A THEORY OF STACKS WITH TWISTED FIELDS AND RESOLUTION OF MODULI OF GENUS TWO STABLE MAPS

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Abstract. We construct a smooth algebraic stack of tuples consisting of genus two nodal curves, line bundles, and twisted fields. It leads to a desingularization of the moduli of genus two stable maps to projective spaces. The construction is based on systematical application of the theory of stacks with twisted fields (STF), which has its prototype appeared in [9, 10] and is fully developed in this article. The results of this article are the second step of a series of works toward the resolutions of the moduli of stable maps of higher genera.

Contents

1. Introduction 1
2. A theory of stacks with twisted fields 3
3. Recursive constructions 15
4. Applications of the STF theory to $\mathfrak{P}_2$ and $\overline{M}_2(\mathbb{P}^n, d)$ 19
References 36

1. Introduction

This paper is the second of a series, sequel to [10]. The series aims to resolve the singularities of the moduli $\overline{M}_g(\mathbb{P}^n, d)$ of degree d stable maps from genus g nodal curves into projective spaces $\mathbb{P}^n$, which possess arbitrary singularities when all g and d are considered (see [16]). The problem of resolution of singularities is arguably one of the hardest problems in algebraic geometry ([6, 7, 11, 12]).

For $g = 2$, the only resolution of $\overline{M}_2(\mathbb{P}^n, d)$ so far is provided in [9] via a huge sequence of blowups. In higher genus cases, a direct blowup construction of a possible resolution of $\overline{M}_g(\mathbb{P}^n, d)$ may seem formidable. It thus calls for a more abstract and geometric approach. Our goal is to construct a new moduli with smooth irreducible components and normal crossing boundaries that dominates $\overline{\mathfrak{M}}_g(\mathbb{P}^n, d)$ properly and birationally onto the primary component (the component whose general points have smooth domain curves).

To this end, we consider the smooth Artin stack $\mathfrak{P}_g$ of pairs (C, L) where C are genus g nodal curves and $L \longrightarrow C$ are line bundles (i.e. the relative Picard stack), along with the morphism

$$\overline{\mathfrak{M}}_g(\mathbb{P}^n, d) \longrightarrow \mathfrak{P}_g, \qquad [C, \mathbf{u}] \mapsto [C, \mathbf{u}^*\mathscr{O}_{\mathbb{P}^n}(1)].$$

We hope to introduce a novel smooth Artin stack $\widetilde{\mathfrak{P}}_g^{\mathrm{tf}}$ of tuples (C, L, η) where $(C, L) \in \mathfrak{P}_g$ and η are the extra structure (called twisted fields) added to (C, L), along with a canonical forgetful morphism $\widetilde{\mathfrak{P}}_g^{\mathrm{tf}} \longrightarrow \mathfrak{P}_g$. We then take

$$\widetilde{M}_g^{\mathrm{tf}}(\mathbb{P}^n, d) := \overline{M}_g(\mathbb{P}^n, d) \times_{\mathfrak{P}_g} \widetilde{\mathfrak{P}}_g^{\mathrm{tf}}$$

to be the moduli of degree d stable maps from genus g nodal curves into projective spaces $\mathbb{P}^n$ with twisted fields. We aim to demonstrate that $\widetilde{M}_g^{\mathrm{tf}}(\mathbb{P}^n, d)$ is a smooth Deligne-Mumford stack with the desired desingularization property aforementioned. The $g=1$ case of this program is accomplished in [10]. In this paper, we carry out the program when $g=2$.

To make our approach systematic, we develop the theory of stacks with twisted fields (STF). The STF theory is an abstraction based upon a thorough analysis on the combinatorial and geometric structures of $\mathfrak{P}_g$. To summarize it, let $\mathfrak{M}$ be a smooth stack (e.g. $\mathfrak{P}_g$) and Γ be a finite set of graphs (e.g. the set of dual graphs of nodal curves appearing in $\mathfrak{P}_g$). We assume that $\mathfrak{M}$ admits a Γ-stratification (e.g. the stratification of $\mathfrak{P}_g$ indexed by Γ together with local smooth divisors corresponding to the edges of the graphs); see Definition 2.13 for details. We then introduce a treelike structure Λ on $(\mathfrak{M},\Gamma)$ in Definition 2.16, which assigns a rooted tree to each connected component of the strata of $\mathfrak{M}$ in a suitable way. Such a treelike structure encodes a hidden blowing up process to be performed on the stack $\mathfrak{M}$, although the STF theory does not rely on the actual blowing up process. With all the above devices at hand, we construct a new smooth stack $\mathfrak{M}^{\rm tf}_\Lambda$, properly and birationally dominating $\mathfrak{M}$. The following theorem is the key statement of the STF theory. It is a restatement of Theorem 2.18 (p_1), and the notation is elaborated in Section 2.

Theorem 1.1. *Let Γ be a set of finite graphs, $\mathfrak{M}$ be a smooth algebraic stack with a Γ-stratification $\mathfrak{M}=\bigsqcup_{\gamma\in\Gamma}\mathfrak{M}_\gamma$ as in Definition 2.13, Λ be a treelike structure on $(\mathfrak{M},\Gamma)$ as in Definition 2.16, $\pi_0(\mathfrak{M}_\gamma)$ be the set of the connected components $\mathfrak{N}_\gamma$ of $\mathfrak{M}_\gamma$, and $\overline{\Lambda}_{\mathfrak{N}_\gamma}$ be the set of equivalence classes of rooted level trees (c.f. Definitions 2.10 & 2.12) associated with $\mathfrak{N}_\gamma$ as in (2.13). Then, the following fiber products of open subsets of the projectivization of the direct sums of certain line bundles as in (2.14):*

$$\mathfrak{N}^{\rm tf}_{\gamma,[\mathfrak{t}]}=\Bigg(\prod_{i\in[\![\mathbf{m},0)\!)_\mathfrak{t}}\Big(\Big(\mathring{\mathbb{P}}\big(\bigoplus_{e\in\mathfrak{E}_i^\perp(\mathfrak{t})}L_{\geq_{\mathfrak{N}_\gamma}e}\big)\Big)\Big/\mathfrak{N}_\gamma\Big)\Bigg)\xrightarrow{\varpi}\mathfrak{N}_\gamma,\qquad \gamma\in\Gamma,\ \mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma),\ [\mathfrak{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma},$$

can be glued together in a canonical way to form a smooth algebraic stack $\mathfrak{M}^{\rm tf}$:

$$\mathfrak{M}^{\rm tf}=\mathfrak{M}^{\rm tf}_\Lambda=\bigsqcup_{\gamma\in\Gamma;\ \mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma);\ [\mathfrak{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}}\mathfrak{N}^{\rm tf}_{\gamma,[\mathfrak{t}]}\ .$$

Moreover, the stratawise projections $\varpi\colon\mathfrak{N}^{\rm tf}_{\gamma,[\mathfrak{t}]}\longrightarrow\mathfrak{N}_\gamma$ together give rise to a proper and birational morphism $\varpi\colon\mathfrak{M}^{\rm tf}\longrightarrow\mathfrak{M}$.

The stack $\mathfrak{M}^{\rm tf}_\Lambda$ is called the stack with twisted fields of $\mathfrak{M}$ with respect to Λ, and it enjoys several desirable properties as stated in Theorem 2.18 and Proposition 2.19. In particular, Theorem 2.18 (p_4) is crucial for resolving the moduli $\overline{M}_g(\mathbb{P}^n,d)$.

Theorem 2.18 and Proposition 2.19 further suggest a possible recursive construction:

$$\cdots\longrightarrow(\mathfrak{M}^{\rm tf}_\Lambda)^{\rm tf}_{\Lambda'}\longrightarrow\mathfrak{M}^{\rm tf}_\Lambda\longrightarrow\mathfrak{M}.$$

The key observation is that the new smooth stack $\mathfrak{M}^{\rm tf}_\Lambda$ naturally comes with some choices of the sets Γ' of graphs and the corresponding Γ'-stratifications. Upon introducing a suitable treelike structure Λ' on $(\mathfrak{M}^{\rm tf}_\Lambda,\Gamma')$, one can obtain a newer stack $(\mathfrak{M}^{\rm tf}_\Lambda)^{\rm tf}_{\Lambda'}$, and the construction keeps on. For example, we apply the STF theory to the smooth stack $\mathfrak{P}_2$ eight times to obtain $\widetilde{\mathfrak{P}}^{\rm tf}_2$ in Corollary 4.34, which leads to the main application of the STF theory in this paper:

Theorem 1.2. *There exits a smooth algebraic stack $\widetilde{\mathfrak{P}}^{\rm tf}_2$ of tuples consisting of nodal curves of genus two, line bundles, and twisted fields, along with a proper and birational forgetful morphism $\widetilde{\mathfrak{P}}^{\rm tf}_2\longrightarrow\mathfrak{P}_2$, such that the Deligne-Mumford moduli stack of genus two stable maps with twisted fields given by*

$$\widetilde{M}^{\rm tf}_2(\mathbb{P}^n,d):=\overline{M}_2(\mathbb{P}^n,d)\times_{\mathfrak{P}_2}\widetilde{\mathfrak{P}}^{\rm tf}_2$$

satisfies that

(1) $\widetilde{M}_2^{\mathrm{tf}}(\mathbb{P}^n,d)$ *has smooth irreducible components and admits at worst normal crossing singularities;*
(2) *the morphism* $\widetilde{M}_2^{\mathrm{tf}}(\mathbb{P}^n,d)\longrightarrow \overline{M}_2(\mathbb{P}^n,d)$ *is proper;*
(3) *the induced morphism* $\widetilde{M}_2^{\mathrm{tf}}(\mathbb{P}^n,d)^{\mathrm{pri}}\longrightarrow \overline{M}_2(\mathbb{P}^n,d)^{\mathrm{pri}}$ *between the primary components (whose general points are stable maps with smooth domain curves) is birational; and*
(4) *for any irreducible component N of* $\widetilde{\mathfrak{M}}_2^{\mathrm{tf}}(\mathbb{P}^n,d)$, *with*
$$\tilde{\pi}_N:\tilde{\mathcal{C}}|_N\longrightarrow N\subset \widetilde{M}_2^{\mathrm{tf}}(\mathbb{P}^n,d),\qquad \tilde{\mathfrak{f}}_N:\tilde{\mathcal{C}}|_N\longrightarrow \mathbb{P}^n,$$
denoting the restriction of the pullback of the universal family $\pi:\mathcal{C}\longrightarrow\overline{M}_2(\mathbb{P}^n,d)$, $\mathfrak{f}:\mathcal{C}\longrightarrow\mathbb{P}^n$ *of* $\overline{M}_2(\mathbb{P}^n,d)$ *to N, the direct image sheaf* $(\tilde{\pi}_N)_*(\tilde{\mathfrak{f}}_N)^*\mathscr{O}_{\mathbb{P}^n}(k)$ *is locally free for all* $k\geqslant 1$.

The eight-step construction of the stack $\widetilde{\mathfrak{P}}_2^{\mathrm{tf}}$ is provided in §4.2-§4.9. The proof of the properties (1)-(4) of Theorem 1.2 is provided in §4.10.

We remark that the stack with twisted fields $\widetilde{\mathfrak{P}}_2^{\mathrm{tf}}$ may not be isomorphic to the blowup stack $\widetilde{\mathfrak{P}}_2$ constructed in [9]. In fact, if we change the order of some rounds and phases of the sequential blowups in [9] (interchanging $\mathsf{r}_1\mathsf{p}_3$ and $\mathsf{r}_1\mathsf{p}_4$ and interchanging $\mathsf{r}_3\mathsf{p}_1$ and $\mathsf{r}_3\mathsf{p}_2$ to be precise), the resulting stack should be isomorphic to $\widetilde{\mathfrak{P}}_2^{\mathrm{tf}}$. Indeed, several blowups in [9] can be performed in various orders, and in general, they should lead to different resolutions of $\overline{M}_2(\mathbb{P}^n,d)$.

We also remark that if we take $N=\widetilde{M}_2^{\mathrm{tf}}(\mathbb{P}^n,d)^{\mathrm{pri}}$ in Part (4) of Theorem 1.2, then
$$\Big\langle\, e\big((\tilde{\pi}_{\widetilde{M}_2^{\mathrm{tf}}(\mathbb{P}^n,d)^{\mathrm{pri}}})_*(\tilde{\mathfrak{f}}_{\widetilde{M}_2^{\mathrm{tf}}(\mathbb{P}^n,d)^{\mathrm{pri}}})^*\mathscr{O}_{\mathbb{P}^n}(k)\big)\,,\,[\widetilde{M}_2^{\mathrm{tf}}(\mathbb{P}^n,d)^{\mathrm{pri}}]\,\Big\rangle$$
should equal the expected reduced genus 2 Gromov-Witten (GW) invariants of the corresponding complete intersection, parallel to [17, (1.4)] and [15, (1.7)]. The reduced genus 1 GW-invariants, as well as its comparison with the standard genus 1 GW-invariants, are introduced in [20, 19] and further studied in [15, 17, 4, 5, 13, 14], and lead to important results such as A. Zinger's proof [21] of the prediction of [3] for genus 1 GW-invariants of a quintic 3-fold.

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2. A theory of stacks with twisted fields

2.1. **Graphs and levels.** Throughout the article, we use the following definition of the graphs adapted from [18, §5.1].

Definition 2.1. *A finite graph γ, or simply a graph when the context is clear, is a finite set* $\mathrm{HE}(\gamma)$ *of half-edges along with*

- *a set* $\mathrm{Ver}(\gamma)$ *of disjoint subsets* $v\subset\mathrm{HE}(\gamma)$ *known as the vertices such that*
$$\bigsqcup_{v\in\mathrm{Ver}(\gamma)} v=\mathrm{HE}(\gamma)$$
and
- *a set* $\mathrm{Edg}(\gamma)$ *of disjoint subsets* $e\subset\mathrm{HE}(\gamma)$ *known as the edges such that*
$$|e|=2\quad\forall\ e\in\mathrm{Edg}(\gamma);\qquad \bigsqcup_{e\in\mathrm{Edg}(\gamma)} e=\mathrm{HE}(\gamma).$$

Definition 2.1 naturally allows graphs that have self loops and multiple edges with the same endpoints, which is convenient for our purpose. For every half-edge $\hbar\in\mathrm{HE}(\gamma)$, Definition 2.1 implies that there exist a unique vertex $v(\hbar)$ and a unique edge $e(\hbar)$ containing $\hbar$, respectively.

Definition 2.2. *Let γ, γ' be graphs. We say γ' is a* subgraph *of γ if there exist injections*

$$f_{\gamma,\gamma'}: \mathrm{HE}(\gamma') \hookrightarrow \mathrm{HE}(\gamma), \qquad f^{\mathrm{e}}_{\gamma,\gamma'}: \mathrm{Edg}(\gamma') \hookrightarrow \mathrm{Edg}(\gamma), \qquad f^{\mathrm{v}}_{\gamma,\gamma'}: \mathrm{Ver}(\gamma') \hookrightarrow \mathrm{Ver}(\gamma)$$

satisfying

$$\begin{aligned} f^{\mathrm{e}}_{\gamma,\gamma'}(e) &= \{f_{\gamma,\gamma'}(\hbar^+),\, f_{\gamma,\gamma'}(\hbar^-)\} && \forall\, e = \{\hbar^+, \hbar^-\} \in \mathrm{Edg}(\gamma'); \\ f^{\mathrm{v}}_{\gamma,\gamma'}(v) &\supset \{f_{\gamma,\gamma'}(\hbar) : \forall\, \hbar \in v\} && \forall\, v \in \mathrm{Ver}(\gamma'). \end{aligned}$$

Definition 2.3. *Given a graph γ and two vertices v and w of γ, a* path *from v to w is a sequence of pairwise distinct half-edges of γ:*

$$\hbar_1^+,\, \hbar_1^-,\, \hbar_2^+,\, \hbar_2^-,\, \cdots,\, \hbar_m^+,\, \hbar_m^-$$

such that $v(\hbar_1^+) = v$, $v(\hbar_m^-) = w$, $\{\hbar_i^+, \hbar_i^-\} \in \mathrm{Edg}(\gamma)$ for all $1 \leqslant i \leqslant m$, and $\{\hbar_i^-, \hbar_{i+1}^+\} \in \mathrm{Ver}(\gamma)$ for all $1 \leqslant i \leqslant m-1$.

A path defined in this way cannot contain repetitive edges.

Definition 2.4. *A graph γ is said to be* connected *if it is non-empty and for any two vertices v and w, there exists a path form v to w. The set of all connected graphs is denoted by $\mathbf{G}$.*

Given $\gamma \in \mathbf{G}$ and $E \subset \mathrm{Edg}(\gamma)$, we write the set of all half-edges of E as $\mathrm{HE}(E) = \bigsqcup_{e \in E} e$. If $E \neq \varnothing$, there exists a unique partition $P(E)$ of E, with each $E' \in P(E)$ corresponding to a connected subgraph $\gamma(E')$ of γ, satisfying $\mathrm{Edg}(\gamma(E')) = E'$ and $\mathrm{Ver}(\gamma(E')) \cap \mathrm{Ver}(\gamma(E'')) = \varnothing$ for all distinct $E', E'' \in P(E)$. If $E = \varnothing$, we set $P(E) = \varnothing$.

Definition 2.5. *With notation as above, let $\gamma_{(E)}$ be the graph obtained from γ by* contracting the edges in *E:*

$$\mathrm{HE}\big(\gamma_{(E)}\big) = \mathrm{HE}(\gamma) \backslash \mathrm{HE}(E), \qquad \mathrm{Edg}\big(\gamma_{(E)}\big) = \mathrm{Edg}(\gamma) \backslash E,$$
$$\mathrm{Ver}\big(\gamma_{(E)}\big) = \{v \in \mathrm{Ver}(\gamma) : v \cap \mathrm{HE}(E) = \varnothing\} \sqcup \Big\{ \bigcup_{\hbar \in \mathrm{HE}(E')} v(\hbar) \backslash \mathrm{HE}(E) : E' \in P(E) \Big\}.$$

Such an operation is called an edge contraction. *For $e \in \mathrm{Edg}(\gamma)$, we simply write $\gamma_{(e)} = \gamma_{(\{e\})}$.*

By definition, $\gamma_{(\varnothing)} = \gamma$ for any $\gamma \in \mathbf{G}$. If γ is connected, then every graph obtained from γ via edge contraction is still connected.

Given $\gamma, \gamma' \in \mathbf{G}$, we define

$$\gamma' \prec \gamma \qquad \Longleftrightarrow \qquad \exists\ \ E \subset \mathrm{Edg}(\gamma') \ \text{ s.t. } \ E \neq \varnothing,\ \gamma = \gamma'_{(E)}. \tag{2.1}$$

This gives rise to a partial order $\prec$ on $\mathbf{G}$.

Aside from the edge contractions, we introduce two more graph operations that will be used in §4 to describe the treelike structures of each step.

Definition 2.6. *Given $\gamma \in \mathbf{G}$ and $V \subset \mathrm{Ver}(\gamma)$, let γ^{ds}_V be the graph obtained from γ by* dissolving the vertices in *V:*

$$\mathrm{HE}(\gamma^{\mathrm{ds}}_V) = \mathrm{HE}(\gamma), \quad \mathrm{Edg}(\gamma^{\mathrm{ds}}_V) = \mathrm{Edg}(\gamma), \quad \mathrm{Ver}(\gamma^{\mathrm{ds}}_V) = (\mathrm{Ver}(\gamma) \backslash V) \sqcup \bigsqcup_{\hbar \in v,\, v \in V} \{\hbar\}.$$

Such an operation is called a vertex dissolution.

Definition 2.7. *Given $\gamma \in \mathbf{G}$ and $V \subset \mathrm{Ver}(\gamma)$, let γ^{id}_V be the graph obtained from γ by* identifying the vertices in *V:*

$$\mathrm{HE}(\gamma^{\mathrm{id}}_V) = \mathrm{HE}(\gamma), \quad \mathrm{Edg}(\gamma^{\mathrm{id}}_V) = \mathrm{Edg}(\gamma), \quad \mathrm{Ver}(\gamma^{\mathrm{id}}_V) = (\mathrm{Ver}(\gamma) \backslash V) \sqcup \{\hbar \in v :\ v \in V\}.$$

Such an operation is called a vertex identification.

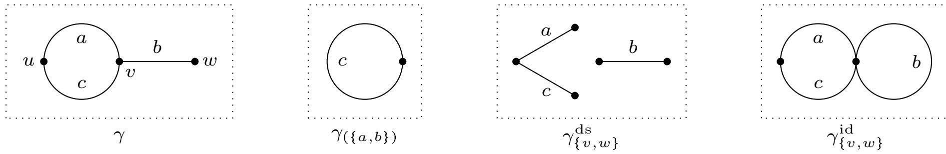


FIGURE 1. Edge contraction, vertex dissolution, and vertex identification

Intuitively, dissolving a vertex $v\in\mathrm{Ver}(\gamma)$ means removing v from $\mathrm{Ver}(\gamma)$ and assigning to each edge e with $e\cap v\neq\varnothing$ a distinct new vertex, whereas identifying the vertices in a subset V of $\mathrm{Ver}(\gamma)$ means "gluing" the vertices in V into a single vertex. If γ is connected, a graph obtained from γ via vertex dissolution may become disconnected, but that via vertex identification is connected. Figure 1 provides illustrations for Definitions 2.5-2.7.

Definition 2.8. *For a connected graph γ, its* first Betti number *$b_1(\gamma)$ is given by*

$$b_1(\gamma) = |\mathrm{Edg}(\gamma)| - |\mathrm{Ver}(\gamma)| + 1.$$

The graphs with $b_1=0$ are of our particular interest. They play crucial roles in the STF theory.

Definition 2.9. *A* rooted tree

$$\tau = (\gamma, o)$$

consists of a connected graph γ with $b_1(\gamma)=0$ as well as a chosen vertex $o\in\mathrm{Ver}(\gamma)$ known as the root. *We write*

$$\mathrm{HE}(\tau):=\mathrm{HE}(\gamma), \qquad \mathrm{Edg}(\tau):=\mathrm{Edg}(\gamma), \qquad \mathrm{Ver}(\tau):=\mathrm{Ver}(\gamma).$$

When the root is clear, we simply write $\tau=\gamma$. The single vertex edge-less rooted tree is denoted by $\tau_\bullet$.

