Asymptotics For High Dimensional Regression *M*-Estimates: Fixed Design Results

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Abstract

We investigate the asymptotic distributions of coordinates of regression M-estimates in the moderate p/n regime, where the number of covariates p grows proportionally with the sample size n. Under appropriate regularity conditions, we establish the coordinate-wise asymptotic normality of regression M-estimates assuming a fixed-design matrix. Our proof is based on the second-order Poincaré inequality (Chatterjee, 2009) and leave-one-out analysis (El Karoui et al., 2011). Some relevant examples are indicated to show that our regularity conditions are satisfied by a broad class of design matrices. We also show a counterexample, namely the ANOVA-type design, to emphasize that the technical assumptions are not just artifacts of the proof. Finally, the numerical experiments confirm and complement our theoretical results.

1 Introduction

High-dimensional statistics has a long history (Huber, 1973; Wachter, 1976, 1978) with considerable renewed interest over the last two decades. In many applications, the researcher collects data which can be represented as a matrix, called a design matrix and denoted by $X \in \mathbb{R}^{n \times p}$, as well as a response vector $y \in \mathbb{R}^n$ and aims to study the connection between X and y. The linear model is among the most popular models as a starting point of data analysis in various fields. A linear model assumes that

$$y = X\beta^* + \epsilon, \tag{1}$$

where $\beta^* \in \mathbb{R}^p$ is the coefficient vector which measures the marginal contribution of each predictor and ϵ is a random vector which captures the unobserved errors.

The aim of this article is to provide valid inferential results for features of β^* . For example, a researcher might be interested in testing whether a given predictor has a negligible effect on the response, or equivalently whether $\beta_j^* = 0$ for some j. Similarly, linear contrasts of β^* such as $\beta_1^* - \beta_2^*$ might be of interest in the case of the group comparison problem in which the first two predictors represent the same feature but are collected from two different groups.

An M-estimator, defined as

$$\hat{\beta}(\rho) = \underset{\beta \in \mathbb{R}^p}{\operatorname{arg\,min}} \frac{1}{n} \sum_{i=1}^n \rho(y_i - x_i^T \beta)$$
(2)

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where ρ denotes a loss function, is among the most popular estimators used in practice (Relles, 1967; Huber, 1973). In particular, if $\rho(x) = \frac{1}{2}x^2$, $\hat{\beta}(\rho)$ is the famous Least Square Estimator (LSE). We intend to explore the distribution of $\hat{\beta}(\rho)$, based on which we can achieve the inferential goals mentioned above.

The most well-studied approach is the asymptotic analysis, which assumes that the scale of the problem grows to infinity and use the limiting result as an approximation. In regression problems, the scale parameter of a problem is the sample size n and the number of predictors p. The classical approach is to fix p and let n grow to infinity. It has been shown (Relles, 1967; Yohai, 1972; Huber, 1972, 1973) that $\hat{\beta}(\rho)$ is consistent in terms of L_2 norm and asymptotically normal in this regime. The asymptotic variance can be then approximated by the bootstrap (Bickel & Freedman, 1981). Later on, the studies are extended to the regime in which both n and p grow to infinity but p/n converges to 0 (Yohai & Maronna, 1979; Portnoy, 1984, 1985, 1986, 1987; Mammen, 1989). The consistency, in terms of the L_2 norm, the asymptotic normality and the validity of the bootstrap still hold in this regime. Based on these results, we can construct a 95% confidence interval for β_{0j} simply as $\hat{\beta}_j(\rho) \pm 1.96\sqrt{\operatorname{Var}(\hat{\beta}_j(\rho))}$ where $\operatorname{Var}(\hat{\beta}_j(\rho))$ is calculated by bootstrap. Similarly we can calculate p-values for the hypothesis testing procedure.

We ask whether the inferential results developed under the low-dimensional assumptions and the software built on top of them can be relied on for moderate and highdimensional analysis? Concretely, if in a study n = 50 and p = 40, can the software built upon the assumption that $p/n \simeq 0$ be relied on when p/n = .8? Results in random matrix theory (Marčenko & Pastur, 1967) already offer an answer in the negative side for many PCA-related questions in multivariate statistics. The case of regression is more subtle: For instance for least-squares, standard degrees of freedom adjustments effectively take care of many dimensionality-related problems. But this nice property does not extend to more general regression M-estimates.

Once these questions are raised, it becomes very natural to analyze the behavior and performance of statistical methods in the regime where p/n is fixed. Indeed, it will help us to keep track of the inherent statistical difficulty of the problem when assessing the variability of our estimates. In other words, we assume in the current paper that $p/n \to \kappa > 0$ while let n grows to infinity. Due to identifiability issues, it is impossible to make inference on β^* if p > n without further structural or distributional assumptions. We discuss this point in details in Section 2.3. Thus we consider the regime where $p/n \to \kappa \in (0, 1)$. We call it the moderate p/n regime. This regime is also the natural regime in random matrix theory (Marčenko & Pastur, 1967; Wachter, 1978; Johnstone, 2001; Bai & Silverstein, 2010). It has been shown that the asymptotic results derived in this regime sometimes provide an extremely accurate approximations to finite sample distributions of estimators at least in certain cases (Johnstone, 2001) where n and p are both small.

1.1 Qualitatively Different Behavior of Moderate p/n Regime

First, $\hat{\beta}(\rho)$ is no longer consistent in terms of L_2 norm and the risk $\mathbb{E}\|\hat{\beta}(\rho) - \beta^*\|^2$ tends to a non-vanishing quantity determined by κ , the loss function ρ and the error distribution through a complicated system of non-linear equations (El Karoui et al., 2011; El Karoui, 2013, 2015; Bean et al., 2012). This L_2 -inconsistency prohibits the use of standard perturbation-analytic techniques to assess the behavior of the estimator. It also leads to qualitatively different behaviors for the residuals in moderate dimensions; in contrast to the low-dimensional case, they cannot be relied on to give accurate information about the distribution of the errors. However, this seemingly negative result does not exclude the possibility of inference since $\hat{\beta}(\rho)$ is still consistent in terms of $L_{2+\nu}$ norms for any $\nu > 0$ and in particular in L_{∞} norm. Thus, we can at least hope to perform inference on each coordinate.

Second, classical optimality results do not hold in this regime. In the regime $p/n \rightarrow 0$, the maximum likelihood estimator is shown to be optimal (Huber, 1964, 1972; Bickel &

Doksum, 2015). In other words, if the error distribution is known then the M-estimator associated with the loss $\rho(\cdot) = -\log f_{\epsilon}(\cdot)$ is asymptotically efficient, provided the design is of appropriate type, where $f_{\epsilon}(\cdot)$ is the density of entries of ϵ . However, in the moderate p/n regime, it has been shown that the optimal loss is no longer the log-likehood but an other function with a complicated but explicit form (Bean et al., 2013), at least for certain designs. The suboptimality of maximum likelihood estimators suggests that classical techniques fail to provide valid intuition in the moderate p/n regime.

Third, the joint asymptotic normality of $\hat{\beta}(\rho)$, as a p-dimensional random vector, may be violated for a fixed design matrix X. This has been proved for least-squares by Huber (1973) in his pioneering work. For general M-estimators, this negative result is a simple consequence of the results of El Karoui et al. (2011): They exhibit an ANOVA design (see below) where even marginal fluctuations are not Gaussian. By contrast, for random design, they show that $\hat{\beta}(\rho)$ is jointly asymptotically normal when the design matrix is elliptical with general covariance by using the non-asymptotic stochastic representation for $\hat{\beta}(\rho)$ as well as elementary properties of vectors uniformly distributed on the uniform sphere in \mathbb{R}^p ; See section 2.2.3 of El Karoui et al. (2011) or the supplementary material of Bean et al. (2013) for details. This does not contradict Huber (1973)'s negative result in that it takes the randomness from both X and ϵ into account while Huber (1973)'s result only takes the randomness from ϵ into account. Later, El Karoui (2015) shows that each coordinate of $\beta(\rho)$ is asymptotically normal for a broader class of random designs. This is also an elementary consequence of the analysis in El Karoui (2013). However, to the best of our knowledge, beyond the ANOVA situation mentioned above, there are no distributional results for fixed design matrices. This is the topic of this article.

Last but not least, bootstrap inference fails in this moderate-dimensional regime. This has been shown by Bickel and Freedman (1983) for least-squares and residual bootstrap in their influential work. Recently, El Karoui and Purdom (2015) studied the results to general M-estimators and showed that all commonly used bootstrapping schemes, including pairs-bootstrap, residual bootstrap and jackknife, fail to provide a consistent variance estimator and hence valid inferential statements. These latter results even apply to the marginal distributions of the coordinates of $\hat{\beta}(\rho)$. Moreover, there is no simple, design independent, modification to achieve consistency (El Karoui & Purdom, 2015).

1.2 Our Contributions

In summary, the behavior of the estimators we consider in this paper is completely different in the moderate p/n regime from its counterpart in the low-dimensional regime. As discussed in the next section, moving one step further in the moderate p/n regime is interesting from both the practical and theoretical perspectives. The main contribution of this article is to establish coordinate-wise asymptotic normality of $\hat{\beta}(\rho)$ for certain fixed design matrices X in this regime under technical assumptions. The following theorem informally states our main result.

Theorem (Informal Version of Theorem 3.1 in Section 3). Under appropriate conditions on the design matrix X, the distribution of ϵ and the loss function ρ , as $p/n \to \kappa \in (0, 1)$, while $n \to \infty$,

$$\max_{1 \le j \le p} d_{\mathrm{TV}} \left(\mathcal{L} \left(\frac{\hat{\beta}_j(\rho) - \mathbb{E}\hat{\beta}_j(\rho)}{\sqrt{\mathrm{Var}(\hat{\beta}_j(\rho))}} \right), N(0, 1) \right) = o(1)$$

where $d_{\mathrm{TV}}(\cdot, \cdot)$ is the total variation distance and $\mathcal{L}(\cdot)$ denotes the law.

It is worth mentioning that the above result can be extended to finite dimensional linear contrasts of $\hat{\beta}$. For instance, one might be interested in making inference on $\beta_1^* - \beta_2^*$ in the problems involving the group comparison. The above result can be extended to give the asymptotic normality of $\hat{\beta}_1 - \hat{\beta}_2$.

Besides the main result, we have several other contributions. First, we use a new approach to establish asymptotic normality. Our main technique is based on the second-order Poincaré inequality (SOPI), developed by Chatterjee (2009) to derive, among many

other results, the fluctuation behavior of linear spectral statistics of random matrices. In contrast to classical approaches such as the Lindeberg-Feller central limit theorem, the second-order Poincaré inequality is capable of dealing with nonlinear and potentially implicit functions of independent random variables. Moreover, we use different expansions for $\hat{\beta}(\rho)$ and residuals based on double leave-one-out ideas introduced in El Karoui et al. (2011), in contrast to the classical perturbation-analytic expansions. See aforementioned paper and follow-ups. An informal interpretation of the results of Chatterjee (2009) is that if the Hessian of the nonlinear function of random variables under consideration is sufficiently small, this function acts almost linearly and hence a standard central limit theorem holds.

Second, to the best of our knowledge this is the first inferential result for fixed (non ANOVA-like) design in the moderate p/n regime. Fixed designs arise naturally from an experimental design or a conditional inference perspective. That is, inference is ideally carried out without assuming randomness in predictors; see Section 2.2 for more details. We clarify the regularity conditions for coordinate-wise asymptotic normality of $\hat{\beta}(\rho)$ explicitly, which are checkable for LSE and also checkable for general M-estimators if the error distribution is known. We also prove that these conditions are satisfied with by a broad class of designs.

The ANOVA-like design described in Section 3.3.4 exhibits a situation where the distribution of $\hat{\beta}_j(\rho)$ is not going to be asymptotically normal. As such the results of Theorem 3.1 below are somewhat surprising.

For complete inference, we need both the asymptotic normality and the asymptotic bias and variance. Under suitable symmetry conditions on the loss function and the error distribution, it can be shown that $\hat{\beta}(\rho)$ is unbiased (see Section 3.2.1 for details) and thus it is left to derive the asymptotic variance. As discussed at the end of Section 1.1, classical approaches, e.g. bootstrap, fail in this regime. For least-squares, classical results continue to hold and we discuss it in section 5 for the sake of completeness. However, for M-estimators, there is no closed-form result. We briefly touch upon the variance estimation in Section 3.4.2. The derivation for general situations is beyond the scope of this paper and left to the future research.

1.3 Outline of Paper

The rest of the paper is organized as follows: In Section 2, we clarify details which are mentioned in the current section. In Section 3, we state the main result (Theorem 3.1) formally and explain the technical assumptions. Then we show several examples of random designs which satisfy the assumptions with high probability. In Section 4, we introduce our main technical tool, second-order Poincaré inequality (Chatterjee, 2009), and apply it on M-estimators as the first step to prove Theorem 3.1. Since the rest of the proof of Theorem 3.1 is complicated and lengthy, we illustrate the main ideas in Appendix A. The rigorous proof is left to Appendix B. In Section 5, we provide reminders about the theory of least-squares estimation for the sake of completeness, by taking advantage of its explicit form. In Section 6, we display the numerical results. The proof of other results are stated in Appendix C and more numerical experiments are presented in Appendix D.

2 More Details on Background

2.1 Moderate p/n Regime: a more informative type of asymptotics?

In Section 1, we mentioned that the ratio p/n measures the difficulty of statistical inference. The moderate p/n regime provides an approximation of finite sample properties with the difficulties fixed at the same level as the original problem. Intuitively, this regime should capture more variation in finite sample problems and provide a more accurate approximation. We will illustrate this via simulation. Consider a study involving 50 participants and 40 variables; we can either use the asymptotics in which p is fixed to be 40, n grows to infinity or p/n is fixed to be 0.8, and n grows to infinity to perform approximate inference. Current software rely on low-dimensional asymptotics for inferential tasks, but there is no evidence that they yield more accurate inferential statements than the ones we would have obtained using moderate dimensional asymptotics. In fact, numerical evidence (Johnstone, 2001; El Karoui et al., 2013; Bean et al., 2013) show that the reverse is true.

We exhibit a further numerical simulation showing that. Consider a case that n = 50, ϵ has i.i.d. entries and X is one realization of a matrix generated with i.i.d. gaussian (mean 0, variance 1) entries. For $\kappa \in \{0.1, 0.2, \dots, 0.9\}$ and different error distributions, we use the Kolmogorov-Smirnov (KS) statistics to quantify the distance between the finite sample distribution and two types of asymptotic approximation of the distribution of $\hat{\beta}_1(\rho)$.

Specifically, we use the Huber loss function $\rho_{\text{Huber},k}$ with default parameter k = 1.345 (Huber, 2011), i.e.

$$\rho_{\text{Huber},k}(x) = \begin{cases} \frac{1}{2}x^2 & |x| \le k\\ k(|x| - \frac{1}{2}k) & |x| > k \end{cases}$$

Specifically, we generate three design matrices $X^{(0)}$, $X^{(1)}$ and $X^{(2)}$: $X^{(0)}$ for small sample case with a sample size n = 50 and a dimension $p = n\kappa$; $X^{(1)}$ for low-dimensional asymptotics (p fixed) with a sample size n = 1000 and a dimension $p = 50\kappa$; and $X^{(2)}$ for moderate-dimensional asymptotics (p/n fixed) with a sample size n = 1000 and a dimension $p = n\kappa$. Each of them is generated as one realization of an i.i.d. standard gaussian design and then treated as fixed across K = 100 repetitions. For each design matrix, vectors ϵ of appropriate length are generated with i.i.d. entries. The entry has either a standard normal distribution, or a t_3 -distribution, or a standard Cauchy distribution, i.e. t_1 . Then we use ϵ as the response, or equivalently assume $\beta^* = 0$, and obtain the M-estimators $\hat{\beta}^{(0)}, \hat{\beta}^{(1)}, \hat{\beta}^{(2)}$. Repeating this procedure for K = 100 times results in K replications in three cases. Then we extract the first coordinate of each estimator, denoted by $\{\hat{\beta}_{k,1}^{(0)}\}_{k=1}^{K}, \{\hat{\beta}_{k,1}^{(2)}\}_{k=1}^{K}$. Then the two-sample Kolmogorov-Smirnov statistics can be obtained by

$$\mathrm{KS}_{1} = \sqrt{\frac{n}{2}} \max_{x} |\hat{F}_{n}^{(0)}(x) - \hat{F}_{n}^{(1)}(x)|, \quad \mathrm{KS}_{2} = \sqrt{\frac{n}{2}} \max_{x} |\hat{F}_{n}^{(0)}(x) - \hat{F}_{n}^{(2)}(x)|,$$

where $\hat{F}_n^{(r)}$ is the empirical distribution of $\{\hat{\beta}_{k,1}^{(r)}\}_{k=1}^K$. We can then compare the accuracy of two asymptotic regimes by comparing KS₁ and KS₂. The smaller the value of KS_i, the better the approximation.

Figure 1 displays the results for these error distributions. We see that for gaussian errors and even t_3 errors, the p/n-fixed/moderate-dimensional approximation is uniformly more accurate than the widely used p-fixed/low-dimensional approximation. For Cauchy errors, the low-dimensional approximation performs better than the moderatedimensional one when p/n is small but worsens when the ratio is large especially when p/n is close to 1. Moreover, when p/n grows, the two approximations have qualitatively different behaviors: the p-fixed approximation becomes less and less accurate while the p/n-fixed approximation does not suffer much deterioration when p/n grows. The qualitative and quantitative differences of these two approximations reveal the practical importance of exploring the p/n-fixed asymptotic regime. (See also Johnstone (2001).)

2.2 Random vs fixed design?

As discussed in Section 1.1, assuming a fixed design or a random design could lead to qualitatively different inferential results.

In the random design setting, X is considered as being generated from a super population. For example, the rows of X can be regarded as an i.i.d. sample from a distribution known, or partially known, to the researcher. In situations where one uses techniques such as cross-validation (Stone, 1974), pairs bootstrap in regression (Efron &



Figure 1: Axpproximation accuracy of *p*-fixed asymptotics and p/n-fixed asymptotics: each column represents an error distribution; the x-axis represents the ratio κ of the dimension and the sample size and the y-axis represents the Kolmogorov-Smirnov statistic; the red solid line corresponds to *p*-fixed approximation and the blue dashed line corresponds to p/n-fixed approximation.

Efron, 1982) or sample splitting (Wasserman & Roeder, 2009), the researcher effectively assumes exchangeability of the data $(x_i^T, y_i)_{i=1}^n$. Naturally, this is only compatible with an assumption of random design. Given the extremely widespread use of these techniques in contemporary machine learning and statistics, one could argue that the random design setting is the one under which most of modern statistics is carried out, especially for prediction problems. Furthermore, working under a random design assumption forces the researcher to take into account two sources of randomness as opposed to only one in the fixed design case. Hence working under a random design assumption should yield conservative confidence intervals for β_i^* .

In other words, in settings where the researcher collects data without control over the values of the predictors, the random design assumption is arguably the more natural one of the two.

However, it has now been understood for almost a decade that common random design assumptions in high-dimension (e.g. $x_i = \Sigma^{1/2} z_i$ where $z_{i,j}$'s are i.i.d with mean 0 and variance 1 and a few moments and Σ "well behaved") suffer from considerable geometric limitations, which have substantial impacts on the performance of the estimators considered in this paper (El Karoui et al., 2011). As such, confidence statements derived from that kind of analysis can be relied on only after performing a few graphical tests on the data (see El Karoui (2010)). These geometric limitations are simple consequences of the concentration of measure phenomenon (Ledoux, 2001).

On the other hand, in the fixed design setting, X is considered a fixed matrix. In this case, the inference only takes the randomness of ϵ into consideration. This perspective is popular in several situations. The first one is the experimental design. The goal is to study the effect of a set of factors, which can be controlled by the experimenter, on the response. In contrast to the observational study, the experimenter can design the experimental condition ahead of time based on the inference target. For instance, a one-way ANOVA design encodes the covariates into binary variables (see Section 3.3.4 for details) and it is fixed prior to the experiment. Other examples include two-way ANOVA designs, factorial designs, Latin-square designs, etc. (Scheffe, 1999).

Another situation which is concerned with fixed design is the survey sampling where the inference is carried out conditioning on the data (Cochran, 1977). Generally, in order to avoid unrealistic assumptions, making inference conditioning on the design matrix X is necessary. Suppose the linear model (1) is true and identifiable (see Section 2.3 for details), then all information of β^* is contained in the conditional distribution $\mathcal{L}(y|X)$ and hence the information in the marginal distribution $\mathcal{L}(X)$ is redundant. The conditional inference framework is more robust to the data generating procedure due to the irrelevance of $\mathcal{L}(X)$.

Also, results based on fixed design assumptions may be preferable from a theoretical point of view in the sense that they could potentially be used to establish corresponding results for certain classes of random designs. Specifically, given a marginal distribution $\mathcal{L}(X)$, one only has to prove that \mathcal{X} satisfies the assumptions for fixed design with high probability.

In conclusion, fixed and random design assumptions play complementary roles in moderate-dimensional settings. We focus on the least understood of the two, the fixed design case, in this paper.

2.3 Modeling and Identification of Parameters

The problem of identifiability is especially important in the fixed design case. Define $\beta^*(\rho)$ in the population as

$$\beta^*(\rho) = \underset{\beta \in \mathbb{R}^p}{\operatorname{arg\,min}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}\rho(y_i - x_i^T \beta).$$
(3)

One may ask whether $\beta^*(\rho) = \beta^*$ regardless of ρ in the fixed design case. We provide an affirmative answer in the following proposition by assuming that ϵ_i has a symmetric distribution around 0 and ρ is even.

Proposition 2.1. Suppose X has a full column rank and $\epsilon_i \stackrel{d}{=} -\epsilon_i$ for all *i*. Further assume ρ is an even convex function such that for any i = 1, 2, ... and $\alpha \neq 0$,

$$\frac{1}{2} \left(\mathbb{E}\rho(\epsilon_i - \alpha) + \mathbb{E}\rho(\epsilon_i + \alpha) \right) > \mathbb{E}\rho(\epsilon_i).$$
(4)

Then $\beta^*(\rho) = \beta^*$ regardless of the choice of ρ .

The proof is left to Appendix C. It is worth mentioning that Proposition 2.1 only requires the marginals of ϵ to be symmetric but does not impose any constraint on the dependence structure of ϵ . Further, if ρ is strongly convex, then for all $\alpha \neq 0$,

$$\frac{1}{2}\left(\rho(x-\alpha)+\rho(x+\alpha)\right)>\rho(x).$$

As a consequence, the condition (4) is satisfied provided that ϵ_i is non-zero with positive probability.

If ϵ is asymmetric, we may still be able to identify β^* if ϵ_i are i.i.d. random variables. In contrast to the last case, we should incorporate an intercept term as a shift towards the centroid of ρ . More precisely, we define $\alpha^*(\rho)$ and $\beta^*(\rho)$ as

$$(\alpha^*(\rho), \beta^*(\rho)) = \underset{\alpha \in \mathbb{R}, \beta \in \mathbb{R}^p}{\operatorname{arg\,min}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}\rho(y_i - \alpha - x_i^T \beta).$$

Proposition 2.2. Suppose (1, X) is of full column rank and ϵ_i are *i.i.d.* such that $\mathbb{E}\rho(\epsilon_1 - \alpha)$ as a function of α has a unique minimizer $\alpha(\rho)$. Then $\beta^*(\rho)$ is uniquely defined with $\beta^*(\rho) = \beta^*$ and $\alpha^*(\rho) = \alpha(\rho)$.

The proof is left to Appendix C. For example, let $\rho(z) = |z|$. Then the minimizer of $\mathbb{E}\rho(\epsilon_1 - a)$ is a median of ϵ_1 , and is unique if ϵ_1 has a positive density. It is worth pointing out that incorporating an intercept term is essential for identifying β^* . For instance, in the least-square case, $\beta^*(\rho)$ no longer equals to β^* if $\mathbb{E}\epsilon_i \neq 0$. Proposition 2.2 entails that the intercept term guarantees $\beta^*(\rho) = \beta^*$, although the intercept term itself depends on the choice of ρ unless more conditions are imposed.

If ϵ_i 's are neither symmetric nor i.i.d., then β^* cannot be identified by the previous criteria because $\beta^*(\rho)$ depends on ρ . Nonetheless, from a modeling perspective, it is popular and reasonable to assume that ϵ_i 's are symmetric or i.i.d. in many situations. Therefore, Proposition 2.1 and Proposition 2.2 justify the use of M-estimators in those cases and M-estimators derived from different loss functions can be compared because they are estimating the same parameter.

3 Main Results

3.1 Notation and Assumptions

Let $x_i^T \in \mathbb{R}^{1 \times p}$ denote the *i*-th row of X and $X_j \in \mathbb{R}^{n \times 1}$ denote the *j*-th column of X. Throughout the paper we will denote by $X_{ij} \in \mathbb{R}$ the (i, j)-th entry of X, by $X_{[j]} \in \mathbb{R}^{n \times (p-1)}$ the design matrix X after removing the *j*-th column, and by $x_{i,[j]}^T \in \mathbb{R}^{1 \times (p-1)}$ the vector x_i^T after removing *j*-th entry. The M-estimator $\hat{\beta}(\rho)$ associated with the loss function ρ is defined as

$$\hat{\beta}(\rho) = \operatorname*{arg\,min}_{\beta \in \mathbb{R}^p} \frac{1}{n} \sum_{k=1}^n \rho(y_k - x_k^T \beta) = \operatorname*{arg\,min}_{\beta \in \mathbb{R}^p} \frac{1}{n} \sum_{k=1}^n \rho(\epsilon_k - x_k^T (\beta - \beta^*)) \tag{5}$$

We define $\psi = \rho'$ to be the first derivative of ρ . We will write $\hat{\beta}(\rho)$ simply $\hat{\beta}$ when no confusion can arise.

When the original design matrix X does not contain an intercept term, we can simply replace X by $(\mathbf{1}, X)$ and augment β into a (p+1)-dimensional vector $(\alpha, \beta^T)^T$. Although being a special case, we will discuss the question of intercept in Section 3.2.2 due to its important role in practice.

Equivariance and reduction to the null case

Notice that our target quantity $\frac{\hat{\beta}_j - \mathbb{E}\hat{\beta}_j}{\sqrt{\operatorname{Var}(\hat{\beta}_j)}}$ is invariant to the choice of β^* , provided that β^* is identifiable as discussed in Section 2.3, we can assume $\beta^* = 0$ without loss of generality. In this case, we assume in particular that the design matrix X has full column rank. Then $y_k = \epsilon_k$ and

$$\hat{\beta} = \operatorname*{argmin}_{\beta \in \mathbb{R}^p} \frac{1}{n} \sum_{k=1}^n \rho(\epsilon_k - x_k^T \beta).$$

Similarly we define the leave-j-th-predictor-out version as

$$\hat{\beta}_{[j]} = \underset{\beta \in \mathbb{R}^{p-1}}{\operatorname{arg\,min}} \frac{1}{n} \sum_{k=1}^{n} \rho(\epsilon_k - x_{k,[j]}^T \beta).$$

Based on these notations we define the full residuals R_k as

$$R_k = \epsilon_k - x_k^T \hat{\beta}, \quad k = 1, 2, \dots, n$$

and the leave-*j*-th-predictor-out residual as

$$r_{k,[j]} = \epsilon_k - x_{k,[j]}^T \hat{\beta}_{[j]}, \quad k = 1, 2, \dots, n, \quad j = 1, \dots, p.$$

Three $n \times n$ diagonal matrices are defined as

$$D = \operatorname{diag}(\psi'(R_k))_{k=1}^n, \quad \tilde{D} = \operatorname{diag}(\psi''(R_k))_{k=1}^n, \quad D_{[j]} = \operatorname{diag}(\psi'(r_{k,[j]}))_{k=1}^n.$$
(6)

We say a random variable Z is σ^2 -sub-gaussian if for any $\lambda \in \mathbb{R}$,

$$\mathbb{E}e^{\lambda Z} \le e^{\frac{\lambda^2 \sigma^2}{2}}.$$

In addition, we use $J_n \subset \{1, \ldots, p\}$ to represent the indices of parameters which are of interest. Intuitively, more entries in J_n would require more stringent conditions for the asymptotic normality.

Finally, we adopt Landau's notation $(O(\cdot), o(\cdot), O_p(\cdot), o_p(\cdot))$. In addition, we say $a_n = \Omega(b_n)$ if $b_n = O(a_n)$ and similarly, we say $a_n = \Omega_p(b_n)$ if $b_n = O_p(a_n)$. To simplify the logarithm factors, we use the symbol polyLog(n) to denote any factor that can be upper bounded by $(\log n)^{\gamma}$ for some $\gamma > 0$. Similarly, we use $\frac{1}{\text{polyLog}(n)}$ to denote any factor that can be lower bounded by $\frac{1}{(\log n)^{\gamma'}}$ for some $\gamma' > 0$.

3.2 Technical Assumptions and main result

Before stating the assumptions, we need to define several quantities of interest. Let

$$\lambda_{+} = \lambda_{\max}\left(\frac{X^{T}X}{n}\right), \quad \lambda_{-} = \lambda_{\min}\left(\frac{X^{T}X}{n}\right)$$

be the largest (resp. smallest) eigenvalue of the matrix $\frac{X^T X}{n}$. Let $e_i \in \mathbb{R}^n$ be the *i*-th canonical basis vector and

$$h_{j,0} \triangleq (\psi(r_{1,[j]}), \dots, \psi(r_{n,[j]}))^T, \quad h_{j,1,i} \triangleq (I - D_{[j]}X_{[j]}(X_{[j]}^T D_{[j]}X_{[j]})^{-1}X_{[j]}^T)e_i.$$

Finally, let

$$\Delta_C = \max\left\{ \max_{j \in J_n} \frac{|h_{j,0}^T X_j|}{\|h_{j,0}\|_2}, \max_{i \le n, j \in J_n} \frac{|h_{j,1,i}^T X_j|}{\|h_{j,1,i}\|_2} \right\},\ Q_j = \operatorname{Cov}(h_{j,0})$$

Based on the quantities defined above, we state our technical assumptions on the design matrix X followed by the main result. A detailed explanation of the assumptions follows.

A1 $\rho(0) = \psi(0) = 0$ and there exists positive numbers $K_0 = \Omega\left(\frac{1}{\text{polyLog}(n)}\right), K_1, K_2 = O(\text{polyLog}(n))$, such that for any $x \in \mathbb{R}$,

$$K_0 \le \psi'(x) \le K_1, \quad \left| \frac{d}{dx} (\sqrt{\psi'}(x)) \right| = \frac{|\psi''(x)|}{\sqrt{\psi'(x)}} \le K_2;$$

A2 $\epsilon_i = u_i(W_i)$ where $(W_1, \ldots, W_n) \sim N(0, I_{n \times n})$ and u_i are smooth functions with $\|u_i'\|_{\infty} \leq c_1$ and $\|u_i''\|_{\infty} \leq c_2$ for some $c_1, c_2 = O(\text{polyLog}(n))$. Moreover, assume $\min_i \operatorname{Var}(\epsilon_i) = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right)$.

A3 $\lambda_{+} = O(\text{polyLog}(n)) \text{ and } \lambda_{-} = \Omega\left(\frac{1}{\text{polyLog}(n)}\right);$

- **A**4 min_{j \in J_n} $\frac{X_j^T Q_j X_j}{\operatorname{tr}(Q_j)} = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right);$
- $\mathbf{A5} \ \mathbb{E}\Delta_C^8 = O \ (\mathrm{polyLog}(\mathbf{n})).$

Theorem 3.1. Under assumptions A1 - A5, as $p/n \to \kappa$ for some $\kappa \in (0,1)$, while $n \to \infty$,

$$\max_{j \in J_n} d_{\mathrm{TV}} \left(\mathcal{L} \left(\frac{\hat{\beta}_j - \mathbb{E}\hat{\beta}_j}{\sqrt{\mathrm{Var}(\hat{\beta}_j)}} \right), N(0, 1) \right) = o(1),$$

where $d_{\text{TV}}(P,Q) = \sup_{A} |P(A) - Q(A)|$ is the total variation distance.

We provide several examples where our assumptions hold in Section 3.3. We also provide an example where the asymptotic normality does not hold in Section 3.3.4. This shows that our assumptions are not just artifacts of the proof technique we developed, but that there are (probably many) situations where asymptotic normality will not hold, even coordinate-wise.

3.2.1 Discussion of Assumptions

Now we discuss assumptions A1 - A5. Assumption A1 implies the boundedness of the first-order and the second-order derivatives of ψ . The upper bounds are satisfied by most loss functions including the L_2 loss, the smoothed L_1 loss, the smoothed Huber loss, etc. The non-zero lower bound K_0 implies the strong convexity of ρ and is required for technical reasons. It can be removed by considering first a ridge-penalized M-estimator

and taking appropriate limits as in El Karoui (2013, 2015). In addition, in this paper we consider the smooth loss functions and the results can be extended to non-smooth case via approximation.

Assumption A_2 was proposed in Chatterjee (2009) when deriving the second-order Poincaré inequality discussed in Section 4.1. It means that the results apply to non-Gaussian distributions, such as the uniform distribution on [0,1] by taking $u_i = \Phi$, the cumulative distribution function of standard normal distribution. Through the gaussian concentration (Ledoux, 2001), we see that A2 implies that ϵ_i are c_1^2 -sub-gaussian. Thus A2 controls the tail behavior of ϵ_i . The boundedness of u'_i and u''_i are required only for the direct application of Chatterjee's results. In fact, a look at his proof suggests that one can obtain a similar result to his Second-Order Poincaré inequality involving moment bounds on $u'_i(W_i)$ and $u''_i(W_i)$. This would be a way to weaken our assumptions to permit to have the heavy-tailed distributions expected in robustness studies. Since we are considering strongly convex loss-functions, it is not completely unnatural to restrict our attention to light-tailed errors. Furthermore, efficiency - and not only robustness - questions are one of the main reasons to consider these estimators in the moderate-dimensional context. The potential gains in efficiency obtained by considering regression M-estimates (Bean et al., 2013) apply in the light-tailed context, which further justify our interest in this theoretical setup.

Assumption A3 is completely checkable since it only depends on X. It controls the singularity of the design matrix. Under A1 and A3, it can be shown that the objective function is strongly convex with curvature (the smallest eigenvalue of the Hessian matrix) lower bounded by $\Omega\left(\frac{1}{\text{polyLog}(n)}\right)$ everywhere.

