ADIABATIC LIMITS OF ANTI-SELF-DUAL CONNECTIONS ON COLLAPSED K3 SURFACES

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ABSTRACT. We prove a convergence result for a family of Yang-Mills connections over an elliptic $K3$ surface M as the fibers collapse. In particular, assume M is projective, admits a section, and has singular fibers of Kodaira type I_1 and type II. Let Ξ_{t_k} be a sequence of $SU(n)$ connections on a principal $SU(n)$ bundle over M, that are anti-self-dual with respect to a sequence of Ricci flat metrics collapsing the fibers of M. Given certain non-degeneracy assumptions on the spectral covers induced by $\bar{\partial}_{\Xi_{t_k}}$, we show that away from a finite number of fibers, the curvature $F_{\Xi_{t_k}}$ is locally bounded in C^0, the connections converge along a subsequence (and modulo unitary gauge change) in L^p_1 to a limiting L^p_1 connection Ξ_0, and the restriction of Ξ_0 to any fiber is $C^{1,\alpha}$ gauge equivalent to a flat connection with holomorphic structure determined by the sequence of spectral covers. Additionally, we relate the connections Ξ_{t_k} to a converging family of special Lagrangian multi-sections in the mirror HyperKähler structure, addressing a conjecture of Fukaya in this setting.

1. Introduction

The adiabatic limit of anti-self-dual connections on 4-manifolds has been extensively studied by many authors, with various interesting applications to problems in gauge theory, geometry, and physics. In [22, 23], Dostoglou and Salamon proved the Atiyah-Floer conjecture (see [5]) by showing that the adiabatic limits of self-dual connections on the product of $\mathbb{R}$ and the mapping cylinder of a principal $SO(3)$-bundle over a compact Riemann surface of higher genus (greater than one) produce holomorphic curves in the moduli space of flat connections on the $SO(3)$-bundle. Later, the behavior of anti-self-dual $SU(n)$-connections along the adiabatic degenerations of the product of two compact Riemann surfaces of higher genus was studied in [11] and [60] respectively, which gave mathematical rigorous proofs of the reduction from the 4-dimensional Yang-Mills theory to 2-dimensional sigma models discovered by physicists (cf. [9]). Based on previous works of gauge theory on higher dimensional manifolds [21, 65], [12] generalized the 4-dimensional case to complex anti-self-dual connections on products of

* Supported in part by NSF RTG grant DMS-1344991.
† Supported in part by a grant from the Hellman Foundation.
‡ Supported by the Simons Foundation's program Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics (grant #488620).

Calabi-Yau surfaces. The Atiyah-Floer conjecture was studied in [25] for principal $PU(n)$-bundles.

Another motivation for the study of adiabatic limits of anti-self-dual connections arises in the context of the mirror symmetry. In [64], Strominger, Yau and Zaslow proposed a conjecture, called the SYZ conjecture, for constructing mirror Calabi-Yau manifolds via dual special Lagrangian fibrations. Gross, Wilson, Kontsevich, Soibelman and Todorov [45, 54, 55] proposed an alternative version of the SYZ conjecture by using the collapsing of Ricci-flat Kähler metrics. Motivated by the study of homological mirror symmetry, a gauge theory analogue of the collapsing of Ricci-flat Kähler metrics was conjectured by Fukaya (Conjecture 5.5 in [34]), which relates the adiabatic limits of anti-self-dual connections on Calabi-Yau manifolds to special Lagrangian cycles on the mirror Calabi-Yau manifolds. This conjecture was studied in the preprints [31, 58] for Hermitian-Yang-Mills connections on 2-dimensional complex torus, and in [14] for the case of Hermitian-Yang-Mills connections on higher dimensional semi-flat Calabi-Yau manifolds. The present paper proves a version of Fukaya's conjecture for anti-self-dual connections on elliptically fibered K3 surfaces.

Let M be a projective elliptically fibered $K3$ surface, $f : M \to N \cong \mathbb{CP}^1$, admitting a section $\sigma : N \to M$. Let α be an ample class on M, $\alpha_t = t\alpha + f^*c_1(\mathcal{O}_{\mathbb{CP}^1}(1))$, $t \in (0,1]$, and let $\omega_t \in \alpha_t$ be the unique Ricci-flat Kähler-Einstein metric in this class (from [77]). We denote by g_t the corresponding Riemannian metric of ω_t, which is a HyperKähler metric. The limit behavior of ω_t as $t \to 0$ was studied by Gross and Wilson in [45], for K3 surfaces with only type I_1 singular fibers. This was generalized to any elliptically fibered $K3$ surface in [66, 41, 43]. More precisely, if $N_0 \subseteq N$ denotes the complement of the discriminant locus of f, i.e. for any $w \in N_0$ the fiber $M_w = f^{-1}(w)$ is a smooth elliptic curve, then it is proved in [41] that ω_t converges to $f^*\omega$ in the locally C^∞-sense on $M_{N_0} = f^{-1}(N_0)$, where ω is a Kähler metric on N_0 with Ricci curvature $\mathrm{Ric}(\omega) = \omega_{WP}$ (obtained previously by [62, 66]), and ω_{WP} denotes the Weil-Petersson metric of the fibers of f. Furthermore, (M, ω_t) converges to a compact metric space Y homeomorphic to N in the Gromov-Hausdorff sense [43].

Assume that $f : M \to N$ has only singular fibers of Kodaira type I_1 and type II. Let P be a principal $SU(n)$-bundle on M, and $(\mathcal{V}, H)$ be the smooth Hermitian vector bundle of rank n obtained by the twisted product, i.e. $\mathcal{V} \cong P \times_\rho \mathbb{C}^n$ where ρ is the standard $SU(n)$ representation on $\mathbb{C}^n$. Assume that there is a family of anti-self-dual connections Ξ_t on P with respect to g_t, for $t \in (0,1]$. This is equivalent to the curvature F_{Ξ_t} satisfying

$$F_{\Xi_t} \wedge \omega_t = 0, \quad \text{and} \quad F_{\Xi_t} \wedge \Omega = 0,$$

where Ω is a holomorphic symplectic form on M. For each $t \in (0,1]$, Ξ_t induces a holomorphic structure on $\mathcal{V}$, and we denote the resulting holomorphic bundle of rank n as V_t.

Under some non-degeneracy assumptions on the behavior of V_t, the main result of this paper, Theorem 3.1, asserts that for any sequence $t_k \to 0$, there exists a Zariski open subset $N^o \subset N_0$ such that $u_k(\Xi_{t_k})$ converges subsequentially to Ξ_0 in the locally $C^{0,\alpha}$-sense on M_{N^o}, for some sequence of unitary gauge transformations u_k on P. Furthermore, the restriction of the limit Ξ_0 to any fiber is unitary gauge equivalent to a smooth flat $SU(n)$-connection induced by a holomorphic curve in M, which can be regarded as a multi-section of f. Furthermore, Ξ_0 is the Fourier-Mukai transform of a certain flat $U(1)$-connection on the multi-section. We refer the reader to Theorem 3.1 and Theorem 3.2 for more precise statements. By performing the HyperKähler rotation, we can use this result to show a version of Fukaya's conjecture, relating the connections Ξ_{t_k} to a converging family of special Lagrangian multi-sections in the mirror HyperKähler structure.

In comparison to previous results on the adiabatic limits of anti-self-dual connections, including, for example [22, 11, 60, 32], one essential difficulty we encounter is that the moduli space $\mathfrak{M}_E(n)$ of flat $SU(n)$-connections on a smooth elliptic curve is not smooth, and actually, the whole $\mathfrak{M}_E(n)$ is degenerated, i.e. there is no smooth point (cf. [59]). Specifically, since every flat connection is gauge equivalent to a reducible connection, Poincaré type inequalities may not follow, creating immense analytic difficulties. The same issue also appears for the case of $T^4 = \mathbb{C}^2/\mathbb{Z}^4$ as in [31, 58]. To overcome this, we take a totally different approach from [31, 58], which is inspired by the study of collapsing of Einstein 4-manifolds [1, 15]. In addition we adapt some of the arguments from [22, 23], as suggested in [34].

Fortunately, in the literature there is a very satisfactory theory about the moduli spaces of semi-stable holomorphic bundles of rank n on elliptic curves in algebraic geometry. In the proof of Theorem 3.1, we utilize the well understood results of holomorphic bundles on elliptic fibered surfaces in [30, 29, 28], as opposed to the pseudo-holomorphic curve theory in symplectic geometry used in [22, 58]. Additionally, in the course of our analysis, we obtain a Poincaré type inequality for the curvatures of $SU(n)$-connections on smooth elliptic curves, which relies on the earlier work of the first two named authors (cf. [17]). This enables us to generalize certain arguments of [22] to the present case. Finally, the small energy estimates for sufficiently collapsed Einstein 4-manifolds developed in [1] can be adapted to the case of Yang-Mills connections on collapsed 4-manifolds, which is used to finish the proof of the main theorem.

Here we outline the paper briefly. Section 2 reviews the background notions, and preliminary results, which are needed for the main theorem. We recall the standard background on gauge theory in Section 2.1, and the theory of holomorphic vectors bundles on elliptic curves in Section 2.2. Section 2.3 reviews the previous work about the gauge fixing on elliptic curves by the first two named authors, which is one essential ingredient in the proof of the main result of the present paper. Section 2.4 recalls the work of

Friedman-Morgan-Witten [30, 29], where the relationship between holomorphic bundles and spectral covers on elliptic surfaces is established. This work is the algebro-geometric input needed to overcome the difficulty of non-smoothness of the moduli spaces of flat connections. In Section 2.5, we set up some notations for the collapsing of Ricci-flat Kähler Einstein metrics on K3 surfaces, and leave more detailed discussions to the Appendix. Section 2.6 reviews the notion of Fourier-Mukai transform. We adapt the small energy estimates for sufficiently collapsed Einstein 4-manifolds by Anderson [1] to the present case in Section 2.7.

Section 3 is devoted to the main theorems of this paper. We state the main theorems, and in Section 3.1, we apply the main theorems to the SYZ mirror symmetry for K3 surfaces, which proves a version of Fukaya's conjecture in [34]. Section 4 contains the proof of Theorem 3.1 assuming some important a priori estimates, which are established in the sections that follow. Section 5 contains the key analytic result of the paper, namely the Poincaré type inequality mentioned above. In Section 6, we obtain a C^0-bound for curvature under the assumption of a certain decay rate of curvatures as the fibers collapse. Section 7 studies the relationship between the energy of curvature and the spectral covers. In Section 8, we use a blowup argument to prove the desired curvature decay rate, thereby completing the proof of Theorem 3.1. Section 9 proves Theorem 3.2.

Finally, the appendix has some results of independent interest, where we study the collapsing rate of Ricci-flat Kähler-Einstein metrics on general Abelian fibered Calabi-Yau manifolds. Here we improve on the previous results of [41, 43, 69].

Acknowledgements: We would like to thank Mark Haskins for introducing the authors to the question, and some valuable comments. The work was initiated when the second and the third named author attended the First Annual Meeting 2017 of the Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics. We thank the Simons Foundation and the organisers of the meeting for providing this opportunity. We also thank Simon Donaldson, Mark Gross, Valentino Tosatti, Yuuji Tanaka, and Michael Singer for some discussions.

2. Preliminaries

In this section, we review the various notions, and preliminary results, which are needed for the main theorem. Although there is quite a bit of background to cover, we find it necessary to provide all the important details before we can state our results.

Let M be a projective, elliptically fibered $K3$ surface. Denote the fibration by $f : M \to N \cong \mathbb{CP}^1$. Assume f admits a section $\sigma : N \to M$, and furthermore assume f has only singular fibers of Kodaira type I_1 and type II. Let I denote the holomorphic structure on M for which f is holomorphic.

We denote by S_N the discriminant locus f, and $N_0 = N \backslash S_N$ the regular locus. The preimage of the regular locus is denoted by $M_0 := f^{-1}(N_0)$. For any point $w \in N$, the fiber over this point is written $M_w := f^{-1}(w)$. Additionally, for any subset $U \subset N$, we use the notation $M_U := f^{-1}(U)$.

Let P be a principal $SU(n)$-bundle on M, and $\mathcal{V}$ be the smooth vector bundle of rank n equipped with an Hermitian metric H induced by P, i.e. $\mathcal{V} = P \times_\rho \mathbb{C}^n$, where ρ is the standard unitary representation of $SU(n)$ on $\mathbb{C}^n$. Note that first Chern class of $\mathcal{V}$ vanishes, i.e. $c_1(\mathcal{V}) = 0$.

For computing norms it is convenient to use a fixed Kähler form ω on M, which lies in a fixed Kähler class α. Unless otherwise specified, all norms are computed with respect to ω and H. We let $\langle \cdot, \cdot \rangle_w$ denote the inner product of the space of forms induced by $\omega|_{M_w}$ on the fiber M_w, and $\| \cdot \|_w$ the respective L^2-norm on M_w.

Throughout the paper, we let C denote constants, which only depend on fixed background data, whose value may change from line to line. The constants may depend on a compact or open sets contained in N, and this dependence is either explicitly stated, or clear from context.

2.1. **Anti-self-dual connections.** We begin by recalling the standard background on anti-self-dual connections, and readers are referred to texts [6, 20, 27, 52] for details.

Given the definition of P above, let Ξ be a connection on P, or an $SU(n)$-connection of $\mathcal{V}$. If the curvature F_Ξ satisfies

$$F_\Xi^{0,2} = 0, \quad \text{or equivalently} \quad F_\Xi = F_\Xi^{1,1},$$

then Ξ induces a holomorphic structure on $\mathcal{V}$. We denote the resulting holomorphic bundle as V_Ξ, and $\bar{\partial}_\Xi$ the corresponding Cauchy-Riemann operator. Specifically, we can write the covariant derivative $d_\Xi : C^\infty(\wedge^q T^*M \otimes \mathcal{V}) \to C^\infty(\wedge^{q+1} T^*M \otimes \mathcal{V})$ as $d_\Xi = \partial_\Xi + \bar{\partial}_\Xi$, and the Cauchy-Riemann operator is the $(0,1)$-component. By construction Ξ is the unique Chern connection induced by H and $\bar{\partial}_\Xi$.

Let $\mathcal{A}^{1,1}$ be the space of all unitary connections with vanishing $(0,2)$-component of curvatures on P, so for any $\Xi \in \mathcal{A}^{1,1}$, we have $F_\Xi^{0,2} = 0$. If $\mathcal{G}$ denotes the unitary gauge group, i.e. the space of unitary automorphisms of $\mathcal{V}$ covering the identity on M, then $\mathcal{G}$ acts on $\mathcal{A}^{1,1}$ by

$$u(\Xi) = \Xi + u^{-1}(d_\Xi u),$$

for $u \in \mathcal{G}$ and $\Xi \in \mathcal{A}^{1,1}$. The $\mathcal{G}$-action extends to an action of the complex gauge group $\mathcal{G}_\mathbb{C}$, which consists all automorphisms of $\mathcal{V}$ covering the identity on M, on $\mathcal{A}^{1,1}$ by

$$g(\Xi) = \Xi + g^{-1}(\bar{\partial}_\Xi g) - (g^{-1}(\bar{\partial}_\Xi g))^*,$$

for $g \in \mathcal{G}_\mathbb{C}$, where $(\cdot)^*$ denotes the conjugate transpose. Any two connections Ξ_1 and $\Xi_2 \in \mathcal{A}^{1,1}$ induce isomorphic holomorphic structures on $\mathcal{V}$ if and only if $\Xi_1 = g(\Xi_2)$ for a certain $g \in \mathcal{G}_\mathbb{C}$. Therefore the quotient space $\mathcal{A}^{1,1}/\mathcal{G}_\mathbb{C}$ parameterizes the holomorphic structures on $\mathcal{V}$.

Note that if $g \in \mathcal{G}_{\mathbb{C}}$ is an Hermitian gauge, i.e. $g = g^*$, then for any $\Xi \in \mathcal{A}^{1,1}$, the curvature transforms via

$$\begin{aligned} F_{g(\Xi)} &= F_\Xi + \partial_\Xi(g^{-1}(\bar{\partial}_\Xi g)) - \bar{\partial}_\Xi((\partial_\Xi g)g^{-1}) \\ &\quad + \partial_\Xi g g^{-2}\bar{\partial}_\Xi g - g^{-1}\bar{\partial}_\Xi g \partial_\Xi g g^{-1}, \end{aligned}$$

where $F_{g(\Xi)}$ is the curvature of the connection $g(\Xi)$. The transformation of Ξ to $g(\Xi)$ by a Hermitian gauge g is equivalent to fixing the holomorphic structure on a bundle V, and then changing the Hermitian metric (see [19] for details).

Given a Kähler class α on M, choose a Kähler form $\omega \in \alpha$, and let g be the corresponding Riemannian metric.

Definition 2.1. *An $SU(n)$-connection Ξ is called anti-self-dual with respect to the Kähler metric ω if Ξ satisfies the equation*

$$\star_{\mathrm{g}} F_\Xi = -F_\Xi, \tag{2.1}$$

where $\star_{\mathrm{g}}$ denotes the Hodge star operator of g.

For any anti-self-dual connection, Chern-Weil theory gives

$$\int_M |F_\Xi|^2_\omega \omega^2 = -\int_M \mathrm{tr}(F_\Xi \wedge F_\Xi) = 8\pi^2 c_2(\mathcal{V}). \tag{2.2}$$

Furthermore, anti-self-dual connections are absolute minima of the Yang-Mills functional on P, and thus satisfy the Yang-Mills equations

$$d_\Xi F_\Xi = 0, \quad \text{and} \quad d^*_\Xi F_\Xi = 0.$$

This implies the following Weitzenböck formula for the curvature of Ξ

$$0 = \Delta_\Xi F_\Xi = \nabla^*_\Xi \nabla_\Xi F_\Xi + R_\omega \# F_\Xi + F_\Xi \# F_\Xi. \tag{2.3}$$

Here R_ω denotes the Riemannian curvature of ω, and $S\#T$ denotes some algebraic bilinear expression involving the tensors S and T, where the exact form is not important for the present paper.

In complex dimension 2, a connection Ξ is anti-self-dual if and only if it is Hermitian-Yang-Mills [20], which is given by the following set of equations

$$F^{1,1}_\Xi \wedge \omega = 0, \qquad \text{and} \qquad F^{0,2}_\Xi = 0. \tag{2.4}$$

Thus an anti-self-dual connection Ξ induces a holomorphic structure on $\mathcal{V}$, and we denote the resulting holomorphic vector bundle as V_Ξ.

For a given Kähler class α on M, a holomorphic vector bundle V is called α-stable (respectively α-semi-stable), if for any proper torsion-free coherent subsheaf $\mathcal{F}$, the following inequality holds

$$\frac{c_1(\mathcal{F}) \cdot \alpha}{\mathrm{rank}(\mathcal{F})} < \frac{c_1(V) \cdot \alpha}{\mathrm{rank}(V)} \quad (\text{respectively} \quad \le).$$

Fundamental work of Donaldson, Uhlenbeck, and Yau, asserts the equivalence between stability and the existence of Hermitian-Yang-Mills connections (cf. [19, 72]). In particular, we state the following Theorem, restricted to the $SU(n)$ case.

Theorem 2.2 (Donaldson [19], Uhlenbeck-Yau [72]). *Let $(\mathcal{V}, H)$ be the smooth Hermitian bundle induced by a principal $SU(n)$-bundle P, α be a Kähler class on M, and $\omega \in \alpha$ a Kähler metric. If the holomorphic bundle V determined by a $\mathcal{G}_{\mathbb{C}}$-orbit O in $\mathcal{A}^{1,1}$ is α-stable, then O contains an anti-self-dual connection (equivalently a Hermitian-Yang-Mills connection). Furthermore, this connection is unique up to unitary gauge transformations. Conversely, if Ξ is an anti-self-dual connection with respect to ω, and the holomorphic bundle V_Ξ induced by Ξ is irreducible, then V_Ξ is α-stable.*

Note that if ω is a Ricci-flat Kähler-Einstein metric, then the corresponding Riemannian metric g is a HyperKähler metric, and $(\omega, \mathrm{Re}(\Omega), \mathrm{Im}(\Omega))$ is a HyperKähler triple (cf. [42]), where Ω is a holomorphic symplectic form such that

$$\omega^2 = \mathrm{Re}(\Omega)^2 = \mathrm{Im}(\Omega)^2, \quad \omega \wedge \Omega = 0, \quad \text{and} \quad \mathrm{Re}(\Omega) \wedge \mathrm{Im}(\Omega) = 0.$$

Complex structures making g HyperKähler are parameterized by S^2, and any anti-self-dual connection Ξ with respect to g is also a Hermitian-Yang-Mills connection with respect to any such complex structure. In the HyperKähler case, the anti-self-dual equation (2.1) and the Hermitian-Yang-Mills equation (2.4) are equivalent to the following system

$$F_\Xi \wedge \omega = 0, \quad \text{and} \quad F_\Xi \wedge \Omega = 0. \tag{2.5}$$

For the remainder of the paper, we mainly work with the above equations, as they are the most applicable to our setup.

The above equations (2.5) are given with respect to the complex structure I making $f : M \to N$ holomorphic. By the HyperKähler rotation, we have another complex structure J such that the holomorphic symplectic form $\Omega_J = \mathrm{Im}(\Omega) + i\omega$, and the Kähler form $\omega_J = \mathrm{Re}(\Omega)$. If Ξ is an anti-self-dual connection with respect to g, then Ξ also satisfies $F_\Xi \wedge \omega_J = 0$, and $F_\Xi \wedge \Omega_J = 0$. Thus Ξ induces a holomorphic bundle structure on $\mathcal{V}$ with respect to the complex structure J, denoted as $V_{\Xi,J}$, and Ξ is a Hermitian-Yang-Mills connection on $V_{\Xi,J}$.

We conclude this section by recalling Uhlenbeck's compactness theorems, which are divided into the cases of weak and strong compactness.

Theorem 2.3 (Uhlenbeck [71, 75]). *Let K be a compact subset of M.*

i) [Weak compactness] *If Ξ_k is a sequence of unitary connections on $P|_K$ such that $\|F_{\Xi_k}\|_{L^p} \leq C$, for $p > 2$, then there exists a sequence of unitary gauge transformations $u_k \in \mathcal{G}^{2,p}$ so that $u_k(\Xi_k)$ converges along a subsequence in $L^p_{1,loc}$ to a L^p_1-unitary connection Ξ_∞ on K, where $\mathcal{G}^{2,p}$ denotes the space of L^p_2-unitary gauge changes.*

ii) [Strong compactness] *If we further assume that Ξ_k is anti-self-dual with respect to a Riemannian metric g_k, and g_k converges smoothly to a smooth Riemannian metric g_∞ locally on K, then $u_k(\Xi_k)$ converges to Ξ_∞ in the locally C^∞-sense, and Ξ_∞ is anti-self-dual with respect to g_∞.*

2.2. **Gauge theory on elliptic curves.** While working with bundles over M, we need several preliminary results dealing with the restriction of a bundle to a fixed elliptic fiber, which we detail here.

Fix a point $w \in N_0$, and consider the fiber $M_w = E$, a smooth elliptic curve with period τ, i.e. $E = \mathbb{C}/\mathrm{Span}_{\mathbb{Z}}\{1, \tau\}$. Equip E with the flat metric $\omega_w^F := i\mathrm{Im}(\tau)^{-1}\, dz \wedge d\bar{z}$. Let V be a holomorphic vector bundle of rank n with trivial determinant line bundle $\bigwedge^n V \cong \mathcal{O}_E$, let $\bar{\partial}$ be the Cauchy-Riemann operator, and fix a Hermitian metric H on V. Let A_{ch} be the unique Chern connection determined by the holomorphic structure and the Hermitian metric H, i.e. $A_{ch} = (\partial H)H^{-1}$ under a certain local holomorphic trivialization. Recall that $\|\cdot\|_w$ denotes the L^2 norm on E.

Proposition 2.4. *There exists a $\delta > 0$, dependent only on E and V, so that if A is in the complexified gauge orbit of A_{ch} and satisfies $\|F_A\|_w < \delta$, then the holomorphic bundle V is semi-stable.*

Proof. This proposition follows from the fact, proven by Råde, that the critical values of the Yang-Mills functional (the L^2 norm of the curvature) are discrete, and that in real dimension 2 and 3 the Yang-Mills flow converges in L^2_1 [61]. If A satisfies $\|F_A\|_w < \delta$ for δ sufficiently small, then the Yang-Mills flow starting at A must converge to a flat connection A_0, by discreteness of critical values. Thus $\|F_{A(t)}\|_w \to 0$, where $A(t)$ denotes the flow of connections. Furthermore, the Yang-Mills flow preserves the complex gauge equivalence class of A, so $A(t)$ all define isomorphic holomorphic structures on V. As a result, V admits an approximate Hermitian-Einstein structure, and is semi-stable [52]. $\square$

Although the Yang-Mills flow preserves the complex gauge equivalence class of A, it is not immediately clear whether the limiting flat connection A_0 is contained in the complexified gauge orbit, or only strictly in the closure. To better understand this, we turn to Atiyah's classification of semi-stable bundles on an elliptic curve.

Let $0 \in E$ the identity of the group law. Denote the trivial line bundle by $\mathcal{O}_E$, and given a point $q \in E$, let $\mathcal{O}_E(q - 0)$ be the line bundle associated to the divisor $q - 0$. Define $\mathcal{I}_r$ inductively, with $\mathcal{I}_1 = \mathcal{O}_E$ and $\mathcal{I}_r$ the unique nontrivial extension of $\mathcal{I}_{r-1}$ by $\mathcal{O}_E$.

Theorem 2.5 (Atiyah [4])**.** *Any semi-stable bundle V over E with trivial determinant bundle is isomorphic to a direct sum of bundles of the form $\mathcal{O}_E(q - 0) \otimes \mathcal{I}_r$, i.e.*

$$V \cong \bigoplus_{j=1}^{\ell} \mathcal{O}_E(q_j - 0) \otimes \mathcal{I}_{r_j}.$$

Definition 2.6. *A semi-stable bundle V is called regular if it is of the form $V \cong \bigoplus_{j=1}^{\ell} \mathcal{O}_E(q_j - 0) \otimes \mathcal{I}_{r_j}$ with $q_j \neq q_i$ for any $j \neq i$.*

Now, in our setting one (and only one) of two things can happen. Either V is isomorphic a direct sum of line bundles $V = \oplus \mathcal{O}_E(q-0)$, and the limiting flat connection A_0 is in the complex gauge orbit of A, or V is isomorphic to a direct sum of bundles of the form $\mathcal{O}_E(q-0) \otimes \mathcal{I}_r$, with at least one $r > 1$. In the latter case, $\mathcal{O}_E(q-0) \otimes \mathcal{I}_r$ is strictly semi-stable, since $\mathcal{O}_E(q-0) \subset \mathcal{O}_E(q-0) \otimes \mathcal{I}_r$ has degree zero but $\mathcal{O}_E(q-0) \otimes \mathcal{I}_r$ does not split holomorphically. As a result V does not admit a flat connection, and so A is not complex gauge equivalent to A_0.

Note that if $V \cong \bigoplus_{j=1}^{\ell} \mathcal{O}_E(q_j - 0) \otimes \mathcal{I}_{r_j}$, then V is *S-equivalent* to the flat bundle $\bigoplus_{j=1}^{\ell} \mathcal{O}_E(q_j - 0)^{\oplus r_j}$ (see [28] for the precise definition of S-equivalence). Every S-equivalence class corresponds to a divisor $\sum_{j=1}^{\ell} r_j q_j$ in the complete linear system $|n0|$. Conversely, any divisor $\sum_{j=1}^{\ell} r_j q_j \in |n0|$ on E induces an S-equivalence class of semi-stable bundles with trivial determinant, which contains $\bigoplus_{j=1}^{\ell} \mathcal{O}_E(q_j - 0)^{\oplus r_j}$. Therefore, the moduli space of S-equivalence classes of semi-stable bundles with trivial determinant is given by the complete linear system $|n0| \cong \mathbb{CP}^{n-1}$.

Furthermore, the moduli space of flat line bundles on E is the dual torus $\check{E} \cong H^{0,1}(E)/H^1(E, \mathbb{Z})$, and we identify E and $\check{E}$ by $q \mapsto \mathcal{O}_E(q-0)$. Another way to state this is that a point $q \in E$ corresponds to a flat connection $\pi(\mathrm{Im}\tau)^{-1}(q d\bar{z} - \bar{q} dz)$ on the trivial Hermitian bundle $E \times \mathbb{C}$. Therefore the flat bundle structure of $\bigoplus_{j=1}^{n} \mathcal{O}_E(q_j - 0)$ is given by the flat connection

$$A_0 = \pi(\mathrm{Im}\tau)^{-1}(\mathrm{diag}\{q_1, \cdots, q_n\} d\bar{z} - \mathrm{diag}\{\bar{q}_1, \cdots, \bar{q}_n\} dz), \tag{2.6}$$

where $\sum_{j=1}^{n} q_j \in |n0|$. Note that the above connection has this form in a global unitary frame for V. Let $\mathfrak{M}_E(n)$ denote the moduli space of flat $SU(n)$ connections on V, which is naturally identified with $|n0|$, the moduli space of S-equivalence classes of semi-stable bundles with trivial determinant.

We note that from the perspective of algebraic geometry, the linear system $|n0|$ is a well behaved object. On the other hand, from the perspective of symplectic geometry, the moduli space $\mathfrak{M}_E(n)$ is quite complicated. In particular, any flat $SU(n)$-connection on E is degenerate, the virtual dimension of $\mathfrak{M}_E(n)$ is zero, and the whole space $\mathfrak{M}_E(n)$ is regarded as singular, i.e. there is no smooth point (cf. [58, 60]). If we let $\mathcal{A}$ denote the space of all unitary connections on the trivial bundle on E, and $\mathcal{G}$ the unitary gauge group, then following Atiyah-Bott [6], one can construct $\mathfrak{M}_E(n)$ as the symplectic reduction $\mathfrak{M}_E(n) = \{A \in \mathcal{A} | F_A = 0\}/\mathcal{G}$. Using this construction

$\mathfrak{M}_E(n)$ is in the singular locus of $\mathcal{A}/\mathcal{G}$. Such ill behavior of $\mathfrak{M}_E(n)$ prevents us to generalize the arguments in [11, 22, 32, 59] directly, where the moduli space of flat connections on Riemann surfaces of higher genus are considered. Instead we follow an algebro-geometric approach combined with estimates for the above non-linear partial differential equations.

2.3. **Gauge fixing.** In this section we continue to work on a single elliptic curve (E,ω). Let V be a regular, semi-stable, holomorphic vector bundle of rank n which admits a flat connection A_0, equipped with a Hermitian metric H. Suppose A is another connection in the complex gauge orbit of A_0, i.e. $A = g(A_0)$ for some $g \in \mathcal{G}_{\mathbb{C}}$. It will be important for us to know under what conditions we have control over the C^0 norm of g. Since the action of a fixed unitary gauge transformation will not affect this norm, without loss of generality we assume that $A = e^s(A_0)$ for a trace free Hermitian endomorphism s.

In general it is not reasonable to expect direct control of s. For example, if e^s were a diagonal matrix of constants $c_1, ..., c_n$ in the trivial frame, then $e^s(A_0)$ will also be a flat connection. However, one eigenvalue c_i can be arbitrarily large while still preserving the condition that s be trace free, so s cannot be controlled. What does end up being true is that under a small curvature assumption, there exists a normalized endomorphism $\hat{s}$, which may be distinct from s, that nevertheless gives the same connection under the complexified gauge group action, and is uniformly controlled in C^0. The key result of the first two named authors is as follows.

Theorem 2.7 (Datar-Jacob [17])**.** *Let $e^s(A_0)$ be a connection on V given by the action of a trace free Hermitian endomorphism s. There exists constants $\epsilon_0 > 0$, and $C_0 > 0$, depending only on ω, A_0, and H, so that if*

$$\|F_{e^s(A_0)}\|^2_{C^0(E)} \leq \epsilon_0,$$

then there exists another trace free Hermitian endomorphism $\hat{s}$ satisfying that $\hat{s}$ is perpendicular to the Kernel of d_{A_0}, in addition to

$$e^s(A_0) = e^{\hat{s}}(A_0) \qquad \text{and} \qquad \|\hat{s}\|_{C^0(E)} \leq C_0.$$

We remark that the assumptions that V be regular and admit a flat connection are critical, as they imply that the holomorphic automorphism group of V is precisely n dimensional [30]. The idea of the proof is that the linearization of the complex gauge group action of a Hermitian endomorphism on A_0 is $\star d_{A_0} s$. Restricting to endomorphisms perpendicular to the Kernel of d_{A_0}, a Poincaré inequality gives that the linearized map is invertible with bounded inverse. Thus, if $e^s(A_0)$ is sufficiently close to A_0, via the contraction mapping principle the results of the theorem hold. In order for the theorem to hold under the small curvature assumption, a connectedness argument is applied. We direct the reader to [17] for further details.

2.4. **Spectral covers.** We now discuss holomorphic vector bundles over our elliptic fibration M, as opposed to a single elliptic curve.