Every rooted tree τ determines a unique partial order $\prec$ on $\mathrm{HE}(\tau)$, known as the tree order, so that $\hbar\prec\hbar'$ if and only if $\hbar\neq\hbar'$ and the unique path from o to $v(\hbar)$ contains $\hbar'$. Thus, every $e\in\mathrm{Edg}(\tau)$ can be written as

$$e = \{\hbar_e^+, \hbar_e^-\} \qquad \text{with} \qquad \hbar_e^- \prec \hbar_e^+.$$

The tree order on $\mathrm{HE}(\tau)$ induces partial orders $\prec$ on $\mathrm{Edg}(\tau)$ and $\mathrm{Ver}(\tau)$ by requiring

$$e\prec e' \iff \big(\hbar\prec\hbar' \ \forall\,\hbar\in e,\ \hbar'\in e'\big), \qquad v\prec v' \iff \big(\exists\,\hbar\in v,\ \hbar'\in v' \text{ s.t. } \hbar\prec\hbar'\big),$$

respectively. We still call the induced orders on $\mathrm{Edg}(\tau)$ and $\mathrm{Ver}(\tau)$ the tree orders. The subsets of the maximal edges, minimal edges, maximal vertices, and minimal vertices with respect to the tree orders are denoted by

$$\mathrm{Edg}(\tau)_{\max}, \quad \mathrm{Edg}(\tau)_{\min}, \quad \mathrm{Ver}(\tau)_{\max}, \quad \mathrm{Ver}(\tau)_{\min}\,,$$

respectively. The minimal vertices are known as the leaves in the graph theory.

Definition 2.10. *A* rooted level tree

$$\mathfrak{t} = (\,\tau_{\mathfrak{t}}\,,\ \ell_{\mathfrak{t}}\,)$$

is a tuple consisting of a rooted tree $\tau_{\mathfrak{t}}=(\gamma_{\mathfrak{t}},o_{\mathfrak{t}})$ and a map

$$\ell = \ell_{\mathfrak{t}}:\ \mathrm{HE}(\tau_{\mathfrak{t}}) \longrightarrow \mathbb{R}_{\leqslant 0}, \tag{2.2}$$

called a level map, *that satisfies*

$$\ell^{-1}(0) = o_{\mathfrak{t}}\,; \qquad \ell(\hbar) = \ell(\hbar') \quad \text{whenever } v(\hbar)=v(\hbar'); \qquad \ell(\hbar_e^-) < \ell(\hbar_e^+) \quad \forall\ e\in\mathrm{Edg}(\tau_{\mathfrak{t}}).$$

We write

$$\mathrm{HE}(\mathfrak{t}) := \mathrm{HE}(\tau_\mathfrak{t}), \qquad \mathrm{Edg}(\mathfrak{t}) := \mathrm{Edg}(\tau_\mathfrak{t}), \qquad \mathrm{Ver}(\mathfrak{t}) := \mathrm{Ver}(\tau_\mathfrak{t}).$$

The level map ℓ determines a map on $\mathrm{Ver}(\tau_\mathfrak{t})$, which is still denoted by ℓ, that is given by

$$\ell\colon \mathrm{Ver}(\tau_\mathfrak{t}) \longrightarrow \mathbb{R}_{\leqslant 0}, \qquad \ell(v) = \ell(\hbar) \quad \forall\ \hbar \in v,$$

which is also called a level map *when the context is clear. The image of ℓ is denoted by $\mathrm{Im}(\ell)$, whose elements are called* levels *of $\mathfrak{t}$. The set of all rooted level trees is denoted by $\mathbf{T}$.*

Remark 2.11. In Definition 2.10, the level maps on $\mathrm{HE}(\tau_\mathfrak{t})$ and $\mathrm{Ver}(\tau_\mathfrak{t})$ determine each other, and a rooted level tree defined in this way is consistent with that introduced in [10, §2.1]. We remark that a rooted level tree is a *level graph* with the root as the unique *top level vertex* in [1, 2]. The relation between the STF theory and the theory of twisted/multi-scale differentials ([1, 2]) appears to be beyond the combinatorial resemblance, which will be revealed in the succeeding works on the resolution of $\overline{M}_g(\mathbb{P}^n, d)$.

For every level $i \in \mathrm{Im}(\ell)$, we denote by $i^\sharp$ and $i^\flat$ the levels immediately above and below i (if such levels exist), respectively:

$$i^\sharp = \min\big\{ j \in \mathrm{Im}(\ell) : j > i \big\}, \qquad i^\flat = \max\big\{ h \in \mathrm{Im}(\ell) : h < i \big\}. \tag{2.3}$$

Among the levels of $\mathfrak{t}$, a particularly important one is

$$\mathbf{m} = \mathbf{m}(\mathfrak{t}) := \max\{\, \ell(v) : v \in \mathrm{Ver}(\gamma)_{\min} \,\} \quad \in \mathrm{Im}(\ell).$$

For $i \in \mathrm{Im}(\ell)$, we write

$$\begin{aligned}
&\mathfrak{E}_i = \mathfrak{E}_i(\mathfrak{t}) = \big\{ e \in \mathrm{Edg}(\tau_\mathfrak{t}) \colon \ell(\hbar_e^+) > i,\ \ell(\hbar_e^-) \leqslant i \big\}, && \mathfrak{E}_{\geqslant i} = \mathfrak{E}_{\geqslant i}(\mathfrak{t}) = \bigcup_{j \geqslant i} \mathfrak{E}_j,\\
&\mathfrak{E}_i^\perp = \mathfrak{E}_i^\perp(\mathfrak{t}) = \big\{ e \in \mathfrak{E}_i \colon \ell(\hbar_e^-) = i \big\}, && \mathfrak{E}_{\geqslant i}^\perp = \mathfrak{E}_{\geqslant i}^\perp(\mathfrak{t}) = \bigcup_{j \geqslant i} \mathfrak{E}_j^\perp.\\
&\big(\mathfrak{E}_{\mathbf{m};\min}^\perp\big)^{\succeq} = \bigcup_{e \in \mathfrak{E}_\mathbf{m}^\perp \cap \mathrm{Edg}(\tau_\mathfrak{t})_{\min}} \big\{ e' \in \mathrm{Edg}(\tau_\mathfrak{t}) : e' \succeq e \big\}.
\end{aligned} \tag{2.4}$$

The notation "$\perp$" in $\mathfrak{E}_i^\perp$ intuitively suggests the lower half-edges of the edges in $\mathfrak{E}_i^\perp$ "stop" at the level i, with "|" representing the edges and "$-$" representing the level i. The set $(\mathfrak{E}_{\mathbf{m};\min}^\perp)^{\succeq}$ consists of the edges of the paths from the root o to the minimal vertices on level $\mathbf{m}$; see Figure 2 for illustration.

With $\mathfrak{t}$ as above, we write

$$[\![i, j]\!]_\mathfrak{t} = \mathrm{Im}(\mathfrak{t}) \cap [i, j], \quad [\![i, j)\!)_\mathfrak{t} = \mathrm{Im}(\mathfrak{t}) \cap [i, j) \qquad \forall\ i, j \in \mathrm{Im}(\ell).$$

For $k \in \mathbb{Z}_{>0}$, let

$$\mathbf{n}_k = \mathbf{n}_k(\mathfrak{t}) = \min\big(\{\, i \in [\![\mathbf{m}, 0)\!)_\mathfrak{t} : |\mathfrak{E}_i \cap (\mathfrak{E}_{\mathbf{m};\min}^\perp)^{\succeq}| \leqslant k \,\} \sqcup \{0\} \big) \ \in [\![\mathbf{m}, 0]\!]_\mathfrak{t}. \tag{2.5}$$

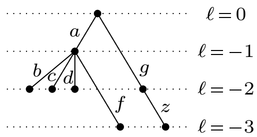
Intuitively, $\mathbf{n}_k$ is the lowest level on which there are at most k vertices that are contained in the paths from the root o to the minimal vertices on level $\mathbf{m}$; see Figure 2 for illustration.

Each subset

$$\begin{aligned}
\mathbb{I}' &= I'_+ \sqcup E'_\mathbf{m} \sqcup E'_-\\
\subset \mathbb{I}(\mathfrak{t}) &= I_+(\mathfrak{t}) \sqcup E_\mathbf{m}(\mathfrak{t}) \sqcup E_-(\mathfrak{t})\\
&:= [\![\mathbf{m}, 0)\!)_\mathfrak{t} \sqcup \big(\mathfrak{E}_\mathbf{m} \backslash \mathfrak{E}_\mathbf{m}^\perp \big) \sqcup \big(\mathrm{Edg}(\mathfrak{t}) \backslash \mathfrak{E}_{\geqslant \mathbf{m}} \big)
\end{aligned}$$

determines a rooted level tree

$$\mathfrak{t}_{(\mathbb{I}')} = (\, \tau_{(\mathbb{I}')} \,,\, \ell_{(\mathbb{I}')} \,) = (\, \gamma_{(\mathbb{I}')} \,,\, o_{(\mathbb{I}')} \,,\, \ell_{(\mathbb{I}')} \,) \tag{2.6}$$



$\mathfrak{t} = (\,\tau\,,\ \ell\colon \mathrm{Ver}(\tau) \longrightarrow \mathbb{R}_{\leqslant 0}\,)$

$$\begin{aligned}
&\mathfrak{E}_{-1} = \{a,g\}, && \mathfrak{E}_{-2} = \{b,c,d,f,g\},\\
&\mathfrak{E}^{\perp}_{-1} = \{a\}, && \mathfrak{E}^{\perp}_{-2} = \{b,c,d,g\},\\
&\mathbf{m} = -2, && (\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\geq} = \{a,b,c,d\},\\
&\mathbf{n}_1 = \mathbf{n}_2 = -1, && \mathbf{n}_3 = -2.
\end{aligned}$$

FIGURE 2. A rooted level tree

as follows:

- the rooted tree $\tau_{(\mathbb{I}')}$ is obtained from $\tau_{\mathfrak{t}}$ by contracting the edges in

$$\Big\{\, e\in\mathfrak{E}^{\perp}_{\geqslant\mathbf{m}}\sqcup E'_{\mathbf{m}}\,:\ [\![\,\max\{\ell(\hbar_e^-),\mathbf{m}\},\ \ell(\hbar_e^+)]\!)_{\mathfrak{t}}\subset I'_+\ \Big\}\sqcup E'_-;$$

- the level map $\ell_{(\mathbb{I}')}$ is such that for any $\hbar\in\mathrm{HE}\big(\gamma_{(\mathbb{I}')}\big)$,

$$\ell_{(\mathbb{I}')}(\hbar)=\begin{cases}\min\big\{i\in[\![\mathbf{m},0]\!]_{\mathfrak{t}}\backslash I'_+\,:\,i\geqslant\ell(\hbar)\big\} & \text{if } \hbar=\hbar_e^-,\ e\in E_{\mathbf{m}},\\ \min\big\{i\in\mathrm{Im}(\ell)\backslash I'_+\,:\,i\geqslant\ell(\hbar)\big\} & \text{otherwise.}\end{cases}$$

The above construction of $\mathfrak{t}_{(\mathbb{I}')}$ implies that

$$\begin{aligned}
\mathbf{m}(\mathfrak{t}_{(\mathbb{I}')}) &= \min\big([\![\mathbf{m},0]\!]_{\mathfrak{t}}\backslash I'_+\big), & E_{\mathbf{m}}\big(\mathfrak{t}_{(\mathbb{I}')}\big) &= \big\{e\in E_{\mathbf{m}}\backslash E'_{\mathbf{m}}\colon \ell(\hbar_e^+)>\mathbf{m}(\mathfrak{t}_{(\mathbb{I}')})\big\},\\
I_+\big(\mathfrak{t}_{(\mathbb{I}')}\big) &= I_+\backslash I'_+, & E_-\big(\mathfrak{t}_{(\mathbb{I}')}\big) &= \big(E_-\backslash E'_-\big)\sqcup\big\{e\in E_{\mathbf{m}}\backslash E'_{\mathbf{m}}\colon \ell(\hbar_e^+)\leqslant\mathbf{m}(\mathfrak{t}_{(\mathbb{I}')})\big\}.
\end{aligned}$$

In particular, we have

$$E_{\mathbf{m}}\big(\mathfrak{t}_{(\mathbb{I}')}\big)\sqcup E_-\big(\mathfrak{t}_{(\mathbb{I}')}\big) = \big(E_{\mathbf{m}}\backslash E'_{\mathbf{m}}\big)\sqcup\big(E_-\backslash E'_-\big).$$

Intuitively, $\mathfrak{t}_{(\mathbb{I}')}$ is obtained from $\mathfrak{t}$ by contracting E'_-, then lifting the lower half-edges of the elements of $E'_{\mathbf{m}}$ to the level $\mathbf{m}$, and finally contracting the levels in I'_+.

Definition 2.12. *Two rooted level trees $\mathfrak{t}=(\tau,\ell)$ and $\mathfrak{t}'=(\tau',\ell')$ are said to be* equivalent, *denoted by $\mathfrak{t}\sim\mathfrak{t}'$, if the following conditions are satisfied:*

(E1) $\tau=\tau'$;
(E2) $\ell^{-1}[\![\mathbf{m},0]\!]_{\mathfrak{t}}=(\ell')^{-1}[\![\mathbf{m}',0]\!]_{\mathfrak{t}'}$;
(E3) there exists a (unique) order preserving bijection $\alpha\colon[\![\mathbf{m},0]\!]_{\mathfrak{t}}\longrightarrow[\![\mathbf{m}',0]\!]_{\mathfrak{t}'}$ *such that* $\alpha\circ\ell=\ell'$ *on* $\ell^{-1}[\![\mathbf{m},0]\!]_{\mathfrak{t}}$.

This gives rise to an equivalence relation on the set $\mathbf{T}$ of the rooted level trees. We denote by $[\mathfrak{t}]$ the equivalence class containing $\mathfrak{t}\in\mathbf{T}$ and by

$$\overline{\mathbf{T}}:=\big\{\,[\mathfrak{t}]\,:\ \mathfrak{t}\in\mathbf{T}\,\big\}$$

the set of such equivalence classes.

Given $[\mathfrak{t}],[\mathfrak{t}']\in\overline{\mathbf{T}}$, we set

$$[\mathfrak{t}']\prec[\mathfrak{t}] \qquad\Longleftrightarrow\qquad \exists\ \ \mathbb{J}\subset\mathbb{I}(\mathfrak{t}')\ \ \text{s.t.}\ \ \mathbb{I}'\neq\varnothing,\ [\mathfrak{t}'_{(\mathbb{J})}]=[\mathfrak{t}]. \tag{2.7}$$

This gives rise to a partial order on $\overline{\mathbf{T}}$. Notice that $[\mathfrak{t}']\prec[\mathfrak{t}]$ implies that $\gamma_{[\mathfrak{t}']}\preceq\gamma_{[\mathfrak{t}]}$.

2.2. **Γ-stratification and Treelike structures.** In this subsection, we describe the stacks to which the theory of stacks with twisted fields (STF) can be applied.

Recall that $\mathbf{G}$ denotes the set of connected graphs, which is endowed with a partial order (2.1) given by the edge contractions. We say two graphs γ and γ' are isomorphic and write

$$\gamma \simeq \gamma'$$

if there exists a bijection $\phi\colon \mathrm{HE}(\gamma) \longrightarrow \mathrm{HE}(\gamma')$ satisfying

$$\mathrm{Ver}(\gamma') = \big\{\{\phi(\hbar) : \hbar \in v\} : v \in \mathrm{Ver}(\gamma)\big\}, \qquad \mathrm{Edg}(\gamma') = \big\{\{\phi(\hbar) : \hbar \in e\} : e \in \mathrm{Edg}(\gamma)\big\}.$$

The graph isomorphism gives an equivalence relation on $\mathbf{G}$ that is compatible with the partial order (2.1) on $\mathbf{G}$. Also recall that $\tau_\bullet$ denotes the (connected) edge-less rooted tree. For any topological space X, let $\pi_0(X)$ be the set of all connected components of X. We do not consider $\varnothing$ as a connected space, thus we take $\pi_0(\varnothing) = \varnothing$.

Definition 2.13. *Let $\Gamma \subset \mathbf{G}$ be a nonempty subset and $\mathfrak{M}$ be a smooth algebraic stack. A stratification*

$$\mathfrak{M} = \bigsqcup_{\gamma \in \Gamma} \mathfrak{M}_\gamma$$

by substacks is called a Γ-stratification of $\mathfrak{M}$ if

- *for every $\gamma \in \Gamma$, there exists a set $\underline{\mathfrak{V}}_\gamma$ of affine smooth charts of $\mathfrak{M}$ satisfying $\mathfrak{M}_\gamma \subset \bigcup_{\mathcal{V} \in \underline{\mathfrak{V}}_\gamma} \mathcal{V}$ and*
- *for every $\gamma \in \Gamma$ and every $\mathcal{V} \in \underline{\mathfrak{V}}_\gamma$, there exists a subset $\{\zeta_e^{\mathcal{V}}\}_{e \in \mathrm{Edg}(\gamma)}$ of local parameters on $\mathcal{V}$, known as the modular parameters,*

such that for every $\gamma, \gamma' \in \Gamma \backslash \{\tau_\bullet\}$ (i.e. $\mathrm{Edg}(\gamma), \mathrm{Edg}(\gamma') \neq \varnothing$) and every $\mathcal{V}' \in \underline{\mathfrak{V}}_{\gamma'}$,

$$\begin{aligned} &\mathfrak{M}_\gamma \cap \mathcal{V}' = \varnothing \qquad \text{if } \gamma \not\succeq \gamma', \\ &\pi_0(\mathfrak{M}_\gamma \cap \mathcal{V}') \subset \big\{\{\zeta_e^{\mathcal{V}'} = 0 \ \forall\, e \in \mathrm{Edg}(\gamma') \backslash E;\ \zeta_e^{\mathcal{V}'} \neq 0 \ \forall\, e \in E\} : E \subset \mathrm{Edg}(\gamma'),\ \gamma'_{(E)} \simeq \gamma\,\big\} \quad \text{if } \gamma \succeq \gamma'. \end{aligned} \tag{2.8}$$

In Definition 2.13, some strata $\mathfrak{M}_\gamma$ may be disconnected or empty. If $\mathfrak{M}_\gamma \neq \varnothing$ and $\gamma \neq \tau_\bullet$, then (2.8) implies that for every $x \in \mathfrak{M}_\gamma$, there exists $\mathcal{V} \in \underline{\mathfrak{V}}_\gamma$ containing x such that

$$\mathfrak{M}_\gamma \cap \mathcal{V} = \{\, \zeta_e^{\mathcal{V}} = 0 \ \ \forall\, e \in \mathrm{Edg}(\gamma)\,\}.$$

Remark 2.14. When we consider a Γ-stratification, it is often handy to allow isomorphic graphs $\gamma \simeq \gamma'$ to index the same stratum: $\mathfrak{M}_\gamma = \mathfrak{M}_{\gamma'}$. In such a situation, rigorously we should take Γ as a set of equivalence classes $[\gamma]$ of graphs $\gamma \in \mathbf{G}$ with respect to graph isomorphism. However, writing the strata of $\mathfrak{M}$ as $\mathfrak{M}_{[\gamma]}$ would make the notation in §3 and §4 too complicated, so by abuse of notation, we will still write $\mathfrak{M} = \bigsqcup_{\gamma \in \Gamma} \mathfrak{M}_\gamma$ even if the graphs in Γ are considered up to graph isomorphism. Similarly, when the context is clear, we will still write $\gamma' \prec \gamma$ even if the graphs $\gamma, \gamma' \in \mathbf{G}$ are considered up to graph isomorphism.

For any subset $\mathfrak{N} \subset \mathfrak{M}$, we denote by

$$\mathrm{Cl}_{\mathfrak{M}}(\mathfrak{N}) \ (\subset \mathfrak{M}\,)$$

the closure of $\mathfrak{N}$ in $\mathfrak{M}$. The lemma below follows directly from (2.8).

Lemma 2.15. *Let Γ and $\mathfrak{M}$ be as in Definition 2.13. For every $\gamma, \gamma' \in \Gamma$ and every $\mathfrak{N}_{\gamma'} \in \pi_0(\mathfrak{M}_{\gamma'})$ such that $\mathrm{Cl}_{\mathfrak{M}}(\mathfrak{M}_\gamma) \cap \mathfrak{N}_{\gamma'} \neq \varnothing$, we have $\gamma' \preceq \gamma$ (up to graph isomorphism).*

If $\mathfrak{M}$ is endowed with a Γ-stratification, we define the boundary of $\mathfrak{M}$ to be

$$\Delta := \bigsqcup_{\gamma \in \Gamma,\, |\mathrm{Edg}(\gamma)| > 0} \mathfrak{M}_\gamma. \tag{2.9}$$

By (2.8), Δ is of lower dimension. Thus, Γ must contain the connected edge-less graph $\tau_\bullet$, and

$$\mathfrak{M} = \mathfrak{M}_{\tau_\bullet} \sqcup \Delta.$$

For every $\mathfrak{M}_\gamma \subset \Delta$, every $\mathfrak{N}_\gamma \in \pi_0(\mathfrak{M}_\gamma)$, and every $e \in \mathrm{E}_{\mathfrak{N}_\gamma}$, we denote by

$$L_e \longrightarrow \mathfrak{N}_\gamma \tag{2.10}$$

the line bundle such that on each chart $\mathcal{V} \in \underline{\mathfrak{V}}_\gamma$ with $\mathfrak{N}_\gamma \cap \mathcal{V} \neq \varnothing$, the line bundle $L_e/(\mathfrak{N}_\gamma \cap \mathcal{V})$ is the restriction of normal bundle of the local divisor $\{\zeta_e^{\mathcal{V}}=0\}$ to $\mathfrak{N}_\gamma \cap \mathcal{V}$. As shown in [10, Lemma 3.1], the restriction of

$$\tfrac{\partial}{\partial \zeta_e} := (\mathrm{d}\zeta_e)^\vee \ \in \Gamma(\mathcal{V}; T\mathfrak{M})$$

to $\mathfrak{N}_\gamma \cap \mathcal{V}$ gives a nowhere vanishing section of $L_e/(\mathfrak{N}_\gamma \cap \mathcal{V})$. In §2.3, the line bundles (2.10) will be "twisted" in (2.14) to define the twisted fields in Theorem 2.18 (p_1).

