}$, it acts on $B^{p,q}$ by the Schouten bracket. We denote its action by ad_A. i.e.

$$\mathrm{ad}_A(B) = [\![A, B]\!].$$

An element A is in the center of the Schouten algebra $\oplus_{p,q} B^{p,q}$ with respect to the Schouten bracket $[\![-,-]\!]$ if and only if $\mathrm{ad}_A \equiv 0$. Similarly, an element A in $H^q(\mathfrak{g}^{p,0})$ is in the center of the Schouten algebra $\oplus_{p,q} H^q(\mathfrak{g}^{p,0})$ if $\mathrm{ad}_A(B)$ is equal to zero on the cohomology level for any B in $\oplus_{p,q} H^q(\mathfrak{g}^{p,0})$.

Let $\mathfrak{t} = \mathfrak{g}/\mathfrak{c}$. Below are some facts shown in Sections 2 and 3 of [16]. Since $\mathfrak{g}$ is 2-step nilpotent, $\mathfrak{t}$ is abelian. As a vector space, $\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{1,0}$, and $\mathfrak{g}^{*(1,0)} = \mathfrak{t}^{*(1,0)} \oplus \mathfrak{c}^{*(1,0)}$. The only non-trivial Lie brackets in $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ are of the form $[\mathfrak{t}^{1,0}, \mathfrak{t}^{0,1}] \subset \mathfrak{c}^{1,0} \oplus \mathfrak{c}^{0,1}$.

Explicitly, there exists a real basis $\{X_k, JX_k : 1 \leq k \leq n\}$ for $\mathfrak{t}$ and $\{Z_\ell, JZ_\ell : 1 \leq \ell \leq m\}$ a real basis for $\mathfrak{c}$. The corresponding complex bases for $\mathfrak{t}^{1,0}$ and $\mathfrak{c}^{1,0}$ are respectively composed of the following elements:

$$T_k = \frac{1}{2}(X_k - iJX_k) \quad \text{and} \quad W_\ell = \frac{1}{2}(Z_\ell - iJZ_\ell). \tag{18}$$

The structure equations of $\mathfrak{g}$ are determined by

$$[\![\overline{T}_k, T_j]\!] = \sum_\ell E^\ell_{kj} W_\ell - \sum_\ell \overline{E}^\ell_{jk} \overline{W}_\ell \tag{19}$$

for some constants E^ℓ_{kj}. Let $\{\omega^k : 1 \leq k \leq n\}$ be the dual basis for $\mathfrak{t}^{*(1,0)}$, and let $\{\rho^\ell : 1 \leq \ell \leq m\}$ be the dual basis for $\mathfrak{c}^{*(1,0)}$. The dual structure equations for (19) are

$$d\rho^\ell = \sum_{i,j} E^\ell_{ji} \omega^i \wedge \overline{\omega}^j \quad \text{and} \quad d\omega = 0. \tag{20}$$

Equivalently,

$$d\overline{\rho}^\ell = -\sum_{i,j} \overline{E}^\ell_{ji} \omega^j \wedge \overline{\omega}^i \quad \text{and} \quad d\overline{\omega} = 0. \tag{21}$$

It follows that

$$[\![T_j, \overline{\rho}^\ell]\!] = \mathcal{L}_{T_j} \overline{\rho}^\ell = \iota_{T_j} d\overline{\rho}^\ell = -\sum_i \overline{E}^\ell_{ji} \overline{\omega}^i. \tag{22}$$

By Cartan Formula (1),

$$\overline{\partial} T_j = \sum_{k,\ell} E^\ell_{kj} \overline{\omega}^k \wedge W_\ell, \tag{23}$$

The consequence of the above computation is twofold. One is about the structure of Lie algebroid differential $\overline{\partial}$ on $\wedge^\bullet(\mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}^*_{\mathbb{C}})$. Another is about structure of Schouten bracket on the same space. On differential, the above computation could be summarized as below.

Lemma 1 $\mathfrak{g}^{*(0,1)} \oplus \mathfrak{c}^{1,0} \subseteq \ker \overline{\partial}$ *and* $\overline{\partial}\mathfrak{t}^{1,0} \subseteq \mathfrak{t}^{*(0,1)} \otimes \mathfrak{c}^{1,0}$*. Moreover,*

$$\overline{\partial}(\mathfrak{t}^{k,0} \otimes \mathfrak{c}^{\ell,0} \otimes \mathfrak{t}^{*(0,a)} \otimes \mathfrak{c}^{*(0,b)}) \subseteq \mathfrak{t}^{k-1,0} \otimes \mathfrak{c}^{\ell+1,0} \otimes \mathfrak{t}^{*(0,a+1)} \otimes \mathfrak{c}^{*(0,b)}.$$

In subsequent presentation, we suppress the notations for the vector spaces, and simply keep track of the quadruple of indices to indicate the composition of the components involved. In particular, every element in the Schouten algebra $\oplus_{p,q} B^{p,q}$ decomposes into a sum of different types according to their indices $(k, \ell; a, b)$, with $k + \ell = p$ and $a + b = q$.