We assume that $f : M \to N$ has only singular fibers of Kodaira type I_1 and type II. Then M coincides with the Weierstrass model $\check{f} : \check{M} \to N$, i.e. $M = \check{M}$ and $f = \check{f}$ (cf. Definition 18 of Chapter 7 in [28]). Let V be a holomorphic vector bundle V of rank n on M such that the determinant line bundle $\bigwedge^n V$ is trivial, i.e. $\bigwedge^n V \cong \mathcal{O}_M$. If the restriction of V on the generic fiber of f is regular semi-stable, then a multi-valued section of f is constructed in [30], which is called the spectral cover associated to V. More precisely, we have the following theorem.

Theorem 2.8 ([30])**.** *Assume that the restriction of V on the generic fiber of f is semi-stable and regular. Then there exists a divisor*

$$D_V \in |n\sigma(N) + ml|,$$

called the spectral cover associated to V, where l denotes effective divisor class of the fibers of f, $m \in \mathbb{Z}$ satisfies $0 \leq m \leq c_2(V)$, and for a generic $w \in N_0$,

$$V|_{M_w} \cong \bigoplus_{j=1}^{\ell} \mathcal{O}_{M_w}(q_j - 0) \otimes \mathcal{I}_{r_j}, \quad D_V \cap M_w = \sum_{j=1}^{\ell} r_j q_j \in |n\sigma(w)|.$$

We recall the construction in [30]. Since $h^0(M_w, \mathcal{O}_{M_w}(n\sigma(w))) = n$ for any fiber M_w, the push forward $f_*\mathcal{O}_M(n\sigma)$ is a vector bundle of rank n on N, and more precisely,

$$f_*\mathcal{O}_M(n\sigma) = \mathcal{O}_N \oplus L^{-2} \oplus \cdots \oplus L^{-n},$$

where $L^{-1} = \sigma^*\mathcal{O}_M(\sigma)$ by Lemma 4.1 of [30]. We denote $p : \mathcal{P}_{n-1} \to N$ the projection bundle, so $\mathcal{P}_{n-1} = \mathbb{P}f_*\mathcal{O}_M(n\sigma)$ (cf. Section 4.1 of [30]). For any $w \in N$, the fiber $p^{-1}(w)$ is the complete linear system $|n\sigma(w)| \cong \mathbb{CP}^{n-1}$, i.e. $p^{-1}(w) = |n\sigma(w)|$, and is identified as the coarse moduli space for semi-stable bundles of rank n on M_w (cf. Section 1 of [30]). Since the restriction of V to the generic fiber is semi-stable, there is a non-empty Zariski open subset $N' \subset N$ such that for any $w \in N'$, $V|_{M_w}$ is semi-stable, which defines a point $\varrho(V|_{M_w}) \in |n\sigma(w))|$ by Theorem 1.2 in [30]. Then Lemma 4.2 of [30] defines a section

$$\mathcal{A}_V : N' \to p^{-1}(N'), \quad \text{by} \quad \mathcal{A}_V(w) = \varrho(V|_{M_w}),$$

and by Lemma 6.1 in [30], $\mathcal{A}_V$ extends to N as a section of $\mathcal{P}_{n-1}$, denoted still by $\mathcal{A}_V : N \to \mathcal{P}_{n-1}$.

Section 4.3 in [30] constructs an n-sheeted branched covering $\varrho : \mathcal{T} \to \mathcal{P}_{n-1}$, which admits a $\mathbb{CP}^{n-2}$–fibration $r : \mathcal{T} \to M$. For any smooth fiber M_w, $\mathcal{T}_w = r^{-1}(M_w) \to M_w$ coincides with the construction in Section 2.1 of [30] as follows. Let $\Pi_w \subset M_w^{\otimes n}$ be the subset such that $(q_1, \cdots, q_n) \in \Pi_w$ if and only if the divisor $q_1 + \cdots + q_n$ is linearly equivalent to $n\sigma(w)$. If $\mathbb{S}_n$ denotes the symmetric group, and $\mathbb{S}_{n-1} \subset \mathbb{S}_n$ is the subgroup fixing the last element, then $\mathbb{S}_n$ acts on Π_w, and the quotient $\Pi_w/\mathbb{S}_n = |n\sigma(w)| \cong \mathbb{CP}^{n-1}$. Also $\mathcal{T}_w = \Pi_w/\mathbb{S}_{n-1}$, $r|_{\mathcal{T}_w} : \mathcal{T}_w \to M_w$ is given by $(q_1, \cdots, q_{n-1}, q_n) \mapsto q_n$, and $\varrho|_{\mathcal{T}_w} : \mathcal{T}_w \to |n\sigma(w)|$ is a branched n-sheeted cover such that $\varrho|_{\mathcal{T}_w}$ is unbranched over $q_1 + \cdots + q_n \in |n\sigma(w)|$ if and only if $q_i \neq q_j$ for any $i \neq j$. Clearly, $r|_{\mathcal{T}_w}(\varrho|_{\mathcal{T}_w}^{-1}(q_1 + \cdots + q_n)) = \{q_1, \cdots, q_n\} \subset M_w$ for any $q_1 + \cdots + q_n \in |n\sigma(w)|$.

The spectral cover D_V is defined as the scheme-theoretic inverse image of $\mathcal{A}_V(N)$, i.e. $D_V = \varrho^{-1}(\mathcal{A}_V(N))$, which is a subscheme of $\mathcal{T}$, and $p \circ \varrho|_{D_V} : D_V \to N$ is finite and flat of degree n (cf. Definition 5.3 in [30]). By Lemma 5.4 of [30], $r|_{D_V}$ embeds D_V in M as an effective Cartier divisor, and $f \circ r|_{D_V} = p \circ \varrho|_{D_V}$. Therefore, we always regard D_V as a divisor of M in the present paper. Furthermore, Lemma 5.4 in [30] shows that $\mathcal{O}_M(D_V) \cong \mathcal{O}_M(n\sigma(N)) \otimes f^*\mathcal{L}_V$ where $\mathcal{L}_V = \mathcal{A}_V^*\mathcal{O}_{\mathcal{P}_{n-1}}(1)$. Thus

$$D_V \in |n\sigma(N) + ml|,$$

where l denotes the effective divisor class of the fibers of f, and $m = \deg\mathcal{L}_V \in \mathbb{Z}$.

The arguments in Section 6.1 of [30] show that

$$0 \leq m = \deg\mathcal{L}_V \leq c_2(V), \tag{2.7}$$

which is sketched as follows. Since the restriction of V to the generic fiber is regular semi-stable, there are only finite possible fibers such that the restrictions of V are unstable. Lemma 6.2 of [30] proves that by preforming finite allowable elementary modifications to V, one obtains a new bundle V' such that the restriction of V' to any fiber is semi-stable. Furthermore $c_2(V') \leq c_2(V)$, and equality holds if and only if $V' = V$, i.e. there is no elementary modification preformed.

The proof of Corollary 6.3 in [30] shows that there is a coherent sheaf V_0, whose restriction on any fiber is regular semi-stable, and a morphism $\psi : V_0 \to V'$, which is an isomorphism on $f^{-1}(U)$ for a nonempty Zariski open set $U \subset N$. The cokernel coherent sheaf Q is a torsion sheaf supported on finite fibers, and admits a filtration by degree zero sheaves. Consequently, $c_2(V_0) = c_2(V')$. Note that V_0 is isomorphic to V on $f^{-1}(U')$ for a nonempty Zariski open set $U' \subset N$, as the above two processes only change the restrictions of V on finite fibers. Therefore we have $\mathcal{A}_{V_0} = \mathcal{A}_V$, $D_{V_0} = D_V$, and $\mathcal{L}_{V_0} = \mathcal{L}_V$. By Proposition 5.15 of [30], $\deg\mathcal{L}_{V_0} = c_2(V_0)$, and we obtain the inequality (2.7).

The spectral cover D_V gives a criterion of V being stable.

Theorem 2.9 (Theorem 7.4 of [30])**.** *If D_V is reduced and irreducible, then V is stable with respect to $f^*c_1(\mathcal{O}_{\mathbb{CP}^1}(1)) + t\alpha$, for all $0 < t \leq (\frac{n^3}{4}c_2(V))^{-1}$, where α is an ample class on M.*

This theorem can be used to construct stable bundles on M as follows. If $D \in |n\sigma(N) + ml|$, $m > 2n$, is an effective reduced and irreducible divisor, then Lemma 5.4 in [30] asserts that D is the spectral cover of a unique section $\mathcal{A}$ of $\mathcal{P}_{n-1}$, which satisfies $m = \deg \mathcal{A}^*\mathcal{O}_{\mathcal{P}_{n-1}}(1)$. A holomorphic vector bundle V is constructed from $\mathcal{A}$ (cf. Definition 5.2 in [30]) such that the restriction of V on every fiber is regular semi-stable with trivial determinant line bundle, and D is the spectral cover of V, i.e. $D_V = D$.

We recall the construction in Section 5.1 of [30] by assuming that D is smooth, and does not intersect with any singular set of the singular fibers of f. If $\tilde{M} = D \times_N M$ denotes the base change, which is smooth, then there are morphisms $\tilde{f} : \tilde{M} \to D$ and $\nu_D : \tilde{M} \to M$ such that $f \circ \nu_D = f|_D \circ \tilde{f}$. We regard $\tilde{M} = D \times_N M \subset M \times_N M$ via the natural embedding $D \hookrightarrow M$. Then $\Sigma_D = \nu_D^* \sigma$ and $\Delta = \tilde{M} \cap \Delta_0$ are divisors, where Δ_0 is the diagonal of $M \times_N M$. For any $w \in N_0$, and $q_j(w) \in M_w \cap D$, we have $\tilde{M}_{(w,q_j(w))} = M_w$, $\Sigma_D \cap \tilde{M}_{(w,q_j(w))} = \{\sigma(w)\}$, and $\Delta \cap \tilde{M}_{(w,q_j(w))} = \{q_j(w)\}$. Lemma 5.5 of [30] asserts that the push forward $(\nu_D)_* \mathcal{O}_{\tilde{M}}(\Delta - \Sigma_D)$ satisfies that its restriction on every fiber is regular semi-stable with trivial determinant line bundle. Furthermore, for any line bundle $\tilde{L}$ on D, $(\nu_D)_*(\mathcal{O}_{\tilde{M}}(\Delta - \Sigma_D) \otimes \tilde{f}^* \tilde{L})$ also satisfies the required conditions.

Conversely, if V is a holomorphic vector bundle whose restriction of V on every fiber is regular semi-stable with trivial determinant line bundle, and D is the spectral cover of V, then

$$V = (\nu_D)_*(\mathcal{O}_{\tilde{M}}(\Delta - \Sigma_D) \otimes \tilde{f}^* \tilde{L})$$

for a certain line bundle $\tilde{L}$ on D by Proposition 5.7 in [30]. Now, since $\deg L = -\sigma^2 = 2$, Proposition 5.12 of [30] asserts that one can choose V via a suitable $\tilde{L}$ on D such that the first Chern class $c_1(V) = 0$, and therefore, V has trivial determinant line bundle on M. Now Theorem 7.4 of [30] shows that V is stable with respect to $f^*c_1(\mathcal{O}_{\mathbb{CP}^1}(1)) + t\alpha$ for $0 < t \ll 1$. In summary, we have

Theorem 2.10. *If $D \in |n\sigma(N) + ml|$, $m > 2n$, is an effective reduced and irreducible divisor, then there exists a holomorphic vector bundle V of rank n with $c_1(V) = 0$ on M such that the restriction of V on every fiber is regular semi-stable, and D is the spectral cover of V, i.e. $D_V = D$. Furthermore, V is stable with respect to $f^*c_1(\mathcal{O}_{\mathbb{CP}^1}(1)) + t\alpha$, for all $0 < t \leq (\frac{n^3}{4} c_2(V))^{-1}$, where α is an ample class on M.*

2.5. **Collapsing of Ricci-flat Kähler-Einstein metrics.** We now introduce some preliminary results on our family of collapsing base metrics on M, and highlight a new decay estimate necessary for our main theorem. The reader is directed to Appendix A for a proof of this particular asymptotic decay.

Let α be an ample class on M, $\alpha_t = t\alpha + f^*c_1(\mathcal{O}_{\mathbb{CP}^1}(1))$, $t \in (0,1]$, and $\omega_t \in \alpha_t$ the unique Ricci-flat Kähler-Einstein metric, which satisfies the complex Monge-Ampère equation

$$\omega_t^2 = c_t t \Omega \wedge \overline{\Omega}.$$

Here Ω is a holomorphic symplectic form on M, and c_t tends to a positive number c_0 when $t \to 0$.

For any $t \in (0,1]$, there exists a family of Kähler metrics ω_t^{SF} on M_0, such that $\omega_t^{SF}|_{M_w}$ is the flat metric in the class $t\alpha|_{M_w}$. Such metrics are called *semi-flat*, and we recall their construction here. Note that M_0 is obtained by the quotient of the holomorphic cotangent bundle T^*N_0 by a lattice subbundle Λ. More precisely, we have a covering map $p : T^*N_0 \to M_0$, so that $p(\Lambda) = \sigma(N_0)$, and the pull-back $p^*\Omega$ is the canonical holomorphic symplectic form on T^*N_0. If $U \subset N_0$ is a small open disk, we can choose a holomorphic coordinate w on U so that $\Lambda \cap T^*U = \mathrm{Span}_{\mathbb{Z}}\{dw, \tau(w)dw\}$, where $\tau(w)$ is the period of the elliptic curve M_w. Under the trivialization $T^*U \cong U \times \mathbb{C}$ given by $zdw \mapsto (w,z)$, we see $p^*\Omega = dw \wedge dz$. Note that the $(1,1)$-form

$$i\partial\overline{\partial}\mathrm{Im}(\tau)^{-1}(\mathrm{Im}(z))^2 = \frac{i}{2}W(dz+bdw)\wedge\overline{(dz+bdw)}$$

is invariant under the translation of any local constant section of Λ (cf. Section 3 in [41]), where

$$W = \mathrm{Im}(\tau)^{-1} \qquad \text{and} \qquad b = -\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\frac{\partial\tau}{\partial w}.$$

Thus the above $(1,1)$-form can be regarded as living on $f^{-1}(U)$. The semi-flat metric is defined as

$$\omega_t^{SF} = \frac{i}{2}\left(tW(dz+bdw)\wedge\overline{(dz+bdw)} + W^{-1}dw\wedge d\bar{w}\right). \tag{2.8}$$

For simplicity we denote $\omega^{SF} := \omega_1^{SF}$, which we use as a fixed base metric. We denote

$$\theta = dz + bdw. \tag{2.9}$$

We now state our decay result for ω_t as $t \to 0$, which is contained in Theorem A.1 (see Appendix A below). Given $U \subset N_0$, [41] asserts that there exists a local section σ_0 such that for any $\ell \geq 0$,

$$\|T_{\sigma_0}^*\omega_t - \omega_t^{SF}\|_{C^\ell_{\mathrm{loc}}(M_U,\omega_t^{SF})} \to 0,$$

when $t \to \infty$, where T_{σ_0} denotes the fiberwise translation by σ_0 (cf. Lemma 4.7 in [41]). Theorem A.1 shows that there is a $(1,1)$-form χ_t satisfying $\chi_t \to 0$ in C^∞ as $t \to 0$, so that $T_{\sigma_0}^*\omega_t$ approaches to $\omega_t^{SF} + f^*\chi_t$ faster than any polynomial rate, i.e.

$$T_{\sigma_0}^*\omega_t = \omega_t^{SF} + f^*\chi_t + o(t^{\frac{\nu}{2}}),$$

for any $\nu \gg 1$.

In the proof of the main theorem we need a slightly stronger statement. The difference between $T_{\sigma_0}^*\omega_t$ and ω_t^{SF} can be written out in components in the fiber and base directions:

$$T_{\sigma_0}^*\omega_t - \omega_t^{SF} = \varphi_{t,z\bar{z}}dz\wedge d\bar{z} + \varphi_{t,w\bar{w}}dw\wedge d\bar{w} + \varphi_{t,w\bar{z}}dw\wedge d\bar{z} + \varphi_{t,z\bar{w}}dz\wedge d\bar{w}.$$

We need the following important lemma, which is a direct consequence of Lemma A.2.

Lemma 2.11. *For any $\nu \gg 1$ and $\ell \geq 0$, there is a constant $C_{\ell,\nu} > 0$ such that on $M_{U'}$, $U' \subset U$,*

$$\|\varphi_{t,w\bar{w}} - \chi_{t,w\bar{w}}\|_{C^0} \leq C_{0,\nu} t^{\frac{\nu}{2}},$$

$$\|\frac{\partial}{\partial z}\varphi_{t,w\bar{w}}\|_{C^\ell} + \|\frac{\partial}{\partial \bar{z}}\varphi_{t,w\bar{w}}\|_{C^\ell} + \|\varphi_{t,z\bar{z}}\|_{C^\ell} + \|\varphi_{t,z\bar{w}}\|_{C^\ell} + \|\varphi_{t,w\bar{z}}\|_{C^\ell} \leq C_{\ell,\nu} t^{\frac{\nu}{2}},$$

and $\chi_{t,w\bar{w}} \to 0$ in the C^∞-sense when $t \to 0$. Here $\chi_t = \chi_{t,w\bar{w}} dw \wedge d\bar{w}$, and the C^ℓ-norms are calculated using the fixed Kähler metric ω^{SF} on M_U.

In this section we also recall the blow-up limit of $t^{-1}\omega_t$, which shows up in the analysis to follow. Let $t_k \to 0$ and $w_k \to w_0$ in $U \subset N_0$. By [41],

$$(M, t_k^{-1}\omega_{t_k}, p_k) \to (\mathbb{C} \times M_{w_0}, \omega_\infty = \omega_{w_0}^F + \frac{i}{2} W^{-1}(w_0) d\tilde{w} \wedge d\bar{\tilde{w}}, p_0),$$

in the C^∞-Cheeger-Gromov sense, where $w_k = f(p_k)$, $p_k \to p_0 \in M_{w_0}$, $\omega_{w_0}^F$ is the flat Kähler metric representing $\alpha|_{M_{w_0}}$, i.e. $\omega_{w_0}^F = \omega^{SF}|_{M_{w_0}}$, and $\tilde{w}$ denotes the coordinate of $\mathbb{C}$. More precisely, if $D_r = \{\tilde{w} \in \mathbb{C} | |\tilde{w}| < r\}$, we define smooth embeddings $\Phi_{k,r} : D_r \times M_{w_0} \to M_U$ by

$$(\tilde{w}, a_1 + a_2\tau(w_0)) \mapsto (w_k + \sqrt{t_k}\tilde{w}, a_1 + a_2\tau(w_k + \sqrt{t_k}\tilde{w})), \quad a_1, a_2 \in \mathbb{R}/\mathbb{Z},$$

where we identify M_U with $(U \times \mathbb{C})/\mathrm{Span}_{\mathbb{Z}}\{1, \tau\}$. If $z = a_1 + a_2\tau(w_0)$, then $a_1 + a_2\tau(w_k + \sqrt{t_k}\tilde{w}) = z + h_k$, where

$$h_k = i(2\mathrm{Im}\tau(w_0))^{-1}(\bar{z} - z)(\tau(w_k + \sqrt{t_k}\tilde{w}) - \tau(w_0)),$$

which satisfies that $\|h_k\|_{C^\ell} \to 0$ when $t_k \to 0$. Therefore

$$\Phi_{k,r}^*(dz + bdw) = dz + dh_k + \sqrt{t_k}(b - \mathrm{Im}h_k(\mathrm{Im}\tau)^{-1}\partial_w\tau)d\tilde{w} \to dz,$$

in the C^∞-sense. Clearly, $d\Phi_{k,r}^{-1} I d\Phi_{k,r} \to I_\infty$, where I is the complex structure of M and I_∞ denotes the complex structure of $\mathbb{C} \times M_{w_0}$, and

$$\Phi_{k,r}^* t_k^{-1}\omega_{t_k}^{SF} \to \omega_\infty = \frac{i}{2}\left(W(w_0)dz \wedge d\bar{z} + W^{-1}(w_0)d\tilde{w} \wedge d\bar{\tilde{w}}\right), \tag{2.10}$$

in the C^∞-sense on $D_r \times M_{w_0}$. Furthermore,

$$(T_{\sigma_0} \circ \Phi_{k,r})^* t_k^{-1}\omega_{t_k} = \Phi_{k,r}^* t_k^{-1} T_{\sigma_0}^*\omega_{t_k} \to \omega_\infty, \tag{2.11}$$

in the C^∞-sense, on $D_r \times M_{w_0}$, when $t_k \to 0$ by [41].

2.6. **Fourier-Mukai transform.** In this section, we recall a notion, called the Fourier-Mukai transform (cf. [7, 56, 14, 13] etc.), and we present a little variant of the standard construction for the convenience of the proof of Theorem 3.2.

Let $N^o \subset N_0$ be a Zariski open subset, and $D^o \subset M_{N^o}$ be a smooth curve such that $f|_{D^o} : D^o \to N^o$ is a unbranched n-sheets cover. Note that the moduli space of flat $U(1)$-connections on D^o is the cohomology group $H^1(D^o, \mathcal{U}_c(1)) \cong H^1(D^o, U(1))$, where $\mathcal{U}_c(1)$ is the $U(1)$-valued locally constant sheaf. For any $\Theta \in H^1(D^o, \mathcal{U}_c(1))$, the Fourier-Mukai transform takes the pair (D^o, Θ) to a unitary gauge equivalent class $\mathcal{FM}(D^o, \Theta)$ of $U(n)$-connections on M_{N^o}. We review the construction as the following.

If $\tilde{M}^o = D^o \times_{N^o} M_{N^o}$ is the base change, then the projection $\tilde{f} : \tilde{M}^o \to D^o$ is a fibration with the fiber $\tilde{M}^o_p = M_{f(p)}$, and $\nu_D : \tilde{M}^o \to M_{N^o}$ is a unbranched n-sheets cover satisfying $f \circ \nu_D = f|_{D^o} \circ \tilde{f}$. We embed $\tilde{M}^o = D^o \times_{N^o} M_{N^o} \hookrightarrow M_{N^o} \times_{N^o} M_{N^o}$ via the natural inclusion $D^o \hookrightarrow M_{N^o}$. Let $\Sigma = \nu_D^* \sigma$ and $\Delta = \tilde{M}^o \cap \Delta_0$, where Δ_0 denotes the diagonal of $M_{N^o} \times_{N^o} M_{N^o}$. For any $x \in N^o$, and $q(x) \in M_x \cap D^o$, we have $\tilde{M}^o_{(x,q(x))} = M_x$, $\Sigma \cap \tilde{M}^o_{(x,q(x))} = \{\sigma(x)\}$, and $\Delta \cap \tilde{M}^o_{(x,q(x))} = \{q(x)\}$. We regard Σ as the zero section of $\tilde{f}$, which is used to identify the fibers with elliptic curves, and view Δ as the pull back the multi-section D^o, which is a section of $\tilde{f}$.

There is a $U(1)$-connection A^o on the smooth trivial line bundle $\tilde{M}^o \times \mathbb{C}$, which is obtained by the restriction of the Poincaré line bundle (cf. [7]) on $M_{N^o} \times_{N^o} M_{N^o}$ by identifying M_{N^o} with the Jacobian $\check{M}_{N^o}$. We exhibit A^o explicitly.

If $U \subset D^o$ is an open disc such that $f|_U : U \to f(U)$ is biholomorphic, we choose the coordinate w such that $\tilde{M}^o_U \cong T^*U/\mathrm{Span}_{\mathbb{Z}}\{dw, \tau dw\}$, where $\tau(w)$ is the period of $\tilde{M}^o_w$. Here the section $\Sigma \equiv 0$ under this identification. If z denotes the coordinate on the fiber, then the holomorphic symplectic form $\nu_D^* \Omega = dw \wedge dz$, and $\Delta \cap \tilde{M}^o_U$ is given by a holomorphic function $q = q(w)$ on U, i.e. $\Delta \cap \tilde{M}^o_U = \{(w, q(w))\} \subset U \times \mathbb{C}/\mathrm{Span}_{\mathbb{Z}}\{1, \tau\}$. We have the $U(1)$-connection

$$A^o = \pi(\mathrm{Im}(\tau))^{-1}(q\bar{\theta} - \bar{q}\theta), \tag{2.12}$$

on $\tilde{M}^o_U$, where θ is defined by (2.9).

If y_1 and y_2 are real functions defined on $U \times \mathbb{C}$ by $z = y_1 + \tau y_2$, then dy_1 and dy_2 are well-defined 1-forms on $\tilde{M}^o_U$. Note that $\tilde{M}^o_U$ is diffeomorphic to $U \times (\mathbb{R}/\mathbb{Z})^2$, and we can regard y_1 and y_2 as the angle coordinates of $\mathbb{R}/\mathbb{Z}$. We have the decomposition of the cotangent bundle $T^*\tilde{M}^o_U = \mathrm{Span}_{\mathbb{R}}\{dy_1, dy_2\} \oplus \mathrm{Span}_{\mathbb{R}}\{dx_1, dx_2\}$, where $w = x_1 + ix_2$. Since $dz = dy_1 + \tau dy_2 + y_2 d\tau$, $2i\mathrm{Im}(\tau) y_2 = z - \bar{z}$, we have $\theta = dy_1 + \tau dy_2$. If we write $q = q_1 + \tau q_2$, then

$$A^o = 2\pi i(q_2 dy_1 - q_1 dy_2). \tag{2.13}$$

If we choose another basis of the lattice $\mathrm{Span}_{\mathbb{Z}}\{1, \tau\}$, and let y_1' and y_2' be the corresponding angle coordinates, then $y_j' = \sum c_{ji} y_i$ and $q_j' = \sum c_{ji} q_i$ with $\det(c_{ji}) = 1$ and $c_{ji} \in \mathbb{Z}$. A direct calculation shows that A^o is independent of the choice of the basis, and therefore A^o is a global defined $U(1)$-connection on the trivial line bundle $\tilde{M}^o \times \mathbb{C}$.

Let $\check{y}_1$ and $\check{y}_2$ be the dual coordinates of y_1 and y_2 on the dual space $(\mathbb{R}^2)^*$, i.e. if we view $(\mathbb{R}^2)^*$ as the cotangent space, then $\check{y}_1$ and $\check{y}_2$ are coordinates with respect to the basis dy_1 and dy_2. We identify $\mathbb{R}^2/\mathbb{Z}^2$ with the dual torus $(\mathbb{R}^2)^*/(\mathbb{Z}^2)^*$ via the symplectic form $\omega = dy_2 \wedge dy_1$, i.e. $v \mapsto \omega(v, \cdot)$. Then $q = (q_1, q_2) \in \mathbb{R}^2/\mathbb{Z}^2$ is mapped to $\check{q} = (q_2, -q_1)$ in $(\mathbb{R}^2)^*/(\mathbb{Z}^2)^*$. The

Poincaré line bundle is a line bundle on $U \times \mathbb{R}^2/\mathbb{Z}^2 \times (\mathbb{R}^2)^*/(\mathbb{Z}^2)^*$ with the $U(1)$-connection

$$A_P = 2\pi i(\check{y}_1 dy_1 + \check{y}_2 dy_2).$$

Thus $A^o = A_P|_{U\times\mathbb{R}^2/\mathbb{Z}^2\times\{\check{q}\}}$, which coincides with the constructions [7, 56].

Let $\{U_\lambda|\lambda \in \Lambda\}$ be a locally finite open cover of N^o such that any intersection $U_{\lambda_1} \cap \cdots \cap U_{\lambda_h}$ is contractible. On any U_λ, there are holomorphic functions $q_1, \cdots, q_n$, such that $D^o \cap M_{U_\lambda} = \{(w, q_1(w)), \cdots, (w, q_n(w))|w \in U_\lambda\}$. Furthermore, $D^o \cap M_{U_\lambda} = U_\lambda^1 \cup \cdots \cup U_\lambda^n$ is a disjoint union of open sets biholomorphic to U_λ where $U_\lambda^j = \{(w, q_j(w))\}$, and $\{U_\lambda^j|\lambda \in \Lambda, j = 1, \cdots, n\}$ is an open cover of D^o such that any intersections are contractible.

If $\Theta \in H^1(D^o, \mathcal{U}_c(1))$, then we let $\{g_{\mu\lambda}^{ij_i}\} \in \mathcal{C}^1(\{U_\lambda^j\}, \mathcal{U}_c(1))$ be the cocycle representing Θ, where $U_\mu^i \cap U_\lambda^{j_i} \neq \emptyset$, and $g_{\mu\lambda}^{ij_i}$ are $U(1)$-valued constant functions on $U_\mu^i \cap U_\lambda^{j_i}$. If $U_\mu^i \cap U_\lambda^j \cap U_\nu^k \neq \emptyset$, then $g_{\mu\lambda}^{ij} g_{\lambda\nu}^{jk} g_{\nu\lambda}^{ki} = 1$. We identify $\tilde{f}^* g_{\mu\lambda}^{ij_i} = g_{\mu\lambda}^{ij_i}$, and regard $g_{\mu\lambda}^{ij_i}$ as $U(1)$-valued constant functions on $\tilde{M}_{U_\mu^i}^o \cap \tilde{M}_{U_\lambda^{j_i}}^o$. Note that $(g_{\mu\lambda}^{ij_i})^{-1} A^o g_{\mu\lambda}^{ij_i} + (g_{\mu\lambda}^{ij_i})^{-1} dg_{\mu\lambda}^{ij_i} = A^o$. If L_Θ denotes the line bundle on $\tilde{M}^o$ given by the cocycle $\{(\tilde{M}_{U_\mu^i}^o \cap \tilde{M}_{U_\lambda^{j_i}}^o, g_{\mu\lambda}^{ij_i})\}$, then A^o induces a $U(1)$-connection on L_Θ locally given by (2.12) denoted still by A^o.

The pushforward $(\nu_D)_* L_\Theta$ is a rank n bundle on M_{N^o} given by the transitions $g_{\mu\lambda} = \mathrm{diag}\{g_{\mu\lambda}^{1,j_1}, \cdots, g_{\mu\lambda}^{n,j_n}\}$ on $M_{U_\mu} \cap M_{U_\lambda}$, where $U_\mu^i \cap U_\lambda^{j_i} \neq \emptyset$. There is a natural $U(n)$-connection Ξ on $(\nu_D)_* L_\Theta$ induced by A^o given locally by

$$\begin{aligned} \Xi|_{M_{U_\lambda}} &= \mathrm{diag}\{(\nu_D)_* A^o|_{\tilde{M}_{U_\lambda^1}^o}, \cdots, (\nu_D)_* A^o|_{\tilde{M}_{U_\lambda^n}^o}\} \\ &= \pi(\mathrm{Im}(\tau))^{-1}(\mathrm{diag}\{q_1, \cdots, q_n\}\bar{\theta} - \mathrm{diag}\{\bar{q}_1, \cdots, \bar{q}_n\}\theta), \end{aligned}$$

which satisfies $g_{\mu\lambda}^{-1} \Xi|_{M_{U_\mu}} g_{\mu\lambda} + g_{\mu\lambda}^{-1} dg_{\mu\lambda} = \Xi|_{M_{U_\lambda}}$. If $\{g_{\mu\lambda}^{\prime ij_i}\}$ is an another cocycle representing Θ, then there is a cycle $\{s_\lambda^j\} \in \mathcal{C}^0(\{U_\lambda^j\}, \mathcal{U}_c(1))$ such that $g_{\mu\lambda}^{\prime ij_i} s_\lambda^{j_i} = s_\mu^i g_{\mu\lambda}^{ij_i}$ when $U_\mu^i \cap U_\lambda^{j_i} \neq \emptyset$. If we define $s_\lambda = \mathrm{diag}\{s_\lambda^1, \cdots, s_\lambda^n\}$ on M_{U_λ}, then $g'_{\mu\lambda} s_\lambda = s_\mu g_{\mu\lambda}$, and $\{s_\lambda|\lambda \in \Lambda\}$ induces a unitary gauge change of $(\nu_D)_* L_\Theta$.

Definition 2.12. *The Fourier-Mukai transform $\mathcal{FM}(D^o, \Theta)$ of (D^o, Θ) is defined as the unitary gauge equivalent class $[\Xi]$ of the $U(n)$-connection Ξ on $(\nu_D)_* L_\Theta$, i.e.*

$$\mathcal{FM}(D^o, \Theta) = [\Xi].$$

For any $t \in (0, 1]$, the semi-flat metric ω_t^{SF} is HyperKähler, and by using the HyperKähler rotation, we can find a new complex structure and a symplectic form such that D^o is a special lagrangian submanifold. In [56], it is shown that the connection obtained by the Fourier-Mukai transform of a special lagrangian section satisfies the deformed Hermitian-Yang-Mills equation, and in the case of dimension 2, the standard Hermitian-Yang-Mills equation. In the present case, the bundle $(\nu_D)_* L_\Theta$ with the connection Ξ

splits locally, and therefore, it is a corollary of [56] that Ξ is an anti-self-dual connection. We give a direct calculation proof of this assertion.