Definition 2.16. *Let Γ and $\mathfrak{M}$ be as in Definition 2.13. We call the indexed family*

$$\Lambda = \Lambda_{(\mathfrak{M},\Gamma)} := \big(\mathcal{T}_{\mathfrak{N}_\gamma} := (\,\tau_{\mathfrak{N}_\gamma},\, \mathrm{E}_{\mathfrak{N}_\gamma},\, \beta_{\mathfrak{N}_\gamma}\,)\big)_{\gamma\in\Gamma,\, \mathfrak{N}_\gamma \in \pi_0(\mathfrak{M}_\gamma)} \tag{2.11}$$

a treelike structure *on $(\mathfrak{M},\Gamma)$ if it assigns to each pair $\big(\gamma \in \Gamma\,;\, \mathfrak{N}_\gamma \in \pi_0(\mathfrak{M}_\gamma)\big)$ a unique tuple $\mathcal{T}_{\mathfrak{N}_\gamma}$ consisting of*

- *a rooted tree $\tau_{\mathfrak{N}_\gamma}$ with the root $o_{\mathfrak{N}_\gamma}$,*
- *a subset $\mathrm{E}_{\mathfrak{N}_\gamma} \subset \mathrm{Edg}(\gamma)$, and*
- *a bijection $\beta_{\mathfrak{N}_\gamma}\colon \mathrm{Edg}(\tau_{\mathfrak{N}_\gamma}) \longrightarrow \mathrm{E}_{\mathfrak{N}_\gamma}$ (which induces a bijection $\beta_{\mathfrak{N}_\gamma} : \mathrm{HE}(\tau_{\mathfrak{N}_\gamma}) \longrightarrow \mathrm{HE}(\mathrm{E}_{\mathfrak{N}_\gamma})$)*

such that for every $\gamma' \in \Gamma$ and every $\mathfrak{N}_{\gamma'} \in \pi_0(\mathfrak{M}_{\gamma'})$, all of the following conditions hold.

(a) *For every $\gamma \in \Gamma$ and every $\mathfrak{N}_\gamma \in \pi_0(\mathfrak{M}_\gamma)$, if $\mathrm{Cl}_{\mathfrak{M}}(\mathfrak{N}_\gamma) \cap \mathfrak{N}_{\gamma'} \neq \varnothing$, then $\tau_{\mathfrak{N}_\gamma} \geq \tau_{\mathfrak{N}_{\gamma'}}$ up to graph isomorphism.*

(b) *If $\mathrm{E}_{\mathfrak{N}_{\gamma'}} \neq \varnothing$ (i.e. $\tau_{\mathfrak{N}_{\gamma'}} \neq \tau_\bullet$), then for every $e \in \mathrm{Edg}(\gamma')$,*

(b_1) *if $e \notin \beta_{\mathfrak{N}_{\gamma'}}\big(\mathrm{Edg}(\tau_{\mathfrak{N}_{\gamma'}})_{\max} \cap \mathrm{Edg}(\tau_{\mathfrak{N}_{\gamma'}})_{\min}\big)$, then we have*

- *the graph $\gamma := \gamma'_{(e)}$ is contained in Γ,*
- *for every $\mathcal{V}' \in \underline{\mathfrak{V}}_{\gamma'}$ with $\mathcal{V}' \cap \mathfrak{N}_{\gamma'} \neq \varnothing$,*

$$\{\, \zeta_{e'}^{\mathcal{V}'}=0 \ \ \forall\, e' \in \mathrm{Edg}(\gamma')\backslash\{e\}\ ;\ \zeta_e^{\mathcal{V}'}\neq 0 \,\} \in \pi_0(\mathfrak{M}_\gamma \cap \mathcal{V}'), \qquad \text{and}$$

- *for every $\mathfrak{N}_\gamma \in \pi_0(\mathfrak{M}_\gamma)$ with $\mathrm{Cl}_{\mathfrak{M}}(\mathfrak{N}_\gamma) \cap \mathfrak{N}_{\gamma'} \neq \varnothing$, the tuple $\mathcal{T}_{\mathfrak{N}_\gamma}$ is (up to graph isomorphism) equal to*

$$\begin{cases} \mathcal{T}_{\mathfrak{N}_{\gamma'}} & \text{if } e \notin \mathrm{E}_{\mathfrak{N}_{\gamma'}}, \\ \big(\,(\tau_{\mathfrak{N}_{\gamma'}})_{(\mathfrak{e})}\,,\ \mathrm{E}_{\mathfrak{N}_{\gamma'}}\backslash\{e\}\,,\ \beta_{\mathfrak{N}_{\gamma'}}|_{\mathrm{Edg}(\tau_{\mathfrak{N}_\gamma})}\,\big) & \text{if } \mathfrak{e} := \beta_{\mathfrak{N}_{\gamma'}}^{-1}(e) \in \mathrm{Edg}(\tau_{\mathfrak{N}_{\gamma'}})\backslash \mathrm{Edg}(\tau_{\mathfrak{N}_{\gamma'}})_{\min}, \\ \big(\,(\tau_{\mathfrak{N}_{\gamma'}})_{(\mathfrak{E}_{\mathfrak{e}}^{\wedge})}\,,\ \mathrm{E}_{\mathfrak{N}_{\gamma'}}\backslash \beta_{\mathfrak{N}_{\gamma'}}(\mathfrak{E}_{\mathfrak{e}}^{\wedge})\,,\ \beta_{\mathfrak{N}_{\gamma'}}|_{\mathrm{Edg}(\tau_{\mathfrak{N}_\gamma})}\,\big) & \text{if } \mathfrak{e} := \beta_{\mathfrak{N}_{\gamma'}}^{-1}(e) \in \mathrm{Edg}(\tau_{\mathfrak{N}_{\gamma'}})_{\min}\backslash \mathrm{Edg}(\tau_{\mathfrak{N}_{\gamma'}})_{\max}, \end{cases}$$

where $\mathfrak{E}_{\mathfrak{e}}^{\wedge} := \{\, \mathfrak{e}' \in \mathrm{Edg}(\tau_{\mathfrak{N}_{\gamma'}}) \,:\, v(\hbar_{\mathfrak{e}'}^+) \leq v(\hbar_{\mathfrak{e}}^+)\,\}\ \big(\subset \mathrm{Edg}(\tau_{\mathfrak{N}_{\gamma'}})\big)$;

(b_2) *if $e \in \beta_{\mathfrak{N}_{\gamma'}}\big(\mathrm{Edg}(\tau_{\mathfrak{N}_{\gamma'}})_{\max} \cap \mathrm{Edg}(\tau_{\mathfrak{N}_{\gamma'}})_{\min}\big)$, then for every $E \subset \mathrm{Edg}(\gamma')$ containing e with $\gamma := \gamma'_{(E)} \in \Gamma$ and every $\mathfrak{N}_\gamma \in \pi_0(\mathfrak{M}_\gamma)$ with $\mathrm{Cl}_{\mathfrak{M}}(\mathfrak{N}_\gamma) \cap \mathfrak{N}_{\gamma'} \neq \varnothing$, we have $\tau_{\mathfrak{N}_\gamma} = \tau_\bullet$.*

We remark that in Definition 2.16, if $\mathfrak{e}$ is a *minimal* edge of $\tau_{\mathfrak{N}_{\gamma'}}$, then $(\tau_{\mathfrak{N}_{\gamma'}})_{(\mathfrak{E}_{\mathfrak{e}}^{\wedge})}$ that is obtained from $\tau_{\mathfrak{N}_{\gamma'}}$ by contracting all the edges "below" the vertex $v(\hbar_{\mathfrak{e}}^+)$ can equivalently be obtained by dissolving the vertex $v(\hbar_{\mathfrak{e}}^+)$ and then taking the connected component containing the root of $\tau_{\mathfrak{N}_{\gamma'}}$.

Given $\gamma' \in \Gamma$ and $E = \{e_1, \dots e_n\} \subset \mathrm{Edg}(\gamma')$, contracting the edges of E one by one in any order

$$e_1,\ e_2,\ \dots,\ e_n \tag{2.12}$$

yields the same graph $\gamma := \gamma'_{(E)}$. If $\gamma \in \Gamma$, then for all $\mathfrak{N}_{\gamma'} \in \pi_0(\mathfrak{M}_{\gamma'})$ and $\mathfrak{N}_\gamma \in \pi_0(\mathfrak{M}_\gamma)$ with $\mathrm{Cl}_{\mathfrak{M}}(\mathfrak{N}_\gamma) \cap \mathfrak{N}_{\gamma'} \neq \varnothing$, we can follow Definition 2.16 and obtain a new tuple, which is exactly $\mathcal{T}_{\mathfrak{N}_\gamma}$, from $\mathcal{T}_{\mathfrak{N}_{\gamma'}}$ by contracting $e_1, \ldots, e_n$ with respect to the order (2.12). The tuple $\mathcal{T}_{\mathfrak{N}_\gamma}$, however, is independent of the choice of the order (2.12).

Example 2.17. Introduced in [8] and further studied in [10], the smooth stack $\mathfrak{M}_1^{\mathrm{wt}}$ of genus 1 stable weighted curves has a natural treelike structure. Recall that $\mathfrak{M}_1^{\mathrm{wt}}$ consists of the pairs $(C, \mathbf{w})$ of genus 1 nodal curves C and weights $\mathbf{w} \in H^2(C, \mathbb{Z})$ satisfying $\mathbf{w}(\Sigma) \geqslant 0$ for all irreducible $\Sigma \subset C$. A pair $(C, \mathbf{w})$ is stable if every rational irreducible component of C with weight 0 contains at least three nodal points. Let $\Gamma = \{\gamma \in \mathbf{G} : b_1(\gamma) \leqslant 1\}$. The stack $\mathfrak{M}_1^{\mathrm{wt}}$ has a natural Γ-stratification by dual graphs (c.f. Definition 4.5). Given $\gamma \in \Gamma$, the connected components $\mathfrak{N}_\gamma$ of the stratum $\mathfrak{M}_{1;\gamma}^{\mathrm{wt}}$ are indexed by the distribution of the weights on the irreducible components (or equivalently on the vertices of γ).

To each $\mathfrak{N}_\gamma$, we assign a rooted tree $\tau_{\mathfrak{N}_\gamma}$ that is obtained from γ by

- contracting all the edges in the smallest connected genus 1 subgraph γ_{cor} of γ,
- then dissolving all the vertices with $\mathbf{w}(v) > 0$, and
- finally taking the connected component containing the vertex that is the image of γ_{cor}.

Obviously, the edges of $\tau_{\mathfrak{N}_\gamma}$ form a subset $\mathrm{E}_{\mathfrak{N}_\gamma}$ of $\mathrm{Edg}(\gamma)$, so $\beta_{\mathfrak{N}_\gamma}$ is simply the inclusion. We remark that $\tau_{\mathfrak{N}_\gamma}$ is called the *weighted dual tree* of $\mathfrak{N}_\gamma$ in [10, §2.2].

The above construction of $\tau_{\mathfrak{N}_\gamma}$, $\mathrm{E}_{\mathfrak{N}_\gamma}$, and $\beta_{\mathfrak{N}_\gamma}$ gives rise to a treelike structure on $(\mathfrak{M}_1^{\mathrm{wt}}, \Gamma)$. In fact, it is straightforward to verify that $\tau_{\mathfrak{N}_\gamma}$ is a rooted tree satisfying the condition (a) of Definition 2.16 and that the graph $\gamma_{(e)}$ is in Γ for every edge e. It is also straightforward that for every $\gamma' \preceq \gamma$ and every chart $\mathcal{V} \longrightarrow \mathfrak{M}_1^{\mathrm{wt}}$ centered at a point of $\mathfrak{M}_{1;\gamma'}^{\mathrm{wt}}$,

$$\mathfrak{M}_{1;\gamma}^{\mathrm{wt}} \cap \mathcal{V} = \bigsqcup_{E \subset \mathrm{Edg}(\gamma'),\ \gamma'_{(E)} \simeq \gamma} \left\{ \zeta_e = 0 \ \forall\, e \in \mathrm{Edg}(\gamma') \backslash E\, ;\ \zeta_e \neq 0 \ \forall\, e \in E \right\},$$

where each ζ_e is a parameter corresponding to the smoothing of the node labeled by e. If $\mathrm{E}_{\mathfrak{N}_\gamma} \neq \varnothing$, let $\mathfrak{N}_{\gamma_{(e)}}$ be a connected component of $\mathfrak{M}_{1;\gamma_{(e)}}^{\mathrm{wt}}$ satisfying $\mathrm{Cl}_{\mathfrak{M}_1^{\mathrm{wt}}}(\mathfrak{N}_{\gamma_{(e)}}) \cap \mathfrak{N}_\gamma \neq \varnothing$; such $\mathfrak{N}_{\gamma_{(e)}}$ exists by the deformation of nodal curves. If e is not an edge of $\tau_{\mathfrak{N}_\gamma}$, then obviously $\mathcal{T}_{\mathfrak{N}_{\gamma_{(e)}}} = \mathcal{T}_{\mathfrak{N}_\gamma}$. If an edge e of $\tau_{\mathfrak{N}_\gamma}$ is not minimal, then the two endpoints of e are both of weight 0, hence their image in $\gamma_{(e)}$ is of weight 0, which implies $\tau_{\mathfrak{N}_{\gamma_{(e)}}} = (\tau_{\mathfrak{N}_\gamma})_{(e)}$. If an edge e of $\tau_{\mathfrak{N}_\gamma}$ is minimal, the construction of $\tau_{\mathfrak{N}_\gamma}$ implies that $v(\hbar_e^-)$ is positively weighted and so is its image $v_{(e)}$ in $\gamma_{(e)}$. Therefore, $v_{(e)}$ is a minimal vertex of $\tau_{\mathfrak{N}_{\gamma_{(e)}}}$. Every vertex v' of $\tau_{\mathfrak{N}_\gamma}$ with $v' \prec v(\hbar_e^+)$ is thus not in $\tau_{\mathfrak{N}_{\gamma_{(e)}}}$. In sum, the conditions of Definition 2.16 are all satisfied.

After choosing this treelike structure, we can construct the stack with twisted fields $\mathfrak{M}_1^{\mathrm{tf}}$ as in Theorem 2.18 (p_1) below, which is the same as that in [10, (2.13)].

2.3. **Stacks with twisted fields.** We are ready to present the main statement of the STF theory. Given a direct sum of line bundles $V = \oplus_i L_i'$ (over an arbitrary base), we write

$$\mathring{\mathbb{P}}(V) = \left\{ \big(x, [v_i]\big) \in \mathbb{P}(V) : v_i \neq 0 \ \forall\, i \right\}.$$

Given $k \in \mathbb{Z}_{>0}$ and morphisms $M_i \longrightarrow S$ with $1 \leqslant i \leqslant k$, we write

$$\prod_{1 \leqslant i \leqslant k} (M_i/S) := M_1 \times_S M_2 \times_S \cdots \times_S M_k.$$

Let Γ, $\mathfrak{M}$, and Λ be as in Definition 2.16. Given $\gamma \in \Gamma$ and $\mathfrak{N}_\gamma \in \pi_0(\mathfrak{M}_\gamma)$, there exists a unique tuple $(\tau_{\mathfrak{N}_\gamma}, \mathrm{E}_{\mathfrak{N}_\gamma}, \beta_{\mathfrak{N}_\gamma}) \in \Lambda$, which in turn determines a subset of $\overline{\mathbf{T}}$:

$$\overline{\Lambda}_{\mathfrak{N}_\gamma} := \left\{ [\mathfrak{t}] = [\tau_{\mathfrak{t}}, \ell_{\mathfrak{t}}] \in \overline{\mathbf{T}} \ :\ \tau_{\mathfrak{t}} = \tau_{\mathfrak{N}_\gamma} \right\}. \tag{2.13}$$

For every $e\in\mathrm{Edg}(\tau_{\mathfrak{N}_\gamma})$, let

$$L_{\geq_{\mathfrak{N}_\gamma} e} := \bigotimes_{e'\geq_{\mathfrak{N}_\gamma} e} L_{\beta_{\mathfrak{N}_\gamma}(e')} \longrightarrow \mathfrak{N}_\gamma\,, \tag{2.14}$$

where L_e are the line bundles as in (2.10) and $\geq_{\mathfrak{N}_\gamma}$ is the tree order on $\tau_{\mathfrak{N}_\gamma}$.

Theorem 2.18. *With notation as above, we have the following conclusions:*

(p_1) *the following disjoint union of the fiber products over the strata of $\mathfrak{M}$:*

$$\mathfrak{M}^{\rm tf}=\mathfrak{M}^{\rm tf}_\Lambda=\bigsqcup_{\gamma\in\Gamma;\ \mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma);\ [\mathsf{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}}(\mathfrak{N}_\gamma)^{\rm tf}_{[\mathsf{t}]}\,,\qquad\text{where}$$

$$(\mathfrak{N}_\gamma)^{\rm tf}_{[\mathsf{t}]}=\Big(\prod_{i\in[\![\mathbf{m},0)\!)_{\mathsf{t}}}\Big(\Big(\mathring{\mathbb{P}}\big(\bigoplus_{e\in\mathfrak{E}_i^\perp(\mathsf{t})}L_{\geq_{\mathfrak{N}_\gamma}e}\big)\Big)\Big/\mathfrak{N}_\gamma\Big)\Big)\xrightarrow{\ \varpi\ }\mathfrak{N}_\gamma,$$

has a canonical smooth algebraic stack structure, known as the stack with twisted fields over $\mathfrak{M}$ with respective to Λ, *and determines a proper and birational morphism $\varpi:\mathfrak{M}^{\rm tf}\longrightarrow\mathfrak{M}$ known as the* forgetful morphism;

(p_2) *for any $[\mathsf{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}$ and $x\in\mathfrak{N}^{\rm tf}_{\gamma,[\mathsf{t}]}$, there exist a smooth chart $\mathfrak{U}_x\longrightarrow\mathfrak{M}^{\rm tf}$ containing x, called a* twisted chart, *a set of special edges $\{e_i\in\mathfrak{E}_i^\perp(\mathsf{t})\}_{i\in[\![\mathbf{m},0)\!)_{\mathsf{t}}}$, and a subset of local parameters called the* twisted parameters:

$$\begin{aligned}\xi^{\mathfrak{U}_x}_s,\quad s\in\widetilde{\mathbb{I}}(\mathsf{t})&:=\mathbb{I}(\mathsf{t})\sqcup\big(\mathfrak{E}^\perp_{\geqslant\mathbf{m}}(\mathsf{t})\backslash\{e_i\}_{i\in[\![\mathbf{m},0)\!)_{\mathsf{t}}}\big)\\&=[\![\mathbf{m},0)\!)_{\mathsf{t}}\sqcup E_{\mathbf{m}}(\mathsf{t})\sqcup E_-(\mathsf{t})\sqcup\big(\mathfrak{E}^\perp_{\geqslant\mathbf{m}}(\mathsf{t})\backslash\{e_i\}_{i\in[\![\mathbf{m},0)\!)_{\mathsf{t}}}\big)\end{aligned}$$

centered at x such that

$$\xi^{\mathfrak{U}_x}_s\in\Gamma(\mathscr{O}^*_{\mathfrak{U}_x})\quad\forall\,s\in\mathfrak{E}^\perp_{\geqslant\mathbf{m}}(\mathsf{t})\backslash\{e_i\}_{i\in[\![\mathbf{m},0)\!)_{\mathsf{t}}},\qquad(\mathfrak{N}_\gamma)^{\rm tf}_{[\mathsf{t}]}\cap\mathfrak{U}_x=\{\,\xi^{\mathfrak{U}_x}_s{=}0\ \forall\,s\in\mathbb{I}(\mathsf{t})\,\};$$

moreover, for every $\gamma'\in\Gamma$, every $\mathfrak{N}_{\gamma'}\in\pi_0(\mathfrak{M}_{\gamma'})$, and every $[\mathsf{t}']\in\overline{\Lambda}_{\mathfrak{N}_{\gamma'}}$,

- *if $[\mathsf{t}']\geq[\mathsf{t}]$ and $\mathrm{Cl}_{\mathfrak{M}}(\mathfrak{N}_{\gamma'})\cap\mathfrak{N}_\gamma\neq\varnothing$, then $\pi_0\big((\mathfrak{N}_{\gamma'})^{\rm tf}_{[\mathsf{t}']}\cap\mathfrak{U}_x\big)$ is a subset of*

$$\begin{aligned}\big\{\,\{\,\xi^{\mathfrak{U}_x}_s{=}0\ \forall\,s\in\mathbb{I}(\mathsf{t}')\,;\ \xi^{\mathfrak{U}_x}_s\neq0\ \forall\,s\in\mathbb{J}\,;\ \varpi^*\zeta_e=0\ \forall\,e\in(\mathrm{Edg}(\gamma)\backslash\mathrm{E}_{\mathfrak{N}_\gamma})\backslash E\,;\\ \varpi^*\zeta_e\neq0\ \forall\,e\in(\mathrm{Edg}(\gamma)\backslash\mathrm{E}_{\mathfrak{N}_\gamma})\cap E\,\}\ :\ \mathbb{J}\subset\mathbb{I}(\mathsf{t}),\ \mathsf{t}_{(\mathbb{J})}\sim\mathsf{t}',\ E\subset\mathrm{Edg}(\gamma),\ \gamma_{(E)}\simeq\gamma'\,\big\},\end{aligned}$$

- *if $[\mathsf{t}']\not\geq[\mathsf{t}]$ or $\mathrm{Cl}_{\mathfrak{M}}(\mathfrak{N}_{\gamma'})\cap\mathfrak{N}_\gamma=\varnothing$, then $(\mathfrak{N}_{\gamma'})^{\rm tf}_{[\mathsf{t}']}\cap\mathfrak{U}_x=\varnothing$;*

(p_3) *with notation as in (p_2), if the set of modular parameters $\{\zeta^{\mathcal{V}}_e\}$ on $\mathcal{V}$ centered at $\varpi(x)$ extends to a set of local parameters on $\mathcal{V}$ centered at $\varpi(x)$:*

$$\{\zeta^{\mathcal{V}}_e\}_{e\in\mathrm{Edg}(\gamma)}\sqcup\{\varsigma_j\}_{j\in J}$$

for some index set J, then

$$\{\xi^{\mathfrak{U}_x}_s\}_{s\in\widetilde{\mathbb{I}}(\mathsf{t})}\sqcup\{\varpi^*\zeta^{\mathcal{V}}_e\}_{e\in\mathrm{Edg}(\gamma)\backslash\mathrm{E}_{\mathfrak{N}_\gamma}}\sqcup\{\varpi^*\varsigma_j\}_{j\in J}$$

is a set of local parameters on $\mathfrak{U}_x$ centered at x; and

(p_4) *for any $[\mathsf{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}$, $x\in(\mathfrak{N}_\gamma)^{\rm tf}_{[\mathsf{t}]}$, $\mathcal{V}\in\underline{\mathfrak{V}}_\gamma$ containing $\varpi(x)$, and $e\in\mathfrak{E}_{\mathbf{m}}(\mathsf{t})$, we have*

$$\varpi^*\big(\prod_{e'\geq_{\mathfrak{N}_\gamma}e}\zeta^{\mathcal{V}}_{\beta_{\mathfrak{N}_\gamma}(e')}\big)=u_e\cdot\prod_{i\in[\![\mathbf{m},0)\!)_{\mathsf{t}}}\xi^{\mathfrak{U}_x}_i,$$

where $u_e\in\Gamma(\mathscr{O}_{\mathfrak{U}_x})$ is a unit if $e\in\mathfrak{E}^\perp_{\mathbf{m}}(\mathsf{t})$ and is equal to $\xi^{\mathfrak{U}_x}_e$ (up to a unit) if $e\in E_{\mathbf{m}}(\mathsf{t})$ $\big(=\mathfrak{E}_{\mathbf{m}}(\mathsf{t})\backslash\mathfrak{E}^\perp_{\mathbf{m}}(\mathsf{t})\big)$.