With this notation, Lemma 1 above is summarized as

$$\overline{\partial}(k, \ell; a, b) \subseteq (k-1, \ell+1; a+1, b). \tag{24}$$

Note that in our notations, whenever anyone of the four indices $(k, \ell; a, b)$ is less than zero, we mean the trivial vector space $\{0\}$. Due to the assumption that $\dim \mathfrak{c}^{1,0} = 1$, the space $(k-1, \ell+1; a+1, b)$ is trivial when $\ell \geq 1$.

Next, we turn our attention to the structure of the Schouten bracket. In particular, we are interested in the adjoint action of elements in $\mathfrak{c}^{1,0}$ and $\mathfrak{t}^{1,0}$. Since the complex algebra $\mathfrak{g}^{1,0}$ is abelian, it is clear that for any W in $\mathfrak{c}^{1,0}$, the action of ad_W on $\mathfrak{g}^{1,0}$ is identically zero. Since for any $\overline{\omega} \in \mathfrak{t}^{*(0,1)}$, $d\overline{\omega} = 0$. It follows that $\mathrm{ad}_W \overline{\omega} = 0$. For any $\overline{\rho} \in \mathfrak{c}^{*(0,1)}$, $d\overline{\rho} \in \mathfrak{t}^{*(1,1)}$. it follows that $\mathrm{ad}_W \overline{\rho} = 0$ as well. Therefore, the action of ad_W on $\mathfrak{g}^{*(0,1)}$ is identically zero; and subsequently $\mathrm{ad}_W \equiv 0$ on $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{*(0,1)}$. For future reference we summarize our observation on the central vector field W in a lemma.

Lemma 2 *The space* $\mathfrak{c}^{1,0}$ *is in the center of the Schouten algebra* $(\oplus_{p,q\geq 0} B^{p,q}, [\![-,-]\!])$.

Similarly, for any $T \in \mathfrak{t}^{1,0}$, the restriction of ad_T on $\mathfrak{g}^{1,0} \oplus \mathfrak{t}^{*(0,1)}$ is identically zero, and the image of ad_T acting on $\mathfrak{c}^{*(0,1)}$ is contained in $\mathfrak{t}^{*(0,1)}$.

Lemma 3 *For any* $T \in \mathfrak{t}^{1,0}$*,* $\mathfrak{g}^{1,0} \oplus \mathfrak{t}^{*(0,1)} \subseteq \ker \mathrm{ad}_T$*, and* $\mathrm{ad}_T(\mathfrak{c}^{*(0,1)}) \subseteq \mathfrak{t}^{*(0,1)}$.

Lemma 4 See also [3, Lemma 7]. *Suppose that* $\Lambda_1 = W \wedge T$ *and* $\Lambda_2 = T_1 \wedge T_2$ *where* $W \in \mathfrak{c}^{1,0}$ *and* T*,* T_1*,* $T_2 \in \mathfrak{t}^{1,0}$*, then*

$$\mathrm{ad}_{\Lambda_1}(k,\ell;a,b) \subseteq (k,\ell+1;a+1,b-1). \tag{25}$$

$$\mathrm{ad}_{\Lambda_2}(k,\ell;a,b) \subseteq (k+1,\ell;a+1,b-1). \tag{26}$$

Proof: Whenever V_1, V_2 are in $\mathfrak{g}^{1,0}$ and Φ in $B^{p,q}$, then

$$[\![V_1 \wedge V_2, \Phi]\!] = V_1 \wedge [\![V_2, \Phi]\!] - V_2 \wedge [\![V_1, \Phi]\!],$$

The lemma in question now follows the previous two. ∎

In particular, for any $\overline{\rho} \in \mathfrak{c}^{*(0,1)}$,

$$[\![\Lambda_1, \overline{\rho}]\!] \in \mathfrak{c}^{1,0} \otimes \mathfrak{t}^{*(0,1)} \quad \text{and} \quad [\![\Lambda_2, \overline{\rho}]\!] \in \mathfrak{t}^{1,0} \otimes \mathfrak{t}^{*(0,1)}. \tag{27}$$