Assumption A4 is controlling the left tail of quadratic forms. It is fundamentally connected to aspects of the concentration of measure phenomenon (Ledoux, 2001). This condition is proposed and emphasized under the random design setting by El Karoui et al. (2013). Essentially, it means that for a matrix Q_j , which does not depend on X_j , the quadratic form $X_i^T Q_i X_j$ should have the same order as $tr(Q_j)$.

Assumption A5 is proposed by El Karoui (2013) under the random design settings. It is motivated by leave-one-predictor-out analysis. Note that Δ_C is the maximum of linear contrasts of X_j , whose coefficients do not depend on X_j . It is easily checked for design matrix X which is a realization of a random matrix with i.i.d sub-gaussian entries for instance.

Remark 3.2. In certain applications, it is reasonable to make the following additional assumption:

A6 ρ is an even function and ϵ_i 's have symmetric distributions.

Although assumption A6 is not necessary to Theorem 3.1, it can simplify the result. Under assumption A6, when X is full rank, we have, if $\stackrel{d}{=}$ denotes equality in distribution,

$$\hat{\beta} - \beta^* = \operatorname*{arg\,min}_{\eta \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^n \rho(\epsilon_i - x_i^T \eta) = \operatorname*{arg\,min}_{\eta \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^n \rho(-\epsilon_i + x_i^T \eta)$$
$$\stackrel{d}{=} \operatorname*{arg\,min}_{\eta \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^n \rho(\epsilon_i + x_i^T \eta) = \beta^* - \hat{\beta}.$$

This implies that $\hat{\beta}$ is an unbiased estimator, provided it has a mean, which is the case here. Unbiasedness is useful in practice, since then Theorem 3.1 reads

$$\max_{j \in J_n} d_{\mathrm{TV}} \left(\mathcal{L} \left(\frac{\hat{\beta}_j - \beta_j^*}{\sqrt{\mathrm{Var}(\hat{\beta}_j)}} \right), N(0, 1) \right) = o(1) \; .$$

For inference, we only need to estimate the asymptotic variance.

3.2.2 An important remark concerning Theorem 3.1

When J_n is a subset of $\{1, \ldots, p\}$, the coefficients in J_n^c become nuisance parameters. Heuristically, in order for identifying $\beta_{J_n}^*$, one only needs the subspaces $\operatorname{span}(X_{J_n})$ and $\operatorname{span}(X_{J_n^c})$ to be distinguished and X_{J_n} has a full column rank. Here X_{J_n} denotes the sub-matrix of X with columns in J_n . Formally, let

$$\hat{\Sigma}_{J_n} = \frac{1}{n} X_{J_n}^T (I - X_{J_n^c} (X_{J_n^c}^T X_{J_n^c})^- X_{J_n^c}^T) X_{J_n}$$

where A^- denotes the generalized inverse of A, and

$$\tilde{\lambda}_{+} = \lambda_{\max}\left(\hat{\Sigma}_{J_{n}}\right), \quad \tilde{\lambda}_{-} = \lambda_{\min}\left(\hat{\Sigma}_{J_{n}}\right).$$

Then Σ_{J_n} characterizes the behavior of X_{J_n} after removing the effect of $X_{J_n^c}$. In particular, we can modify the assumption **A**3 by

A3*
$$\tilde{\lambda}_+ = O(\text{polyLog}(n)) \text{ and } \tilde{\lambda}_- = \Omega\left(\frac{1}{\text{polyLog}(n)}\right).$$

Then we are able to derive a stronger result in the case where $|J_n| < p$ than Theorem 3.1 as follows.

Corollary 3.3. Under assumptions A1-2, A4-5 and A3*, as $p/n \rightarrow \kappa$ for some $\kappa \in (0,1)$,

$$\max_{j \in J_n} d_{\mathrm{TV}} \left(\mathcal{L} \left(\frac{\hat{\beta}_j - \mathbb{E}\hat{\beta}_j}{\sqrt{\mathrm{Var}(\hat{\beta}_j)}} \right), N(0, 1) \right) = o(1).$$

It can be shown that $\hat{\lambda}_{+} \leq \lambda_{+}$ and $\hat{\lambda}_{-} \geq \lambda_{-}$ and hence the assumption $\mathbf{A3}^{*}$ is weaker than $\mathbf{A3}$. It is worth pointing out that the assumption $\mathbf{A3}^{*}$ even holds when $X_{J_{n}}^{c}$ does not have full column rank, in which case $\beta_{J_{n}}^{*}$ is still identifiable and $\hat{\beta}_{J_{n}}$ is still well-defined, although $\beta_{J_{n}^{*}}^{*}$ and $\hat{\beta}_{J_{n}^{c}}$ are not; see Appendix C-2 for details.

3.3 Examples

Throughout this subsection (except subsubsection 3.3.4), we consider the case where X is a realization of a random matrix, denoted by Z (to be distinguished from X). We will verify that the assumptions A3-A5 are satisfied with high probability under different regularity conditions on the distribution of Z. This is a standard way to justify the conditions for fixed design (Portnoy, 1984, 1985) in the literature on regression M-estimates.

3.3.1 Random Design with Independent Entries

First we consider a random matrix Z with i.i.d. sub-gaussian entries.

Proposition 3.4. Suppose Z has i.i.d. mean-zero σ^2 -sub-gaussian entries with $\operatorname{Var}(Z_{ij}) = \tau^2 > 0$ for some $\sigma = O(\operatorname{polyLog}(n))$ and $\tau = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right)$, then, when X is a realization of Z, assumptions A3-A5 for X are satisfied with high probability over Z for $J_n = \{1, \ldots, p\}$.

In practice, the assumption of identical distribution might be invalid. In fact the assumptions A4, A5 and the first part of A3 ($\lambda_{+} = O$ (polyLog(n))) are still satisfied with high probability if we only assume the independence between entries and boundedness of certain moments. To control λ_{-} , we rely on Litvak et al. (2005) which assumes symmetry of each entry. We obtain the following result based on it.

Proposition 3.5. Suppose Z has independent σ^2 -sub-gaussian entries with

$$Z_{ij} \stackrel{d}{=} -Z_{ij}, \quad \operatorname{Var}(Z_{ij}) > \tau^2$$

for some $\sigma = O(\text{polyLog}(n))$ and $\tau = \Omega\left(\frac{1}{\text{polyLog}(n)}\right)$, then, when X is a realization of Z, assumptions A3-A5 for X are satisfied with high probability over Z for $J_n = \{1, \ldots, p\}$.

Under the conditions of Proposition 3.5, we can add an intercept term into the design matrix. Adding an intercept allows us to remove the mean-zero assumption for Z_{ij} 's. In fact, suppose Z_{ij} is symmetric with respect to μ_j , which is potentially non-zero, for all *i*, then according to section 3.2.2, we can replace Z_{ij} by $Z_{ij} - \mu_j$ and Proposition 3.6 can be then applied.

Proposition 3.6. Suppose $Z = (\mathbf{1}, \tilde{Z})$ and $\tilde{Z} \in \mathbb{R}^{n \times (p-1)}$ has independent σ^2 -subgaussian entries with

$$\tilde{Z}_{ij} - \mu_j \stackrel{d}{=} \mu_j - \tilde{Z}_{ij}, \quad \operatorname{Var}(\tilde{Z}_{ij}) > \tau^2$$

for some $\sigma = O(\text{polyLog}(n))$, $\tau = \Omega\left(\frac{1}{\text{polyLog}(n)}\right)$ and arbitrary μ_j . Then, when X is a realization of Z, assumptions $A3^*$, A4 and A5 for X are satisfied with high probability over Z for $J_n = \{2, \ldots, p\}$.

3.3.2 Dependent Gaussian Design

To show that our assumptions handle a variety of situations, we now assume that the observations, namely the rows of Z, are i.i.d. random vectors with a covariance matrix Σ . In particular we show that the Gaussian design, i.e. $z_i \stackrel{i.i.d.}{\sim} N(0, \Sigma)$, satisfies the assumptions with high probability.

Proposition 3.7. Suppose $z_i \overset{i.i.d.}{\sim} N(0, \Sigma)$ with $\lambda_{\max}(\Sigma) = O(\operatorname{polyLog}(n))$ and $\lambda_{\min}(\Sigma) = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right)$, then, when X is a realization of Z, assumptions A3-A5 for X are satisfied with high probability over Z for $J_n = \{1, \ldots, p\}$.

This result extends to the matrix-normal design (Muirhead, 1982)[Chapter 3], i.e. $(Z_{ij})_{i \leq n, j \leq p}$ is one realization of a *np*-dimensional random variable Z with multivariate gaussian distribution

$$\operatorname{vec}(Z) \triangleq (z_1^T, z_2^T, \dots, z_n^T) \sim N(0, \Lambda \otimes \Sigma),$$

and \otimes is the Kronecker product. It turns out that assumptions A3 - A5 are satisfied if both Λ and Σ are well-behaved.

Proposition 3.8. Suppose Z is matrix-normal with $vec(Z) \sim N(0, \Lambda \otimes \Sigma)$ and

$$\lambda_{\max}(\Lambda), \lambda_{\max}(\Sigma) = O\left(\operatorname{polyLog}(n)\right), \quad \lambda_{\min}(\Lambda), \lambda_{\min}(\Sigma) = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right) \ .$$

Then, when X is a realization of Z, assumptions A3-A5 for X are satisfied with high probability over Z for $J_n = \{1, ..., p\}$.

In order to incorporate an intercept term, we need slightly more stringent condition on Λ . Instead of assumption A3, we prove that assumption A3^{*} - see subsubsection 3.2.2 - holds with high probability.

Proposition 3.9. Suppose Z contains an intercept term, i.e. $Z = (\mathbf{1}, \mathbf{Z})$ and \mathbf{Z} satisfies the conditions of Proposition 3.8. Further assume that

$$\frac{\max_{i} |(\Lambda^{-\frac{1}{2}} \boldsymbol{1})_{i}|}{\min_{i} |(\Lambda^{-\frac{1}{2}} \boldsymbol{1})_{i}|} = O\left(\text{polyLog}(n)\right).$$
(7)

Then, when X is a realization of Z, assumptions $A3^*$, A4 and A5 for X are satisfied with high probability over Z for $J_n = \{2, \ldots, p\}$.

When $\Lambda = I$, the condition (7) is satisfied. Another non-trivial example is the exchangeable case where Λ_{ij} are all equal for $i \neq j$. In this case, **1** is an eigenvector of Λ and hence it is also an eigenvector of $\Lambda^{-\frac{1}{2}}$. Thus $\Lambda^{-\frac{1}{2}}\mathbf{1}$ is a multiple of **1** and the condition (7) is satisfied.

3.3.3 Elliptical Design

Furthermore, we can move from Gaussian-like structure to generalized elliptical models where $z_i = \zeta_i \Sigma^{1/2} Z_i$ where $\{\zeta_i, Z_{ij} : i = 1, \ldots, n; j = 1, \ldots, p\}$ are independent random variables, Z_{ij} having for instance mean 0 and variance 1. The elliptical family is quite flexible in modeling data. It represents a type of data formed by a common driven factor and independent individual effects. It is widely used in multivariate statistics (Anderson (1962); Tyler (1987)) and various fields, including finance (Cizek et al., 2005) and biology (Posekany et al., 2011). In the context of high-dimensional statistics, this class of model was used to refute universality claims in random matrix theory (El Karoui, 2009). In robust regression, El Karoui et al. (2011) used elliptical models to show that the limit of $\|\hat{\beta}\|_2^2$ depends on the distribution of ζ_i and hence the geometry of the predictors. As such, studies limited to Gaussian-like design were shown to be of very limited statistical interest. See also the deep classical inadmissibility results (Baranchik, 1973; Jurečkovà & Klebanov, 1997). However, as we will show in the next proposition, the common factors ζ_i do not distort the shape of the asymptotic distribution. A similar phenomenon happens in the random design case - see El Karoui et al. (2013); Bean et al. (2013).

Proposition 3.10. Suppose Z is generated from an elliptical model, i.e.

$$Z_{ij} = \zeta_i \mathcal{Z}_{ij}$$

where ζ_i are independent random variables taking values in [a, b] for some $0 < a < b < \infty$ and Z_{ij} are independent random variables satisfying the conditions of Proposition 3.4 or Proposition 3.5. Further assume that $\{\zeta_i : i = 1, ..., n\}$ and $\{Z_{ij} : i = 1, ..., n; j = 1, ..., p\}$ are independent. Then, when X is a realization of Z, assumptions A3-A5 for X are satisfied with high probability over Z for $J_n = \{1, ..., p\}$.

Thanks to the fact that ζ_i is bounded away from 0 and ∞ , the proof of Proposition 3.10 is straightforward, as shown in Appendix C. However, by a more refined argument and assuming identical distributions ζ_i , we can relax this condition.

Proposition 3.11. Under the conditions of Proposition 3.10 (except the boundedness of ζ_i) and assume ζ_i are *i.i.d.* samples generated from some distribution F, independent of n, with

$$P\left(\zeta_1 \ge t\right) \le c_1 e^{-c_2 t^{\alpha}},$$

for some fixed $c_1, c_2, \alpha > 0$ and $F^{-1}(q) > 0$ for any $q \in (0, 1)$ where F^{-1} is the quantile function of F and is continuous. Then, when X is a realization of Z, assumptions A3-A5 for X are satisfied with high probability over Z for $J_n = \{1, \ldots, p\}$.

3.3.4 A counterexample

Consider a one-way ANOVA situation. In other words, let the design matrix have exactly 1 non-zero entry per row, whose value is 1. Let $\{k_i\}_{i=1}^n$ be integers in $\{1, \ldots, p\}$. And let $X_{i,j} = 1(j = k_i)$. Furthermore, let us constrain $n_j = |\{i : k_i = j\}|$ to be such that $1 \le n_j \le 2\lfloor p/n \rfloor$. Taking for instance $k_i = (i \mod p)$ is an easy way to produce such a matrix. The associated statistical model is just $y_i = \epsilon_i + \beta_{k_i}^*$.

It is easy to see that

$$\hat{\beta}_j = \operatorname*{arg\,min}_{\beta \in \mathbb{R}} \sum_{i:k_i=j} \rho(y_i - \beta_j) = \operatorname*{arg\,min}_{\beta \in \mathbb{R}} \sum_{i:k_i=j} \rho(\epsilon_i - (\beta_j - \beta_j^*)) \ .$$

This is of course a standard location problem. In the moderate-dimensional setting we consider, n_j remains finite as $n \to \infty$. So $\hat{\beta}_j$ is a non-linear function of finitely many random variables and will in general not be normally distributed.

For concreteness, one can take $\rho(x) = |x|$, in which case $\hat{\beta}_j$ is a median of $\{y_i\}_{\{i:k_i=j\}}$. The cdf of $\hat{\beta}_j$ is known exactly by elementary order statistics computations (see David and Nagaraja (1981)) and is not that of a Gaussian random variable in general. In fact, the ANOVA design considered here violates the assumption **A**3 since $\lambda_- = \min_j n_j/n = O(1/n)$. Further, we can show that the assumption **A**5 is also violated, at least in the least-square case; see Section 5.1 for details.

3.4 Comments and discussions

3.4.1 Asymptotic Normality in High Dimensions

In the *p*-fixed regime, the asymptotic distribution is easily defined as the limit of $\mathcal{L}(\hat{\beta})$ in terms of weak topology (Van der Vaart, 1998). However, in regimes where the dimension p grows, the notion of asymptotic distribution is more delicate. a conceptual question arises from the fact that the dimension of the estimator $\hat{\beta}$ changes with n and thus there is no well-defined distribution which can serve as the limit of $\mathcal{L}(\hat{\beta})$, where $\mathcal{L}(\cdot)$ denotes the law. One remedy is proposed by Mallows (1972). Under this framework, a triangular array $\{W_{n,j}, j = 1, 2, \ldots, p_n\}$, with $\mathbb{E}W_{n,j} = 0, \mathbb{E}W_{n,j}^2 = 1$, is called jointly asymptotically normal if for any deterministic sequence $a_n \in \mathbb{R}^{p_n}$ with $||a_n||_2 = 1$,

$$\mathcal{L}\left(\sum_{j=1}^{p_n} a_{n,j} W_{n,j}\right) \to N(0,1).$$

When the zero mean and unit variance are not satisfied, it is easy to modify the definition by normalizing random variables.

Definition 3.12 (joint asymptotic normality).

 $\{W_n : W_n \in \mathbb{R}^{p_n}\}$ is jointly asymptotically normal if and only if for any sequence $\{a_n : a_n \in \mathbb{R}^{p_n}\},\$

$$\mathcal{L}\left(\frac{a_n^T(W_n - \mathbb{E}W_n)}{\sqrt{a_n^T \operatorname{Cov}(W_n)a_n}}\right) \to N(0, 1).$$

The above definition of asymptotic normality is strong and appealing but was shown not to hold for least-squares in the moderate p/n regime (Huber, 1973). In fact, Huber (1973) shows that $\hat{\beta}^{LS}$ is jointly asymptotically normal only if

$$\max(X(X^TX)^{-1}X^T)_{i,i} \to 0.$$

When $p/n \to \kappa \in (0, 1)$, provided X is full rank,

$$\max_{i} (X(X^{T}X)^{-1}X^{T})_{i,i} \ge \frac{1}{n} \operatorname{tr}(X(X^{T}X)^{-1}X^{T}) = \frac{p}{n} \to \kappa > 0.$$

In other words, in moderate p/n regime, the asymptotic normality cannot hold for all linear contrasts, even in the case of least-squares.

In applications, however, it is usually not necessary to consider all linear contrasts but instead a small subset of them, e.g. all coordinates or low dimensional linear contrasts such as $\beta_1^* - \beta_2^*$. We can naturally modify Definition 3.12 and adapt to our needs by imposing constraints on a_n . A popular concept, which we use in Section 1 informally, is called coordinate-wise asymptotic normality and defined by restricting a_n to be the canonical basis vectors, which have only one non-zero element. An equivalent definition is stated as follows.

Definition 3.13 (coordinate-wise asymptotic normal).

 $\{W_n : W_n \in \mathbb{R}^{p_n}\}$ is coordinate-wise asymptotically normal if and only if for any sequence $\{j_n : j_n \in \{1, \ldots, p_n\}\},\$

$$\mathcal{L}\left(\frac{W_{n,j_n} - \mathbb{E}W_{n,j_n}}{\sqrt{\operatorname{Var}(W_{n,j_n})}}\right) \to N(0,1).$$

A more convenient way to define the coordinate-wise asymptotic normality is to introduce a metric $d(\cdot, \cdot)$, e.g. Kolmogorov distance and total variation distance, which induces the weak convergence topology. Then W_n is coordinate-wise asymptotically normal if and only if

$$\max_{j} d\left(\mathcal{L}\left(\frac{W_{n,j} - \mathbb{E}W_{n,j}}{\sqrt{\operatorname{Var}(W_{n,j})}}\right), N(0,1)\right) = o(1).$$

3.4.2 Discussion about inference and technical assumptions

Variance and bias estimation

To complete the inference, we need to compute the bias and variance. As discussed in Remark 3.2, the M-estimator is unbiased if the loss function and the error distribution are symmetric. For the variance, it is easy to get a conservative estimate via resampling methods such as Jackknife as a consequence of Efron-Stein's inequality; see El Karoui (2013) and El Karoui and Purdom (2015) for details. Moreover, by the variance decomposition formula,

$$\operatorname{Var}(\hat{\beta}_j) = \mathbb{E}\left[\operatorname{Var}(\hat{\beta}_j|X)\right] + \operatorname{Var}\left[\mathbb{E}(\hat{\beta}_j|X)\right] \ge \mathbb{E}\left[\operatorname{Var}(\hat{\beta}_j|X)\right],$$

the unconditional variance, when X is a random design matrix, is a conservative estimate. The unconditional variance can be calculated by solving a non-linear system; see El Karoui (2013) and Donoho and Montanari (2016).

However, estimating the exact variance is known to be hard. El Karoui and Purdom (2015) show that the existing resampling schemes, including jacknife, pairs-bootstrap, residual bootstrap, etc., are either too conservative or too anti-conservative when p/n is large. The challenge, as mentioned in El Karoui (2013); El Karoui and Purdom (2015), is due to the fact that the residuals $\{R_i\}$ do not mimic the behavior of $\{\epsilon_i\}$ and that the resampling methods effectively modifies the geometry of the dataset from the point of view of the statistics of interest. We believe that variance estimation in moderate p/n regime should rely on different methodologies from the ones used in low-dimensional estimation.

Technical assumptions

On the other hand, we assume that ρ is strongly convex. One remedy would be adding a ridge regularized term as in El Karoui (2013) and the new problem is amenable to analysis with the method we used in this article. However, the regularization term introduces a non-vanishing bias, which is as hard to be derived as the variance. For unregularized Mestimators, the strong convexity is also assumed by other works (El Karoui, 2013; Donoho & Montanari, 2016). However, we believe that this assumption is unnecessary and can be removed at least for well-behaved design matrices. Another possibility, for errors that have more than 2 moments is to just add a small quadratic term to the loss function, e.g. $\lambda x^2/2$ with a small λ . Finally, we recall that in many situations, least-squares is actually more efficient than ℓ_1 -regression (see numerical work in Bean et al. (2013)) in moderate dimensions. This is for instance the case for double-exponential errors if p/n is greater than .3 or so. As such working with strongly convex loss functions is as problematic for moderate-dimensional regression M-estimates as it would be in the low-dimensional setting.

To explore traditional robustness questions, we will need to weaken the requirements of Assumption A2. This requires substantial work and an extension of the main results

of Chatterjee (2009). Because the technical part of the paper is already long, we leave this interesting statistical question to future works.

4 Proof Sketch

Since the proof of Theorem 3.1 is somewhat technical, we illustrate the main idea in this section.

First notice that the M-estimator $\hat{\beta}$ is an implicit function of independent random variables $\epsilon_1, \ldots, \epsilon_n$, which is determined by

$$\frac{1}{n}\sum_{i=1}^{n}x_{i}\psi(\epsilon_{i}-x_{i}\hat{\beta})=0.$$
(8)

The Hessian matrix of the loss function in (5) is $\frac{1}{n}X^TDX \succeq D_0\lambda_-I_p$ under the notation introduced in section 3.1. The assumption **A**3 then implies that the loss function is strongly convex, in which case $\hat{\beta}$ is unique. Then $\hat{\beta}$ can be seen as a non-linear function of ϵ_i 's. A powerful central limit theorem for this type of statistics is the second-order Poincaré inequality (SOPI), developed in Chatterjee (2009) and used there to re-prove central limit theorems for linear spectral statistics of large random matrices. We recall one of the main results for the convenience of the reader.

Proposition 4.1 (SOPI; Chatterjee, 2009). Let $\mathscr{W} = (\mathscr{W}_1, \ldots, \mathscr{W}_n) = (u_1(W_1), \ldots, u_n(W_n))$ where $W_i \stackrel{i.i.d.}{\sim} N(0,1)$ and $||u'_i||_{\infty} \leq c_1, ||u''_i||_{\infty} \leq c_2$. Take any $g \in C^2(\mathbb{R}^n)$ and let $\nabla_i g$, ∇g and $\nabla^2 g$ denote the *i*-th partial derivative, gradient and Hessian of g. Let

$$\kappa_0 = \left(\mathbb{E}\sum_{i=1}^n \left|\nabla_i g(\mathscr{W})\right|^4\right)^{\frac{1}{2}}, \quad \kappa_1 = (\mathbb{E}\|\nabla g(\mathscr{W})\|_2^4)^{\frac{1}{4}}, \quad \kappa_2 = (\mathbb{E}\|\nabla^2 g(\mathscr{W})\|_{\mathrm{op}}^4)^{\frac{1}{4}},$$

and $U = g(\mathcal{W})$. If U has finite fourth moment, then

$$d_{\mathrm{TV}}\left(\mathcal{L}\left(\frac{U-\mathbb{E}U}{\sqrt{\mathrm{Var}(U)}}\right), N(0,1)\right) \leq \frac{2\sqrt{5}(c_1c_2\kappa_0+c_1^3\kappa_1\kappa_2)}{\mathrm{Var}(U)}.$$

From (8), it is not hard to compute the gradient and Hessian of $\hat{\beta}_j$ with respect to ϵ . Recalling the definitions in Equation (6) on p. 8, we have

Lemma 4.2. Suppose $\psi \in C^2(\mathbb{R}^n)$, then

$$\frac{\partial \hat{\beta}_j}{\partial \epsilon^T} = e_j^T (X^T D X)^{-1} X^T D \tag{9}$$

$$\frac{\partial \hat{\beta}_j}{\partial \epsilon \partial \epsilon^T} = G^T \operatorname{diag}(e_j^T (X^T D X)^{-1} X^T \tilde{D}) G$$
(10)

where e_j is the *j*-th cononical basis vectors in \mathbb{R}^p and

$$G = I - X(X^T D X)^{-1} X^T D.$$

Recalling the definitions of K_i 's in Assumption A1 on p. 9, we can bound κ_0 , κ_1 and κ_2 as follows.

Lemma 4.3. Let $\kappa_{0j}, \kappa_{1j}, \kappa_{2j}$ defined as in Proposition 4.1 by setting $\mathscr{W} = \epsilon$ and $g(\mathscr{W}) = \hat{\beta}_j$. Let

$$M_j = \mathbb{E} \| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} \|_{\infty},$$
(11)

then

$$\kappa_{0j}^2 \le \frac{K_1^2}{(nK_0\lambda_-)^{\frac{3}{2}}} \cdot M_j, \quad \kappa_{1j}^4 \le \frac{K_1^2}{(nK_0\lambda_-)^2}, \quad \kappa_{2j}^4 \le \frac{K_2^4}{(nK_0\lambda_-)^{\frac{3}{2}}} \cdot \left(\frac{K_1}{K_0}\right)^4 \cdot M_j.$$

As a consequence of the second-order Poincaré inequality, we can bound the total variation distance between $\hat{\beta}_j$ and a normal distribution by M_j and $\operatorname{Var}(\hat{\beta}_j)$. More precisely, we prove the following Lemma.

Lemma 4.4. Under assumptions A1-A3,

$$\max_{j} d_{\mathrm{TV}}\left(\mathcal{L}\left(\frac{\hat{\beta}_{j} - \mathbb{E}\hat{\beta}_{j}}{\sqrt{\mathrm{Var}(\hat{\beta}_{j})}}\right), N(0, 1)\right) = O_{p}\left(\frac{\max_{j}(nM_{j}^{2})^{\frac{1}{8}}}{n \cdot \min_{j} \mathrm{Var}(\hat{\beta}_{j})} \cdot \mathrm{polyLog}(n)\right).$$

Lemma 4.4 is the key to prove Theorem 3.1. To obtain the coordinate-wise asymptotic normality, it is left to establish an upper bound for M_j and a lower bound for $\operatorname{Var}(\hat{\beta}_j)$. In fact, we can prove that

Lemma 4.5. Under assumptions A1 - A5,

$$\max_{j} M_{j} = O\left(\frac{\text{polyLog}(\mathbf{n})}{n}\right), \quad \min_{j} \operatorname{Var}(\hat{\beta}_{j}) = \Omega\left(\frac{1}{n \cdot \operatorname{polyLog}(\mathbf{n})}\right).$$

Then Lemma 4.4 and Lemma 4.5 together imply that

$$\max_{j} d_{\mathrm{TV}}\left(\mathcal{L}\left(\frac{\hat{\beta}_{j} - \mathbb{E}\hat{\beta}_{j}}{\sqrt{\mathrm{Var}(\hat{\beta}_{j})}}\right), N(0, 1)\right) = O\left(\frac{\mathrm{polyLog}(\mathbf{n})}{n^{\frac{1}{8}}}\right) = o(1).$$

Appendix A, provides a roadmap of the proof of Lemma 4.5 under a special case where the design matrix X is one realization of a random matrix with i.i.d. sub-gaussian entries. It also serves as an outline of the rigorous proof in Appendix B.

4.1 Comment on the Second-Order Poincaré inequality

Notice that when g is a linear function such that $g(z) = \sum_{i=1}^{n} a_i z_i$, then the Berry-Esseen inequality (Esseen, 1945) implies that

$$d_K\left(\mathcal{L}\left(\frac{W-\mathbb{E}W}{\sqrt{\operatorname{Var}(W)}}\right), N(0,1)\right) \preceq \frac{\sum_{i=1}^n |a_i|^3}{\left(\sum_{i=1}^n a_i^2\right)^{\frac{3}{2}}},$$

where

$$d_K(F,G) = \sup_x |F(x) - G(x)|.$$

On the other hand, the second-order Poincaré inequality implies that

$$d_K\left(\mathcal{L}\left(\frac{W - \mathbb{E}W}{\sqrt{\operatorname{Var}(W)}}\right), N(0, 1)\right) \le d_{\operatorname{TV}}\left(\mathcal{L}\left(\frac{W - \mathbb{E}W}{\sqrt{\operatorname{Var}(W)}}\right), N(0, 1)\right) \preceq \frac{\left(\sum_{i=1}^n a_i^4\right)^{\frac{1}{2}}}{\sum_{i=1}^n a_i^2} \,.$$

This is slightly worse than the Berry-Esseen bound and requires stronger conditions on the distributions of variates but provides bounds for TV metric instead of Kolmogorov metric. This comparison shows that second-order Poincaré inequality can be regarded as a generalization of the Berry-Esseen bound for non-linear transformations of independent random variables.

5 Least-Squares Estimator

The Least-Squares Estimator is a special case of an M-estimator with $\rho(x) = \frac{1}{2}x^2$. Because the estimator can then be written explicitly, the analysis of its properties is extremely simple and it has been understood for several decades (see arguments in e.g. Huber (1973)[Lemma 2.1] and Huber (1981)[Proposition 2.2]). In this case, the hat matrix $H = X(X^TX)^{-1}X^T$ captures all the problems associated with dimensionality in the problem. In particular, proving the asymptotic normality simply requires an application of the Lindeberg-Feller theorem.

It is however somewhat helpful to compare the conditions required for asymptotic normality in this simple case and the ones we required in the more general setup of Theorem 3.1. We do so briefly in this section.

5.1 Coordinate-Wise Asymptotic Normality of LSE

Under the linear model (1), when X is full rank,

$$\hat{\beta}^{LS} = \beta^* + (X^T X)^{-1} X^T \epsilon,$$

thus each coordinate of $\hat{\beta}^{LS}$ is a linear contrast of ϵ with zero mean. Instead of assumption **A**2, which requires ϵ_i to be sub-gaussian, we only need to assume $\max_i \mathbb{E}|\epsilon_i|^3 < \infty$, under which the Berry-Essen bound for non-i.i.d. data (Esseen, 1945) implies that

$$d_K\left(\mathcal{L}\left(\frac{\hat{\beta}_j - \beta_j^*}{\sqrt{\operatorname{Var}(\hat{\beta}_j)}}\right), N(0, 1)\right) \preceq \frac{\|e_j(X^T X)^{-1} X^T\|_3^3}{\|e_j^T (X^T X)^{-1} X^T\|_2^3} \leq \frac{\|e_j(X^T X)^{-1} X^T\|_\infty}{\|e_j(X^T X)^{-1} X^T\|_2}.$$

This motivates us to define a matrix specific quantity $S_i(X)$ such that

$$S_j(X) = \frac{\|e_j^T(X^T X)^{-1} X^T\|_{\infty}}{\|e_j^T(X^T X)^{-1} X^T\|_2}$$
(12)

then the Berry-Esseen bound implies that $\max_{j \in J_n} S_j(X)$ determines the coordinate-wise asymptotic normality of $\hat{\beta}^{LS}$.

Theorem 5.1. If $\mathbb{E} \max_i |\epsilon_i|^3 < \infty$, then

$$\max_{j\in J_n} d_K\left(\frac{\hat{\beta}_{LS,j} - \beta_{0,j}}{\sqrt{\operatorname{Var}(\hat{\beta}_{LS,j})}}, N(0,1)\right) \le A \cdot \frac{\mathbb{E}|\epsilon_i|^3}{(\mathbb{E}\epsilon_i^2)^{\frac{3}{2}}} \cdot \max_{j\in J_n} S_j(X),$$

where A is an absolute constant and $d_K(\cdot, \cdot)$ is the Kolmogorov distance, defined as

$$d_K(F,G) = \sup_x |F(x) - G(x)|.$$

It turns out that $\max_{j \in J_n} S_j(X)$ plays in the least-squares setting the role of Δ_C in assumption **A5**. Since it has been known that a condition like $S_j(X) \to 0$ is necessary for asymptotic normality of least-square estimators (Huber (1973)[Proposition 2.2]), this shows in particular that our Assumption **A5**, or a variant, is also needed in the general case. See Appendix C-4.1 for details.

5.2 Discussion

Naturally, checking the conditions for asymptotic normality is much easier in the leastsquares case than in the general case under consideration in this paper. In particular:

- 1. Asymptotic normality conditions can be checked for a broader class of random design matrices. See Appendix C-4.2 for details.
- 2. For orthogonal design matrices, i.e $X^T X = c \operatorname{Id}$ for some c > 0, $S_j(X) = \frac{\|X_j\|_{\infty}}{\|X_j\|_2}$. Hence, the condition $S_j(X) = o(1)$ is true if and only if no entry dominates the j - th row of X.
- 3. The ANOVA-type counterexample we gave in Section 3.3.4 still provides a counterexample. The reason now is different: namely the sum of finitely many independent random variables is evidently in general non-Gaussian. In fact, in this case, $S_j(X) = \frac{1}{\sqrt{n_j}}$ is bounded away from 0.

Inferential questions are also extremely simple in this context and essentially again dimensionindependent for the reasons highlighted above. Theorem 5.1 naturally reads,

$$\frac{\hat{\beta}_j - \beta_j^*}{\sigma \sqrt{e_j^T (X^T X)^{-1} e_j}} \xrightarrow{d} N(0, 1).$$
(13)

Estimating σ is still simple under minimal conditions provided $n - p \to \infty$: see Bickel and Freedman (1983)[Theorem 1.3] or standard computations concerning the normalized residual sum-of-squares (using variance computations for the latter may require up to 4 moments for ϵ_i 's). Then we can replace σ in (13) by $\hat{\sigma}$ with

$$\hat{\sigma}^2 = \frac{1}{n-p} \sum_{k=1}^n R_k^2$$

where $R_k = y_k - x_k^T \hat{\beta}$ and construct confidence intervals for β_j^* based on $\hat{\sigma}$. If n - p does not tend to ∞ , the normalized residual sum of squares is evidently not consistent even in the case of Gaussian errors, so this requirement may not be dispensed of.