Proof. The statement that $\mathfrak{M}^{\mathrm{tf}}$ is a smooth algebraic stack in (p_1), as well as the construction of the smooth chart in (p_2), follow from the same argument as in [10, Corollary 3.7]. The key reason is as follows. For every $\gamma\in\Gamma$, $\mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma)$ with $\tau_{\mathfrak{N}_\gamma}\neq\tau_\bullet$, $[\mathfrak{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}$, and $\mathbb{I}'\subset\mathbb{I}(\mathfrak{t})$ with

$$I'_+\big(:=\mathbb{I}'\cap[\![\mathbf{m},0)\!)_{\mathfrak{t}}\big)\subsetneq[\![\mathbf{m},0)\!)_{\mathfrak{t}}, \tag{2.15}$$

we take

$$\widetilde{E}_{(\mathbb{I}')}:=\mathrm{Edg}(\tau_{\mathfrak{N}_\gamma})\backslash\mathrm{Edg}(\mathfrak{t}_{(\mathbb{I}')}),$$

which is the set of the edges contracted from $\mathrm{Edg}(\mathfrak{t})=\mathrm{Edg}(\tau_{\mathfrak{N}_\gamma})$ to obtain $\mathfrak{t}_{(\mathbb{I}')}$ as in (2.6). By (2.15), $\widetilde{E}_{(\mathbb{I}')}\cap\mathrm{Edg}(\tau_{\mathfrak{N}_\gamma})_{\max}\cap\mathrm{Edg}(\tau_{\mathfrak{N}_\gamma})_{\min}=\varnothing$. Thus, by Definition 2.16 (b_1), for every $x\in\mathfrak{N}_\gamma$ and every $\mathcal{V}\in\underline{\mathfrak{V}}_\gamma$ containing x, the chart $\mathcal{V}\longrightarrow\mathfrak{M}$ induces a chart

$$\begin{aligned}\mathcal{V}_{\mathfrak{t}_{(\mathbb{I}')}}:=\Big\{\,\zeta^{\mathcal{V}}_{\beta_{\mathfrak{N}_\gamma}(\mathfrak{e})}=0\ \ \forall\,\mathfrak{e}\in\mathrm{Edg}(\mathfrak{t}_{(\mathbb{I}')})\ \big(=\mathrm{Edg}(\tau_{\mathfrak{N}_\gamma})\backslash\widetilde{E}_{(\mathbb{I}')}\big)\,;\ \ \zeta^{\mathcal{V}}_{\beta_{\mathfrak{N}_\gamma}(\mathfrak{e})}\neq0\ \ \forall\,\mathfrak{e}\in\widetilde{E}_{(\mathbb{I}')}\,\Big\}\\ \longrightarrow\bigsqcup_{E\subset\mathrm{Edg}(\gamma),\ E\cap\mathrm{E}_{\mathfrak{N}_\gamma}=\beta_{\mathfrak{N}_\gamma}(\widetilde{E}_{(\mathbb{I}')})}\mathfrak{M}_{\gamma_{(E)}}\qquad(\subset\mathfrak{M})\,,\end{aligned}$$

which is the analogue of [10, (3.2)]. Definition 2.16 (b_1) also implies that

$$\begin{aligned}&\mathcal{T}_{\mathfrak{N}_{\gamma_{(E)}}}=\big(\,\tau_{\mathfrak{t}_{(\mathbb{I}')}},\,\mathrm{E}_{\mathfrak{N}_\gamma}\backslash E,\,\beta_{\mathfrak{N}_\gamma}|_{\mathrm{Edg}(\mathfrak{t}_{(\mathbb{I}')})}\,\big)\\ &\forall\ E\subset\mathrm{Edg}(\gamma),\ \mathfrak{N}_{\gamma_{(E)}}\in\pi_0(\mathfrak{M}_{\gamma_{(E)}})\ \ \text{with}\ \ E\cap\mathrm{E}_{\mathfrak{N}_\gamma}=\beta_{\mathfrak{N}_\gamma}\big(\widetilde{E}_{(\mathbb{I}')}\big),\ \mathrm{Cl}_{\mathfrak{M}}(\mathfrak{N}_{\gamma_{(E)}})\cap\mathfrak{N}_\gamma\neq\varnothing.\end{aligned}$$

Thus, we can define the locus $\mathfrak{U}_{x;[\mathfrak{t}_{(\mathbb{I}')}]}\subset\mathfrak{U}_x\subset\mathbb{A}^{\widetilde{\mathbb{I}}(\mathfrak{t})}$ mimicking the paragraph containing [10, (3.10)], and define the morphism

$$\Phi_{x;(\mathbb{I}')}:\mathfrak{U}_{x;[\mathfrak{t}_{(\mathbb{I}')}]}\longrightarrow\mathcal{V}_{\mathfrak{t}_{(\mathbb{I}')}}\times_{\mathfrak{M}}\mathfrak{M}^{\mathrm{tf}}$$

as in [10, (3.14)]. Such $\Phi_{x;(\mathbb{I}')}$ can also be defined for $\mathbb{I}'\subset\mathbb{I}(\mathfrak{t})$ that does not satisfy (2.15) and for $\mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma)$ with $\tau_{\mathfrak{N}_\gamma}=\tau_\bullet$, because no twisted field is added to these strata by Definition 2.16 (a) and (b_2). Therefore, the above morphisms $\Phi_{x;(\mathbb{I}')}$ together give rise to a map

$$\Phi_x:\mathfrak{U}_x\longrightarrow\mathfrak{M}^{\mathrm{tf}};$$

see the display below [10, (3.15)]. As proved in [10, Corollary 3.7], these Φ_x form smooth charts of the algebraic stack $\mathfrak{M}^{\mathrm{tf}}$. The twisted parameters in (p_2) are written as ε_i, u_e, and z_e in [10, (3.9)]. The local expressions of the strata of $\mathfrak{M}^{\mathrm{tf}}$ in (p_2) follow from [10, (3.10) & Lemma 3.2]. The last and the first equations in [10, (3.12)] imply (p_3) and (p_4), respectively; see also [10, Remark 3.8] for (p_4).

In the remainder of the proof of Theorem 2.18, we will show that ϖ is birational and proper, as stated in (p_1). Notice that the only level map on the connected edge-less graph $\tau_\bullet$ is the empty function, hence ϖ restricts to the identity map on the pullback of $\mathfrak{M}_{\tau_\bullet}=\mathfrak{M}\backslash\Delta$, where Δ is the boundary of $\mathfrak{M}$ as in (2.9). By the local expressions of the strata of $\mathfrak{M}$ and $\mathfrak{M}^{\mathrm{tf}}$ in (2.8) and (p_2), respectively, we see that Δ and its the pullback are of lower dimension in $\mathfrak{M}$ and $\mathfrak{M}^{\mathrm{tf}}$, respectively. Thus, ϖ is birational.

It is straightforward that ϖ is of finite type. To establish the properness of ϖ, let $(\mathbf{D},0)$ be a nonsingular pointed curve with the complement and the inclusion respectively denoted by

$$\mathbf{D}^*:=\mathbf{D}\backslash\{0\}\overset{\iota}{\hookrightarrow}\mathbf{D}.$$

We will show that for any $f:\mathbf{D}\longrightarrow\mathfrak{M}$ and any $F:\mathbf{D}^*\longrightarrow\mathfrak{M}^{\mathrm{tf}}$ with $\varpi\circ F=f\circ\iota$, there exists a unique $\psi:\mathbf{D}\longrightarrow\mathfrak{M}^{\mathrm{tf}}$ so that the following diagram commutes:

$$\begin{tikzcd} \mathbf{D}^* \arrow[r, "F"] \arrow[d, hook, "\iota"'] & \mathfrak{M}^{\rm tf} \arrow[d, "\varpi"] \\ \mathbf{D} \arrow[r, "f"] \arrow[ur, dashed, "\psi"] & \mathfrak{M} \end{tikzcd}$$

Deleting a discrete subset of $\mathbf{D}^*$ if necessary, we assume that there exist $\gamma,\gamma'\in\Gamma$, $\mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma)$, and $\mathfrak{N}_{\gamma'}\in\pi_0(\mathfrak{M}_{\gamma'})$ such that

$$f(t)\in\mathfrak{N}_\gamma \quad \forall\, t\in\mathbf{D}^*; \qquad f(0)\in\mathfrak{N}_{\gamma'}.$$

This implies $\mathrm{Cl}_{\mathfrak{M}}(\mathfrak{N}_\gamma)\cap\mathfrak{N}_{\gamma'}\neq\varnothing$, hence by Lemma 2.15, we see that $\gamma'\preceq\gamma$. There thus exists

$$E\subset\mathrm{Edg}(\gamma') \quad \text{with} \quad \gamma\simeq\gamma'_{(E)}.$$

The treelike structure Λ on $(\mathfrak{M},\Gamma)$ then determines the tuples

$$\big(\,\tau:=\tau_{\mathfrak{N}_\gamma}\,,\ \mathrm{E}:=\mathrm{E}_{\mathfrak{N}_\gamma}\subset\mathrm{Edg}(\gamma)\,,\ \beta:=\beta_{\mathfrak{N}_\gamma}\colon\mathrm{Edg}(\tau)\longrightarrow\mathrm{E}\,\big) \qquad \text{and}$$
$$\big(\,\tau':=\tau_{\mathfrak{N}_{\gamma'}}\,,\ \mathrm{E}':=\mathrm{E}_{\mathfrak{N}_{\gamma'}}\subset\mathrm{Edg}(\gamma')\,,\ \beta':=\beta_{\mathfrak{N}_{\gamma'}}\colon\mathrm{Edg}(\tau')\longrightarrow\mathrm{E}'\,\big)$$

as in Definition 2.16. Let

$$\mathrm{E}^{\rm ctr}:=\mathrm{E}'\cap E=\mathrm{E}'\backslash\mathrm{E}, \qquad \mathfrak{E}^{\rm ctr}:=(\beta')^{-1}(\mathrm{E}^{\rm ctr})=\mathrm{Edg}(\tau')\backslash\mathrm{Edg}(\tau).$$

Deleting a discrete subset of $\mathbf{D}^*$ again if necessary, we further assume that there exists a fixed $[\mathfrak{t}]=[\tau_{\mathfrak{N}_\gamma},\ell]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}$ such that

$$F(t)=\big(\,f(t)\,;\,\big([\mu_e(t)]_{e\in\mathfrak{E}_i^\perp(\mathfrak{t})}\big)_{i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}}}\big)\in(\mathfrak{N}_\gamma)^{\rm tf}_{[\mathfrak{t}]} \qquad \forall\, t\in\mathbf{D}^*.$$

Here we fix a rooted level tree $\mathfrak{t}$ representing $[\mathfrak{t}]$.

With notation as in Definition 2.13, let $\mathcal{V}\in\underline{\mathfrak{V}}_{\gamma'}$ be a chart containing $f(0)$ and $\{\zeta_e=\zeta_e^{\mathcal{V}}\}_{e\in\mathrm{Edg}(\gamma')}$ be modular parameters on $\mathcal{V}$ centered at $f(0)$. For every $e\in\mathrm{E}^{\rm ctr}$, there exist a unique integer $n_e\in\mathbb{Z}_{>0}$ and a unique nowhere vanishing function a_e on $\mathbf{D}\cap f^{-1}(\mathcal{V})$ so that

$$\zeta_e\big(f(t)\big)=t^{n_e}a_e(t) \qquad \forall\ t\in\mathbf{D}\cap f^{-1}(\mathcal{V}). \tag{2.16}$$

Similarly, for every level $i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}}$, we can choose an edge $\mathfrak{e}_i\in\mathfrak{E}_i^\perp(\mathfrak{t})$ such that for every $e\in\mathfrak{E}_i^\perp(\mathfrak{t})$, the specialization of $\mu_e/\mu_{\mathfrak{e}_i}$ at $t=0$ exists. There thus exist a unique integer $n_e\in\mathbb{Z}_{\geqslant 0}$ and a unique nowhere vanishing function a_e on the whole $\mathbf{D}\cap f^{-1}(\mathcal{V})$ satisfying $n_{\mathfrak{e}_i}=0$, $a_{\mathfrak{e}_i}\equiv 1$, and the level-$i$ twisted fields of $F(t)$ can be written as

$$[\mu_e(t)]_{e\in\mathfrak{E}_i^\perp(\mathfrak{t})}=\Big[\,t^{n_e}a_e(t)\bigotimes_{e'\succeq_{\mathfrak{t}}e}\tfrac{\partial}{\partial\zeta_{e'}}\big|_{f(t)}\,\Big]_{e\in\mathfrak{E}_i^\perp(\mathfrak{t})} \qquad \forall\ t\in\mathbf{D}^*\cap f^{-1}(\mathcal{V}). \tag{2.17}$$

Since $\mathfrak{E}_{\geqslant\mathbf{m}}(\mathfrak{t})=\bigsqcup_{i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}}}\mathfrak{E}_i^\perp(\mathfrak{t})$, we have assigned to each $e\in\mathfrak{E}_{\geqslant\mathbf{m}}(\mathfrak{t})\sqcup\mathfrak{E}^{\rm ctr}$ an integer n_e and a nowhere vanishing function a_e on $\mathbf{D}\cap f^{-1}(\mathcal{V})$ via (2.16) and (2.17). Let

$$s_e:=-\sum_{\substack{e'\in\mathfrak{E}_{\geqslant\mathbf{m}}(\mathfrak{t})\sqcup\mathfrak{E}^{\rm ctr},\\ e'\succeq_{\tau'}e}}n_{e'} \qquad \forall\ e\in\mathfrak{E}_{\geqslant\mathbf{m}}(\mathfrak{t})\sqcup\mathfrak{E}^{\rm ctr}\ \big(\subset\mathrm{Edg}(\tau')\big).$$

Note that each s_e is non-positive, so they can be used to define levels on τ' as follows.

By (2.16) and (2.17), there exists a rooted level tree

$$\mathfrak{t}'=(\,\tau'\,,\,\ell'\,)\,,$$

unique up to equivalence, satisfying

$$[\mathfrak{t}']\preceq[\mathfrak{t}], \qquad \big(\mathfrak{E}_{\geqslant\mathbf{m}}(\mathfrak{t})\subset\big)\ \ \mathfrak{E}_{\geqslant\mathbf{m}'}(\mathfrak{t}')\subset\mathfrak{E}_{\geqslant\mathbf{m}}(\mathfrak{t})\sqcup\mathfrak{E}^{\rm ctr}\ \ \big(\subset\mathrm{Edg}(\tau')\big), \tag{2.18}$$

$$\ell'(\hbar_e^-)=s_e \qquad \forall\ e\in\mathfrak{E}_{\geqslant\mathbf{m}'}(\mathfrak{t}'). \tag{2.19}$$

where $\mathbf{m}'=\mathbf{m}(\mathfrak{t}')$. Since the underlying rooted tree of $\mathfrak{t}'$ is $\tau'=(\gamma_{\mathfrak{N}_{\gamma'}},o)$, we have

$$[\mathfrak{t}'] \in \overline{\Lambda}_{\mathfrak{N}_{\gamma'}}.$$

We next show that there exists a *unique* twisted field

$$\eta := \Big(f(0);\ \Big(\big[\lambda_e \bigotimes_{e'\succeq_{\tau'}e} \tfrac{\partial}{\partial\zeta_{e'}}\big|_{f(0)}\big]_{e\in\mathfrak{E}_j^{\perp}(\mathfrak{t}')}\Big)_{j\in[\![\mathbf{m}',0)\!)_{\mathfrak{t}'}} \Big) \in (\mathfrak{N}_{\gamma'})^{\mathrm{tf}}_{[\mathfrak{t}']}\big|_{f(0)} \tag{2.20}$$

over $f(0)$ such that $F(t) \rightsquigarrow \eta$. Here we write $z \rightsquigarrow z_0$ if z specializes to z_0 for any scheme S and any points z and z_0 of S.

Fix a twisted field η over $f(0)$. Since $[\mathfrak{t}']\preceq[\mathfrak{t}]$, there exists a subset $\mathbb{J}\subset\mathbb{I}(\mathfrak{t}')$ such that $[\mathfrak{t}'_{(\mathbb{J})}]=[\mathfrak{t}]$; see (2.7). Let $\mathfrak{U}_\eta$ be the twisted chart with the twisted parameters $\xi_s:=\xi_s^{\mathfrak{U}_\eta}$ as in (p_2) and set

$$\xi_{\mathsf{e}_j}\equiv 1 \qquad \forall\ j\in[\![\mathbf{m}',0)\!)_{\mathfrak{t}'}. \tag{2.21}$$

Then, ξ_s is defined for all $s\in[\![\mathbf{m}',0)\!)_{\mathfrak{t}'}\sqcup\mathrm{Edg}(\mathfrak{t}')$. Let

$$\Psi_{\eta;(\mathbb{J})} : \mathcal{U}'_{\eta;[\mathfrak{t}]} \longrightarrow \mathfrak{U}_\eta \cap \big\{ \xi_s\neq 0\ \forall\, s\in\mathbb{J}\ ;\ \xi_s=0\ \forall\, s\in\mathbb{I}(\mathfrak{t}')\backslash\mathbb{J} \big\}$$

be the isomorphism constructed in [10, (3.17)], where $\mathcal{U}'_{\eta;[\mathfrak{t}]}$ is an open subset of $\mathfrak{N}^{\mathrm{tf}}_{\gamma,[\mathfrak{t}]}$ containing the image $F\big(\mathbf{D}^*\cap f^{-1}(\mathcal{V})\big)$. Let

$$\varepsilon_j(t):=\xi_j\big(\Psi_{\eta;(\mathbb{J})}(F(t))\big)\ \ \forall\, j\in[\![\mathbf{m}',0)\!)_{\mathfrak{t}'}\,;\quad u_e(t):=\xi_e\big(\Psi_{\eta;(\mathbb{J})}(F(t))\big)\ \ \forall\, e\in\mathfrak{E}_{\geqslant\mathbf{m}'}(\mathfrak{t}')\,. \tag{2.22}$$

Then, $F(t)\rightsquigarrow\eta$ if and only if

$$\begin{aligned} &\Big((\varepsilon_j(t))_{j\in[\![\mathbf{m}',0)\!)_{\mathfrak{t}'}}\,,\ (u_e(t))_{e\in\mathfrak{E}^{\perp}_{\geqslant\mathbf{m}'}(\mathfrak{t}')}\,,\ (u_e(t))_{e\in E_{\mathbf{m}'}(\mathfrak{t}')} \Big) \\ &\rightsquigarrow \big(\underline{0}\,,\ (\lambda_e)_{e\in\mathfrak{E}^{\perp}_{\geqslant\mathbf{m}'}(\mathfrak{t}')}\,,\ \underline{0}\big), \qquad\qquad \text{where}\ \ \underline{0}=(0,\ldots,0),\quad \lambda_{\mathsf{e}_j}=1\ \forall\ j\in[\![\mathbf{m}',0)\!)_{\mathfrak{t}'}. \end{aligned} \tag{2.23}$$

Indeed, by [10, (3.21)] and Cases 1-3 of the proof of [10, Lemma 3.2], as well as the assumption (2.19) above, we conclude inductively over the levels of $\mathfrak{t}'$ in the descending order that there exist nowhere vanishing functions $b_j(t)$, $j\in\mathbb{J}=\mathbb{J}\cap[\![\mathbf{m}',0)\!)_{\mathfrak{t}'}$, on $\mathbf{D}\cap f^{-1}(\mathcal{V})$ such that

$$\varepsilon_j(t) = \begin{cases} t^{(j^\sharp-j)}b_j(t) & \text{if } j\in\mathbb{J}_+; \\ 0 & \text{if } j\notin\mathbb{J}_+ \end{cases} \tag{2.24}$$

Similarly, by [10, (3.22)] and Cases A and B of the proof of [10, Lemma 3.2], as well as (2.19) and (2.24), we conclude inductively over the levels of $\mathfrak{t}'$ in the descending order that there exist nowhere vanishing functions $b_e(t)$, $e\in\mathfrak{E}_{\geqslant\mathbf{m}'}(\mathfrak{t}')$, on $\mathbf{D}\cap f^{-1}(\mathcal{V})$ such that

$$u_e(t) = \begin{cases} b_e(t) & \text{if } e\in\mathfrak{E}^{\perp}_{\geqslant\mathbf{m}'}(\mathfrak{t}'); \\ t^{(\mathbf{m}'-s_e)}b_e(t) & \text{if } e\in E_{\mathbf{m}'}(\mathfrak{t}'). \end{cases} \tag{2.25}$$

We set

$$\lambda_e := b_e(0) \qquad \forall\ \in\mathfrak{E}^{\perp}_{\geqslant\mathbf{m}'}(\mathfrak{t}'). \tag{2.26}$$

Since the functions b_e are all nowhere vanishing, all λ_e are nonzero. Moreover, by (2.21), the second equation in (2.22, (2.25), and (2.26), we obtain $\lambda_{\mathsf{e}_j}=1$ for all $j\in[\![\mathbf{m}',0)\!)_{\mathfrak{t}'}$. Thus, λ_e can be used to define η in (2.20).