5 Computation of the map d_2

Now on a 2-step nilmanifold M with abelian complex structure, if $\dim \mathfrak{c}^{1,0} = 1$, then

$$\mathfrak{g}^{2,0} = (\mathfrak{t}^{1,0} \otimes \mathfrak{c}^{1,0}) \oplus \mathfrak{t}^{2,0}.$$

Therefore, an invariant bi-vector Λ decomposes into a sum of two types.

$$\Lambda = \Lambda_1 + \Lambda_2$$

where $\Lambda_1 \in \mathfrak{t}^{1,0} \otimes \mathfrak{c}^{1,0}$ and $\Lambda_2 \in \mathfrak{t}^{2,0}$. Since the complex structure is abelian, $[\![\Lambda, \Lambda]\!] = 0$. Therefore, all bi-vectors are Poisson. In subsequence computation, we assume that Λ is holomorphic.

For completeness we quickly review a computation of the map d_2 [3]. Recall that

$$E_2^{p,q} = \frac{\text{kernel of } \mathrm{ad}_\Lambda : H^q(\mathfrak{g}^{p,0}) \to H^q(\mathfrak{g}^{p+1,0})}{\text{image of } \mathrm{ad}_\Lambda : H^q(\mathfrak{g}^{p-1,0}) \to H^q(\mathfrak{g}^{p,0})}. \tag{28}$$

An element in $E_2^{p,q}$ is represented by an element $\Upsilon \in \mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)}$ such that it is in $H^q(\mathfrak{g}^{p,0})$ and in the kernel of ad_Λ. Equivalently, Υ is $\overline{\partial}$-closed and $\mathrm{ad}_\Lambda \Upsilon$ is $\overline{\partial}$-exact. Therefore, there exists $\Gamma \in \mathfrak{g}^{p+1,0} \otimes \mathfrak{g}^{*(0,q-1)}$ such that

$$\overline{\partial}\Upsilon = 0 \quad \text{and} \quad \mathrm{ad}_\Lambda \Upsilon = \overline{\partial}\Gamma. \tag{29}$$

By definition, $d_2[\Upsilon]$ is represented by $\mathrm{ad}_\Lambda \Gamma$.

When $\dim_{\mathbb{C}} \mathfrak{c}^{1,0} = 1$, the space $(k, \ell; a, b)$ is equal to the zero set except when $b \in \{0, 1\}$ and $\ell \in \{0, 1\}$. Under this assumption, we consider the components of Υ. By Lemma 4, its component in $(k, \ell; a, 0)$ is mapped to zero by ad_Λ. In such case, Γ could be chosen to be zero, and hence d_2 maps this component of Υ to zero.

The non-trivial case is when Υ has a component in $(k, \ell; a, 1)$. By (25) and (26), $\mathrm{ad}_\Lambda(\Upsilon)$ is contained in the direct sum of the following two types of subspaces

$$(k, \ell + 1; a + 1, 0), \qquad (k + 1, \ell; a + 1, 0).$$

By Lemma 1 or equivalently Identity (24), for Γ to be a solution of (29), it must come from the direct sum of the following two types of subspaces.

$$(k + 1, \ell; a, 0), \qquad (k + 2, \ell - 1; a, 0).$$

By (25) and (26), elements of these types are in the kernel of ad_{Λ_1} and ad_{Λ_2}. We conclude that $\mathrm{ad}_\Lambda \Gamma = 0$, and hence $d_2[\Upsilon] = 0$. So we recover a theorem in [3].

Theorem 3 [3, Theorem 2] *Suppose that M is a 2-step nilmanifold with an abelian complex structure. Suppose that the center of the Lie algebra of the simply connected covering space is real two-dimensional, then for any invariant holomorphic Poisson structure, the spectral sequence of the Poisson bi-complex degenerates on the second page.*

6 Computation of the map d_1

Next, we push the work in the previous section to study first page degeneracy. First of all, we explore the necessary condition for the spectral sequence of the bi-complex of $\Lambda = \Lambda_1 + \Lambda_2$ to degenerate on the first page.

Consider the adjoint action of Λ on the zero-th row on the first page of the spectral sequence.

$$\mathrm{ad}_\Lambda : H^0(\mathfrak{g}^{p,0}) \to H^0(\mathfrak{g}^{p+1,0}).$$

Since the complex structure is abelian, this map is identically zero. As $\Lambda = \Lambda_1 + \Lambda_2$, by linearity of the Schouten bracket $\mathrm{ad}_\Lambda = \mathrm{ad}_{\Lambda_1} + \mathrm{ad}_{\Lambda_2}$.