Proposition 2.13. *If $\Theta \in H^1(D^o, \mathcal{U}_c(1))$, then for any $\Xi \in \mathcal{FM}(D^o, \Theta)$, Ξ is an anti-self-dual connection with respect to the semi-flat HyperKähler structure (ω_t^{SF}, Ω), $t \in (0,1]$, i.e. the curvature F_Ξ satisfies that*

$$F_\Xi \wedge \omega_t^{SF} = 0, \quad \textit{and} \quad F_\Xi \wedge \Omega = 0.$$

Proof. Since the anti-self-dual equation is unitary gauge invariant, we only need to verify the split case, i.e. $\Xi|_{M_{U_\lambda}} = \mathrm{diag}\{A^o|_{\tilde{M}^o_{U^1_\lambda}}, \cdots, A^o|_{\tilde{M}^o_{U^n_\lambda}}\}$, where we identify M_{U_λ} and $\tilde{M}^o_{U^j_\lambda}$ via ν_D. The curvature

$$F_\Xi|_{M_{U_\lambda}} = \mathrm{diag}\{F_{A^o}|_{\tilde{M}^o_{U^1_\lambda}}, \cdots, F_{A^o}|_{\tilde{M}^o_{U^n_\lambda}}\},$$

and thus we need to prove that F_{A^o} satisfies the anti-self-dual equation.

By $\overline{\partial}\tau = 0$, we have $0 = \overline{\partial}\tau_1 + i\overline{\partial}\tau_2$, where $\tau = \tau_1 + i\tau_2$, and $\partial_{\bar{w}}\bar{\tau} = \partial_{\bar{w}}\tau_1 - i\partial_{\bar{w}}\tau_2 = -2i\partial_{\bar{w}}\tau_2$. Thus

$$F^{0,2}_{A^o} = \pi\overline{\partial}(\tau_2^{-1} q\bar{\theta}) = \frac{\pi q}{2i\tau_2^2}\partial_{\bar{w}}\bar{\tau} d\bar{z} \wedge d\bar{w} + \frac{\pi q}{\tau_2^2}\partial_{\bar{w}}\tau_2 d\bar{z} \wedge d\bar{w} = 0,$$

which is equivalent to $F_{A^o} \wedge \Omega = 0$. By (2.13),

$$F_{A^o} = dA^o = 2\pi i \sum_{j=1,2} (\partial_{x_j} q_2 dx_j \wedge dy_1 - \partial_{x_j} q_1 dx_j \wedge dy_2),$$

and by (2.8),

$$\omega_t^{SF} = t dy_1 \wedge dy_2 + W^{-1} dx_1 \wedge dx_2.$$

Thus

$$F_{A^o} \wedge \omega_t^{SF} = 0,$$

and we obtain the conclusion. $\square$

Finally, we remark that the split Ξ obtained by the Fourier-Mukai transform is T^2-invariant, and thus reduces to solutions of the Hitchin equation [49] and the Poisson metric equation [16] on the base N^o.

2.7. **Small energy estimates on collapsed K3 surfaces.** Finally, we review small energy estimates for curvatures of anti-self-dual connections with respect to collapsed metrics.

As above, for $t \in (0,1]$, let $\omega_t \in \alpha_t = t\alpha + f^* c_1(\mathcal{O}_{\mathbb{CP}^1}(1))$ be the unique Ricci-flat Kähler-Einstein metric in α_t, and consider Ξ_t, a family of anti-self-dual connections on P with with respect to ω_t.

For any $p \in M$, and $r > 0$, we define the local energy of the curvature F_{Ξ_t} as

$$\mathcal{E}_t(p, r) = \frac{r^4}{\mathrm{Vol}_{\omega_t}(B_{\omega_t}(p, r))} \int_{B_{\omega_t}(p,r)} |F_{\Xi_t}|^2_{\omega_t} \, \omega_t^2. \tag{2.14}$$

This energy is a continuous function of p and r. By the Bishop-Gromov comparison Theorem, for $r_1 \leq r_2$ it holds

$$\mathcal{E}_t(p, r_1) \leq \mathcal{E}_t(p, r_2) \qquad \text{and} \qquad \mathcal{E}_t(p, 0) = 0.$$

We have the following small energy estimate for curvatures of anti-self-dual connections, which is essentially Theorem 4.4 in [1].

Lemma 2.14. *There exists a universal constant $\varepsilon > 0$, independent of t, such that if*

$$\mathcal{E}_t(p, r) \leq \varepsilon,$$

for p and r satisfying $p \in M_K$ and $B_{\omega_t}(p, r) \subset M_{K'}$ (for fixed compact subsets $K \subset K' \subset N_0$), then

$$\sup_{B_{\omega_t}(p, r/2)} |F_{\Xi_t}|_{\omega_t} \leq \frac{C_{K'}\varepsilon^{\frac{1}{2}}}{r^2}$$

for a constant $C_{K'} > 0$.

Proof. By Lemma 4.4 of [41], the curvature R_{ω_t} is bounded by a uniform constant $c_{K'}$ on $M_{K'}$. The Weitzenböck formula (2.3) implies the Bochner formula

$$\Delta_{\omega_t} |F_{\Xi_t}|_{\omega_t} \geq -|F_{\Xi_t}|^2_{\omega_t} - c_{K'} |F_{\Xi_t}|_{\omega_t}.$$

One can now carry over the exact argument from [1], consisting of Moser iteration with the local Sobolev inequality

$$\frac{c_S}{3} \Big(\frac{B_{\omega_t}(p, r)}{r^4}\Big)^{\frac{1}{4}} \|\xi\|_{L^4(\omega_t)} \leq \|d\xi\|_{L^2(\omega_t)}$$

for any compactly supported function ξ on $B_{\omega_t}(p, r)$, where c_S is a universal constant (cf. (4.1) and Theorem 4.1 in [1]). If we keep track of the extra $c_{K'}$ term, because this term is of lower order, it does not affect the choice of the uniform constant τ, which is thus independent of K and K'. $\square$

Choose $\varepsilon \ll 1$ such that $C_{K'}\varepsilon^{\frac{1}{2}} \leq 4$. This allows us to make the following definition.

Definition 2.15. *For any $t \in (0, 1]$, we define $R_t(p) > 0$ be the minimal number such that*

$$\mathcal{E}_t(p, R_t(p)) = \varepsilon.$$

In particular, for any compact set $K \subset N_0$, and $p \in M_K$, as long as $R_t(p)$ is small enough, it holds

$$|F_{\Xi_t}|_{\omega_t}(p) \leq 4R_t(p)^{-2}, \tag{2.15}$$

and for any $r \geq R_t(p)$,

$$\mathcal{E}_t(p, r) \geq \varepsilon.$$

3. The main theorems

In this section, we present the main theorems of this paper, and demonstrate its applications to SYZ mirror symmetry of K3 surfaces.

Theorem 3.1. *Let M be a projective elliptically fibered $K3$ surface with fibration $f : M \to N \cong \mathbb{CP}^1$. Assume f has a section $\sigma : N \to M$, and assume it has only singular fibers of Kodaira type I_1 and type II. Let Ω be a holomorphic symplectic form on M, and let $\omega_t \in \alpha_t$ be the unique Ricci-flat Kähler-Einstein metric in $\alpha_t = t\alpha + f^*c_1(\mathcal{O}_{\mathbb{CP}^1}(1))$, $t \in (0,1]$, where α is an ample class on M. Let P be a principal $SU(n)$-bundle on M, and let $\mathcal{V}$ be the smooth vector bundle of rank n equipped with a Hermitian metric H induced by P, i.e. $\mathcal{V} = P \times_\rho \mathbb{C}^n$.*

Assume there exists a family of anti-self-dual $SU(n)$-connections Ξ_t on P with respect to (ω_t, Ω), i.e.

$$F_{\Xi_t} \wedge \omega_t = 0, \quad \textit{and} \quad F_{\Xi_t} \wedge \Omega = 0,$$

with $t \in (0,1]$. Let V_t denote the holomorphic bundle of $\mathcal{V}$ equipped with the holomorphic structure induced by Ξ_t. Furthermore, assume:

i) *The restriction of V_t to a generic fiber of f is semi-stable and regular.*
ii) *Let $D_t \in |n\sigma(N) + ml|$ be the corresponding spectral cover of V_t, where $0 < m \leq c_2(\mathcal{V})$. As $t \to 0$,*

$$D_t \to D_0 \quad \textit{in} \quad |n\sigma(N) + ml|.$$

iii) *The limit D_0 can be written*

$$D_0 = D_0^o + D_0',$$

where $D_0^o \in |n\sigma(N) + m'l|$ is reduced, for some $0 \leq m' \leq m$, and $D_0' \in |(m-m')l|$ consists of all irreducible components of D_0 supported on fibers.

Then the following holds:

i) *For any sequence $t_k \to 0$, and any $p > 2$, there exists a Zariski open subset $N^o \subset N_0$, a subsequence (still denoted t_k), a sequence of L_2^p unitary gauge changes $u_k \in \mathcal{G}^{2,p}$ of $P|_{M_{N^o}}$, and a L_1^p $SU(n)$-connection Ξ_0 on $P|_{M_{N^o}}$ so that on M_{N^o}*

$$u_k(\Xi_{t_k}) \to \Xi_0$$

in the locally L_1^p sense. Here the norms are calculated using a fixed Kähler metric on M, and the Hermitian metric H on $\mathcal{V}$.
ii) *The curvature $F_{\Xi_{t_k}}$ of Ξ_{t_k} is locally bounded, i.e. for any compact subset $K \subset N^o$, there exists a constant C_K so that*

$$\|F_{\Xi_{t_k}}\|_{C^0(M_K)} \leq C_K.$$

iii) *For any $w \in N^o$ and $0 < \alpha < 1$, there is a $C^{1,\alpha}$ unitary gauge u_∞ on M_w so that $u_\infty(\Xi_0|_{M_w})$ is a smooth flat connection. This* *limiting connection satisfies that the bundle $\mathcal{V}|_{M_w}$ equipped with the holomorphic structure induced by $u_\infty(\Xi_0|_{M_w})$ is bi-holomorphic to*

$$\bigoplus_{q\in D_0^o\cap M_w} \mathcal{O}_{M_w}(q-\sigma(w)).$$

Remark 1. We remark that $D_0' \in |(m-m')l|$ is supported on fibers over a finite number of points, and we refer to these fibers as type III bubbles, which is the terminology used in the previous relevant works [22, 58, 60].

Remark 2. There is a topological constraint on $\mathcal{V}$ built into the above theorem, namely that

$$c_2(\mathcal{V}) \geq 2n-2.$$

To see this, note that if $\sigma(N)$ is not an irreducible component of D_0^o, then $D_0^o\cdot\sigma(N) = -2n+m' \geq 0$. Otherwise, $(D_0^o-\sigma(N))\cdot\sigma(N) = -2n+2+m' \geq 0$. In both cases, we have $m' \geq 2n-2$, which implies the inequality for the second Chern number.

Let us demonstrate a case in which the hypotheses of Theorem 3.1 hold. For a given $m \in \mathbb{N}$ and $s \in (0,1]$, let D_s be a family of effective reduced irreducible divisors in the complete linear system $|n\sigma(N)+ml|$ such that as $s \to 0$,

$$D_s \to D_0 = D_0^o + \sum_j D_j \quad \text{in} \quad |n\sigma(N)+ml|,$$

where D_0^o is reduced and irreducible, $D_0^o \in |n\sigma(N)+m'l|$ for some $m' \leq m$, and $\sum_j D_j \in |(m-m')l|$. For example, we can take $D_s \equiv D$ for some fixed divisor. By Theorem 2.10, we can construct a family of holomorphic bundles V_s of rank n satisfying $c_1(V_s) = 0$, the restriction of V_s to any fiber M_w is semi-stable and regular, and D_s is the spectral cover of V_s. Furthermore, Proposition 5.15 of [30] asserts that $c_2(V_s) = m$, and therefore, all of V_s are smoothly isomorphic to the same smooth bundle, since $SU(n)$ is simply connected. Now, Theorem 7.4 of [30] shows that for any s the bundle V_s is stable with respect to $f^*c_1(\mathcal{O}_{\mathbb{CP}^1}(1)) + t\alpha$ for $0 < t \ll 1$ and $t \leq s$. As a result, by Theorem 2.2 (and taking a diagonal sequence) we obtain a family of anti-self-dual connections Ξ_t, for which the hypotheses of Theorem 3.1 are verified.

Theorem 3.2. *Under the setup of Theorem 3.1, the unitary gauge equivalent class of the limit connection Ξ_0 is the Fourier-Mukai transform of a $\Theta \in H^1(D_0^o \cap M_{N^o}, \mathcal{U}_c(1))$, i.e.*

$$\Xi_0 \in \mathcal{FM}(D_0^o \cap M_{N^o}, \Theta),$$

where $\mathcal{U}_c(1)$ is the $U(1)$-valued locally constant sheaf.

3.1. **Strominger-Yau-Zaslow mirror symmetry with anti-self-dual connections.** We now apply Theorem 3.1 to Fukaya's Conjecture 5.5 in [34], which relates the adiabatic limits of anti-self-dual connections to special Lagrangian cycles on the mirror Calabi-Yau manifolds. While describing the mirror symmetry background, we first consider the more general setup where M is any projective elliptically fibered $K3$ surface admitting a section.

We normalize α_t by multiplying a constant, so that the normalized class $\tilde{\alpha}_t$ satisfies $\tilde{\alpha}_t^2 = [\mathrm{Re}\Omega]^2 = [\mathrm{Im}\Omega]^2$. Let $\tilde{\omega}_t \in \tilde{\alpha}_t$ be the Ricci-flat Kähler-Einstein metric in this class, and so $(\tilde{\omega}_t, \mathrm{Re}\Omega, \mathrm{Im}\Omega)$ is a HyperKähler triple. Using the HyperKähler rotation, we have a family of complex structures J_t with corresponding Kähler form and the holomorphic symplectic from

$$\omega_{J_t} = \mathrm{Im}\Omega \quad \text{and} \quad \Omega_{J_t} = \tilde{\omega}_t + i\mathrm{Re}\Omega.$$

Using $\Omega|_{M_w} = 0$ and $\Omega|_{\sigma(N)} = 0$, under J_t the fibration f becomes a special Lagrangian fibration, and the section σ is a special Lagrangian section with respect to ω_{J_t} and Ω_{J_t}.

Mirror symmetry for K3 surfaces is well understood (cf. [3, 18, 44, 40, 2]), and in particular the SYZ mirror symmetry of K3 surfaces was studied in Section 7 of Gross [40] and in Gross-Wilson [44]. For the reader's convenience we elaborate further on this setup. Let $[\sigma]$ denotes the class of the section $\sigma(N)$ in $H^2(M, \mathbb{Z})$ and l the fiber class. Then we have the following intersection pairings:

$$l^2 = 0, \qquad [\sigma] \cdot l = 1, \qquad [\sigma]^2 = -2, \qquad [\omega_{J_t}] \cdot [\sigma] = 0,$$
$$[\mathrm{Im}\Omega_{J_t}] \cdot [\sigma] = 0, \qquad [\omega_{J_t}] \cdot l = 0, \qquad \text{and} \qquad [\mathrm{Im}\Omega_{J_t}] \cdot l = 0.$$

Now, the SYZ construction from Section 7 of [40] uses the choice of a B-field $\mathbb{B} \in l^{\perp}/l \otimes \mathbb{R}/\mathbb{Z}$. However, Gross' assumptions are slightly different than those of the present paper. Namely, Gross assumes the K3 surface M is generic, i.e. the Picard group $\mathrm{Pic}(M) \cong \mathbb{Z}$, while in our case we have $\dim \mathrm{Pic}(M) \geq 2$. Nevertheless, the proof of Theorem 7.3 of [40] shows that, in our case, if we further assume that $[\sigma] + (1 + \frac{1}{2}[\omega_{J_t}]^2)l$ is an ample class on M, and the B-field $\mathbb{B}$ vanishes, then the SYZ mirror of $(M, \tilde{\omega}_t, \Omega_{J_t})$ is $f : M \to N$ equipped with the HyperKähler structure $(\check{\omega}_t, \check{\Omega}_t)$ and the B-field $\check{\mathbb{B}}_t$ satisfying

$$[\check{\Omega}_t] = (l\cdot[\mathrm{Re}\Omega_{J_t}])^{-1}([\sigma]+(1+\frac{1}{2}[\omega_{J_t}]^2)l-i[\omega_{J_t}]), \quad [\check{\omega}_t] = (l\cdot[\mathrm{Re}\Omega_{J_t}])^{-1}[\mathrm{Im}\Omega_{J_t}],$$

$$\text{and} \quad \check{\mathbb{B}}_t = (l \cdot [\mathrm{Re}\Omega_{J_t}])^{-1}[\mathrm{Re}\Omega_{J_t}] - [\sigma] + \mathrm{mod}(l),$$

on the cohomological level.

We study the case that $[\sigma] + (1 + \frac{1}{2}[\omega_{J_t}]^2)l$ is not necessarily ample. Recall that the Weierstrass model $\check{f} : \check{M} \to N$ of $f : M \to N$ is obtained by contracting the irreducible components of singular fibers of f, which do not intersect with the section σ (cf. Chapter 7 in [28]). Denote by $\pi : M \to \check{M}$ the contraction morphism. Since π contracts finitely many (-2)-curves, $\check{M}$ has only orbifold A-D-E singularities.

Proposition 3.3. *Normalize Ω so that $[\mathrm{Im}\Omega]^2 = 4$. The SYZ mirror of $(M, \omega_{J_t}, \Omega_{J_t})$ with vanishing B-field is $(M, (l \cdot \tilde{\alpha}_t)^{-1}\check{\omega}, (l \cdot \tilde{\alpha}_t)^{-1}\check{\Omega})$ with the B-field $\check{\mathbb{B}}_t$, where*

$$\check{\Omega} = \pi^*\omega_{\check{M}} - i\mathrm{Im}\Omega, \quad \check{\omega} = \mathrm{Re}\Omega, \quad \text{and}$$

$$\check{\mathbb{B}}_t = (l \cdot \tilde{\alpha}_t)^{-1}\tilde{\alpha}_t - [\sigma] + \mathrm{mod}(l).$$

Here $\omega_{\check{M}}$ is the Ricci-flat Kähler-Einstein metric, possibly in the orbifold sense, such that $\pi^*\omega_{\check{M}} \in c_1(\mathcal{O}_M(\sigma(N) + 3l))$.

Proof. Firstly, note that $([\sigma] + 3l)^2 = 4 > 0$. Now, let D be an irreducible curve such that $([\sigma] + 3l) \cdot [D] \leq 0$. If $[D] \cdot l > 0$, then $[\sigma] \cdot [D] < 0$. Thus $D = \sigma$, and $([\sigma]+3l)\cdot[D] = 1 > 0$, which is a contradiction. We obtain that $[D] \cdot l \leq 0$, and D is an irreducible component of a fiber. Thus $[D] \cdot l = 0$, and $[\sigma] \cdot [D] \leq 0$, which implies that $[\sigma] \cdot [D] = 0$, and D is an irreducible component of a singular fiber of f which does not intersect with σ. Therefore $[\sigma]+3l$ is nef and big, and an irreducible curve D satisfies $([\sigma]+3l)\cdot[D] = 0$ if and only if D is an irreducible component of a singular fiber of f which does not intersect with σ. There is an ample class $\alpha_{\check{M}}$ on the Weierstrass model $\check{M}$ such that $[\sigma] + 3l = \pi^*\alpha_{\check{M}}$, and by [53], there exists a unique Ricci-flat Kähler-Einstein metric $\omega_{\check{M}} \in \alpha_{\check{M}}$ on $\check{M}$ in the orbifold sense.

Since $[\pi^*\omega_{\check{M}}]^2 = ([\sigma] + 3l)^2 = [\mathrm{Im}\Omega]^2 = [\mathrm{Re}\Omega]^2$, $(\pi^*\omega_{\check{M}}, \mathrm{Re}\Omega, \mathrm{Im}\Omega)$ is a HyperKähler triple on $\pi^{-1}(\check{M}_{reg})$. By using the HyperKähler rotation, we can find new complex structure K, and define a family of HyperKähler structures

$$\check{\Omega}_t = (l \cdot \tilde{\alpha}_t)^{-1}(\pi^*\omega_{\check{M}} - i\mathrm{Im}\Omega), \quad \check{\omega}_t = (l \cdot \tilde{\alpha}_t)^{-1}\mathrm{Re}\Omega,$$

which satisfy

$$[\check{\Omega}_t] = (l \cdot [\mathrm{Re}\Omega_{J_t}])^{-1}([\sigma] + 3l - i[\omega_{J_t}]), \text{ and } [\check{\omega}_t] = (l \cdot [\mathrm{Re}\Omega_{J_t}])^{-1}[\mathrm{Im}\Omega_{J_t}].$$

By letting

$$\check{\mathbb{B}}_t = (l \cdot [\mathrm{Re}\Omega_{J_t}])^{-1}[\mathrm{Re}\Omega_{J_t}] - [\sigma] + \mathrm{mod}(l),$$

the proof of Theorem 7.3 in [40] shows that $(M, \check{\omega}_t, \check{\Omega}_t)$ with $\check{\mathbb{B}}_t$ is the SYZ mirror of $(M, \omega_{J_t}, \Omega_{J_t})$, i.e. $(f : M_{N_0} \to N_0, \check{\omega}_t, \check{\Omega}_t)$ is the dual special Lagrangian fibration of $(f : M_{N_0} \to N_0, \omega_{J_t}, \Omega_{J_t})$. □

We now assume that M satisfies the hypotheses of Theorem 3.1, which gives $M = \check{M}$ and π is the identity. We can now see how Theorem 3.1 applies to Conjecture 5.5 in [34]. In our setup, the anti-self-dual connection Ξ_t and the complex structure J_t induce a holomorphic structure on $\mathcal{V}$ for any $t \in (0, 1]$, and Ξ_t satisfies the Hermitian-Yang-Mills equation

$$F_{\Xi_t} \wedge \omega_{J_t} = 0, \quad \text{and} \quad F_{\Xi_t} \wedge \Omega_{J_t} = 0.$$

The spectral cover D_t and the limit D_0 are special Lagrangian cycles with respect to the mirror HyperKähler structure $(\check{\omega}, \check{\Omega})$. We now rephrase Theorem 3.1 and Theorem 3.2 in the context of SYZ mirror symmetry.

Theorem 3.4. *Under the assumptions of Theorem 3.1, for any sequence $t_k \to 0$ and any $p > 2$, there exists an open dense subset $N^o \subset N_0$, a subsequence (still denoted t_k), a sequence of L^p_2 unitary gauge changes u_k of P, and a L^p_1 $SU(n)$-connection Ξ_0 on $P|_{M_{N^o}}$ so that*

$$u_k(\Xi_{t_k}) \to \Xi_0$$

in the locally L^p_1 sense on M_{N^o}. Here the norms are calculated by using a fixed metric on M.

For any $w \in N^o$, the restriction of Ξ_0 to the fiber M_w, denoted $\Xi_0|_{M_w}$, is $C^{1,\alpha}$ gauge equivalent to a smooth flat $SU(n)$-connection

$$u_\infty(\Xi_0|_{M_w}) = \frac{\pi}{\mathrm{Im}(\tau)}(\mathrm{diag}\{q_1(w), \cdots, q_n(w)\}d\bar{z} - \mathrm{diag}\{\bar{q}_1(w), \cdots, \bar{q}_n(w)\}dz),$$

where $u_\infty \in \mathcal{G}^{1,\alpha}(M_w)$, $M_w \cong \mathbb{C}/\Lambda_\tau$, $\Lambda_\tau = \mathrm{Span}_{\mathbb{Z}}\{1, \tau\}$, $\sigma(w) = 0$, and z denotes the coordinate on $\mathbb{C}$. As w varies, $\{q_1(w), \cdots, q_n(w)\} \subset M_w$ forms a special Lagrangian multisection of $f^{-1}(N^o) \to N^o$ with respect to the SYZ mirror HyperKähler structure $(\check{\omega}, \check{\Omega})$, and its closure D^o_0 is a special Lagrangian cycle, i.e.

$$\check{\omega}|_{D^o_0} \equiv 0, \quad \text{and} \quad \mathrm{Im}\check{\Omega}|_{D^o_0} \equiv 0.$$

The family of special Lagrangian submanifolds D_t with respect to $(\check{\omega}, \check{\Omega})$ converges to D^o_0 on $f^{-1}(N^o)$ in the locally Hausdorff sense. Furthermore, the unitary gauge equivalent class of the limit connection Ξ_0 is the Fourier-Mukai transform of a flat $U(1)$-connection Θ on $D^o_0 \cap M_{N^o}$, i.e.

$$\Xi_0 \in \mathcal{FM}(D^o_0 \cap M_{N^o}, \Theta).$$

Conversely, if D is a smooth special Lagrangian submanifold with respect to $(\check{\omega}, \check{\Omega})$ on M such that D represents $n[\sigma]+ml \in H_2(M, \mathbb{Z})$ for some $m \in \mathbb{N}$, and Θ is a flat $U(1)$-connection on D, then D is a smooth holomorphic curve in M. The argument in Section 3.1 shows that there is a stable bundle V of rank n with respect to $f^*c_1(\mathcal{O}_{\mathbb{CP}^1}(1)) + t\alpha$ for $0 < t \ll 1$. The anti-self-dual connections Ξ_t on V are also Hermitian-Yang-Mills with respect to $(\omega_{J_t}, \Omega_{J_t})$.

In the context of mirror symmetry, a special Lagrangian submanifold with a flat $U(1)$-connection is called an A-cycle, and a Hermitian-Yang-Mills connection on a complex submanifold is called a B-cycle (cf. [56, 51, 73]). The correspondence between B-cycles and A-cycles is motivated by the study of homological mirror symmetry via the SYZ construction in [7, 33, 34], and the extended mirror symmetry with bundles [56, 73]. Theorem 3.4 says that in the current case, the adiabatic limit of B-cycles is corresponding to an A-cycle on the mirror K3 surface.

3.2. **Remarks.** We conclude this section with a few more remarks.

Remark 3. Note that the Levi-Civita connection of the Ricci-flat Kähler-Einstein metric ω_t is an anti-self-dual connection. However Theorem 3.1 does not apply to this case due to the following. If M_w is a smooth fiber, then the restriction of the tangent bundle of M satisfies a short exact sequence

$$0 \to TM_w \to TM|_{M_w} \to f^*T_wN \to 0,$$

and $TM|_{M_w}$ is S-equivalent to $\mathcal{O}_{M_w} \oplus \mathcal{O}_{M_w}$. Thus the special cover of TM is $D_{TM} = 2\sigma(N)$, and is not reduced. Consequently, the hypotheses of Theorem 3.1 are not satisfied.

The curvature F_{Ξ_t} in Theorem 3.1 behaves very differently from the curvature of the Ricci-flat Kähler-Einstein metric ω_t. In the metric case, the curvature R_{ω_t} of ω_t is bounded away from the singular fibers along the collapsing of ω_t, i.e.

$$\sup_{M_K} |R_{\omega_t}|_{\omega_t} \leq C_K,$$

for any compact subset $K \subset N_0$, by [45, 41]. Furthermore, there is a more general result in [15] that asserts the boundedness of curvatures of sufficiently collapsed Ricci-flat Riemannian Einstein metrics g on 4-manifolds away from finite metric balls. The readers are referred to [15] for details.

In Theorem 3.1, it is shown that the curvature F_{Ξ_t} is bounded with respect to any fixed metric on M_U. However, F_{Ξ_t} can not be bounded with respect to the collapsed metric ω_t as the following demonstrates. If it were bounded, then Proposition 7.1 of Section 7 shows that on any $U \subset N^o$,

$$\begin{aligned}\int_U \sum_{j=1,2} \|\partial_{x_j} A_{0,t}\|_w^2 dx_1 dx_2 &\leq C(\|F_{\Xi_t}\|^2_{L^2(M_U,\omega_t)} + t) \\ &\leq C(\sup_{M_U} |F_{\Xi_t}|^2_{\omega_t} \mathrm{Vol}_{\omega_t}(M_U) + t) \\ &\leq Ct \to 0,\end{aligned}$$

where x_1 and x_2 are coordinates on U, which implies $\partial_{x_j} A_0 \equiv 0$, $j = 1, 2$. Thus $\partial_{x_j}(\mathrm{Im}(\tau)^{-1} q_i(w)) \equiv 0$, $j = 1, 2$, and $q_i(w) = c_i(\tau(w) - \bar{\tau}(w))$ for constants $c_i \in \mathbb{C}$, $i = 1, \cdots, n$. Note that $q_i(w)$ is holomorphic, and $\tau(w)$ is not constant as the fibration f is a Weierstrass fibration. We have $c_i = 0$ and $q_i(w) \equiv 0$, $i = 1, \cdots, n$. Hence $D_0^o \cap M_U = n\sigma(U)$, which contradicts the assumption of D_0^o being reduced.

Remark 4. Theorem 3.1 is a compactness result, i.e. the convergence of Ξ_t occurs along subsequences t_k. The convergence along the parameter t may hold under certain stronger assumptions, for example the following. For any $t \ll 1$, we assume that $V_t|_{M_w}$ is regular semi-stable for any $w \in N$. As in Section 2.4, Proposition 5.7 of [30] shows that

$$V_t = (\nu_{D_t})_*(\mathcal{O}_{\tilde{M}}(\Delta_t - \Sigma_{D_t}) \otimes \tilde{f}^* \tilde{L}_t)$$

for a line bundle $\tilde{L}_t$ on D_t. If we assume further that $\tilde{L}_t$ converges to a $\tilde{L}_0$ on D_0 as divisors along the convergence of D_t to D_0, then we expect that Ξ_t converges away from finite fibers without passing to any subsequence, which would be left for the future study.

Remark 5. There are many more questions that the authors would like to investigate in the future. Firstly, we would like to understand what are the corresponding algebraic geometric descriptions of the type I and type II bubbles in the proof of Proposition 4.1. Secondly, we like to have an explicit formula for the second Chern number $c_2(\mathcal{V})$ via the bubbles and the limit special cover D_0. Here a certain bubble tree convergence is expected.

Finally, we like to study the metric geometry of the moduli space of anti-self-dual Yang-Mills connections on collapsed K3 surfaces, inspired by the F-theory/heterotic string theory duality as in [29]. For any $0 < t \leq (\frac{n^3}{4}c)^{-1}$, let $\mathfrak{M}_t(n,c)$ be the moduli space of anti-self-dual connections on $\mathcal{V}$ with respect to the HyperKähler structure (ω_t, Ω), where $c = c_2(\mathcal{V})$, which is not empty (cf. Theorem 2.9). By Theorem 7.10 in [52], (ω_t, Ω) induces a HyperKähler structure $(\omega_{\mathfrak{M},t}, \Omega_{\mathfrak{M},t})$ on the regular locus $\mathfrak{M}_t(n,c)^o$ of $\mathfrak{M}_t(n,c)$. Furthermore, it is expected that there is a holomorphic lagrangian fibration $\mathfrak{f} : \mathfrak{M}_t(n,c)^o \to \mathfrak{U} \subset |n\sigma(N) + ml|$ (cf. Section 2.4 of [29]). For example, if $D \in |n\sigma(N) + ml|$ is smooth, then the fiber $\mathfrak{f}^{-1}(D)$ is the Jacobian $\mathfrak{J}(D)$ of D, which parameterises the flat $U(1)$-connections on D. We would like to investigate the degeneration behavior of $(\omega_{\mathfrak{M},t}, \Omega_{\mathfrak{M},t})$ when $t \to 0$ in future study.

4. The proof of Theorem 3.1

In this section we prove Theorem 3.1, assuming some important estimates which will be proved in the subsequent sections. We begin with a bubbling result, which gives a decay estimate for curvature away from a finite set. This set may depend on the chosen sequence of times $t_k \to 0$.

Since we are interested in the behavior of the restriction of the connections Ξ_{t_k} to a fiber M_w, we use the notation $A_{t_k}(w) = \Xi_{t_k}|_{M_w}$. In general we write this fiberwise connection as A_{t_k}, as the dependence on w is clear from context.

Proposition 4.1. *If Ξ_t is a family of anti-self-dual connections on P with respect to (ω_t, Ω), then for any sequence $t_k \to 0$, there is a Zariski open subset $N_1 \subset N_0$, and a subsequence (still denoted t_k), so that the curvature $F_{\Xi_{t_k}}$ satisfies*

$$\sup_{M_K} |F_{\Xi_{t_k}}|_{\omega_{t_k}} \leq \frac{\epsilon_k}{t_k}$$

on any compact subset $K \subset N_1$. Here the constants ϵ_k may depend on K, and satisfy $\epsilon_k \to 0$ as $k \to \infty$. Consequently, for any $w \in K$ and $t_k \ll 1$,

$$\|F_{A_{t_k}}\|_{C^0(\omega^{SF}|_{M_w})} \to 0,$$

and $V_{t_k}|_{M_w}$ is semi-stable.

Note that the above assumptions are slightly weaker than those used in Theorem 3.1. To prove the proposition, we follow a bubbling argument similar to arguments seen previously (for example [22]), however we present the details here for completeness.