6 Numerical Results

As seen in the previous sections and related papers, there are five important factors that affect the distribution of $\hat{\beta}$: the design matrix X, the error distribution $\mathcal{L}(\epsilon)$, the sample size n, the ratio κ , and the loss function ρ . The aim of this section is to assess the quality of the agreement between the asymptotic theoretical results of Theorem 3.1 and the empirical, finite-dimensional properties of $\hat{\beta}(\rho)$. We also perform a few simulations where some of the assumptions of Theorem 3.1 are violated to get an intuitive sense of whether those assumptions appear necessary or whether they are simply technical artifacts associated with the method of proof we developed. As such, the numerical experiments we report on in this section can be seen as a complement to Theorem 3.1 rather than only a simple check of its practical relevance.

The design matrices we consider are one realization of random design matrices of the following three types:

- (i.i.d. design) : $X_{ij} \stackrel{i.i.d.}{\sim} F;$
- (elliptical design) : $X_{ij} = \zeta_i \tilde{X}_{ij}$, where $\tilde{X}_{ij} \stackrel{i.i.d.}{\sim} N(0,1)$ and $\zeta_i \stackrel{i.i.d.}{\sim} F$. In addition, $\{\zeta_i\}$ is independent of $\{\tilde{X}_{ij}\}$;
- (partial Hadamard design) : a matrix formed by a random set of p columns of a $n \times n$ Hadamard matrix, i.e. a $n \times n$ matrix whose columns are orthogonal with entries restricted to ± 1 .

Here we consider two candidates for F in i.i.d. design and elliptical design: standard normal distribution N(0, 1) and t-distribution with two degrees of freedom (denoted t_2). For the error distribution, we assume that ϵ has i.i.d. entries with one of the above two distributions, namely N(0, 1) and t_2 . The t-distribution violates our assumption A2.

To evaluate the finite sample performance, we consider the sample sizes $n \in \{100, 200, 400, 800\}$ and $\kappa \in \{0.5, 0.8\}$. In this section we will consider a Huber loss with k = 1.345 (Huber, 1981), i.e.

$$\rho(x) = \begin{cases} \frac{1}{2}x^2 & |x| \le k \\ kx - \frac{k^2}{2} & |x| > k \end{cases}$$

k = 1.345 is the default in R and yields 95% relative efficiency for Gaussian errors in low-dimensional problems. We also carried out the numerical work for L_1 -regression, i.e. $\rho(x) = |x|$. See Appendix D for details.

6.1 Asymptotic Normality of A Single Coordinate

First we simulate the finite sample distribution of $\hat{\beta}_1$, the first coordinate of $\hat{\beta}$. For each combination of sample size n (100, 200, 400 and 800), type of design (i.i.d, elliptical and Hadamard), entry distribution F (normal and t_2) and error distribution $\mathcal{L}(\epsilon)$ (normal and t_2), we run 50 simulations with each consisting of the following steps:

(Step 1) Generate one design matrix X;

- (Step 2) Generate the 300 error vectors ϵ ;
- (Step 3) Regress each $Y = \epsilon$ on the design matrix X and end up with 300 random samples of $\hat{\beta}_1$, denoted by $\hat{\beta}_1^{(1)}, \ldots, \hat{\beta}_1^{(300)}$;
- (Step 4) Estimate the standard deviation of $\hat{\beta}_1$ by the sample standard error \widehat{sd} ;
- (Step 5) Construct a confidence interval $\mathcal{I}^{(k)} = \left[\hat{\beta}_1^{(k)} 1.96 \cdot \widehat{\mathrm{sd}}, \hat{\beta}_1^{(k)} + 1.96 \cdot \widehat{\mathrm{sd}}\right]$ for each $k = 1, \dots, 300;$
- (Step 6) Calculate the empirical 95% coverage by the proportion of confidence intervals which cover the true $\beta_1 = 0$.

Finally, we display the boxplots of the empirical 95% coverages of β_1 for each case in Figure 2. It is worth mentioning that our theories cover two cases: 1) i.i.d design with normal entries and normal errors (orange bars in the first row and the first column), see Proposition 3.4; 2) elliptical design with normal factors ζ_i and normal errors (orange bars in the second row and the first column), see Proposition 3.10.

We first discuss the case $\kappa = 0.5$. In this case, there are only two samples per parameter. Nonetheless, we observe that the coverage is quite close to 0.95, even with a sample size as small as 100, in both cases that are covered by our theories. For other cases, it is interesting to see that the coverage is valid and most stable in the partial hadamard design case and is not sensitive to the distribution of multiplicative factor in elliptical design case even when the error has a t₂ distribution. For i.i.d. designs, the coverage is still valid and stable when the entry is normal. By contrast, when the entry has a t₂ distribution, the coverage has a large variation in small samples. The average coverage is still close to 0.95 in the i.i.d. normal design case but is slightly lower than 0.95 in the i.i.d. t₂ design case. In summary, the finite sample distribution of $\hat{\beta}_1$ is more sensitive to the entry distribution than the error distribution. This indicates that the assumptions on the design matrix are not just artifacts of the proof but are quite essential.

The same conclusion can be drawn from the case where $\kappa = 0.8$ except that the variation becomes larger in most cases when the sample size is small. However, it is worth pointing out that even in this case where there is 1.25 samples per parameter, the sample distribution of $\hat{\beta}_1$ is well approximated by a normal distribution with a moderate sample size $(n \ge 400)$. This is in contrast to the classical rule of thumb which suggests that 5-10 samples are needed per parameter.

6.2 Asymptotic Normality for Multiple Marginals

Since our theory holds for general J_n , it is worth checking the approximation for multiple coordinates in finite samples. For illustration, we consider 10 coordinates, namely $\hat{\beta}_1 \sim \hat{\beta}_{10}$, simultaneously and calculate the minimum empirical 95% coverage. To avoid the finite sample dependence between coordinates involved in the simulation, we estimate the empirical coverage independently for each coordinate. Specifically, we run 50 simulations with each consisting of the following steps:

- (Step 1) Generate one design matrix X;
- (Step 2) Generate the 3000 error vectors ϵ ;
- (Step 3) Regress each $Y = \epsilon$ on the design matrix X and end up with 300 random samples of $\hat{\beta}_j$ for each j = 1, ..., 10 by using the (300(j-1)+1)-th to 300j-th response vector Y;
- (Step 4) Estimate the standard deviation of $\hat{\beta}_j$ by the sample standard error \hat{sd}_j for j = 1, ..., 10;
- (Step 5) Construct a confidence interval $\mathcal{I}_{j}^{(k)} = \left[\hat{\beta}_{j}^{(k)} 1.96 \cdot \widehat{\mathrm{sd}}_{j}, \hat{\beta}_{j}^{(k)} + 1.96 \cdot \widehat{\mathrm{sd}}_{j}\right]$ for each $j = 1, \ldots, 10$ and $k = 1, \ldots, 300$;



Figure 2: Empirical 95% coverage of $\hat{\beta}_1$ with $\kappa = 0.5$ (left) and $\kappa = 0.8$ (right) using Huber_{1.345} loss. The x-axis corresponds to the sample size, ranging from 100 to 800; the y-axis corresponds to the empirical 95% coverage. Each column represents an error distribution and each row represents a type of design. The orange solid bar corresponds to the case F = Normal; the blue dotted bar corresponds to the case $F = t_2$; the red dashed bar represents the Hadamard design.

(Step 6) Calculate the empirical 95% coverage by the proportion of confidence intervals which cover the true $\beta_j = 0$, denoted by C_j , for each j = 1, ..., 10,

(Step 7) Report the minimum coverage $\min_{1 \le j \le 10} C_j$.

If the assumptions A1 - A5 are satisfied, $\min_{1 \le j \le 10} C_j$ should also be close to 0.95 as a result of Theorem 3.1. Thus, $\min_{1 \le j \le 10} C_j$ is a measure for the approximation accuracy for multiple marginals. Figure 3 displays the boxplots of this quantity under the same scenarios as the last subsection. In two cases that our theories cover, the minimum coverage is increasingly closer to the true level 0.95. Similar to the last subsection, the approximation is accurate in the partial hadamard design case and is insensitive to the distribution of multiplicative factors in the elliptical design case. However, the approximation is very inaccurate in the i.i.d. t₂ design case. Again, this shows the evidence that our technical assumptions are not artifacts of the proof.

On the other hand, the figure 3 suggests using a conservative variance estimator, e.g. the Jackknife estimator, or corrections on the confidence level in order to make simultaneous inference on multiple coordinates. Here we investigate the validity of Bonferroni correction by modifying the step 5 and step 6. The confidence interval after Bonferroni correction is obtained by

$$\mathcal{I}_{j}^{(k)} = \left[\hat{\beta}_{j}^{(k)} - z_{1-\alpha/20} \cdot \widehat{\mathrm{sd}}_{j}, \hat{\beta}_{j}^{(k)} + z_{1-\alpha/20} \cdot \widehat{\mathrm{sd}}_{j}\right]$$
(14)

where $\alpha = 0.05$ and z_{γ} is the γ -th quantile of a standard normal distribution. The proportion of k such that $0 \in \mathcal{I}_{j}^{(k)}$ for all $j \leq 10$ should be at least 0.95 if the marginals are all close to a normal distribution. We modify the confidence intervals in step 5 by (14) and calculate the proportion of k such that $0 \in \mathcal{I}_{j}^{(k)}$ for all j in step 6. Figure 4 displays the boxplots of this coverage. It is clear that the Bonferroni correction gives the valid coverage except when $n = 100, \kappa = 0.8$ and the error has a t₂ distribution.



Figure 3: Mininum empirical 95% coverage of $\hat{\beta}_1 \sim \hat{\beta}_{10}$ with $\kappa = 0.5$ (left) and $\kappa = 0.8$ (right) using Huber_{1.345} loss. The x-axis corresponds to the sample size, ranging from 100 to 800; the y-axis corresponds to the minimum empirical 95% coverage. Each column represents an error distribution and each row represents a type of design. The orange solid bar corresponds to the case F = Normal; the blue dotted bar corresponds to the case $F = t_2$; the red dashed bar represents the Hadamard design.

7 Conclusion

We have proved coordinate-wise asymptotic normality for regression M-estimates in the moderate-dimensional asymptotic regime $p/n \rightarrow \kappa \in (0, 1)$, for fixed design matrices under appropriate technical assumptions. Our design assumptions are satisfied with high probability for a broad class of random designs. The main novel ingredient of the proof is the use of the second-order Poincaré inequality. Numerical experiments confirm and complement our theoretical results.



Figure 4: Empirical 95% coverage of $\hat{\beta}_1 \sim \hat{\beta}_{10}$ after Bonferroni correction with $\kappa = 0.5$ (left) and $\kappa = 0.8$ (right) using Huber_{1.345} loss. The x-axis corresponds to the sample size, ranging from 100 to 800; the y-axis corresponds to the empirical uniform 95% coverage after Bonferroni correction. Each column represents an error distribution and each row represents a type of design. The orange solid bar corresponds to the case F = Normal; the blue dotted bar corresponds to the case $F = t_2$; the red dashed bar represents the Hadamard design.

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APPENDIX

A Proof Sketch of Lemma 4.5

In this Appendix, we provide a roadmap for proving Lemma 4.5 by considering a special case where X is one realization of a random matrix Z with i.i.d. mean-zero σ^2 -subgaussian entries. Random matrix theory (Geman, 1980; Silverstein, 1985; Bai & Yin, 1993) implies that $\lambda_+ = (1 + \sqrt{\kappa})^2 + o_p(1) = O_p(1)$ and $\lambda_- = (1 - \sqrt{\kappa})^2 + o_p(1) = \Omega_p(1)$. Thus, the assumption **A**3 is satisfied with high probability. Thus, the Lemma 4.4 in p. 16 holds with high probability. It remains to prove the following lemma to obtain Theorem 3.1.

Lemma A.1. Let Z be a random matrix with i.i.d. mean-zero σ^2 -sub-gaussian entries and X be one realization of Z. Then under assumptions A1 and A2,

$$\max_{1 \le j \le p} M_j = O_p\left(\frac{\text{polyLog}(n)}{n}\right), \quad \min_{1 \le j \le p} \text{Var}(\hat{\beta}_j) = \Omega_p\left(\frac{1}{n \cdot \text{polyLog}(n)}\right),$$

where M_j is defined in (11) in p.16 and the randomness in $o_p(\cdot)$ and $O_p(\cdot)$ comes from Z.

A-1 Upper Bound of M_i

First by Proposition E.3,

$$\lambda_+ = O_p(1), \quad \lambda_- = \Omega_p(1).$$

In the rest of the proof, the symbol \mathbb{E} and Var denotes the expectation and the variance conditional on Z. Let $\tilde{Z} = D^{\frac{1}{2}}Z$, then $M_j = \mathbb{E} \|e_j^T (\tilde{Z}^T \tilde{Z})^{-1} \tilde{Z}^T\|_{\infty}$. Let $\tilde{H}_j = I - \tilde{Z}_{[j]} (\tilde{Z}_{[j]}^T \tilde{Z}_{[j]})^{-1} \tilde{Z}_{[j]}^T$, then by block matrix inversion formula (see Proposition E.1), which we state as Proposition E.1 in Appendix E.

$$(\tilde{Z}^T \tilde{Z})^{-1} \tilde{Z}^T = \begin{pmatrix} \tilde{Z}_1^T \tilde{Z}_1 & \tilde{Z}_1^T \tilde{Z}_{[1]} \\ \tilde{Z}_{[1]}^T \tilde{Z}_1 & \tilde{Z}_{[1]}^T \tilde{Z}_{[1]} \end{pmatrix}^{-1} \begin{pmatrix} \tilde{Z}_1 \\ \tilde{Z}_{[1]} \end{pmatrix}$$

$$= \frac{1}{\tilde{Z}_1^T (I - \tilde{H}_1) \tilde{Z}_1} \begin{pmatrix} 1 & -\tilde{Z}_1^T \tilde{Z}_{[1]} (\tilde{Z}_{[1]}^T \tilde{Z}_{[1]})^{-1} \\ * & * \end{pmatrix} \begin{pmatrix} \tilde{Z}_1 \\ \tilde{Z}_{[1]} \end{pmatrix}$$

$$= \frac{1}{\tilde{Z}_1^T (I - \tilde{H}_1) \tilde{Z}_1} \begin{pmatrix} \tilde{Z}_1^T (I - \tilde{H}_1) \\ * \end{pmatrix} .$$

This implies that

$$M_1 = \mathbb{E} \frac{\|\tilde{Z}_1^T(I - \tilde{H}_1)\|_{\infty}}{\tilde{Z}_1^T(I - \tilde{H}_1)\tilde{Z}_1}.$$
 (A-1)

Since $Z^T DZ/n \succeq K_0 \lambda_- I$, we have

$$\frac{1}{\tilde{Z}_1^T (I - \tilde{H}_1) \tilde{Z}_1} = e_1^T (\tilde{Z}^T \tilde{Z})^{-1} e_1 = e_1^T (Z^T D Z)^{-1} e_1 = \frac{1}{n} e_1^T \left(\frac{Z^T D Z}{n}\right)^{-1} e_1 \le \frac{1}{nK_0\lambda_-}$$

and we obtain a bound for M_1 as

$$M_{1} \leq \frac{\mathbb{E} \|\tilde{Z}_{1}^{T}(I - \tilde{H}_{1})\|_{\infty}}{nK_{0}\lambda_{-}} = \frac{\mathbb{E} \|Z_{1}^{T}D^{\frac{1}{2}}(I - \tilde{H}_{1})\|_{\infty}}{nK_{0}\lambda_{-}}$$

Similarly,

$$M_{j} \leq \frac{\mathbb{E} \|Z_{j}^{T} D^{\frac{1}{2}} (I - \tilde{H}_{j})\|_{\infty}}{nK_{0}\lambda_{-}} = \frac{\mathbb{E} \|Z_{j}^{T} D^{\frac{1}{2}} (I - D^{\frac{1}{2}} Z_{[j]}^{T} (Z_{[j]}^{T} DZ_{[j]})^{-1} Z_{[j]} D^{\frac{1}{2}})\|_{\infty}}{nK_{0}\lambda_{-}}.$$
 (A-2)

The vector in the numerator is a linear contrast of Z_j and Z_j has mean-zero i.i.d. subgaussian entries. For any fixed matrix $A \in \mathbb{R}^{n \times n}$, denote A_k by its k-th column, then $A_k^T Z_j$ is $\sigma^2 ||A_k||_2^2$ -sub-gaussian (see Section 5.2.3 of Vershynin (2010) for a detailed discussion) and hence by definition of sub-Gaussianity,

$$P(|A_k^T Z_j| \ge \sigma ||A_k||_2 t) \le 2e^{-\frac{t^2}{2}}.$$

Therefore, by a simple union bound, we conclude that

$$P(||A^T Z_j||_{\infty} \ge \sigma \max_k ||A_k||_2 t) \le 2ne^{-\frac{t^2}{2}}.$$

Let $t = 2\sqrt{\log n}$,

$$P(||A^T Z_j||_{\infty} \ge 2\sigma \max_k ||A_k||_2 \sqrt{\log n}) \le \frac{2}{n} = o(1).$$

This entails that

$$\|A^T Z_j\|_{\infty} = O_p\left(\max_k \|A_k\|_2 \cdot \operatorname{polyLog}(\mathbf{n})\right) = O_p\left(\|A\|_{\operatorname{op}} \cdot \operatorname{polyLog}(\mathbf{n})\right).$$
(A-3)

with high probability. In M_j , the coefficient matrix $(I - H_j)D^{\frac{1}{2}}$ depends on Z_j through D and hence we cannot use (A-3) directly. However, the dependence can be removed by replacing D by $D_{[j]}$ since $r_{i,[j]}$ does not depend on Z_j .

Since Z has i.i.d. sub-gaussian entries, no column is highly influential. In other words, the estimator will not change drastically after removing *j*-th column. This would suggest $R_i \approx r_{i,[j]}$. It is proved by El Karoui (2013) that

$$\sup_{i,j} |R_i - r_{i,[j]}| = O_p\left(\frac{\text{polyLog}(n)}{\sqrt{n}}\right)$$

It can be rigorously proved that

$$\left| \|Z_j^T D(I - \tilde{H}_j)\|_{\infty} - \|Z_j^T D_{[j]}(I - H_j)\|_{\infty} \right| = O_p\left(\frac{\operatorname{polyLog}(n)}{n}\right)$$

where $H_j = I - D_{[j]}^{\frac{1}{2}} Z_{[j]} (Z_{[j]}^T D_{[j]} Z_{[j]})^{-1} Z_{[j]}^T D_{[j]}^{\frac{1}{2}}$; see Appendix A-1 for details. Since $D_{[j]} (I - H_j)$ is independent of Z_j and

$$||D_{[j]}(I - H_j)||_{\text{op}} \le ||D_{[j]}||_{\text{op}} \le K_1 = O (\text{polyLog}(n)),$$

it follows from (A-2) and (A-3) that

$$||Z_j^T D_{[j]}(I - H_j)||_{\infty} = O_p\left(\frac{\text{polyLog}(\mathbf{n})}{n}\right).$$

In summary,

$$M_j = O_p\left(\frac{\text{polyLog}(n)}{n}\right). \tag{A-4}$$

A-2 Lower Bound of $Var(\hat{\beta}_i)$

A-2.1 Approximating $Var(\hat{\beta}_i)$ by $Var(b_i)$

It is shown by El Karoui $(2013)^1$ that

$$\hat{\beta}_j \approx b_j \triangleq \frac{1}{\sqrt{n}} \frac{N_j}{\xi_j} \tag{A-5}$$

¹El Karoui (2013) considers a ridge regularized M estimator, which is different from our setting. However, this argument still holds in our case and proved in Appendix B.

where

$$N_{j} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{ij} \psi(r_{i,[j]}), \quad \xi_{j} = \frac{1}{n} Z_{j}^{T} (D_{[j]} - D_{[j]} Z_{[j]} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} Z_{[j]}^{T} D_{[j]}) Z_{j}.$$

It has been shown by El Karoui (2013) that

$$\max_{j} |\hat{\beta}_{j} - b_{j}| = O_{p} \left(\frac{\text{polyLog}(n)}{n} \right).$$

Thus, $\operatorname{Var}(\hat{\beta}_j) \approx \operatorname{Var}(b_j)$ and a more refined calculation in Appendix A-2.1 shows that

$$|\operatorname{Var}(\hat{\beta}_j) - \operatorname{Var}(b_j)| = O_p\left(\frac{\operatorname{polyLog}(\mathbf{n})}{n^{\frac{3}{2}}}\right).$$

It is left to show that

$$\operatorname{Var}(b_j) = \Omega_p\left(\frac{1}{n \cdot \operatorname{polyLog}(n)}\right).$$
(A-6)

A-2.2 Bounding $Var(b_j)$ via $Var(N_j)$

By definition of b_j ,

$$\operatorname{Var}(b_j) = \Omega_p\left(\frac{\operatorname{polyLog}(\mathbf{n})}{n}\right) \Longleftrightarrow \operatorname{Var}\left(\frac{N_j}{\xi_j}\right) = \Omega_p\left(\operatorname{polyLog}(\mathbf{n})\right)$$

As will be shown in Appendix B-6.4,

$$\operatorname{Var}(\xi_j) = O_p\left(\frac{\operatorname{polyLog}(\mathbf{n})}{n}\right)$$

As a result, $\xi_j \approx \mathbb{E}\xi_j$ and

$$\operatorname{Var}\left(\frac{N_j}{\xi_j}\right) \approx \operatorname{Var}\left(\frac{N_j}{\mathbb{E}\xi_j}\right) = \frac{\operatorname{Var}(N_j)}{(\mathbb{E}\xi_j)^2}$$

As in the previous paper (El Karoui, 2013), we rewrite ξ_j as

$$\xi_{j} = \frac{1}{n} Z_{j}^{T} D_{[j]}^{\frac{1}{2}} (I - D_{[j]}^{\frac{1}{2}} Z_{[j]} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} Z_{[j]}^{T} D_{[j]}^{\frac{1}{2}}) D_{[j]}^{\frac{1}{2}} Z_{j}$$

The middle matrix is idempotent and hence positive semi-definite. Thus,

$$\xi_j \leq \frac{1}{n} Z_j^T D_{[j]} Z_j \leq K_1 \lambda_+ = O_p \left(\text{polyLog}(\mathbf{n}) \right).$$

Then we obtain that

$$\frac{\operatorname{Var}(N_j)}{(\mathbb{E}\xi_j)^2} = \Omega_p\left(\frac{\operatorname{Var}(N_j)}{\operatorname{polyLog}(n)}\right)$$

and it is left to show that

$$\operatorname{Var}(N_j) = \Omega_p\left(\frac{1}{\operatorname{polyLog}(\mathbf{n})}\right). \tag{A-7}$$

A-2.3 Bounding $Var(N_j)$ via $tr(Q_j)$

Recall the definition of N_j (A-5), and that of Q_j (see Section 3.1 in p.8), we have

$$\operatorname{Var}(N_j) = \frac{1}{n} Z_j^T Q_j Z_j$$

Notice that Z_j is independent of $r_{i,[j]}$ and hence the conditional distribution of Z_j given Q_j remains the same as the marginal distribution of Z_j . Since Z_j has i.i.d. sub-gaussian

entries, the Hanson-Wright inequality (Hanson & Wright, 1971; Rudelson & Vershynin, 2013; see Proposition E.2), shown in Proposition E.2, implies that any quadratic form of Z_j , denoted by $Z_j^T Q_j Z_j$ is concentrated on its mean, i.e.

$$Z_j^T Q_j Z_j \approx \mathbb{E}_{Z_j,\epsilon} Z_j^T Q_j Z_j = (\mathbb{E} Z_{1j}^2) \cdot \operatorname{tr}(Q_j).$$

As a consequence, it is left to show that

$$\operatorname{tr}(Q_j) = \Omega_p\left(\frac{n}{\operatorname{polyLog}(\mathbf{n})}\right).$$
 (A-8)

A-2.4 Lower Bound of $tr(Q_j)$

By definition of Q_j ,

$$\operatorname{tr}(Q_j) = \sum_{i=1}^n \operatorname{Var}(\psi(r_{i,[j]})).$$

To lower bounded the variance of $\psi(r_{i,[j]})$, recall that for any random variable W,

$$\operatorname{Var}(W) = \frac{1}{2} \mathbb{E} (W - W')^2.$$
 (A-9)

where W' is an independent copy of W. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function such that $|g'(x)| \ge c$ for all x, then (A-9) implies that

$$\operatorname{Var}(g(W)) = \frac{1}{2} \mathbb{E}(g(W) - g(W'))^2 \ge \frac{c^2}{2} \mathbb{E}(W - W')^2 = c^2 \operatorname{Var}(W).$$
(A-10)

In other words, (A-10) entails that Var(W) is a lower bound for Var(g(W)) provided that the derivative of g is bounded away from 0. As an application, we see that

$$\operatorname{Var}(\psi(r_{i,[j]})) \ge K_0^2 \operatorname{Var}(r_{i,[j]})$$

and hence

$$\operatorname{tr}(Q_j) \ge K_0^2 \sum_{i=1}^n \operatorname{Var}(r_{i,[j]}).$$

By the variance decomposition formula,

$$\operatorname{Var}(r_{i,[j]}) = \mathbb{E}\left(\operatorname{Var}\left(r_{i,[j]}\big|\epsilon_{(i)}\right)\right) + \operatorname{Var}\left(\mathbb{E}\left(r_{i,[j]}\big|\epsilon_{(i)}\right)\right) \ge \mathbb{E}\left(\operatorname{Var}\left(r_{i,[j]}\big|\epsilon_{[i]}\right)\right),$$

where $\epsilon_{(i)}$ includes all but *i*-th entry of ϵ . Given $\epsilon_{(i)}$, $r_{i,[j]}$ is a function of ϵ_i . Using (A-10), we have

$$\operatorname{Var}(r_{i,[j]}|\epsilon_{(i)}) \ge \inf_{\epsilon_i} \left| \frac{\partial r_{i,[j]}}{\partial \epsilon_i} \right|^2 \cdot \operatorname{Var}(\epsilon_i|\epsilon_{(i)}) \ge \inf_{\epsilon_i} \left| \frac{\partial r_{i,[j]}}{\partial \epsilon_i} \right|^2 \cdot \operatorname{Var}(\epsilon_i).$$

This implies that

$$\operatorname{Var}(r_{i,[j]}) \geq \mathbb{E}\left(\operatorname{Var}\left(r_{i,[j]} \middle| \epsilon_{[i]}\right)\right) \geq \mathbb{E}\inf_{\epsilon} \left| \frac{\partial r_{i,[j]}}{\partial \epsilon_i} \right|^2 \cdot \min_{i} \operatorname{Var}(\epsilon_i).$$

Summing $\operatorname{Var}(r_{i,[j]})$ over $i = 1, \ldots, n$, we obtain that

$$\operatorname{tr}(Q_j) = \sum_{i=1}^n \operatorname{Var}(r_{i,[j]}) \ge \mathbb{E}\left(\sum_i \inf_{\epsilon} \left|\frac{\partial r_{i,[j]}}{\partial \epsilon_i}\right|^2\right) \cdot \min_i \operatorname{Var}(\epsilon_i).$$

It will be shown in Appendix B-6.3 that under assumptions A1-A3,

$$\mathbb{E}\sum_{i} \inf_{\epsilon} \left| \frac{\partial r_{i,[j]}}{\partial \epsilon_{i}} \right|^{2} = \Omega_{p} \left(\frac{n}{\text{polyLog}(n)} \right).$$
(A-11)

This proves (A-8) and as a result,

$$\min_{j} \operatorname{Var}(\hat{\beta}_{j}) = \Omega_{p} \left(\frac{1}{n \cdot \operatorname{polyLog}(n)} \right)$$

B Proof of Theorem 3.1

B-1 Notation

To be self-contained, we summarize our notations in this subsection. The model we considered here is

$$y = X\beta^* + \epsilon$$

where $X \in \mathbb{R}^{n \times p}$ be the design matrix and ϵ is a random vector with independent entries. Notice that the target quantity $\frac{\hat{\beta}_j - \mathbb{E}\hat{\beta}_j}{\sqrt{\operatorname{Var}(\hat{\beta}_j)}}$ is shift invariant, we can assume $\beta^* = 0$ without loss of generality provided that X has full column rank; see Section 3.1 for details.

Let $x_i^T \in \mathbb{R}^{1 \times p}$ denote the *i*-th row of X and $X_j \in \mathbb{R}^{n \times 1}$ denote the *j*-th column of X. Throughout the paper we will denote by $X_{ij} \in \mathbb{R}$ the (i, j)-th entry of X, by $X_{(i)} \in \mathbb{R}^{(n-1) \times p}$ the design matrix X after removing the *i*-th row, by $X_{[j]} \in \mathbb{R}^{n \times (p-1)}$ the design matrix X after removing the *j*-th column, by $X_{(i),[j]} \in \mathbb{R}^{(n-1) \times (p-1)}$ the design matrix after removing both *i*-th row and *j*-th column, and by $x_{i,[j]} \in \mathbb{R}^{1 \times (p-1)}$ the vector x_i after removing *j*-th entry. The M-estimator $\hat{\beta}$ associated with the loss function ρ is defined as

$$\hat{\beta} = \underset{\beta \in \mathbb{R}^p}{\operatorname{arg\,min}} \frac{1}{n} \sum_{k=1}^n \rho(\epsilon_k - x_k^T \beta).$$
(B-12)

Similarly we define the leave-j-th-predictor-out version as

$$\hat{\beta}_{[j]} = \underset{\beta \in \mathbb{R}^p}{\operatorname{arg\,min}} \frac{1}{n} \sum_{k=1}^n \rho(\epsilon_k - x_{k,[j]}^T \beta).$$
(B-13)

Based on these notation we define the full residual R_k as

$$R_k = \epsilon_k - x_k^T \hat{\beta}, \quad k = 1, 2, \dots, n \tag{B-14}$$

the leave-j-th-predictor-out residual as

$$r_{k,[j]} = \epsilon_k - x_{k,[j]}^T \hat{\beta}_{[j]}, \quad k = 1, 2, \dots, n, \ j \in J_n.$$
 (B-15)

Four diagonal matrices are defined as

$$D = \operatorname{diag}(\psi'(R_k)), \quad \tilde{D} = \operatorname{diag}(\psi''(R_k)), \quad (B-16)$$

$$D_{[j]} = \text{diag}(\psi'(r_{k,[j]})), \quad \tilde{D}_{[j]} = \text{diag}(\psi''(r_{k,[j]})).$$
 (B-17)

Further we define G and $G_{[j]}$ as

$$G = I - X(X^T D X)^{-1} X^T D, \quad G_{[j]} = I - X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]}.$$
(B-18)

Let J_n denote the indices of coefficients of interest. We say $a \in]a_1, a_2[$ if and only if $a \in [\min\{a_1, a_2\}, \max\{a_1, a_2\}]$. Regarding the technical assumptions, we need the following quantities

$$\lambda_{+} = \lambda_{\max}\left(\frac{X^{T}X}{n}\right), \quad \lambda_{-} = \lambda_{\min}\left(\frac{X^{T}X}{n}\right)$$
 (B-19)

be the largest (resp. smallest) eigenvalue of the matrix $\frac{X^T X}{n}$. Let $e_i \in \mathbb{R}^n$ be the *i*-th canonical basis vector and

$$h_{j,0} = (\psi(r_{1,[j]}), \dots, \psi(r_{n,[j]}))^T, \quad h_{j,1,i} = G_{[j]}^T e_i.$$
 (B-20)

Finally, let

$$\Delta_C = \max\left\{\max_{j\in J_n} \frac{|h_{j,0}^T X_j|}{\|h_{j,0}\|}, \max_{i\leq n, j\in J_n} \frac{|h_{j,1,i}^T X_j|}{\|h_{j,1,i}\|}\right\},\tag{B-21}$$

$$Q_j = \operatorname{Cov}(h_{j,0}). \tag{B-22}$$

We adopt Landau's notation $(O(\cdot), o(\cdot), O_p(\cdot), o_p(\cdot))$. In addition, we say $a_n = \Omega(b_n)$ if $b_n = O(a_n)$ and similarly, we say $a_n = \Omega_p(b_n)$ if $b_n = O_p(a_n)$. To simplify the logarithm factors, we use the symbol polyLog(n) to denote any factor that can be upper bounded by $(\log n)^{\gamma}$ for some $\gamma > 0$. Similarly, we use $\frac{1}{\text{polyLog}(n)}$ to denote any factor that can be lower bounded by $\frac{1}{(\log n)^{\gamma'}}$ for some $\gamma' > 0$.

Finally we restate all the technical assumptions:

A1 $\rho(0) = \psi(0) = 0$ and there exists $K_0 = \Omega\left(\frac{1}{\text{polyLog}(n)}\right), K_1, K_2 = O(\text{polyLog}(n)),$ such that for any $x \in \mathbb{R}$,

$$K_0 \le \psi'(x) \le K_1, \quad \left| \frac{d}{dx} (\sqrt{\psi'}(x)) \right| = \frac{|\psi''(x)|}{\sqrt{\psi'(x)}} \le K_2;$$

A2 $\epsilon_i = u_i(W_i)$ where $(W_1, \ldots, W_n) \sim N(0, I_{n \times n})$ and u_i are smooth functions with $\|u_i'\|_{\infty} \leq c_1$ and $\|u_i''\|_{\infty} \leq c_2$ for some $c_1, c_2 = O(\text{polyLog}(n))$. Moreover, assume $\min_i \operatorname{Var}(\epsilon_i) = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right)$.

A3
$$\lambda_+ = O(\text{polyLog}(n))$$
 and $\lambda_- = \Omega\left(\frac{1}{\text{polyLog}(n)}\right);$

A4 min_{j\in J_n} $\frac{X_j^T Q_j X_j}{\operatorname{tr}(Q_j)} = \Omega\left(\frac{1}{\operatorname{polyLog}(\mathbf{n})}\right);$

A5 $\mathbb{E}\Delta_C^8 = O(\text{polyLog}(n)).$

B-2 Deterministic Approximation Results

In Appendix A, we use several approximations under random designs, e.g. $R_i \approx r_{i,[j]}$. To prove them, we follow the strategy of El Karoui (2013) which establishes the deterministic results and then apply the concentration inequalities to obtain high probability bounds. Note that $\hat{\beta}$ is the solution of

$$0 = f(\beta) \triangleq \frac{1}{n} \sum_{i=1}^{n} x_i \psi(\epsilon_i - x_i^T \beta),$$

we need the following key lemma to bound $\|\beta_1 - \beta_2\|_2$ by $\|f(\beta_1) - f(\beta_2)\|_2$, which can be calculated explicitly.