Since $j^\sharp$ stands for the level of ℓ' immediately above j, we have

$$j^\sharp-j>0 \qquad \forall\ j\in\mathbb{J}_+.$$

Similarly, by (2.4) and (2.19), we have

$$\mathbf{m}'-s_e>0 \qquad \forall\ e\in E_{\mathbf{m}'}(\mathfrak{t}').$$

Thus, by (2.24), (2.25), and (2.26), we see that $F:\mathbf{D}^*\longrightarrow\mathfrak{M}^{\mathrm{tf}}$ extends to $\psi:\mathbf{D}\longrightarrow\mathfrak{M}$ with $\psi(0)=\eta$ as in (2.20).

It remains to prove the uniqueness of ψ. Mimicking the proof of [10, Lemma 3.2], we conclude that for any $[\mathsf{t}^\dagger]=[\tau',\ell^\dagger]\in\overline{\Lambda}_{\mathfrak{N}_{\gamma'}}\backslash[\mathsf{t}']$, at least one of (2.24) and (2.25) no longer holds for $\mathsf{t}^\dagger$. Hence by (2.23), $F(t)$ cannot specialize to $\eta^\dagger$. Consequently, every possible specialization of $F(t)$ at $t=0$ must be in $(\mathfrak{N}_{\gamma'})^{\mathrm{tf}}_{[\mathsf{t}']}\big|_{f(0)}$. Since the functions $u_e(t)$ in (2.25) are uniquely determined by [10, (3.22), (3.26), & (3.28)] as well as the expressions (2.16), (2.17), and (2.24), so are the functions $b_e(t)$. This establishes the uniqueness of η in (2.20) and hence the uniqueness of ψ. □

The following proposition is a restatement of [10, Proposition 3.9] under the current setup.

Proposition 2.19. *Let Γ, $\mathfrak{M}$, and Λ be as in Theorem 2.18. Assume that $\pi:\mathcal{C}\longrightarrow\mathfrak{M}$ is the universal family of $\mathfrak{M}$. Then, the Γ-stratification of $\mathfrak{M}$ in Definition 2.13 induces a stratification $\mathcal{C}=\bigsqcup_{\gamma\in\Gamma}\mathcal{C}_\gamma$ such that the union*

$$\mathcal{C}^{\mathrm{tf}}:=\bigsqcup_{\gamma\in\Gamma;\ \mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma);\ [\mathsf{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}}(\mathcal{N}_\gamma)^{\mathrm{tf}}_{[\mathsf{t}]}\,,\qquad\text{where}$$

$$(\mathcal{N}_\gamma)^{\mathrm{tf}}_{[\mathsf{t}]}:=\bigg(\prod_{i\in[\![\mathbf{m},0)\!)_{\mathsf{t}}}\Big(\mathring{\mathbb{P}}\Big(\big(\bigoplus_{e\in\mathfrak{E}_i^\perp(\mathsf{t})}\pi^*L_{\ge_{\mathfrak{N}_\gamma}e}\big)\Big)\Big/\mathcal{N}_\gamma\Big]\bigg)\longrightarrow\mathcal{N}_\gamma:=\pi^{-1}(\mathfrak{N}_\gamma),$$

has a canonical smooth algebraic stack structure, and the projection

$$\pi^{\mathrm{tf}}:\mathcal{C}^{\mathrm{tf}}\longrightarrow\mathfrak{M}^{\mathrm{tf}}$$

induced by π gives the universal family of $\mathfrak{M}^{\mathrm{tf}}$.

On the conclusion of this section, we remark that in [10], the properness of the forgetful morphism $\mathfrak{M}_1^{\mathrm{tf}}\longrightarrow\mathfrak{M}_1^{\mathrm{wt}}$ (see Example 2.17 for notation) was established via the isomorphism $\mathfrak{M}_1^{\mathrm{tf}}\longrightarrow\widetilde{\mathfrak{M}}_1^{\mathrm{wt}}$ to the blowup stack $\widetilde{\mathfrak{M}}_1^{\mathrm{wt}}$ in [8], as well as the properness of the blowing up $\widetilde{\mathfrak{M}}_1^{\mathrm{wt}}\longrightarrow\mathfrak{M}_1^{\mathrm{wt}}$. In this paper, Theorem 2.18 (p_1) provides a more direct approach to the properness, with a broader setup that includes the genus one case as a special case, yet without involving the comparison with the blowups. This could be an advantage of the STF theory in the higher genera cases.

3. Recursive constructions

Theorem 2.18 and Proposition 2.19 suggest the possibility of adding twisted fields recursively:

$$\cdots\longrightarrow(\mathfrak{M}^{\mathrm{tf}})^{\mathrm{tf}}\longrightarrow\mathfrak{M}^{\mathrm{tf}}\longrightarrow\mathfrak{M}.$$

The tricky part is choosing appropriate stratification Γ' and treelike structure Λ' in each step. Given a Γ-stratification on $\mathfrak{M}$ and a treelike structure Λ on $(\mathfrak{M},\Gamma)$, we provide a possible stratification known as the derived stratification on $\mathfrak{M}^{\mathrm{tf}}$ and another possible stratification on $\mathfrak{M}$ known as a grafted stratification, respectively in this section. They will be used in the description of $\widetilde{\mathfrak{P}}_2^{\mathrm{tf}}$ and $\widetilde{M}_2(\mathbb{P}^n,d)^{\mathrm{tf}}$ of Theorem 1.2.

3.1. Derived stratification.

Recall $\mathbf{G}$ denotes the set of connected graphs. Let $\mathfrak{M}$ be a smooth stack endowed with a Γ-stratification $\mathfrak{M}=\bigsqcup_{\gamma\in\Gamma}\mathfrak{M}_\gamma$ as in Definition 2.13 and a treelike structure Λ as in Definition 2.16 that assigns to every pair $\big(\gamma\in\Gamma,\ \mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma)\big)$ a unique tuple

$$\big(\tau\,,\,\mathrm{E}\,,\,\beta\big):=(\tau_{\mathfrak{N}_\gamma},\mathrm{E}_{\mathfrak{N}_\gamma},\beta_{\mathfrak{N}_\gamma}).$$

Given a rooted level tree $\mathsf{t}=(\tau,\ell)$ with $[\mathsf{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}$ and a level $i\in\mathrm{Im}(\ell)$, recall that $i^\sharp$ and $i^\flat$ stand for the levels immediately above and below i (if they exist), respectively. We also recall that

$(\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\succeq}$ denotes the set of the edges of the paths of τ from the root o to the level $\mathbf{m}=\mathbf{m}(\mathfrak{t})$ *minimal* vertices; see (2.4).

Definition 3.1. *Let Γ, $\mathfrak{M}$, and Λ be as in Definition 2.16. For every $\gamma\in\Gamma$, $\mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma)$, and $[\mathfrak{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}$, we construct the* derived graph of γ with respect to $[\mathfrak{t}]$:

$$\rho_{\gamma,[\mathfrak{t}]}\in\mathbf{G}$$

as follows. For an arbitrary $\mathfrak{t}$ representing $[\mathfrak{t}]$,

(1) firstly, we replace every $e=\{\hbar^+_e,\hbar^-_e\}\in(\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\succeq}\subset\mathrm{Edg}(\mathfrak{t})$ with a new list of half-edges

$$\hbar^+_{e;\ell(\hbar^+_e)^\flat},\ \hbar^-_{e;\ell(\hbar^+_e)^\flat},\ \hbar^+_{e;(\ell(\hbar^+_e)^\flat)^\flat},\ \ldots,\ \hbar^-_{e;\ell(\hbar^-_e)^\sharp},\ \hbar^+_{e;\ell(\hbar^-_e)},\ \hbar^-_{e;\ell(\hbar^-_e)};$$

(2) then, for each $j\in[\![\mathbf{m},0)\!)_{\mathfrak{t}}$ and $\bullet=+,-$, we set

$$\hbar^\bullet_j:=\big\{\,\beta(\hbar^\bullet_{e;j})\ :\ e\in\mathfrak{E}_j\cap(\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\succeq}\,\big\}\quad\big(\subset\mathrm{HE}(\mathrm{E})\subset\mathrm{HE}(\gamma)\,\big)$$

and consider the sets $\hbar^\pm_j$ as new half-edges;

(3) next, we set

$$\begin{aligned}
&\mathsf{e}_i:=\{\hbar^+_i,\hbar^-_i\}\quad\forall\ i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}},\\
&\mathsf{v}'_i:=\{\hbar^-_i,\hbar^+_{i^\flat}\}\sqcup\bigsqcup_{e\in\mathfrak{E}^{\perp}_i\cap(\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\succeq}}\big(v(\beta(\hbar^-_e))\setminus(\hbar^-_i\sqcup\hbar^+_{i^\flat})\,\big)\quad\forall\ i\in(\!(\mathbf{m},0)\!)_{\mathfrak{t}},\\
&\mathsf{v}'_{\mathbf{m}}:=\{\hbar^-_{\mathbf{m}}\}\sqcup\bigsqcup_{e\in\mathfrak{E}^{\perp}_{\mathbf{m}}\cap(\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\succeq}}\big(v(\beta(\hbar^-_e))\setminus\hbar^-_{\mathbf{m}}\big),\qquad \mathsf{v}'_0:=\{\hbar^+_{0^\flat}\}\sqcup\bigsqcup_{e\in\mathfrak{E}^{\perp}_{0^\flat}\cap(\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\succeq}}\big(v(\beta(\hbar^+_e))\setminus\hbar^+_{0^\flat}\,\big)
\end{aligned}$$

and take $\rho'_{\gamma,[\mathfrak{t}]}$ to be the graph satisfying

$$\begin{aligned}
\mathrm{HE}\big(\rho'_{\gamma,[\mathfrak{t}]}\big)&=\mathrm{HE}\big(\mathrm{Edg}(\gamma)\backslash\beta((\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\succeq})\big)\sqcup\big\{\hbar^\pm_i:i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}}\big\},\\
\mathrm{Edg}\big(\rho'_{\gamma,[\mathfrak{t}]}\big)&=\big(\mathrm{Edg}(\gamma)\backslash\beta((\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\succeq})\big)\sqcup\big\{\mathsf{e}_i:\ i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}}\big\},\\
\mathrm{Ver}\big(\rho'_{\gamma,[\mathfrak{t}]}\big)&=\big(\,\mathrm{Ver}(\gamma)\setminus\{\,v(\beta(\hbar)):\hbar\in\mathrm{HE}((\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\succeq})\,\}\,\big)\sqcup\big\{\mathsf{v}'_i:\ i\in[\![\mathbf{m},0]\!]_{\mathfrak{t}}\big\};
\end{aligned}$$

(4) finally, we define $\rho_{\gamma,[\mathfrak{t}]}$ via edge contraction:

$$\rho_{\gamma,[\mathfrak{t}]}:=\big(\rho'_{\gamma,[\mathfrak{t}]}\big)_{\left(\beta(\,\mathfrak{E}^{\perp}_{\geqslant\mathbf{m}}\setminus(\mathfrak{E}^{\perp}_{\mathbf{m};\min})^{\succeq}\,)\right)}.$$

Obviously, the construction of $\rho_{\gamma,[\mathfrak{t}]}$ is independent of the choice of $\mathfrak{t}$ representing $[\mathfrak{t}]$. By Definition 3.1, we observe that if $\mathbf{m}=0$ (i.e. $\mathfrak{t}$ is edge-less), then $\rho_{\gamma,[\mathfrak{t}]}=\gamma$. If $\mathbf{m}<0$, then

$$\begin{aligned}
\mathrm{HE}\big(\rho_{\gamma,[\mathfrak{t}]}\big)&=\mathrm{HE}\big(\mathrm{Edg}(\gamma)\backslash\beta(\mathfrak{E}^{\perp}_{\geqslant\mathbf{m}})\big)\sqcup\big\{\hbar^\pm_i:i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}}\big\},\\
\mathrm{Edg}\big(\rho_{\gamma,[\mathfrak{t}]}\big)&=\big(\mathrm{Edg}(\gamma)\backslash\beta(\mathfrak{E}^{\perp}_{\geqslant\mathbf{m}})\big)\sqcup\big\{\mathsf{e}_i:\ i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}}\big\},\\
\mathrm{Ver}\big(\rho_{\gamma,[\mathfrak{t}]}\big)&=\big(\,\mathrm{Ver}(\gamma)\setminus\{\,v(\beta(\hbar)):\hbar\in\mathrm{HE}(\mathfrak{E}^{\perp}_{\geqslant\mathbf{m}})\,\}\,\big)\sqcup\big\{\mathsf{v}_i:\ i\in[\![\mathbf{m},0]\!]_{\mathfrak{t}}\big\}
\end{aligned}\tag{3.1}$$

where the vertices v_i are the images of v'_i, i.e. they are determined by

$$\mathsf{v}_0\ni\hbar^+_{0^\flat};\qquad \mathsf{v}_i\ni\hbar^-_i\quad\forall\ i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}}.\tag{3.2}$$

The new edges e_i and vertices v_i (if exist) are called the exceptional edges and vertices, respectively.

With notation as above, let

$$\Gamma^{\mathrm{der}}_{\Lambda}=\big\{\,\rho_{\gamma,[\mathfrak{t}]}\ :\ \gamma\in\Gamma,\ \mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma),\ [\mathfrak{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}\,\big\}\quad(\subset\mathbf{G}).\tag{3.3}$$

An example of derive graphs is illustrated in Figure 3.

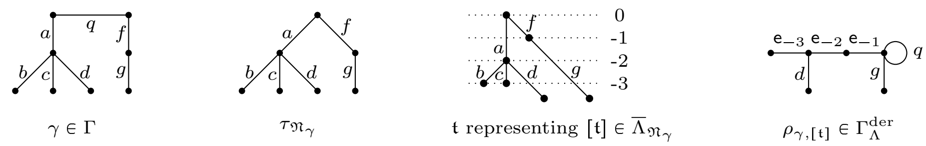


FIGURE 3. An example of derived graphs

Corollary 3.2. *Let* Γ, $\mathfrak{M}$, Λ, *and* $\mathfrak{M}^{\mathrm{tf}}$ *be as in Theorem 2.18. Assume that for every* $\gamma\in\Gamma$ *and every* $\mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma)$, *we have* $\mathrm{E}_{\mathfrak{N}_\gamma}=\mathrm{Edg}(\gamma')$ *for some connected subgraph* γ' *of* γ. *Then,*

$$\mathfrak{M}^{\mathrm{tf}} = \bigsqcup_{\gamma'\in\Gamma^{\mathrm{der}}_{\Lambda}} \mathfrak{M}^{\mathrm{tf}}_{\gamma'}, \qquad \text{where} \quad \mathfrak{M}^{\mathrm{tf}}_{\gamma'} = \bigsqcup_{\gamma\in\Gamma,\, \mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma),\, [\mathfrak{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma},\, \rho_{\gamma,[\mathfrak{t}]}=\gamma'} (\mathfrak{N}_\gamma)^{\mathrm{tf}}_{[\mathfrak{t}]}\,, \tag{3.4}$$

is a $\Gamma^{\mathrm{der}}_{\Lambda}$*-stratification of* $\mathfrak{M}^{\mathrm{tf}}$, *known as the* derived stratification of $\mathfrak{M}^{\mathrm{tf}}$ with respect to $(\mathfrak{M},\Gamma,\Lambda)$. *Moreover, for any* $\gamma'\in\Gamma^{\mathrm{der}}_{\Lambda}$, *the connected components of* $\mathfrak{M}^{\mathrm{tf}}_{\gamma'}$ *are given by the RHS of (3.4).*

Proof. It is direct to verify the statements using the sets

$$\underline{\mathfrak{U}}_{\gamma'} := \big\{ \mathfrak{U}_x : x\in\mathfrak{M}^{\mathrm{tf}}_{\gamma'} \big\}, \qquad \gamma'\in\Gamma^{\mathrm{der}}_{\Lambda}$$

of affine smooth charts of $\mathfrak{M}^{\mathrm{tf}}$, the subsets

$$\big\{\xi^{\mathfrak{U}_x}_{s} : i\in\mathbb{I}(\mathfrak{t})=[\![\mathbf{m},0]\!)_{\mathfrak{t}}\sqcup E_{\mathbf{m}}(\mathfrak{t})\sqcup E_{-}(\mathfrak{t})\big\}, \qquad \gamma'\in\Gamma^{\mathrm{der}}_{\Lambda}, \quad \mathfrak{U}_x\in\underline{\mathfrak{U}}_{\gamma'},$$

of local parameters, and the local equations of $(\mathfrak{N}_\gamma)^{\mathrm{tf}}_{[\mathfrak{t}]}$ as in Theorem 2.18 (p_2). The assumption that each $\tau_{\mathfrak{N}_\gamma}$ is identified with a *connected* subgraph of γ via the bijection $\beta_{\mathfrak{N}_\gamma}$ guarantees that for any derived graph $\rho_{\gamma,[\mathfrak{t}]}\in\Gamma^{\mathrm{der}}_{\Lambda}$, contracting any special edges e_i from $\rho_{\gamma,[\mathfrak{t}]}$ does not create graphs that are not in $\Gamma^{\mathrm{der}}_{\Lambda}$. □

3.2. **Grafted stratification.** For every graph γ and every vertex $v\in\mathrm{Ver}(\gamma)$, let $(\gamma,v)_{\mathrm{gft}}$ be the graph given by

$$\mathrm{HE}\big((\gamma,v)_{\mathrm{gft}}\big) = \mathrm{HE}(\gamma)\sqcup\{\hbar^+_{\mathrm{gft}},\hbar^-_{\mathrm{gft}}\}, \qquad \mathrm{Edg}\big((\gamma,v)_{\mathrm{gft}}\big) = \mathrm{Edg}(\gamma)\sqcup\big\{\, e_{\mathrm{gft}}:=\{\hbar^+_{\mathrm{gft}},\hbar^-_{\mathrm{gft}}\}\,\big\}$$
$$\mathrm{Ver}\big((\gamma,v)_{\mathrm{gft}}\big) = \big\{v\sqcup\{\hbar^+_{\mathrm{gft}}\}\big\}\sqcup\big\{v_{\mathrm{gft}}:=\{\hbar^-_{\mathrm{gft}}\}\big\}\sqcup\mathrm{Ver}(\gamma)\backslash\{v\}\,.$$

Intuitively, $(\gamma,v)_{\mathrm{gft}}$ is obtained from γ by *grafting* an extra vertex v_{gft} onto the chosen vertex v via an extra edge e_{gft}. If $\tau=(\tau,o)$ is a rooted tree, then we write $\tau_{\mathrm{gft}}:=(\tau,o)_{\mathrm{gft}}$.

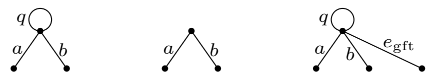
We continue with the setup in the first paragraph of §3.1. Recall that $\tau_\bullet$ denotes the single-vertex rooted tree.

Corollary 3.3. *Let* Γ, $\mathfrak{M}$, Λ *be as in Definition 2.16 so that* $(\mathfrak{M},\Gamma)$ *is endowed with a treelike structure* Λ, *and* $\Delta\subset\mathfrak{M}$ *be the boundary of* $\mathfrak{M}$ *(w.r.t.* Γ*). Assume that*

- *there exists* $K\subset\Delta(\subset\mathfrak{M})$ *satisfying that for every* $\gamma\in\Gamma\backslash\{\tau_\bullet\}$, $\mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma)$, *and* $\mathcal{V}\in\underline{\mathfrak{V}}_\gamma$, *there exists* $\kappa^{\mathcal{V}}\in\Gamma(\mathscr{O}_{\mathcal{V}})$ *such that* $\{\zeta^{\mathcal{V}}_e\}_{e\in\mathrm{Edg}(\gamma)}\sqcup\{\kappa^{\mathcal{V}}\}$ *is a subset of local parameters on* $\mathcal{V}$ *and*

$$K\cap\mathfrak{N}_{\gamma'}\cap\mathcal{V} \text{ is either } \{\kappa^{\mathcal{V}}=0\}\cap\mathfrak{N}_{\gamma'}\cap\mathcal{V} \text{ or } \varnothing \qquad \forall\,\gamma'\in\Gamma,\ \mathfrak{N}_{\gamma'}\in\pi_0(\mathfrak{M}_{\gamma'});$$

- *for every* $\gamma\in\Gamma$ *and every* $\mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma)$, *the bijection* $\beta_{\mathfrak{N}_\gamma}:\mathrm{HE}(\tau_{\mathfrak{N}_\gamma})\longrightarrow\mathrm{HE}(\mathrm{E}_{\mathfrak{N}_\gamma})\ \big(\subset\mathrm{HE}(\gamma)\big)$ *identifies* $\tau_{\mathfrak{N}_\gamma}$ *with a connected subgraph of* γ.