Since $\overline{\rho}$ is $\overline{\partial}$-closed, it represents an element in $H^1(\mathbb{C})$. If ad_Λ is identically zero, then in particular, $\mathrm{ad}_{\Lambda_1}\overline{\rho} + \mathrm{ad}_{\Lambda_2}\overline{\rho}$ is $\overline{\partial}$-exact.

With respect to the type decomposition $(k, \ell; a, b)$, $\overline{\rho}$ is type $(0, 0; 0, 1)$. By (25), $\mathrm{ad}_{\Lambda_1}\overline{\rho}$ is type $(0, 1; 1, 0)$. By (26), $\mathrm{ad}_{\Lambda_2}\overline{\rho}$ is type $(1, 0; 1, 0)$. As $\mathrm{ad}_{\Lambda_1}\overline{\rho}$ and $\mathrm{ad}_{\Lambda_2}\overline{\rho}$ are of different types, $\mathrm{ad}_{\Lambda_1}\overline{\rho} + \mathrm{ad}_{\Lambda_2}\overline{\rho}$ is $\overline{\partial}$-exact if and only if $\mathrm{ad}_{\Lambda_1}\overline{\rho}$ and $\mathrm{ad}_{\Lambda_2}\overline{\rho}$ are both $\overline{\partial}$-exact.

However, by Lemma 1 or equivalently Identity (24) the image of the $\overline{\partial}$-operator is of type $(k, 1; a, b)$. A type $(1, 0; 1, 0)$ element such as $\mathrm{ad}_{\Lambda_2}\overline{\rho}$ could be $\overline{\partial}$-exact only if it is identically zero.

By Lemma 4 and Formula (26), $\mathfrak{t}^{*(0,1)} \oplus \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{1,0}$ is in the kernel of ad_{Λ_2}. If $\mathrm{ad}_{\Lambda_2}\overline{\rho} = 0$ in addition, then ad_{Λ_2} vanishes on $\mathfrak{g}^{*(0,1)} \oplus \mathfrak{g}^{1,0}$. It follows that the action of ad_{Λ_2} on $B^{p,q}$ is identically zero for all $p, q \geq 0$.

Lemma 5 *If the spectral sequence of the bi-complex of* $\Lambda = \Lambda_1 + \Lambda_2$ *degenerates on the first page, then* Λ_2 *is the center of the Schouten algebra* $\oplus_{p,q} B^{p,q}$.

Now we proceed by assuming that Λ_2 satisfies the necessary condition in the last lemma. In particular, $\mathrm{ad}_\Lambda = \mathrm{ad}_{\Lambda_1} + \mathrm{ad}_{\Lambda_2} = \mathrm{ad}_{\Lambda_1}$.

Since $\mathfrak{c}^{1,0}$ is one-dimensional, $\Lambda_1 = W \wedge T$ for some $T \in \mathfrak{t}^{1,0}$. ad_{Λ_1} is $\overline{\partial}$-exact if there exists V in $\mathfrak{t}^{1,0}$ such that $[\![\Lambda_1, \overline{\rho}]\!] = \overline{\partial} V$. Under this condition, we compute ad_{Λ_1}.

Lemma 6 *Suppose that* $\mathrm{ad}_{\Lambda_1}\overline{\rho}$ *is* $\overline{\partial}$*-exact, then the map* ad_{Λ_1} *sends any element in* $H^q(\mathfrak{g}^{p,0})$ *to a* $\overline{\partial}$*-exact element.*

Proof: Given the dimension constraint on $\mathfrak{c}^{1,0}$, (25) shows that the action of ad_{Λ_1} is non-trivial only possibly when it acts on components of type $(k, 0; a, 1)$. Therefore, if Υ represents a class in $H^q(\mathfrak{g}^{p,0})$, ad_{Λ_1} could possibly be non-trivial only when Υ is of type $(p, 0; q-1, 1)$.

Given such Υ, there exist finitely many $\overline{\Omega}_j$ in $\mathfrak{t}^{*(0,q-1)}$ and the same number of Θ_j in $\mathfrak{t}^{p,0}$ such that

$$\Upsilon = \overline{\rho} \wedge \sum_j (\overline{\Omega}_j \wedge \Theta_j).$$

Therefore, $[\![\Lambda_1, \Upsilon]\!]$ is equal to

$$[\![\Lambda_1, \overline{\rho} \wedge \sum_j (\overline{\Omega}_j \wedge \Theta_j)]\!] = W \wedge [\![T, \overline{\rho} \wedge \sum_j (\overline{\Omega}_j \wedge \Theta_j)]\!] - T \wedge [\![W, \overline{\rho} \wedge \sum_j (\overline{\Omega}_j \wedge \Theta_j)]\!].$$