Proof. Suppose that there exists a sequence of points $p_k \in M$ so that $f(p_k) \to w$ in N_0, and furthermore

$$\liminf_{k\to\infty} t_k |F_{\Xi_{t_k}}|_{\omega_{t_k}}(p_k) > 0. \tag{4.1}$$

We claim that there is a universal constant $\varepsilon > 0$ such that for any neighborhood U_w of w,

$$\int_{M_{U_w}} |F_{\Xi_{t_k}}|^2_{\omega_{t_k}} \omega^2_{t_k} \geq \varepsilon,$$

for $k \gg 1$. Once this is demonstrated, by (2.2) there can only be a finite number of such w.

By [41], for some $p \in M_w$ we have

$$(M, t_k^{-1}\omega_{t_k}, p_k) \to (M_x \times \mathbb{C}, \omega_\infty = \omega_w^F + \frac{i}{2} W^{-1}(w) d\tilde{w} \wedge d\bar{\tilde{w}}, p)$$

in the pointed C^∞-Cheeger-Gromov sense, where ω_w^F is the flat Kähler metric representing $\alpha|_{M_w}$, i.e. $\omega_w^F = \omega^{SF}|_{M_w}$, and $\tilde{w}$ denotes the scaled coordinate of $\mathbb{C}$ (see Section 2.4). More precisely, if $D_r = \{\tilde{w} \in \mathbb{C} || \tilde{w}| < r\}$, there are smooth embeddings $\Phi_{t_k,r} : M_w \times D_r \to M_U$ such that

$$\Phi^*_{t_k,r} t_k^{-1} \omega_{t_k} \to \omega_\infty, \quad \Phi^*_{t_k,r} I \Phi_{t_k,r,*} \to I_\infty,$$

in the C^∞-sense on $M_w \times D_r$, where I (resp. I_∞) denotes the complex structure on M (resp. $M_w \times \mathbb{C}$).

We have two cases. In the first case, for any compact subset $K \subset M_w \times \mathbb{C}$, there is a constant $C_K > 0$ such that

$$|F_{\Xi_{t_k}}|_{t_k^{-1}\omega_{t_k}} = t_k |F_{\Xi_{t_k}}|_{\omega_{t_k}} \leq C_K,$$

on $\Phi_{t_k,r}(K)$, $r \gg 1$. By passing a subsequence, Uhlenbeck's strong compactness theorem shows that there is a sequence of unitary gauge transformations $u_{K,k}$, and an anti-self-dual $SU(n)$-connection Ξ_∞ on $M_w \times \mathbb{C}$ such that $u_{K,k}(\Phi^*_{t_k,r}\Xi_{t_k})$ converges to Ξ_∞ in the locally C^∞-sense on K. Thus, in the C^0-sense on K,

$$\Phi^*_{t_k,r} |F_{\Xi_{t_k}}|_{t_k^{-1}\omega_{t_k}} \to |F_{\Xi_\infty}|_{\omega_\infty}, \quad \text{and} \quad |F_{\Xi_\infty}|_{\omega_\infty}(p) > 0.$$

By [74], there is a constant $\mu = \mu(n)$ depending only on the group $SU(n)$, such that

$$\int_{M_x \times \mathbb{C}} |F_{\Xi_\infty}|^2_{\omega_\infty} \omega^2_\infty \geq \mu.$$

Furthermore if $n = 2$, we know $\mu(2) = 4\pi^2$. This is called the bubble of type II in [22]. By choosing K large enough,

$$\int_{M_{U_w}} |F_{\Xi_{t_k}}|^2_{\omega_{t_k}} \omega^2_{t_k} \geq \int_{\Phi_{t_k,r}(K)} |F_{\Xi_{t_k}}|^2_{t_k^{-1}\omega_{t_k}} t_k^{-2} \omega^2_{t_k} \geq \frac{\mu}{2},$$

for $k \gg 1$.

The second case is that there are $p'_k \in M$ such that

$$d_{t_k^{-1}\omega_{t_k}}(p_k, p'_k) < C < \infty, \quad \text{and,} \quad t_k |F_{\Xi_{t_k}}|_{\omega_{t_k}}(p'_k) \to \infty,$$

when $k \to \infty$. In order to perform the bubbling argument, recall the following point choosing lemma.

Lemma 4.2 (Lemma 9.3 in [22]). *Let (Y, d_Y) be a complete metric space, and ζ be a continuous non-negative function. For any $y \in Y$, there exist $y' \in Y$ and $0 < \rho \leq 1$ such that*

$$d_Y(y, y') \leq 1, \quad \sup_{B_{d_Y}(y', \rho)} \zeta \leq 2\zeta(y'), \quad \text{and} \quad 2\rho\zeta(y') \geq \zeta(y).$$

We apply this lemma to $\zeta = |F_{\Xi_{t_k}}|_{t_k^{-1}\omega_{t_k}}$, $y = p_k'$, and obtain $y' = p_k''$ and $0 \leq \rho \leq 1$. We further rescale the metric, and $(M, |F_{\Xi_{t_k}}|^{-1}_{\omega_{t_k}}(p_k'')\omega_{t_k}, p_k'')$ converges to the standard Euclidean space $(\mathbb{C}^2, \omega_E, 0)$ in the smooth Cheeger-Gromov sense by passing to a subsequence. The same argument as above shows that Ξ_{t_k} smoothly converges to an non-trivial anti-self-dual $SU(n)$-connection Ξ'_∞ on $\mathbb{C}^2$ by passing to certain unitary gauge changes and subsequences. We now have

$$\int_{\mathbb{C}^2} |F_{\Xi'_\infty}|^2_{\omega_E} \omega_E^2 \geq \tau,$$

where τ is the constant in Lemma 2.14. This is called a bubble of type I, and is standard in the study of Yang-Mills fields on 4-manifolds (cf. [20, 27]). Just as above,

$$\int_{M_{U_w}} |F_{\Xi_{t_k}}|^2_{\omega_{t_k}} \omega_{t_k}^2 \geq \int_{\Phi_{K,k}(K)} |F_{\Xi_{t_k}}|^2_{t_k^{-1}\omega_{t_k}} t_k^{-2}\omega_{t_k}^2 \geq \frac{\tau}{2},$$

for $k \gg 1$, where K satisfies that $p_k' \in \Phi_{K,k}(K)$. We obtain the claim by letting $\varepsilon = \frac{1}{2}\min\{\mu, \tau\}$.

Let S_1 be the set of points $x \in N_0$ for which there is a sequence $p_k \in M$ such that $f(p_k) \to w$ in N_0, and (4.1) is satisfied. By (2.2)

$$8\pi^2 c_2(\mathcal{V}) = \lim_{k\to\infty} \int_M |F_{\Xi_{t_k}}|^2_{\omega_{t_k}} \omega_{t_k}^2 \geq \sharp(S_1)\varepsilon,$$

and as a result S_1 is a finite set. Therefore $N_1 = N_0 \backslash S_1$ is a Zariski open subset, and for any compact subset $K \subset N_1$,

$$\sup_{M_K} t_k |F_{\Xi_{t_k}}|_{\omega_{t_k}} \leq \epsilon_k \to 0,$$

when $k \to \infty$.

Since $\Phi^*_{t_k,r} t_k^{-1}\omega_{t_k}$ converges smoothly to ω_∞ on $M_w \in \mathbb{C}$ for $w \in K$, we have

$$\|F_{A_{t_k}}\|_{C^0(\omega^F)} \leq 2\|F_{A_{t_k}}\|_{C^0(t_k^{-1}\omega_{t_k}|_{M_w})} \leq 2\sup_{M_K} |F_{\Xi_{t_k}}|_{t_k^{-1}\omega_{t_k}} \to 0.$$

By Proposition 2.4, $V_{t_k}|_{M_w}$ is semi-stable, where as above V_{t_k} denotes $\mathcal{V}$ equipped with the holomorphic structure induced by Ξ_{t_k}. $\square$

Restricting to a fiber M_w, by the above proposition, weak Uhlenbeck compactness gives that for any $p > 2$, there exists a sequence of unitary gauge $u_{w,k}$ such that along a subsequence of times, $u_{w,k}(A_{t_k})$ converges in L_1^p to a flat L_1^p-connection $\Xi_{\infty,w}$ on M_w. In other words, we have fiberwise convergence of Ξ_{t_k} up to gauge changes. However, it is not clear yet that Ξ_{t_k} has any limit when $t_k \to 0$ on M_K. For this, we need the stronger assumptions in Theorem 3.1, and further results and estimates.

We now work under the setup of Theorem 3.1, and consider a sequence of connections Ξ_{t_k} where $t_k \to 0$ as $k \to \infty$. Before we turn to the key estimates, we need to describe the explicit form of the holomorphic structure of the bundle V_t in a local trivialization.

Note that $f|_{D_0^o} : D_0^o \to N$ is an n-sheeted branched covering. If $S_{D_0^o}$ denotes the subset of D_0^o consisting all singular points of D_0^o and all branch points of $f|_{D_0^o}$, then $f(S_{D_0^o})$ is a finite subset of N. We define a Zariski open subset

$$N^o = N_1 \backslash (f(D_0 - D_0^o) \cup f(S_{D_0^o})). \tag{4.2}$$

On N^o, $f|_{D_0^o} : f|_{D_0^o}^{-1}(N^o) \to N^o$ is an n-sheeted unbranched covering, since D_0^o is reduced. For any $w \in N^o$, $D_0^o \cap M_w$ consists n distinct points in M_w, i.e. $D_0^o \cap M_w = \{q_1, \cdots, q_n\}$ where $q_i \neq q_j$ for any $i \neq j$. The trivial bundle $\mathcal{V}|_{M_w}$ equipped with the holomorphic structure induced by $D_0^o \cap M_w$ is isomorphic to the flat holomorphic bundle

$$\mathcal{O}_{M_w}(q_1 - \sigma(w)) \oplus \cdots \oplus \mathcal{O}_{M_x}(q_n - \sigma(w)).$$

Since D_t converges to D_0 and $D_0 - D_0^o \in |(m-m')l|$ is supported on fibers, for any compact subset $K \subset N^o$ we have that $f : D_t \cap M_K \to K$ is an n-sheeted unbranched covering for $t \ll 1$. For any $w \in K$, $D_t \cap M_w = \{q_{1,t}, \cdots, q_{n,t}\}$ such that $q_{i,t} \neq q_{j,t}$ for any $i \neq j$, and $q_{i,t} \to q_i$ when $t \to 0$. Furthermore, $V_t|_{M_w}$ is semi-stable, which implies that $V_t|_{M_w}$ is regular by Proposition 6.4 in [30], and

$$V_t|_{M_w} \cong \mathcal{O}_{M_w}(q_{1,t} - \sigma(w)) \oplus \cdots \oplus \mathcal{O}_{M_w}(q_{n,t} - \sigma(w)).$$

For any $t \ll 1$, there is a Zariski open subset $N_t^o \supset K$ such that $V_t|_{M_w}$, $w \in N_t^o$, is regular semi-stable. Proposition 5.7 of [30] asserts that

$$V_t|_{M_{N_t^o}} = (\nu_{D_t})_*(\mathcal{O}_{\tilde{M}_{N_t^o}}(\Delta_t - \Sigma_{D_t}) \otimes \tilde{f}^* \tilde{L}_t)$$

for a certain line bundle $\tilde{L}_t$ on $D_t \cap M_{N_t^o}$. Here, as in Section 2.4,

$$\nu_{D_t} : \tilde{M}_{N_t^o} = D_t \times_{N_t^o} M \to M_{N_t^o},$$

$\Sigma_{D_t} = \nu_{D_t}^* \sigma$, and $\Delta_t = \tilde{M}_{N_t^o} \cap \Delta_0$ for the diagonal Δ_0 of $M \times_{N_t^o} M$ via the natural embedding $\tilde{M}_{N_t^o} = D_t \times_{N_t^o} M \hookrightarrow M \times_{N_t^o} M$.

Let $U \subset K \subset N_t^o$ be an open subset biholomorphic to the unit disk, and w be a coordinate on U. Then $M_U \cong (U \times \mathbb{C})/\Lambda$ for lattice subbundle $\Lambda = \mathrm{Span}_{\mathbb{Z}}\{1, \tau\}$, where $\tau = \tau(w)$ varies holomorphically and is the period of the elliptic curve M_w. Furthermore under this identification the section

σ satisfies $\sigma \equiv 0$. If z is the coordinate on $\mathbb{C}$, we define real functions y_1 and y_2 on $U \times \mathbb{C}$ by $z = y_1 + \tau y_2$. Then dy_1 and dy_2 are well-defined 1-forms on M_U, and we have the decomposition of cotangent bundle $T^*M_U \cong \mathrm{Span}_{\mathbb{R}}\{dy_1, dy_2\} \oplus \mathrm{Span}_{\mathbb{R}}\{dx_1, dx_2\}$, where $w = x_1 + ix_2$. Let $\theta = dy_1 + \tau dy_2$, whose restriction $\theta|_{M_w} = dz$ on any fiber M_w. Note that $\overline{\partial}\tau = 0$, $d\tau = \partial\tau$ and $0 = \overline{\partial}\tau_1 + i\overline{\partial}\tau_2$, where $\tau = \tau_1 + i\tau_2$. Thus $dz = dy_1 + \tau dy_2 + y_2 d\tau$, $2i\tau_2 y_2 = z - \bar{z}$, and $\theta = dz - \frac{z-\bar{z}}{2i\tau_2}\partial_w \tau dw = dz + bdw$.

We fix the trivializations $P|_{M_U} \cong M_U \times SU(n)$ and $\mathcal{V}|_{M_U} \cong M_U \times \mathbb{C}^n$. The unitary gauge group consists of $SU(n)$ valued functions, in other words $\mathcal{G} = C^\infty(M_U, SU(n))$, and the complex gauge group is $\mathcal{G}_{\mathbb{C}} = C^\infty(M_U, SL(n,\mathbb{C}))$ under this trivialization. The respective Lie algebras are $\mathfrak{g} = C^\infty(M_U, \mathfrak{su}(n))$ and $\mathfrak{g}_{\mathbb{C}} = C^\infty(M_U, \mathfrak{sl}(n,\mathbb{C}))$. Note that there is the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ induced by $\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{su}(n) + i\mathfrak{su}(n)$, and if $s \in \mathfrak{g}_{\mathbb{C}}$ is Hermitian (given by $s^* = s$), then $s \in i\mathfrak{g}$. Therefore any complex gauge g can be written as $g = \exp(v + s)$, for a certain $v \in \mathfrak{g}$ and an $s \in i\mathfrak{g}$.

Note that $D_0^o \cap M_U$ (resp. $D_t \cap M_U$) is given by n distinct holomorphic functions $q_j(w)$ (resp. $q_{j,t}(w)$), and for any j, $q_{j,t}(w) \to q_j(w)$ in the C^∞-sense as $t \to 0$. Thus $D_t \cap M_U$ consists of n distinct unit disks, and because $\tilde{L}_t|_{D_t \cap M_U}$ is holomorphically trivial, we obtain

$$V_t|_{M_U} \cong \bigoplus_{j=1}^{n} \mathcal{O}_{M_U}(q_{j,t}(U) - \sigma(U)).$$

We define the background connections on the trivial bundle $\mathcal{V}|_{M_U}$

$$A_{0,t} = \pi(\mathrm{Im}(\tau))^{-1}(\mathrm{diag}\{q_{1,t}, \cdots, q_{n,t}\}\bar{\theta} - \mathrm{diag}\{\bar{q}_{1,t}, \cdots, \bar{q}_{n,t}\}\theta), \tag{4.3}$$

$$A_0 = \pi(\mathrm{Im}(\tau))^{-1}(\mathrm{diag}\{q_1, \cdots, q_n\}\bar{\theta} - \mathrm{diag}\{\bar{q}_1, \cdots, \bar{q}_n\}\theta). \tag{4.4}$$

Thus $A_{0,t} \to A_0$ in the C^∞-sense when $t \to 0$, $V_t|_{M_w}$ is isomorphic to $\mathcal{V}|_{M_w}$ equipped with the holomorphic structure induced by the flat connection $A_{0,t}|_{M_w}$, and $A_0|_{M_w}$ induces the holomorphic bundle structure $\bigoplus_{i=1}^{n} \mathcal{O}_{M_w}(q_i(w) - \sigma(w))$.

Lemma 4.3. *The unitary connection $A_{0,t}$ on $\mathcal{V}|_{M_U}$ induces the holomorphic structure isomorphic to $V_t|_{M_U}$.*

Proof. In general, if L is a holomorphic line bundle, and h determines a Hermitian metric on L in a local holomorphic trivialization, then the unique Chern connection is given by $A_h = \partial \log h$. If ρ is a local unitary frame, i.e. $|\rho|_h^2 = h|\rho|^2 \equiv 1$, then we have smooth trivialization of L via $\rho \mapsto 1$, and under such trivialization, A_h is transformed to $A = \overline{\partial}\log\rho - \partial\log\bar{\rho}$. A different choice of ρ gives a unitary gauge transformation of A.

Note that the holomorphic line bundle $\mathcal{O}_{M_U}(q_{j,t}(U) - \sigma(U))$ is given by the multiplier $\{e_1 \equiv 1, e_\tau = \exp(-2\pi i q_{j,t}(w))\}$, i.e. $\mathcal{O}_{M_U}(q_{j,t}(U) - \sigma(U))$ is obtained by the quotient of $U \times \mathbb{C} \times \mathbb{C}$ via

$$(w, z, \xi) \sim (w, z+1, e_1\xi), \quad (w, z, \xi) \sim (w, z+\tau, e_\tau\xi)$$

(cf. Section 6 in Chapter 2 of [38]). On $U \times \mathbb{C}$, if we let

$$h = \exp \pi\big(\mathrm{Im}(\tau)^{-1}(z-\bar{z})(q_{j,t}-\bar{q}_{j,t})\big),$$

then $h(w,z+1)=h(w,z)$ and $h(w,z+\tau)=|\exp(2\pi i q_{j,t}(w))|^2 h(w,z)$, and thus h defines a Hermitian metric on $\mathcal{O}_{M_U}(q_{j,t}(U)-\sigma(U))$. If

$$\rho = \exp\big(-\pi\mathrm{Im}(\tau)^{-1}(z-\bar{z})q_{j,t}\big),$$

then $h|\rho|^2=1$, $\rho(w,z+1)=\rho(w,z)$ and $\rho(w,z+\tau)=e_\tau\rho(w,z)$. Thus ρ is a global unitary frame, and under the trivialization induced by ρ, the Chern connection $\Xi_{0,t,j}=\Xi_j^{1,0}+\Xi_j^{0,1}$ is given by $\Xi_j^{1,0}=-\overline{\Xi_j^{0,1}}$ and

$$\Xi_j^{0,1}=\bar{\partial}\log\rho=\pi\mathrm{Im}(\tau)^{-1}q_{j,t}d\bar{z}-\pi(z-\bar{z})q_{j,t}\bar{\partial}\mathrm{Im}(\tau)^{-1}=\pi\mathrm{Im}(\tau)^{-1}q_{j,t}\bar{\theta},$$

by

$$\bar{\theta}=d\bar{z}-\frac{z-\bar{z}}{2i\mathrm{Im}(\tau)}\partial_{\bar{w}}\bar{\tau}d\bar{w}=d\bar{z}+\frac{z-\bar{z}}{\mathrm{Im}(\tau)}\partial_{\bar{w}}\mathrm{Im}(\tau)d\bar{w}.$$

We obtain the desired conclusion. □

Since Ξ_t and $A_{0,t}$ induce the same holomorphic structure on $\mathcal{V}|_{M_U}$ over M_U, there is a complex gauge $g\in\mathcal{G}_{\mathbb{C}}$ such that $g(\Xi_t)=A_{0,t}$. Note that gg^* is Hermitian, and $gg^*=e^{2s_t}$ for some $s_t\in C^\infty(M_U,\mathfrak{sl}(n,\mathbb{C}))$ with $s_t^*=s_t$. If we let $u=e^{-s_t}g$, then $u^*=u^{-1}$, i.e. u is a unitary gauge, and $g=e^{s_t}u$. Therefore, by a further unitary gauge change if necessary, we assume that

$$e^{s_t}(\Xi_t)=A_{0,t} \tag{4.5}$$

for a Hermitian gauge e^{s_t} on M_U.

In order to prove the main theorem, we need to improve the curvature estimates of Proposition 4.1.

Proposition 4.4. *For any compact set $K\subset N^o$, there exists a constant C_K such that*

$$\sup_{M_K}|F_{\Xi_{t_k}}|_{\omega_{t_k}}\le C_K t_k{}^{-\frac{1}{2}}.$$

The proof of this proposition can be found in Section 7. This implies the subsequence of connections Ξ_{t_k} satisfies (6.3), which is the main assumption of Proposition 6.1 in Section 6. Thus we can apply Proposition 6.1 to Ξ_{t_k} and achieve uniform C^0 control of the curvature, from which we conclude:

Proposition 4.5. *Along the sequence of connection Ξ_{t_k}, there exists a constant $C_1>0$ such that*

$$\|F_{A_{t_k}}\|_{C^0(M_w)}\le C_1 t_k,\quad \textit{and}\quad \|F_{\Xi_{t_k}}\|_{C^0(M_K)}\le C_1,$$

for any $w\in K$. Consequently, for any $p>2$, by the weak Uhlenbeck compactness theorem [71] there exists a subsequence (still denoted t_k), a sequence of unitary gauge transformations $u_k\in\mathcal{G}^{2,p}$, and a limiting L_1^p connection Ξ_∞, so that

$$u_k(\Xi_{t_k})\to\Xi_\infty$$

in $L_1^p(M_K)$. Here all norms are calculated by using a fixed Kähler metric on M.

In order to prove Theorem 3.1, we also need a generalization of Theorem 1.1 in [17], which is a direct consequence of Lemma 5.4.

Proposition 4.6. *For any $w \in K$ and $0 < \alpha < 1$, there exists a constant $C_2 > 0$ so that*

$$\|A_{t_k} - A_{0,t_k}\|_{C^{0,\alpha}(M_w)} \leq C_2 t_k.$$

Granted these three propositions, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Proposition 4.5 and the Sobolev embedding theorem, there exists $u_k \in \mathcal{G}^{1,\alpha}$ and a limiting $C^{0,\alpha}$-connection Ξ_0, so that

$$u_k(\Xi_{t_k}) \to \Xi_0$$

in $C^{0,\alpha}(M_K)$. Thus, for any $w \in K$, the restriction $\Xi_0|_{M_w}$ of Ξ_0 is a $C^{0,\alpha}$-connection on M_w, and $u_k(\Xi_{t_k})|_{M_w}$ converges to $\Xi_0|_{M_w}$ in the $C^{0,\alpha}$-sense. Proposition 4.6, along with the fact that $A_{0,t} \to A_0$ in the C^∞-sense, gives

$$A_{t_k} \to A_0,$$

on M_w in the $C^{0,\alpha}$-sense, where A_0 is given by (4.4).

Since

$$du_k = u_k \Xi_{t_k}|_{M_w} - u_k(\Xi_{t_k})|_{M_w} u_k,$$

and the u_k are unitary, we have a C^1-bound for u_k, i.e. $\|u_k\|_{C^1(M_w)} \leq C$. As a result, the $C^{0,\alpha}$-convergence of $u_k(\Xi_{t_k})|_{M_w}$ and $\Xi_{t_k}|_{M_w}$ imply the $C^{1,\alpha}$-bound of u_k, i.e. $\|u_k\|_{C^{1,\alpha}(M_w)} \leq C'$. Thus by passing a subsequence, for $\alpha' < \alpha$ we have u_k converges to a $C^{1,\alpha'}$-unitary gauge u_∞ in the $C^{1,\alpha'}$-sense, which satisfies that $u_\infty(\Xi_0|_{M_w}) = A_0$. This concludes the theorem. □

5. A Poincaré inequality for F_{A_t}

We continue to work under the setup of Theorem 3.1, and choose a sequence of connections Ξ_{t_k}. We work on the fiber M_w over a point $w \in N^o$, which is away from the discriminant locus of f, the bubbling points, and the ramification points and singularities of the spectral cover. As above we let A_{t_k} denote the restriction of the anti-self-dual connection Ξ_{t_k} to the smooth fiber M_w. The goal of this section is to derive a Poincaré type inequality for the curvature $F_{A_{t_k}}$, when $F_{A_{t_k}}$ is sufficiently small in the C^0-sense. The following proposition is the key analytic input to overcome the difficulty of the non-smoothness of the moduli spaces of flat connections on elliptic curves.

For notational simplicity we drop the subscript k, and denote our connections by A_t. We do this because, aside from being used to define N^o, the explicit sequence of times t_k does not have any bearing on the results in this section.

Proposition 5.1. *For any compact set $K \subset N^o$, there are constants $\epsilon_K > 0$ and $C_K > 0$ such that if*

$$\|F_{A_t}\|_{C^0(M_w,\omega^{SF})} \leq \epsilon_K$$

for a certain $t \in (0,1]$ and $w \in K$, then

$$\|F_{A_t}\|_w \leq C_K \|d^*_{A_t} F_{A_t}\|_w.$$

We begin by recalling part of our setup, as described in Theorem 3.1. Fix an open subset $U \subset N^o$ biholomorphic to a disk in $\mathbb{C}$, satisfying $f^{-1}(U) \cong (U \times \mathbb{C})/\mathrm{Span}_{\mathbb{Z}}\{1,\tau\}$, where τ is a holomorphic function on U. Fix trivializations $P|_{M_U} \cong M_U \times SU(n)$ and $\mathcal{V}|_{M_U} \cong M_U \times \mathbb{C}^n$. In Section 4 we define the connections $A_{0,t} = \mathrm{diag}\{\alpha_{t,1}, \cdots, \alpha_{t,n}\}$ and $A_0 = \mathrm{diag}\{\alpha_{0,1}, \cdots, \alpha_{0,n}\}$ associated to the spectral covers, where

$$\alpha_{t,j} = \pi \mathrm{Im}(\tau)^{-1}(q_{j,t}(w)\bar{\theta} - \bar{q}_{j,t}(w)\theta), \quad \alpha_{0,j} = \pi \mathrm{Im}(\tau)^{-1}(q_j(w)\bar{\theta} - \bar{q}_j(w)\theta),$$

and $\theta|_{M_w} = dz$. Here all points vary holomorphically in the base, and satisfy

$$\sum_{j=1}^n q_{j,t}(w) \equiv 0, \qquad \sum_{j=1}^n q_j(w) \equiv 0.$$

We also have that $q_{j,t}$ converges to q_j as $t \to 0$ as holomorphic functions. Furthermore, for any $w \in U$,

$$q_{i,t}(w) \neq q_{j,t}(w) \mathrm{mod}(\mathbb{Z} + \tau(w)\mathbb{Z}), \quad q_i(w) \neq q_j(w) \mathrm{mod}(\mathbb{Z} + \tau(w)\mathbb{Z})$$

if $i \neq j$. The connections $d_{A_{0,t}}$ and d_{A_0} act on $\eta \in C^\infty(M_w, \mathfrak{sl}(n,\mathbb{C}))$ via

$$d_{A_{0,t}}\eta = d\eta + [A_{0,t}, \eta], \quad d_{A_0}\eta = d\eta + [A_0, \eta].$$

Note that if $d_{A_{0,t}}\eta = 0$, then $d\eta_{jj} = 0$ and $d\eta_{ij} + (\alpha_{t,i} - \alpha_{t,j})\eta_{ij} = 0$, which implies that $\eta_{ij} = 0$ for $i \neq j$, and η_{jj} are constants. Therefore $\ker d_{A_{0,t}} = \{\mathrm{diag}\{\eta_1, \cdots, \eta_n\} \in \mathfrak{sl}(n,\mathbb{C})\}$, and the same argument gives also $\ker d_{A_0} = \{\mathrm{diag}\{\eta_1, \cdots, \eta_n\} \in \mathfrak{sl}(n,\mathbb{C})\}$.

Since $A_{0,t}$ is flat ($F_{A_{0,t}} = d^2_{A_{0,t}} = 0$), we have a de Rham complex

$$C^\infty(M_w, \mathfrak{sl}(n,\mathbb{C})) \xrightarrow{d_{A_{0,t}}} C^\infty(T^*M_w \otimes \mathfrak{sl}(n,\mathbb{C})) \xrightarrow{d_{A_{0,t}}} C^\infty(\wedge^2 T^*M_w \otimes \mathfrak{sl}(n,\mathbb{C})).$$

Furthermore, there is a well behaved Hodge theory (cf. [6]). If $\star_w$ denotes the Hodge star operator with respect to the flat metric $\omega_w^F := \omega^{SF}|_{M_w}$, then $d^*_{A_{0,t}} = -\star_w d_{A_{0,t}} \star_w$ is the adjoint of $d_{A_{0,t}}$, and $d^*_{A_{0,t}} d_{A_{0,t}} + d_{A_{0,t}} d^*_{A_{0,t}}$ is the Hodge Laplacian. If we denote $\mathcal{H}^q_{A_{0,t}}(M_w, \mathfrak{sl}(n,\mathbb{C}))$ the space of $\mathfrak{sl}(n,\mathbb{C})$ valued harmonic q-forms, the Hodge theory asserts an orthogonal decomposition

$$C^\infty(\wedge^q T^*M_w \otimes \mathfrak{sl}(n,\mathbb{C})) \cong \mathcal{H}^q_{A_{0,t}}(M_w, \mathfrak{sl}(n,\mathbb{C})) \oplus \mathrm{Im} d_{A_{0,t}} \oplus \mathrm{Im} d^*_{A_{0,t}},$$

for $q = 0, 1, 2$.

If we replace $\mathfrak{sl}(n,\mathbb{C})$ by the subalgebra $\mathfrak{su}(n)$, then we have the subcomplex $(C^\infty(\wedge^q T^*M_w \otimes \mathfrak{su}(n)), d_{A_{0,t}})$, the harmonic space of $\mathfrak{su}(n)$ valued q-forms $\mathcal{H}^q_{A_{0,t}}(M_w, \mathfrak{su}(n))$, and the respective Hodge decomposition. Note that we have the connection $A_t \in C^\infty(T^*M_w \otimes \mathfrak{su}(n))$ and the curvature $F_{A_t} \in C^\infty(\wedge^2 T^*M_w \otimes \mathfrak{su}(n))$. The virtual dimension of the moduli space $\mathfrak{M}_{M_w}(n)$ of flat $SU(n)$-connections on M_w is zero due to the Euler number of the complex $(C^\infty(\wedge^q T^*M_w \otimes \mathfrak{su}(n)), d_{A_{0,t}})$ vanishing, and thus the whole $\mathfrak{M}_{M_w}(n)$ is regarded as degenerated, which causes many difficulties in the global analysis. However, the flat connection $A_{0,t}$ belongs to the regular part of $\mathfrak{M}_{M_w}(n)$, and $\mathcal{H}^1_{A_{0,t}}(M_w, \mathfrak{su}(n))$ is the tangent space at $A_{0,t}$. The infinitesimal deformation space under the action of the unitary gauge group is $\mathrm{Im} d_{A_{0,t}} \cap C^\infty(T^*M_w \otimes \mathfrak{su}(n))$, and by using the decomposition $\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n)$, the space $\mathrm{Im} d^*_{A_{0,t}} \cap C^\infty(T^*M_w \otimes \mathfrak{su}(n))$ is identified with the infinitesimal deformation space induced by Hermitian gauges. The readers are referred to [59] for details of the above discussion.

We denote by $\Delta_{A_{0,t}} = -d^*_{A_{0,t}} d_{A_{0,t}}$ the Laplacian operator acting on $C^\infty(M_w, \mathfrak{sl}(n,\mathbb{C}))$, and have $\ker \Delta_{A_{0,t}} = \ker d_{A_{0,t}}$, $\mathrm{Im}\Delta_{A_{0,t}} = \mathrm{Im} d^*_{A_{0,t}}$, and $\ker \Delta_{A_{0,t}} \perp \mathrm{Im} d^*_{A_{0,t}}$ by the Hodge decomposition. We need a uniform estimate for the lower bounds of the first eigenvalue of $\Delta_{A_{0,t}}$.

Lemma 5.2. *For any $w \in U$ and $t \in (0,1]$, if $\lambda_{w,t}$ is the first eigenvalue of $-\Delta_{A_{0,t}}$ on the fiber M_w, then there is a constant $C_1 > 0$ independent of t and w such that*

$$\lambda_{w,t} \geq C_1.$$

Proof. If the above bound does not hold, there are sequences w_k and t_k such that $t_k \to t_0$ in $[0,1]$, $w_k \to w_0$ in U, and

$$\lambda_{w_k,t_k} \to 0$$

when $k \to \infty$. Let $\psi_k \in C^\infty(M_{w_k}, \mathfrak{sl}(n,\mathbb{C}))$ be a normalized eigenvector of $\Delta_{A_{0,t_k}}$, i.e. $\Delta_{A_{0,t_k}} \psi_k = -\lambda_{w_k,t_k} \psi_k$ and $\|\psi_k\|_{w_k} = 1$.