$\gamma\in\Gamma$ $\qquad\tau_{\mathfrak{N}_\gamma}$ $\qquad\gamma_{\mathrm{gft}}\in\Gamma_{\mathrm{gft}}$

FIGURE 4. An example of grafted graphs

Then, by setting

$$\begin{aligned}
&\Gamma_{\mathrm{gft}}=\Gamma_{\mathrm{gft}}(K):=\{\tau_\bullet\}\sqcup\big\{\,\big(\gamma,\,\beta_{\mathfrak{N}_\gamma}(o_{\mathfrak{N}_\gamma})\big)_{\mathrm{gft}}:\,\gamma\in\Gamma\backslash\{\tau_\bullet\},\ \mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma)\,\big\},\\
&(\mathfrak{N}_\gamma)_{\mathrm{gft}}:=K\cap\mathfrak{N}_\gamma\qquad\forall\,\gamma\in\Gamma,\ \mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma),\qquad\qquad(3.5)\\
&(\mathfrak{M}_{\mathrm{gft}})_{\gamma'}:=\bigsqcup_{(\gamma\,,\,\beta_{\mathfrak{N}_\gamma}(o_{\mathfrak{N}_\gamma}))_{\mathrm{gft}}=\gamma'}(\mathfrak{N}_\gamma)_{\mathrm{gft}}\quad\forall\,\gamma'\in\Gamma_{\mathrm{gft}}\backslash\{\tau_\bullet\},\qquad(\mathfrak{M}_{\mathrm{gft}})_{\tau_\bullet}:=\mathfrak{M}\backslash\Big(\bigsqcup_{\gamma'\in\Gamma_{\mathrm{gft}}\backslash\tau_\bullet}(\mathfrak{M}_{\mathrm{gft}})_{\gamma'}\Big),
\end{aligned}$$

we observe that

$$\mathfrak{M}=\bigsqcup_{\gamma'\in\Gamma_{\mathrm{gft}}}(\mathfrak{M}_{\mathrm{gft}})_{\gamma'}$$

is a Γ_{gft}-stratification of $\mathfrak{M}$, known as the grafted stratification *of $(\mathfrak{M},\Gamma)$ (with respect to K). Furthermore, if*

$$K\cap\mathfrak{N}_{\gamma'}\cap\mathcal{V}\;=\;\{\kappa^{\mathcal{V}}=0\}\cap\mathfrak{N}_{\gamma'}\cap\mathcal{V}\qquad\forall\,\gamma'\in\Gamma,\ \mathfrak{N}_{\gamma'}\in\pi_0(\mathfrak{M}_{\gamma'})\ \text{with}\ \tau_{\mathfrak{N}_{\gamma'}}\neq\tau_\bullet,\tag{3.6}$$

then the treelike structure Λ on $(\mathfrak{M},\Gamma)$ induces a treelike structure Λ_{gft} on $(\mathfrak{M},\Gamma_{\mathrm{gft}})$ satisfying for every $\mathfrak{N}\in\pi_0(\mathfrak{M}_{\mathrm{gft}})_{\tau_\bullet}$, we have

$$\mathcal{T}_{\mathfrak{N}}=(\tau_\bullet,\varnothing,-);$$

for every $\gamma'\in\Gamma_{\mathrm{gft}}\backslash\{\tau_\bullet\}$ and every $\mathfrak{N}=(\mathfrak{N}_\gamma)_{\mathrm{gft}}\in\pi_0\big((\mathfrak{M}_{\mathrm{gft}})_{\gamma'}\big)$ (which implies $\gamma\in\Gamma\backslash\{\tau_\bullet\}$ and $K\cap\mathfrak{N}_\gamma\neq\varnothing$), we have

$$\begin{gathered}
\mathcal{T}_{\mathfrak{N}}=\big(\tau_{\mathfrak{N}}=(\tau_{\mathfrak{N}_\gamma})_{\mathrm{gft}}\,,\,\mathrm{E}_{\mathfrak{N}}=\mathrm{E}_{\mathfrak{N}_\gamma}\sqcup\{e_{\mathrm{gft}}\}\,,\,\beta_{\mathfrak{N}}\colon\mathrm{Edg}\big((\tau_{\mathfrak{N}_\gamma})_{\mathrm{gft}}\big)\longrightarrow\mathrm{E}_{\mathfrak{N}_\gamma}\sqcup\{e_{\mathrm{gft}}\}\big),\\
\text{where}\quad\beta_{\mathfrak{N}}|_{\mathrm{Edg}(\tau_{\mathfrak{N}_\gamma})}=\beta_{\mathfrak{N}_\gamma},\ \ \beta_{\mathfrak{N}}(e_{\mathrm{gft}})=e_{\mathrm{gft}}.
\end{gathered}$$

An example of grafted graphs is illustrated in Figure 4.

Proof of Corollary 3.3. For every $\gamma'\in\Gamma_{\mathrm{gft}}\backslash\{\tau_\bullet\}$, every nonempty $(\mathfrak{N}_\gamma)_{\mathrm{gft}}\in\pi_0\big((\mathfrak{M}_{\mathrm{gft}})_{\gamma'}\big)$, and every $x\in(\mathfrak{N}_\gamma)_{\mathrm{gft}}$, we have $x\in\mathfrak{N}_\gamma$, thus for any chart $\mathcal{V}\in\underline{\mathfrak{V}}_\gamma$ containing x, we take the subset of local parameters

$$\zeta_e^{\mathcal{V}},\quad e\in\mathrm{Edg}(\gamma)\qquad\text{and}\qquad\zeta_{e_{\mathrm{gft}}}^{\mathcal{V}}:=\kappa^{\mathcal{V}}.$$

Using these charts and local parameters, we see that the last equation in (3.5) gives a Γ_{gft}-stratification of $\mathfrak{M}$ satisfying Definition 2.13. The connected components of each $(\mathfrak{M}_{\mathrm{gft}})_{\gamma'}$ are described in the first equation of the last line of (3.5).

It remains to show that Λ_{gft} is a treelike structure on $(\mathfrak{M},\Gamma_{\mathrm{gft}})$ if (3.6) holds. Notice that for every $\gamma'\in\Gamma_{\mathrm{gft}}$ and every $\mathfrak{N}\in\pi_0((\mathfrak{M}_{\mathrm{gft}})_{\gamma'})$ with $\tau_{\mathfrak{N}}\neq\tau_\bullet$, the edge e_{gft} is both maximal and minimal in $\mathrm{Edg}(\tau_{\mathfrak{N}})$. Thus, for every $E\in\mathrm{Edg}(\gamma')$ containing e_{gft} such that $\widetilde{\gamma}:=\gamma'_{(E)}$ belongs to Γ_{gft}, the first line of (3.5) implies $\widetilde{\gamma}=\tau_\bullet$, hence $\tau_{\mathfrak{N}_{\widetilde{\gamma}}}=\tau_\bullet$ for any $\mathfrak{N}_{\widetilde{\gamma}}\in\pi_0(\mathfrak{M}_{\widetilde{\gamma}})$. This confirms the condition (b_2) of Definition 2.16. The conditions (a) and (b_1) of Definition 2.16 follow directly from the assumption that Λ is a treelike structure on $(\mathfrak{M},\Gamma)$. $\square$

4. Applications of the STF theory to $\mathfrak{P}_2$ and $\overline{M}_2(\mathbb{P}^n,d)$

Let $\mathfrak{P}_g$ be the genus g relative Picard stack of the stable pairs (C,L) where C is a genus g nodal curve and L is a line bundle over C. A pair (C,L) is said to be stable if every rational irreducible component $\Sigma\subset C$ with L/Σ as a trivial line bundle contains at least three nodal points. It is a well-known fact that $\mathfrak{P}_g$ is a smooth algebraic stack.

In §4.2-4.9, we apply the STF theory to $\mathfrak{P}_2$ recursively in eight times and obtain $\widetilde{\mathfrak{P}}_2^{\mathrm{tf}}$, which in turn gives rise to the resolution $\widetilde{M}_2^{\mathrm{tf}}(\mathbb{P}^n,d)$ of $\overline{M}_2(\mathbb{P}^n,d)$ as in Theorem 1.2. The proof of the properties (1)-(4) of Theorem 1.2 is provided in §4.10.

As stated in §1, the stack $\widetilde{\mathfrak{P}}_2^{\mathrm{tf}}$ may not be isomorphic to the blowup stack $\widetilde{\mathfrak{P}}_2$ of [9]. The eight-step construction of $\widetilde{\mathfrak{P}}_2^{\mathrm{tf}}$ in §4 corresponds to the blowup construction of [9] as follows:

$$\underbrace{\mathsf{r}_1\mathsf{p}_1}_{\S4.2}\to\underbrace{\mathsf{r}_1\mathsf{p}_2}_{\S4.3}\to\underbrace{\big(\mathsf{r}_1\mathsf{p}_4\to\mathsf{r}_1\mathsf{p}_3\big)}_{\S4.4}\to\underbrace{\mathsf{r}_1\mathsf{p}_5}_{\S4.5}\to\underbrace{\mathsf{r}_2}_{\S4.6}\to\underbrace{\big(\mathsf{r}_3\mathsf{p}_2\to\mathsf{r}_3\mathsf{p}_1\big)}_{\S4.7}\to\underbrace{\mathsf{r}_3\mathsf{p}_3}_{\S4.8}\to\underbrace{\mathsf{r}_3\mathsf{p}_4}_{\S4.9}.$$

In fact, several blowups in [9] can be performed in various orders and should generally give rise to different resolutions of $\overline{M}_2(\mathbb{P}^n,d)$.

4.1. **Notation.** The following notation and terminology will be used repeatedly in the recursive construction.

Definition 4.1. *We call a tuple*

$$\gamma^\star=\big(\gamma,\ p_g\colon \mathrm{Ver}(\gamma)\longrightarrow\mathbb{Z}_{\geqslant0},\ \mathbf{w}\colon \mathrm{Ver}(\gamma)\longrightarrow\mathbb{Z}_{\geqslant0}\big)$$

of a connected graph γ and two functions p_g and $\mathbf{w}$ a connected decorated graph, or simply a decorated graph when the context is clear. The arithmetic genus p_a of a decorated graph is

$$p_a(\gamma,p_g,\mathbf{w}):=b_1(\gamma)+\sum_{v\in\mathrm{Ver}(\gamma)}p_g(v).$$

We denote by $\mathbf{G}^\star$ the set of all connected decorated graphs. For $k\in\mathbb{Z}_{\geqslant0}$, let

$$\mathbf{G}^\star_k:=\{\gamma^\star\in\mathbf{G}^\star:\ p_a(\gamma^\star)=k\}.$$

With $\gamma^\star$ as above, for every subgraph γ' of γ, we write

$$\mathbf{w}(\gamma'):=\sum_{v\in\mathrm{Ver}(\gamma')}\mathbf{w}(v). \tag{4.1}$$

Here we identify $v\in\mathrm{Ver}(\gamma')$ with its image in γ; see Definition 2.2.

We say two decorated graphs $\gamma^\star$ and $(\gamma')^\star$ are isomorphic and write

$$\gamma^\star\simeq(\gamma')^\star$$

if $\gamma\simeq\gamma'$ and the graph isomorphism is compatible with the corresponding p_g, p_g', $\mathbf{w}$, and $\mathbf{w}'$.

Given $\gamma\in\mathbf{G}$ and $E\subset\mathrm{Edg}(\gamma)$, recall that $\gamma_{(E)}$ denotes the graph obtained from γ via edge contraction; see Definition 2.5. The subset E determines a subgraph γ' of γ with $\mathrm{Edg}(\gamma')=E$. Let $\{\gamma_1',\ldots,\gamma_k'\}$ be the set of the connected components of γ', which are contracted to distinct vertices $\overline{v}_1,\ldots,\overline{v}_k$ in $\gamma_{(E)}$. Thus, there exists a surjection

$$\pi_{\mathrm{ver};(E)}:\ \mathrm{Ver}(\gamma)\twoheadrightarrow\mathrm{Ver}\big(\gamma_{(E)}\big),\quad v\mapsto\begin{cases}v & \text{if } v\notin\mathrm{Ver}(\gamma');\\ \overline{v}_i & \text{if } v\in\mathrm{Ver}(\gamma_i'),\ i=1,\ldots,k.\end{cases}$$

Definition 4.2. *Every decoration $\gamma^\star$ of γ determines a decoration $\gamma^\star_{(E)}$ of $\gamma_{(E)}$ by setting*

$$p_g(\overline{v}_i) = b_1(\gamma'_i) + \sum_{v\,\in\,\pi^{-1}_{\mathrm{ver};(E)}(\overline{v}_i)} p_g(v), \qquad \mathbf{w}(\overline{v}_i) = \sum_{v\,\in\,\pi^{-1}_{\mathrm{ver};(E)}(\overline{v}_i)} \mathbf{w}(v), \qquad \forall\ i=1,\ldots,k,$$

which is called the induced decoration *of $\gamma_{(E)}$.*

Definition 4.3. *Let $\gamma^\star$ be a decorated graph. The* core *$\gamma^\star_{\mathrm{cor}}$ is the smallest connected subgraph of $\gamma^\star$ with $p_a(\gamma^\star_{\mathrm{cor}})=p_a(\gamma^\star)$, together with the restrictions of p_g and $\mathbf{w}$ to $\mathrm{Ver}(\gamma')$.*

Let Γ, $\mathfrak{M}$, and Λ be as in Definition 2.16. Given $\gamma\in\Gamma$, $\mathfrak{N}_\gamma\in\pi_0(\mathfrak{M}_\gamma)$, and $[\mathfrak{t}]\in\overline{\Lambda}_{\mathfrak{N}_\gamma}$, let $\rho_{\gamma,[\mathfrak{t}]}$ be the derived graph as in Definition 3.1.

Definition 4.4. *A decoration $\gamma^\star$ of γ determines a decoration $\rho^\star_{\gamma,[\mathfrak{t}]}$ of $\rho_{\gamma,[\mathfrak{t}]}$, called the* induced decoration *of $\rho_{\gamma,[\mathfrak{t}]}$, such that:*

- *if $\mathbf{m}(\mathfrak{t})=0$ (i.e. $\mathfrak{t}$ is edge-less), then $\rho_{\gamma,[\mathfrak{t}]}=\gamma$, so we set $\rho^\star_{\gamma,[\mathfrak{t}]}=\gamma^\star$;*
- *if $\mathbf{m}(\mathfrak{t})<0$, then with $\mathsf{v}_i\in\mathrm{Ver}(\rho_{\gamma,[\mathfrak{t}]})$, $i\in[\![\mathbf{m},0]\!]_{\mathfrak{t}}$, denoting the exceptional vertices as in (3.2), let*

$$p_g(\mathsf{v}_0) = \sum_{v\in\{v(\beta(\hbar)):\,\hbar\in\mathrm{HE}(\mathfrak{E}^\perp_{\geqslant\mathbf{m}})\}} p_g(v)\,, \qquad \mathbf{w}(\mathsf{v}_0) = \sum_{v\in\{v(\beta(\hbar)):\,\hbar\in\mathrm{HE}(\mathfrak{E}^\perp_{\geqslant\mathbf{m}})\}} \mathbf{w}(v)\,, \qquad p_g(\mathsf{v}_i)=\mathbf{w}(\mathsf{v}_i)=0 \quad \forall\ i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}}\,,$$

and let the restrictions of p_g and $\mathbf{w}$ to $\mathrm{Ver}(\rho_{\gamma,[\mathfrak{t}]})\backslash\{\mathsf{v}_i : i\in[\![\mathbf{m},0]\!]_{\mathfrak{t}}\}$ be determined by $\gamma^\star$.

Definition 4.5. *Let C be a nodal curve. The* dual graph *γ_C of C is the graph whose vertices v and edges e correspond to the irreducible components C_v and the nodes q_e of C, respectively.*

Definition 4.6. *Given $(C,L)\in\mathfrak{P}_g$, its* decorated dual graph *is a tuple*

$$\gamma^\star_{C,L} := \big(\ \gamma_C\ ,\ p_g\colon\mathrm{Ver}(\gamma_C)\longrightarrow\mathbb{Z}_{\geqslant 0}\ ,\ \mathbf{w}\colon\mathrm{Ver}(\gamma)\longrightarrow\mathbb{Z}_{\geqslant 0}\ \big)$$

such that p_g and $\mathbf{w}$ assign to each $v\in\mathrm{Ver}(\gamma_C)$ the geometric genus of the irreducible component C_v and the degree of L/C_v, respectively.

4.2. **The first step of the recursive construction.** The initial package of our recursive construction is given by

$$\mathfrak{M}^1 = \mathfrak{P}_2, \qquad \Gamma^1 = \{\,[\gamma] :\ \gamma\in\mathbf{G},\ b_1(\gamma)\leqslant 2\,\},$$

where the first Betti number $b_1(\gamma)$ of a connected graph γ is described in Definition 2.8 and the equivalence class is given by the graph isomorphism (see the paragraph before Definition 2.13). As explained in Remark 2.14, it is convenient to fix a representative of each element of Γ^1 and write $\gamma\in\Gamma^1$ instead of $[\gamma]\in\Gamma^1$.

For any $(C,L)\in\mathfrak{M}^1$, we take an affine smooth chart

$$(C,L)\in\mathcal{V}_{C,L}\longrightarrow\mathfrak{M}^1.$$

As in [8, §4.3] and [10, §3.1], there exists a set of regular functions $\{\zeta^{\mathcal{V}}_e\}_{e\in\mathrm{Edg}(\gamma_C)}$ so that for each $e\in\mathrm{Edg}(\gamma_C)$, the locus

$$\{\,\zeta^{\mathcal{V}}_e = 0\,\}\subset\mathcal{V}_{C,L} \tag{4.2}$$

is where the node labeled by e is not smoothed out.

Lemma 4.7. *With notation as above, there exists a Γ^1-stratification of $\mathfrak{M}^1$ given by*

$$\mathfrak{M}^1 = \bigsqcup_{\gamma\in\Gamma^1}\mathfrak{M}^1_\gamma := \bigsqcup_{\gamma\in\Gamma^1}\{\,(C,L)\in\mathfrak{M}^1 :\ \gamma_C\simeq\gamma\,\}.$$

Proof. Consider the aforementioned charts of $\mathfrak{M}^1$ and the subsets of local parameters:

$$\underline{\mathfrak{V}}_\gamma := \big\{\mathcal{V}_{C,L} : \gamma_C \simeq \gamma\big\}, \quad \gamma\in\Gamma^1; \qquad \{\zeta_e^{\mathcal{V}} : e\in\mathrm{Edg}(\gamma)\}, \quad \gamma\in\Gamma^1,\ \mathcal{V}\in\underline{\mathfrak{V}}_\gamma, \tag{4.3}$$

respectively. By the deformation of the nodal curves, we know that

$$\mathrm{Cl}_{\mathfrak{M}^1}(\mathfrak{M}^1_\gamma) = \bigsqcup_{\gamma_1\in\Gamma^1,\,\gamma_1\preceq\gamma} \mathfrak{M}^1_{\gamma_1} \qquad \forall\ \gamma\in\Gamma^1.$$

Notice that $\gamma_1\preceq\gamma$ above is also up to graph isomorphism. Therefore,

$$\bigsqcup_{\gamma\in\Gamma^1,\,\gamma\not\succeq\gamma'} \mathfrak{M}^1_\gamma = \bigcup_{\gamma\in\Gamma^1,\,\gamma\not\succeq\gamma'} \mathrm{Cl}_{\mathfrak{M}^1}(\mathfrak{M}^1_\gamma) \qquad \forall\ \gamma'\in\Gamma^1. \tag{4.4}$$

By the deformation of $(C,L)\in\mathfrak{M}^1=\mathfrak{P}_2$, we also know that $\sum_{v\in\mathrm{Ver}(\gamma_C)} c_1(L/C_v)$ is constant within each connected component of $\mathfrak{P}_2$. Taking the stability of $\mathfrak{P}_2$ into account, we see that for every $\mathfrak{N}\in\pi_0(\mathfrak{P}_2)$, there exist only finitely many $\gamma\in\Gamma^1$ such that $\mathfrak{M}^1_\gamma\cap\mathfrak{N}\neq\varnothing$. Thus, the RHS of (4.4) is a finite union in each connected component of $\mathfrak{M}^1$, hence

$$\bigsqcup_{\gamma\in\Gamma^1,\,\gamma\not\succeq\gamma'} \mathfrak{M}^1_\gamma \ \text{ is closed in } \mathfrak{M}^1.$$

Consequently, we may shrink the charts in (4.3) if necessary so that

$$\mathfrak{M}^1_\gamma \cap \mathcal{V}' = \varnothing \qquad \forall\ \gamma,\gamma'\in\Gamma^1 \text{ with } \gamma\not\succeq\gamma',\ \forall\ \mathcal{V}'\in\underline{\mathfrak{V}}_{\gamma'}. \tag{4.5}$$

In addition, since the local parameters $\zeta_e^{\mathcal{V}}$ correspond to the smoothing of the nodes, we conclude that for every $\gamma,\gamma'\in\Gamma^1$ with $\gamma\succeq\gamma'$ and every chart $\mathcal{V}'\in\underline{\mathfrak{V}}_{\gamma'}$,

$$\mathfrak{M}^1_\gamma \cap \mathcal{V}' = \bigcup_{E\subset\mathrm{Edg}(\gamma'),\ \gamma'_{(E)}\simeq\gamma} \big\{\, \zeta_e^{\mathcal{V}}=0\ \ \forall\, e\in\mathrm{Edg}(\gamma')\backslash E\,,\ \ \zeta_e^{\mathcal{V}}\neq 0\ \ \forall\, e\in E\,\big\}. \tag{4.6}$$

Lemma 4.7 then follows from (4.5), (4.6), and Definition 2.13. □

One can mimic Example 2.17 and construct a treelike structure Λ' on $(\mathfrak{M}^1,\Gamma^1)$ likewise, which gives rise to a stack with twisted fields $\mathfrak{M}^1_{\Lambda'}$. However, such $(\mathfrak{M}^1,\Gamma^1,\Lambda')$ does not satisfy the hypothesis of Corollary 3.2, so the derived stratification of $\mathfrak{M}^1_{\Lambda'}$ with respect to $(\mathfrak{M}^1,\Gamma^1,\Lambda')$ does not exist, and the proposed recursive construction cannot proceed further. Thus, we introduce a slightly different stratification on $\mathfrak{M}^1$ as follows.

For every decoration $\gamma^\star$ of $\gamma\in\Gamma^1$, we denote by

$$\gamma_{\mathrm{vic}} = \gamma^{\mathrm{id}}_{\mathrm{Ver}(\gamma^\star_{\mathrm{cor}})} \in \mathbf{G}$$

the (connected) graph obtained from γ by applying the *vertex identification* to $\mathrm{Ver}(\gamma^\star_{\mathrm{cor}})$; see Definitions 2.7 and 4.3 for terminology. The image of the vertices of $\gamma^\star_{\mathrm{cor}}$ in γ_{vic} is denoted by $o_{\gamma_{\mathrm{vic}}}$. The decoration $\gamma^\star$ of γ further induces a (unique) decoration $\gamma^\star_{\mathrm{vic}}$ of γ_{vic} by setting

$$p_g(o_{\gamma_{\mathrm{vic}}}) = \sum_{v\in\mathrm{Ver}(\gamma^\star_{\mathrm{cor}})} p_g(v) \qquad \text{and} \qquad \mathbf{w}(o_{\gamma_{\mathrm{vic}}}) = \sum_{v\in\mathrm{Ver}(\gamma^\star_{\mathrm{cor}})} \mathbf{w}(v) \tag{4.7}$$

while leaving the decorations of the vertices of $\mathrm{Ver}(\gamma^\star_{\mathrm{vic}})\backslash\{o_{\gamma_{\mathrm{vic}}}\}=\mathrm{Ver}(\gamma)\backslash\mathrm{Ver}(\gamma^\star_{\mathrm{cor}})$ unchanged.