By Lemma 2, the last term on the right hand side is identical zero. By Lemma 3, the first term on the right hand side is equal to

$$W \wedge [\![T, \overline{\rho}]\!] \wedge \sum_j (\overline{\Omega}_j \wedge \Theta_j) = [\![W \wedge T, \overline{\rho}]\!] \wedge \sum_j (\overline{\Omega}_j \wedge \Theta_j)$$

If $\mathrm{ad}_{\Lambda_1}\overline{\rho}$ is $\overline{\partial}$-exact, then there exists V in $\mathfrak{t}^{1,0}$ such that $[\![W \wedge T, \overline{\rho}]\!] = [\![\Lambda_1, \overline{\rho}]\!] = \overline{\partial} V$. Therefore,

$$[\![\Lambda_1, \Upsilon]\!] = [\![W \wedge T, \overline{\rho}]\!] \wedge \sum_j (\overline{\Omega}_j \wedge \Theta_j) = \overline{\partial} V \wedge (\sum_j (\overline{\Omega}_j \wedge \Theta_j)). \tag{30}$$

Given that $\overline{\rho}$ is $\overline{\partial}$-close,

$$\overline{\partial}\Upsilon = \overline{\partial}(\overline{\rho} \wedge \sum_j (\overline{\Omega}_j \wedge \Theta_j)) = -\overline{\rho} \wedge \overline{\partial}(\sum_j (\overline{\Omega}_j \wedge \Theta_j)). \tag{31}$$

Since $\overline{\partial}(\sum_j (\overline{\Omega}_j \wedge \Theta_j))$ is contained in $\mathfrak{t}^{*(0,q)} \otimes \mathfrak{t}^{p-1,0} \otimes \mathfrak{c}^{1,0}$, the exterior product on the right of the equality (31) is equal to zero only if $\overline{\partial}(\sum_j (\overline{\Omega}_j \wedge \Theta_j)) = 0$. Therefore, Υ is $\overline{\partial}$-closed if and only if $\sum_j (\overline{\Omega}_j \wedge \Theta_j)$ is $\overline{\partial}$-closed. It follows that Identity (30) is further transformed to

$$[\![\Lambda_1, \Upsilon]\!] = \overline{\partial}(\ V \wedge \sum_j (\overline{\Omega}_j \wedge \Theta_j)\ ). \tag{32}$$

It shows that the image of any $\overline{\partial}$-closed element via ad_{Λ_1} is $\overline{\partial}$-exact when $\mathrm{ad}_{\Lambda_1}\overline{\rho}$ is $\overline{\partial}$-exact. ■

As a consequence of the last two lemmas, we conclude that for the nilmanifolds and complex structures as specified, the necessary condition for the bi-complex of Λ to degenerate on the first page is also sufficient.

Theorem 4 *Let $M = G/\Gamma$ be a 2-step nilmanifold with abelian complex structure. Let $\mathfrak{c}$ be the center of the Lie algebra $\mathfrak{g}$ of the simply connected Lie group G. Let*

$\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{1,0}$ *be the space of invariant* $(1,0)$*-vectors. Assume that* $\dim \mathfrak{c}^{1,0} = 1$*. Suppose that* $\Lambda = \Lambda_1 + \Lambda_2$ *is an invariant holomorphic Poisson bivector with* $\Lambda_1 \in \mathfrak{t}^{1,0} \otimes \mathfrak{c}^{1,0}$ *and* $\Lambda_2 \in \mathfrak{t}^{2,0}$*. Let* $\overline{\rho}$ *span* $\mathfrak{c}^{*(0,1)}$*. The spectral sequence of the bi-complex of* Λ *degenerates on the first page if and only if*

- $\mathrm{ad}_{\Lambda_1}\overline{\rho}$ *is* $\overline{\partial}$*-exact; and*
- Λ_2 *is in the center of the Schouten algebra* $\oplus_{p,q} B^{p,q}$*.*

7 Condition for $\overline{\partial}$-exactness

In this section, we explore deeper in the situation when there exists V such that

$$[\![W \wedge T, \overline{\rho}]\!] = \overline{\partial} V. \tag{33}$$

On the left of the equality above,

$$[\![W \wedge T, \overline{\rho}]\!] = W \wedge [\![T, \overline{\rho}]\!] = W \wedge \iota_T d\overline{\rho}. \tag{34}$$

To compute $\overline{\partial} V$, we applies the Cartan Formula to evaluate $\overline{\partial} V$ on a generic element $\overline{T}$ in $\mathfrak{t}^{0,1}$ and on ρ.