We regard M_w as the 2-torus T^2 equipped with the complex structure I_w, and the Kähler metric ω^F_w as a metric on T^2 with respect to I_w. Since $\tau(w_k) \to \tau(w_0)$, we have that $I_{w_k} \to I_{w_0}$ and $\omega^F_{w_k} \to \omega^F_{w_0}$ in the C^∞-sense. Note that $A_{0,t_k} \to A_{0,t_0}$ in the C^∞-sense, and if $t_0 = 0$, then $A_{0,t_0} = A_0$. Standard elliptic estimates show that $\|\psi_k\|_{C^\ell} \leq C_\ell$ for constants $C_\ell > 0$ independent of k, where the C^ℓ-norms are calculated by using any fixed metric on T^2. By passing to a subsequence, we have that $\psi_k \to \psi_\infty$ smoothly on T^2, $\|\psi_\infty\|_{w_0} = 1$, and $\Delta_{A_{0,t_0}} \psi_\infty = 0$. Thus $\psi_\infty \in \ker \Delta_{A_{0,t_0}}$ and can be represented as $\mathrm{diag}\{\eta_1, \cdots, \eta_n\} \in \mathfrak{sl}(n,\mathbb{C})$.

Since $\psi_k \perp \ker \Delta_{A_{0,t_k}}$, for any $\psi \in \ker \Delta_{A_{0,t_0}} = \ker \Delta_{A_{0,t_k}}$ we have

$$0 = \langle \psi_k, \psi \rangle_{w_k} \to \langle \psi_\infty, \psi \rangle_{w_0}.$$

So $\langle \psi_\infty, \psi \rangle_{w_0} = 0$ yet $\|\psi_\infty\|_{w_0} = 1$. This is a contradiction, and we obtain the conclusion. □

Again restricting our attention to a single fiber M_w for $w \in U$, we can compute the norm of the fiberwise component of the curvature F_{A_t} with respect to the semi-flat metric

$$\|F_{A_t}\|^2_{C^0(M_w,\omega_t^{SF})} = \frac{1}{t^2}\|F_{A_t}\|^2_{C^0(M_w,\omega^{SF})}.$$

Because the error terms relating ω_t and ω_t^{SF} decay fast enough (see Theorem A.1), we have

$$\|F_{A_t}\|^2_{C^0(M_w,\omega^{SF})} \leq Ct^2\|F_{A_t}\|^2_{C^0(M_w,\omega_t)} \leq Ct^2\|F_{\Xi_t}\|^2_{C^0(M_w,\omega_t)}.$$

We assume that there is a constant $0 < \epsilon \ll 1$, which is determined later, such that for a certain t small enough it holds

$$\|F_{A_t}\|_{C^0(M_w,\omega^{SF})} \leq \epsilon, \tag{5.1}$$

for $w \in U$. By Proposition 4.1, there is a sequence $t_k \to 0$ such that

$$\|F_{A_{t_k}}\|^2_{C^0(M_w,\omega^{SF})} \leq Ct_k^2\|F_{\Xi_{t_k}}\|^2_{C^0(M_w,\omega_{t_k})} \leq \epsilon_k \to 0.$$

Here we used that U is away from the bubbling set. Therefore, for any fixed $\epsilon > 0$, if we take t to be some time $t_k \ll 1$ such that $\epsilon_k < \epsilon$, then (5.1) holds.

Recall by (4.5) that there exists a Hermitian gauge transformation e^{-s_t} so that $e^{-s_t}(A_t) = A_{0,t}$. Although given above, we include the definition of this action here to emphasize that we are working exclusively on a fiber:

$$e^{-s_t}(A_t) = A_t + e^{-s_t}\bar{\partial}_{A_t}e^{s_t} + \left(e^{-s_t}\bar{\partial}_{\ A_t}e^{s_t}\right)^*. \tag{5.2}$$

Given inequality (5.1), the assumptions of Theorem 6.1 from [17] are satisfied, which yields a new sequence of Hermitian gauge transformations $e^{\hat{s}_t}$ which are perpendicular to the kernel of $d_{A_{0,t}}$, bounded in C^0, and define the same connection $e_*^{-\hat{s}_t}A_t = A_{0,t}$.

For the remainder of this section we work on the fiber M_w, and so we may drop it from adorning norms when it is clear from context. Similarly, all norms in this section are computed with respect to H and ω_w^F.

Lemma 5.3. *Given* (5.1)*, for every $w \in U$ the Hermitian endomorphism $\hat{s}_t$ satisfies*

$$\|\hat{s}_t\|_{C^0(M_w,\omega^{SF})} \leq C_2\epsilon \tag{5.3}$$

for a uniform constant C_2.

Proof. To begin, we use that $\hat{s}_t$ is uniformly bounded in C^0. Following Appendix A of [50], the fact that $A_{0,t}$ is flat, along with a standard formula for curvatures related by a complex gauge transformation, yields

$$-\Delta_w|\hat{s}_t|^2 \leq -|\partial_{A_{0,t}}\hat{s}_t|^2 + \mathrm{Tr}\left(e^{\hat{s}_t} \star_w F_{A_t}e^{-\hat{s}_t}\hat{s}_t\right), \tag{5.4}$$

where Δ_w is the Laplacian with respect to the flat Kähler metric ω_w^F. Integrating the above equality over M_w, and using Lemma 5.2 along with the fact that $\hat{s}_t$ is perpendicular to the kernel of $d_{A_{0,t}}$, gives

$$\|\hat{s}_t\|^2_w \leq C\|d_{A_{0,t}}\hat{s}_t\|^2_w \leq C\epsilon\|\hat{s}_t\|_w.$$

Therefore $\|\hat{s}_t\|_w \leq C\epsilon$. Now we argue $\|\hat{s}_t\|_{C^0(M_w)}$ is also bounded by $C\epsilon$.

Note that (5.4) implies

$$-\Delta_w |\hat{s}_t|^2 \leq C\epsilon|\hat{s}_t|.$$

Now, suppose the desired bound does not hold, so we can find a sequence of constants $C_t \to \infty$ so $\|\hat{s}_t\|_{C^0} \geq C_t\epsilon$. Set $\phi_t = |\hat{s}_t|^2/\|\hat{s}_t\|^2_{C^0}$. For t small enough it holds

$$-\Delta_w \phi_t \leq \frac{C\epsilon|\hat{s}_t|}{\|\hat{s}_t\|^2_{C^0}} \leq \frac{C}{C_t} \leq 1.$$

If y_t denotes the point in M_w realizing $\sup|\hat{s}_t|^2$, in a fixed neighborhood of radius a of y_t we see ϕ_t is a C^2 function satisfying $-\Delta_w\phi_t \leq 1$, $0 \leq \phi_t \leq 1$, and $\phi_t(y_t) = 1$. Let u_t be a C^2 function satisfying both $\Delta_w u_t = -1$ and $u_t(y_t) = 1$. By making a smaller if necessary we can guarantee that u_t is strictly positive on $B_a(y_t)$, and this choice will only depend on ω^F_w. Thus we have $-\Delta_w(\phi_t - u_t) \leq 0$ and $\phi_t(y_t) - u_t(y_t) = 0$. Applying the mean value inequality to $\phi_t - u_t$ gives

$$0 \leq \int_{B_a(y_t)} (\phi_t - u_t).$$

By the positivity of u_t, there exists a constant $\delta > 0$ independent of t so that

$$\delta \leq \int_{B_a(y_t)} u_t \leq \int_{B_a(y_t)} \phi_t.$$

Rearranging terms gives

$$\|\hat{s}_t\|^2_{C^0} \leq \frac{1}{\delta}\int_{B_a(y_t)} |\hat{s}_t|^2 \leq \frac{1}{\delta}\|\hat{s}_t\|^2_w \leq C\epsilon^2,$$

which is our desired bound. □

The above lemma has some strong consequences, which we now detail. First we need a few key formulas on M_w. The complex gauge action by a Hermitian endomorphism (5.2) gives

$$A_t = e^{\hat{s}_t}_* A_{0,t} = A_{0,t} + e^{\hat{s}_t}\bar{\partial}_{A_{0,t}} e^{-\hat{s}_t} + \left(e^{\hat{s}_t}\bar{\partial}_{A_{0,t}} e^{-\hat{s}_t}\right)^*.$$

For a given s define $\mathrm{ad}_s := [s, \cdot]$, and let $\Upsilon(s) \in \mathrm{End}(\mathrm{End}(V_t))$ denote the power series

$$\Upsilon(s) = \frac{e^{\mathrm{ad}_s} - 1}{\mathrm{ad}_s} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!}(\mathrm{ad}_s)^m.$$

Note that the first term from the power series $\Upsilon(\hat{s}_t)$ is the identity, allowing us to write $\Upsilon(\hat{s}_t) = Id + \tilde{\Upsilon}(\hat{s}_t)$. Now, recall the standard formula for the derivative of the exponential map

$$e^{\hat{s}_t}\bar{\partial}_{A_{0,t}} e^{-\hat{s}_t} = -\Upsilon(\hat{s}_t)\bar{\partial}_{A_{0,t}}\hat{s}_t.$$

Following Appendix A in [50] we see

$$\begin{aligned} A_t &= A_{0,t} - \bar{\partial}_{A_{0,t}}\hat{s}_t + \partial_{A_{0,t}}\hat{s}_t - \tilde{\Upsilon}(\hat{s}_t)\bar{\partial}_{A_{0,t}}\hat{s}_t + \tilde{\Upsilon}(-\hat{s}_t)\partial_{A_{0,t}}\hat{s}_t \\ &= A_{0,t} - i \star_w d_{A_{0,t}}\hat{s}_t + o(\hat{s}_t, \nabla_{A_{0,t}}\hat{s}_t), \end{aligned} \tag{5.5}$$

and

$$
\begin{aligned}
F_{A_t} \;&=\; F_{A_{0,t}} + \Upsilon(-\hat{s}_t)\bar{\partial}_{A_{0,t}}\partial_{A_{0,t}}\hat{s}_t - \Upsilon(\hat{s}_t)\partial_{A_{0,t}}\bar{\partial}_{A_{0,t}}\hat{s}_t \\
&\quad + \bar{\partial}_{A_{0,t}}\Upsilon(-\hat{s}_t)\wedge\partial_{A_{0,t}}\hat{s}_t - \partial_{A_{0,t}}\Upsilon(\hat{s}_t)\wedge\bar{\partial}_{A_{0,t}}\hat{s}_t \\
&\quad - \Upsilon(-\hat{s}_t)\partial_{A_{0,t}}\hat{s}_t\wedge\Upsilon(\hat{s}_t)\bar{\partial}_{A_{0,t}}\hat{s}_t + \Upsilon(\hat{s}_t)\bar{\partial}_{A_{0,t}}\hat{s}_t\wedge\Upsilon(-\hat{s}_t)\partial_{A_{0,t}}\hat{s}_t.
\end{aligned} \tag{5.6}
$$

This formula, along with the fact that $A_{0,t}$ is flat, leads to the following characterization of the curvature F_{A_t}

$$F_{A_t} = -i\, d_{A_{0,t}} \star_w d_{A_{0,t}}\hat{s}_t + T_1(\hat{s}_t, \nabla^2_{A_{0,t}}\hat{s}_t) + T_2(\partial_{A_{0,t}}\hat{s}_t, \bar{\partial}_{A_{0,t}}\hat{s}_t). \tag{5.7}$$

Thus we conclude

$$\star_w F_{A_t} = -i\Delta_{A_{0,t}}\hat{s}_t + T_1 + T_2$$

where the tensors T_1 and T_2 satisfy

$$|T_1| \le C\epsilon|\nabla^2_{A_{0,t}}\hat{s}_t| \qquad \text{and} \qquad |T_2| \le |\nabla_{A_{0,t}}\hat{s}_t|^2. \tag{5.8}$$

Lemma 5.4. *Given* (5.1) *and* (5.3)*, the following bound holds*

$$\|A_t - A_{0,t}\|_{C^{0,\alpha}(M_w,\omega^{SF})} \le C_3\epsilon, \quad \|\nabla_{A_{0,t}}\hat{s}_t\|_{C^{0,\alpha}(M_w,\omega^{SF})} \le C_3\epsilon \tag{5.9}$$

for any $0 < \alpha < 1$, by choosing ϵ small enough. Here the constant C_3 depends on $U \subset N^o$.

Proof. We begin the proof with the standard elliptic a priori estimate (cf. [36, 10])

$$
\begin{aligned}
\|\hat{s}_t\|_{L^p_2} \;&\le\; C\left(\|\Delta_{A_{0,t}}\hat{s}_t\|_{L^p} + \|\hat{s}_t\|_{L^p}\right) \\
&\le\; C\left(\|F_{A_t}\|_{L^p} + \|T_1\|_{L^p} + \|T_2\|_{L^p} + \|\hat{s}_t\|_{L^p}\right) \\
&\le\; C\left(\epsilon + \|T_1\|_{L^p} + \|T_2\|_{L^p}\right)
\end{aligned}
$$

where we have used (5.1) and (5.3) in the last inequality. We also use the assumption that $A_{0,t} \to A_0$ smoothly, and therefore all derivatives of $A_{0,t}$ are bounded independent of t. Thus all constants in the above inequality are independent of t.

The necessary bound for T_1 follows immediately $\|T_1\|_{L^p} \le C\epsilon\|\hat{s}_t\|_{L^p_2}$. For T_2 we use the interpolation inequality for tensors from [46] (see also Section 7.6 in [8])

$$\left(\int_{M_w} |\nabla_{A_{0,t}}\hat{s}_t|^{2p}\right)^{\frac{1}{p}} \le (\sqrt{2} + 2p - 2)\|\hat{s}_t\|_{C^0}\left(\int_{M_w} |\nabla^2_{A_{0,t}}\hat{s}_t|^p\right)^{\frac{1}{p}}.$$

This implies $\|T_2\|_{L^p} \le C\epsilon\|\hat{s}_t\|_{L^p_2}$. Thus

$$\|\hat{s}_t\|_{L^p_2} \le C\left(\epsilon + \epsilon\|\hat{s}_t\|_{L^p_2}\right)$$

and for ϵ small enough

$$\|\hat{s}_t\|_{L^p_2} \le C\epsilon. \tag{5.10}$$

By Morrey's inequality, for large enough p we can conclude

$$\|\nabla_{A_{0,t}}\hat{s}_t\|_{C^{0,\alpha}} \le C\epsilon,$$

and the proof follows from (5.5). □

Comparing this lemma to Theorem 3.11 of [59], the bound of (5.9) is stronger, i.e. we have ϵ instead of $\epsilon^{\frac{1}{2}}$, due to our assumption that $A_{0,t}$ and A_0 are regular.

We now turn to the proof of the main proposition of this section.

Proof of Proposition 5.1. Once again we begin with the standard elliptic a priori estimate

$$\|\hat{s}_t\|_{L^2_2} \leq C \left(\|\Delta_{A_{0,t}} \hat{s}_t\|_w + \|\hat{s}_t\|_w \right).$$

Since $\hat{s}_t$ is perpendicular to the the kernel of $d_{A_{0,t}}$, we have a stronger inequality

$$\|\hat{s}_t\|_{L^2_2} \leq C \|\Delta_{A_{0,t}} \hat{s}_t\|_w$$

(cf. [36, 10]). Again we use the fact that all derivatives of $A_{0,t}$ and A_0 are bounded independent of t.

Next, we recall (5.8). Applying the the interpolation inequality for tensors from the previous lemma for $p = 2$, we have

$$\|T_1 + T_2\|_w \leq C\epsilon \|\hat{s}_t\|_{L^2_2} \leq C\epsilon \|\Delta_{A_{0,t}} \hat{s}_t\|_w.$$

Let F^o_t denote the projection of $\star_w F_{A_t}$ onto the kernel of $\Delta_{A_{0,t}}$, and set $F^\perp_t = \star_w F_{A_t} - F^o_t$. Because $\Delta_{A_{0,t}} \hat{s}_t$ is perpendicular to the kernel of $\Delta_{A_{0,t}}$, we can conclude

$$\begin{aligned}
\|F^\perp_t\|_w &\geq \|\Delta_{A_{0,t}} \hat{s}_t\|_w - \|F^\perp_t - \Delta_{A_{0,t}} \hat{s}_t\|_w \\
&= \|\Delta_{A_{0,t}} \hat{s}_t\|_w - \|(T_1 + T_2)^\perp\|_w \\
&\geq (1 - C\epsilon) \|\Delta_{A_{0,t}} \hat{s}_t\|_w \\
&\geq \frac{1}{2} \|\Delta_{A_{0,t}} \hat{s}_t\|_w.
\end{aligned}$$

We take ϵ small enough such that $C\epsilon < \frac{1}{2}$.

Now, since $(\Delta_{A_{0,t}} \hat{s}_t)^o = 0$, we also have

$$\|F^o_t\|_w \leq \|(T_1 + T_2)^o\|_w \leq C\epsilon \|\Delta_{A_{0,t}} \hat{s}_t\|_w \leq 2C\epsilon \|F^\perp_t\|_w,$$

which implies

$$\begin{aligned}
\|F_{A_t}\|_w &\leq \|F^o_t\|_w + \|F^\perp_t\|_w \\
&\leq (1 + 2C\epsilon) \|F^\perp_t\|_w \\
&\leq 2\|F^\perp_t\|_w.
\end{aligned}$$

Thus, applying the Poincaré inequality to $F^\perp_t$ and Lemma 5.2, we can conclude

$$\|F_{A_t}\|_w \leq 2\|F^\perp_t\|_w \leq C \|d^*_{A_{0,t}} F_{A_t}\|_w.$$

The proposition now follows from Lemma 5.4, which allows us to bound the difference between the connections A_t and $A_{0,t}$

$$\begin{aligned} \|F_{A_t}\|_w &\leq C\|d^*_{A_{0,t}}F_{A_t}\|_w \\ &\leq C\|d^*_{A_t}F_{A_t}\|_w + C\|A_t - A_{0,t}\|_{C^0}\|F_{A_t}\|_w \\ &\leq C\|d^*_{A_t}F_{A_t}\|_w + C\epsilon\|F_{A_t}\|_w. \end{aligned}$$

We choose further that $C\epsilon < \frac{1}{2}$, and obtain

$$\|F_{A_t}\|_w \leq 2C\|d^*_{A_t}F_{A_t}\|_w.$$

For any $K \subset N^o$, we cover K by finite open disks U_β, i.e. $K \subset \bigcup U_\beta \subset N^o$, and apply the above arguments to any U_β. By letting $\epsilon_K = \min\{\epsilon\}$ over the covering, and C_K the maximum constant over the covering, the proposition is proved. □

A corollary is the following Sobolev inequality.

Corollary 5.5. *For any $p \geq 2$, there exists a cosntant C_p so that*

$$\|F_{A_t}\|_{L^p(M_w)} \leq C_p\|d^\star_{A_t}F_{A_t}\|_w.$$

Proof. In dimension two we have the Sobolev inequality

$$\|\xi\|_{L^p} \leq C_p(\|\nabla_{A_{0,t}}\xi\|_w + \|\xi\|_w) \leq C_p(\|\nabla_{A_t}\xi\|_w + \|(A_t - A_{0,t})\xi\|_w + \|\xi\|_w),$$

for any smooth section ξ of $\mathrm{End}(\mathcal{V})$ and some constant C_p independent of $w \in U$ and t. Applying this to $\xi = \star_w F_{A_t}$, we obtain

$$\|F_{A_t}\|_{L^p} \leq C_p(\|d^\star_{A_t}F_{A_t}\|_w + (1+\epsilon)\|F_{A_t}\|_w) \leq 2C_pC_K\|d^\star_{A_t}F_{A_t}\|_w,$$

by Proposition 5.1. □

6. C^0 bounds on curvature

The main goal of this section is to prove Proposition 6.1, which establishes C^0 control for the curvature of a family of connections. It is a conditional result relying on assumption (6.3). To avoid confusion, we note that this result is applied twice. In Section 8, in the proof of Proposition 4.4, it is applied to a family of connections in scaled coordinates, for which (6.3) can be verified directly. Once Proposition 4.4 is established, assumption (6.3) holds for our main sequence of connections Ξ_{t_k} from the statement of Theorem 3.1, and so Proposition 6.1 can be used to establish Proposition 4.5.

As above, let $U \subset\subset N^o$ be an open subset, compactly contained in N_0, and biholomorphic to a disk in $\mathbb{C}$. We have $f^{-1}(U) \cong (U \times \mathbb{C})/\mathrm{Span}_{\mathbb{Z}}\{1, \tau\}$, where the period τ is holomorphic on U. Let w denote the complex coordinate on U, and z the coordinate on $\mathbb{C}$. Furthermore, we fix a trivialization $P|_{M_U} \cong M_U \times SU(n)$ and $\mathcal{V}|_{M_U} \cong M_U \times \mathbb{C}^n$. Under such trivialization, the Hermitian metric H is the absolute value $|\cdot|$, the connection Ξ_t is a matrix valued 1-form, and the curvature F_{Ξ_t} is a matrix valued 2-form, i.e. $\Xi_t \in C^\infty(T^*M_U, \mathfrak{su}(n))$ and $F_{\Xi_t} \in C^\infty(\wedge^2 T^*M_U, \mathfrak{su}(n))$.

Define real coordinates (x_1, x_2) on U satisfying $w = x_1 + ix_2$, and recall that we have the decomposition $T^*M_U \cong \mathrm{Span}_{\mathbb{R}}\{dy_1, dy_2\} \oplus \mathrm{Span}_{\mathbb{R}}\{dx_1, dx_2\}$, where $z = y_1 + \tau y_2$, and z is the coordinate on $\mathbb{C}$. In these coordinates we write

$$\Xi_t = A_t + B_{t,1}dx_1 + B_{t,2}dx_2, \tag{6.1}$$

where A_t is a connection on the restriction to the fiber $\mathcal{V}|_{M_w}$, and $B_{t,i}$ is a section in $\Gamma(U, \Omega^0(M_w, \mathfrak{su}(n)))$ for $i = 1, 2$. Given this decomposition, the curvature can be written as

$$F_{\Xi_t} = F_{A_t} - \kappa_{t,1} \wedge dx_1 - \kappa_{t,2} \wedge dx_2 - F_{B,t}dx_1 \wedge dx_2. \tag{6.2}$$

Here F_{A_t} is the curvature of A_t, the mixed terms are given by

$$\kappa_{t,i} = \frac{\partial}{\partial x_i}A_t - d_{A_t}B_{t,i} \qquad \text{for} \qquad i = 1, 2,$$

and the curvature in the base direction can be expressed as

$$F_{B,t} = \frac{\partial}{\partial x_2}B_{t,1} - \frac{\partial}{\partial x_1}B_{t,2} - [B_{t,1}, B_{t,2}].$$

Because of the uniform equivalence

$$C_U^{-1}\omega_t^{SF} \leq \omega_t \leq C_U\omega_t^{SF}, \quad \text{and} \quad \omega_t^{SF}|_{M_w} = t\omega^{SF}|_{M_w},$$

the norms of the different curvature components satisfy

$$|F_{A_t}|_{\omega^{SF}} = t|F_{A_t}|_{\omega_t^{SF}}, \quad |\kappa_{t,i}|_{\omega^{SF}} = \sqrt{t}|\kappa_{t,i}|_{\omega_t^{SF}}, \quad |F_{B,t}|_{\omega^{SF}} = |F_{B,t}|_{\omega_t^{SF}}.$$

We now state the main assumption of this section. Assume that there is a constant $C_1 > 0$, so that for a $t \in (0, 1]$ it holds

$$\sup_{M_U} |F_{\Xi_t}|_{\omega_t} \leq C_1 t^{-\frac{1}{2}}. \tag{6.3}$$

This implies

$$\sup_{M_U} |F_{A_t}|_{\omega^{SF}} \leq C_1 t^{\frac{1}{2}}, \quad \sup_{M_U} |\kappa_{t,i}|_{\omega^{SF}} \leq C_1, \quad \sup_{M_U} |F_{B,t}|_{\omega^{SF}} \leq C_1 t^{-\frac{1}{2}}.$$

We assume that $t \ll 1$ small enough such that $C_1 t^{\frac{1}{2}} < \epsilon_K$, where ϵ_K is the small constant controlling the curvature in Proposition 5.1, and $U \subset K$. Thus by Proposition 5.1, we see that the curvature F_{A_t} satisfies the Poincaré type inequality

$$\|F_{A_t}\|_w \leq C_2\|d_{A_t}^* F_{A_t}\|_w. \tag{6.4}$$

This inequality, along with assumption (6.3), are instrumental in the following:

Proposition 6.1. *Let $\nabla_{x_i} = \partial_{x_i} + B_{t,i}$ for $i = 1, 2$ denote covariant differentiation in the base direction. If (6.3) and (6.4) hold for $t \ll 1$, for $U' \subset\subset U$ we have the following inequalities:*

i)

$$\|F_{A_t}\|_{C^0(M_{U'}, \omega^{SF})} \leq C_3 t, \quad \|F_{B,t}\|_{C^0(M_{U'}, \omega^{SF})} \leq C_3,$$

ii)

$$\|\nabla_{x_i} F_{A_t}\|_{L^2(M_{U'},\omega^{SF})} \leq C_3 t^{\frac{1}{2}},$$

iii)

$$\|F_{\Xi_t}\|_{C^0(M_{U'},\omega^{SF})} \leq C_3,$$

where the constant C_3 may depend on the distance from U' to ∂U, but is independent of t.

As above let $\star_w$ denote the Hodge star operator on the fiber M_w with respect to the flat metric $\omega_w^F := \omega^{SF}|_{M_w} = i\mathrm{Im}(\tau)^{-1}\, dz \wedge d\bar{z}$. Then $\star_w^2 = -1$, $\star_w dz = -idz$ and $\star_w d\bar{z} = id\bar{z}$. We write the anti-self-dual equation under the decomposition (6.2).

Lemma 6.2. *The curvature of Ξ_t satisfies*

$$\star_w \kappa_{t,1} = \kappa_{t,2} \tag{6.5}$$

and

$$t^{-1}(1 + G_0 + G_1) \star_w F_{A_t} - (W + G_2) F_{B,t} = \sum_{j=1}^{2} \kappa_{t,j} \# G_3, \tag{6.6}$$

where G_1, G_2, G_3 are smooth functions depending on t such that

$$t^{-\frac{\nu}{2}}(\|G_1\|_{C^0(\omega^{SF})} + \|\frac{\partial}{\partial z} G_1\|_{C^\ell(\omega^{SF})} + \|\frac{\partial}{\partial \bar{z}} G_1\|_{C^\ell(\omega^{SF})} + \sum_{j=2,3} \|G_j\|_{C^\ell(\omega^{SF})}) \to 0,$$

for any $\nu \in \mathbb{N}$, and G_0 is a function on U such that $\|G_0\|_{C^\ell(U)} \to 0$, when $t \to 0$.

Proof. We first demonstrate that (6.5) follows from $F_{\Xi_t}^{0,2} = F_{\Xi_t}^{2,0} = 0$. Note that

$$2(\kappa_{t,1} \wedge dx_1 + \kappa_{t,2} \wedge dx_2) = (\kappa_{t,1} - i\kappa_{t,2}) \wedge dw + (\kappa_{t,1} + i\kappa_{t,2}) \wedge d\bar{w}.$$

This implies, using $\star_w dz = -idz$ and $\star_w d\bar{z} = id\bar{z}$, that

$$\star_w(\kappa_{t,1} - i\kappa_{t,2}) = i(\kappa_{t,1} - i\kappa_{t,2}) = i\kappa_{t,1} + \kappa_{t,2}$$

and

$$\star_w(\kappa_{t,1} + i\kappa_{t,2}) = -i(\kappa_{t,1} + i\kappa_{t,2}) = -i\kappa_{t,1} + \kappa_{t,2}.$$

Adding these two equations together proves (6.5).

We now concentrate on (6.6). Using $F_{\Xi_t} \wedge \omega_t = 0$, along with the decompositions (2.8) and (6.2), we see

$$\begin{aligned}
0 = F_{\Xi_t} \wedge \omega_t &= F_{\Xi_t} \wedge \omega_t^{SF} + F_{\Xi_t} \wedge i\partial\bar{\partial}\varphi_t \\
&= \frac{i}{2}(W^{-1} + 2\varphi_{t,w\bar{w}}) F_{A_t} \wedge dw \wedge d\bar{w} \\
&\quad -\frac{i}{2}(tW + 2\varphi_{t,z\bar{z}}) F_{B,t}\, dx_1 \wedge dx_2 \wedge \theta \wedge \bar{\theta} \\
&\quad +(\kappa_{t,1} \wedge dx_1 + \kappa_{t,2} \wedge dx_2) \wedge \mathrm{Im}\,(2\varphi_{t,w\bar{z}} dw \wedge d\bar{z})\,.
\end{aligned}$$

Next, note that $\theta = dy_1 + \tau dy_2 = dz + bdw$,

$$dx_1 \wedge dx_2 = \frac{i}{2} dw \wedge d\bar{w} \qquad \text{and} \qquad F_{A_t} = \frac{i}{2}(\star_w F_{A_t}) W \theta \wedge \bar{\theta}.$$

Thus, dividing out by the volume form $dz \wedge dw \wedge d\bar{z} \wedge d\bar{w} = \theta \wedge dw \wedge \bar{\theta} \wedge d\bar{w}$, the above equation can be rewritten as

$$\begin{aligned} 0 &= (1 + 2\varphi_{t,w\bar{w}} W) \star_w F_{A_t} - (tW + 2\varphi_{t,z\bar{z}}) F_{B,t} \\ &\quad + \sum_{i=1}^{2} \kappa_{t,i} \# (\varphi_{t,z\bar{w}} + \varphi_{t,w\bar{z}}) . \end{aligned}$$

We set $G_0 = 2\chi_{t,w\bar{w}} W$, $G_1 = 2(\varphi_{t,w\bar{w}} - \chi_{t,w\bar{w}}) W$, $G_2 = 2t^{-1}\varphi_{t,z\bar{z}}$, and $G_3 = t^{-1}(\varphi_{t,z\bar{w}} + \varphi_{t,w\bar{z}})$. The proof now follows from Lemma 2.11. $\square$

Next we turn to a Bochner type formula for F_{A_t}.

Lemma 6.3. *If we denote $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$, then*

$$\begin{aligned} \Delta \|F_{A_t}\|_w^2 &\geq \frac{1}{4} \sum_{i=1,2} \|\nabla_{x_i} F_{A_t}\|_w^2 + \frac{\delta}{t} \|d_{A_t}^* F_{A_t}\|_w^2 \\ &\quad - C_4' t \Big(\sum_{j=1,2} \|[\kappa_{t,j}, \kappa_{t,j}]\|_w^2 + t^\nu \Big) \\ &\geq \frac{1}{4} \sum_{i=1,2} \|\nabla_{x_i} F_{A_t}\|_w^2 + \frac{\delta}{t} \|d_{A_t}^* F_{A_t}\|_w^2 - C_4 t, \end{aligned}$$

for constants $\delta > 0$, $C_4 > 0$ and $C_4' > 0$.

Proof. Note we can write the mixed and base curvature terms as

$$\nabla_{x_1} d_{A_t} - d_{A_t} \nabla_{x_1} = \kappa_{t,1}, \quad \nabla_{x_2} d_{A_t} - d_{A_t} \nabla_{x_2} = \kappa_{t,2}, \quad [\nabla_{x_1}, \nabla_{x_2}] = F_{B,t}.$$

By the Bianchi identity $d_{\Xi_t} F_{\Xi_t} = 0$, and so

$$d_{A_t} F_{t,B} = \nabla_{x_1} \kappa_{t,2} - \nabla_{x_2} \kappa_{t,1}, \quad \nabla_{x_1} F_{A_t} = d_{A_t} \kappa_{t,1}, \quad \text{and} \quad \nabla_{x_2} F_{A_t} = d_{A_t} \kappa_{t,2}.$$

Recall that $\star_w dz = -idz$, $\star_w d\bar{z} = id\bar{z}$ and $\star_w \frac{i}{2} W dz \wedge d\bar{z} = 1$. Also, $\star_w$ is independent of w when acting on 1-forms, and $\partial_{x_i} \star_w = -W^{-1}(\partial_{x_i} W) \star_w$ in the other cases. By the above formulas, we derive

$$\begin{aligned} (\nabla_{x_1}^2 + \nabla_{x_2}^2) F_{A_t} &= \nabla_{x_1} d_{A_t} \kappa_{t,1} + \nabla_{x_2} d_{A_t} \kappa_{t,2} \\ &= d_{A_t} (\nabla_{x_1} \kappa_{t,1} + \nabla_{x_2} \kappa_{t,2}) + \sum_{j=1,2} [\kappa_{t,j}, \kappa_{t,j}]. \end{aligned}$$

By (6.5), we also have

$$\nabla_{x_1} \kappa_{t,1} = - \star_w \nabla_{x_1} \kappa_{t,2}, \quad \text{and} \quad \nabla_{x_2} \kappa_{t,2} = \star_w \nabla_{x_2} \kappa_{t,1}.$$

Hence, using (6.6), we obtain a Weitzenböck type formula for F_{A_t}:

$$
\begin{aligned}
(6.7)\, (\nabla_{x_1}^2 + \nabla_{x_2}^2) F_{A_t} &= d_{A_t} \star_w (\nabla_{x_2}\kappa_{t,1} - \nabla_{x_1}\kappa_{t,2}) + \sum_{j=1,2} [\kappa_{t,j}, \kappa_{t,j}] \\
&= -d_{A_t} \star_w d_{A_t} F_{B,t} + \sum_{j=1,2} [\kappa_{t,j}, \kappa_{t,j}] \\
&= -t^{-1} d_{A_t} \star_w d_{A_t} (G_4 \star_w F_{A_t}) + \sum_{j=1,2} [\kappa_{t,j}, \kappa_{t,j}] \\
&\quad + d_{A_t} \star_w d_{A_t} (\sum_{i=1,2} \kappa_{t,i} \# G_5),
\end{aligned}
$$

where

$$G_4 = (W + G_2)^{-1}(1 + G_0 + G_1), \quad \text{and} \quad G_5 = (W + G_2)^{-1} G_3.$$

Note that for any differential form α, $d_{A_t}\alpha = d^f\alpha$, where d^f denotes the differential along the fiber direction, i.e. $d^f = \partial_{y_1}(\cdot) dy_1 + \partial_{y_2}(\cdot) dy_2$, and $\nabla_{x_i}\alpha = \partial_{x_i}\alpha$.