Lemma B.1. [El Karoui (2013), Proposition 2.1] For any β_1 and β_2 ,

$$\|\beta_1 - \beta_2\|_2 \le \frac{1}{K_0 \lambda_-} \|f(\beta_1) - f(\beta_2)\|_2.$$

Proof. By the mean value theorem, there exists $\nu_i \in]\epsilon_i - x_i^T \beta_1, \epsilon_i - x_i^T \beta_2[$ such that

$$\psi(\epsilon_i - x_i^T \beta_1) - \psi(\epsilon_i - x_i^T \beta_2) = \psi'(\nu_i) \cdot x_i^T (\beta_2 - \beta_1).$$

Then

$$\|f(\beta_{1}) - f(\beta_{2})\|_{2} = \left\|\frac{1}{n}\sum_{i=1}^{n}\psi'(\nu_{i})x_{i}x_{i}^{T}(\beta_{1} - \beta_{2})\right\|_{2}$$
$$\geq \lambda_{\min}\left(\frac{1}{n}\sum_{i=1}^{n}\psi'(\nu_{i})x_{i}x_{i}^{T}\right) \cdot \|\beta_{1} - \beta_{2}\|_{2}$$
$$\geq K_{0}\lambda_{-}\|\beta_{1} - \beta_{2}\|_{2}.$$

Е		

Based on Lemma B.1, we can derive the deterministic results informally stated in Appendix A. Such results are shown by El Karoui (2013) for ridge-penalized M-estimates and here we derive a refined version for unpenalized M-estimates. Throughout this subsection, we only assume assumption A1. This implies the following lemma,

Lemma B.2. Under assumption A_1 , for any x and y,

$$|\psi(x)| \le K_1|x|, \quad |\sqrt{\psi'(x)} - \sqrt{\psi'(y)}| \le K_2|x-y|, \quad |\psi'(x) - \psi'(y)| \le 2\sqrt{K_1}K_2|x-y| \triangleq K_3|x-y|$$

To state the result, we define the following quantities.

$$T = \frac{1}{\sqrt{n}} \max\left\{ \max_{i} \|x_i\|_2, \max_{j \in J_n} \|X_j\|_2 \right\}, \quad \mathcal{E} = \frac{1}{n} \sum_{i=1}^n \rho(\epsilon_i), \quad (B-23)$$

$$U = \left\| \frac{1}{n} \sum_{i=1}^{n} x_i(\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i)) \right\|_2, \quad U_0 = \left\| \frac{1}{n} \sum_{i=1}^{n} x_i \mathbb{E}\psi(\epsilon_i) \right\|_2.$$
(B-24)

The following proposition summarizes all deterministic results which we need in the proof.

Proposition B.3. Under Assumption A1,

(i) The norm of M estimator is bounded by

$$\|\hat{\beta}\|_2 \le \frac{1}{K_0\lambda_-}(U+U_0);$$

(ii) Define b_j as

$$b_j = \frac{1}{\sqrt{n}} \frac{N_j}{\xi_j}$$

where

$$N_{j} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{ij} \psi(r_{i,[j]}), \quad \xi_{j} = \frac{1}{n} X_{j}^{T} (D_{[j]} - D_{[j]} X_{[j]} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]}) X_{j},$$

Then

$$\max_{j\in J_n} |b_j| \le \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{2K_1}}{K_0\lambda_-} \cdot \Delta_C \cdot \sqrt{\mathcal{E}},$$

(iii) The difference between $\hat{\beta}_j$ and b_j is bounded by

$$\max_{j\in J_n} |\hat{\beta}_j - b_j| \le \frac{1}{n} \cdot \frac{2K_1^2 K_3 \lambda_+ T}{K_0^4 \lambda_-^{\frac{7}{2}}} \cdot \Delta_C^3 \cdot \mathcal{E}.$$

(iv) The difference between the full and the leave-one-predictor-out residual is bounded by

,

$$\max_{j \in J_n} \max_{i} |R_i - r_{i,[j]}| \le \frac{1}{\sqrt{n}} \left(\frac{2K_1^2 K_3 \lambda_+ T^2}{K_0^4 \lambda_-^{\frac{7}{2}}} \cdot \Delta_C^3 \cdot \mathcal{E} + \frac{\sqrt{2}K_1}{K_0^{\frac{3}{2}} \lambda_-} \cdot \Delta_C^2 \cdot \sqrt{\mathcal{E}} \right).$$

Proof. (i) By Lemma B.1,

$$\|\hat{\beta}\|_{2} \leq \frac{1}{K_{0}\lambda_{-}} \|f(\hat{\beta}) - f(0)\|_{2} = \frac{\|f(0)\|_{2}}{K_{0}\lambda_{-}},$$

since $\hat{\beta}$ is a zero of $f(\beta)$. By definition,

$$f(0) = \frac{1}{n} \sum_{i=1}^{n} x_i \psi(\epsilon_i) = \frac{1}{n} \sum_{i=1}^{n} x_i (\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i)) + \frac{1}{n} \sum_{i=1}^{n} x_i \mathbb{E}\psi(\epsilon_i).$$

This implies that

$$\|f(0)\|_2 \le U + U_0.$$

(ii) First we prove that

$$\xi_j \ge K_0 \lambda_-. \tag{B-25}$$

Since all diagonal entries of $D_{[j]}$ is lower bounded by K_0 , we conclude that

$$\lambda_{\min}\left(\frac{X^T D_{[j]}X}{n}\right) \ge K_0 \lambda_-.$$

Note that ξ_j is the Schur's complement (Horn & Johnson, 2012, chapter 0.8) of $\frac{X^T D_{[j]} X}{n}$, we have

$$\xi_j^{-1} = e_j^T \left(\frac{X^T D_{[j]} X}{n} \right)^{-1} e_j \le \frac{1}{K_0 \lambda_-},$$

which implies (B-25). As for N_j , we have

$$N_j = \frac{X_j^T h_{j,0}}{\sqrt{n}} = \frac{\|h_{j,0}\|_2}{\sqrt{n}} \cdot \frac{X_j^T h_{j,0}}{\|h_{j,0}\|_2}.$$
 (B-26)

The the second term is bounded by Δ_C by definition, see (B-21). For the first term, the assumption A1 that $\psi'(x) \leq K_1$ implies that

$$\rho(x) = \rho(x) - \rho(0) = \int_0^x \psi(y) dy \ge \int_0^x \frac{\psi'(y)}{K_1} \cdot \psi(y) dy = \frac{1}{2K_1} \psi^2(x).$$

Here we use the fact that $sign(\psi(y)) = sign(y)$. Recall the definition of $h_{j,0}$, we obtain that

$$\frac{\|h_{j,0}\|_2}{\sqrt{n}} = \sqrt{\frac{\sum_{i=1}^n \psi(r_{i,[j]})^2}{n}} \le \sqrt{2K_1} \cdot \sqrt{\frac{\sum_{i=1}^n \rho(r_{i,[j]})}{n}}$$

Since $\hat{\beta}_{[j]}$ is the minimizer of the loss function $\sum_{i=1}^{n} \rho(\epsilon_i - x_{i,[j]}^T \beta_{[j]})$, it holds that

$$\frac{1}{n}\sum_{i=1}^n \rho(r_{i,[j]}) \le \frac{1}{n}\sum_{i=1}^n \rho(\epsilon_i) = \mathcal{E}.$$

Putting together the pieces, we conclude that

$$|N_j| \le \sqrt{2K_1} \cdot \Delta_C \sqrt{\mathcal{E}}.$$
 (B-27)

By definition of b_j ,

$$|b_j| \leq \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{2K_1}}{K_0 \lambda_-} \Delta_C \sqrt{\mathcal{E}}.$$

(iii) The proof of this result is almost the same as El Karoui (2013). We state it here for the sake of completeness. Let $\tilde{\mathbf{b}}_{\mathbf{j}} \in \mathbb{R}^p$ with

$$(\tilde{\mathbf{b}}_{\mathbf{j}})_j = b_j, \quad (\tilde{\mathbf{b}}_{\mathbf{j}})_{[j]} = \hat{\beta}_{[j]} - b_j (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]} X_j$$
(B-28)

where the subscript j denotes the j-th entry and the subscript [j] denotes the subvector formed by all but j-th entry. Furthermore, define γ_j with

$$(\gamma_j)_j = -1, \quad (\gamma_j)_{[j]} = (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]} X_j.$$
 (B-29)

Then we can rewrite $\mathbf{\tilde{b}_{j}}$ as

$$(\tilde{\mathbf{b}}_{\mathbf{j}})_j = -b_j(\gamma_j)_j, \quad (\tilde{\mathbf{b}}_{\mathbf{j}})_{[j]} = \hat{\beta}_{[j]} - b_j(\gamma_j)_{[j]}.$$

By definition of $\hat{\beta}_{[j]},$ we have $[f(\hat{\beta}_{[j]})]_{[j]}=0$ and hence

$$[f(\tilde{\mathbf{b}}_{\mathbf{j}})]_{[j]} = [f(\tilde{\mathbf{b}}_{\mathbf{j}})]_{[j]} - [f(\hat{\beta}_{[j]})]_{[j]} = \frac{1}{n} \sum_{i=1}^{n} x_{i,[j]} \left[\psi(\epsilon_i - x_i^T \tilde{\mathbf{b}}_{\mathbf{j}}) - \psi(\epsilon_i - x_{i,[j]}^T \hat{\beta}_{[j]}) \right].$$
(B-30)

By mean value theorem, there exists $\nu_{i,j} \in]\epsilon_i - x_i^T \tilde{\mathbf{b}}_{\mathbf{j}}, \epsilon_i - x_{i,[j]}^T \hat{\beta}_{[j]}[$ such that

$$\begin{aligned} \psi(\epsilon_{i} - x_{i}^{T}\tilde{\mathbf{b}}_{j}) &- \psi(\epsilon_{i} - x_{i,[j]}^{T}\hat{\beta}_{[j]}) = \psi'(\nu_{i,j})(x_{i,[j]}^{T}\hat{\beta}_{[j]} - x_{i}^{T}\tilde{\mathbf{b}}_{j}) \\ &= \psi'(\nu_{i,j})(x_{i,[j]}^{T}\hat{\beta}_{[j]} - x_{i,[j]}^{T}(\tilde{\mathbf{b}}_{j})_{[j]} - X_{ij}b_{j}) \\ &= \psi'(\nu_{i,j}) \cdot b_{j} \cdot \left[x_{i,[j]}^{T}(X_{[j]}^{T}D_{[j]}X_{[j]})^{-1}X_{[j]}^{T}D_{[j]}X_{j} - X_{ij}\right] \end{aligned}$$

Let

$$d_{i,j} = \psi'(\nu_{i,j}) - \psi'(r_{i,[j]})$$
(B-31)

and plug the above result into (B-30), we obtain that

$$\begin{split} [f(\tilde{\mathbf{b}}_{\mathbf{j}})]_{[j]} &= \frac{1}{n} \sum_{i=1}^{n} x_{i,[j]} \cdot \left(\psi'(r_{i,[j]}) + d_{i,j} \right) \cdot b_{j} \cdot \left[x_{i,[j]}^{T} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]} X_{j} - X_{ij} \right] \\ &= b_{j} \cdot \frac{1}{n} \sum_{i=1}^{n} \psi'(r_{i,[j]}) x_{i,[j]} \left[x_{i,[j]}^{T} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]} X_{j} - X_{ij} \right] \\ &+ b_{j} \cdot \frac{1}{n} \sum_{i=1}^{n} d_{i,j} x_{i,[j]} (x_{i,[j]}^{T} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]} X_{j} - X_{ij}) \\ &= b_{j} \cdot \frac{1}{n} \left[X_{[j]}^{T} D_{[j]} X_{[j]} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]} X_{j} - X_{[j]}^{T} D_{[j]} X_{j} \right] \\ &+ b_{j} \cdot \frac{1}{n} \sum_{i=1}^{n} d_{i,j} x_{i,[j]} \cdot x_{i}^{T} \gamma_{j} \\ &= b_{j} \cdot \frac{1}{n} \left(\sum_{i=1}^{n} d_{i,j} x_{i,[j]} x_{i}^{T} \right) \gamma_{j}. \end{split}$$

Now we calculate $[f(\tilde{\mathbf{b}}_{\mathbf{j}})]_j$, the *j*-th entry of $f(\tilde{\mathbf{b}}_{\mathbf{j}})$. Note that

$$\begin{split} &[f(\tilde{\mathbf{b}}_{\mathbf{j}})]_{j} = \frac{1}{n} \sum_{i=1}^{n} X_{ij} \psi(\epsilon_{i} - x_{i}^{T} \tilde{\mathbf{b}}_{\mathbf{j}}) \\ &= \frac{1}{n} \sum_{i=1}^{n} X_{ij} \psi(r_{i,[j]}) + b_{j} \cdot \frac{1}{n} \sum_{i=1}^{n} X_{ij} (\psi'(r_{i,[j]}) + d_{i,j}) \cdot \left[x_{i,[j]}^{T} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]} X_{j} - X_{ij} \right] \\ &= \frac{1}{n} \sum_{i=1}^{n} X_{ij} \psi(r_{i,[j]}) + b_{j} \cdot \frac{1}{n} \sum_{i=1}^{n} \psi'(r_{i,[j]}) X_{ij} \left[x_{i,[j]}^{T} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]} X_{j} - X_{ij} \right] \\ &\quad + b_{j} \cdot \left(\frac{1}{n} \sum_{i=1}^{n} d_{i,j} X_{ij} x_{i}^{T} \right) \gamma_{j} \\ &= \frac{1}{\sqrt{n}} N_{j} + b_{j} \cdot \left(\frac{1}{n} X_{j}^{T} D_{[j]} X_{[j]} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]} X_{j} - \frac{1}{n} \sum_{i=1}^{n} \psi'(r_{i,[j]}) X_{ij}^{2} \right) \\ &\quad + b_{j} \cdot \left(\frac{1}{n} \sum_{i=1}^{n} d_{i,j} X_{ij} x_{i}^{T} \right) \gamma_{j} \\ &= \frac{1}{\sqrt{n}} N_{j} - b_{j} \cdot \xi_{j} + b_{j} \cdot \left(\frac{1}{n} \sum_{i=1}^{n} d_{i,j} X_{ij} x_{i}^{T} \right) \gamma_{j} \\ &= b_{j} \cdot \left(\frac{1}{n} \sum_{i=1}^{n} d_{i,j} X_{ij} x_{i}^{T} \right) \gamma_{j} \end{split}$$

where the second last line uses the definition of b_j . Putting the results together, we obtain that

$$f(\tilde{\mathbf{b}}_{\mathbf{j}}) = b_j \cdot \left(\frac{1}{n} \sum_{i=1}^n d_{i,j} x_i x_i^T\right) \cdot \gamma_j.$$

This entails that

$$\|f(\tilde{\mathbf{b}}_{\mathbf{j}})\|_{2} \le |b_{j}| \cdot \max_{i} |d_{i,j}| \cdot \lambda_{+} \cdot \|\gamma_{j}\|_{2}.$$
(B-32)

Now we derive a bound for $\max_i |d_{i,j}|$, where $d_{i,j}$ is defined in (B-36). By Lemma B.2,

$$|d_{i,j}| = |\psi'(\nu_{i,j}) - \psi'(r_{i,[j]})| \le K_3 |\nu_{i,j} - r_{i,[j]}| = K_3 |x_{i,[j]}^T \hat{\beta}_{[j]} - x_i^T \tilde{\mathbf{b}}_{\mathbf{j}}|.$$

By definition of $\tilde{\mathbf{b}}_{\mathbf{j}}$ and $h_{j,1,i}$,

$$\begin{aligned} & |x_{i,[j]}^{T}\hat{\beta}_{[j]} - x_{i}^{T}\tilde{\mathbf{b}}_{\mathbf{j}}| = |b_{j}| \cdot \left| x_{i,[j]}^{T} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]} X_{j} - X_{ij} \right| \\ & = |b_{j}| \cdot |e_{i}^{T} (I - X_{[j]} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]}) X_{j}| \\ & = |b_{j}| \cdot |h_{j,1,i}^{T} X_{j}| \leq |b_{j}| \cdot \Delta_{C} \|h_{j,1,i}\|_{2}, \end{aligned}$$
(B-33)

where the last inequality is derived by definition of Δ_C , see (B-21). Since $h_{j,1,i}$ is the *i*-th column of matrix $I - D_{[j]}X_{[j]}(X_{[j]}^T D_{[j]}X_{[j]})^{-1}X_{[j]}^T$, its L_2 norm is upper bounded by the operator norm of this matrix. Notice that

$$I - D_{[j]}X_{[j]}(X_{[j]}^T D_{[j]}X_{[j]})^{-1}X_{[j]}^T = D_{[j]}^{\frac{1}{2}} \left(I - D_{[j]}^{\frac{1}{2}}X_{[j]}(X_{[j]}^T D_{[j]}X_{[j]})^{-1}X_{[j]}^T D_{[j]}^{\frac{1}{2}} \right) D_{[j]}^{-\frac{1}{2}}.$$

The middle matrix in RHS of the displayed atom is an orthogonal projection matrix and hence

$$\|I - D_{[j]}X_{[j]}(X_{[j]}^T D_{[j]}X_{[j]})^{-1}X_{[j]}^T\|_{\text{op}} \le \|D_{[j]}^{\frac{1}{2}}\|_{\text{op}} \cdot \|D_{[j]}^{-\frac{1}{2}}\|_{\text{op}} \le \left(\frac{K_1}{K_0}\right)^{\frac{1}{2}}.$$
 (B-34)

Therefore,

$$\max_{i,j} \|h_{j,1,i}\|_2 \le \max_{j \in J_n} \|I - D_{[j]} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T \|_{\text{op}} \le \left(\frac{K_1}{K_0}\right)^{\frac{1}{2}}, \quad (B-35)$$

and thus

$$\max_{i} |d_{i,j}| \le K_3 \sqrt{\frac{K_1}{K_0}} \cdot |b_j| \cdot \Delta_C.$$
(B-36)

As for γ_j , we have

$$K_{0}\lambda_{-} \|\gamma_{j}\|_{2}^{2} \leq \gamma_{j}^{T} \left(\frac{X^{T}D_{[j]}X}{n}\right)\gamma_{j}$$

= $(\gamma_{j})_{j}^{2} \cdot \frac{X_{j}^{T}D_{j}X_{j}}{n} + (\gamma_{j})_{[j]}^{T} \left(\frac{X_{[j]}^{T}D_{[j]}X_{[j]}}{n}\right)(\gamma_{j})_{[j]} + 2\gamma_{j}\frac{X_{j}^{T}D_{[j]}X_{[j]}}{n}(\gamma_{j})_{[j]}$

Recall the definition of γ_j in (B-37), we have

$$(\gamma_j)_{[j]}^T \left(\frac{X_{[j]}^T D_{[j]} X_{[j]}}{n}\right) (\gamma_j)_{[j]} = \frac{1}{n} X_j^T D_{[j]} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]} X_j$$

and

$$\gamma_j \frac{X_j^T D_{[j]} X_{[j]}}{n} (\gamma_j)_{[j]} = -\frac{1}{n} X_j^T D_{[j]} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]} X_j.$$

As a result,

$$\begin{split} & K_{0}\lambda_{-} \|\gamma_{j}\|_{2}^{2} \\ & \leq \frac{1}{n}X_{j}^{T}D_{[j]}^{\frac{1}{2}}(I - D_{[j]}^{\frac{1}{2}}X_{[j]}(X_{[j]}^{T}D_{[j]}X_{[j]})^{-1}X_{[j]}^{T}D_{[j]}^{\frac{1}{2}})D_{[j]}^{\frac{1}{2}}X_{j} \\ & \leq \frac{\|D_{[j]}^{\frac{1}{2}}X_{j}\|_{2}^{2}}{n} \cdot \left\|I - D_{[j]}^{\frac{1}{2}}X_{[j]}(X_{[j]}^{T}D_{[j]}X_{[j]})^{-1}X_{[j]}^{T}D_{[j]}^{\frac{1}{2}}\right\|_{op} \\ & \leq \frac{\|D_{[j]}^{\frac{1}{2}}X_{j}\|_{2}^{2}}{n} \leq \frac{K_{1}\|X_{j}\|_{2}^{2}}{n} \leq T^{2}K_{1}, \end{split}$$

where T is defined in (B-23). Therefore we have

$$\|\gamma_j\|_2 \le \sqrt{\frac{K_1}{K_0\lambda_-}}T.$$
(B-37)

Putting (B-32), (B-36), (B-37) and part (ii) together, we obtain that

$$\begin{split} \|f(\tilde{\mathbf{b}}_{\mathbf{j}})\|_{2} &\leq \lambda_{+} \cdot |b_{j}| \cdot K_{3} \sqrt{\frac{K_{1}}{K_{0}}} \Delta_{C} |b_{j}| \cdot \sqrt{\frac{K_{1}}{K_{0}\lambda_{-}}} T \\ &\leq \lambda_{+} \cdot \frac{1}{n} \frac{2K_{1}}{(K_{0}\lambda_{-})^{2}} \Delta_{C}^{2} \mathcal{E} \cdot K_{3} \sqrt{\frac{K_{1}}{K_{0}}} \Delta_{C} \cdot \sqrt{\frac{K_{1}}{K_{0}\lambda_{-}}} T \\ &= \frac{1}{n} \cdot \frac{2K_{1}^{2}K_{3}\lambda_{+}T}{K_{0}^{3}\lambda_{-}^{\frac{5}{2}}} \cdot \Delta_{C}^{3} \cdot \mathcal{E}. \end{split}$$

By Lemma B.1,

$$\|\hat{\beta} - \tilde{\mathbf{b}}_{\mathbf{j}}\|_2 \leq \frac{\|f(\hat{\beta}) - f(\tilde{\mathbf{b}}_{\mathbf{j}})\|_2}{K_0\lambda_-} = \frac{\|f(\tilde{\mathbf{b}}_{\mathbf{j}})\|_2}{K_0\lambda_-} \leq \frac{1}{n} \cdot \frac{2K_1^2K_3\lambda_+T}{K_0^4\lambda_-^{\frac{7}{2}}} \cdot \Delta_C^3 \cdot \mathcal{E}.$$

Since $\hat{\beta}_j - b_j$ is the *j*-th entry of $\hat{\beta} - \tilde{\mathbf{b}}_{\mathbf{j}}$, we have

$$|\hat{\beta}_j - b_j| \le \|\hat{\beta} - \tilde{\mathbf{b}}_{\mathbf{j}}\|_2 \le \frac{1}{n} \cdot \frac{2K_1^2 K_3 \lambda_+ T}{K_0^4 \lambda_-^{\frac{7}{2}}} \cdot \Delta_C^3 \cdot \mathcal{E}.$$

(iv) Similar to part (iii), this result has been shown by El Karoui (2013). Here we state a refined version for the sake of completeness. Let $\tilde{\mathbf{b}}_{\mathbf{j}}$ be defined as in (B-28), then

$$|R_{i} - r_{i,[j]}| = |x_{i}^{T}\hat{\beta} - x_{i,[j]}^{T}\hat{\beta}_{[j]}| = |x_{i}^{T}(\hat{\beta} - \tilde{\mathbf{b}}_{\mathbf{j}}) + x_{i}^{T}\tilde{\mathbf{b}}_{\mathbf{j}} - x_{i,[j]}^{T}\hat{\beta}_{[j]}|$$

$$\leq ||x_{i}||_{2} \cdot ||\hat{\beta} - \tilde{\mathbf{b}}_{\mathbf{j}}||_{2} + |x_{i}^{T}\tilde{\mathbf{b}}_{\mathbf{j}} - x_{i,[j]}^{T}\hat{\beta}_{[j]}|.$$

Note that $||x_i||_2 \leq \sqrt{n}T$, by part (iii), we have

$$\|x_i\|_2 \cdot \|\hat{\beta} - \tilde{\mathbf{b}}_{\mathbf{j}}\|_2 \le \frac{1}{\sqrt{n}} \frac{2K_1^2 K_3 \lambda_+ T^2}{K_0^4 \lambda_-^{\frac{7}{2}}} \cdot \Delta_C^3 \cdot \mathcal{E}.$$
 (B-38)

On the other hand, similar to (B-36), by (B-33),

$$|x_i^T \tilde{\mathbf{b}}_{\mathbf{j}} - x_{i,[j]}^T \hat{\beta}_{[j]}| \le \sqrt{\frac{K_1}{K_0}} \cdot |b_j| \cdot \Delta_C \le \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{2}K_1}{K_0^{\frac{3}{2}} \lambda_-} \cdot \Delta_C^2 \cdot \sqrt{\mathcal{E}}.$$
 (B-39)

Therefore,

$$|R_i - r_{i,[j]}| \leq \frac{1}{\sqrt{n}} \left(\frac{2K_1^2 K_3 \lambda_+ T^2}{K_0^4 \lambda_-^{\frac{7}{2}}} \cdot \Delta_C^3 \cdot \mathcal{E} + \frac{\sqrt{2}K_1}{K_0^{\frac{3}{2}} \lambda_-} \cdot \Delta_C^2 \cdot \sqrt{\mathcal{E}} \right).$$

B-3 Summary of Approximation Results

Under our technical assumptions, we can derive the rate for approximations via Proposition B.3. This justifies all approximations in Appendix A.

Theorem B.4. Under the assumptions A1 - A5,

(i)
$$T \le \lambda_+ = O\left(\text{polyLog}(\mathbf{n})\right);$$

$$\max_{j \in J_n} |\hat{\beta}_j| \le \|\hat{\beta}\|_2 = O_{L^4} \left(\text{polyLog}(\mathbf{n}) \right);$$

(iii)

$$\max_{j \in J_n} |b_j| = O_{L^2} \left(\frac{\text{polyLog}(n)}{\sqrt{n}} \right);$$

(iv)

$$\max_{j\in J_n} |\hat{\beta}_j - b_j| = O_{L^2}\left(\frac{\text{polyLog}(n)}{n}\right);$$

(v)

$$\max_{j \in J_n} \max_i |R_i - r_{i,[j]}| = O_{L^2} \left(\frac{\text{polyLog}(n)}{\sqrt{n}} \right).$$

Proof. (i) Notice that $X_j = Xe_j$, where e_j is the *j*-th canonical basis vector in \mathbb{R}^p , we have

$$\frac{\|X_j\|^2}{n} = e_j^T \frac{X^T X}{n} e_j \le \lambda_+.$$

Similarly, consider the X^T instead of X, we conclude that

$$\frac{\|x_i\|^2}{n} \le \lambda_{\max}\left(\frac{XX^T}{n}\right) = \lambda_+.$$

Recall the definition of T in (B-23), we conclude that

$$T \leq \sqrt{\lambda_+} = O(\text{polyLog}(n)).$$

(ii) Since $\epsilon_i = u_i(W_i)$ with $||u'_i||_{\infty} \leq c_1$, the gaussian concentration property (Ledoux, 2001, chapter 1.3) implies that ϵ_i is c_1^2 -sub-gaussian and hence $\mathbb{E}|\epsilon_i|^k = O(c_1^k)$ for any finite k > 0. By Lemma B.2, $|\psi(\epsilon_i)| \leq K_1 |\epsilon_i|$ and hence for any finite k,

$$\mathbb{E}|\psi(\epsilon_i)|^k \le K_1^k \mathbb{E}|\epsilon_i|^k = O(c_1^k).$$

By part (i) of Proposition B.3, using the convexity of x^4 and hence $\left(\frac{a+b}{2}\right)^4 \leq \frac{a^4+b^4}{2}$,

$$\mathbb{E}\|\hat{\beta}\|_{2}^{4} \leq \frac{1}{(K_{0}\lambda_{-})^{4}} \mathbb{E}(U+U_{0})^{4} \leq \frac{8}{(K_{0}\lambda_{-})^{4}} (\mathbb{E}U^{4}+U_{0}^{4}).$$

Recall (B-24) that $U = \left\| \frac{1}{n} \sum_{i=1}^{n} x_i(\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i)) \right\|_2$,

$$\begin{aligned} U^4 &= (U^2)^2 = \frac{1}{n^4} \left(\sum_{i,i'=1}^n x_i^T x_{i'} (\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i))(\psi(\epsilon_{i'}) - \mathbb{E}\psi(\epsilon_{i'})) \right)^2 \\ &= \frac{1}{n^4} \left(\sum_{i=1}^n \|x_i\|_2^2 (\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i))^2 + \sum_{i \neq i'} |x_i^T x_{i'}| (\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i))(\psi(\epsilon_{i'}) - \mathbb{E}\psi(\epsilon_{i'})) \right)^2 \end{aligned}$$

$$= \frac{1}{n^4} \left\{ \sum_{i=1}^n \|x_i\|_2^4 (\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i))^4 + \sum_{i \neq i'} (2|x_i^T x_{i'}|^2 + \|x_i\|_2^2 \|x_{i'}\|_2^2) (\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i))^2 (\psi(\epsilon_{i'}) - \mathbb{E}\psi(\epsilon_{i'}))^2 \\ + \sum_{\text{others}} |x_i^T x_{i'}| \cdot |x_k^T x_{k'}| \cdot (\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i)) (\psi(\epsilon_{i'}) - \mathbb{E}\psi(\epsilon_{i'})) (\psi(\epsilon_k) - \mathbb{E}\psi(\epsilon_k)) (\psi(\epsilon_{k'}) - \mathbb{E}\psi(\epsilon_{k'})) \right\}$$

Since $\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i)$ has a zero mean, we have

$$\mathbb{E}(\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i))(\psi(\epsilon_{i'}) - \mathbb{E}\psi(\epsilon_{i'}))(\psi(\epsilon_k) - \mathbb{E}\psi(\epsilon_k))(\psi(\epsilon_{k'}) - \mathbb{E}\psi(\epsilon_{k'})) = 0$$

for any $(i,i') \neq (k,k')$ or (k',k) and $i \neq i'.$ As a consequence,

$$\mathbb{E}U^{4} = \frac{1}{n^{4}} \left(\sum_{i=1}^{n} \|x_{i}\|_{2}^{4} \mathbb{E}(\psi(\epsilon_{i}) - \mathbb{E}\psi(\epsilon_{i}))^{4} + \sum_{i \neq i'} (2|x_{i}^{T}x_{i'}|_{2}^{2} + \|x_{i}\|_{2}^{2} \|x_{i'}\|_{2}^{2}) \mathbb{E}(\psi(\epsilon_{i}) - \mathbb{E}\psi(\epsilon_{i}))^{2} \mathbb{E}(\psi(\epsilon_{i'}) - \mathbb{E}\psi(\epsilon_{i'}))^{2} \right) \\ \leq \frac{1}{n^{4}} \left(\sum_{i=1}^{n} \|x_{i}\|_{2}^{4} \mathbb{E}(\psi(\epsilon_{i}) - \mathbb{E}\psi(\epsilon_{i}))^{4} + 3 \sum_{i \neq i'} \|x_{i}\|_{2}^{2} \mathbb{E}(\psi(\epsilon_{i}) - \mathbb{E}\psi(\epsilon_{i}))^{2} \mathbb{E}(\psi(\epsilon_{i'}) - \mathbb{E}\psi(\epsilon_{i'}))^{2} \right).$$

For any *i*, using the convexity of x^4 , hence $(\frac{a+b}{2})^4 \leq \frac{a^4+b^4}{2}$, we have

$$\mathbb{E}(\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i))^4 \le 8\mathbb{E}\left(\psi(\epsilon_i)^4 + (\mathbb{E}\psi(\epsilon_i))^4\right) \le 16\mathbb{E}\psi(\epsilon_i)^4 \le 16\max_i \mathbb{E}\psi(\epsilon_i)^4.$$

By Cauchy-Schwartz inequality,

$$\mathbb{E}(\psi(\epsilon_i) - \mathbb{E}\psi(\epsilon_i))^2 \le \mathbb{E}\psi(\epsilon_i)^2 \le \sqrt{\mathbb{E}\psi(\epsilon_i)^4} \le \sqrt{\max_i \mathbb{E}\psi(\epsilon_i)^4}.$$

Recall (B-23) that $||x_i||_2^2 \le nT^2$ and thus,

$$\mathbb{E}U^4 \leq \frac{1}{n^4} \left(16n \cdot n^2 T^4 + 3n^2 \cdot n^2 T^4 \right) \cdot \max_i \mathbb{E}\psi(\epsilon_i)^4$$
$$\leq \frac{1}{n^4} \cdot (16n^3 + 3n^4) T^4 \max_i \mathbb{E}\psi(\epsilon_i)^4 = O\left(\text{polyLog}(\mathbf{n})\right)$$

On the other hand, let $\mu^T = (\mathbb{E}\psi(\epsilon_1), \dots, \mathbb{E}\psi(\epsilon_n))$, then $\|\mu\|_2^2 = O(n \cdot \text{polyLog}(n))$ and hence by definition of U_0 in (B-24),

$$U_{0} = \frac{\|\mu^{T}X\|_{2}}{n} = \frac{1}{n}\sqrt{\mu^{T}XX^{T}\mu} \le \sqrt{\frac{\|\mu\|_{2}^{2}}{n}} \cdot \lambda_{+} = O\left(\text{polyLog}(n)\right).$$

In summary,

$$\mathbb{E}\|\hat{\beta}\|_2^4 = O\left(\mathrm{polyLog}(\mathbf{n})\right).$$

(iii) By mean-value theorem, there exists $a_x \in (0, x)$ such that

$$\rho(x) = \rho(0) + x\psi(0) + \frac{x^2}{2}\psi'(a_x).$$

By assumption A1 and Lemma B.2, we have

$$\rho(x) = \frac{x^2}{2}\psi'(a_x) \le \frac{x^2}{2} \|\psi'\|_{\infty} \le \frac{K_3 x^2}{2},$$

where K_3 is defined in Lemma B.2. As a result,

$$\mathbb{E}\rho(\epsilon_i)^8 \le \left(\frac{K_3}{2}\right)^8 \mathbb{E}\epsilon_i^{16} = O(c_1^{16}).$$

Recall the definition of \mathcal{E} in (B-23) and the convexity of x^8 , we have

$$\mathbb{E}\mathcal{E}^8 \le \frac{1}{n} \sum_{i=1}^n \mathbb{E}\rho(\epsilon_i)^8 = O(c_1^{16}) = O\left(\text{polyLog}(n)\right). \tag{B-40}$$