The above description of γ_{vic} implies

$$\mathrm{Edg}(\gamma_{\mathrm{vic}})=\mathrm{Edg}(\gamma); \quad \mathrm{Edg}\big((\gamma^\star_{\mathrm{vic}})_{\mathrm{cor}}\big)=\mathrm{Edg}(\gamma^\star_{\mathrm{cor}}); \quad (\gamma_{(E)})_{\mathrm{vic}}=(\gamma_{\mathrm{vic}})_{(E)}\ \forall\, E\subset\mathrm{Edg}(\gamma). \tag{4.8}$$

Let

$$\Gamma^1_{\mathrm{vic}} = \{\, \gamma_{\mathrm{vic}} :\ \gamma^\star=(\gamma,p_g,\mathbf{w})\in\mathbf{G}^\star_2,\ \gamma\in\Gamma^1\,\}.$$

The following statement follows immediately from (4.8) and Lemma 4.7.

Corollary 4.8. *The Γ^1-stratification of $\mathfrak{M}^1$ in Lemma 4.7 determines a Γ^1_{vic}-stratification:*

$$\mathfrak{M}^1 = \bigsqcup_{\gamma'\in\Gamma^1_{\mathrm{vic}}} \mathfrak{M}^1_{\mathrm{vic};\gamma'} := \bigsqcup_{\gamma'\in\Gamma^1_{\mathrm{vic}}} \left\{ (C,L)\in\mathfrak{M}^1 : (\gamma^\star_{C,L})_{\mathrm{vic}} \simeq \gamma' \right\}.$$

For every $\gamma^\star\in\mathbf{G}^\star_2$, let

$$\mathfrak{N}^1_{\gamma^\star} := \left\{(C,L)\in\mathfrak{M}^1 : \gamma^\star_{C,L}\simeq\gamma^\star\right\} \subset \mathfrak{M}^1.$$

Then, for every $\gamma'\in\Gamma^1_{\mathrm{vic}}$, we have

$$\pi_0(\mathfrak{M}^1_{\mathrm{vic};\gamma'}) = \left\{\mathfrak{N}^1_{\gamma^\star} :\ \gamma^\star\in\mathbf{G}^\star_2,\ \gamma_{\mathrm{vic}}\simeq\gamma'\right\}.$$

Fix a graph $\gamma\in\Gamma^1$ and a decoration $\gamma^\star$ of γ. Let

$$\mathfrak{N}^1 := \mathfrak{N}^1_{\gamma^\star} \in \pi_0(\mathfrak{M}^1_{\mathrm{vic};\gamma_{\mathrm{vic}}}). \tag{4.9}$$

We will describe the tuple $(\tau_{\mathfrak{N}^1}, \mathrm{E}_{\mathfrak{N}^1}, \beta_{\mathfrak{N}^1})$ to provide the proposed treelike structure Λ^1 on $(\mathfrak{M}^1,\Gamma^1_{\mathrm{vic}})$. Let $\gamma^\star_{\mathrm{cor}}$ be the core of $\gamma^\star$ as in Definition 4.3 and $\mathbf{w}(\gamma^\star_{\mathrm{cor}})$ be as in (4.1).

- If $\mathbf{w}(\gamma^\star_{\mathrm{cor}})>0$, we set $\tau_{\mathfrak{N}^1}=\tau_\bullet$ and $o_{\mathfrak{N}^1}=o_{\gamma_{\mathrm{vic}}}$.
- If $\mathbf{w}(\gamma^\star_{\mathrm{cor}})=0$, let

  $$\gamma_{\mathrm{ctr}} = (\gamma_{\mathrm{vic}})_{\left(\mathrm{Edg}((\gamma^\star_{\mathrm{vic}})_{\mathrm{cor}})\right)} = \gamma_{(\mathrm{Edg}(\gamma^\star_{\mathrm{cor}}))}$$

  be the graph obtained via edge contraction; the second equality above follows from (4.8). We then dissolve all $v\in\mathrm{Ver}(\gamma_{\mathrm{ctr}})$ satisfying $\mathbf{w}(v)>0$ (see Definition 2.6 for terminology) and take $\tau_{\mathfrak{N}^1}$ to be the unique connected component that contains $o_{\gamma_{\mathrm{vic}}}$. Thus, $\tau_{\mathfrak{N}^1}$ is a rooted tree with the root $o_{\gamma_{\mathrm{vic}}}$.

Since $\tau_{\mathfrak{N}^1}$ is obtained via contraction of the edges in $\gamma^\star_{\mathrm{cor}}$, dissolution of vertices not in $\gamma^\star_{\mathrm{cor}}$, and/or taking a connected component of a graph, we conclude that $\tau_{\mathfrak{N}^1}$ is a *connected* subgraph of γ_{vic}. In particular,

$$\mathrm{Edg}(\tau_{\mathfrak{N}^1}) \subset \mathrm{Edg}(\gamma)\backslash\mathrm{Edg}(\gamma^\star_{\mathrm{cor}}) = \mathrm{Edg}(\gamma_{\mathrm{vic}})\backslash\mathrm{Edg}\big((\gamma^\star_{\mathrm{vic}})_{\mathrm{cor}}\big). \tag{4.10}$$

Let

$$\Lambda^1 = \Big((\tau_{\mathfrak{N}^1},\, \mathrm{E}_{\mathfrak{N}^1} := \mathrm{Edg}(\tau_{\mathfrak{N}^1}),\, \mathrm{Id}) \Big)_{\gamma'\in\Gamma^1_{\mathrm{vic}},\ \mathfrak{N}^1\in\pi_0(\mathfrak{M}^1_{\mathrm{vic};\gamma'})}.$$

Lemma 4.9. *The set Λ^1 give a treelike structure on $(\mathfrak{M}^1,\Gamma^1_{\mathrm{vic}})$ as in Definition 2.16.*

Proof. The proof is parallel to its genus 1 counterpart that is studied in Example 2.17, with $\mathfrak{M}^{\mathrm{wt}}_1$ replaced by $\mathfrak{P}_2$. The key point is that the positively weighted vertices of $\tau_{\mathfrak{N}^1}$ are and only are the minimal vertices (i.e. leaves), which is the same as in the genus 1 case. We omit further details. □

Corollary 4.8, Lemma 4.9, and Theorem 2.18 together imply the following:

Corollary 4.10. *Let $(\mathfrak{M}^1)^{\mathrm{tf}}_{\Lambda^1}$ be constructed as in Theorem 2.18 (p_1). Then, $(\mathfrak{M}^1)^{\mathrm{tf}}_{\Lambda^1}$ is a smooth algebraic stack and the forgetful morphism $(\mathfrak{M}^1)^{\mathrm{tf}}_{\Lambda^1}\longrightarrow\mathfrak{M}^1=\mathfrak{P}_2$ is proper and birational.*

4.3. **The second step of the recursive construction.** In this step, we take

$$\mathfrak{M}^2 = (\mathfrak{M}^1)^{\mathrm{tf}}_{\Lambda^1}, \qquad \Gamma^2_{\mathrm{vic}} = (\Gamma^1_{\mathrm{vic}})^{\mathrm{der}}_{\Lambda^1},$$

where $(\Gamma^1_{\mathrm{vic}})^{\mathrm{der}}_{\Lambda^1}$ is the set of the derived graphs of Γ^1_{vic} with respect to the rooted level trees given by the treelike structure Λ^1; see (3.3). Since for every $\gamma'\in\Gamma^1_{\mathrm{vic}}$ and every $\mathfrak{N}^1\in\pi_0(\mathfrak{M}^1_{\mathrm{vic};\gamma'})$, the rooted tree $\tau_{\mathfrak{N}^1}$ is a connected subgraph of γ', Corollary 3.2 thus gives rise to the following statement.

Lemma 4.11. *The stack $\mathfrak{M}^2$ comes equipped with a Γ^2_{vic}-stratification, which is the derived stratification of $\mathfrak{M}^2$ with respect to $(\mathfrak{M}^1,\Gamma^1_{\mathrm{vic}},\Lambda^1)$.*

The graphs of Γ^2_{vic} are in the form

$$\gamma^2 := \rho_{\gamma_{\mathrm{vic}},[\mathfrak{t}^1]} \in \Gamma^2_{\mathrm{vic}}, \qquad \text{where} \;\; \gamma_{\mathrm{vic}} \in \Gamma^1_{\mathrm{vic}}, \;\; [\mathfrak{t}^1] \in \overline{\Lambda}^1_{\mathfrak{N}^1}, \;\; \mathfrak{N}^1 \in \pi_0(\mathfrak{M}^1_{\mathrm{vic};\gamma_{\mathrm{vic}}}); \tag{4.11}$$

see (4.9) for $\mathfrak{N}^1$. The vertex that is the image of $o_{\gamma_{\mathrm{vic}}}$ in γ^2 is denoted by o_{γ^2}. The decoration $\gamma^\star_{\mathrm{vic}}$ of γ_{vic} induces the decoration $(\gamma^2)^\star$ of γ^2 as in Definition 4.4. By Corollary 3.2, the connected components of $\mathfrak{M}^2_{\gamma^2}$ are in the form

$$\mathfrak{N}^2 := (\mathfrak{N}^1)^{\mathrm{tf}}_{[\mathfrak{t}^1]} \;\in \pi_0\big(\mathfrak{M}^2_{\gamma^2}\big). \tag{4.12}$$

For each $\mathfrak{N}^2$ as above, we will describe the tuple $(\tau_{\mathfrak{N}^2}, \mathrm{E}_{\mathfrak{N}^2}, \beta_{\mathfrak{N}^2})$ to provide the proposed treelike structure Λ^2 on $(\mathfrak{M}^2, \Gamma^2_{\mathrm{vic}})$.

Let

$$\begin{aligned} \mathrm{E}^1 &:= \mathfrak{E}^{\perp}_{\geqslant \mathbf{m}}(\mathfrak{t}^1) \subset \mathrm{Edg}(\tau_{\mathfrak{N}^1}) = \mathrm{E}_{\mathfrak{N}^1} \\ &\subset \mathrm{Edg}(\gamma) \backslash \mathrm{Edg}(\gamma^\star_{\mathrm{cor}}) = \mathrm{Edg}(\gamma_{\mathrm{vic}}) \backslash \mathrm{Edg}\big((\gamma^\star_{\mathrm{vic}})_{\mathrm{cor}}\big), \end{aligned} \tag{4.13}$$

which follows from (4.10). Consider the graph $\gamma_{(\mathrm{E}^1)}$ obtained from γ via edge contraction, which is endowed with the induced decoration $\gamma^\star_{(\mathrm{E}^1)}$ from $\gamma^\star$ as in Definition 4.2. Then,

$$\mathrm{Edg}(\gamma_{(\mathrm{E}^1)}) \subset \mathrm{Edg}(\gamma^2), \quad \mathrm{Edg}\big(\gamma^\star_{(\mathrm{E}^1);\mathrm{cor}}\big) = \mathrm{Edg}\big((\gamma^2)^\star_{\mathrm{cor}}\big), \quad \mathbf{w}\big(\gamma^\star_{(\mathrm{E}^1);\mathrm{cor}}\big) = \mathbf{w}(o_{\gamma^2}) \geqslant 1. \tag{4.14}$$

The first two relations follow from (3.1) and (4.13), and the last equality follows from the choice of $\mathbf{m}(\mathfrak{t}^1)$ and Definitions 4.2 and 4.4. Further, by the construction of $\tau_{\mathfrak{N}^1}$ in §4.2, we see that $\gamma^\star_{(\mathrm{E}^1)}$ contains *at most* one pair of the smallest disjoint connected decorated subgraphs $\gamma^\star_{(\mathrm{E}^1);\pm}$ satisfying

$$p_a\big(\gamma^\star_{(\mathrm{E}^1);\pm}\big) = 1, \qquad \mathbf{w}\big(\gamma^\star_{(\mathrm{E}^1);\pm}\big) = 0. \tag{4.15}$$

We are ready to construct $\tau_{\mathfrak{N}^2}$.

- If $\gamma^\star_{(\mathrm{E}^1)}$ does not contain two *disjoint* connected decorated subgraphs $\gamma^\star_{(\mathrm{E}^1);\pm}$ satisfying (4.15), then we set $\tau_{\mathfrak{N}^2} = \tau_\bullet$ and $o_{\mathfrak{N}^2} = o_{\gamma^2}$.
- If $\gamma^\star_{(\mathrm{E}^1)}$ contains a unique pair of disjoint connected decorated subgraphs $\gamma^\star_{(\mathrm{E}^1);\pm}$ satisfying (4.15), then they are decorated subgraphs of $\gamma^\star_{(\mathrm{E}^1);\mathrm{cor}}$, and the (shortest) path in $\gamma_{(\mathrm{E}^1)}$ connecting $\gamma^\star_{(\mathrm{E}^1);+}$ and $\gamma^\star_{(\mathrm{E}^1);-}$ must contain a vertex with $\mathbf{w}(v) > 0$ by the last inequality in (4.14).
  - First, let $\gamma_{(\mathrm{E}^1);\mathrm{ctr}}$ be the graph obtained from $\gamma_{(\mathrm{E}^1)}$ by contracting the edges of $\gamma^\star_{(\mathrm{E}^1);\pm}$ into two distinct vertices $o_{\mathrm{ctr},\pm}$, respectively;
  - then, let $\widehat{\gamma}_{(\mathrm{E}^1);\mathrm{ctr}}$ be the graph obtained from $\gamma_{(\mathrm{E}^1);\mathrm{ctr}}$ by identifying the vertices $o_{\mathrm{ctr},+}$ and $o_{\mathrm{ctr},-}$ as a same vertex $\widehat{o}_{\mathrm{ctr}}$;
  - finally, we dissolve all $v \in \mathrm{Ver}(\widehat{\gamma}_{(\mathrm{E}^1);\mathrm{ctr}})$ satisfying $\mathbf{w}(v) > 0$ and take $\tau_{\mathfrak{N}^2}$ to be the unique connected component that contains $\widehat{o}_{\mathrm{ctr}}$.

  In this way, we see that $\tau_{\mathfrak{N}^2}$ is a rooted tree with the root $o_{\mathfrak{N}^2} = \widehat{o}_{\mathrm{ctr}}$.

Notice that $\tau_{\mathfrak{N}^2}$ is obtained via contraction of the edges of $(\gamma^2)^\star_{\mathrm{cor}}$ by (4.14), identification of vertices of $\gamma_{(\mathrm{E}^1);\mathrm{cor}}$, vertex dissolution, and taking a connected component of a graph, thus

$$\mathrm{Edg}(\tau_{\mathfrak{N}^2}) \;\subset\; \mathrm{Edg}(\gamma_{(\mathrm{E}^1)}) = \mathrm{Edg}(\gamma^2) \backslash \{\, \mathrm{e}^1_i : i \in [\![\mathbf{m}, 0]\!)_{\mathfrak{t}^1} \} \;\subset\; \mathrm{Edg}(\gamma^2), \tag{4.16}$$

and moreover, $\mathrm{Edg}(\tau_{\mathfrak{N}^2})$ is the set of the edges of a *connected* subgraph $\tau'_{\mathfrak{N}^2}$ of γ^2 that satisfies

$$o_{\mathfrak{N}^2} \subset o_{\gamma^2} \in \mathrm{Ver}(\tau'_{\mathfrak{N}^2})$$

(recall that every vertex is a set of half-edges). We set

$$\Lambda^2 = \Big(\big(\tau_{\mathfrak{N}^2},\, \mathrm{E}_{\mathfrak{N}^2} := \mathrm{Edg}(\tau_{\mathfrak{N}^2}),\, \mathrm{Id}\big) \Big)_{\gamma^2 \in \Gamma^2_{\mathrm{vic}},\, \mathfrak{N}^2 \in \pi_0(\mathfrak{M}^2_{\gamma^2})}.$$

The following lemma is the analogue of Lemma 4.9 in this step.

Lemma 4.12. *The set* Λ^2 *gives a treelike structure on* $(\mathfrak{M}^2, \Gamma^2_{\mathrm{vic}})$ *as in Definition 2.16.*

Lemmas 4.11 and 4.12, along with Theorem 2.18, give rise to the following statement.

Corollary 4.13. *Let* $(\mathfrak{M}^2)^{\mathrm{tf}}_{\Lambda^2}$ *be constructed as in Theorem 2.18* (p_1)*. Then,* $(\mathfrak{M}^2)^{\mathrm{tf}}_{\Lambda^2}$ *is a smooth algebraic stack and the composite forgetful morphism* $(\mathfrak{M}^2)^{\mathrm{tf}}_{\Lambda^2} \longrightarrow \mathfrak{M}^1 = \mathfrak{P}_2$ *is proper and birational.*

4.4. **The third step of the recursive construction.** In this step, we take

$$\mathfrak{M}^3 = (\mathfrak{M}^2)^{\mathrm{tf}}_{\Lambda^2}, \qquad \Gamma^3_{\mathrm{vic}} = (\Gamma^2_{\mathrm{vic}})^{\mathrm{der}}_{\Lambda^2}.$$

Analogous to Lemma 4.11, we have the following statement.

Lemma 4.14. *The stack* $\mathfrak{M}^3$ *comes equipped with a* Γ^3_{vic}*-stratification, which is the derived stratification of* $\mathfrak{M}^3$ *with respect to* $(\mathfrak{M}^2, \Gamma^2_{\mathrm{vic}}, \Lambda^2)$.

The graphs of Γ^3_{vic} are in the form

$$\gamma^3 := \rho_{\gamma^2,[\mathsf{t}^2]} \in \Gamma^3_{\mathrm{vic}}, \qquad \text{where} \;\; \gamma^2 \in \Gamma^2_{\mathrm{vic}}, \;\; [\mathsf{t}^2] \in \overline{\Lambda}^2_{\mathfrak{N}^2}, \;\; \mathfrak{N}^2 \in \pi_0\big(\mathfrak{M}^2_{\gamma^2}\big); \tag{4.17}$$

see (4.11) for γ^2 and (4.12) for $\mathfrak{N}^2$. The vertex that is the image of o_{γ^2} in γ^3 is denoted by o_{γ^3}. The decoration $(\gamma^2)^\star$ of γ^2 induces the decoration $(\gamma^3)^\star$ of γ^3 as in Definition 4.4. By Corollary 3.2, for every γ^3 as in (4.17), the connected components of $\mathfrak{M}^3_{\gamma^3}$ are in the form

$$\mathfrak{N}^3 := (\mathfrak{N}^2)^{\mathrm{tf}}_{[\mathsf{t}^2]} \in \pi_0\big(\mathfrak{M}^3_{\gamma^3}\big). \tag{4.18}$$

For each $\mathfrak{N}^3$ as above, we will describe the tuple $(\tau_{\mathfrak{N}^3}, \mathrm{E}_{\mathfrak{N}^3}, \beta_{\mathfrak{N}^3})$ to provide the proposed treelike structure Λ^3 on $(\mathfrak{M}^3, \Gamma^3_{\mathrm{vic}})$.

Let

$$\begin{aligned} \mathrm{E}^2 &:= \mathfrak{E}^{\perp}_{\geqslant \mathbf{m}}(\mathsf{t}^2) \subset \mathrm{Edg}(\tau_{\mathfrak{N}^2}) = \mathrm{E}_{\mathfrak{N}^2} \\ &\subset \mathrm{Edg}(\gamma_{(\mathrm{E}^1)}) = \mathrm{Edg}(\gamma^2) \backslash \{\mathsf{e}^1_i : i \in [\![\mathbf{m}, 0]\!)_{\mathsf{t}^1}\} \subset \mathrm{Edg}(\gamma^2), \end{aligned} \tag{4.19}$$

which follows from (4.16). Consider the graph and its induced decoration

$$\gamma_{(\underline{\mathrm{E}})^2} := \big(\gamma_{(\mathrm{E}^1)}\big)_{(\mathrm{E}^2)} = \gamma_{(\mathrm{E}^1 \sqcup \mathrm{E}^2)} \qquad \text{and} \qquad \gamma^\star_{(\underline{\mathrm{E}})^2},$$

respectively. Analogous to (4.14), we have

$$\mathrm{Edg}(\gamma_{(\underline{\mathrm{E}})^2}) \subset \mathrm{Edg}(\gamma^3), \quad \mathrm{Edg}\big(\gamma^\star_{(\underline{\mathrm{E}})^2;\mathrm{cor}}\big) = \mathrm{Edg}\big((\gamma^3)^\star_{\mathrm{cor}}\big), \quad \mathbf{w}\big(\gamma^\star_{(\underline{\mathrm{E}})^2;\mathrm{cor}}\big) = \mathbf{w}(o_{\gamma^3}) \geqslant 1. \tag{4.20}$$

In addition, by the construction of $\tau_{\mathfrak{N}^2}$ in §4.3, we see that $\gamma^\star_{(\underline{\mathrm{E}})^2}$ contains at most one *smallest* connected decorated subgraph

$$\gamma^\star_{(\underline{\mathrm{E}})^2;+} \qquad \text{with} \quad p_a\big(\gamma^\star_{(\underline{\mathrm{E}})^2;+}\big) = 1, \quad \mathbf{w}\big(\gamma^\star_{(\underline{\mathrm{E}})^2;+}\big) = 0. \tag{4.21}$$

We are ready to construct the rooted tree $\tau_{\mathfrak{N}^3}$.

- If $\gamma^\star_{(\underline{\mathrm{E}})^2}$ does not contain any connected decorated subgraph $\gamma^\star_{(\underline{\mathrm{E}})^2;+}$ satisfying (4.21), then we set $\tau_{\mathfrak{N}^3} = \tau_\bullet$ and $o_{\mathfrak{N}^3} = o_{\gamma^3}$.
- If $\gamma^\star_{(\underline{\mathrm{E}})^2}$ contains a unique smallest connected decorated subgraph $\gamma^\star_{(\underline{\mathrm{E}})^2;+}$ satisfying (4.21), it is a subgraph of $\gamma^\star_{(\underline{\mathrm{E}})^2;\mathrm{cor}}$ by Definition 4.3. We contract the edges of $\gamma^\star_{(\underline{\mathrm{E}})^2;+}$ from $\gamma^\star_{(\underline{\mathrm{E}})^2}$ into one vertex o_{ctr}, then dissolve the vertices v with $\mathbf{w}(v) > 0$ and take $\tau_{\mathfrak{N}^3}$ to be the unique connected component that contains o_{ctr}. In this way, we see that $\tau_{\mathfrak{N}^3}$ is a rooted tree with the root $o_{\mathfrak{N}^3} = o_{\mathrm{ctr}}$.