Since $\|F_{A_t}\|_w^2 = \int_{M_w} \operatorname{tr} F_{A_t} \wedge \star_w F_{A_t}$, a direct calculation shows

$$\partial_{x_i}^2 \|F_{A_t}\|_w^2 = \|\nabla_{x_i} F_{A_t}\|_w^2 + 2\mathrm{Re}\langle \nabla_{x_i}^2 F_{A_t}, F_{A_t}\rangle_w + T_i,$$

where the term T_i arises from derivative on the fiber metric, and satisfies

$$
\begin{aligned}
|T_i| &\leq C(|\partial_{x_i} \star_w| \|\nabla_{x_i} F_{A_t}\|_w \|F_{A_t}\|_w + |\partial_{x_i}^2 \star_w| \|F_{A_t}\|_w^2) \\
&\leq \frac{1}{2}\|\nabla_{x_i} F_{A_t}\|_w^2 + C\|F_{A_t}\|_w^2.
\end{aligned}
$$

Using the notation $\|\nabla_x F_{A_t}\|_w^2 = \sum\limits_{i=1,2} \|\nabla_{x_i} F_{A_t}\|_w^2$, the above calculations give

$$\Delta \|F_{A_t}\|_w^2 = \|\nabla_x F_{A_t}\|_w^2 + 2\mathrm{Re}\langle (\nabla_{x_1}^2 + \nabla_{x_2}^2) F_{A_t}, F_{A_t}\rangle_w + T_1 + T_2.$$

To this equality, we can now apply (6.7). Using $d_{A_t}^* = -\star_w d_{A_t} \star_w$, we see

$$
\begin{aligned}
\mathrm{Re}\langle (\nabla_{x_1}^2 + \nabla_{x_2}^2) F_{A_t}, F_{A_t}\rangle_w &= t^{-1}\mathrm{Re}\langle G_4 d_{A_t}^* F_{A_t}, d_{A_t}^* F_{A_t}\rangle_w \\
&\quad + \mathrm{Re}\langle \sum_{j=1,2} [\kappa_{t,j}, \kappa_{t,j}], F_{A_t}\rangle_w \\
&\quad - t^{-1}\mathrm{Re}\langle \star_w (d^f G_4) \star_w F_{A_t}, d_{A_t}^* F_{A_t}\rangle_w \\
&\quad + \mathrm{Re}\langle \star_w d_{A_t} (\sum_{i=1,2} \kappa_{t,i} \# G_5), d_{A_t}^* F_{A_t}\rangle_w.
\end{aligned}
$$

Next, note that for a constant $\delta > 0$, we have

$$\mathrm{Re}\langle G_4 d_{A_t}^* F_{A_t}, d_{A_t}^* F_{A_t}\rangle_w \geq 8\delta \|d_{A_t}^* F_{A_t}\|_w^2.$$

Using (6.3) to bound the mixed terms, and the Poincaré inequality (6.4), we have

$$\begin{aligned}|\langle \sum_{j=1,2}[\kappa_{t,j},\kappa_{t,j}],F_{A_t}\rangle_w| &\leq C\sum_{j=1,2}\|[\kappa_{t,j},\kappa_{t,j}]\|_w\|F_{A_t}\|_w\\ &\leq C\sum_{j=1,2}\|[\kappa_{t,j},\kappa_{t,j}]\|_w\|d^*_{A_t}F_{A_t}\|_w\\ &\leq Ct\sum_{j=1,2}\|[\kappa_{t,j},\kappa_{t,j}]\|^2_w+\frac{\delta}{t}\|d^*_{A_t}F_{A_t}\|^2_w.\end{aligned}$$

Because $d^fW=0$, $d^fG_0=0$, and $d^fG_4=o(t^\nu)$ for $\nu\gg 1$, it follows that

$$\begin{aligned}|t^{-1}\mathrm{Re}\langle \star_w(d^fG_4)\star_w F_{A_t},d^*_{A_t}F_{A_t}\rangle_w| &\leq C\|F_{A_t}\|_w\|d^*_{A_t}F_{A_t}\|_w\\ &\leq Ct\|F_{A_t}\|^2_w+\frac{\delta}{t}\|d^*_{A_t}F_{A_t}\|^2_w.\end{aligned}$$

Finally, $|d^fG_5|_{\omega^{SF}}=o(t^\nu)$ for any $\nu\gg 1$, and so

$$\begin{aligned}|\langle \star_w d_{A_t}(\sum_{i=1,2}\kappa_{t,i}\# G_5),d^*_{A_t}F_{A_t}\rangle_w| &\leq C\|d^*_{A_t}F_{A_t}\|_w(t^\nu+\sum_{i=1,2}\|d_{A_t}\kappa_{t,i}\|_w)\\ &= C\|d^*_{A_t}F_{A_t}\|_w(t^\nu+\sum_{i=1,2}\|\nabla_{x_i}F_{A_t}\|_w)\\ &\leq Ct(t^\nu+\|\nabla_xF_{A_t}\|^2_w)+\frac{\delta}{t}\|d^*_{A_t}F_{A_t}\|^2_w.\end{aligned}$$

Putting everything together

$$\begin{aligned}\mathrm{Re}\langle(\nabla^2_{x_1}+\nabla^2_{x_2})F_{A_t},F_{A_t}\rangle_w &\geq \frac{4\delta}{t}\|d^*_{A_t}F_{A_t}\|^2_w-Ct(t^\nu+\|F_{A_t}\|^2_w+\|\nabla_xF_{A_t}\|^2_w\\ &\quad+\sum_{j=1,2}\|[\kappa_{t,j},\kappa_{t,j}]\|^2_w),\end{aligned}$$

which implies

$$\begin{aligned}\Delta\|F_{A_t}\|^2_w &\geq \|\nabla_xF_{A_t}\|^2_w+\frac{4\delta}{t}\|d^*_{A_t}F_{A_t}\|^2_w-\frac{1}{2}\|\nabla_xF_{A_t}\|^2_w-2C\|F_{A_t}\|^2_w\\ &\quad -Ct(t^\nu+\|F_{A_t}\|^2_w+\|\nabla_xF_{A_t}\|^2_w+\sum_{j=1,2}\|[\kappa_{t,j},\kappa_{t,j}]\|^2_w).\end{aligned}$$

The Poincaré inequality (6.4), along with Young's inequality, gives

$$\Delta\|F_{A_t}\|^2_w\geq\frac{1}{4}\|\nabla_xF_{A_t}\|^2_w+\frac{\delta}{t}\|d^*_{A_t}F_{A_t}\|^2_w-Ct(\sum_{j=1,2}\|[\kappa_{t,j},\kappa_{t,j}]\|^2_w+t^\nu).$$

□

We need the following elementary lemma, and we include the proof for the reader's convenience (cf. Sublemma 6.48 in [32]). As in the previous lemma, let $\Delta=\partial^2_{x_1}+\partial^2_{x_2}$ denote the coordinate Laplacian in the base.

Lemma 6.4. *Let ζ be a non-negative real valued function satisfying*

$$\Delta\zeta \geq \frac{\delta}{t}\zeta - t$$

on a disk $U \subset \mathbb{C}$. Then for an open subset $U' \subset\subset U$, there exists a constant C_5, which depends on the distance from U' to ∂U, such that

$$\sup_{U'} |\zeta| \leq C_5 t^2.$$

Proof. For any point $w_0 \in U'$, let $d = \sup\{|w - w_0| \| w \in U\}$, and let a be a positive number such that $4a^2d^2 + 4a < \delta$. Consider the function $\xi = \zeta \exp(-\frac{a|w-w_0|^2}{\sqrt{t}})$. If ξ achieves its maximum w_1 on ∂U, then

$$\zeta(w_0) = \xi(w_0) \leq \xi(w_1) = \zeta(w_1)\exp(-\frac{a|w_1 - w_0|^2}{\sqrt{t}}) \leq C\exp(-\frac{ar^2}{\sqrt{t}}),$$

where r is the distance from w_0 to ∂U. For t small enough the right hand side is smaller than Ct^2.

Otherwise, at an interior maximum w_1, we see

$$0 = \partial_w \xi(w_1) = (-\frac{a(\bar{w}_1 - \bar{w}_0)}{\sqrt{t}}\zeta(w_1) + \partial_w\zeta(w_1))\exp(-\frac{a|w_1 - w_0|^2}{\sqrt{t}}),$$

and $\partial_{\bar{w}}\xi(w_1) = 0$. Furthermore, since $\Delta = 2\partial_w\partial_{\bar{w}}$, at this maximum point

$$\begin{aligned}
0 &\geq \Delta\xi(w_1) \\
&= 2\left(\partial_w\partial_{\bar{w}}\zeta(w_1) - \frac{a^2|w_1 - w_0|^2 + a\sqrt{t}}{t}\zeta(w_1)\right)\exp(-\frac{a|w_1 - w_0|^2}{\sqrt{t}}) \\
&\geq \left(\frac{\delta}{t}\zeta(w_1) - 2\frac{a^2d^2 + a}{t}\zeta(w_1) - t\right)\exp(-\frac{a|w_1 - w_0|^2}{\sqrt{t}}) \\
&\geq \left(\frac{\delta}{2t}\zeta(w_1) - t\right)\exp(-\frac{a|w_1 - w_0|^2}{\sqrt{t}}).
\end{aligned}$$

Thus

$$\xi(w_1) \leq \zeta(w_1) \leq 2\delta^{-1}t^2,$$

and so

$$\zeta(w_0) = \xi(w_0) \leq \xi(w_1) \leq 2\delta^{-1}t^2.$$

□

Lemma 6.5. *For any $w \in U' \subset\subset U$,*

$$\|F_{A_t}\|_w \leq C_6 t, \quad \text{and} \quad \|\nabla_{x_i}F_{A_t}\|_{L^2(U',\omega^{SF})} \leq C_6 t^{\frac{1}{2}},$$

for a constant $C_6 > 0$ independent of t and w.

Proof. Lemma 6.3 and Lemma 5.2 imply

$$\Delta\|F_{A_t}\|_w^2 \geq \frac{1}{4}\|\nabla_x F_{A_t}\|_w^2 + \frac{\delta}{t}\|d_{A_t}^* F_{A_t}\|_w^2 - Ct \geq \frac{\delta'}{t}\|F_{A_t}\|_w^2 - Ct.$$

Thus by Lemma 6.4,

$$\|F_{A_t}\|_w^2 \leq Ct^2.$$

Let ϑ be a smooth non-negative function on U such that $\vartheta \equiv 1$ on U', and $U' \subset \mathrm{supp}(\vartheta) \subset U$. By Lemma 6.3,

$$\begin{aligned}\int_{U'} \frac{1}{4}\|\nabla_x F_{A_t}\|_w^2 dx_1 dx_2 &\leq \int_U \vartheta \Delta \|F_{A_t}\|_w^2 dx_1 dx_2 + Ct \\ &\leq \int_U \max\{0, \Delta\vartheta\}\|F_{A_t}\|_w^2 dx_1 dx_2 + C_{22}t \\ &\leq C(\int_U \|F_{A_t}\|_w^2 dx_1 dx_2 + t) \\ &\leq Ct,\end{aligned}$$

and we obtain the second estimate. □

Proof of Proposition 6.1. Firstly, we prove the C^0-estimate of F_{A_t}. Assume that there is a sequence $t_k \to 0$ such that

$$t_k^{-1} \sup_{M_{w_k}} |F_{A_{t_k}}|_{\omega^{SF}} \to \infty,$$

where $w_k \to w_0$ in U'.

In Section 2.4, we saw that for $D_r = \{\tilde{w} \in \mathbb{C} | |\tilde{w}| < r\}$, one can define smooth embeddings $\Phi_{k,r} : D_r \times M_{w_0} \to M_U$ by

$$(\tilde{w}, a_1 + a_2\tau(w_0)) \mapsto (w_k + \sqrt{t_k}\tilde{w}, a_1 + a_2\tau(w_k + \sqrt{t_k}\tilde{w})), \quad a_1, a_2 \in \mathbb{R}/\mathbb{Z},$$

using the identification of M_U with $(U \times \mathbb{C})/\mathrm{Span}_{\mathbb{Z}}\{1, \tau\}$. We also demonstrated that $d\Phi_{k,r}^{-1} I d\Phi_{k,r} \to I_\infty$, where I is the complex structure of M, and I_∞ denotes the complex structure of $\mathbb{C} \times M_{w_0}$. Furthermore, as $t_k \to 0$, we have both

$$\Phi_{k,r}^* t_k^{-1} \omega_{t_k}^{SF} \to \omega_\infty \quad \text{and} \quad (T_{\sigma_0} \circ \Phi_{k,r})^* t_k^{-1} \omega_{t_k} = \Phi_{k,r}^* t_k^{-1} T_{\sigma_0}^* \omega_{t_k} \to \omega_\infty$$

in the C^∞-sense on $D_r \times M_{w_0}$. For any t_k, we identify $D_r \times M_{w_0}$ with $\Phi_{k,r}(D_r \times M_{w_0})$ by $\Phi_{k,r}$. We have the curvature bound

$$|F_{\Xi_{t_k}}|_{t_k^{-1}\omega_{t_k}^{SF}} \leq C t_k^{\frac{1}{2}}, \text{ and } |F_{\Xi_{t_k}}|_{\omega_\infty} \leq 2C t_k^{\frac{1}{2}},$$

by (6.3).

Since Ξ_{t_k} is Yang-Mills, by the strong Uhlenbeck compactness theorem (cf. Theorem 2.3), there exists a subsequence and a family of unitary gauges u_{t_k}, such that

$$\Xi'_{t_k} = u_{t_k}(\Xi_{t_k}) \to \Xi_\infty$$

in the locally C^∞-sense on $D_r \times M_{w_0}$, where Ξ_∞ is a flat $SU(n)$-connection. Note that $F_{\Xi'_{t_k}} = u_{t_k} F_{\Xi_{t_k}} u_{t_k}^{-1}$, and so

$$|F_{\Xi'_{t_k}}|_{t_k^{-1}\omega_{t_k}^{SF}} = |F_{\Xi_{t_k}}|_{t_k^{-1}\omega_{t_k}^{SF}} \leq C t_k^{\frac{1}{2}} \quad \text{and} \quad |F_{\Xi'_{t_k}}|_{\omega_\infty} \leq 2C t_k^{\frac{1}{2}}.$$

Furthermore we have $\|F_{\Xi'_{t_k}}\|_{C^\ell(\omega_\infty)} \to 0$ for any $\ell \geq 0$, when $t_k \to 0$. Now, recall the Weitzenböck formula

$$0 = \Delta_{\Xi'_{t_k}} F_{\Xi'_{t_k}} = \nabla^*_{\Xi'_{t_k}} \nabla_{\Xi'_{t_k}} F_{\Xi'_{t_k}} + R_{t_k^{-1}\omega_{t_k}} \# F_{\Xi'_{t_k}} + F_{\Xi'_{t_k}} \# F_{\Xi'_{t_k}},$$

which is an elliptic partial differential equation with smooth coefficients. The L^p-estimate for elliptic equations (cf. [36], and the appendix of [10]) gives

$$\|F_{\Xi'_{t_k}}\|_{L^p_2(\omega_\infty)} \leq C\|F_{\Xi'_{t_k}}\|_{L^p(\omega_\infty)} \leq Ct_k^{\frac{1}{2}},$$

for any $p > 2$.

We have $w - w_k = \sqrt{t_k}\tilde{w}$ through $\Phi_{k,r}$, and let $\tilde{w} = \tilde{x}_1 + i\tilde{x}_2$. By (6.7),

$$\begin{aligned}(\nabla^2_{x_1} + \nabla^2_{x_2})F_{A'_{t_k}} &= -t_k^{-1}d_{A'_{t_k}} \star_w d_{A'_{t_k}}(G_4 \star_w F_{A'_{t_k}}) + \sum_{ij}\kappa'_{t_k,i}\#\kappa'_{t_k,j} \\ &\quad + d_{A'_{t_k}} \star_w d_{A'_{t_k}}(\sum_{i=1,2}\kappa'_{t_k,i}\#G_5),\end{aligned} \tag{6.8}$$

where $\nabla_{x_j} = \partial_{x_j} + B'_{t_k,j}$, $G_4 = (W + G_2)^{-1}(1 + G_0 + G_1)$ and $G_5 = (W + G_2)^{-1}G_3$. Recall

$$\|G_1\|_{C^0} + \|d^f G_1\|_{C^\ell} + \|G_j\|_{C^\ell} \leq Ct_k^\nu$$

for $\nu \gg 1$. Let $z = \tilde{y}_1 + i\tilde{y}_2$, and set $\nabla_{A'_{t_k},y_j} = \partial_{\tilde{y}_j} + A'_{t_k,j}$. By the Weitzenböck formula,

$$d_{A'_{t_k}}d^*_{A'_{t_k}}F_{A'_{t_k}} = \nabla^*_{A'_{t_k}}\nabla_{A'_{t_k}}F_{A'_{t_k}} + F_{A'_{t_k}}\#F_{A'_{t_k}}.$$

The connection Laplacian above is given by

$$\nabla^*_{A'_{t_k}}\nabla_{A'_{t_k}} = -W^{-1}(\nabla^2_{A'_{t_k},\tilde{y}_1} + \nabla^2_{A'_{t_k},\tilde{y}_2}),$$

since $|\partial_{\tilde{y}_j}|^2_{\omega^{SF}} = W$.

We want to bound terms on the right hand side of (6.8). Scaling gives $B'_{t_k,i}dx_i = \sqrt{t_k}B'_{t_k,i}d\tilde{x}_i$ and $\kappa'_{t_k,i}dx_i = \sqrt{t_k}\kappa'_{t_k,i}d\tilde{x}_i$, in addition to

$$F_{B,t_k}dx_1 \wedge dx_2 = t_k F_{B,t_k}d\tilde{x}_1 \wedge d\tilde{x}_2.$$

This leads to the following control of the mixed terms

$$|\sqrt{t_k}\kappa'_{t_k,i}|_{\omega_\infty} \leq 2Ct_k^{\frac{1}{2}}, \qquad \|\sqrt{t_k}\kappa'_{t_k,i}\|_{C^\ell(\omega_\infty)} \to 0,$$

and

$$\|\sqrt{t_k}\kappa'_{t_k,i}\|_{L^p_2(\omega_\infty)} \leq \|F_{\Xi'_{t_k}}\|_{L^p_2(\omega_\infty)} \leq Ct_k^{\frac{1}{2}}.$$

Additionally, writing $\nabla_{\tilde{x}_j} = \partial_{\tilde{x}_j} + \sqrt{t_k}B'_{t_k,j}$, we have

$$\nabla^2_{\tilde{x}_1} + \nabla^2_{\tilde{x}_2} = t_k(\nabla^2_{x_1} + \nabla^2_{x_2}).$$

The bound $|\partial^\ell_{y_j}G_5| \leq C$ gives

$$\|t_k^{\frac{1}{2}}d_{A'_{t_k}} \star_w d_{A'_{t_k}}(\sum_{i=1,2}\kappa'_{t_k,i}\#G_5)\|_{L^p(\omega_\infty)} \leq Ct_k^{\frac{1}{2}}$$

for any $p > 2$. Furthermore

$$\|\sum_{ij}\kappa'_{t_k,i}\#\kappa'_{t_k,j}\|_{C^0(\omega_\infty)} \leq C.$$

Now, if we write $G_4 = W^{-1}(1+G_0) + G_6$, then

$$\frac{1}{2}W^{-1}(w_0) \le G_4 \le 2W^{-1}(w_0), \quad |\partial^\ell_{\tilde{y}_j} G_6| \le Ct_k^\nu,$$

and

$$\begin{aligned} d_{A'_{t_k}} d^*_{A'_{t_k}} G_4 F_{A'_{t_k}} &= G_4 d_{A'_{t_k}} d^*_{A'_{t_k}} F_{A'_{t_k}} + d^f G_6 \# \nabla_{A'_{t_k}} F_{A'_{t_k}} \\ &\quad + \partial^2_{\tilde{y}_i \tilde{y}_j} G_6 \# F_{A'_{t_k}}. \end{aligned}$$

We define the operator

$$\mathcal{D}_k = \nabla^2_{\tilde{x}_1} + \nabla^2_{\tilde{x}_2} - G_4 \nabla^*_{A'_{t_k}} \nabla_{A'_{t_k}} = \nabla^2_{\tilde{x}_1} + \nabla^2_{\tilde{x}_2} + W^{-1} G_4 (\nabla^2_{A'_{t_k}, \tilde{y}_1} + \nabla^2_{A'_{t_k}, \tilde{y}_2}),$$

which is a uniformly elliptic operator of order two. Then $F_{A'_{t_k}}$ satisfies the following elliptic equation

$$\begin{aligned} (6.9)\qquad & \mathcal{D}_k F_{A'_{t_k}} - d^f G_6 \# \nabla_{A'_{t_k}} F_{A'_{t_k}} - \partial^2_{\tilde{y}_i \tilde{y}_j} G_6 \# F_{A'_{t_k}} \\ =\ & G_4 F_{A'_{t_k}} \# F_{A'_{t_k}} + t_k \sum_{ij} \kappa'_{t_k,i} \# \kappa'_{t_k,j} + t_k d_{A'_{t_k}} \star_w d_{A'_{t_k}} (\sum_{i=1,2} \kappa'_{t_k,i} \# G_5) \\ =\ & G_7. \end{aligned}$$

By the L^p-estimate for elliptic equations, for any $p > 2$,

$$\|F_{A'_{t_k}}\|_{L^p_2(D_{r'} \times M_{w_0})} \le C(\|F_{A'_{t_k}}\|_{L^2(D_r \times M_{w_0})} + \|G_7\|_{L^p(D_r \times M_{w_0})}),$$

for a $r' < r$. We obtain

$$\|F_{A'_{t_k}}\|_{L^p_2(D_{r'} \times M_{w_0})} \le Ct_k,$$

since

$$\|G_7\|_{L^p(D_r \times M_{w_0})} \le C(\|F_{A'_{t_k}}\|^2_{C^0(D_r \times M_{w_0})} + t_k) \le Ct_k,$$

and

$$\|F_{A'_{t_k}}\|^2_{L^2(D_r \times M_{w_0})} = \int_{D_r} \|F_{A'_{t_k}}\|^2_w d\tilde{x}_1 d\tilde{x}_2 \le Ct_k^2$$

by Lemma 6.5. The Sobolev embedding theorem gives

$$\|F_{A'_{t_k}}\|_{C^{1,\alpha}(D_{r'} \times M_{w_0})} \le Ct_k,$$

and thus

$$\|F_{A_{t_k}}\|_{C^0(M_{w_k})} = \|F_{A'_{t_k}}\|_{C^0(M_{w_k})} \le \|F_{A'_{t_k}}\|_{C^{1,\alpha}(D_{r'} \times M_{w_0})} \le Ct_k,$$

which is a contradiction.

Therefore we obtain the C^0-estimate, i.e.

$$\|F_{A_t}\|_{C^0(M_{U'}, \omega^{SF})} \le Ct,$$

for a constant $C > 0$, and

$$\|F_{B,t}\|_{C^0(M_{U'}, \omega^{SF})} \le C(t^{-1}\|F_{A_t}\|_{C^0(M_{U'}, \omega^{SF})} + \|\kappa_{t,j}\|_{C^0(M_{U'}, \omega^{SF})}) \le C,$$

by (6.6). $\square$

7. Further estimates for small fiberwise curvature

We continue our discussion of the previous section, and prove further estimates under the exact same setup. Let $U \subset\subset N^o$ be an open subset, biholomorphic to a disk in $\mathbb{C}$, and $M_U \cong (U\times\mathbb{C})/\mathrm{Span}_{\mathbb{Z}}\{1,\tau\}$. Fix a trivialization $P|_{M_U} \cong M_U \times SU(n)$ and $\mathcal{V}|_{M_U} \cong M_U \times \mathbb{C}^n$. Under such trivialization, the Hermitian metric H is the absolute value $|\cdot|$, the connection Ξ_t is a matrix valued 1-form, and the curvature F_{Ξ_t} is a matrix valued 2-form. Assume that for $t \ll 1$, (6.3) and (6.4) hold, and thus all conclusions of Section 6 hold.

Recall that a fiberwise flat connection

$$A_{0,t} = \pi(\mathrm{Im}(\tau))^{-1}(\mathrm{diag}\{q_{1,t},\cdots,q_{n,t}\}\bar{\theta} - \mathrm{diag}\{\bar{q}_{1,t},\cdots,\bar{q}_{n,t}\}\theta) \tag{7.1}$$

is induced by $D_t\cap M_U$ (see Section 3.3), i.e. $D_t\cap M_w = \{q_{1,t}(w),\cdots,q_{n,t}(w)\}$. The goal of this section is the following proposition, which shows the relationship between the energy of curvature and the spectral covers. Here, as above, the coordinate derivative in the base is computed in our fixed frame.

Proposition 7.1. *If (6.3) and (6.4) hold for $t \ll 1$, we have the following inequalities. For $U' \subset\subset U$,*

$$\|F_{\Xi_t}\|^2_{L^2(M_{U'},\omega_t)} \leq C_1\Big(t + \int_{U'}\sum_{j=1,2}\|\partial_{x_j}A_{0,t}\|^2_w dx_1dx_2\Big), \quad \textit{and}$$

$$\|F_{\Xi_t}\|^2_{L^2(M_{U'},\omega_t)} \geq C_1^{-1}\Big(\int_{U'}\sum_{j=1,2}\|\partial_{x_j}A_{0,t}\|^2_w dx_1dx_2 - t\Big),$$

where the constant C_1 may depend on the distance from U' to ∂U, but is independent of t.

The proof rests on several important lemmas.

Lemma 7.2. *There exists a constant C_2 such that for all $t \ll 1$,*

$$\sup_{M_{U'}}|\nabla_{A_{0,t}}F_{A_t}|_{\omega^{SF}} \leq C_2 t^{\frac{1}{2}}.$$

Proof. By (5.9), it suffices to prove the above bound for $\nabla_{A_t}F_{A_t}$. We argue by contradiction. Let $t_k \to 0$ such that

$$\lim_{k\to\infty} t_k^{-\frac{1}{2}}\sup_{M_{U'}}|\nabla_{A_{t_k}}F_{A_{t_k}}|_{\omega^{SF}} = \infty.$$

Let $p_k \in M_{U'}$ be the points where the supremum is attained, and in addition let $f(p_k) := w_k \to w_0 \in U$. As in Section 2.5, we consider the rescaled metrics $\hat{\omega}_k = t_k^{-1}\omega_{t_k}$ and the embeddings $\Phi_{k,r}: D_r\times M_{w_0} \to M_U$ defined by

$$(\tilde{w}, a_1 + a_2\tau(w_0)) \mapsto (w_k + \sqrt{t_k}\tilde{w}, a_1 + a_2\tau(w_k+\sqrt{t_k}\tilde{w})), \quad a_1, a_2 \in \mathbb{R}/\mathbb{Z},$$

where $D_r = \{\tilde{w}\in\mathbb{C}||\tilde{w}| < r\}$. We have seen that if I is the complex structure of M, and I_∞ the complex structure of $\mathbb{C}\times M_{w_0}$, then $d\Phi_{k,r}^{-1} I d\Phi_{k,r} \to I_\infty$, and in addition

$$\Phi_{k,r}^* t_k^{-1}\omega_{t_k}^{SF} \to \omega_\infty \quad \text{and} \quad \Phi_{k,r}^*\hat\omega_k \to \omega_\infty$$

in the C^∞-sense on $D_r \times M_{w_0}$. Here ω_∞ is a flat Kähler metric on $D_r \times M_{w_0}$. Denote by $\hat\Xi_k$ the pull-back of Ξ_{t_k} by $\Phi_{k,r}$, and identify $D_r \times M_{w_0}$ with $\Phi_{k,r}(D_r \times M_{w_0})$ via $\Phi_{k,r}$. By our hypothesis,

$$\sup_{D_r\times M_{w_0}} t_k^{-\frac{1}{2}}|\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|_{\omega_\infty} = \infty, \tag{7.2}$$

while by (6.3) we have the curvature bounds

$$|F_{\hat\Xi_k}|_{t_k^{-1}\omega_{t_k}^{SF}} \le Ct_k^{\frac{1}{2}} \;\text{ and }\; |F_{\hat\Xi_k}|_{\hat\omega_k} \le 2Ct_k^{\frac{1}{2}}.$$

Since $\hat\omega_k$ is equivalent to a fixed metric, standard Yang-Mills theory gives the first derivative bound $|\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|_{\hat\omega_k} \le C$ (for instance see [76]), but this is of course not enough to obtain a contradiction. So following [76], as in the proof of Lemma 2.14, we consider the the Bochner formula

$$0 = \Delta_{\hat\omega_k}|F_{\hat\Xi_k}|^2_{\hat\omega_k} - 2|\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|^2_{\hat\omega_k} + F_{\hat\Xi_k}\# F_{\hat\Xi_k}\# F_{\hat\Xi_k} + R_{\hat\omega_k}\# F_{\hat\Xi_k}\# F_{\hat\Xi_k}.$$

We have seen that the curvature of the base metric satisfies $|R_{\omega_t}|^2_{\omega_t} \le C$ on a compact subset of N_0, and scaling only improves this bound $|R_{\hat\omega_k}|^2_{\hat\omega_k} \le Ct_k^2$. Rearranging terms, and multiplying by a positive function χ yields

$$2\chi|\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|^2_{\hat\omega_k} \le \chi\Delta_{\hat\omega_k}|F_{\hat\Xi_k}|^2_{\hat\omega_k\hat\Xi_k} + \chi|F_{\hat\Xi_k}|^3_{\hat\omega_k} + C\chi|F_{\hat\Xi_k}|^2_{\hat\omega_k}.$$

If η is a positive bump function supported in $D_{r/2}$ and satisfying $\eta \equiv 1$ in $D_{r/4}$, we specify $\chi = f^{-1}(\eta)$. Integrating the above inequality gives

$$\begin{aligned}\int_{D_{\frac{r}{4}}\times M_{w_0}} |\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|^2_{\hat\omega_k}\hat\omega_k^2 \;&\le\; \frac{1}{2}\int_{D_{\frac{r}{2}}\times M_{w_0}} \Delta_{\hat\omega_k}\chi|F_{\hat\Xi_k}|^2_{\hat\omega_k}\hat\omega_k^2 + C\int_{D_{\frac{r}{2}}\times M_{w_0}} t_k \\ &\le\; Ct_k, \end{aligned} \tag{7.3}$$

where the constant C depends on r, which again we take to be fixed.