Under assumption A5, by Cauchy-Schwartz inequality,

$$\mathbb{E}(\Delta_C \sqrt{\mathcal{E}})^2 = \mathbb{E}\Delta_C^2 \mathcal{E} \le \sqrt{\mathbb{E}\Delta_C^4} \cdot \sqrt{\mathbb{E}\mathcal{E}^2} = O\left(\text{polyLog}(n)\right).$$

Under assumptions A1 and A3,

$$\frac{\sqrt{2K_1}}{K_0\lambda_-} = O\left(\text{polyLog}(\mathbf{n})\right)$$

Putting all the pieces together, we obtain that

$$\max_{j \in J_n} |b_j| = O_{L^2} \left(\frac{\text{polyLog}(n)}{\sqrt{n}} \right).$$

(iv) Similarly, by Holder's inequality,

$$\mathbb{E}(\Delta_C^3 \mathcal{E})^2 = \mathbb{E}\Delta_C^6 \mathcal{E}^2 \le \left(\mathbb{E}\Delta_C^8\right)^{\frac{3}{4}} \cdot \left(\mathbb{E}\mathcal{E}^8\right)^{\frac{1}{4}} = O\left(\text{polyLog}(\mathbf{n})\right),$$

and under assumptions A1 and A3,

$$\frac{2K_1^2K_3\lambda_+T}{K_0^4\lambda_-^{\frac{7}{2}}} = O\left(\text{polyLog}(\mathbf{n})\right).$$

Therefore,

$$\max_{j \in J_n} |\hat{\beta}_j - b_j| = O_{L^2} \left(\frac{\text{polyLog}(n)}{n} \right).$$

(v) It follows from the previous part that

$$\mathbb{E}(\Delta_C^2 \cdot \sqrt{\mathcal{E}})^2 = O\left(\text{polyLog}(\mathbf{n})\right).$$

Under assumptions A1 and A3, the multiplicative factors are also O(polyLog(n)), i.e.

$$\frac{2K_1^2K_3\lambda_+T^2}{K_0^4\lambda_-^{\frac{7}{2}}} = O\left(\operatorname{polyLog}(\mathbf{n})\right), \quad \frac{\sqrt{2K_1}}{K_0^{\frac{3}{2}}\lambda_-} = O\left(\operatorname{polyLog}(\mathbf{n})\right).$$

Therefore,

$$\max_{j \in J_n} \max_{i} |R_i - r_{i,[j]}| = O_{L^2} \left(\frac{\text{polyLog}(n)}{\sqrt{n}} \right).$$

B-4 Controlling Gradient and Hessian

Proof of Lemma 4.2. Recall that $\hat{\beta}$ is the solution of the following equation

$$\frac{1}{n}\sum_{i=1}^{n}x_{i}\psi(\epsilon_{i}-x_{i}^{T}\hat{\beta})=0.$$
 (B-41)

Taking derivative of (B-41), we have

$$X^T D\left(I - X\frac{\partial \hat{\beta}}{\partial \epsilon^T}\right) = 0 \Longrightarrow \frac{\partial \hat{\beta}}{\partial \epsilon^T} = (X^T D X)^{-1} X^T D.$$

This establishes (9). To establishes (10), note that (9) can be rewritten as

$$(X^T D X) \frac{\partial \hat{\beta}}{\partial \epsilon^T} = X^T D. \tag{B-42}$$

Fix $k \in \{1, \dots, n\}$. Note that

$$\frac{\partial R_i}{\partial \epsilon_k} = \frac{\partial \epsilon_i}{\partial \epsilon_k} - x_i^T \frac{\partial \hat{\beta}}{\partial \epsilon_k} = I(i=k) - x_i^T (X^T D X)^{-1} X^T D.$$

Recall that $G = I - X(X^T D X)^{-1} X^T D$, we have

$$\frac{\partial R_i}{\partial \epsilon_k} = e_i^T G e_k, \tag{B-43}$$

where e_i is the *i*-th canonical basis of \mathbb{R}^n . As a result,

$$\frac{\partial D}{\partial \epsilon_k} = \tilde{D} \operatorname{diag}(Ge_k). \tag{B-44}$$

Taking derivative of (B-42), we have

$$X^{T} \frac{\partial D}{\partial \epsilon_{k}} X \frac{\partial \hat{\beta}}{\partial \epsilon^{T}} + (X^{T} D X) \frac{\partial \hat{\beta}}{\partial \epsilon_{k} \partial \epsilon^{T}} = X^{T} \frac{\partial D}{\partial \epsilon_{k}}$$
$$\Longrightarrow \frac{\partial \hat{\beta}}{\partial \epsilon_{k} \partial \epsilon^{T}} = (X^{T} D X)^{-1} X^{T} \frac{\partial D}{\partial \epsilon_{k}} \left(I - X (X^{T} D X)^{-1} X^{T} D\right)$$
$$\Longrightarrow \frac{\partial \hat{\beta}}{\partial \epsilon_{k} \partial \epsilon^{T}} = (X^{T} D X)^{-1} X^{T} \tilde{D} \operatorname{diag}(G e_{k}) G,$$

where $G = I - X(X^T D X)^{-1} X^T D$ is defined in (B-18) in p.30. Then for each $j \in \{1, \dots, p\}$ and $k \in \{1, \dots, n\}$,

$$\frac{\partial \hat{\beta}_j}{\partial \epsilon_k \partial \epsilon^T} = e_j^T (X^T D X)^{-1} X^T \tilde{D} \operatorname{diag}(G e_k) G = e_k^T G^T \operatorname{diag}(e_j^T (X^T D X)^{-1} X^T \tilde{D}) G$$

where we use the fact that $a^T \operatorname{diag}(b) = b^T \operatorname{diag}(a)$ for any vectors a, b. This implies that

$$\frac{\partial \hat{\beta}_j}{\partial \epsilon \partial \epsilon^T} = G^T \operatorname{diag}(e_j^T (X^T D X)^{-1} X^T \tilde{D}) G$$

Proof of Lemma 4.3. Throughout the proof we are using the simple fact that $||a||_{\infty} \leq ||a||_2$. Based on it, we found that

$$\left\| e_{j}^{T} (X^{T} D X)^{-1} X^{T} D^{\frac{1}{2}} \right\|_{\infty} \leq \left\| e_{j}^{T} (X^{T} D X)^{-1} X^{T} D^{\frac{1}{2}} \right\|_{2}$$

$$= \sqrt{e_{j}^{T} (X^{T} D X)^{-1} X^{T} D X (X^{T} D X)^{-1} e_{j}}$$

$$= \sqrt{e_{j}^{T} (X^{T} D X)^{-1} e_{j}} \leq \frac{1}{(nK_{0}\lambda_{-})^{\frac{1}{2}}}.$$
(B-45)

Thus for any m > 1, recall that $M_j = \mathbb{E} \left\| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} \right\|_{\infty}$,

$$\mathbb{E} \left\| e_{j}^{T} (X^{T} D X)^{-1} X^{T} D^{\frac{1}{2}} \right\|_{\infty}^{m} \\
\leq \mathbb{E} \left\| e_{j}^{T} (X^{T} D X)^{-1} X^{T} D^{\frac{1}{2}} \right\|_{\infty} \cdot \left\| e_{j}^{T} (X^{T} D X)^{-1} X^{T} D^{\frac{1}{2}} \right\|_{2}^{m-1} \\
\leq \frac{M_{j}}{(nK_{0}\lambda_{-})^{\frac{m-1}{2}}}.$$
(B-46)

We should emphasize that we cannot use the naive bound that

$$\mathbb{E} \left\| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} \right\|_{\infty}^m \leq \mathbb{E} \left\| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} \right\|_2^m \leq \frac{1}{(nK_0\lambda_-)^{\frac{m}{2}}}, \quad (B-47)$$

$$\Longrightarrow \left\| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} \right\|_{\infty} = O_{L^m} \left(\frac{\operatorname{polyLog}(n)}{\sqrt{n}} \right)$$

since it fails to guarantee the convergence of TV distance. We will address this issue after deriving Lemma 4.4.

By contrast, as proved below,

$$\left\| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} \right\|_{\infty} = O_p(M_j) = O_p\left(\frac{\text{polyLog}(\mathbf{n})}{n}\right) \ll \frac{1}{\sqrt{nK_0\lambda_-}}.$$
 (B-48)

Thus (B-46) produces a slightly tighter bound

$$\left\| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} \right\|_{\infty} = O_{L^m} \left(\frac{\operatorname{polyLog}(\mathbf{n})}{n^{\frac{m+1}{2m}}} \right)$$

It turns out that the above bound suffices to prove the convergence. Although (B-48) implies the possibility to sharpen the bound from $n^{-\frac{m+1}{2m}}$ to n^{-1} using refined analysis, we do not explore this to avoid extra conditions and notation.

• Bound for κ_{0j}

First we derive a bound for κ_{0j} . By definition,

$$\kappa_{0j}^2 = \mathbb{E} \left\| \frac{\partial \hat{\beta}_j}{\partial \epsilon^T} \right\|_4^4 \le \mathbb{E} \left(\left\| \frac{\partial \hat{\beta}_j}{\partial \epsilon^T} \right\|_\infty^2 \cdot \left\| \frac{\partial \hat{\beta}_j}{\partial \epsilon^T} \right\|_2^2 \right).$$

By Lemma 4.2 and (B-46) with m = 2,

$$\mathbb{E} \left\| \frac{\partial \hat{\beta}_j}{\partial \epsilon^T} \right\|_{\infty}^2 \leq \mathbb{E} \left\| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} \right\|_{\infty}^2 \cdot K_1 = \frac{K_1 M_j}{(nK_0 \lambda_-)^{\frac{1}{2}}}.$$

On the other hand, it follows from (B-45) that

$$\left\|\frac{\partial\hat{\beta}_{j}}{\partial\epsilon^{T}}\right\|_{2}^{2} = \left\|e_{j}^{T}(X^{T}DX)^{-1}X^{T}D\right\|_{2}^{2} \le K_{1} \cdot \left\|e_{j}^{T}(X^{T}DX)^{-1}X^{T}D^{\frac{1}{2}}\right\|_{2}^{2} \le \frac{K_{1}}{nK_{0}\lambda_{-}}.$$
 (B-49)

Putting the above two bounds together we have

$$\kappa_{0j}^2 \le \frac{K_1^2}{(nK_0\lambda_-)^{\frac{3}{2}}} \cdot M_j.$$
 (B-50)

• Bound for κ_{1j}

As a by-product of (B-49), we obtain that

$$\kappa_{1j}^4 = \mathbb{E} \left\| \frac{\partial \hat{\beta}_j}{\partial \epsilon^T} \right\|_2^4 \le \frac{K_1^2}{(nK_0\lambda_-)^2}.$$
 (B-51)

• Bound for κ_{2j}

Finally, we derive a bound for κ_{2j} . By Lemma 4.2, κ_{2j} involves the operator norm of a symmetric matrix with form $G^T M G$ where M is a diagonal matrix. Then by the triangle inequality,

$$\|G^T M G\|_{op} \le \|M\|_{op} \cdot \|G^T G\|_{op} = \|M\|_{op} \cdot \|G\|_{op}^2.$$

Note that

$$D^{\frac{1}{2}}GD^{-\frac{1}{2}} = I - D^{\frac{1}{2}}X(X^{T}DX)^{-1}X^{T}D^{\frac{1}{2}}$$

is a projection matrix, which is idempotent. This implies that

$$\left\| D^{\frac{1}{2}}GD^{-\frac{1}{2}} \right\|_{op} = \lambda_{\max} \left(D^{\frac{1}{2}}GD^{-\frac{1}{2}} \right) \le 1.$$

Write G as $D^{-\frac{1}{2}}(D^{\frac{1}{2}}GD^{-\frac{1}{2}})D^{\frac{1}{2}}$, then we have

$$\|G\|_{op} \le \left\|D^{-\frac{1}{2}}\right\|_{op} \cdot \left\|D^{\frac{1}{2}}GD^{-\frac{1}{2}}\right\|_{op} \cdot \left\|D^{\frac{1}{2}}\right\|_{op} \le \sqrt{\frac{K_1}{K_0}}.$$

Returning to κ_{2j} , we obtain that

$$\begin{split} \kappa_{2j}^4 &= \mathbb{E} \left\| G^T \operatorname{diag}(e_j^T (X^T D X)^{-1} X^T \tilde{D}) G \right\|_{op}^4 \\ &\leq \mathbb{E} \left(\left\| e_j^T (X^T D X)^{-1} X^T \tilde{D} \right\|_{\infty}^4 \cdot \|G\|_{op}^8 \right) \\ &\leq \mathbb{E} \left(\left\| e_j^T (X^T D X)^{-1} X^T \tilde{D} \right\|_{\infty}^4 \right) \left(\frac{K_1}{K_0} \right)^4 \\ &= \mathbb{E} \left(\left\| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} D^{-\frac{1}{2}} \tilde{D} \right\|_{\infty}^4 \right) \cdot \left(\frac{K_1}{K_0} \right)^4 \end{split}$$

Assumption A1 implies that

$$\forall i, \ \frac{|\psi''(R_i)|}{\sqrt{\psi'(R_i)}} \le K_2 \ \& \text{ hence } \|D^{-\frac{1}{2}}\tilde{D}\|_{\text{op}} \le K_2.$$

Therefore,

$$\left\| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} D^{-\frac{1}{2}} \tilde{D} \right\|_{\infty}^4 \le K_2^4 \cdot \left\| e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}} \right\|_{\infty}^4$$

By (B-46) with m = 4,

$$\kappa_{2j}^4 \le \frac{K_2^4}{(n\lambda_-)^{\frac{3}{2}}} \cdot \left(\frac{K_1}{K_0}\right)^4 \cdot M_j.$$
(B-52)

Proof of Lemma 4.4. By Theorem B.4, for any j,

 $\mathbb{E}\hat{\beta}_j^4 \le \mathbb{E}\|\hat{\beta}\|_2^4 < \infty.$

Then using the second-order Poincaré inequality (Proposition 4.1),

$$\begin{split} & \max_{j \in J_n} d_{TV} \left(\mathcal{L}\left(\frac{\hat{\beta}_j - \mathbb{E}\hat{\beta}_j}{\sqrt{\operatorname{Var}(\hat{\beta}_j)}}\right), N(0, 1) \right) = O\left(\frac{c_1 c_2 \kappa_{0j} + c_1^3 \kappa_{1j} \kappa_{2j}}{\operatorname{Var}(\hat{\beta}_j)}\right) \\ &= O\left(\frac{\frac{M_j^{\frac{1}{2}}}{n^{\frac{3}{4}}} + \frac{M_j^{\frac{1}{4}}}{n^{\frac{5}{4}}}}{\operatorname{Var}(\hat{\beta}_j)} \cdot \operatorname{polyLog}(n)\right) = O\left(\frac{(nM_j^2)^{\frac{1}{4}} + (nM_j^2)^{\frac{1}{8}}}{n\operatorname{Var}(\hat{\beta}_j)} \cdot \operatorname{polyLog}(n)\right). \end{split}$$

It follows from (B-45) that $nM_j^2 = O$ (polyLog(n)) and the above bound can be simplified as

$$\max_{j \in J_n} d_{TV} \left(\mathcal{L} \left(\frac{\hat{\beta}_j - \mathbb{E}\hat{\beta}_j}{\sqrt{\operatorname{Var}(\hat{\beta}_j)}} \right), N(0, 1) \right) = O \left(\frac{(nM_j^2)^{\frac{1}{8}}}{n\operatorname{Var}(\hat{\beta}_j)} \cdot \operatorname{polyLog}(n) \right).$$

Remark B.5. If we use the naive bound (B-47), by repeating the above derivation, we obtain a worse bound for $\kappa_{0,j} = O(\frac{\text{polyLog}(n)}{n})$ and $\kappa_2 = O(\frac{\text{polyLog}(n)}{\sqrt{n}})$, in which case,

$$\max_{j \in J_n} d_{TV} \left(\mathcal{L} \left(\frac{\hat{\beta}_j - \mathbb{E} \hat{\beta}_j}{\sqrt{\operatorname{Var}(\hat{\beta}_j)}} \right), N(0, 1) \right) = O \left(\frac{\operatorname{polyLog}(\mathbf{n})}{n \operatorname{Var}(\hat{\beta}_j)} \right).$$

However, we can only prove that $\operatorname{Var}(\hat{\beta}_j) = \Omega(\frac{1}{n})$. Without the numerator $(nM_j^2)^{\frac{1}{8}}$, which will be shown to be $O(n^{-\frac{1}{8}}\operatorname{polyLog}(n))$ in the next subsection, the convergence cannot be proved.

B-5 Upper Bound of M_i

As mentioned in Appendix A, we should approximate D by $D_{[j]}$ to remove the functional dependence on X_j . To achieve this, we introduce two terms, $M_j^{(1)}$ and $M_j^{(2)}$, defined as

$$M_j^{(1)} = \mathbb{E}(\|e_j^T (X^T D X)^{-1} X^T D_{[j]}^{\frac{1}{2}}\|_{\infty}), \quad M_j^{(2)} = \mathbb{E}(\|e_j^T (X^T D_{[j]} X)^{-1} X^T D_{[j]}^{\frac{1}{2}}\|_{\infty}).$$

We will first prove that both $|M_j - M_j^{(1)}|$ and $|M_j^{(1)} - M_j^{(2)}|$ are negligible and then derive an upper bound for $M_j^{(2)}$.

B-5.1 Controlling $|M_j - M_j^{(1)}|$

By Lemma B.2,

$$||D^{\frac{1}{2}} - D^{\frac{1}{2}}_{[j]}||_{\infty} \le K_2 \max_i |R_i - r_{i,[j]}| \triangleq K_2 \mathcal{R}_j,$$

and by Theorem B.4,

$$\sqrt{\mathbb{E}\mathcal{R}_j^2} = O\left(\frac{\text{polyLog}(\mathbf{n})}{\sqrt{n}}\right).$$

Then we can bound $|M_j - M_j^{(1)}|$ via the fact that $||a||_{\infty} \leq ||a||_2$ and algebra as follows.

$$|M_{j} - M_{j}^{(1)}| \leq \mathbb{E}(||e_{j}^{T}(X^{T}DX)^{-1}X^{T}(D^{\frac{1}{2}} - D_{[j]}^{\frac{1}{2}})||_{\infty})$$

$$\leq \mathbb{E}(||e_{j}^{T}(X^{T}DX)^{-1}X^{T}(D^{\frac{1}{2}} - D_{[j]}^{\frac{1}{2}})||_{2})$$

$$\leq \sqrt{\mathbb{E}(||e_{j}^{T}(X^{T}DX)^{-1}X^{T}(D^{\frac{1}{2}} - D_{[j]}^{\frac{1}{2}})||_{2}^{2})}$$

$$= \sqrt{\mathbb{E}(e_{j}^{T}(X^{T}DX)^{-1}X^{T}(D^{\frac{1}{2}} - D_{[j]}^{\frac{1}{2}})^{2}X(X^{T}DX)^{-1}e_{j})}.$$

By Lemma B.2,

$$|\sqrt{\psi'(R_i)} - \sqrt{\psi'(r_{i,[j]})}| \le K_2 |R_i - r_{i,[j]}| \le K_2 \mathcal{R}_j,$$

thus

$$(D^{\frac{1}{2}} - D^{\frac{1}{2}}_{[j]})^2 \preceq K_2^2 \mathcal{R}_j^2 I \preceq \frac{K_2^2}{K_0} \mathcal{R}_j^2 D.$$

This entails that

$$\begin{split} |M_j - M_j^{(1)}| &\leq K_2 K_0^{-\frac{1}{2}} \sqrt{\mathbb{E}(\mathcal{R}_j^2 \cdot e_j^T (X^T D X)^{-1} X^T D X (X^T D X)^{-1} e_j)} \\ &= K_2 K_0^{-\frac{1}{2}} \sqrt{\mathbb{E}(\mathcal{R}_j^2 \cdot e_j^T (X^T D X)^{-1} e_j)} \\ &\leq \frac{K_2}{\sqrt{n} K_0 \sqrt{\lambda_-}} \sqrt{\mathbb{E}(\mathcal{R}_j^2)} = O\left(\frac{\text{polyLog}(n)}{n}\right). \end{split}$$

B-5.2 Bound of $|M_j^{(1)} - M_j^{(2)}|$

First we prove a useful lemma.

Lemma B.6. For any symmetric matrix N with $||N||_{op} < 1$,

$$(I - (I + N)^{-1})^2 \preceq \frac{N^2}{(1 - ||N||_{\text{op}})^2}$$

Proof. First, notice that

$$I - (I + N)^{-1} = (I + N - I)(I + N)^{-1} = N(I + N)^{-1},$$

and therefore

$$(I - (I + N)^{-1})^2 = N(I + N)^{-2}N.$$

Since $\|N\|_{\text{op}} < 1$, I + N is positive semi-definite and

$$(I+N)^{-2} \leq \frac{1}{(1-\|N\|_{\text{op}})^2}I$$

Therefore,

$$N(I+N)^{-2}N \preceq \frac{N^2}{(1-\|N\|_{\rm op})^2}.$$

г		

We now back to bounding $|M_j^{(1)} - M_j^{(2)}|$. Let $A_j = X^T D_{[j]} X$, $B_j = X^T (D - D_{[j]}) X$. By Lemma B.2,

$$|D - D_{[j]}||_{\infty} \le K_3 \max_i |R_i - r_{i,[j]}| = K_3 \mathcal{R}_j$$

and hence

$$\|B_j\|_{\rm op} \le K_3 \mathcal{R}_j \cdot n\lambda_+ I \triangleq n\eta_j.$$

where $\eta_j = K_3 \lambda_+ \cdot \mathcal{R}_j$. Then by Theorem B.4.(v),

$$\mathbb{E}(\eta_j^2) = O\left(\frac{\text{polyLog}(\mathbf{n})}{n}\right).$$

Using the fact that $\|a\|_{\infty} \leq \|a\|_2$, we obtain that

$$\begin{split} |M_{j}^{(1)} - M_{j}^{(2)}| &\leq \mathbb{E}(\|e_{j}^{T}A_{j}^{-1}X^{T}D_{[j]}^{\frac{1}{2}} - e_{j}^{T}(A_{j} + B_{j})^{-1}X^{T}D_{[j]}^{\frac{1}{2}}\|_{\infty}) \\ &\leq \sqrt{\mathbb{E}(\|e_{j}^{T}A_{j}^{-1}X^{T}D_{[j]}^{\frac{1}{2}} - e_{j}^{T}(A_{j} + B_{j})^{-1}X^{T}D_{[j]}^{\frac{1}{2}}\|_{2}^{2})} \\ &= \sqrt{\mathbb{E}\left[e_{j}^{T}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})X^{T}D_{[j]}X(A_{j}^{-1} - (A_{j} + B_{j})^{-1})e_{j}\right]} \\ &= \sqrt{\mathbb{E}\left[e_{j}^{T}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})A_{j}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})e_{j}\right]} \end{split}$$

The inner matrix can be rewritten as

$$(A_{j}^{-1} - (A_{j} + B_{j})^{-1})A_{j}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})$$

= $A_{j}^{-\frac{1}{2}}(I - (I + A_{j}^{-\frac{1}{2}}B_{j}A_{j}^{-\frac{1}{2}})^{-1})A_{j}^{-\frac{1}{2}}A_{j}A_{j}^{-\frac{1}{2}}(I - (I + A_{j}^{-\frac{1}{2}}B_{j}A_{j}^{-\frac{1}{2}})^{-1})A_{j}^{-\frac{1}{2}}$
= $A_{j}^{-\frac{1}{2}}(I - (I + A_{j}^{-\frac{1}{2}}B_{j}A_{j}^{-\frac{1}{2}})^{-1})^{2}A_{j}^{-\frac{1}{2}}.$ (B-53)

Let
$$N_j = A_j^{-\frac{1}{2}} B_j A_j^{-\frac{1}{2}}$$
, then
 $\|N_j\|_{\text{op}} \le \|A_j^{-\frac{1}{2}}\|_{\text{op}} \cdot \|B_j\|_{\text{op}} \cdot \|A_j^{-\frac{1}{2}}\|_{\text{op}} \le (nK_0\lambda_-)^{-\frac{1}{2}} \cdot n\eta_j \cdot (nK_0\lambda_-)^{-\frac{1}{2}} = \frac{\eta_j}{K_0\lambda_-}.$

On the event $\{\eta_j \leq \frac{1}{2}K_0\lambda_-\}, \|N_j\|_{\text{op}} \leq \frac{1}{2}$. By Lemma B.6,

$$(I - (I + N_j)^{-1})^2 \leq 4N_j^2$$

This together with (B-53) entails that

$$\begin{split} e_j^T (A_j^{-1} - (A_j + B_j)^{-1}) A_j (A_j^{-1} - (A_j + B_j)^{-1}) e_j &= e_j^T A_j^{-\frac{1}{2}} (I - (I + N_j)^{-1})^2 A_j^{-\frac{1}{2}} e_j \\ &\leq 4 e_j^T A_j^{-\frac{1}{2}} N_j^2 A_j^{-\frac{1}{2}} e_j = e_j^T A_j^{-1} B_j A_j^{-1} B_j A_j^{-1} e_j \leq \|A_j^{-1} B_j A_j^{-1} B_j A_j^{-1} \|_{\text{op}}. \\ &\text{Since } A_j \succeq n K_0 \lambda_- I, \text{ and } \|B_j\|_{\text{op}} \leq n \eta_j, \text{ we have} \end{split}$$

$$\|A_j^{-1}B_jA_j^{-1}B_jA_j^{-1}\|_{\rm op} \le \|A_j^{-1}\|_{\rm op}^3 \cdot \|B_j\|_{\rm op}^2 \le \frac{1}{n} \cdot \frac{1}{(K_0\lambda_-)^3} \cdot \eta_j^2.$$

Thus,

$$\mathbb{E}\left[e_{j}^{T}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})A_{j}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})e_{j} \cdot I\left(\eta_{j} \leq \frac{K_{0}\lambda_{-}}{2}\right)\right]$$

$$\leq \mathbb{E}\left[e_{j}^{T}A_{j}^{-1}B_{j}A_{j}^{-1}B_{j}A_{j}^{-1}e_{j}\right] \leq \frac{1}{n} \cdot \frac{1}{(K_{0}\lambda_{-})^{3}} \cdot \mathbb{E}\eta_{j}^{2} = O\left(\frac{\text{polyLog}(n)}{n^{2}}\right).$$

On the event $\{\eta_j > \frac{1}{2}K_0\lambda_-\}$, since $nK_0\lambda_-I \preceq A_j \preceq nK_1\lambda_+I$ and $A_j + B_j \succeq nK_0\lambda_-I$,

$$\begin{split} &|e_{j}^{T}(A_{j}^{-1}-(A_{j}+B_{j})^{-1})A_{j}(A_{j}^{-1}-(A_{j}+B_{j})^{-1})e_{j}|\\ \leq &nK_{1}\lambda_{+}\cdot|e_{j}^{T}(A_{j}^{-1}-(A_{j}+B_{j})^{-1})^{2}e_{j}|\\ \leq &nK_{1}\lambda_{+}\cdot\left(2|e_{j}^{T}A_{j}^{-2}e_{j}|+2|e_{j}^{T}(A_{j}+B_{j})^{-2}e_{j}|\right)\\ \leq &\frac{4nK_{1}\lambda_{+}}{(nK_{0}\lambda_{-})^{2}} = \frac{1}{n}\cdot\frac{4K_{1}\lambda_{+}}{(K_{0}\lambda_{-})^{2}}. \end{split}$$

This together with Markov inequality implies htat

$$\mathbb{E}\left[e_j^T (A_j^{-1} - (A_j + B_j)^{-1})A_j (A_j^{-1} - (A_j + B_j)^{-1})e_j \cdot I\left(\eta_j > \frac{K_0\lambda_-}{2}\right)\right]$$

$$\leq \frac{1}{n} \cdot \frac{4K_1\lambda_+}{(K_0\lambda_-)^2} \cdot P\left(\eta_j > \frac{K_0\lambda_-}{2}\right)$$

$$\leq \frac{1}{n} \cdot \frac{4K_1\lambda_+}{(K_0\lambda_-)^2} \cdot \frac{4}{(K_0\lambda_-)^2} \cdot \mathbb{E}\eta_j^2$$

$$= O\left(\frac{\text{polyLog}(n)}{n^2}\right).$$

Putting pieces together, we conclude that

$$\begin{split} |M_{j}^{(1)} - M_{j}^{(2)}| &\leq \sqrt{\mathbb{E}\left[e_{j}^{T}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})A_{j}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})e_{j}\right]} \\ &\leq \sqrt{\mathbb{E}\left[e_{j}^{T}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})A_{j}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})e_{j} \cdot I\left(\eta_{j} > \frac{K_{0}\lambda_{-}}{2}\right)\right]} \\ &+ \sqrt{\mathbb{E}\left[e_{j}^{T}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})A_{j}(A_{j}^{-1} - (A_{j} + B_{j})^{-1})e_{j} \cdot I\left(\eta_{j} \le \frac{K_{0}\lambda_{-}}{2}\right)\right]} \\ &= O\left(\frac{\operatorname{polyLog}(n)}{n}\right). \end{split}$$

B-5.3 Bound of $M_j^{(2)}$

Similar to (A-1), by block matrix inversion formula (See Proposition E.1),

$$e_j^T (X^T D_{[j]} X)^{-1} X^T D_{[j]}^{\frac{1}{2}} = \frac{X_j^T D_{[j]}^{\frac{1}{2}} (I - H_j)}{X_j^T D_{[j]}^{\frac{1}{2}} (I - H_j) D_{[j]}^{\frac{1}{2}} X_j},$$

where $H_j = D_{[j]}^{\frac{1}{2}} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]}^{\frac{1}{2}}$. Recall that $\xi_j \ge K_0 \lambda_-$ by (B-25), so we have $X_j^T D_{[j]}^{\frac{1}{2}} (I - H_j) D_{[j]}^{\frac{1}{2}} X_j = n\xi_j \ge n\lambda_-.$

As for the numerator, recalling the definition of $h_{j,1,i}$, we obtain that

$$\begin{split} \|X_{j}^{T}D_{[j]}^{\frac{1}{2}}(I-H_{j})\|_{\infty} &= \left\|\frac{1}{n}X_{j}^{T}(I-D_{[j]}X_{[j]}(X_{[j]}^{T}D_{[j]}X_{[j]})^{-1}X_{[j]}) \cdot D_{[j]}^{\frac{1}{2}}\right\|_{\infty} \\ &\leq \sqrt{K_{1}} \cdot \left\|\frac{1}{n}X_{j}^{T}(I-D_{[j]}X_{[j]}(X_{[j]}^{T}D_{[j]}X_{[j]})^{-1}X_{[j]})\right\|_{\infty} \\ &= \sqrt{K_{1}}\max_{i}\left|h_{j,1,i}^{T}X_{j}\right| \leq \sqrt{K_{1}}\Delta_{C}\max_{i}\|h_{j,1,i}\|_{2}. \end{split}$$

As proved in (B-35),

$$\max_{i} \|h_{j,1,i}\|_{2} \le \left(\frac{K_{1}}{K_{0}}\right)^{\frac{1}{2}}.$$

This entails that

$$\|X_j^T D_{[j]}^{\frac{1}{2}} (I - H_j)\|_{\infty} \leq \frac{K_1}{\sqrt{K_0}} \cdot \Delta_C = O_{L^1} \left(\text{polyLog}(\mathbf{n}) \right).$$

Putting the pieces together we conclude that

$$M_j^{(2)} \le \frac{\mathbb{E} \|X_j^T D_{[j]}^{\frac{1}{2}} (I - H_j)\|_{\infty}}{n\lambda_-} = O\left(\frac{\operatorname{polyLog}(n)}{n}\right).$$

B-5.4 Summary

Based on results from Section B.5.1 - Section B.5.3, we have

$$M_j = O\left(\frac{\text{polyLog}(n)}{n}\right)$$

Note that the bounds we obtained do not depend on j, so we conclude that

$$\max_{j \in J_n} M_j = O\left(\frac{\text{polyLog}(n)}{n}\right).$$

B-6 Lower Bound of $Var(\hat{\beta}_i)$

B-6.1 Approximating $Var(\hat{\beta}_i)$ by $Var(b_i)$

By Theorem B.4,

$$\max_{j} \mathbb{E}(\hat{\beta}_{j} - b_{j})^{2} = O\left(\frac{\text{polyLog}(n)}{n^{2}}\right), \quad \max_{j} \mathbb{E}b_{j}^{2} = O\left(\frac{\text{polyLog}(n)}{n}\right).$$

Using the fact that

$$\hat{\beta}_j^2 - b_j^2 = (\hat{\beta}_j - b_j + b_j)^2 - b_j^2 = (\hat{\beta}_j - b_j)^2 + 2(\hat{\beta}_j - b_j)b_j,$$

we can bound the difference between $\mathbb{E} \hat{\beta}_j^2$ and $\mathbb{E} b_j^2$ by

$$\begin{aligned} \left| \mathbb{E}\hat{\beta}_{j}^{2} - \mathbb{E}b_{j}^{2} \right| &= \mathbb{E}(\hat{\beta}_{j} - b_{j})^{2} + 2|\mathbb{E}(\hat{\beta}_{j} - b_{j})b_{j}| \leq \mathbb{E}(\hat{\beta}_{j} - b_{j})^{2} + 2\sqrt{\mathbb{E}(\hat{\beta}_{j} - b_{j})^{2}}\sqrt{\mathbb{E}b_{j}^{2}} = O\left(\frac{\text{polyLog}(\mathbf{n})}{n^{\frac{3}{2}}}\right).\end{aligned}$$
Similarly, since $|a^{2} - b^{2}| = |a - b| \cdot |a + b| \leq |a - b| (|a - b| + 2|b|)$

Similarly, since $|a^2 - b^2| = |a - b| \cdot |a + b| \le |a - b|(|a - b| + 2|b|)$,

$$|(\mathbb{E}\hat{\beta}_j)^2 - (\mathbb{E}b_j)^2| \le \mathbb{E}|\hat{\beta}_j - b_j| \cdot \left(\mathbb{E}|\hat{\beta}_j - b_j| + 2\mathbb{E}|b_j|\right) = O\left(\frac{\text{polyLog}(n)}{n^{\frac{3}{2}}}\right).$$

Putting the above two results together, we conclude that

$$\left|\operatorname{Var}(\hat{\beta}_j) - \operatorname{Var}(b_j)\right| = O\left(\frac{\operatorname{polyLog}(n)}{n^{\frac{3}{2}}}\right).$$
 (B-54)

Then it is left to show that

$$\operatorname{Var}(b_j) = \Omega\left(\frac{1}{n \cdot \operatorname{polyLog}(n)}\right).$$

B-6.2 Controlling $Var(b_i)$ by $Var(N_i)$

Recall that

$$b_j = \frac{1}{\sqrt{n}} \frac{N_j}{\xi_j}$$

where

$$N_{j} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{ij} \psi(r_{i,[j]}), \quad \xi_{j} = \frac{1}{n} X_{j}^{T} (D_{[j]} - D_{[j]} X_{[j]} (X_{[j]}^{T} D_{[j]} X_{[j]})^{-1} X_{[j]}^{T} D_{[j]}) X_{j}.$$

Then

$$n\operatorname{Var}(b_j) = \mathbb{E}\left(\frac{N_j}{\xi_j} - \mathbb{E}\frac{N_j}{\xi_j}\right)^2 = \mathbb{E}\left(\frac{N_j - \mathbb{E}N_j}{\xi_j} + \frac{\mathbb{E}N_j}{\xi_j} - \mathbb{E}\frac{N_j}{\xi_j}\right)^2.$$

Using the fact that $(a+b)^2 - (\frac{1}{2}a^2 - b^2) = \frac{1}{2}(a+2b)^2 \ge 0$, we have

$$n\operatorname{Var}(b_j) \ge \frac{1}{2}\mathbb{E}\left(\frac{N_j - \mathbb{E}N_j}{\xi_j}\right)^2 - \mathbb{E}\left(\frac{\mathbb{E}N_j}{\xi_j} - \mathbb{E}\frac{N_j}{\xi_j}\right)^2 \triangleq \frac{1}{2}I_1 - I_2.$$
(B-55)

B-6.3 Controlling I_1

The Assumption A4 implies that

$$\operatorname{Var}(N_j) = \frac{1}{n} X_j^T Q_j X_j = \Omega\left(\frac{\operatorname{tr}(\operatorname{Cov}(h_{j,0}))}{n \operatorname{polyLog}(n)}\right).$$

It is left to show that $\operatorname{tr}(\operatorname{Cov}(h_{j,0}))/n = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right)$. Since this result will also be used later in Appendix C, we state it in the following the lemma.