$$\begin{aligned} \overline{\partial} V(\rho, \overline{T}) &= -V([\![\rho, \overline{T}]\!]) = V([\![\overline{T}, \rho]\!]) = V(\iota_{\overline{T}} d\rho) \\ &= d\rho(\overline{T}, V) = -d\rho(V, \overline{T}) = -(\iota_V d\rho)(\overline{T}). \end{aligned}$$

Therefore, $\overline{\partial} V = -W \wedge \iota_V d\rho$. Now comparing (33) with (34), we obtain a rather simple identity below.

$$\iota_T d\overline{\rho} = -\iota_V d\rho. \tag{35}$$

Since the complex structure J is abelian, $d\rho$ is a type $(1,1)$-form. So is $d\overline{\rho}$. We could treat their contractions with elements in $\mathfrak{t}^{1,0}$ as linear maps.

$$d\rho, \quad d\overline{\rho} \quad : \mathfrak{t}^{1,0} \to \mathfrak{t}^{*(0,1)}. \tag{36}$$

Suppose that $d\rho$ has a non-trivial kernel and $\iota_T d\overline{\rho} = 0$. In such case, $\mathrm{ad}_{W \wedge T}\overline{\rho} = 0$. However, since $d\overline{\omega}^j = 0$ for all j, $\mathrm{ad}_{W \wedge T}\overline{\omega}^j = 0$. As the complex structure is abelian, then the adjoint action of $W \wedge T$ on $\mathfrak{g}^{1,0}$ is identically zero. Therefore, $\mathrm{ad}_{W \wedge T}$ is identically zero on $B^{p,q}$ for all $p, q \geq 0$. It yields the trivial situation when both ad_{Λ_1} and ad_{Λ_2} on $B^{p,q}$ are identically zero. In such case, the action of ad_Λ is trivial and hence as action on $B^{p,q}$, $\overline{\partial}_\Lambda = \overline{\partial} + \mathrm{ad}_\Lambda = \overline{\partial}$. The Poisson cohomology is simply the direct sum of Dolbeault cohomology.

$$H^n_\Lambda(M) = \oplus_{p+q=n} H^q(\mathfrak{g}^{p,0}) = \oplus_{p+q=n} H^q(M, \Theta^p).$$

On the other hand, $d\rho(\mathfrak{t}^{1,0})$ is a proper subspace of $\mathfrak{t}^{*(0,1)}$ if $d\rho$ degenerates. If T is not in the kernel of $d\overline{\rho}$ but $\iota_T d\overline{\rho}$ is in the complement of $d\rho(\mathfrak{t}^{1,0})$ then the equation (35) does not necessarily have a solution. In the next section, we will present an example to demonstrate that such situation does occur. We summarize our discussion when $d\rho$ degenerates as below.

Theorem 5 *Let $M = G/\Gamma$ be a 2-step nilmanifold with abelian complex structure. Let $\mathfrak{c}$ be the center of the Lie algebra $\mathfrak{g}$ of the simply connected Lie group G. Let $\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{1,0}$ be the space of invariant $(1,0)$-vectors. Suppose that* $\dim \mathfrak{c}^{1,0} = 1$. *Let W span $\mathfrak{c}^{1,0}$ and ρ be its dual. If $d\overline{\rho}$ degenerates and T is in its kernel, then $\Lambda = W \wedge T$ is a holomorphic Poisson structure such that the spectral sequence of its associated bi-complex degenerates on the first page.*

If $d\overline{\rho}$ is non-degenerate, so would $d\rho$. Therefore, for any T the equation (35) always has a unique solution.

Theorem 6 *Let $M = G/\Gamma$ be a 2-step nilmanifold with abelian complex structure. Let $\mathfrak{c}$ be the center of the Lie algebra $\mathfrak{g}$ of the simply connected Lie group G. Let $\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{1,0}$ be the space of invariant $(1,0)$-vectors. Assume that* $\dim \mathfrak{c}^{1,0} = 1$. *Suppose that $\Lambda = \Lambda_1 + \Lambda_2$ is an invariant holomorphic Poisson bivector with $\Lambda_1 \in \mathfrak{t}^{1,0} \otimes \mathfrak{c}^{1,0}$ and $\Lambda_2 \in \mathfrak{t}^{2,0}$. Let $\overline{\rho}$ span $\mathfrak{c}^{*(0,1)}$. The spectral sequence of the bi-complex of Λ degenerates on the second page. If in addition, the map $d\overline{\rho}$ is non-degenerate and if Λ_2 is in the center of the Schouten algebra $\oplus_{p,q} \mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)}$, the spectral sequence degenerates on its first page.*

By setting $\Lambda_2 = 0$ in Theorem 6, we derive Theorem 1 as given in the Introduction.