We next turn to the higher order Bochner formula for Yang-Mills connections:

$$\begin{aligned}0 =&\Delta_{\hat\omega_k}|\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|^2_{\hat\omega_k} - 2|\nabla^2_{\hat\Xi_k}F_{\hat\Xi_k}|^2_{\hat\omega_k} + \nabla_{\hat\Xi_t}F_{\hat\Xi_k}\#\nabla_{\hat\Xi_k}F_{\hat\Xi_k}\# F_{\hat\Xi_k} \\ &+ R_{\hat\omega_k}\#\nabla_{\hat\Xi_k}F_{\hat\Xi_k}\#\nabla_{\hat\Xi_k}F_{\hat\Xi_k} + \nabla_{\hat\omega_k}R_{\hat\omega_k}\# F_{\hat\Xi_k}\#\nabla F_{\hat\Xi_k}.\end{aligned}$$

Since $|\nabla_{\hat\omega_k}R_{\hat\omega_k}|_{\hat\omega_k} \le t_k|\nabla_{\omega_{t_k}}R_{\omega_{t_k}}|_{\omega_{t_k}} \le C_U t_k$, we have

$$-\Delta_{\hat\omega_k}|\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|^2_{\hat\omega_k} \le C(t_k^{\frac{1}{2}}|\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|^2_{\hat\omega_k} + t_k^{\frac{3}{2}}|\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|_{\hat\omega_k}).$$

Set

$$\psi_k := |\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|^2_{\hat\omega_k}/\sup_{D_r\times M_{w_0}}|\nabla_{\hat\Xi_k}F_{\hat\Xi_k}|^2_{\hat\omega_k}.$$

The above Bochner formula, in addition to our hypothesis (7.2), gives

$$-\Delta_{\hat\omega_k}\psi_k \le C(t_k^{\frac{1}{2}} + t_k) \le 1,$$

for $k \gg 1$. We now follow the argument used in Lemma 5.3. Let $\hat{p}_k$ be the pullbacks of the points p_k via $\Phi_{k,r}$. These are the points realizing the supremum of $|\nabla_{\hat{\Xi}_k} F_{\hat{\Xi}_k}|^2_{\hat{\omega}_k}$, so that $\psi_k(\hat{p}_k) = 1$. Now construct a sequence of functions u_k solving $\Delta_{\hat{\omega}_k} u_k = -1$ and $u_k(\hat{p}_k) = 1$. Working on a small ball $B_{\hat{\omega}_k}(\hat{p}_k, r_0)$, we can assume that $u_k > \varepsilon_0$ for some $\varepsilon_0 > 0$ independent of k. Then since $-\Delta(\psi_k - u_k) \leq 0$, by the mean value inequality, there exists a $\delta > 0$ depending only ε_0 and r_0 such that

$$\delta < \int_{B_{\hat{\omega}_k}(\hat{p}_k, r_0)} u_k \leq \int_{B_{\hat{\omega}_k}(\hat{p}_k, r_0)} \psi_k \leq \int_{D_{r/4} \times M_{w_0}} \psi_k \leq \frac{C_8 t_k}{\sup\limits_{D_r \times M_{w_0}} |\nabla_{\hat{\Xi}_k} F_{\hat{\Xi}_k}|^2_{\hat{\omega}_k}}$$

where the final inequality follows from (7.3). This contradicts (7.2), completing the proof. □

Next, we have a $C^{1,\alpha}$-estimate for A_t.

Lemma 7.3. *For all $w \in U'$, and for all $t \ll 1$, $0 < \alpha < 1$,*

$$\|A_t - A_{0,t}\|_{C^{1,\alpha}(M_w)} \leq C_3 t^{\frac{1}{2}} \quad \text{and} \quad \|\nabla^2_{A_{0,t}} \hat{s}_t\|_{C^{0,\alpha}(M_w)} \leq C_4 t^{\frac{1}{2}},$$

for constants C_3 and C_4 independent of w and t.

Proof. We begin by recalling inequality (5.10), which follows from Proposition 6.1, and properties of $\hat{s}_t$

$$\|\hat{s}_t\|_{L^p_2(M_w)} \leq Ct.$$

We would like to extend the above estimate to the case of $p = \infty$. To accomplish this, we turn to the higher order elliptic a priori estimate

$$\begin{aligned} \|\hat{s}_t\|_{L^p_3(M_w)} &\leq C\left(\|\Delta_{A_{0,t}} \hat{s}_t\|_{L^p_1(M_w)} + \|\hat{s}_t\|_{L^p(M_w)}\right) \\ &\leq C\left(\|\Delta_{A_{0,t}} \hat{s}_t\|_{L^p_1(M_w)} + t\right). \end{aligned}$$

Taking one fiber derivative of (5.7), and using the fact that $\|\hat{s}_t\|_{C^0(M_w)}$ and $\|\nabla_{A_{0,t}} \hat{s}_t\|_{C^0(M_w)}$ are controlled by t, we see that

$$\|\Delta_{A_{0,t}} \hat{s}_t\|_{L^p_1(M_w)} \leq \|\nabla_{A_{0,t}} F_{A_t}\|_{L^p(M_w)} + t\|\hat{s}_t\|_{L^p_3(M_w)} + t\|\hat{s}_t\|_{L^p_2(M_w)}.$$

Thus, for t small enough

$$\|\hat{s}_t\|_{L^p_3(M_w)} \leq C(t + \|\nabla_{A_{0,t}} F_{A_t}\|_{L^p(M_w)}) \leq Ct^{\frac{1}{2}}.$$

By Morrey's inequality we have

$$\|\nabla^2_{A_{0,t}} \hat{s}_t\|_{C^{0,\alpha}(M_w)} \leq Ct^{\frac{1}{2}}. \tag{7.4}$$

□

If we let $\Xi^0_t = e^{-\hat{s}_t}(\Xi_t)$, then $\Xi^0_t|_{M_w} = A_{0,t}$, and we write

$$\Xi^0_t = A_{0,t} + B^0_{t,1} dx_1 + B^0_{t,2} dx_2, \quad \text{and} \quad F_{\Xi^0_t} = -\kappa^0_{t,1} dx_1 - \kappa^0_{t,2} dx_2 - F^0_{B,t} dx_1 \wedge dx_2,$$

where

$$\kappa^0_{t,j} = \partial_{x_j} A_{0,t} - d_{A_{0,t}} B^0_{t,j}.$$

Note that we still have $F^{0,2}_{\Xi^0_t} = 0$, which implies

$$\star_w \kappa^0_{t,1} = \kappa^0_{t,2}, \tag{7.5}$$

and thus

$$\star_w \partial_{x_1} A_{0,t} - \partial_{x_2} A_{0,t} = \star_w d_{A_{0,t}} B^0_{t,1} - d_{A_{0,t}} B^0_{t,2}.$$

Since

$$\star_w \partial_{x_1} A_{0,t} - \partial_{x_2} A_{0,t} \in \ker \Delta_{A_{0,t}},\ \ d_{A_{0,t}} B^0_{t,2} \in \mathrm{Im} d_{A_{0,t}},\ \text{ and } \star_w d_{A_{0,t}} B^0_{t,1} \in \mathrm{Im} d^*_{A_{0,t}},$$

we have $\star_w \partial_{x_1} A_{0,t} = \partial_{x_2} A_{0,t}$ and $d_{A_{0,t}} B^0_{t,j} = 0$ by the Hodge decomposition. As a result we obtain

$$\kappa^0_{t,j} = \partial_{x_j} A_{0,t}. \tag{7.6}$$

A direct calculation shows

$$\begin{aligned} \kappa_{t,j} - \kappa^0_{t,j} &= \partial_{x_j}(A_t - A_{0,t}) - d_{A_t} B_{t,j} \\ &= \nabla_{x_j}(A_t - A_{0,t}) - [B_{t,j}, A_t - A_{0,t}] \\ &\quad - d_{A_{0,t}} B_{t,j} + [A_{0,t} - A_t, B_{t,j}] \\ &= \nabla_{x_j}(A_t - A_{0,t}) - d_{A_{0,t}} B_{t,j}. \end{aligned} \tag{7.7}$$

Now, by (6.5), (7.5) and (7.7),

$$\star_w \nabla_{x_1}(A_t - A_{0,t}) - \nabla_{x_2}(A_t - A_{0,t}) = \star_w d_{A_{0,t}} B_{t,1} - d_{A_{0,t}} B_{t,2},$$

and since $\star_w d_{A_{0,t}} B_{t,1} \perp d_{A_{0,t}} B_{t,2}$, i.e. $\langle \star_w d_{A_{0,t}} B_{t,1}, d_{A_{0,t}} B_{t,2} \rangle_w = 0$, we have

$$\|d_{A_{0,t}} B_{t,j}\|_w \le \sum_{i=1,2} \|\nabla_{x_i}(A_t - A_{0,t})\|_w,$$

for any $w \in U$. Consequently, for $j = 1, 2$

$$\|\kappa_{t,j} - \kappa^0_{t,j}\|_w \le 2 \sum_{i=1,2} \|\nabla_{x_i}(A_t - A_{0,t})\|_w. \tag{7.8}$$

Furthermore, if we decompose $B_{t,j} = B^o_{t,j} + B^\perp_{t,j}$, where $B^o_{t,j} \in \ker d_{A_{0,t}}$ and $B^\perp_{t,j} \perp \ker d_{A_{0,t}}$, then

$$\|B^\perp_{t,j}\|_w \le C \|d_{A_{0,t}} B_{t,j}\|_w \le C \sum_{i=1,2} \|\nabla_{x_i}(A_t - A_{0,t})\|_w, \tag{7.9}$$

by Lemma 5.2. We need one more Lemma before we are ready to prove Proposition 7.1.

Lemma 7.4. *On $U' \subset\subset U$, we have*

$$\int_U \sum_{j=1,2} \|\nabla_{x_j}(A_t - A_{0,t})\|^2_w dx_1 dx_2 \le C_5 (t^2 + \int_U \sum_{j=1,2} \|\nabla_{x_j} F_{A_t}\|^2_w dx_1 dx_2),$$

for a constant $C_5 > 0$. Consequently, by ii) of Proposition 6.1,

$$\int_U \sum_{j=1,2} \|\kappa_{t,j} - \kappa^0_{t,j}\|^2_w dx_1 dx_2 \le C_6 t.$$

Proof. We denote two important terms by

$$\Lambda = \sum_{j=1,2} \|\nabla_{x_j}(A_t - A_{0,t})\|_w, \quad \Theta = \sum_{j=1,2} \|\nabla_{x_j} F_{A_t}\|_w.$$

First, for $j = 1, 2$, we decompose $\nabla_{x_j}\hat{s}_t = \nabla_{x_j}\hat{s}_t^o + \nabla_{x_j}\hat{s}_t^\perp$, where $\nabla_{x_j}\hat{s}_t^\perp$ is perpendicular to the kernel of $d_{A_{0,t}}$,and $\nabla_{x_j}\hat{s}_t^o \in \ker d_{A_{0,t}}$. Recall that $\ker d_{A_{0,t}} = \{\mathrm{diag}\{\eta_1, \cdots, \eta_n\} \in \mathfrak{sl}(n, \mathbb{C})\}$, and as a volume form $\omega^{SF}|_{M_w} = dv$ is independent of w under the identification $M_w \cong T^2$. For any $\eta \in \ker d_{A_{0,t}}$, since $[B_{t,j}^o, \eta] = 0$,

$$\nabla_{x_j}\eta = \partial_{x_j}\eta + [B_{t,j}, \eta] = [B_{t,j}^\perp, \eta].$$

Thus

$$0 = \partial_{x_j}\langle \hat{s}_t, \eta\rangle_w = \langle \nabla_{x_j}\hat{s}_t, \eta\rangle_w + \langle \hat{s}_t, \nabla_{x_j}\eta\rangle_w = \langle \nabla_{x_j}\hat{s}_t^o, \eta\rangle_w + \langle \hat{s}_t, [B_{t,j}^\perp, \eta]\rangle_w,$$

and by (7.9)

$$\|\nabla_{x_j}\hat{s}_t^o\|_w \le C\|\hat{s}_t\|_{C^0}\|B_{t,j}^\perp\|_w \le Ct\Lambda.$$

Along with Lemma 5.2, this implies

$$\|\nabla_{x_j}\hat{s}_t\|_w \le C(\|\nabla_{x_j}\hat{s}_t^\perp\|_w + t\Lambda) \le C(\|d_{A_{0,t}}\nabla_{x_j}\hat{s}_t\|_w + t\Lambda).$$

Since

$$d_{A_t}\nabla_{x_j}\hat{s}_t = d_{A_{0,t}}\nabla_{x_j}\hat{s}_t + [A_t - A_{0,t}, \nabla_{x_j}\hat{s}_t], \quad \text{and} \quad \|A_t - A_{0,t}\|_{C^0} \le Ct,$$

we obtain

$$\|\nabla_{x_j}\hat{s}_t\|_w \le C(\|d_{A_t}\nabla_{x_j}\hat{s}_t\|_w + t\Lambda).$$

Next, take the derivative of (5.5) in the base direction to see

$$\|\nabla_{x_j}(A_t - A_{0,t})\|_w^2 \quad \le \quad 2\|\nabla_{x_j}(\Upsilon(\hat{s}_t))d_{A_t}\hat{s}_t\|_w^2 + 2\|\Upsilon(\hat{s}_t)\nabla_{x_j}(d_{A_t}\hat{s}_t)\|_w^2.$$

We concentrate on the two terms on the right hand side above separately. By Lemma 5.4 and Proposition 6.1, $\hat{s}_t$, $\nabla_{A_{0,t}}\hat{s}_t$ and $A_t - A_{0,t}$ are bounded in C^0 by t, and so the first term satisfies

$$\|\nabla_{x_j}(\Upsilon(\hat{s}_t))d_{A_t}\hat{s}_t\|_w^2 \le t^2C\|\nabla_{x_j}\hat{s}_t\|_w^2 \le t^2C(\|d_{A_t}\nabla_{x_j}\hat{s}_t\|_w^2 + t^2\Lambda^2).$$

To bound the second of the two terms, note that $\kappa_{t,j}$ is bounded, and $\nabla_{x_j}d_{A_t} - d_{A_t}\nabla_{x_j} = \kappa_{t,j}$. Thus

$$\|\Upsilon(\hat{s}_t)\nabla_{x_j}(d_{A_t}\hat{s}_t)\|_w^2 \le C\|\hat{s}_t\|_w^2 + 2\|d_{A_t}\nabla_{x_j}\hat{s}_t\|_w^2 \le Ct^2 + 2\|d_{A_t}\nabla_{x_j}\hat{s}_t\|_w^2,$$

from which we conclude

$$\Lambda^2 \le 2\sum_{j=1,2}\|\nabla_{x_j}(A_t - A_{0,t})\|_w^2 \le 6\sum_{j=1,2}\|d_{A_t}\nabla_{x_j}\hat{s}_t\|_w^2 + Ct^2.$$

Therefore it suffices to bound $\|d_{A_t}\nabla_{x_j}\hat{s}_t\|_w^2$.

Integration by parts, along with Lemma 5.2, gives

$$\begin{aligned}\int_{M_w} |d_{A_t}\nabla_{x_j}\hat{s}_t|^2\omega^{SF} &\le \int_{M_w} |\nabla_{x_j}\hat{s}_t||\Delta_{A_t}\nabla_{x_j}\hat{s}_t|\omega^{SF} \\ &\le \|\nabla_{x_j}\hat{s}_t\|_w\|\Delta_{A_t}\nabla_{x_j}\hat{s}_t\|_w \\ &\le C(\|d_{A_t}\nabla_{x_j}\hat{s}_t\|_w + t\Lambda)\|\Delta_{A_t}\nabla_{x_j}\hat{s}_t\|_w\end{aligned}$$

and so

$$\|d_{A_t}\nabla_{x_j}\hat{s}_t\|_w^2 \le C\|\Delta_{A_t}\nabla_{x_j}\hat{s}_t\|_w^2 + t^2\Lambda^2. \tag{7.10}$$

Thus we obtain

$$\Lambda^2 \le C(\sum_{j=1,2}\|\Delta_{A_t}\nabla_{x_j}\hat{s}_t\|_w^2 + t^2).$$

In order to bound $\Delta_{A_t}\nabla_{x_j}\hat{s}_t$, we turn to the equality (5.6) for the curvature of A_t, using the fact that $A_{0,t}$ is flat,

$$\begin{aligned}F_{A_t} &= i\,d_{A_t}\star_w d_{A_t}\hat{s}_t - \tilde{\Upsilon}(\hat{s}_t)\bar{\partial}_{A_t}\partial_{A_t}\hat{s}_t + \tilde{\Upsilon}(-\hat{s}_t)\partial_{A_t}\bar{\partial}_{A_t}\hat{s}_t \\ &\quad -\bar{\partial}_{A_t}\tilde{\Upsilon}(\hat{s}_t)\wedge\partial_{A_t}\hat{s}_t + \partial_{A_t}\tilde{\Upsilon}(-\hat{s}_t)\wedge\bar{\partial}_{A_t}\hat{s}_t \\ &\quad -\Upsilon(\hat{s}_t)\partial_{A_t}\hat{s}_t\wedge\Upsilon(-\hat{s}_t)\bar{\partial}_{A_t}\hat{s}_t + \Upsilon(-\hat{s}_t)\bar{\partial}_{A_t}\hat{s}_t\wedge\Upsilon(\hat{s}_t)\partial_{A_t}\hat{s}_t.\end{aligned}$$

We take the derivative of this equation in the base direction, and calculate $\nabla_{x_j}F_{A_t}$. Firstly,

$$\begin{aligned}\nabla_{x_j}d_{A_t}\star_w d_{A_t}\hat{s}_t &= d_{A_t}\nabla_{x_j}\star_w d_{A_t}\hat{s}_t + \kappa_{t,j}\#d_{A_t}\hat{s}_t \\ &= d_{A_t}\star_w d_{A_t}\nabla_{x_j}\hat{s}_t + d_{A_t}[\star_w\kappa_{t,j},\hat{s}_t] + \kappa_{t,j}\#d_{A_t}\hat{s}_t \\ &= d_{A_t}\star_w d_{A_t}\nabla_{x_j}\hat{s}_t \pm [\nabla_{x_i}F_{A_t},\hat{s}_t] + \kappa_{t,j}\#d_{A_t}\hat{s}_t\end{aligned}$$

by $\nabla_{x_i}F_{A_t} = d_{A_t}\kappa_{t,i} = \pm d_{A_t}\star_w\kappa_{t,j}$, which implies

$$\begin{aligned}&|\nabla_{x_j}d_{A_t}\star_w d_{A_t}\hat{s}_t - d_{A_t}\star_w d_{A_t}\nabla_{x_j}\hat{s}_t| \\ &\qquad\le C(|\nabla_{A_{0,t}}\hat{s}_t| + |A_t - A_{0,t}||\hat{s}_t| + |\nabla_{x_i}F_{A_t}||\hat{s}_t|), \\ &\qquad\le Ct(1 + \sum_{i=1,2}|\nabla_{x_i}F_{A_t}|).\end{aligned}$$

As a result, we have

$$\|\nabla_{x_j}d_{A_t}\star_w d_{A_t}\hat{s}_t - d_{A_t}\star_w d_{A_t}\nabla_{x_j}\hat{s}_t\|_w \le Ct(1+\Theta).$$

Secondly, note that $\nabla_{A_t} = \nabla_{A_{0,t}} + (A_t - A_{0,t})$, and

$$\nabla_{A_t}^2 = \nabla_{A_{0,t}}^2 + (A_t - A_{0,t})\#\nabla_{A_{0,t}} + \nabla_{A_{0,t}}(A_t - A_{0,t}) + (A_t - A_{0,t})\#(A_t - A_{0,t}).$$

A direct calculation shows

$$\begin{aligned}
&\|\nabla_{x_j}(\tilde{\Upsilon}(\hat{s}_t)\bar{\partial}_{A_t}\partial_{A_t}\hat{s}_t)\|_w \\
&\qquad \le C(\|\nabla_{x_j}\hat{s}_t\|_w\|\nabla^2_{A_t}\hat{s}_t\|_{C^0} + \|\nabla_{x_j}\bar{\partial}_{A_t}\partial_{A_t}\hat{s}_t\|_w\|\hat{s}_t\|_{C^0}) \\
&\qquad \le C\|\nabla_{x_j}\hat{s}_t\|_w(\|\nabla^2_{A_{0,t}}\hat{s}_t\|_{C^0} + \|A_t - A_{0,t}\|_{C^1}\|\hat{s}_t\|_{C^1}) \\
&\qquad\quad + C(\|\Delta_{A_t}\nabla_{x_j}\hat{s}_t\|_w + 1 + t\Theta))\|\hat{s}_t\|_{C^0} \\
&\qquad \le C(t\|\Delta_{A_t}\nabla_{x_j}\hat{s}_t\|_w + t^{\frac{1}{2}}\|\nabla_{x_j}\hat{s}_t\|_w + t + t^2\Theta) \\
&\qquad \le C(t^{\frac{1}{2}}\|\Delta_{A_t}\nabla_{x_j}\hat{s}_t\|_w + t + t\Lambda + t^2\Theta),
\end{aligned}$$

where we used Lemma 7.3. For the later terms, we have

$$\begin{aligned}
&\|\nabla_{x_j}(\bar{\partial}_{A_t}\tilde{\Upsilon}(\hat{s}_t) \wedge \partial_{A_t}\hat{s}_t)\|_w + \|\nabla_{x_j}(\Upsilon(\hat{s}_t)\partial_{A_t}\hat{s}_t \wedge \Upsilon(-\hat{s}_t)\bar{\partial}_{A_t}\hat{s}_t)\|_w \\
&\quad \le C(\|\nabla_{A_{0,t}}\hat{s}_t\|_{C^0} + \|A_t - A_{0,t}\|_{C^0}\|\hat{s}_t\|_{C^0})(\|\nabla_{x_j}\hat{s}_t\|_w \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \|d_{A_t}\nabla_{x_j}\hat{s}_t\|_w + \|\hat{s}_t\|_w) \\
&\quad \le C(t^2 + t\|\Delta_{A_t}\nabla_{x_j}\hat{s}_t\|_w + t^2\Lambda).
\end{aligned}$$

Returning to (7.10), we put everything together to see

$$\|\nabla_{x_j}F_{A_t} - id_{A_t}\star_w d_{A_t}\nabla_{x_j}\hat{s}_t\|_w \le C(t^{\frac{1}{2}}\|\Delta_{A_t}\nabla_{x_j}\hat{s}_t\|_w + t\Lambda + t\Theta + t),$$

$$\|\Delta_{A_t}\nabla_{x_j}\hat{s}_t\|_w \le C(\Theta + t + t\Lambda),$$

and

$$\|d_{A_t}\nabla_{x_j}\hat{s}_t\|_w \le C(\Theta + t + t\Lambda).$$

Thus we conclude

$$\Lambda^2 \le C(\Theta^2 + t^2),$$

proving the lemma. □

Now, we are ready to prove Proposition 7.1.

Proof of Proposition 7.1. Note that we have

$$\|F_{\Xi_t}\|^2_{L^2(M_{U'},\omega_t)} \le 2\int_{M_{U'}}(t^{-1}|F_{A_t}|^2_{\omega^{SF}} + \sum_{j=1,2}|\kappa_{t,j}|^2_{\omega^{SF}} + t|F_{B,t}|^2_{\omega^{SF}})(\omega^{SF})^2.$$

By (6.6), we have

$$t|F_{B,t}|^2_{\omega^{SF}} \le C(t^{-1}|F_{A_t}|^2_{\omega^{SF}} + t\sum_{j=1,2}|\kappa_{t,j}|^2_{\omega^{SF}}),$$

which in turn implies

$$\begin{aligned}
\|F_{\Xi_t}\|^2_{L^2(M_{U'},\omega_t)} &\le C\int_{U'}(t^{-1}\|F_{A_t}\|_w^2+\sum_{j=1,2}\|\kappa_{t,j}\|_w^2)dx_1dx_2 \\
&\le C(t+\sum_{j=1,2}\int_{U'}\|\kappa_{t,j}-\kappa^0_{t,j}\|_w^2dx_1dx_2 \\
&\quad +\sum_{j=1,2}\int_{U'}\|\partial_{x_j}A_{0,t}\|_w^2dx_1dx_2) \\
&\le C(t+\sum_{j=1,2}\int_{U'}\|\partial_{x_j}A_{0,t}\|_w^2dx_1dx_2)
\end{aligned}$$

For the second inequality above we used $\|F_{A_t}\|_w^2\le Ct^2$ and $\kappa^0_{t,j}=\partial_{x_j}A_{0,t}$.

Finally,

$$\begin{aligned}
\|F_{\Xi_t}\|^2_{L^2(M_{U'},\omega_t)} &\ge \frac{1}{2}\int_{U'}\sum_{j=1,2}\|\kappa_{t,j}\|_w^2dx_1dx_2 \\
&\ge \frac{1}{2}(\sum_{j=1,2}\int_{U'}\|\partial_{x_j}A_{0,t}\|_w^2dx_1dx_2 \\
&\quad -\sum_{j=1,2}\int_{U'}\|\kappa_{t,j}-\kappa^0_{t,j}\|_w^2dx_1dx_2) \\
&\ge C(\sum_{j=1,2}\int_{U'}\|\partial_{x_j}A_{0,t}\|_w^2dx_1dx_2-t),
\end{aligned}$$

and we obtain the conclusion. □

We finish this section by a lemma that is needed in the proof of Theorem 3.2.

Lemma 7.5.

$$\sum_{j=1,2}\|[\kappa_{t,j},\kappa_{t,j}]\|^2_{L^2(M_{U'},\omega^{SF})}\le C_7t,$$

for a constant $C_7>0$.

Proof. Recall that $\kappa^0_{t,j}=\partial_{x_j}A_{0,t}$ by (7.6), and thus

$$[\kappa^0_{t,j},\kappa^0_{t,j}]=0,\quad j=1,2.$$

We have

$$[\kappa_{t,j},\kappa_{t,j}]=2[\kappa^0_{t,j},\kappa_{t,j}-\kappa^0_{t,j}]+[\kappa_{t,j}-\kappa^0_{t,j},\kappa_{t,j}-\kappa^0_{t,j}],$$

and by $|\kappa_{t,j}|\le C$,

$$|[\kappa_{t,j},\kappa_{t,j}]|\le C|\kappa_{t,j}-\kappa^0_{t,j}|.$$

Lemma 7.4 shows that

$$\int_U\sum_{j=1,2}\|[\kappa_{t,j},\kappa_{t,j}]\|_w^2dx_1dx_2\le C\int_U\sum_{j=1,2}\|\kappa_{t,j}-\kappa^0_{t,j}\|_w^2dx_1dx_2\le Ct.$$

We obtain the conclusion. □

8. Proof of Proposition 4.4

Now, we have the tools to verify assumption (6.3) along our main subsequence of times t_k, which is chosen in Proposition 4.1.

Proof of Proposition 4.4. We work via contradiction, and assume the Proposition is false, in other words assumption (6.3) fails for our sequence Ξ_{t_k}. By passing to a subsequence, there exists a sequence of points $p'_k \in M_K$ so that

$$t_k^{\frac{1}{2}}|F_{\Xi_{t_k}}|_{\omega_{t_k}}(p'_k) \to \infty, \tag{8.1}$$

and $f(p'_k)$ converges to a point $x \in K$, as $t_k \to 0$.

Applying Lemma 4.2, we can pick new points near p_k to carry out our argument. Specifically, if $r = \frac{1}{2}\mathrm{dist}_\omega(x, N\backslash K)$, there exists a sequence of real numbers $0 < \rho_k < r$ and a sequence $p_k \in M$ so that $d_{\omega_{t_k}}(p_k, p'_k) \leq r$,

$$\sup_{B_{\omega_{t_k}}(p_k,\rho_k)} |F_{\Xi_{t_k}}|_{\omega_{t_k}} \leq 2|F_{\Xi_{t_k}}|_{\omega_{t_k}}(p_k),$$

and

$$2\rho_k|F_{\Xi_{t_k}}|_{\omega_{t_k}}(p_k) \geq r|F_{\Xi_{t_k}}|_{\omega_{t_k}}(p'_k).$$

If we set $\delta_k := t_k^{-\frac{1}{2}}|F_{\Xi_{t_k}}|^{-1}_{\omega_{t_k}}(p_k)$, then (8.1) and the above inequalities give $\delta_k \to 0$, and

$$\rho_k\delta_k^{-1} \geq rt_k^{\frac{1}{2}}|F_{\Xi_{t_k}}|_{\omega_{t_k}}(p'_k) \to \infty.$$

Furthermore, define

$$\tilde{t}_k := t_k\delta_k^{-2} = {t_k}^2|F_{\Xi_{t_k}}|^2_{\omega_{t_k}}(p_k) \leq \epsilon_k^2 \to 0,$$

which goes to zero as $t_k \to 0$ by Proposition 4.1.

We now consider the scaled metric $\tilde{\omega}_{\tilde{t}_k} = \delta_k^{-2}\omega_{t_k}$, and claim that $\tilde{\omega}_{\tilde{t}_k}$ satisfies the same collapsing properties of ω_{t_k}. If $\tilde{w} = \delta_k^{-1}w$ denotes the scaled coordinate on $D_r = \{|w| < r\delta_k\} = \{|\tilde{w}| < r\}$, where $f(p_k)$ is given by $w = 0$, then

$$\delta_k^{-2}\omega^{SF}_{t_k} = \frac{i}{2}\left(\tilde{t}_kW(dz + \tilde{b}d\tilde{w}) \wedge \overline{(dz + \tilde{b}d\tilde{w})} + W^{-1}d\tilde{w} \wedge d\bar{\tilde{w}}\right),$$

where $\tilde{b} = -\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\frac{\partial\tau}{\partial\tilde{w}}$. For a certain fiberwise translation T_{σ_0}, we write

$$\begin{aligned} T^*_{\sigma_0}\delta_k^{-2}\omega_{t_k} - \delta_k^{-2}\omega^{SF}_{t_k} &= \delta_k^{-2}\varphi_{t_k,z\bar{z}}dz \wedge d\bar{z} + \varphi_{t_k,w\bar{w}}d\tilde{w} \wedge d\bar{\tilde{w}} \\ &\quad + \delta_k^{-1}\varphi_{t_k,w\bar{z}}d\tilde{w} \wedge d\bar{z} + \delta_k^{-1}\varphi_{t_k,z\bar{w}}dz \wedge d\bar{\tilde{w}}. \end{aligned}$$

By Lemma 2.11, for $\nu \gg 1$,

$$\|\delta_k^{-2}\varphi_{t_k,z\bar{z}}\|_{C^\ell_{\mathrm{loc}}} + \|\delta_k^{-1}\varphi_{t_k,z\bar{w}}\|_{C^\ell_{\mathrm{loc}}} + \|\delta_k^{-1}\varphi_{t_k,w\bar{z}}\|_{C^\ell_{\mathrm{loc}}} \leq C_\ell\tilde{t}_k^\nu,$$

and

$$\|\frac{\partial}{\partial z}\varphi_{t_k,w\bar{w}}\|_{C^\ell_{\mathrm{loc}}} + \|\frac{\partial}{\partial\bar{z}}\varphi_{t_k,w\bar{w}}\|_{C^\ell_{\mathrm{loc}}} \leq C_\ell\tilde{t}_k^\nu, \qquad \|\varphi_{t_k,w\bar{w}} - \chi_{t_k,w\bar{w}}\|_{C^0_{\mathrm{loc}}} \leq C_0\tilde{t}_k^\nu.$$

Here we used $t_k \leq \tilde{t}_k$, and that $\chi_{t_k,w\bar{w}}$ is a function on D_r that satisfies $\chi_{t_k,w\bar{w}} \to 0$ in the C^∞-sense as $t_k \to 0$. The $C^\ell_{\rm loc}$-norms are calculated in coordinates z and $\tilde{w}$.

Working in the scaled metrics, we have that $d_{\omega_{\tilde{t}_k}}(p_k, p) \leq \rho_k \delta_k^{-1}$ for any $p \in B_{\omega_{t_k}}(p_k, \rho_k)$, so the radius of the disk approaches infinity. In particular this implies that on $B_{\tilde{\omega}_{\tilde{t}_k}}(p_k, \rho_k\delta_k^{-1})$, we have the bound

$$|F_{\Xi_{t_k}}|_{\tilde{\omega}_{\tilde{t}_k}} = \delta_k^2 |F_{\Xi_{t_k}}|_{\omega_{t_k}} \leq 2\delta_k^2 |F_{\Xi_{t_k}}|_{\omega_{t_k}}(p_k) = 2t_k^{-1}|F_{\Xi_{t_k}}|^{-1}_{\omega_{t_k}}(p_k) = 2\tilde{t}_k^{-\frac{1}{2}}.$$

Now, because the energy $\mathcal{E}_{t_k}(p, R_{t_k}(p_k))$ is scale invariant,

$$\begin{aligned} \varepsilon &= \mathcal{E}_{t_k}(p_k, R_{t_k}(p_k)) \\ &= \frac{\delta_k^{-4} R_{t_k}(p_k)^4}{\mathrm{Vol}(B_{\tilde{\omega}_{\tilde{t}_k}}(p_k, \delta_{t_k}^{-1} R_{t_k}(p_k)))} \int_{B_{\tilde{\omega}_{\tilde{t}_k}}(p_k, \delta_{t_k}^{-1} R_{t_k}(p_k))} |F_{\Xi_{t_k}}|^2_{\tilde{\omega}_{\tilde{t}_k}} \tilde{\omega}^2_{\tilde{t}_k}. \end{aligned}$$

Additionally, note that

$$\delta_k^{-1} R_{t_k}(p_k) = t_k{}^{\frac{1}{2}} |F_{\Xi_{t_k}}|_{\omega_{t_k}}(p_k) R_{t_k}(p_k) \leq 4 t_k^{\frac{1}{2}} |F_{\Xi_{t_k}}|^{\frac{1}{2}}_{\omega_{t_k}}(p_k) = 4\tilde{t}_k^{\frac{1}{4}},$$

since $|F_{\Xi_{t_k}}|_{\omega_{t_k}}(p_k) \leq 4R^{-2}_{t_k}(p_k)$ by (2.15). Thus, on $B_{\tilde{\omega}_{\tilde{t}_k}}(p_k, \rho_k\delta_k^{-1})$ we have

$$|F_{\Xi_{t_k}}|_{\tilde{\omega}_{\tilde{t}_k}} \leq 2\tilde{t}_k^{-\frac{1}{2}} \tag{8.2}$$

and

$$\varepsilon \leq \frac{4^4 \tilde{t}_k}{\mathrm{Vol}(B_{\tilde{\omega}_{\tilde{t}_k}}(p_k, 4\tilde{t}_k^{\frac{1}{4}}))} \int_{B_{\tilde{\omega}_{\tilde{t}_k}}(p_k, 4\tilde{t}_k^{\frac{1}{4}})} |F_{\Xi_{t_k}}|^2_{\tilde{\omega}_{\tilde{t}_k}} \tilde{\omega}^2_{\tilde{t}_k}. \tag{8.3}$$

Inequality (8.2) gives assumption (6.3) for our connections in scaled coordinates (with scaled parameter $\tilde{t}$). Also (6.4) is also satisfied since the scaling does not effect the fiber direction. Thus Proposition 6.1 holds in scaled coordinates, which in turn allows us to conclude Proposition 7.1 as well.