Lemma B.7. Under assumptions A1 - A3,

$$\frac{\operatorname{tr}(\operatorname{Cov}(\psi(h_{j,0})))}{n} \ge \frac{K_0^4}{K_1^2} \cdot \left(\frac{n-p+1}{n}\right)^2 \cdot \min_i \operatorname{Var}(\epsilon_i) = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right).$$

Proof. The (A-10) implies that

$$\operatorname{Var}(\psi(r_{i,[j]})) \ge K_0^2 \operatorname{Var}(r_{i,[j]}).$$
(B-56)

Note that $r_{i,[j]}$ is a function of ϵ , we can apply (A-10) again to obtain a lower bound for $\operatorname{Var}(r_{i,[j]})$. In fact, by variance decomposition formula, using the independence of ϵ'_i s,

$$\operatorname{Var}(r_{i,[j]}) = \mathbb{E}\left(\operatorname{Var}\left(r_{i,[j]}|\epsilon_{(i)}\right)\right) + \operatorname{Var}\left(\mathbb{E}\left(r_{i,[j]}|\epsilon_{(i)}\right)\right) \ge \mathbb{E}\left(\operatorname{Var}\left(r_{i,[j]}|\epsilon_{(i)}\right)\right),$$

where $\epsilon_{(i)}$ includes all but the *i*-th entry of ϵ . Apply A-10 again,

$$\operatorname{Var}\left(r_{i,[j]}\big|\epsilon_{(i)}\right) \geq \inf_{\epsilon_{i}}\left|\frac{\partial r_{i,[j]}}{\partial \epsilon_{i}}\right|^{2} \cdot \operatorname{Var}(\epsilon_{i}),$$

and hence

$$\operatorname{Var}(r_{i,[j]}) \ge \mathbb{E}\operatorname{Var}\left(r_{i,[j]}\big|\epsilon_{(i)}\right) \ge \mathbb{E}\inf_{\epsilon} \left|\frac{\partial r_{i,[j]}}{\partial \epsilon_{i}}\right|^{2} \cdot \operatorname{Var}(\epsilon_{i}).$$
(B-57)

Now we compute $\frac{\partial r_{i,[j]}}{\partial \epsilon_i}$. Similar to (B-43) in p.40, we have

$$\frac{\partial r_{k,[j]}}{\partial \epsilon_i} = e_i^T G_{[j]} e_k, \tag{B-58}$$

where $G_{[j]}$ is defined in (B-18) in p.30. When k = i,

$$\frac{\partial r_{i,[j]}}{\partial \epsilon_i} = e_i^T G_{[j]} e_i = e_i^T D_{[j]}^{-\frac{1}{2}} D_{[j]}^{\frac{1}{2}} G_{[j]} D_{[j]}^{-\frac{1}{2}} D_{[j]}^{\frac{1}{2}} e_i = e_i^T D_{[j]}^{\frac{1}{2}} G_{[j]} D_{[j]}^{-\frac{1}{2}} e_i.$$
(B-59)

By definition of $G_{[j]}$,

$$D_{[j]}^{\frac{1}{2}}G_{[j]}D_{[j]}^{-\frac{1}{2}} = I - D_{[j]}^{\frac{1}{2}}X_{[j]}(X_{[j]}^{T}D_{[j]}X_{[j]})^{-1}X_{[j]}^{T}D_{[j]}^{\frac{1}{2}}.$$

Let $\tilde{X}_{[j]} = D_{[j]}^{\frac{1}{2}} X_{[j]}$ and $H_j = \tilde{X}_{[j]} (\tilde{X}_{[j]}^T \tilde{X}_{[j]})^{-1} \tilde{X}_{[j]}^T$. Denote by $\tilde{X}_{(i),[j]}$ the matrix $\tilde{X}_{[j]}$ after removing *i*-th row, then by block matrix inversion formula (See Proposition E.1),

$$\begin{split} e_{i}^{T}H_{j}e_{i} &= \tilde{x}_{i,[j]}^{T}(\tilde{X}_{(i),[j]}^{T}\tilde{X}_{(i),[j]} + \tilde{x}_{i,[j]}\tilde{x}_{i,[j]}^{T})^{-1}\tilde{x}_{i,[j]} \\ &= \tilde{x}_{i,[j]}^{T}\left((\tilde{X}_{(i),[j]}^{T}\tilde{X}_{(i),[j]})^{-1} - \frac{(\tilde{X}_{(i),[j]}^{T}\tilde{X}_{(i),[j]})^{-1}\tilde{x}_{i,[j]}\tilde{x}_{i,[j]}^{T}(\tilde{X}_{(i),[j]}^{T}\tilde{X}_{(i),[j]})^{-1}}{1 + \tilde{x}_{i,[j]}^{T}(\tilde{X}_{(i),[j]}^{T}\tilde{X}_{(i),[j]})^{-1}\tilde{x}_{i,[j]}} \right) \tilde{x}_{i,[j]} \\ &= \frac{\tilde{x}_{i,[j]}^{T}(\tilde{X}_{(i),[j]}^{T}\tilde{X}_{(i),[j]})^{-1}\tilde{x}_{i,[j]}}{1 + \tilde{x}_{i,[j]}^{T}(\tilde{X}_{(i),[j]}^{T}\tilde{X}_{(i),[j]})^{-1}\tilde{x}_{i,[j]}}. \end{split}$$

This implies that

$$e_{i}^{T} D_{[j]}^{\frac{1}{2}} G_{[j]} D_{[j]}^{-\frac{1}{2}} e_{i} = e_{i}^{T} (I - H_{j}) e_{i} = \frac{1}{1 + \tilde{x}_{i,[j]}^{T} (\tilde{X}_{(i),[j]}^{T} \tilde{X}_{(i),[j]})^{-1} \tilde{x}_{i,[j]}}$$

$$= \frac{1}{1 + e_{i}^{T} D_{[j]}^{\frac{1}{2}} X_{[j]} (X_{(i),[j]}^{T} D_{(i),[j]} X_{(i),[j]})^{-1} X_{[j]}^{T} D_{[j]}^{\frac{1}{2}} e_{i}}$$

$$\geq \frac{1}{1 + K_{0}^{-1} e_{i}^{T} D_{[j]}^{\frac{1}{2}} X_{[j]} (X_{(i),[j]}^{T} X_{(i),[j]})^{-1} X_{[j]}^{T} D_{[j]}^{\frac{1}{2}} e_{i}}$$

$$= \frac{1}{1 + K_{0}^{-1} (D_{[j]})_{i,i} \cdot e_{i}^{T} X_{[j]} (X_{(i),[j]}^{T} X_{(i),[j]})^{-1} X_{[j]}^{T} e_{i}}$$

$$\geq \frac{1}{1 + K_{0}^{-1} K_{1} e_{i}^{T} X_{[j]} (X_{(i),[j]}^{T} X_{(i),[j]})^{-1} X_{[j]}^{T} e_{i}}$$

$$\geq \frac{K_{0}}{K_{1}} \cdot \frac{1}{1 + e_{i}^{T} X_{[j]} (X_{(i),[j]}^{T} X_{(i),[j]})^{-1} X_{[j]}^{T} e_{i}}.$$
(B-60)

Apply the above argument to $H_j = X_{[j]}(X_{[j]}^T X_{[j]})^{-1} X_{[j]}^T$, we have

$$\frac{1}{1 + e_i^T X_{[j]}^T (X_{(i),[j]}^T X_{(i),[j]})^{-1} X_{[j]} e_i} = e_i^T (I - X_{[j]} (X_{[j]}^T X_{[j]})^{-1} X_{[j]}^T) e_i.$$

Thus, by (B-56) and (B-57),

$$\operatorname{Var}(\psi(r_{i,[j]})) \ge \frac{K_0^4}{K_1^2} \cdot [e_i^T (I - X_{[j]} (X_{[j]}^T X_{[j]})^{-1} X_{[j]}^T) e_i]^2$$

Summing i over $1, \ldots, n$, we obtain that

$$\frac{\operatorname{tr}(\operatorname{Cov}(h_{j,0}))}{n} \ge \frac{K_0^4}{K_1^2} \cdot \frac{1}{n} \sum_{i=1}^n [e_i^T (I - X_{[j]} (X_{[j]}^T X_{[j]})^{-1} X_{[j]}^T) e_i]^2 \cdot \min_i \operatorname{Var}(\epsilon_i)$$

$$\geq \frac{K_0^4}{K_1^2} \cdot \left(\frac{1}{n} \operatorname{tr}(I - X_{[j]}(X_{[j]}^T X_{[j]})^{-1} X_{[j]}^T)\right)^2 \cdot \min_i \operatorname{Var}(\epsilon_i)$$
$$= \frac{K_0^4}{K_1^2} \cdot \left(\frac{n - p + 1}{n}\right)^2 \cdot \min_i \operatorname{Var}(\epsilon_i)$$

Since $\min_i \operatorname{Var}(\epsilon_i) = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right)$ by assumption **A**2, we conclude that

$$\frac{\operatorname{tr}(\operatorname{Cov}(h_{j,0}))}{n} = \Omega\left(\frac{1}{\operatorname{polyLog}(\mathbf{n})}\right).$$

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In summary,

$$\operatorname{Var}(N_j) = \Omega\left(\frac{1}{\operatorname{polyLog}(\mathbf{n})}\right)$$

Recall that

$$\xi_j = \frac{1}{n} X_j^T (D_{[j]} - D_{[j]} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]}) X_j \le \frac{1}{n} X_j^T D_{[j]} X_j \le K_1 T^2,$$

we conclude that

$$I_1 \ge \frac{\operatorname{Var}(N_j)}{(K_1 T^2)^2} = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right).$$
(B-61)

B-6.4 Controlling I_2

By definition,

$$I_{2} = \mathbb{E}\left(\mathbb{E}N_{j}\left(\frac{1}{\xi_{j}} - \mathbb{E}\frac{1}{\xi_{j}}\right) + \mathbb{E}N_{j}\mathbb{E}\frac{1}{\xi_{j}} - \mathbb{E}\frac{N_{j}}{\xi_{j}}\right)^{2}$$

$$= \operatorname{Var}\left(\frac{\mathbb{E}N_{j}}{\xi_{j}}\right) + \left(\mathbb{E}N_{j}\mathbb{E}\frac{1}{\xi_{j}} - \mathbb{E}\frac{N_{j}}{\xi_{j}}\right)^{2}$$

$$= (\mathbb{E}N_{j})^{2} \cdot \operatorname{Var}\left(\frac{1}{\xi_{j}}\right) + \operatorname{Cov}\left(N_{j}, \frac{1}{\xi_{j}}\right)^{2}$$

$$\leq (\mathbb{E}N_{j})^{2} \cdot \operatorname{Var}\left(\frac{1}{\xi_{j}}\right) + \operatorname{Var}(N_{j})\operatorname{Var}\left(\frac{1}{\xi_{j}}\right)$$

$$= \mathbb{E}N_{j}^{2} \cdot \operatorname{Var}\left(\frac{1}{\xi_{j}}\right). \quad (B-62)$$

By (B-27) in the proof of Theorem B.4,

$$\mathbb{E}N_j^2 \le 2K_1\mathbb{E}(\mathcal{E}\cdot\Delta_C^2) \le 2K_1\sqrt{\mathbb{E}\mathcal{E}^2\cdot\mathbb{E}\Delta_C^4} = O\left(\text{polyLog}(\mathbf{n})\right),$$

where the last equality uses the fact that $\mathcal{E} = O_{L^2}$ (polyLog(n)) as proved in (B-40). On the other hand, let $\tilde{\xi}_j$ be an independent copy of ξ_j , then

$$\operatorname{Var}\left(\frac{1}{\xi_j}\right) = \frac{1}{2} \mathbb{E}\left(\frac{1}{\xi_j} - \frac{1}{\tilde{\xi}_j}\right)^2 = \frac{1}{2} \mathbb{E}\frac{(\xi_j - \tilde{\xi}_j)^2}{\xi_j^2 \tilde{\xi}_j^2}.$$

Since $\xi_j \ge K_0 \lambda_-$ as shown in (B-25), we have

$$\operatorname{Var}\left(\frac{1}{\xi_j}\right) \le \frac{1}{2(K_0\lambda_-)^4} \mathbb{E}(\xi_j - \tilde{\xi}_j)^2 = \frac{1}{(K_0\lambda_-)^4} \cdot \operatorname{Var}(\xi_j).$$
(B-63)

To bound $Var(\xi_j)$, we propose to using the standard Poincaré inequality (Chernoff, 1981), which is stated as follows.

Proposition B.8. Let $W = (W_1, \ldots, W_n) \sim N(0, I_{n \times n})$ and f be a twice differentiable function, then

$$\operatorname{Var}(f(W)) \leq \mathbb{E} \left\| \frac{\partial f(W)}{\partial W} \right\|_{2}^{2}$$

In our case, $\epsilon_i = u_i(W_i)$, and hence for any twice differentiable function g,

$$\operatorname{Var}(g(\epsilon)) \leq \mathbb{E} \left\| \frac{\partial g(\epsilon)}{\partial W} \right\|_{2}^{2} = \mathbb{E} \left\| \frac{\partial g(\epsilon)}{\partial \epsilon} \cdot \frac{\partial \epsilon}{\partial W^{T}} \right\|_{2}^{2} \leq \max_{i} \left\| u_{i}^{\prime} \right\|_{\infty}^{2} \cdot \mathbb{E} \left\| \frac{\partial g(\epsilon)}{\partial \epsilon} \right\|_{2}^{2}.$$

Applying it to ξ_j , we have

$$\operatorname{Var}(\xi_j) \le c_1^2 \cdot \mathbb{E} \left\| \frac{\partial \xi_j}{\partial \epsilon} \right\|_2^2.$$
 (B-64)

For given $k \in \{1, ..., n\}$, using the chain rule and the fact that $dB^{-1} = -B^{-1}dBB^{-1}$ for any square matrix B, we obtain that

$$\begin{aligned} &\frac{\partial}{\partial \epsilon_k} \left(D_{[j]} - D_{[j]} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]} \right) \\ &= \frac{\partial D_{[j]}}{\partial \epsilon_k} - \frac{\partial D_{[j]}}{\partial \epsilon_k} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]} - D_{[j]} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T \frac{\partial D_{[j]}}{\partial \epsilon_k} \\ &+ D_{[j]} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T \frac{\partial D_{[j]}}{\partial \epsilon_k} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]} \\ &= G_{[j]}^T \frac{\partial D_{[j]}}{\partial \epsilon_k} G_{[j]} \end{aligned}$$

where $G_{[j]} = I - X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]}$ as defined in last subsection. This implies that $\partial \mathcal{E}_i = 1 - \pi - \pi - \partial D_{[i]}$

$$\frac{\partial \xi_j}{\partial \epsilon_k} = \frac{1}{n} X_j^T G_{[j]}^T \frac{\partial D_{[j]}}{\partial \epsilon_k} G_{[j]} X_j$$

Then (B-64) entails that

$$\operatorname{Var}(\xi_j) \le \frac{1}{n^2} \sum_{k=1}^n \mathbb{E}\left(X_j^T G_{[j]}^T \frac{\partial D_{[j]}}{\partial \epsilon_k} G_{[j]} X_j\right)^2 \tag{B-65}$$

First we compute $\frac{\partial D_{[j]}}{\partial \epsilon_k}$. Similar to (B-44) in p.40 and recalling the definition of $D_{[j]}$ in (B-17) and that of $G_{[j]}$ in (B-18) in p.30, we have

$$\frac{\partial D_{[j]}}{\partial \epsilon_k} = \tilde{D}_{[j]} \operatorname{diag}(G_{[j]}e_k) \operatorname{diag}(\tilde{D}_{[j]}G_{[j]}e_k),$$

Let $\mathcal{X}_j = G_{[j]} X_j$ and $\tilde{\mathcal{X}}_j = \mathcal{X}_j \circ \mathcal{X}_j$ where \circ denotes Hadamard product. Then

$$X_j^T G_{[j]}^T \frac{\partial D_{[j]}}{\partial \epsilon_k} G_{[j]} X_j = \mathcal{X}_j^T \frac{\partial D_{[j]}}{\partial \epsilon_k} \mathcal{X}_j = \mathcal{X}_j^T \operatorname{diag}(\tilde{D}_{[j]} G_{[j]} e_k) \mathcal{X}_j = \tilde{\mathcal{X}}_j^T \tilde{D}_{[j]} G_{[j]} e_k.$$

Here we use the fact that for any vectors $x, a \in \mathbb{R}^n$,

$$x^{T}$$
 diag $(a)x = \sum_{i=1}^{n} a_{i}x_{i}^{2} = (x \circ x)^{T}a_{i}$

This together with (B-65) imply that

$$\operatorname{Var}(\xi_{j}) \leq \frac{1}{n^{2}} \sum_{k=1}^{n} \mathbb{E}(\tilde{\mathcal{X}}_{j}^{T} \tilde{D}_{[j]} G_{[j]} e_{k})^{2} = \frac{1}{n^{2}} \mathbb{E} \left\| \tilde{\mathcal{X}}_{j}^{T} \tilde{D}_{[j]} G_{[j]} \right\|_{2}^{2} = \frac{1}{n^{2}} \mathbb{E}(\tilde{\mathcal{X}}_{j}^{T} \tilde{D}_{[j]} G_{[j]} G_{[j]} \tilde{D}_{[j]} \tilde{\mathcal{X}}_{j})$$

Note that $G_{[j]}G_{[j]}^T \preceq ||G_{[j]}||_{op}^2 I$, and $\tilde{D}_{[j]} \preceq K_3 I$ by Lemma B.2 in p.(B.2). Therefore we obtain that

$$\operatorname{Var}(\xi_{j}) \leq \frac{1}{n^{2}} \mathbb{E}\left(\left\|G_{[j]}\right\|_{op}^{2} \cdot \tilde{\mathcal{X}}_{j}^{T} \tilde{D}_{[j]}^{2} \tilde{\mathcal{X}}_{j}\right) \leq \frac{K_{3}^{2}}{n^{2}} \cdot \mathbb{E}\left(\left\|G_{[j]}\right\|_{op}^{2} \cdot \|\tilde{\mathcal{X}}_{j}\|_{2}^{2}\right)$$
$$= \frac{K_{3}^{2}}{n^{2}} \mathbb{E}\left(\left\|G_{[j]}\right\|_{op}^{2} \cdot \|\mathcal{X}_{j}\|_{4}^{4}\right) \leq \frac{K_{3}^{2}}{n} \mathbb{E}\left(\left\|G_{[j]}\right\|_{op}^{2} \cdot \|\mathcal{X}_{j}\|_{\infty}^{4}\right)$$

As shown in (B-34),

$$||G_{[j]}||_{\text{op}} \le \left(\frac{K_1}{K_0}\right)^{\frac{1}{2}}$$

On the other hand, notice that the *i*-th row of $G_{[j]}$ is $h_{j,1,i}$ (see (B-20) for definition), by definition of Δ_C we have

$$\|\mathcal{X}_{j}\|_{\infty} = \|G_{[j]}X_{j}\|_{\infty} = \max_{i} |h_{j,1,i}^{T}X_{j}| \le \Delta_{C} \cdot \max \|h_{j,1,i}\|_{2}$$

By (B-35) and assumption A5,

$$\|\mathcal{X}_j\|_{\infty} \leq \Delta_C \cdot \left(\frac{K_1}{K_0}\right)^{\frac{1}{2}} = O_{L^4} \left(\text{polyLog}(\mathbf{n}) \right).$$

This entails that

$$\operatorname{Var}(\xi_j) = O\left(\frac{\operatorname{polyLog}(\mathbf{n})}{n}\right).$$

Combining with (B-62) and (B-63), we obtain that

$$I_2 = O\left(\frac{\text{polyLog}(n)}{n}\right).$$

B-6.5 Summary

Putting (B-55), (B-61) and (B-62) together, we conclude that

$$n\operatorname{Var}(b_j) = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right) - O\left(\frac{1}{n \cdot \operatorname{polyLog}(n)}\right) = \Omega\left(\frac{1}{\operatorname{polyLog}(n)}\right) \Longrightarrow \operatorname{Var}(b_j) = \Omega\left(\frac{\operatorname{polyLog}(n)}{n}\right).$$

Combining with (B-54),

$$\operatorname{Var}(\hat{\beta}_j) = \Omega\left(\frac{\operatorname{polyLog}(\mathbf{n})}{n}\right).$$

C Proof of Other Results

C-1 Proofs of Propositions in Section 2.3

Proof of Proposition 2.1. Let $H_i(\alpha) = \mathbb{E}\rho(\epsilon_i - \alpha)$. First we prove that the conditions imply that 0 is the unique minimizer of $H_i(\alpha)$ for all *i*. In fact, since $\epsilon_i \stackrel{d}{=} -\epsilon_i$,

$$H_i(\alpha) = \mathbb{E}\rho(\epsilon_i - \alpha) = \frac{1}{2} \left(\mathbb{E}\rho(\epsilon_i - \alpha) + \rho(-\epsilon_i - \alpha) \right).$$

Using the fact that ρ is even, we have

$$H_i(\alpha) = \mathbb{E}\rho(\epsilon_i - \alpha) = \frac{1}{2} \left(\mathbb{E}\rho(\epsilon_i - \alpha) + \rho(\epsilon_i + \alpha) \right).$$

By (4), for any $\alpha \neq 0$, $H_i(\alpha) > H_i(0)$. As a result, 0 is the unique minimizer of H_i . Then for any $\beta \in \mathbb{R}^p$

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\rho(y_i - x_i^T\beta) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\rho(\epsilon_i - x_i^T(\beta - \beta^*)) = \frac{1}{n}\sum_{i=1}^{n}H_i(x_i^T(\beta - \beta^*)) \ge \frac{1}{n}\sum_{i=1}^{n}H_i(0).$$

The equality holds iff $x_i^T(\beta - \beta^*) = 0$ for all *i* since 0 is the unique minimizer of H_i . This implies that

$$X(\beta^*(\rho) - \beta^*) = 0.$$

Since X has full column rank, we conclude that

$$\beta^*(\rho) = \beta^*$$

Proof of Proposition 2.2. For any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^p$, let

$$G(\alpha;\beta) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\rho(y_i - \alpha - x_i^T \beta).$$

Since α_{ρ} minimizes $\mathbb{E}\rho(\epsilon_i - \alpha)$, it holds that

$$G(\alpha;\beta) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\rho(\epsilon_i - \alpha - x_i^T(\beta - \beta^*)) \ge \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\rho(\epsilon_i - \alpha_\rho) = G(\alpha_\rho, \beta^*).$$

Note that α_{ρ} is the unique minimizer of $\mathbb{E}\rho(\epsilon_i - \alpha)$, the above equality holds if and only if

$$\alpha + x_i^T (\beta - \beta^*) \equiv \alpha_\rho \Longrightarrow (\mathbf{1} \ X) \left(\begin{array}{c} \alpha - \alpha_\rho \\ \beta - \beta^* \end{array} \right) = 0.$$

Since (1 X) has full column rank, it must hold that $\alpha = \alpha_{\rho}$ and $\beta = \beta^*$.

C-2 Proofs of Corollary 3.3

Proposition C.1. Suppose that ϵ_i are *i.i.d.* such that $\mathbb{E}\rho(\epsilon_1 - \alpha)$ as a function of α has a unique minimizer α_{ρ} . Further assume that $X_{J_n^c}$ contains an intercept term, X_{J_n} has full column rank and

$$span(\{X_j : j \in J_n\}) \cap span(\{X_j : j \in J_n^c\}) = \{0\}$$
(C-66)

Let

$$\beta_{J_n}(\rho) = \operatorname*{arg\,min}_{\beta_{J_n}} \left\{ \min_{\beta_{J_n^c}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}\rho(y_i - x_i^T \beta) \right\}.$$

Then $\beta_{J_n}(\rho) = \beta^*_{J_n}$.

Proof. let

$$G(\beta) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\rho(y_i - x_i^T \beta).$$

For any minimizer $\beta(\rho)$ of G, which might not be unique, we prove that $\beta_{J_n}(\rho) = \beta_{J_n}^*$. It follows by the same argument as in Proposition 2.2 that

$$x_i^T(\beta(\rho) - \beta^*) \equiv \alpha_0 \Longrightarrow X(\beta(\rho) - \beta^*) = \alpha_0 \mathbf{1} \Longrightarrow X_{J_n}(\beta_{J_n}(\rho)) = -X_{J_n^c}(\beta(\rho)_{J_n^c} - \beta^*_{J_n^c}) + \alpha_0 \mathbf{1}.$$

Since $X_{J_n^c}$ contains the intercept term, we have

$$X_{J_n}(\beta_{J_n}(\rho) - \beta_{J_n}^*) \in \operatorname{span}(\{X_j : j \in J_n^c\}).$$

It then follows from (C-68) that

$$X_{J_n}(\beta_{J_n}(\rho) - \beta^*_{J_n}) = 0.$$

Since X_{J_n} has full column rank, we conclude that

$$\beta_{J_n}(\rho) = \beta_{J_n}^*.$$

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The Proposition C.1 implies that $\beta_{J_n}^*$ is identifiable even when X is not of full column rank. A similar conclusion holds for the estimator $\hat{\beta}_{J_n}$ and the residuals R_i . The following two propositions show that under certain assumptions, $\hat{\beta}_{J_n}$ and R_i are invariant to the choice of $\hat{\beta}$ in the presense of multiple minimizers.

Proposition C.2. Suppose that ρ is convex and twice differentiable with $\rho''(x) > c > 0$ for all $x \in \mathbb{R}$. Let $\hat{\beta}$ be any minimizer, which might not be unique, of

$$F(\beta) \triangleq \frac{1}{n} \sum_{i=1}^{n} \rho(y_i - x_i^T \beta)$$

Then $R_i = y_i - x_i \hat{\beta}$ is independent of the choice of $\hat{\beta}$ for any *i*.

Proof. The conclusion is obvious if $F(\beta)$ has a unique minimizer. Otherwise, let $\hat{\beta}^{(1)}$ and $\hat{\beta}^{(2)}$ be two different minimizers of F denote by η their difference, i.e. $\eta = \hat{\beta}^{(2)} - \hat{\beta}^{(1)}$. Since F is convex, $\hat{\beta}^{(1)} + v\eta$ is a minimizer of F for all $v \in [0, 1]$. By Taylor expansion,

$$F(\hat{\beta}^{(1)} + v\eta) = F(\hat{\beta}^{(1)}) + v\nabla F(\hat{\beta}^{(1)})\eta + \frac{v^2}{2}\eta^T \nabla^2 F(\hat{\beta}^{(1)})\eta + o(v^2).$$

Since both $\hat{\beta}^{(1)} + v\eta$ and $\hat{\beta}^{(1)}$ are minimizers of F, we have $F(\hat{\beta}^{(1)} + v\eta) = F(\hat{\beta}^{(1)})$ and $\nabla F(\hat{\beta}^{(1)}) = 0$. By letting v tend to 0, we conclude that

$$\eta^T \nabla^2 F(\hat{\beta}^{(1)}) \eta = 0.$$

The hessian of F can be written as

$$\nabla^2 F(\hat{\beta}^{(1)}) = \frac{1}{n} X^T \operatorname{diag}(\rho''(y_i - x_i^T \hat{\beta}^{(1)})) X \succeq \frac{c X^T X}{n}.$$

Thus, η satisfies that

$$\eta^T \frac{cX^T X}{n} \eta = 0 \Longrightarrow X\eta = 0.$$
 (C-67)

This implies that

$$y - X\hat{\beta}^{(1)} = y - X\hat{\beta}^{(2)}$$

and hence R_i is the same for all *i* in both cases.

Proposition C.3. Suppose that ρ is convex and twice differentiable with $\rho''(x) > c > 0$ for all $x \in \mathbb{R}$. Further assume that X_{J_n} has full column rank and

$$span(\{X_j : j \in J_n\}) \cap span(\{X_j : j \in J_n^c\}) = \{0\}$$
(C-68)

Let $\hat{\beta}$ be any minimizer, which might not be unique, of

$$F(\beta) \triangleq \frac{1}{n} \sum_{i=1}^{n} \rho(y_i - x_i^T \beta)$$

Then $\hat{\beta}_{J_n}$ is independent of the choice of $\hat{\beta}$.

Proof. As in the proof of Proposition C.2, we conclude that for any minimizers $\hat{\beta}^{(1)}$ and $\hat{\beta}^{(2)}$, $X\eta = 0$ where $\eta = \hat{\beta}^{(2)} - \hat{\beta}^{(1)}$. Decompose the term into two parts, we have

$$X_{J_n}\eta_{J_n} = -X_{J_n}^c\eta_{J_n^c} \in \operatorname{span}(\{X_j : j \in J_n^c\}).$$

It then follows from (C-68) that $X_{J_n}\eta_{J_n} = 0$. Since X_{J_n} has full column rank, we conclude that $\eta_{J_n} = 0$ and hence $\hat{\beta}_{J_n}^{(1)} = \hat{\beta}_{J_n}^{(2)}$.

Proof of Corollary 3.3. Under assumption $\mathbf{A3}^*$, X_{J_n} must have full column rank. Otherwise there exists $\alpha \in \mathbb{R}^{|J_n|}$ such that $X_{J_n}\alpha$, in which case $\alpha^T X_{J_n}^T (I - H_{J_n^c}) X_{J_n} \alpha = 0$. This violates the assumption that $\tilde{\lambda}_- > 0$. On the other hand, it also guarantees that

$$\operatorname{span}(\{X_j : j \in J_n\}) \cap \operatorname{span}(\{X_j : j \in J_n^c\}) = \{0\}.$$

This together with assumption A1 and Proposition C.3 implies that $\hat{\beta}_{J_n}$ is independent of the choice of $\hat{\beta}$.

Let $B_1 \in \mathbb{R}^{|J_n^c| \times |J_n|}$, $B_2 \in \mathbb{R}^{|J_n^c| \times |J_n^c|}$ and assume that B_2 is invertible. Let $\tilde{X} \in \mathbb{R}^{n \times p}$ such that

$$\tilde{X}_{J_n} = X_{J_n} - X_{J_n^c} B_1, \quad \tilde{X}_{J_n^c} = X_{J_n^c} B_2.$$

Then $\operatorname{rank}(X) = \operatorname{rank}(\tilde{X})$ and model (1) can be rewritten as

$$y = X\beta^* + \epsilon$$

where

$$\tilde{\beta}_{J_n}^* = \beta_{J_n}^*, \quad \tilde{\beta}_{J_n^c}^* = B_2^{-1} \beta_{J_n^c}^* + B_1 \beta_{J_n}^*.$$

Let $\hat{\beta}$ be an M-estimator, which might not be unique, based on \tilde{X} . Then Proposition C.3 shows that $\tilde{\beta}_{J_n}$ is independent of the choice of $\tilde{\beta}$, and an invariance argument shows that

$$\hat{\hat{\beta}}_{J_n} = \hat{\beta}_{J_n}$$

In the rest of proof, we use $\tilde{\cdot}$ to denote the quantity obtained based on \tilde{X} . First we show that the assumption **A**4 is not affected by this transformation. In fact, for any $j \in J_n$, by definition we have

$$\operatorname{span}(X_{[j]}) = \operatorname{span}(X_{[j]})$$

and hence the leave-*j*-th-predictor-out residuals are not changed by Proposition C.2. This implies that $h_{j,0} = h_{j,0}$ and $\tilde{Q}_j = Q_j$. Recall the definition of $h_{j,0}$, the first-order condition of $\hat{\beta}$ entails that $X^T h_{j,0} = 0$. In particular, $X_{J_n^c}^T h_{j,0} = 0$ and this implies that for any $\alpha \in \mathbb{R}^n$,

$$0 = \operatorname{Cov}(X_{J_n^c}^T h_{j,0}, \alpha^T h_{j,0}) = X_{J_n^c} Q_j \alpha$$

Thus,

$$\frac{\tilde{X}_{j}^{T}\tilde{Q}_{j}\tilde{X}_{j}}{\operatorname{tr}(\tilde{Q}_{j})} = \frac{(X_{j} - X_{J_{n}}^{c}(B_{1})_{j})^{T}Q_{j}(X_{j} - X_{J_{n}}^{c}(B_{1})_{j})}{\operatorname{tr}(Q_{j})} = \frac{X_{j}^{T}Q_{j}X_{j}}{\operatorname{tr}(Q_{j})}.$$

Then we prove that the assumption A5 is also not affected by the transformation. The above argument has shown that

$$\frac{\tilde{h}_{j,0}^T \tilde{X}_j}{\|\tilde{h}_{j,0}\|_2} = \frac{h_{j,0}^T X_j}{\|h_{j,0}\|_2}.$$

On the other hand, let $B = \begin{pmatrix} I_{|J_n|} & 0 \\ -B_1 & B_2 \end{pmatrix}$, then *B* is non-singular and $\tilde{X} = XB$. Let $B_{(j),[j]}$ denote the matrix *B* after removing *j*-th row and *j*-th column. Then $B_{(j),[j]}$ is also non-singular and $\tilde{X}_{[j]} = X_{[j]}B_{(j),[j]}$. Recall the definition of $h_{j,1,i}$, we have

$$\begin{split} \tilde{h}_{j,1,i} &= (I - \tilde{D}_{[j]} \tilde{X}_{[j]} (\tilde{X}_{[j]}^T \tilde{D}_{[j]} \tilde{X}_j)^{-1} \tilde{X}_{[j]}^T) e_i \\ &= (I - D_{[j]} X_{[j]} B_{(j),[j]} (B_{(j),[j]}^T X_{[j]}^T D_{[j]} X_j B_{(j),[j]})^{-1} B_{(j),[j]}^T X_{[j]}) e_i \\ &= (I - D_{[j]} X_{[j]} (X_{[j]}^T D_{[j]} X_j)^{-1} X_{[j]}) e_i \\ &= h_{j,1,i}. \end{split}$$

On the other hand, by definition,

$$X_{[j]}^T h_{j,1,i} = X_{[j]}^T (I - D_{[j]} X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T) e_i = 0.$$

Thus,

$$h_{j,1,i}^T \tilde{X}_j = h_{j,1,i}^T (X_j - X_{J_n}^c (B_1)_j) = h_{j,1,i}^T X_j.$$

In summary, for any $j \in J_n$ and $i \leq n$,

$$\frac{\tilde{h}_{j,1,i}^T \tilde{X}_j}{\|\tilde{h}_{j,1,i}\|_2} = \frac{h_{j,1,i}^T X_j}{\|h_{j,1,i}\|_2}.$$

Putting the pieces together we have

$$\tilde{\Delta}_C = \Delta_C.$$

By Theorem 3.1,

$$\max_{j \in J_n} d_{\mathrm{TV}} \left(\mathcal{L} \left(\frac{\hat{\beta}_j - \mathbb{E} \hat{\beta}_j}{\sqrt{\mathrm{Var}(\hat{\beta}_j)}} \right), N(0, 1) \right) = o(1).$$

provided that \tilde{X} satisfies the assumption A3.