8 Examples

As seen in Theorem 6, invariant holomorphic Poisson structure Λ has two components Λ_1 and Λ_2. If we requires the spectral sequence to degenerate on the first page, the action of ad_{Λ_2} is identically zero. Therefore, in this section we focus on the case when $\Lambda_2 = 0$, and hence $\Lambda = \Lambda_1 = W \wedge T$ where $W \in \mathfrak{c}^{1,0}$ and $T \in \mathfrak{t}^{1,0}$. As the complex structure is abelian, it is obvious that $[\![W \wedge T, W \wedge T]\!] = 0$. In addition, since $\overline{\partial} W = 0$, $\overline{\partial}\Lambda = -W \wedge \overline{\partial} T$. By (22), $\overline{\partial} T$ is in $\mathfrak{c}^{1,0} \otimes \mathfrak{t}^{*(0,1)}$. When $\dim \mathfrak{c}^{1,0} = 1$, $W \wedge \overline{\partial} T = 0$. Therefore, for any $T \in \mathfrak{t}^{1,0}$, the bivector $\Lambda = W \wedge T$ is an invariant holomorphic Poisson structure. We now focus on this type of structures, and present concrete examples to illustrate Theorem 5 as well as Theorem 6.

When $\dim \mathfrak{c}^{1,0} = 1$, the dual structure equations in (20) and (21) are simplified as below.

$$d\rho = \sum_{i,j} E_{ji}\omega^i \wedge \overline{\omega}^j, \qquad \text{and} \qquad d\overline{\rho} = -\sum_{i,j} \overline{E}_{ij}\omega^i \wedge \overline{\omega}^j.$$

Example 1. Consider a one-dimensional central extension of the Heisenberg algebra $\mathfrak{h}_{2n+1}$ of real dimension $2n+1$. Let $\{X_j, Y_j, Z, A : 1 \leq j \leq n\}$ be basis with structure equation

$$[\![X_j, Y_j]\!] = -[\![Y_j, X_j]\!] = Z, \qquad \text{for all} \qquad 1 \leq j \leq n. \tag{37}$$

The real center $\mathfrak{c}$ is spanned by Z and A. Define an almost complex structure by

$$JX_j = Y_j, \quad JY_j = -X_j, \quad JZ = A, \quad JA = -Z.$$

It is an abelian complex structure. Let $W = \frac{1}{2}(Z - iA)$ and $T_j = \frac{1}{2}(X_j - iY_j)$, then the complex structure equation becomes

$$[\![\overline{T}_j, T_j]\!] = -\frac{i}{2}(W + \overline{W}).$$

Therefore, $E_{ii} = -\frac{i}{2} = -\overline{E}_{ii}$. Hence $d\rho$ is non-degenerate and serves as an example for Theorem 6.

Example 2. On the direct sum of two Heisenberg algebras $\mathfrak{h}_{2m+1} \oplus \mathfrak{h}_{2n+1}$, choose a basis $\{X_j, Y_j, Z, A_k, B_k, C; 1 \leq j \leq m, 1 \leq k \leq n\}$ so that the non-zero structure equations are give by

$$[\![X_j, Y_j]\!] = -[\![Y_j, X_j]\!] = Z, \quad [\![A_k, B_k]\!] = -[\![B_k, A_k]\!] = C. \tag{38}$$

Define an almost complex structure J by

$$JX_j = Y_j, \quad JA_k = B_k, \quad JZ = C.$$

We get an abelian complex structure. Let

$$W = \frac{1}{2}(Z - iC), \quad S_j = \frac{1}{2}(X_j - iY_j), \quad T_k = \frac{1}{2}(A_k - iB_k).$$

$\mathfrak{c}^{1,0}$ is spanned by W. It follows that the non-zero complex structure equations are given as below.

$$[\![\overline{S}_j, S_j]\!] = -\frac{i}{2}(W + \overline{W}), \quad [\![\overline{T}_k, T_k]\!] = \frac{1}{2}(W - \overline{W}).$$

Then the structural constants

$$E_{jj} = -\frac{i}{2} = -\overline{E}_{jj}, \quad \text{and} \quad E_{kk} = \frac{1}{2} = \overline{E}_{kk},$$

for all $1 \le j \le m$ and $1 \le k \le n$. It is now obvious that $d\rho$ is non-degenerate and serves as example for Theorem 6.