To achieve our contradiction, we show these bounds force the energy on the right hand side of (8.3) to go to zero. We continue to use the notation $\|\cdot\|_w := \|\cdot\|_{L^2(M_w, \tilde{\omega}^{SF})}$ since scaling does not affect the fiber direction.

Applying Proposition 7.1, on any $K \subset D_r$ we have

$$\|F_{\Xi_{t_k}}\|^2_{L^2(M_K, \tilde{\omega}_{\tilde{t}_k})} \leq C(\tilde{t}_k + \int_K \sum_{j=1,2} \|\partial_{\tilde{x}_j} A_{0,t_k}\|^2_w d\tilde{x}_1 d\tilde{x}_2)$$

for a uniform constant C, where $\tilde{x}_1 + i\tilde{x}_2 = \tilde{w}$. Since $A_{0,t_k} \to A_0$ in the C^∞-sense on M_U, we have

$$\|\partial_{\tilde{x}_j} A_{0,t_k}\|^2_w = \delta_k^2 \|\partial_{x_j} A_{0,t_k}\|^2_w \leq C\delta_k^2,$$

and thus

$$\|F_{\Xi_{t_k}}\|^2_{L^2(M_K, \tilde{\omega}_{\tilde{t}_k})} \leq C(\tilde{t}_k + \delta_k^2 \int_K d\tilde{x}_1 d\tilde{x}_2).$$

Because the radius $\tilde{t}_k^{\frac{1}{4}}$ grows slower than the injectivity radius of the elliptic fibers in the metric $\tilde{\omega}_{\tilde{t}_k}$ (which is roughly $\tilde{t}_k^{\frac{1}{2}}$), we see that for $\tilde{t}_k$ small enough

$$\frac{\tilde{t}_k}{\mathrm{Vol}(B_{\tilde{\omega}_{\tilde{t}_k}}(p, 4\tilde{t}_k^{\frac{1}{4}}))} \leq \frac{C\tilde{t}_k}{\tilde{t}_k \tilde{t}_k^{\frac{1}{2}}} = \frac{C}{\tilde{t}_k^{\frac{1}{2}}}.$$

Also $B_{\tilde{\omega}_{\tilde{t}_k}}(p_k, 4\tilde{t}_k^{\frac{1}{4}}) \subset M_{D_r}$. Thus, returning to (8.3), we have

$$\begin{aligned} \varepsilon &\leq \frac{4^4 \tilde{t}_k}{\mathrm{Vol}(B_{\tilde{\omega}_{\tilde{t}_k}}(p_k, 4\tilde{t}_k^{\frac{1}{4}}))} \int_{B_{\tilde{\omega}_{\tilde{t}_k}}(p_k, 4\tilde{t}_k^{\frac{1}{4}})} |F_{\Xi_{t_k}}|^2_{\tilde{\omega}_{\tilde{t}_k}} \tilde{\omega}^2_{\tilde{t}_k} \\ &\leq \frac{C}{\tilde{t}_k^{\frac{1}{2}}} (\tilde{t}_k + \delta_k^2 \tilde{t}_k^{\frac{1}{2}}) \\ &\leq C(\tilde{t}_k^{\frac{1}{2}} + \delta_k^2). \end{aligned}$$

The right hand side above goes to zero, a contradiction. □

9. The proof of Theorem 3.2

At last, we prove Theorem 3.2 in this section. Under the same setup as in Section 6, the first lemma shows that for any fixed $p \geq 2$,

$$\|F_{B,t}\|_{L^p(M_U, \omega^{SF})} \to 0$$

when $t \to 0$.

Lemma 9.1. *If (6.3) and (6.4) hold for $t \ll 1$, for any $p \geq 2$, we have the following inequalities*

$$\|F_{A_t}\|^p_{L^p(M_U, \omega^{SF})} \leq C_1 t^{1+p}, \quad \textit{and} \quad \|F_{B,t}\|^p_{L^p(M_U, \omega^{SF})} \leq C_1 t^{1+\frac{1}{p}},$$

where the constant C_1 is independent of t.

Proof. By Lemma 6.3,

$$\Delta \|F_{A_t}\|^2_w \geq \frac{\delta}{t} \|F_{A_t}\|^2_w - CtZ_t,$$

where

$$Z_t = \sum_{j=1,2} \|[\kappa_{t,j}, \kappa_{t,j}]\|^2_w + t^\nu,$$

for $\nu \gg 1$ and a constant $C > 0$. Lemma 7.5 implies that

$$\int_U Z_t dx_1 dx_2 \leq Ct.$$

Let η be a smooth function such that $0 \leq \eta \leq 1$ and $\mathrm{supp}(\eta) \subset U$. Then

$$\begin{aligned} \int_U \|F_{A_t}\|^2_w dx_1 dx_2 &\leq t\delta^{-1} \int_U \|F_{A_t}\|^2_w \Delta\eta dx_1 dx_2 + t^2 C \int_U \eta Z_t dx_1 dx_2 \\ &\leq t\tilde{C}\delta^{-1} \int_U \|F_{A_t}\|^2_w dx_1 dx_2 + t^2 C \int_U Z_t dx_1 dx_2, \end{aligned}$$

for a constant $\tilde{C} \geq \sup_U \Delta\eta$. Thus for $t \ll 1$,

$$\int_U \|F_{A_t}\|_w^2 dx_1 dx_2 \leq Ct^3.$$

For any $p \geq 2$,

$$\|F_{A_t}\|^p_{L^p(M_U,\omega^{SF})} \leq Ct^{p-2} \int_U \|F_{A_t}\|_w^2 dx_1 dx_2 \leq Ct^{p+1},$$

by Lemma 6.5, and

$$\|F_{B,t}\|^p_{L^p(M_U,\omega^{SF})} \leq Ct^{1+\frac{1}{p}},$$

by (6.6). $\square$

Recall that for any sequence $t_k \to 0$, a subsequence of Ξ_{t_k} $L^p_1 \cap C^{0,\alpha}$-converges to a $L^p_1 \cap C^{0,\alpha}$-connection Ξ_0 by preforming certain further unitary gauge changes if necessary on M_K in Theorem 3.1, where $K \subset N^o$. Thus the curvature $F_{\Xi_{t_k}}$ L^p-converges to F_{Ξ_0} on M_K.

On any open disc $U \subset K$, we have the decompositions

$$\Xi_0 = \tilde{A}_0 + \tilde{B}_{0,1} dx_1 + \tilde{B}_{0,2} dx_2, \quad \text{and}$$

$$F_{\Xi_0} = F_{\tilde{A}_0} - \tilde{\kappa}_{0,1} \wedge dx_1 - \tilde{\kappa}_{0,2} \wedge dx_2 - F_{\tilde{B},0} dx_1 \wedge dx_2,$$

where $\tilde{\kappa}_{0,j} = \frac{\partial}{\partial x_j} \tilde{A}_0 - d_{\tilde{A}_0} \tilde{B}_{0,j}$. By Lemma 9.1 and the convergence, we obtain that

$$F_{\tilde{A}_0} \equiv 0, \quad F_{\tilde{B},0} \equiv 0, \quad \text{and} \quad \star_w \tilde{\kappa}_{0,1} = \tilde{\kappa}_{0,2}.$$

Thus Ξ_0 is an anti-self-dual connection with respect to (ω^{SF}, Ω), i.e.

$$F_{\Xi_0} \wedge \omega^{SF} = 0, \quad \text{and} \quad F_{\Xi_0} \wedge \Omega = 0.$$

It is standard (cf. Theorem 9.4 of [75]) that by preforming a further unitary gauge change if necessary, we can have that Ξ_0 is smooth.

Lemma 9.2. *There is a unitary gauge u such that*

$$u(\Xi_0) = A_0$$

on M_U, where A_0 is given by (4.4).

Proof. By Theorem 3.1, for any $w \in U$, there is a unitary gauge u_w on M_w such that $u_w(\Xi_0|_{M_w}) = A_0|_{M_w}$, and u_w is smooth since both $\Xi_0|_{M_w}$ and $A_0|_{M_w}$ are smooth. We claim that one can choose u_w depending on w smoothly.

Note that $M_w \cong T^2$ and $P|_{M_w} \cong M_w \times SU(n)$. Let $\mathcal{A}^{\ell,p}$ be the space of L^p_ℓ $SU(n)$-connections on the trivial bundle on T^2, $\ell \geq 1$, and $\mathcal{G}^{\ell+1,p}$ be the $L^p_{\ell+1}$ unitary gauge group. We have identifications $\mathcal{A}^{\ell,p} = L^p_\ell(T^2, \mathfrak{sl}(n))$ and $\mathcal{G}^{\ell+1,p} = L^p_{\ell+1}(T^2, SU(n))$ under the trivialization, and $\mathcal{G}^{\ell+1,p}$ acts on $\mathcal{A}^{\ell,p}$

by $u(A) = u^{-1}Au + u^{-1}du$. If we denote the orbit $O_w = \{u(\Xi_0|_{M_w})|u \in \mathcal{G}^{\ell+1,p}\} \subset \mathcal{A}^{\ell,p}$ for any $w \in U$, then $A_0|_{M_w} \in O_w$. Define the orbit map

$$\Psi : \mathcal{G}^{\ell+1,p} \times U \to \bigcup_{w \in U} O_w \subset \mathcal{A}^{\ell,p}, \quad \text{by} \quad \Psi(u,w) = u(\Xi_0|_{M_w}).$$

For a fixed $w_0 \in U$, let $\varrho_w : O_w \to O_{w_0}$ by $A \mapsto v(A_0|_{M_{w_0}})$, where $v(A_0|_{M_w}) = A$ for a unitary gauge v. If v' is an another unitary gauge such that $v'(A_0|_{M_w}) = A$, then $v'v^{-1}(A_0|_{M_w}) = A_0|_{M_w}$, and thus $v'v^{-1} \in T^{n-1} \subset SU(n)$, i.e. a diagonal matrix. Since $A_0|_{M_{w_0}}$ is a diagonal matrix valued 1-form, we have $v(A_0|_{M_{w_0}}) = v'(A_0|_{M_{w_0}})$, and ϱ_w is well-defined.

Let $\Psi' = \varrho_w \circ \Psi : \mathcal{G}^{\ell+1,p} \times U \to O_{w_0}$ be the the composition. Note that the tangent space $T_{A_0|_{M_{w_0}}} O_{w_0} = \mathrm{Im}(d_{A_0|_{M_{w_0}}})$, and the first partial derivative of Ψ' at (u,w) such that $\Psi'(u,w) = A_0|_{M_{w_0}}$ is $D_1\Psi' = -d_{A_0|_{M_{w_0}}}$. Thus $A_0|_{M_{w_0}}$ is a regular value of Ψ', and $\Psi'^{-1}(A_0|_{M_{w_0}})$ is a smooth submanifold. Furthermore, the projection $\mathcal{G}^{\ell+1,p} \times U \to U$ induces a T^{n-1}-bundle structure on $\Psi'^{-1}(A_0|_{M_{w_0}})$ with fiber $T^{n-1} \subset SU(n)$.

If $\tilde{u} : U \to \Psi'^{-1}(A_0|_{M_{w_0}})$ is a smooth section, then $\tilde{u}(w)(\Xi_0|_{M_w}) = A_0|_{M_w}$, and we can regard $\tilde{u}$ as a smooth unitary gauge change on M_U. Therefore we have

$$\tilde{u}(\Xi_0) = A_0 + B_{0,1}dx_1 + B_{0,2}dx_2,$$

which still satisfies

$$\star_w \kappa_{0,1} = \kappa_{0,2}, \quad \text{with} \quad \kappa_{0,j} = \frac{\partial}{\partial x_j} A_0 - d_{A_0} B_{0,j}, \quad j = 1,2, \quad \text{and}$$

$$0 = F_{B,0} = \frac{\partial}{\partial x_2} B_{0,1} - \frac{\partial}{\partial x_1} B_{0,2} - [B_{0,1}, B_{0,2}].$$

Note that $\frac{\partial}{\partial x_j} A_0 \in \ker \Delta_{A_0}$, $j = 1,2$, on any M_w, and

$$\star_w \frac{\partial}{\partial x_1} A_0 - \frac{\partial}{\partial x_2} A_0 = \star_w d_{A_0} B_{0,1} - d_{A_0} B_{0,2}.$$

By the Hodge decomposition, $\ker \Delta_{A_0}$, $\mathrm{Im}(d^*_{A_0})$ and $\mathrm{Im}(d_{A_0})$ are orthogonal to each other. Thus

$$d_{A_0} B_{0,j} \equiv 0, \quad j = 1,2,$$

on any M_w, and $B_{0,j}|_{M_w}$ is a diagonal matrix in $\mathfrak{sl}(n)$. If we write $B_{0,j} = i\mathrm{diag}\{b_{j,1}, \cdots, b_{j,n}\}$, then $\frac{\partial}{\partial x_2} B_{0,1} = \frac{\partial}{\partial x_1} B_{0,2}$ implies that there are real functions ϑ_ℓ on U such that $b_{1,\ell}dx_1 + b_{2,\ell}dx_2 = -d\vartheta_\ell$, $\ell = 1, \cdots, n$. If $\tilde{v} = \mathrm{diag}\{\exp(i\vartheta_1), \cdots, \exp(i\vartheta_n)\}$, and we regard $\tilde{v}$ as a unitary gauge change on M_U, then

$$\tilde{v}(\tilde{u}(\Xi_0)) = A_0.$$

We obtain the conclusion by letting $u = \tilde{v} \cdot \tilde{u}$. □

Proof of Theorem 3.2. Let $\{U_\lambda | \lambda \in \Lambda\}$ be an open cover of N^o such that any intersection $U_{\lambda_1} \cap \cdots \cap U_{\lambda_h}$ is contractible. For any U_λ, $D_0^o \cap M_{U_\lambda} = U_\lambda^1 \cup \cdots \cup U_\lambda^n$ is a disjoint union of open sets biholomorphic to U_λ, and

$\{U_\lambda^j|\lambda \in \Lambda, j = 1, \cdots, n\}$ is an open cover of $D_0^o \cap M_{N^o}$ such that any intersections are contractible.

On any M_{U_λ}, there is a unitary gauge u_λ such that $u_\lambda(\Xi_0) = A_0$ by Lemma 9.2. Recall that

$$A_0 = \mathrm{diag}\{\alpha_1, \cdots, \alpha_n\}, \quad \alpha_j = \pi(\mathrm{Im}(\tau))^{-1}(q_j\bar{\theta} - \bar{q}_j\theta),$$

where $\{(w, q_j(w))\} = U_\lambda^j$ is one component of $D_0^o \cap M_{U_\lambda}$, and α_j is not unitary gauge equivalent to α_i if $j \neq i$. On any intersection $M_{U_\lambda \cap U_\mu}$, $A_0 = u_\mu \cdot u_\lambda^{-1}(A_0)$. Thus $u_\mu \cdot u_\lambda^{-1}|_{M_w} \in T^{n-1} \subset SU(n)$ for any $w \in U_\lambda \cap U_\mu$. We can write $u_\mu \cdot u_\lambda^{-1} = \mathrm{diag}\{g_{\mu\lambda}^{1j_1}, \cdots, g_{\mu\lambda}^{nj_n}\}$, where $g_{\mu\lambda}^{ij_i}$ is a $U(1)$-valued function on $U_\lambda \cap U_\mu$, and is the unitary gauge change between α_i on M_{U_μ} and α_{j_i} on M_{U_λ}. Hence we have that $U_\mu^i \cap U_\lambda^{j_i} \neq \emptyset$, and $d \log g_{\mu\lambda}^{ij_i} = 0$, which implies that $g_{\mu\lambda}^{ij_i}$, $i = 1, \cdots, n$, are $U(1)$-valued constant functions on $U_\lambda \cap U_\mu$. By regarding $g_{\mu\lambda}^{ij_i}$ as a function on $U_\mu^i \cap U_\lambda^{j_i}$, we obtain a 1-chain $\{(U_\mu^i \cap U_\lambda^{j_i}, g_{\mu\lambda}^{ij_i})\} \in \mathcal{C}^1(\{U_\lambda^j\}, \mathcal{U}_c(1))$ for the $U(1)$-valued locally constant sheaf $\mathcal{U}_c(1)$ on $D_0^o \cap M_{N^o}$.

If $U_\mu^i \cap U_\lambda^j \cap U_\nu^k \neq \emptyset$, then $U_\mu \cap U_\lambda \cap U_\nu \neq \emptyset$, and by $u_\mu \cdot u_\lambda^{-1} \cdot u_\lambda \cdot u_\nu^{-1} \cdot u_\nu \cdot u_\mu^{-1} = \mathrm{Id}$, we obtain that $g_{\mu\lambda}^{ij} g_{\lambda\nu}^{jk} g_{\nu\mu}^{ki} = 1$. Therefore $\{(U_\mu^i \cap U_\lambda^{j_i}, g_{\mu\lambda}^{ij_i})\}$ satisfies the cocycle condition, and defines a cohomological class $\Theta = [\{(U_\mu^i \cap U_\lambda^{j_i}, g_{\mu\lambda}^{ij_i})\}] \in H^1(D_0^o \cap M_{N^o}, \mathcal{U}_c(1))$, which is equivalent to a flat $U(1)$-connection on $D_0^o \cap M_{N^o}$. From the construction in Subsection 2.6, it is clear that $\Xi_0 \in \mathcal{FM}(D_0^o \cap M_{N^o}, \Theta)$. $\square$

Appendix A. Collapsing rate of Ricci-flat Kähler-Einstein metrics

Here we study the collapsing rate of Ricci-flat Kähler-Einstein metrics on general Calabi-Yau manifolds, which is used in the proof of the main theorem.

Let M be a Calabi-Yau m-manifold, i.e. M is projective with trivial canonical bundle $\mathcal{K}_M \cong \mathcal{O}_M$. Assume M admits a holomorphic fibration $f : M \to N$, where N is smooth projective manifold with $n = \dim_{\mathbb{C}} N < m$. As above, let S_N denotes the discriminant locus f, and $N_0 = N \backslash S_N$ the regular locus. For any $w \in N_0$, the smooth fiber $M_w = f^{-1}(w)$ is a Calabi-Yau manifold of dimension $m - n$. Let α be an ample class on M, and α_0 an ample class on N. Then for $t \in [0, 1)$, $\alpha_t = t\alpha + f^*\alpha_0$ is a family of Kähler classes. Denote by $\omega_t \in \alpha_t$ the unique Ricci-flat Kähler-Einstein metric, which satisfies the complex Monge-Ampère equation

$$\omega_t^m = c_t t^{m-n} (-1)^{\frac{m^2}{2}} \Omega \wedge \overline{\Omega}.$$

Here Ω is a holomorphic volume form on M, and c_t has a positive limit when $t \to 0$.

The behavior of ω_t when $t \to 0$ has been studied intensively in the literature (see cf. [45, 66, 41, 43, 47, 67, 68, 69, 48], among others). We briefly recall some of the important developments, and refer the readers to the above sources for details. Under the assumption that M is an elliptically fibered K3 surface with only singular fibers of Kodaira type I_1, Gross-Wilson first proved that (M, ω_t) converges to a compact metric space homeomorphic to the sphere S^2 [45]. In the case of general fibered Calabi-Yau manifolds, Tosatti proved that ω_t converges to $f^*\omega$ in the current sense [66], where ω is the Kähler metric on N_0 with

$$\mathrm{Ric}(\omega) = \omega_{WP}$$

obtained in [66, 62, 63], and ω_{WP} is the Weil-Petersson metric of the fibers on N_0.

If M is an Abelian fibered Calabi-Yau m-manifold, then Gross-Tosatti-Zhang improved the convergence of ω_t to C^∞ away from the singular fibers [41]. More precisely ω_t converges smoothly to $f^*\omega$ on $f^{-1}(K)$ for any compact $K \subset N_0$ when $t \to 0$, and additionally the curvature of ω_t is locally uniformly bounded on $f^{-1}(N_0)$. The Gromov-Hausdorff convergence of (M, ω_t) is obtained in [43] for the case of one dimensional base N, which generalizes the Gross-Wilson's result to any elliptically fibered K3 surface. In a recent paper of Tosatti-Zhang [69], the Gromov-Hausdorff convergence of (M, ω_t) is generalized to the case when M is a holomorphic symplectic manifold admitting a holomorphic Lagrangian fibration, and ω_t is a Hyper-Kähler metric.

However, despite this later progress, one important property is still missing for the general cases of Calabi-Yau manifolds that appears in the original work of Gross-Wilson. In their setting they show that ω_t approaches a semi-flat Kähler metric exponentially fast on compact subsets away from the singular fibers. This behavior is expected in general. In fact, motivated by physics, Gaiotto-Moore-Neitzke propose a construction of complete HyperKähler metrics on certain compactifications of complex, completely integrable systems, which asserts the exponential approximations by semi-flat Kähler metrics [35]. In particular, the asymptotic behavior of HyperKähler metrics on the Hitchin moduli spaces are studied in several recent papers [57, 24, 26].

The goal of this appendix is to study the asymptotic rate of ω_t for any Abelian fibered Calabi-Yau manifolds. From now on assume any smooth fiber M_w is an Abelian variety. For an open subset $U \subset N_0$ biholomorphic to a polydisk, $f : M_U \to U$ is a family of Abelian varieties, which is isomorphic to $f : (U \times \mathbb{C}^{m-n})/\Lambda \to U$, where $\Lambda \to U$ is a lattice bundle with fiber $\Lambda_w \cong \mathbb{Z}^{2m-2n}$, so that $M_w \cong \mathbb{C}^{m-n}/\Lambda_w$. We denote the universal covering map $p : U \times \mathbb{C}^{m-n} \to M_U$, which satisfies that $f \circ p(w, z) = w$ for all $(w, z) \in U \times \mathbb{C}^{m-n}$.

For completeness we recall the construction of the semi-flat Kähler metric on M_U (cf. [37, 41]). Note that the ample class α gives an ample polarization of type $(d_1, \ldots, d_{m-n})$ of the fiber M_w, where $d_i \in \mathbb{N}$ and $d_1|d_2|\cdots|d_{m-n}$. Then Λ_w is generated by $d_1e_1, \ldots, d_{n-m}e_{m-n}, Z_1, \ldots, Z_{m-n} \in \mathbb{C}^{m-n}$, where $e_1, \ldots, e_{m-n}$ denotes the standard basis for $\mathbb{C}^{m-n}$, and the matrix $Z = [Z_1, \ldots, Z_{m-n}]$ is the period matrix of M_w, which satisfies the Riemann relationship

$$Z = Z^t, \quad \text{and} \quad \mathrm{Im}Z > 0.$$

If $z_1, \cdots, z_{m-n}$ denote the coordinates on $\mathbb{C}^{m-n}$, then on the fiber M_w, the flat Kähler form

$$i \sum_{k,l} (\mathrm{Im}Z)^{-1}_{kl} dz_k \wedge d\bar{z}_l$$

represents $\alpha|_{M_w}$. Using the notation $W_{kl} = (\mathrm{Im}Z)^{-1}_{kl}$, by Section 3 in [41], if

$$\eta(w, z) = -\frac{1}{2} \sum_{k,l=1}^{m-n} W_{kl}(w)(z_k - \bar{z}_k)(z_l - \bar{z}_l),$$

then $i\partial\bar{\partial}\eta$ is invariant under translation by sections of Λ, and therefore, defines a semi-positive $(1,1)$-form on M_U. The semi-flat metric is defined as

$$\omega_t^{SF} = it\partial\bar{\partial}\eta + f^*\omega, \tag{A.1}$$

for any $t \in (0,1]$, which satisfies that $\omega_t^{SF}|_{M_w}$ is the flat metric in the class $t\alpha|_{M_w}$. Again $\omega \in \alpha_0$ is the Kähler metric on N whose Ricci curvature is the Weil-Petersson metric of fibers on the regular part.

The main result of the appendix is the following:

Theorem A.1. *For any $\nu \in \mathbb{N}$, there is a constant $C_\nu > 0$ such that*

$$\|T^*_{\sigma_0}\omega_t - \omega_t^{SF} - f^*\chi_t\|_{C^0_{\mathrm{loc}}(M_U, \omega_t^{SF})} \leq C_\nu t^{\frac{\nu}{2}},$$

for a certain local section σ_0, where χ_t is a $(1,1)$-form on U such that $\chi_t \to 0$ in the C^∞-sense when $t \to 0$, and T_{σ_0} is the fiberwise translation by σ_0.

Note that $\omega_t^{SF} + f^*\chi_t$ is still a semi-flat metric for $0 < t \ll 1$. Thus this theorem asserts that as $t \to 0$, ω_t approaches a semi-flat metric faster than any polynomial rate. We remark that this decay rate is not as fast as the one demonstrated by Gross-Wilson (Theorem 5.6 in [45]), where

$$T^*_{\sigma_0}\omega_t = \omega_t^{SF} + f^*\chi_t + o(e^{-\frac{C'}{\sqrt{t}}})$$

is obtained. However a sufficiently high polynomial decay rate is enough for the proof of the main theorem of the present paper. We leave the exponential rate for future study.

Proof of Theorem A.1. By Proposition 3.1 in [41], for any Kähler metric $\omega_M \in \alpha$, there is a holomorphic section $\sigma_0 : U \to M_U$ such that

$$\omega + t\omega_M = T^*_{-\sigma_0}\omega_t^{SF} + i\partial\overline{\partial}\xi_t.$$

Thus

$$T^*_{\sigma_0}\omega_t = \omega + tT^*_{\sigma_0}\omega_M + i\partial\overline{\partial}\phi_t \circ T_{\sigma_0} = \omega_t^{SF} + i\partial\overline{\partial}\varphi_t,$$

where $\varphi_t = (\phi_t + \xi_t) \circ T_{\sigma_0}$. If we denote $\lambda_t : U \times \mathbb{C}^{m-n} \to U \times \mathbb{C}^{m-n}$ the dilation given by $\lambda_t(w, z) = (w, t^{-\frac{1}{2}} z)$, then $\lambda_t^* it\partial\bar{\partial}\eta = i\partial\bar{\partial}\eta$, and

$$\lambda_t^* p^* \omega_t^{SF} = i\partial\bar{\partial}\eta + f^*\omega.$$

By Proposition 4.3 in [41],

$$\|\lambda_t^* p^* T_{\sigma_0}^* \omega_t\|_{C^\ell_{\mathrm{loc}}} \leq C_\ell$$

for constants $C_\ell > 0$, and by Lemma 4.7 in [41] (also Proposition 3.2 of [69]),

$$\lambda_t^* p^* T_{\sigma_0}^* \omega_t \to i\partial\bar{\partial}\eta + f^*\omega$$

when $t \to 0$, in the locally C^∞-sense.

If we denote $\psi_t = \varphi_t \circ p \circ \lambda_t$, then ψ_t is $t^{\frac{1}{2}}\Lambda$-periodic, i.e.

$$\psi_t(w, z) = \psi_t(w, z + t^{\frac{1}{2}}(a + bZ))$$

where $a + bZ = (a_1 d_1 e_1 + b_1 Z_1, \cdots, a_{m-n} d_{m-n} e_{m-n} + b_{m-n} Z_{m-n})$ for any $a_j, b_j \in \mathbb{Z}$. By the above we can write

$$\lambda_t^* p^* T_{\sigma_0}^* \omega_t = i\partial\bar{\partial}\eta + \omega + i\partial\overline{\partial}\psi_t,$$

and note that $\|i\partial\overline{\partial}\psi_t\|_{C^\ell_{\mathrm{loc}}} \leq C_\ell$, and $i\partial\overline{\partial}\psi_t \to 0$ as $t \to 0$, on $U \times \mathbb{C}^{m-n}$.

Lemma A.2. *Denote*

$$\psi_{t,w_k\bar{w}_l} = \frac{\partial^2 \psi_t}{\partial w_k \partial \bar{w}_l}, \quad \psi_{t,z_k\bar{z}_l} = \frac{\partial^2 \psi_t}{\partial z_k \partial \bar{z}_l}, \quad \text{and} \quad \psi_{t,z_k\bar{w}_l} = \frac{\partial^2 \psi_t}{\partial z_k \partial \bar{w}_l}.$$

For any $\nu \in \mathbb{N}$ and $\ell \geq 0$, there is a constant $C'_{\ell,\nu} > 0$ such that

$$\|\psi_{t,w_k\bar{w}_l} - \chi_{t,kl}\|_{C^0_{\mathrm{loc}}} \leq C'_{0,\nu} t^{\frac{\nu}{2}},$$

and

$$\|\frac{\partial}{\partial z_j}\psi_{t,w_k\bar{w}_l}\|_{C^\ell_{\mathrm{loc}}} + \|\psi_{t,z_k\bar{z}_l}\|_{C^\ell_{\mathrm{loc}}} + \|\psi_{t,z_k\bar{w}_l}\|_{C^\ell_{\mathrm{loc}}} \leq C'_{\ell,\nu} t^{\frac{\nu}{2}},$$

where $\chi_{t,kl}$ are functions on U.

Proof. For any $t \in (0, 1]$, let h_t be a $\sqrt{t}\Lambda$-periodic real function on $U \times \mathbb{C}^{m-n}$ such that

$$\left|\partial^\beta_{\beta_1,\cdots,\beta_{2(m-n)}} h_t\right| \leq C_\beta,$$

where

$$\partial^\beta_{\beta_1,\cdots,\beta_{2(m-n)}} h_t = \frac{\partial^\beta h_t}{\partial^{\beta_1} y_1 \cdots \partial^{\beta_{2(m-n)}} y_{2(m-n)}},$$

and $z_j = y_j + y_{m-n+j} Z_j$, $\beta = \beta_1 + \cdots + \beta_{2(m-n)}$, and C_β is independent of t. For $w \in U$, let $D_w \subset \{w\} \times \mathbb{C}^{m-n}$ be the fundamental domain of the

$\sqrt{t}\Lambda_w$-action. For any p_1 and $p_2 \in D_w$, if we denote by $\gamma \subset D_w$ the line segment connecting p_1 and p_2, then

$$\begin{aligned}
|\partial^{\beta}_{\beta_1,\cdots,\beta_{2(m-n)}} h_t(p_1) - \partial^{\beta}_{\beta_1,\cdots,\beta_{2(m-n)}} h_t(p_2)| \\
&\leq \Big| \int_\gamma \partial_{\dot\gamma} \partial^{\beta}_{\beta_1,\cdots,\beta_{2(m-n)}} h_t(\gamma(s)) ds \Big| \\
&\leq C\sqrt{t} \sum_{j=1}^{2(m-n)} \sup |\partial_{y_j} \partial^{\beta}_{\beta_1,\cdots,\beta_{2(m-n)}} h_t|.
\end{aligned}$$

Since h_t is periodic we can choose p_2 to be a local maximum of $\partial^{\beta-1}_{\beta_1-1,\cdots,\beta_{2(m-n)}} h_t$, which implies $\partial^{\beta}_{\beta_1,\cdots,\beta_{2(m-n)}} h_t(p_2) = 0$. Thus for any $k \geq 1$, we obtain

$$|h_t - \bar{h}_t| \leq C_{0,\nu} t^{\frac{\nu}{2}}, \quad \text{and} \quad |\partial^{\beta}_{\beta_1,\cdots,\beta_{2(m-n)}} h_t| \leq C_{\beta,\nu} t^{\frac{\nu}{2}},$$

for constants $C_{\beta,\nu}$ independent of t, where $\bar{h}_t = \sup\limits_{z\in D_w} h_t$ is a function on U.

The first inequality in the lemma is obtained by letting $h_t = \psi_{t,w_k\bar{w}_l}$ and $\bar{h}_t = \chi_{t,k,l}$, and the second inequality follows by taking

$$h_t = \frac{\partial^\ell \psi_t}{\partial^{\ell_1} y_1 \cdots \partial^{\ell_{2(m-n)}} y_{2(m-n)}}$$

for any $\ell \geq 1$. □

We obtain the desired conclusion by letting $\chi_t = i \sum\limits_{kl} \chi_{t,kl} dw_k \wedge d\bar{w}_l$. Note that the convergence in Lemma A.2 is slightly stronger than Theorem A.1, and we use Lemma 2.11, a simplified version of Lemma A.2, in the proof of Theorem 3.1. □

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