Now let $U\Lambda V$ be the singular value decomposition of $X_{J_n^c}$, where $U \in \mathbb{R}^{n \times p}$, $\Lambda \in \mathbb{R}^{p \times p}$, $V \in \mathbb{R}^{p \times p}$ with $U^T U = V^T V = I_p$ and $\Lambda = \operatorname{diag}(\nu_1, \ldots, \nu_p)$ being the diagonal matrix formed by singular values of $X_{J_n^c}$. First we consider the case where $X_{J_n^c}$ has full column rank, then $\nu_j > 0$ for all $j \leq p$. Let $B_1 = (X_{J_n}^T X_{J_n})^- X_{J_n}^T X_{J_n}$ and $B_2 = \sqrt{n/|J_n^c|} V^T \Lambda^{-1}$. Then

$$\frac{\tilde{X}^T \tilde{X}}{n} = \frac{1}{n} \left(\begin{array}{cc} X_{J_n}^T (I - X_{J_n^c} (X_{J_n^c}^T X_{J_n^c})^{-1} X_{J_n^c}) X_{J_n} & 0\\ 0 & nI \end{array} \right).$$

This implies that

$$\lambda_{\max}\left(\frac{\tilde{X}^T\tilde{X}}{n}\right) = \max\left\{\tilde{\lambda}_{\max}, 1\right\}, \quad \lambda_{\min}\left(\frac{\tilde{X}^T\tilde{X}}{n}\right) = \min\left\{\tilde{\lambda}_{\min}, 1\right\}.$$

The assumption $A3^*$ implies that

$$\lambda_{\max}\left(\frac{\tilde{X}^T\tilde{X}}{n}\right) = O(\text{polyLog}(n)), \quad \lambda_{\min}\left(\frac{\tilde{X}^T\tilde{X}}{n}\right) = \Omega\left(\frac{1}{\text{polyLog}(n)}\right).$$

By Theorem 3.1, we conclude that

Next we consider the case where $X_{J_n}^c$ does not have full column rank. We first remove the redundant columns from $X_{J_n}^c$, i.e. replace $X_{J_n^c}$ by the matrix formed by its maximum linear independent subset. Denote by **X** this matrix. Then $\operatorname{span}(X) = \operatorname{span}(\mathbf{X})$ and $\operatorname{span}(\{X_j : j \notin J_n\}) = \operatorname{span}(\{\mathbf{X}_j : j \notin J_n\})$. As a consequence of Proposition C.1 and C.3, neither $\beta_{J_n}^*$ nor $\hat{\beta}_{J_n}$ is affected. Thus, the same reasoning as above applies to this case.

C-3 Proofs of Results in Section 3.3

First we prove two lemmas regarding the behavior of Q_j . These lemmas are needed for justifying Assumption A4 in the examples.

Lemma C.4. Under assumptions A1 and A2,

$$||Q_j||_{\text{op}} \le c_1^2 \frac{K_3^2 K_1}{K_0}, \quad ||Q_j||_{\text{F}} \le \sqrt{n} c_1^2 \frac{K_3^2 K_1}{K_0}$$

where $Q_j = \operatorname{Cov}(h_{j,0})$ as defined in section B-1.

Proof of Lemma C.4. By definition,

$$\|Q_j\|_{\mathrm{op}} = \sup_{\alpha \in \mathbb{S}^{n-1}} \alpha^T Q_j \alpha$$

where \mathbb{S}^{n-1} is the *n*-dimensional unit sphere. For given $\alpha \in \mathbb{S}^{n-1}$,

$$\alpha^T Q_j \alpha = \alpha^T \operatorname{Cov}(h_{j,0}) \alpha = \operatorname{Var}(\alpha^T h_{j,0})$$

It has been shown in (B-59) in Appendix B-6.3 that

$$\frac{\partial r_{i,[j]}}{\partial \epsilon_k} = e_i^T G_{[j]} e_k$$

where $G_{[j]} = I - X_{[j]} (X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]}$. This yields that

$$\frac{\partial}{\partial \epsilon} \left(\sum_{i=1}^{n} \alpha_i \psi(r_{i,[j]}) \right) = \sum_{i=1}^{n} \alpha_i \psi'(r_{i,[j]}) \cdot \frac{\partial r_{i,[j]}}{\partial \epsilon} = \sum_{i=1}^{n} \alpha_i \psi'(r_{i,[j]}) \cdot e_i^T G_{[j]} = \alpha^T \tilde{D}_{[j]} G_{[j]}.$$

By standard Poincaré inequality (see Proposition B.8), since $\epsilon_i = u_i(W_i)$,

$$\operatorname{Var}\left(\sum_{i=1}^{n} \alpha_{i}\psi(r_{i,[j]})\right) \leq \max_{k} \|u_{k}'\|_{\infty}^{2} \cdot \mathbb{E}\left\|\frac{\partial}{\partial\epsilon}\left(\sum_{i=1}^{n} \alpha_{i}\psi(r_{i,[j]})\right)\right\|^{2}$$
$$\leq c_{1}^{2} \cdot \mathbb{E}\left(\alpha^{T}\tilde{D}_{[j]}G_{[j]}G_{[j]}\tilde{D}_{[j]}\alpha\right) \leq c_{1}^{2}\mathbb{E}\|\tilde{D}_{[j]}G_{[j]}G_{[j]}G_{[j]}\tilde{D}_{[j]}\|_{2}^{2} \leq c_{1}^{2}\mathbb{E}\|\tilde{D}_{j}\|_{\operatorname{op}}^{2}\|G_{[j]}\|_{\operatorname{op}}^{2}.$$

We conclude from Lemma B.2 and (B-34) in Appendix B-2 that

$$\|\tilde{D}_{[j]}\|_{\text{op}} \le K_3, \quad \|G_{[j]}\|_{\text{op}}^2 \le \frac{K_1}{K_0}.$$

Therefore,

$$\|Q_j\|_{\mathrm{op}} = \sup_{\alpha \in \mathbb{S}^{n-1}} \operatorname{Var}\left(\sum_{i=1}^n \alpha_i \psi(R_i)\right) \le c_1^2 \frac{K_3^2 K_1}{K_0}$$

and hence

$$\|Q_j\|_{\rm F} \le \sqrt{n} \|Q_j\|_{\rm op} \le \sqrt{n} \cdot c_1^2 \frac{K_3^2 K_1}{K_0}$$

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Lemma C.5. Under assumptions A1 - A3,

$$\operatorname{tr}(Q_j) \ge K^* n = \Omega(n \cdot \operatorname{polyLog}(n)),$$

where $K^* = \frac{K_0^4}{K_1^2} \cdot \left(\frac{n-p+1}{n}\right)^2 \cdot \min_i \operatorname{Var}(\epsilon_i).$

Proof. This is a direct consequence of Lemma B.7 in p.47.

Throughout the following proofs, we will use several results from the random matrix theory to bound the largest and smallest singular values of Z. The results are shown in Appendix E. Furthermore, in contrast to other sections, the notation $P(\cdot), \mathbb{E}(\cdot), \operatorname{Var}(\cdot)$ denotes the probability, the expectation and the variance with respect to both ϵ and Z in this section.

Proof of Proposition 3.4. By Proposition E.3,

$$\lambda_{+} = (1 + \sqrt{\kappa})^{2} + o_{p}(1) = O_{p}(1), \quad \lambda_{-} = (1 - \sqrt{\kappa})^{2} - o_{p}(1) = \Omega_{p}(1)$$

and thus the assumption A3 holds with high probability. By Hanson-Wright inequality (Hanson & Wright, 1971; Rudelson & Vershynin, 2013; see Proposition E.2), for any given deterministic matrix A,

$$P(|Z_j^T A Z_j - \mathbb{E}Z_j^T A Z_j| \ge t) \le 2 \exp\left[-c \min\left\{\frac{t^2}{\sigma^4 \|A\|_{\mathrm{F}}^2}, \frac{t}{\sigma^2 \|A\|_{\mathrm{op}}}\right\}\right]$$

for some universal constant c. Let $A = Q_j$ and conditioning on $Z_{[j]}$, then by Lemma C.4, we know that

$$||Q_j||_{\text{op}} \le c_1^2 \frac{K_3^2 K_1}{K_0}, \quad ||Q_j||_{\text{F}} \le \sqrt{n} c_1^2 \frac{K_3^2 K_1}{K_0}$$

and hence

$$P\left(Z_{j}^{T}Q_{j}Z_{j} - \mathbb{E}(Z_{j}^{T}Q_{j}Z_{j}|Z_{[j]}) \leq -t \left| Z_{[j]} \right) \leq 2\exp\left[-c\min\left\{\frac{t^{2}}{\sigma^{4} \cdot nc_{1}^{4}K_{3}^{4}K_{1}^{2}/K_{0}^{2}}, \frac{t}{\sigma^{2}c_{1}^{2}K_{3}^{2}K_{1}/K_{0}}\right\}\right]$$
(C-69)

Note that

$$\mathbb{E}(Z_j^T Q_j Z_j | Z_{[j]}) = \operatorname{tr}(\mathbb{E}[Z_j Z_j^T | Z_{[j]}] Q_j) = \mathbb{E}Z_{1j}^2 \operatorname{tr}(Q_j) = \tau^2 \operatorname{tr}(Q_j).$$

By Lemma C.5, we conclude that

$$P\left(\frac{Z_{j}^{T}Q_{j}Z_{j}}{\operatorname{tr}(Q_{j})} \leq \tau^{2} - \frac{t}{nK^{*}} \left| Z_{[j]} \right) \leq P\left(\frac{Z_{j}^{T}Q_{j}Z_{j}}{\operatorname{tr}(Q_{j})} \leq \tau^{2} - \frac{t}{\operatorname{tr}(Q_{j})} \left| Z_{[j]} \right) \right.$$

$$\leq 2 \exp\left[-c \min\left\{ \frac{t^{2}}{\sigma^{4} \cdot nc_{1}^{4}K_{3}^{4}K_{1}^{2}/K_{0}^{2}}, \frac{t}{2\sigma^{2}c_{1}^{2}K_{3}^{2}K_{1}/K_{0}} \right\} \right].$$
(C-70)

Let $t = \frac{1}{2}\tau^2 n K^*$ and take expectation of both sides over $Z_{[j]}$, we obtain that

$$P\left(\frac{Z_j^T Q_j Z_j}{\operatorname{tr}(Q_j)} \le \frac{\tau^2}{2}\right) \le 2 \exp\left[-cn \min\left\{\frac{K^{*2} \tau^4}{4\sigma^4 c_1^4 K_3^4 K_1^2 / K_0^2}, \frac{K^* \tau^2}{2\sigma^2 c_1^2 K_3^2 K_1 / K_0}\right\}\right]$$

and hence

$$P\left(\min_{j\in J_n} \frac{Z_j^T Q_j Z_j}{\operatorname{tr}(Q_j)} \le \frac{\tau^2}{2}\right) \le 2n \exp\left[-cn \min\left\{\frac{K^{*2}\tau^4}{4\sigma^4 c_1^4 K_3^4 K_1^2 / K_0^2}, \frac{K^*\tau^2}{2\sigma^2 c_1^2 K_3^2 K_1 / K_0}\right\}\right] = o(1).$$
(C-71)

This entails that

$$\min_{j \in J_n} \frac{Z_j^T Q_j Z_j}{\operatorname{tr}(Q_j)} = \Omega_p(\operatorname{polyLog}(n)).$$

Thus, assumption A4 is also satisfied with high probability. On the other hand, since Z_j has i.i.d. mean-zero σ^2 -sub-gaussian entries, for any deterministic unit vector $\alpha \in \mathbb{R}^n$, $\alpha^T Z_j$ is σ^2 -sub-gaussian and mean-zero, and hence

$$P(|\alpha^T Z_j| \ge t) \le 2e^{-\frac{t^2}{2\sigma^2}}.$$

Let $\alpha_{j,i} = h_{j,1,i}/||h_{j,1,i}||_2$ and $\alpha_{j,0} = h_{j,0}/||h_{j,0}||_2$. Since $h_{j,1,i}$ and $h_{j,0}$ are independent of Z_j , a union bound then gives

$$P\left(\Delta_C \ge t + 2\sigma\sqrt{\log n}\right) \le 2n^2 e^{-\frac{t^2 + 4\sigma^2 \log n}{2\sigma^2}} = 2e^{-\frac{t^2}{2\sigma^2}}$$

By Fubini's formula (Durrett, 2010, Lemma 2.2.8.),

$$\mathbb{E}\Delta_C^8 = \int_0^\infty 8t^7 P(\Delta_C \ge t)dt \le \int_0^{2\sigma\sqrt{\log n}} 8t^7 dt + \int_{2\sigma\sqrt{\log n}}^\infty 8t^7 P(\Delta_C \ge t)dt$$
$$= (2\sigma\sqrt{\log n})^8 + \int_0^\infty 8(t + 2\sigma\sqrt{\log n})^7 P(\Delta_C \ge t + 2\sigma\sqrt{\log n})dt$$
$$\le (2\sigma\sqrt{\log n})^8 + \int_0^\infty 64(8t^7 + 128\sigma^7(\log n)^{\frac{7}{2}}) \cdot 2e^{-\frac{t^2}{2\sigma^2}}dt$$
$$= O(\sigma^8 \cdot \operatorname{polyLog}(n)) = O\left(\operatorname{polyLog}(n)\right). \tag{C-72}$$

This, together with Markov inequality, guarantees that assumption A5 is also satisfied with high probability.

Proof of Proposition 3.5. It is left to prove that assumption A3 holds with high probability. The proof of assumption A4 and A5 is exactly the same as the proof of Proposition 3.5. By Proposition E.4,

$$\lambda_+ = O_p(1).$$

On the other hand, by Proposition E.7 (Litvak et al., 2005),

$$P\left(\lambda_{\min}\left(\frac{Z^T Z}{n}\right) < c_1\right) \le e^{-c_2 n}.$$
$$\lambda_- = \Omega_p(1).$$

and thus

Proof of Proposition 3.6. Since J_n excludes the intercept term, the proof of assumption A4 and A5 is still the same as Proposition 3.5. It is left to prove assumption A3. Let R_1, \ldots, R_n be i.i.d. Rademacher random variables, i.e. $P(R_i = 1) = P(R_i = -1) = \frac{1}{2}$, and

$$Z^* = \operatorname{diag}(B_1, \dots, B_n)Z.$$

Then $(Z^*)^T Z^* = Z^T Z$. It is left to show that the assumption A3 holds for Z^* with high probability. Note that

$$(Z_i^*)^T = (B_i, B_i \tilde{x}_i^T).$$

For any $r \in \{1, -1\}$ and borel sets $B_1, \ldots, B_p \subset \mathbb{R}$,

$$P(B_{i} = r, B_{i}Z_{i1} \in B_{1}, \dots, B_{i}Z_{i(p-1)} \in B_{p-1})$$

$$= P(B_{i} = r, \tilde{Z}_{i1} \in rB_{1}, \dots, \tilde{Z}_{i(p-1)} \in rB_{p-1})$$

$$= P(B_{i} = r)P(\tilde{Z}_{i1} \in rB_{1}) \dots P(\tilde{Z}_{i(p-1)} \in rB_{p-1})$$

$$= P(B_{i} = r)P(\tilde{Z}_{i1} \in B_{1}) \dots P(\tilde{Z}_{i(p-1)} \in B_{p-1})$$

$$= P(B_{i} = r)P(B_{i}\tilde{Z}_{i1} \in B_{1}) \dots P(B_{i}\tilde{Z}_{i(p-1)} \in B_{p-1})$$

where the last two lines uses the symmetry of \tilde{Z}_{ij} . Then we conclude that Z_i^* has independent dent entries. Since the rows of Z^* are independent, Z^* has independent entries. Since B_i are symmetric and sub-gaussian with unit variance and $B_i \tilde{Z}_{ij} \stackrel{d}{=} \tilde{Z}_{ij}$, which is also symmetric and sub-gaussian with variance bounded from below, Z^* satisfies the conditions of Propsition 3.5 and hence the assumption **A**₃ is satisfied with high probability. \Box

Proof of Proposition 3.8 (with Proposition 3.7 being a special case). Let $Z_* = \Lambda^{-\frac{1}{2}} Z \Sigma^{-\frac{1}{2}}$, then Z_* has i.i.d. standard gaussian entries. By Proposition 3.6, Z_* satisfies assumption A3 with high probability. Thus,

$$\lambda_{+} = \lambda_{\max}\left(\frac{\Sigma^{\frac{1}{2}} Z_{*}^{T} \Lambda Z_{*} \Sigma^{\frac{1}{2}}}{n}\right) \leq \lambda_{\max}(\Sigma) \cdot \lambda_{\max}(\Lambda) \cdot \lambda_{\max}\left(\frac{Z_{*}^{T} Z_{*}}{n}\right) = O_{p}(\operatorname{polyLog}(n)),$$

and

$$\lambda_{-} = \lambda_{\min}\left(\frac{\Sigma^{\frac{1}{2}} Z_{*}^{T} \Lambda Z_{*} \Sigma^{\frac{1}{2}}}{n}\right) \ge \lambda_{\min}(\Sigma) \cdot \lambda_{\min}(\Lambda) \cdot \lambda_{\min}\left(\frac{Z_{*}^{T} Z_{*}}{n}\right) = \Omega_{p}\left(\frac{1}{\operatorname{polyLog}(n)}\right).$$

As for assumption A4, the first step is to calculate $\mathbb{E}(Z_j^T Q_j Z_j | Z_{[j]})$. Let $\tilde{Z} = \Lambda^{-\frac{1}{2}} Z$, then $\operatorname{vec}(\tilde{Z}) \sim N(0, I \otimes \Sigma)$. As a consequence,

$$\tilde{Z}_j | \tilde{Z}_{[j]} \sim N(\tilde{\mu}_j, \sigma_j^2 I)$$

where

$$\tilde{\mu}_j = \tilde{Z}_{[j]} \Sigma_{[j],[j]}^{-1} \Sigma_{[j],j} = \Lambda^{-\frac{1}{2}} Z_{[j]} \Sigma_{[j],[j]}^{-1} \Sigma_{[j],j}.$$

Thus,

$$Z_j | Z_{[j]} \sim N(\mu_j, \sigma_j^2 \Lambda)$$

where $\mu_j = Z_{[j]} \Sigma_{[j],[j]}^{-1} \Sigma_{[j],j}$. It is easy to see that

$$\lambda_{-} \le \min_{j} \sigma_{j}^{2} \le \max_{j} \sigma_{j}^{2} \le \lambda_{+}.$$
(C-73)

It has been shown that $Q_j \mu_j = 0$ and hence

$$Z_{j}^{T}Q_{j}Z_{j} = (Z_{j} - \mu_{j})^{T}Q_{j}(Z_{j} - \mu_{j}).$$

Let $\mathscr{Z}_j = \Lambda^{-\frac{1}{2}}(Z_j - \mu_j)$ and $\tilde{Q}_j = \Lambda^{\frac{1}{2}}Q_j\Lambda^{\frac{1}{2}}$, then $\mathscr{Z}_j \sim N(0, \sigma_j^2 I)$ and

$$Z_j^T Q_j Z_j = \mathscr{Z}_j^T \tilde{Q}_j \mathscr{Z}_j.$$

By Lemma C.4,

$$\|\tilde{Q}_j\|_{\rm op} \le \|\Lambda\|_{\rm op} \cdot \|Q_j\|_{\rm op} \le \lambda_{\max}(\Lambda) \cdot c_1^2 \frac{K_3^2 K_1}{K_0},$$

and hence

$$\|\tilde{Q}_j\|_{\mathrm{F}} \le \sqrt{n}\lambda_{\max}(\Lambda) \cdot c_1^2 \frac{K_3^2 K_1}{K_0}.$$

By Hanson-Wright inequality (Hanson & Wright, 1971; Rudelson & Vershynin, 2013; see Proposition E.2), we obtain a similar inequality to (C-69) as follows:

$$P\left(\left|Z_j^T Q_j Z_j - \mathbb{E}(Z_j^T Q_j Z_j | Z_{[j]})\right| \ge t \left| Z_{[j]} \right)$$
$$\le 2 \exp\left[-c \min\left\{\frac{t^2}{\sigma_j^4 \cdot n\lambda_{\max}(\Lambda)^2 c_1^4 K_3^4 K_1^2 / K_0^2}, \frac{t}{\sigma_j^2 \lambda_{\max}(\Lambda) c_1^2 K_3^2 K_1 / K_0}\right\}\right].$$

On the other hand,

$$\mathbb{E}(Z_j^T Q_j Z_j | Z_{[j]}) = \mathbb{E}(\mathscr{Z}_j^T \tilde{Q}_j \mathscr{Z}_j | Z_{[j]}) = \sigma_j^2 \operatorname{tr}(\tilde{Q}_j).$$

By definition,

$$\operatorname{tr}(\tilde{Q}_j) = \operatorname{tr}(\Lambda^{\frac{1}{2}}Q_j\Lambda^{\frac{1}{2}}) = \operatorname{tr}(\Sigma Q_j) = \operatorname{tr}(Q_j^{\frac{1}{2}}\Lambda Q_j^{\frac{1}{2}}) \ge \lambda_{\min}(\Lambda)\operatorname{tr}(Q_j)$$

By Lemma C.5,

$$\operatorname{tr}(\tilde{Q}_j) \ge \lambda_{\min}(\Lambda) \cdot nK^*.$$

Similar to (C-70), we obtain that

$$P\left(\frac{Z_j^T Q_j Z_j}{\operatorname{tr}(Q_j)} \ge \sigma_j^2 - \frac{t}{nK^*} \middle| Z_{[j]}\right)$$

$$\le 2 \exp\left[-c \min\left\{\frac{t^2}{\sigma_j^4 \cdot n\lambda_{\max}(\Lambda)^2 c_1^4 K_3^4 K_1^2 / K_0^2}, \frac{t}{\sigma_j^2 \lambda_{\max}(\Lambda) c_1^2 K_3^2 K_1 / K_0}\right\}\right].$$

Let $t = \frac{1}{2}\sigma_j^2 n K^*$, we have

$$P\left(\frac{Z_j^T Q_j Z_j}{\operatorname{tr}(Q_j)} \ge \frac{\sigma_j^2}{2}\right) \le 2 \exp\left[-cn \min\left\{\frac{K^{*2}}{4\lambda_{\max}(\Lambda)^2 c_1^4 K_3^4 K_1^2 / K_0^2}, \frac{K^*}{2\lambda_{\max}(\Lambda) c_1^2 K_3^2 K_1 / K_0}\right\}\right] = o\left(\frac{1}{n}\right)$$

and a union bound together with (C-73) yields that

$$\min_{j \in J_n} \frac{Z_j^T Q_j Z_j}{\operatorname{tr}(Q_j)} = \Omega_p\left(\min_j \sigma_j^2 \cdot \frac{1}{\operatorname{polyLog}(n)}\right) = \Omega_p\left(\frac{1}{\operatorname{polyLog}(n)}\right).$$

As for assumption A5, let

$$\alpha_{j,0} = \frac{\Lambda^{\frac{1}{2}} h_{j,0}}{\|h_{j,0}\|_2}, \quad \alpha_{j,i} = \frac{\Lambda^{\frac{1}{2}} h_{j,1,i}}{\|h_{j,1,i}\|_2}$$

then for i = 0, 1, ..., p,

$$\|\alpha_{j,i}\|_2 \le \sqrt{\lambda_{\max}(\Lambda)}.$$

Note that

$$\frac{h_{j,0}^T Z_j}{\|h_{j,0}\|_2} = \alpha_{j,0}^T Z_j, \quad \frac{h_{j,1,i}^T Z_j}{\|h_{j,1,i}\|_2} = \alpha_{j,i}^T Z_j$$

using the same argument as in (C-72), we obtain that

$$\mathbb{E}\Delta_C^8 = O\left(\lambda_{\max}(\Lambda)^4 \cdot \max_j \sigma_j^8 \cdot \operatorname{polyLog}(n)\right) = O\left(\operatorname{polyLog}(n)\right),$$

and by Markov inequality and (C-73),

$$\mathbb{E}(\Delta_C^8|Z) = O_p\left(\mathbb{E}\Delta_C^8\right) = O_p(\text{polyLog}(n)).$$

Proof of Proposition 3.9. The proof that assumptions A4 and A5 hold with high probability is exactly the same as the proof of Proposition 3.8. It is left to prove assumption A3*; see Corollary 3.3. Let $c = (\min_i |(\Lambda^{-\frac{1}{2}} \mathbf{1})_i|)^{-1}$ and $\mathbf{Z} = (c\mathbf{1} \ \tilde{Z})$. Recall the the definition of $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$, we have

$$\tilde{\lambda}_{+} = \lambda_{\max}(\Sigma_{\{1\}}), \quad \tilde{\lambda}_{-} = \lambda_{\min}(\Sigma_{\{1\}}),$$

where

$$\Sigma_{\{1\}} = \frac{1}{n} \tilde{Z}^T \left(I - \frac{\mathbf{1}\mathbf{1}^T}{n} \right) \tilde{Z}.$$

Rewrite $\Sigma_{\{1\}}$ as

$$\Sigma_{\{1\}} = \frac{1}{n} \left(\left(I - \frac{\mathbf{1}\mathbf{1}^T}{n} \right) \tilde{Z} \right)^T \left(\left(I - \frac{\mathbf{1}\mathbf{1}^T}{n} \right) \tilde{Z} \right).$$

It is obvious that

$$\operatorname{span}\left(\left(I-\frac{\mathbf{1}\mathbf{1}^{T}}{n}\right)\tilde{Z}\right)\subset\operatorname{span}(\mathbf{Z}).$$

As a consequence

$$\tilde{\lambda}_{+} \leq \lambda_{\max}\left(\frac{\mathbf{Z}^{T}\mathbf{Z}}{n}\right), \quad \tilde{\lambda}_{-} \geq \lambda_{\min}\left(\frac{\mathbf{Z}^{T}\mathbf{Z}}{n}\right).$$

It remains to prove that

$$\lambda_{\max}\left(\frac{\mathbf{Z}^T\mathbf{Z}}{n}\right) = O_p\left(\operatorname{polyLog}(\mathbf{n})\right), \quad \lambda_{\min}\left(\frac{\mathbf{Z}^T\mathbf{Z}}{n}\right) = \Omega_p\left(\frac{1}{\operatorname{polyLog}(\mathbf{n})}\right).$$

To prove this, we let

$$Z_* = \Lambda^{-\frac{1}{2}} \mathbf{Z} \begin{pmatrix} 1 & 0 \\ 0 & \Sigma^{-\frac{1}{2}} \end{pmatrix} \triangleq (\nu \ \tilde{Z}_*),$$

where $\nu = c \Lambda^{-\frac{1}{2}} \mathbf{1}$ and $\tilde{Z}_* = \Lambda^{-\frac{1}{2}} \tilde{Z} \Sigma^{-\frac{1}{2}}$. Then

$$\lambda_{\max}\left(\frac{\mathbf{Z}^{T}\mathbf{Z}}{n}\right) = \lambda_{\max}\left(\frac{\Sigma^{\frac{1}{2}}Z_{*}^{T}\Lambda Z_{*}\Sigma^{\frac{1}{2}}}{n}\right) \leq \lambda_{\max}(\Sigma) \cdot \lambda_{\max}(\Lambda) \cdot \lambda_{\max}\left(\frac{Z_{*}^{T}Z_{*}}{n}\right),$$

and

$$\lambda_{\min}\left(\frac{\mathbf{Z}^{T}\mathbf{Z}}{n}\right) = \lambda_{\min}\left(\frac{\Sigma^{\frac{1}{2}}Z_{*}^{T}\Lambda Z_{*}\Sigma^{\frac{1}{2}}}{n}\right) \ge \lambda_{\min}(\Sigma) \cdot \lambda_{\min}(\Lambda) \cdot \lambda_{\min}\left(\frac{Z_{*}^{T}Z_{*}}{n}\right).$$

It is left to show that

$$\lambda_{\max}\left(\frac{Z_*^T Z_*}{n}\right) = O_p(\operatorname{polyLog}(n)), \quad \lambda_{\min}\left(\frac{Z_*^T Z_*}{n}\right) = \Omega_p\left(\frac{1}{\operatorname{polyLog}(n)}\right).$$

By definition, $\min_i |\nu_i| = 1$ and $\max_i |\nu_i| = O$ (polyLog(n)), then

$$\lambda_{\max}\left(\frac{Z_*^T Z_*}{n}\right) = \lambda_{\max}\left(\frac{\tilde{Z}_*^T \tilde{Z}_*}{n} + \frac{\nu\nu^T}{n}\right) \le \lambda_{\max}\left(\frac{\tilde{Z}_*^T \tilde{Z}_*}{n}\right) + \frac{\|\nu\|_2^2}{n}.$$

Since \tilde{Z}_* has i.i.d. standard gaussian entries, by Proposition E.3,

$$\lambda_{\max}\left(\frac{\tilde{Z}_*^T \tilde{Z}_*}{n}\right) = O_p(1).$$

Moreover, $\|\nu\|_2^2 \le n \max_i |\nu_i|^2 = O(n \cdot \text{polyLog}(n))$ and thus,

$$\lambda_{\max}\left(\frac{Z_*^T Z_*}{n}\right) = O_p(\text{polyLog}(n)).$$

On the other hand, similar to Proposition 3.6,

$$\mathbf{Z}_* = \operatorname{diag}(B_1, \ldots, B_n) Z_*$$

where B_1, \ldots, B_n are i.i.d. Rademacher random variables. The same argument in the proof of Proposition 3.6 implies that \mathbf{Z}_* has independent entries with sub-gaussian norm bounded by $\|\nu\|_{\infty}^2 \vee 1$ and variance lower bounded by 1. By Proposition E.7, Z_* satisfies assumption A3 with high probability. Therefore, A3* holds with high probability.

Proof of Proposition 3.10. Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and \mathcal{Z} be the matrix with entries \mathcal{Z}_{ij} , then by Proposition 3.4 or Proposition 3.5, \mathcal{Z}_{ij} satisfies assumption A3 with high probability. Notice that

$$\lambda_{+} = \lambda_{\max}\left(\frac{\mathcal{Z}^{T}\Lambda^{2}\mathcal{Z}}{n}\right) \leq \lambda_{\max}(\Lambda)^{2} \cdot \lambda_{\max}\left(\frac{\mathcal{Z}^{T}\mathcal{Z}}{n}\right) = O_{p}(\operatorname{polyLog}(n)),$$

and

$$\lambda_{-} = \lambda_{\min}\left(\frac{\mathcal{Z}^{T}\Lambda^{2}\mathcal{Z}}{n}\right) \geq \lambda_{\min}(\Lambda)^{2} \cdot \lambda_{\min}\left(\frac{\mathcal{Z}^{T}\mathcal{Z}}{n}\right) = \Omega_{p}\left(\frac{1}{\operatorname{polyLog}(n)}\right).$$

Thus Z satisfies assumption A3 with high probability.