Example 3. Consider a real vector space W_{4n+6} spanned by

$$\{X_{4k+1}, X_{4k+2}, X_{4k+3}, X_{4k+4}, Z_1, Z_2; 0 \le k \le n\}.$$

Define a Lie bracket by

$$\begin{aligned} [\![X_{4k+1}, X_{4k+3}]\!] &= -\frac{1}{2}Z_1, & [\![X_{4k+1}, X_{4k+4}]\!] &= -\frac{1}{2}Z_2, \\ [\![X_{4k+2}, X_{4k+3}]\!] &= -\frac{1}{2}Z_2, & [\![X_{4k+2}, X_{4k+3}]\!] &= \frac{1}{2}Z_1. \end{aligned}$$

One could define an abelian complex structure J by

$$JX_{4k+1} = X_{4k+2}, \quad JX_{4k+3} = -X_{4k+4}, \quad JZ_1 = -Z_2.$$

For $0 \leq k \leq n$, define

$$W = \frac{1}{2}(Z_1 + iZ_2), \quad T_{2k+1} = \frac{1}{2}(X_{4k+1} - iX_{4k+2}), \quad T_{2k+2} = \frac{1}{2}(X_{4k+3} + iX_{4k+4}).$$

It is now a straightforward computation to show that the non-zero structure equations are

$$[\![\overline{T}_{2k+1}, T_{2k+2}]\!] = -\frac{1}{2}W, \quad [\![\overline{T}_{2k+2}, T_{2k+1}]\!] - = \frac{1}{2}\overline{W}. \tag{39}$$

Except when

$$E_{2k+1,2k+2} = -\frac{1}{2}, \quad \text{for all} \quad 0 \leq k \leq n,$$

all other structure constants are equal to zero. In particular,

$$d\rho = -\frac{1}{2}\sum_{k=0}^{n} \omega^{2k+2} \wedge \overline{\omega}^{2k+1}, \quad \text{and} \quad d\overline{\rho} = \frac{1}{2}\sum_{k=0}^{n} \omega^{2k+1} \wedge \overline{\omega}^{2k+2}.$$

Treating $d\rho$ and $d\overline{\rho}$ as maps from $\mathfrak{t}^{1,0}$ to $\mathfrak{t}^{0,1}$, their image spaces are transversal. Therefore, given $T \in \mathfrak{t}^{1,0}$ such that there exists $V \in \mathfrak{t}^{1,0}$ with $\iota_T d\overline{\rho} = -\iota_V d\rho$ only if $\iota_T d\overline{\rho} = 0$. It is possible only when T is the complex linear span of $\{T_2, \ldots, T_{2k+2}, \ldots, T_{2n+2}\}$, which is the kernel of $d\overline{\rho}$.

In this case, it simply means that $[\![T, \overline{\rho}]\!] = 0$, and hence $\mathrm{ad}_\Lambda = 0$. This example illustrates the situation in Theorem 5.

Example 4. Consider a real vector space P_{4n+2} spanned by

$$\{X_{4k+1}, X_{4k+2}, X_{4k+3}, X_{4k+4}, Z_1, Z_2; 0 \leq k \leq n\}.$$

Define a Lie bracket by

$$[\![X_{4k+1}, X_{4k+2}]\!] = -\frac{1}{2}Z_1, \quad [\![X_{4k+1}, X_{4k+4}]\!] = -\frac{1}{2}Z_2, \quad [\![X_{4k+2}, X_{4k+3}]\!] = -\frac{1}{2}Z_2.$$

Define an abelian complex structure J on P_{4n+2} by

$$JX_{4k+1} = X_{4k+2}, \quad JX_{4k+3} = -X_{4k+4}, \quad JZ_1 = -Z_2.$$

Define

$$W = \frac{1}{2}(Z_1 + iZ_2), \quad T_{2k+1} = \frac{1}{2}(X_{4k+1} - iX_{4k+2}), \quad T_{2k+2} = \frac{1}{2}(X_{4k+3} + iX_{4k+4}).$$

The non-zero structure equations in terms of complex vectors are

$$[\![\overline{T}_{2k+1}, T_{2k+1}]\!] = \frac{i}{4}(W + \overline{W}); \quad [\![\overline{T}_{2k+1}, T_{2k+2}]\!] = -\frac{1}{4}(W - \overline{W}).$$

Hence

$$E_{2k+1,2k+1} = \frac{i}{4}, \quad E_{2k+1,2k+2} = -\frac{1}{4}, \quad E_{2k+2,2k+1} = -\frac{1}{4}.$$

It follows that $d\rho$ is non-degenerate, and hence (35) is solvable. We have an example for Theorem 6 again.

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