Conditioning on any realization of Λ , the law of \mathcal{Z}_{ij} does not change due to the independence between Λ and \mathcal{Z} . Repeating the arguments in the proof of Proposition 3.4 and Proposition 3.5, we can show that

$$\frac{\mathcal{Z}_{j}^{T}\tilde{Q}_{j}\mathcal{Z}_{j}}{\operatorname{tr}(\tilde{Q}_{j})} = \Omega_{p}\left(\frac{1}{\operatorname{polyLog}(\mathbf{n})}\right), \quad \text{and} \quad \mathbb{E}\max_{i=0,\dots,n;j=1,\dots,p} |\tilde{\alpha}_{j,i}^{T}\mathcal{Z}_{j}|^{8} = O_{p}(\operatorname{polyLog}(\mathbf{n})),$$
(C-74)

where

$$\tilde{Q}_j = \Lambda Q_j \Lambda, \quad \tilde{\alpha}_{j,0} = \frac{\Lambda h_{j,0}}{\|\Lambda h_{j,0}\|_2}, \quad \tilde{\alpha}_{j,1,i} = \frac{\Lambda h_{j,1,i}}{\|\Lambda h_{j,1,i}\|_2}.$$

Then

$$\frac{Z_j^T Q_j Z_j}{\operatorname{tr}(Q_j)} = \frac{Z_j^T \tilde{Q}_j Z_j}{\operatorname{tr}(\tilde{Q}_j)} \cdot \frac{\operatorname{tr}(\Lambda Q_j \Lambda)}{\operatorname{tr}(Q_j)} \ge a^2 \cdot \frac{Z_j^T \tilde{Q}_j Z_j}{\operatorname{tr}(\tilde{Q}_j)} = \Omega_p \left(\frac{1}{\operatorname{polyLog}(n)}\right), \quad (C-75)$$

and

$$\mathbb{E}\Delta_{C}^{8} = \mathbb{E}\left[\max_{i=0,\dots,n;j=1,\dots,p} |\tilde{\alpha}_{j,i}^{T} \mathcal{Z}_{j}|^{8} \cdot \max\left\{\max_{j} \frac{\|\Lambda h_{j,0}\|_{2}}{\|h_{j,0}\|_{2}}, \max_{i,j} \frac{\|\Lambda h_{j,1,i}\|_{2}}{\|h_{j,1,i}\|_{2}}\right\}^{8}\right] \quad (C-76)$$

$$\leq b^{8} \mathbb{E}\left[\max_{i=0,\dots,n;j=1,\dots,p} |\tilde{\alpha}_{j,i}^{T} \mathcal{Z}_{j}|^{8}\right]$$

$$= O_{p}(\operatorname{polyLog}(n)).$$

By Markov inequality, the assumption A5 is satisfied with high probability.

Proof of Proposition 3.11. The concentration inequality of ζ_i plus a union bound imply that

$$P\left(\max_{i} \zeta_{i} > (\log n)^{\frac{2}{\alpha}}\right) \le nc_{1}e^{-c_{2}(\log n)^{2}} = o(1).$$

Thus, with high probability,

$$\lambda_{\max} = \lambda_{\max}\left(\frac{\mathcal{Z}^T \Lambda^2 \mathcal{Z}}{n}\right) \le (\log n)^{\frac{4}{\alpha}} \cdot \lambda_{\max}\left(\frac{\mathcal{Z}^T \mathcal{Z}}{n}\right) = O_p(\text{polyLog}(n)).$$

Let $n' = \lfloor (1-\delta)n \rfloor$ for some $\delta \in (0, 1-\kappa)$. Then for any subset I of $\{1, \ldots, n\}$ with size n', by Proposition E.6 (Proposition E.7), under the conditions of Proposition 3.4 (Proposition 3.5), there exists constants c_3 and c_4 , which only depend on κ , such that

$$P\left(\lambda_{\min}\left(\frac{\mathcal{Z}_I^T \mathcal{Z}_I}{n}\right) < c_3\right) \le e^{-c_4 n}$$

where \mathcal{Z}_I represents the sub-matrix of \mathcal{Z} formed by $\{\mathcal{Z}_i : i \in I\}$, where \mathcal{Z}_i is the *i*-th row of \mathcal{Z} . Then by a union bound,

$$P\left(\min_{|I|=n'}\lambda_{\min}\left(\frac{\mathcal{Z}_{I}^{T}\mathcal{Z}_{I}}{n}\right) < c_{3}\right) \leq {n \choose n'} e^{-c_{4}n}.$$

By Stirling's formula, there exists a constant $c_5 > 0$ such that

$$\binom{n}{n'} = \frac{n!}{n'!(n-n')!} \le c_5 \exp\left\{ \left(-\tilde{\delta}\log\tilde{\delta} - (1-\tilde{\delta})\log(1-\tilde{\delta})\right)n\right\}$$

where $\tilde{\delta} = n'/n$. For sufficiently small δ and sufficiently large n,

$$-\tilde{\delta}\log\tilde{\delta} - (1-\tilde{\delta})\log(1-\tilde{\delta}) < c_4$$

and hence

$$P\left(\min_{|I|=n'} \lambda_{\min}\left(\frac{\mathcal{Z}_I^T \mathcal{Z}_I}{n}\right) < c_3\right) < c_5 e^{-c_6 n} \tag{C-77}$$

for some $c_6 > 0$. By Borel-Cantelli Lemma,

$$\liminf_{n \to \infty} \min_{|I| = \lfloor (1-\delta)n \rfloor} \lambda_{\min} \left(\frac{Z_I^T Z_I}{n} \right) \ge c_3 \quad a.s..$$

On the other hand, since F^{-1} is continuous at δ , then

$$\zeta_{(\lfloor (1-\delta)n \rfloor)} \stackrel{a.s.}{\to} F^{-1}(\delta) > 0.$$

where $\zeta_{(k)}$ is the k-th largest of $\{\zeta_i : i = 1, ..., n\}$. Let I^* be the set of indices corresponding to the largest $\lfloor (1 - \delta)n \rfloor \zeta'_i$ s. Then with probability 1,

$$\liminf_{n \to \infty} \lambda_{\min}\left(\frac{Z^T Z}{n}\right) = \liminf_{n \to \infty} \lambda_{\min}\left(\frac{Z^T \Lambda^2 Z}{n}\right) \ge \liminf_{n \to \infty} \zeta_{\left(\lfloor (1-\delta)n \rfloor\right)} \cdot \liminf_{n \to \infty} \lambda_{\min}\left(\frac{Z^T_{I^*} \Lambda^2_{I^*} Z_{I^*}}{n}\right)$$
$$\ge \liminf_{n \to \infty} \zeta_{\left(\lfloor (1-\delta)n \rfloor\right)} \cdot \liminf_{n \to \infty} \min_{|I| = \lfloor (1-\delta)n \rfloor} \lambda_{\min}\left(\frac{Z^T_{I} Z_{I}}{n}\right)$$
$$\ge c_3 F^{-1}(\delta)^2 > 0.$$

To prove assumption A4, similar to (C-75) in the proof of Proposition 3.10, it is left to show that (A, C, A) = (A, C, A)

$$\min_{j} \frac{\operatorname{tr}(\Lambda Q_{j}\Lambda)}{\operatorname{tr}(Q_{j})} = \Omega_{p} \left(\frac{1}{\operatorname{polyLog}(n)}\right).$$

Furthermore, by Lemma C.5, it remains to prove that

$$\min_{j} \operatorname{tr}(\Lambda Q_{j} \Lambda) = \Omega_{p} \left(\frac{n}{\operatorname{polyLog}(\mathbf{n})} \right).$$

Recalling the equation (B-60) in the proof of Lemma B.7, we have

$$e_i^T Q_j e_i \ge \frac{K_0}{K_1} \cdot \frac{1}{1 + e_i^T Z_{[j]}^T (Z_{(i),[j]}^T Z_{(i),[j]})^{-1} Z_{[j]} e_i}.$$
(C-78)

By Proposition E.5,

$$P\left(\sqrt{\lambda_{\max}\left(\frac{\mathcal{Z}_j^T \mathcal{Z}_j}{n}\right)} > 3C_1\right) \le 2e^{-C_2 n}.$$

On the other hand, apply (C-77) to $\mathcal{Z}_{(i),[j]}$, we have

$$P\left(\min_{|I|=\lfloor(1-\delta)n\rfloor}\lambda_{\min}\left(\frac{(\mathcal{Z}_{(i),[j]})_I^T(\mathcal{Z}_{(i),[j]})_I}{n}\right) < c_3\right) < c_5 e^{-c_6 n}.$$

A union bound indicates that with probability $(c_5 np + 2p)e^{-\min\{C_2, c_6\}n} = o(1)$,

$$\max_{j} \lambda_{\max}\left(\frac{\mathcal{Z}_{[j]}^T \mathcal{Z}_{[j]}}{n}\right) \le 9C_1^2, \quad \min_{i,j} \min_{|I| = \lfloor (1-\delta)n \rfloor} \lambda_{\min}\left(\frac{(\mathcal{Z}_{(i),[j]})_I^T (\mathcal{Z}_{(i),[j]})_I}{n}\right) \ge c_3.$$

This implies that for any j,

$$\lambda_{\max}\left(\frac{Z_{[j]}^T Z_{[j]}}{n}\right) = \lambda_{\max}\left(\frac{Z_{[j]}^T \Lambda^2 \mathcal{Z}_{[j]}}{n}\right) \le \zeta_{(1)}^2 \cdot 9C_1^2$$

and for any i and j,

$$\lambda_{\min}\left(\frac{Z_{(i),[j]}^{T}Z_{(i),[j]}}{n}\right) = \lambda_{\min}\left(\frac{Z_{(i),[j]}^{T}\zeta_{(i)}^{2}Z_{(i),[j]}}{n}\right)$$

$$\geq \min\{\zeta_{(\lfloor(1-\delta)n\rfloor)},\zeta_{(\lfloor(1-\delta)n\rfloor)}+1\}^{2}\cdot\min_{|I|=\lfloor(1-\delta)n\rfloor}\lambda_{\min}\left(\frac{(Z_{(i),[j]})_{I}^{T}\zeta_{(i)}^{2}(Z_{(i),[j]})_{I}}{n}\right)$$

$$\geq c_{3}\min\{\zeta_{(\lfloor(1-\delta)n\rfloor)},\zeta_{(\lfloor(1-\delta)n\rfloor)}+1\}^{2} > 0.$$

Moreover, as discussed above,

$$\zeta_{(1)} \le (\log n)^{\frac{2}{\alpha}}, \min\{\zeta_{(\lfloor (1-\delta)n \rfloor)}, \zeta_{(\lfloor (1-\delta)n \rfloor)} + 1\} \to F^{-1}(\delta)$$

almost surely. Thus, it follows from (C-78) that with high probability,

$$e_i^T Q_j e_i \ge \frac{K_0}{K_1} \cdot \frac{1}{1 + e_i^T Z_{[j]}^T (Z_{(i),[j]}^T Z_{(i),[j]})^{-1} Z_{[j]} e_i}$$

$$\ge \frac{K_0}{K_1} \cdot \frac{1}{1 + e_i^T \frac{Z_{[j]}^T Z_{[j]}}{n} e_i \cdot c_3 (F^{-1}(\delta))^2}$$

$$\ge \frac{K_0}{K_1} \cdot \frac{1}{1 + (\log n)^{\frac{4}{\alpha}} \cdot 9C_1^2 \cdot c_3 (F^{-1}(\delta))^2}.$$

The above bound holds for all diagonal elements of Q_j uniformly with high probability. Therefore,

$$\operatorname{tr}(\Lambda Q_{j}\Lambda) \geq \zeta_{(\lfloor (1-\delta)n \rfloor)}^{2} \cdot \lfloor (1-\delta)n \rfloor \cdot \frac{K_{0}}{K_{1}} \cdot \frac{1}{1 + (\log n)^{\frac{4}{\alpha}} \cdot 9C_{1}^{2} \cdot c_{3}(F^{-1}(\delta))^{2}} = \Omega_{p}\left(\frac{n}{\operatorname{polyLog}(\mathbf{n})}\right)$$

As a result, the assumption A4 is satisfied with high probability. Finally, by (C-76), we obtain that

$$\mathbb{E}\Delta_C^8 \leq \mathbb{E}\left[\max_{i=0,\dots,n;j=1,\dots,p} |\tilde{\alpha}_{j,i}^T \mathcal{Z}_j|^8 \cdot \|\Lambda\|_{\mathrm{op}}^8\right].$$

By Cauchy's inequality,

$$\mathbb{E}\Delta_C^8 \le \sqrt{\mathbb{E}\max_{i=0,\dots,n;j=1,\dots,p} |\tilde{\alpha}_{j,i}^T \mathcal{Z}_j|^{16}} \cdot \sqrt{\mathbb{E}\max_i \zeta_i^{16}}.$$

Similar to (C-72), we conclude that

$$\mathbb{E}\Delta_C^8 = O\left(\text{polyLog}(\mathbf{n})\right)$$

and by Markov inequality, the assumption A5 is satisfied with high probability.

C-4 More Results of Least-Squares (Section 5)

C-4.1 The Relation Between $S_j(X)$ and Δ_C

In Section 5, we give a sufficient and almost necessary condition for the coordinatewise asymptotic normality of the least-square estimator $\hat{\beta}^{LS}$; see Theorem 5.1. In this subsubsection, we show that Δ_C is a generalization of $\max_{j \in J_n} S_j(X)$ for general Mestimators.

Consider the matrix $(X^T D X)^{-1} X^T$, where D is obtain by using general loss functions, then by block matrix inversion formula (see Proposition E.1),

$$e_{1}^{T}(X^{T}DX)^{-1}X^{T} = e_{1}^{T} \begin{pmatrix} X_{1}^{T}DX_{1} & X_{1}^{T}DX_{[1]} \\ X_{[1]}^{T}DX_{1} & X_{[1]}^{T}DX_{[1]} \end{pmatrix}^{-1} \begin{pmatrix} X_{1}^{T} \\ X_{[1]}^{T} \end{pmatrix}$$
$$= \frac{X_{1}^{T}(I - DX_{[1]}(X_{[1]}^{T}DX_{[1]})^{-1}X_{[1]}^{T})}{X_{1}^{T}(D - DX_{[1]}(X_{[1]}^{T}DX_{[1]})^{-1}X_{[1]}^{T}D)X_{1}}$$
$$\approx \frac{X_{1}^{T}(I - D_{[1]}X_{[1]}(X_{[1]}^{T}D_{[1]}X_{[1]})^{-1}X_{[1]}^{T})}{X_{1}^{T}(D - DX_{[1]}(X_{[1]}^{T}DX_{[1]})^{-1}X_{[1]}^{T})X_{1}}$$

where we use the approximation $D \approx D_{[1]}$. The same result holds for all $j \in J_n$, then

$$\frac{\|e_j^T (X^T D X)^{-1} X^T\|_{\infty}}{\|e_j^T (X^T D X)^{-1} X^T\|_2} \approx \frac{\|X_1^T (I - D_{[1]} X_{[1]} (X_{[1]}^T D_{[1]} X_{[1]})^{-1} X_{[1]}^T)\|_{\infty}}{\|X_1^T (I - D_{[1]} X_{[1]} (X_{[1]}^T D_{[1]} X_{[1]})^{-1} X_{[1]}^T)\|_2}$$

Recall that $h_{j,1,i}^T$ is *i*-th row of $I - D_{[1]}X_{[1]}(X_{[1]}^T D_{[1]}X_{[1]})^{-1}X_{[1]}^T$, we have

$$\max_{i} \frac{|h_{j,1,i}^{T}X_{1}|}{\|h_{j,1,i}\|_{2}} \approx \frac{\|e_{j}^{T}(X^{T}DX)^{-1}X^{T}\|_{\infty}}{\|e_{j}^{T}(X^{T}DX)^{-1}X^{T}\|_{2}}.$$

The right-handed side equals to $S_j(X)$ in the least-square case. Therefore, although of complicated form, assumption A5 is not an artifact of the proof but is essential for the asymptotic normality.

C-4.2 Additional Examples

Benefit from the analytical form of the least-square estimator, we can depart from subgaussinity of the entries. The following proposition shows that a random design matrix Z with i.i.d. entries under appropriate moment conditions satisfies $\max_{j \in J_n} S_j(Z) = o(1)$ with high probability. This implies that, when X is one realization of Z, the conditions Theorem 5.1 are satisfied for X with high probability over Z.

Proposition C.6. If $\{Z_{ij} : i \leq n, j \in J_n\}$ are independent random variables with

- 1. $\max_{i \leq n, j \in J_n} (\mathbb{E}|Z_{ij}|^{8+\delta})^{\frac{1}{8+\delta}} \leq M$ for some $\delta, M > 0$;
- 2. $\min_{i < n, j \in J_n} \operatorname{Var}(Z_{ij}) > \tau^2$ for some $\tau > 0$
- 3. P(Z has full column rank) = 1 o(1);
- 4. $\mathbb{E}Z_j \in \text{span}\{Z_j : j \in J_n^c\}$ almost surely for all $j \in J_n$;

where Z_j is the *j*-th column of Z. Then

$$\max_{j \in J_n} S_j(Z) = O_p\left(\frac{1}{n^{\frac{1}{4}}}\right) = o_p(1).$$

A typical practically interesting example is that Z contains an intercept term, which is not in J_n , and Z_j has i.i.d. entries for $j \in J_n$ with continuous distribution and sufficiently many moments, in which case the first three conditions are easily checked and $\mathbb{E}Z_j$ is a multiple of $(1, \ldots, 1)$, which belongs to span $\{Z_j : j \in J_n^c\}$.

In fact, the condition 4 allows Proposition C.6 to cover more general cases than the above one. For example, in a census study, a state-specific fix effect might be added into the model, i.e.

$$y_i = \alpha_{s_i} + z_i^T \beta^* + \epsilon_i$$

where s_i represents the state of subject *i*. In this case, *Z* contains a sub-block formed by z_i and a sub-block with ANOVA forms as mentioned in Example 1. The latter is usually incorporated only for adjusting group bias and not the target of inference. Then condition 4 is satisfied if only Z_{ij} has same mean in each group for each *j*, i.e. $\mathbb{E}Z_{ij} = \mu_{s_i,j}$.

Proof of Proposition C.6. By Sherman-Morison-Woodbury formula,

$$e_j^T (Z^T Z)^{-1} Z^T = \frac{Z_j^T (I - H_j)}{Z_j^T (I - H_j) Z_j}$$

where $H_j = Z_{[j]} (Z_{[j]}^T Z_{[j]})^{-1} Z_{[j]}^T$ is the projection matrix generated by $Z_{[j]}$. Then

$$S_j(Z) = \frac{\|e_j^T(Z^T Z)^{-1} Z^T\|_{\infty}}{\|e_j^T(Z^T Z)^{-1} Z^T\|_2} = \frac{\|Z_j^T(I - H_j)\|_{\infty}}{\sqrt{Z_j^T(I - H_j)Z_j}}.$$
 (C-79)

Similar to the proofs of other examples, the strategy is to show that the numerator, as a linear contrast of Z_j , and the denominator, as a quadratic form of Z_j , are both concentrated around their means. Specifically, we will show that there exists some constants C_1 and C_2 such that

$$\max_{\substack{j \in J_n \\ \operatorname{tr}(A) = n-p+1}} \sup_{\substack{A \in \mathbb{R}^{n \times n}, A^2 = A, \\ \operatorname{tr}(A) = n-p+1}} \left\{ P\left(\|AZ_j\|_{\infty} > C_1 n^{\frac{1}{4}} \right) + P\left(Z_j^T A Z_j < C_2 n\right) \right\} = o\left(\frac{1}{n}\right). \quad (C-80)$$

If (C-80) holds, since H_j is independent of Z_j by assumptions, we have

$$P\left(S_j(Z) \ge \frac{C_1}{\sqrt{C_2}} \cdot n^{-\frac{1}{4}}\right) = P\left(\frac{\|Z_j^T(I - H_j)\|_{\infty}}{\sqrt{Z_j^T(I - H_j)Z_j}} \ge \frac{C_1}{\sqrt{C_2}} \cdot n^{-\frac{1}{4}}\right)$$

$$\leq P\left(\|(I-H_{j})Z_{j}\|_{\infty} > C_{1}n^{\frac{1}{4}}\right) + P\left(Z_{j}^{T}(I-H_{j})Z_{j} < C_{2}n\right)$$

$$= \mathbb{E}\left[P\left(\|(I-H_{j})Z_{j}\|_{\infty} > C_{1}n^{\frac{1}{4}}\right) \left|Z_{[j]}\right] + \mathbb{E}\left[P\left(Z_{j}^{T}(I-H_{j})Z_{j} < C_{2}n\right) \left|Z_{[j]}\right]\right] \quad (C-81)$$

$$\leq \sup_{A \in \mathbb{R}^{n \times n}, A^{2} = A, \operatorname{tr}(A) = n-p+1} P\left(\|AZ_{j}\|_{\infty} > C_{1}n^{\frac{1}{4}}\right) + P\left(Z_{j}^{T}AZ_{j} < C_{2}n\right)$$

$$\leq \max_{j \in J_{n}}\left\{\sup_{A \in \mathbb{R}^{n \times n}, A^{2} = A, \operatorname{tr}(A) = n-p+1} P\left(\|AZ_{j}\|_{\infty} > C_{1}n^{\frac{1}{4}}\right) + P\left(Z_{j}^{T}AZ_{j} < C_{2}n\right)\right\} = o\left(\frac{1}{n}\right)$$

$$(C-82)$$

Thus with probability $1 - o(|J_n|/n) = 1 - o(1)$,

$$\max_{j \in J_n} S_j(Z) \le \frac{C_1}{\sqrt{C_2}} \cdot n^{-\frac{1}{4}}$$

and hence

$$\max_{j\in J_n} S_j(Z) = O_p\left(\frac{1}{n^{\frac{1}{4}}}\right).$$

Now we prove (C-80). The proof, although looks messy, is essentially the same as the proof for other examples. Instead of relying on the exponential concentration given by the sub-gaussianity, we show the concentration in terms of higher-order moments.

In fact, for any idempotent A, the sum square of each row is bounded by 1 since

$$\sum_{i} A_{ij}^2 = (A^2)_{j,j} \le \lambda_{\max}(A^2) = 1.$$

By Jensen's inequality,

$$\mathbb{E}Z_{ij}^2 \le (\mathbb{E}|Z_{ij}|^{8+\delta})^{\frac{2}{8+\delta}}.$$

For any j, by Rosenthal's inequality (Rosenthal, 1970), there exists some universal constant C such that

$$\mathbb{E}\left|\sum_{i=1}^{n} A_{ij} Z_{ij}\right|^{8+\delta} \leq C \left\{\sum_{i=1}^{n} |A_{ij}|^{8+\delta} \mathbb{E}|Z_{ij}|^{8+\delta} + \left(\sum_{i=1}^{n} A_{ij}^2 \mathbb{E}Z_{ij}^2\right)^{4+\delta/2}\right\}$$
$$\leq C \left\{\sum_{i=1}^{n} |A_{ij}|^2 \mathbb{E}|Z_{ij}|^{8+\delta} + \left(\sum_{i=1}^{n} A_{ij}^2 \mathbb{E}Z_{ij}^2\right)^{4+\delta/2}\right\}$$
$$\leq C M^{8+\delta} \left\{\sum_{i=1}^{n} A_{ij}^2 + \left(\sum_{i=1}^{n} A_{ij}^2\right)^{4+\delta/2}\right\} \leq 2CM^{8+\delta}.$$

Let $C_1 = (2CM^{8+\delta})^{\frac{1}{8+\delta}}$, then for given *i*, by Markov inequality,

$$P\left(\left|\sum_{i=1}^{n} A_{ij} Z_{ij}\right| > C_1 n^{\frac{1}{4}}\right) \le \frac{1}{n^{2+\delta/4}}$$

and a union bound implies that

$$P\left(\|AZ_j\|_{\infty} > C_1 n^{\frac{1}{4}}\right) \le \frac{1}{n^{1+\delta/4}} = o\left(\frac{1}{n}\right).$$
 (C-83)

Now we derive a bound for $Z_j^T A Z_j$. Since $p/n \to \kappa \in (0, 1)$, there exists $\tilde{\kappa} \in (0, 1 - \kappa)$ such that $n - p > \tilde{\kappa}n$. Then

$$\mathbb{E}Z_{j}^{T}AZ_{j} = \sum_{i=1}^{n} A_{ii}\mathbb{E}Z_{ij}^{2} > \tau^{2}\operatorname{tr}(A) = \tau^{2}(n-p+1) > \tilde{\kappa}\tau^{2}n.$$
(C-84)

To bound the tail probability, we need the following result:

Lemma C.7 (Bai and Silverstein (2010), Lemma 6.2). Let B be an $n \times n$ nonrandom matrix and $W = (W_1, \ldots, W_n)^T$ be a random vector of independent entries. Assume that $\mathbb{E}W_i = 0$, $\mathbb{E}W_i^2 = 1$ and $\mathbb{E}|W_i|^k \leq \nu_k$. Then, for any $q \geq 1$,

$$E|W^{T}BW - \operatorname{tr}(B)|^{q} \le C_{q} \left(\left(\nu_{4} \operatorname{tr}(BB^{T}) \right)^{\frac{q}{2}} + \nu_{2q} \operatorname{tr}(BB^{T})^{\frac{q}{2}} \right),$$

where C_q is a constant depending on q only.

It is easy to extend Lemma C.7 to non-isotropic case by rescaling. In fact, denote σ_i^2 by the variance of W_i , and let $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$, $Y = (W_1/\sigma_1, \ldots, W_n/\sigma_n)$. Then

$$W^T B W = Y^T \Sigma^{\frac{1}{2}} B \Sigma^{\frac{1}{2}} Y,$$

with $\operatorname{Cov}(Y) = I$. Let $\tilde{B} = \Sigma^{\frac{1}{2}} B \Sigma^{\frac{1}{2}}$, then

$$\tilde{B}\tilde{B}^T = \Sigma^{\frac{1}{2}}B\Sigma B^T \Sigma^{\frac{1}{2}} \preceq \nu_2 \Sigma^{\frac{1}{2}}BB^T \Sigma^{\frac{1}{2}}.$$

This entails that

$$\operatorname{tr}(\tilde{B}\tilde{B}^T) \le nu_2 \operatorname{tr}(\Sigma^{\frac{1}{2}}BB^T\Sigma^{\frac{1}{2}}) = \nu_2 \operatorname{tr}(\Sigma BB^T) \le \nu_2^2 \operatorname{tr}(BB^T).$$

On the other hand,

$$\mathrm{tr}(\tilde{B}\tilde{B}^{T})^{\frac{q}{2}} \le n\lambda_{\max}(\tilde{B}\tilde{B}^{T})^{\frac{q}{2}} = n\nu_{2}^{\frac{q}{2}}\lambda_{\max}\left(\Sigma^{\frac{1}{2}}BB^{T}\Sigma^{\frac{1}{2}}\right)^{\frac{q}{2}} \le n\nu_{2}^{q}\lambda_{\max}(BB^{T})^{\frac{q}{2}}.$$

Thus we obtain the following result

Lemma C.8. Let B be an $n \times n$ nonrandom matrix and $W = (W_1, \ldots, W_n)^T$ be a random vector of independent mean-zero entries. Suppose $\mathbb{E}|W_i|^k \leq \nu_k$, then for any $q \geq 1$,

$$E|W^{T}BW - \mathbb{E}W^{T}BW|^{q} \le C_{q}\nu_{2}^{q}\left(\left(\nu_{4}\operatorname{tr}(BB^{T})\right)^{\frac{q}{2}} + \nu_{2q}\operatorname{tr}(BB^{T})^{\frac{q}{2}}\right),$$

where C_q is a constant depending on q only.

Apply Lemma C.8 with $W = Z_j$, B = A and $q = 4 + \delta/2$, we obtain that

$$E|Z_j^T A Z_j - \mathbb{E}Z_j^T A Z_j|^{4+\delta/2} \le C M^{16+2\delta} \left((\operatorname{tr}(AA^T))^{2+\delta/4} + \operatorname{tr}(AA^T)^{2+\delta/4} \right)$$

for some constant C. Since A is idempotent, all eigenvalues of A is either 1 or 0 and thus $AA^T \preceq I$. This implies that

$$\operatorname{tr}(AA^T) \le n, \quad \operatorname{tr}(AA^T)^{2+\delta/4} \le n$$

and hence

$$E|Z_j^T A Z_j - \mathbb{E}Z_j^T A Z_j|^{4+\delta/2} \le 2CM^{16+2\delta}n^{2+\delta/4}$$

for some constant C_1 , which only depends on M. By Markov inequality,

$$P\left(|Z_j^T A Z_j - \mathbb{E} Z_j^T A Z_j| \ge \frac{\tilde{\kappa} \tau^2 n}{2}\right) \le 2CM^{16+2\delta} \left(\frac{2}{\tilde{\kappa} \tau^2}\right)^{4+\delta/2} \cdot \frac{1}{n^{2+\delta/4}}$$

Combining with (C-84), we conclude that

$$P\left(Z_j^T A Z_j < C_2 n\right) = O\left(\frac{1}{n^{2+\delta/4}}\right) = o\left(\frac{1}{n}\right)$$
(C-85)

where $C_2 = \frac{\tilde{\kappa}\tau^2}{2}$. Notice that both (C-83) and (C-85) do not depend on j and A. Therefore, (C-80) is proved and hence the Proposition.

D Additional Numerical Experiments

In this section, we repeat the experiments in section 6 by using L_1 loss, i.e. $\rho(x) = |x|$. L_1 -loss is not smooth and does not satisfy our technical conditions. The results are displayed below. It is seen that the performance is quite similar to that with the huber loss.



Figure 5: Empirical 95% coverage of $\hat{\beta}_1$ with $\kappa = 0.5$ (left) and $\kappa = 0.8$ (right) using L_1 loss. The x-axis corresponds to the sample size, ranging from 100 to 800; the y-axis corresponds to the empirical 95% coverage. Each column represents an error distribution and each row represents a type of design. The orange solid bar corresponds to the case F = Normal; the blue dotted bar corresponds to the case $F = t_2$; the red dashed bar represents the Hadamard design.



Figure 6: Mininum empirical 95% coverage of $\hat{\beta}_1 \sim \hat{\beta}_{10}$ with $\kappa = 0.5$ (left) and $\kappa = 0.8$ (right) using L_1 loss. The x-axis corresponds to the sample size, ranging from 100 to 800; the y-axis corresponds to the minimum empirical 95% coverage. Each column represents an error distribution and each row represents a type of design. The orange solid bar corresponds to the case F = Normal; the blue dotted bar corresponds to the case $F = t_2$; the red dashed bar represents the Hadamard design.



Figure 7: Empirical 95% coverage of $\hat{\beta}_1 \sim \hat{\beta}_{10}$ after Bonferroni correction with $\kappa = 0.5$ (left) and $\kappa = 0.8$ (right) using L_1 loss. The x-axis corresponds to the sample size, ranging from 100 to 800; the y-axis corresponds to the empirical uniform 95% coverage after Bonferroni correction. Each column represents an error distribution and each row represents a type of design. The orange solid bar corresponds to the case F = Normal; the blue dotted bar corresponds to the case $F = t_2$; the red dashed bar represents the Hadamard design.

E Miscellaneous

In this appendix we state several technical results for the sake of completeness.

Proposition E.1 (Horn & Johnson, 2012, formula (0.8.5.6)). Let $A \in \mathbb{R}^{p \times p}$ be an invertible matrix and write A as a block matrix

$$A = \left(\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right)$$

with $A_{11} \in \mathbb{R}^{p_1 \times p_1}, A_{22} \in \mathbb{R}^{(p-p_1) \times (p-p_1)}$ being invertible matrices. Then

$$A^{-1} = \begin{pmatrix} A_{11} + A_{11}^{-1} A_{12} S^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} S^{-1} \\ -S^{-1} A_{21} A_{11}^{-1} & S^{-1} \end{pmatrix}$$

where $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur's complement.

Proposition E.2 (Rudelson & Vershynin, 2013; improved version of the original form by Hanson & Wright, 1971). Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a random vector with independent mean-zero σ^2 -sub-gaussian components X_i . Then, for every t,

$$P\left(|X^T A X - \mathbb{E} X^T A X| > t\right) \le 2 \exp\left\{-c \min\left(\frac{t^2}{\sigma^4 \|A\|_F^2}, \frac{t}{\sigma^2 \|A\|_{\mathrm{op}}}\right)\right\}$$

Proposition E.3 (Bai & Yin, 1993). If $\{Z_{ij} : i = 1, ..., n, j = 1, ..., p\}$ are *i.i.d.* random variables with zero mean, unit variance and finite fourth moment and $p/n \to \kappa$, then

$$\lambda_{\max}\left(\frac{Z^T Z}{n}\right) \stackrel{a.s.}{\to} (1+\sqrt{\kappa})^2, \quad \lambda_{\min}\left(\frac{Z^T Z}{n}\right) \stackrel{a.s.}{\to} (1-\sqrt{\kappa})^2.$$

Proposition E.4 (Latała, 2005). Suppose $\{Z_{ij} : i = 1, ..., n, j = 1, ..., p\}$ are independent mean-zero random variables with finite fourth moment, then

$$\mathbb{E}\sqrt{\lambda_{\max}\left(Z^{T}Z\right)} \leq C\left(\max_{i}\sqrt{\sum_{j}\mathbb{E}Z_{ij}^{2}} + \max_{j}\sqrt{\sum_{i}\mathbb{E}Z_{ij}^{2}} + \sqrt[4]{\sum_{i,j}\mathbb{E}Z_{ij}^{4}}\right)$$

for some universal constant C. In particular, if $\mathbb{E}Z_{ij}^4$ are uniformly bounded, then

$$\lambda_{\max}\left(\frac{Z^T Z}{n}\right) = O_p\left(1 + \sqrt{\frac{p}{n}}\right).$$

Proposition E.5 (Rudelson & Vershynin, 2010). Suppose $\{Z_{ij} : i = 1, ..., n, j = 1, ..., p\}$ are independent mean-zero σ^2 -sub-gaussian random variables. Then there exists a universal constant $C_1, C_2 > 0$ such that

$$P\left(\sqrt{\lambda_{\max}\left(\frac{Z^T Z}{n}\right)} > C\sigma\left(1 + \sqrt{\frac{p}{n}} + t\right)\right) \le 2e^{-C_2 n t^2}.$$

Proposition E.6 (Rudelson & Vershynin, 2009). Suppose $\{Z_{ij} : i = 1, ..., n, j = 1, ..., p\}$ are *i.i.d.* σ^2 -sub-gaussian random variables with zero mean and unit variance, then for $\epsilon \geq 0$

$$P\left(\sqrt{\lambda_{\min}\left(\frac{Z^T Z}{n}\right)} \le \epsilon (1 - \sqrt{\frac{p-1}{n}})\right) \le (C\epsilon)^{n-p+1} + e^{-cn}$$

for some universal constants C and c.

Proposition E.7 (Litvak et al., 2005). Suppose $\{Z_{ij} : i = 1, ..., n, j = 1, ..., p\}$ are independent σ^2 -sub-gaussian random variables such that

$$Z_{ij} \stackrel{d}{=} -Z_{ij}, \quad \operatorname{Var}(Z_{ij}) > \tau^2$$

for some $\sigma, \tau > 0$, and $p/n \to \kappa \in (0,1)$, then there exists constants $c_1, c_2 > 0$, which only depends on σ and τ , such that

$$P\left(\lambda_{\min}\left(\frac{Z^T Z}{n}\right) < c_1\right) \le e^{-c_2 n}.$$