Notice that $\tau_{\mathfrak{N}^3}$ is obtained via contraction of the edges of

$$\mathrm{Edg}\big(\gamma^\star_{(\underline{\mathrm{E}})^2;+}\big) \subset \mathrm{Edg}\big(\gamma^\star_{(\underline{\mathrm{E}})^2;\mathrm{cor}}\big) = \mathrm{Edg}\big((\gamma^3)^\star_{\mathrm{cor}}\big)$$

(the last expression follows from (4.20)), vertex dissolution, and taking a connected component of a graph, thus

$$\mathrm{Edg}(\tau_{\mathfrak{N}^3}) \subset \mathrm{Edg}\big(\gamma_{(\underline{\mathrm{E}})^2}\big) = \mathrm{Edg}(\gamma^3)\backslash\{\, \mathsf{e}_i^k :\ k=1,2;\ i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k}\} \subset \mathrm{Edg}(\gamma^3), \tag{4.22}$$

and moreover, $\mathrm{Edg}(\tau_{\mathfrak{N}^3})$ is the set of the edges of a *connected* subgraph $\tau'_{\mathfrak{N}^3}$ of γ^3 that satisfies

$$o_{\mathfrak{N}^3} \subset o_{\gamma^3} \in \mathrm{Ver}(\tau'_{\mathfrak{N}^3})$$

We set

$$\Lambda^3 = \big(\, (\tau_{\mathfrak{N}^3},\, \mathrm{E}_{\mathfrak{N}^3} := \mathrm{Edg}(\tau_{\mathfrak{N}^3}),\, \mathrm{Id})\,\big)_{\gamma^3\in\Gamma^3_{\mathrm{vic}},\ \mathfrak{N}^3\in\pi_0(\mathfrak{M}^3_{\gamma^3})}.$$

The following lemma is the analogue of Lemma 4.9 in this step.

Lemma 4.15. *The set Λ^3 gives a treelike structure on $(\mathfrak{M}^3,\Gamma^3_{\mathrm{vic}})$ as in Definition 2.16.*

Lemmas 4.14 and 4.15, along with Theorem 2.18, give rise to the following statement.

Corollary 4.16. *Let $(\mathfrak{M}^3)^{\mathrm{tf}}_{\Lambda^3}$ be constructed as in Theorem 2.18* (p_1)*. Then, $(\mathfrak{M}^3)^{\mathrm{tf}}_{\Lambda^3}$ is a smooth algebraic stack and the composite forgetful morphism $(\mathfrak{M}^3)^{\mathrm{tf}}_{\Lambda^3} \longrightarrow \mathfrak{M}^1 = \mathfrak{P}_2$ is proper and birational.*

4.5. **The fourth step of the recursive construction.** In this step, we take

$$\mathfrak{M}^4 = (\mathfrak{M}^3)^{\mathrm{tf}}_{\Lambda^3}, \qquad \Gamma^4_{\mathrm{vic}} = (\Gamma^3_{\mathrm{vic}})^{\mathrm{der}}_{\Lambda^3}.$$

Similar to Lemma 4.11, we have the following statement.

Lemma 4.17. *The stack $\mathfrak{M}^4$ comes equipped with a Γ^4_{vic}-stratification, which is the derived stratification of $\mathfrak{M}^4$ with respect to $(\mathfrak{M}^3,\Gamma^3_{\mathrm{vic}},\Lambda^3)$.*

The graphs of Γ^4_{vic} are in the form

$$\gamma^4 := \rho_{\gamma^3,[\mathfrak{t}^3]} \in \Gamma^4_{\mathrm{vic}}, \qquad \text{where}\ \ \gamma^3\in\Gamma^3_{\mathrm{vic}},\ \ [\mathfrak{t}^3]\in\overline{\Lambda}^3_{\mathfrak{N}^3},\ \ \mathfrak{N}^3\in\pi_0\big(\mathfrak{M}^3_{\gamma^3}\big); \tag{4.23}$$

see (4.17) for γ^3 and (4.18) for $\mathfrak{N}^3$. Let o_{γ^4} be the image of o_{γ^3} in γ^4, $(\gamma^4)^\star$ be the induced decoration, and

$$\gamma^4_{\mathrm{vi}3} \in \mathbf{G}$$

be the graph obtained from γ^4 by identifying the vertices:

$$o_{\gamma^4} \quad \text{and} \quad \mathsf{v}_i^k,\ \ i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k},\ \ k=1,2,3;$$

the new vertex is denoted by $o_{\gamma^4_{\mathrm{vi}3}}$. The decoration $(\gamma^4)^\star$ further induces the decoration $(\gamma^4_{\mathrm{vi}3})^\star$ of $\gamma^4_{\mathrm{vi}3}$ as in (4.7). Then,

$$\mathrm{Edg}(\gamma^4)=\mathrm{Edg}(\gamma^4_{\mathrm{vi}3});\quad \mathrm{Edg}((\gamma^4)^\star_{\mathrm{cor}})\subset\mathrm{Edg}\big((\gamma^4_{\mathrm{vi}3})^\star_{\mathrm{cor}}\big);\quad (\gamma^4_{(E)})_{\mathrm{vi}3}=(\gamma^4_{\mathrm{vi}3})_{(E)}\ \forall\, E\subset\mathrm{Edg}(\gamma^4).$$

We take

$$\Gamma^4_{\mathrm{vi}3} = \{\, \gamma^4_{\mathrm{vi}3} :\ \gamma^4\in\Gamma^4_{\mathrm{vic}}\,\}.$$

The following statement follows from Lemma 4.17 and the properties of $\mathrm{Edg}(\gamma^4_{\mathrm{vi}3})$ above.

Corollary 4.18. *The Γ^4_{vic}-stratification of $\mathfrak{M}^4$ in Lemma 4.17 determines a $\Gamma^4_{\mathrm{vi}3}$-stratification:*

$$\mathfrak{M}^4 = \bigsqcup_{\gamma'\in\Gamma^4_{\mathrm{vi}3}} \mathfrak{M}^4_{\mathrm{vi}3;\gamma'}, \qquad \text{where}\quad \mathfrak{M}^4_{\mathrm{vi}3;\gamma'} = \bigsqcup_{\gamma^4\in\Gamma^4_{\mathrm{vic}},\,\gamma^4_{\mathrm{vi}3}=\gamma'} \mathfrak{M}^4_{\gamma^4}.$$

By Corollary 3.2, for every $\gamma^4 \in \Gamma^4_{\mathrm{vic}}$ and the corresponding $\gamma^4_{\mathrm{vi}3}$, the connected components of $\mathfrak{M}^4_{\gamma^4}$ and $\mathfrak{M}^4_{\mathrm{vi}3;\gamma^4_{\mathrm{vi}3}}$ are in the form

$$\mathfrak{N}^4 := (\mathfrak{N}^3)^{\mathrm{tf}}_{[\mathfrak{t}^3]} \in \pi_0\big(\mathfrak{M}^4_{\gamma^4}\big) \cap \pi_0\big(\mathfrak{M}^4_{\mathrm{vi}3;\gamma^4_{\mathrm{vi}3}}\big). \tag{4.24}$$

For each $\mathfrak{N}^4$, we will describe the tuple $(\tau_{\mathfrak{N}^4}, \mathrm{E}_{\mathfrak{N}^4}, \beta_{\mathfrak{N}^4})$ to provide Λ^4 on $(\mathfrak{M}^4, \Gamma^4_{\mathrm{vi}3})$.

Let

$$\begin{aligned} \mathrm{E}^3 &:= \mathfrak{E}^{\perp}_{\geqslant \mathbf{m}}(\mathfrak{t}^3) \subset \mathrm{Edg}(\tau_{\mathfrak{N}^3}) = \mathrm{E}_{\mathfrak{N}^3} \\ &\subset \mathrm{Edg}(\gamma_{(\underline{\mathrm{E}})^2}) = \mathrm{Edg}(\gamma^3)\backslash\{\, \mathsf{e}^k_i : i \in [\![\mathbf{m}, 0)\!)_{\mathfrak{t}^k},\ k = 1, 2\,\} \subset \mathrm{Edg}(\gamma^3), \end{aligned} \tag{4.25}$$

which follows from (4.22). Consider the graph and its induced decoration

$$\gamma_{(\underline{\mathrm{E}})^3} := (\gamma_{(\underline{\mathrm{E}})^2})_{(\mathrm{E}^3)} = \gamma_{(\mathrm{E}^1 \sqcup \mathrm{E}^2 \sqcup \mathrm{E}^3)} \qquad \text{and} \qquad \gamma^{\star}_{(\underline{\mathrm{E}})^3},$$

respectively. Analogous to (4.14) and (4.20), we have

$$\begin{aligned} &\mathrm{Edg}(\gamma_{(\underline{\mathrm{E}})^3}) \subset \mathrm{Edg}(\gamma^4) = \mathrm{Edg}(\gamma^4_{\mathrm{vi}3}), \qquad \mathrm{Edg}\big(\gamma^{\star}_{(\underline{\mathrm{E}})^3;\mathrm{cor}}\big) = \mathrm{Edg}\big((\gamma^4)^{\star}_{\mathrm{cor}}\big) \subset \mathrm{Edg}\big((\gamma^4_{\mathrm{vi}3})^{\star}_{\mathrm{cor}}\big), \\ &\mathbf{w}\big(\gamma^{\star}_{(\underline{\mathrm{E}})^3;\mathrm{cor}}\big) = \mathbf{w}(o_{\gamma^4}) = \mathbf{w}(o_{\gamma^4_{\mathrm{vi}3}}) \geqslant 1. \end{aligned} \tag{4.26}$$

The last equality above holds because $\mathbf{w}(\mathsf{v}^k_i) = 0$ for all $i \in [\![\mathbf{m}, 0)\!)_{\mathfrak{t}^k}$ and all $k = 1, 2, 3$; see Definition 4.4.

We are ready to construct the rooted tree $\tau_{\mathfrak{N}^4}$.

- If $\mathbf{w}(\gamma^{\star}_{(\underline{\mathrm{E}})^3;\mathrm{cor}}) > 1$, then we set $\tau_{\mathfrak{N}^4} = \tau_{\bullet}$ and $o_{\mathfrak{N}^4} = o_{\gamma^4_{\mathrm{vi}3}}$.
- If $\mathbf{w}(\gamma^{\star}_{(\underline{\mathrm{E}})^3;\mathrm{cor}}) = 1$, we contract the edges of $\gamma^{\star}_{(\underline{\mathrm{E}})^3;\mathrm{cor}}$ from $\gamma^{\star}_{(\underline{\mathrm{E}})^3}$ into one vertex o_{ctr}, then dissolve the vertices $v \neq o_{\mathrm{ctr}}$ with $\mathbf{w}(v) > 0$ and take $\tau_{\mathfrak{N}^4}$ to be the unique connected component that contains o_{ctr}. In this way, we see that $\tau_{\mathfrak{N}^4}$ is a rooted tree with the root $o_{\mathfrak{N}^4} = o_{\mathrm{ctr}}$.

Notice that $\tau_{\mathfrak{N}^4}$ is obtained via contraction of the edges of

$$\mathrm{Edg}\big(\gamma^{\star}_{(\underline{\mathrm{E}})^3;\mathrm{cor}}\big) = \mathrm{Edg}\big((\gamma^4)^{\star}_{\mathrm{cor}}\big) \subset \mathrm{Edg}\big((\gamma^4_{\mathrm{vi}3})^{\star}_{\mathrm{cor}}\big)$$

(by (4.26)), vertex dissolution, and taking a connected component of a graph, thus

$$\mathrm{Edg}(\tau_{\mathfrak{N}^4}) \subset \mathrm{Edg}\big(\gamma_{(\underline{\mathrm{E}})^3}\big) = \mathrm{Edg}(\gamma^4)\backslash\{\mathsf{e}^k_i : k = 1, 2, 3;\ i \in [\![\mathbf{m}, 0)\!)_{\mathfrak{t}^k}\} \subset \mathrm{Edg}(\gamma^4) = \mathrm{Edg}(\gamma^4_{\mathrm{vi}3}), \tag{4.27}$$

and moreover, $\mathrm{Edg}(\tau_{\mathfrak{N}^4})$ is the set of the edges of a *connected* subgraph $\tau'_{\mathfrak{N}^4}$ of $\gamma^4_{\mathrm{vi}3}$ that satisfies

$$o_{\mathfrak{N}^4} \subset o_{\gamma^4_{\mathrm{vi}3}} \in \mathrm{Ver}(\tau'_{\mathfrak{N}^4}).$$

We set

$$\Lambda^4 = \big(\, (\tau_{\mathfrak{N}^4},\, \mathrm{E}_{\mathfrak{N}^4} := \mathrm{Edg}(\tau_{\mathfrak{N}^4}),\, \mathrm{Id})\,\big)_{\gamma^4_{\mathrm{vi}3} \in \Gamma^4_{\mathrm{vi}3},\ \mathfrak{N}^4 \in \pi_0(\mathfrak{M}^4_{\mathrm{vi}3;\gamma^4_{\mathrm{vi}3}})}.$$

The following lemma is the analogue of Lemma 4.9 in this step. Notice that the core of $\gamma^{\star}_{(\underline{\mathrm{E}})^3}$ is postively weighted in the current situation, which seems to be different from Lemma 4.9 (or Example 2.17). However, by Definition 4.2, if we contract an edge of $\tau_{\mathfrak{N}^4}$ that is both maximal and minimal, then the weight of the new vertex is greater than one, hence the corresponding rooted tree given by Λ^4 is $\tau_{\bullet}$, which is exactly the same as in the proof of Lemma 4.9.

Lemma 4.19. *The set Λ^4 gives a treelike structure on $(\mathfrak{M}^4, \Gamma^4_{\mathrm{vi}3})$ as in Definition 2.16.*

Lemmas 4.17 and 4.19, along with Theorem 2.18, give rise to the following statement.

Corollary 4.20. *Let $(\mathfrak{M}^4)^{\mathrm{tf}}_{\Lambda^4}$ be constructed as in Theorem 2.18 (p_1). Then, $(\mathfrak{M}^4)^{\mathrm{tf}}_{\Lambda^4}$ is a smooth algebraic stack and the composite forgetful morphism $(\mathfrak{M}^4)^{\mathrm{tf}}_{\Lambda^4} \longrightarrow \mathfrak{M}^1 = \mathfrak{P}_2$ is proper and birational.*

4.6. **The fifth step of the recursive construction.** In this step, we take

$$\mathfrak{M}^5 = (\mathfrak{M}^4)^{\mathrm{tf}}_{\Lambda^4}, \qquad \Gamma^5_{\mathrm{vi}_3} = (\Gamma^4_{\mathrm{vi}_3})^{\mathrm{der}}_{\Lambda^4}.$$

Analogous to Lemma 4.11, we have the following statement.

Lemma 4.21. *The stack $\mathfrak{M}^5$ comes equipped with a $\Gamma^5_{\mathrm{vi}_3}$-stratification, which is the derived stratification of $\mathfrak{M}^5$ with respect to $(\mathfrak{M}^4, \Gamma^4_{\mathrm{vi}_3}, \Lambda^4)$.*

The graphs of $\Gamma^5_{\mathrm{vi}_3}$ are in the form

$$\gamma^5 := \rho_{\gamma^4_{\mathrm{vi}_3},[\mathrm{t}^4]} \in \Gamma^5_{\mathrm{vi}_3}, \qquad \text{where} \ \ \gamma^4_{\mathrm{vi}_3} \in \Gamma^4_{\mathrm{vi}_3}, \ \ [\mathrm{t}^4] \in \overline{\Lambda}^4_{\mathfrak{N}^4}, \ \ \mathfrak{N}^4 \in \pi_0\big(\mathfrak{M}^4_{\mathrm{vi}_3;\gamma^4_{\mathrm{vi}_3}}\big); \tag{4.28}$$

see (4.23) for γ^4 (and the succeeding display for $\gamma^4_{\mathrm{vi}_3}$) and (4.24) for $\mathfrak{N}^4$. Let o_{γ^5} be the image of $o_{\gamma^4_{\mathrm{vi}_3}}$ in γ^5 and $(\gamma^5)^\star$ be the derived decoration. By Corollary 3.2, for every γ^5 as in (4.28), the connected components of $\mathfrak{M}^5_{\gamma^5}$ are in the form

$$\mathfrak{N}^5 := (\mathfrak{N}^4)^{\mathrm{tf}}_{[\mathrm{t}^4]} \ \in \pi_0\big(\mathfrak{M}^5_{\gamma^5}\big). \tag{4.29}$$

For each $\mathfrak{N}^5$, we will describe the tuple $(\tau_{\mathfrak{N}^5}, \mathrm{E}_{\mathfrak{N}^5}, \beta_{\mathfrak{N}^5})$ to provide Λ^5 on $(\mathfrak{M}^5, \Gamma^5_{\mathrm{vi}_3})$.

Let

$$\begin{aligned} \mathrm{E}^4 &:= \mathfrak{E}^{\perp}_{\geqslant \mathbf{m}}(\mathrm{t}^4) \subset \mathrm{Edg}(\tau_{\mathfrak{N}^4}) = \mathrm{E}_{\mathfrak{N}^4} \\ &\subset \mathrm{Edg}(\gamma_{(\underline{\mathrm{E}})^3}) = \mathrm{Edg}(\gamma^4) \backslash \{\, \mathrm{e}^k_i : i \in [\![\mathbf{m}, 0)\!)_{\mathrm{t}^k},\ k = 1, 2, 3\} \subset \mathrm{Edg}(\gamma^4), \end{aligned} \tag{4.30}$$

which follows from (4.27). With $\gamma \in \Gamma^1$ and $[\mathrm{t}^1] \in \overline{\Lambda}^1_{\tau_{\mathfrak{N}^1}}$ as in §4.2, consider the derived graph of γ with respect to $[\mathrm{t}^1]$ and its induced decoration

$$\widetilde{\gamma}^2 := \rho_{\gamma;[\mathrm{t}^1]} \qquad \text{and} \qquad (\widetilde{\gamma}^2)^\star,$$

respectively. By (3.1) and (4.13), we have

$$\mathrm{Edg}(\gamma_{(\mathrm{E}^1)}) \subset \mathrm{Edg}(\widetilde{\gamma}^2).$$

Thus by (4.19), (4.25), and (4.30),

$$\mathrm{E}^k \subset \mathrm{Edg}(\widetilde{\gamma}^2), \qquad k = 2, 3, 4. \tag{4.31}$$

Let

$$\widetilde{\gamma}^2_{(\underline{\mathrm{E}})^4} := \widetilde{\gamma}^2_{(\mathrm{E}^2 \sqcup \mathrm{E}^3 \sqcup \mathrm{E}^4)}$$

and $(\widetilde{\gamma}^2_{(\underline{\mathrm{E}})^4})^\star$ be its induced decoration. Let

$$\widehat{\tau}^5 \qquad \text{and} \qquad (\widehat{\tau}^5)^\star$$

be the rooted tree obtained from $\widetilde{\gamma}^2_{(\underline{\mathrm{E}})^4}$ by contracting the edges of $(\widetilde{\gamma}^2_{(\underline{\mathrm{E}})^4})^\star_{\mathrm{cor}}$ into the root $o_{\widehat{\tau}^5}$ and its induced decoration, respectively. Analogous to (4.14), (4.20), and (4.26), we have

$$\mathrm{Edg}(\widehat{\tau}^5) \subset \mathrm{Edg}(\gamma^5), \qquad o_{\widehat{\tau}^5} \subset o_{\gamma^5}. \tag{4.32}$$

We are ready to construct $\tau_{\mathfrak{N}^5}$. Recall that $\mathbf{n}_1(\mathrm{t}^1) \in [\![\mathbf{m}, 0]\!]_{\mathrm{t}^1}$ as in (2.5) is the lowest level of t^1 on which there exists a unique vertex $v \in \mathrm{Ver}(\mathrm{t}^1)$ satisfying $v \succeq v'$ for all level $\mathbf{m}$ minimal vertices v' of t^1.

- If $\mathbf{n}_1(\mathrm{t}^1) = 0$, or $\mathbf{n}_1(\mathrm{t}^1) < 0$ but $\mathbf{w}\big((\widetilde{\gamma}^2)^\star_{\mathrm{cor}}\big) < \mathbf{w}(o_{\widehat{\tau}^5})$, we set $\tau_{\mathfrak{N}^5} = \tau_\bullet$ and $o_{\mathfrak{N}^5} = o_{\gamma^5}$.
- If $\mathbf{n}_1(\mathrm{t}^1) < 0$ and $\mathbf{w}\big((\widetilde{\gamma}^2)^\star_{\mathrm{cor}}\big) = \mathbf{w}(o_{\widehat{\tau}^5})$, we dissolve $\mathrm{v}^1_{\mathbf{n}_1(\mathrm{t}^1)}$ and all the vertices $v \neq o_{\widehat{\tau}^5}$ with $\mathbf{w}(v) > 0$ in $\widehat{\tau}^5$ and take $\tau_{\mathfrak{N}^5}$ to be the unique connected component that contains $o_{\widehat{\tau}^5}$. Thus, $\tau_{\mathfrak{N}^5}$ is a rooted tree with $o_{\mathfrak{N}^5} = o_{\widehat{\tau}^5}$.

Notice that the above construction of $\tau_{\mathfrak{N}^5}$ is independent of the choice of $\mathfrak{t}^1$ representing $[\mathfrak{t}^1]$. By (4.32), we have

$$\mathrm{Edg}(\tau_{\mathfrak{N}^5}) \subset \mathrm{Edg}(\hat{\tau}^5) \subset \mathrm{Edg}(\gamma^5)\backslash\{\mathsf{e}_i^k : k=2,3,4;\ i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k}\} \subset \mathrm{Edg}(\gamma^5), \tag{4.33}$$

and moreover, $\mathrm{Edg}(\tau_{\mathfrak{N}^5})$ is the set of the edges of a *connected* subgraph $\tau'_{\mathfrak{N}^5}$ of γ^5 that satisfies

$$o_{\mathfrak{N}^5} \subset o_{\gamma^5} \in \mathrm{Ver}(\tau'_{\mathfrak{N}^5}).$$

We set

$$\Lambda^5 = \big((\tau_{\mathfrak{N}^5},\, \mathrm{E}_{\mathfrak{N}^5} := \mathrm{Edg}(\tau_{\mathfrak{N}^5}),\, \mathrm{Id}) \big)_{\gamma^5\in\Gamma^5_{\mathrm{vi}3},\, \mathfrak{N}^5\in\pi_0(\mathfrak{M}^5_{\gamma^5})}.$$

Lemma 4.22. *The set Λ^5 gives a treelike structure on $(\mathfrak{M}^5, \Gamma^5_{\mathrm{vi}3})$ as in Definition 2.16.*

Proof. The proof of Lemma 4.22 is still parallel to that of Lemma 4.9. The only part that needs attention is the minimal vertex $\mathsf{v}^1_{\mathbf{n}_1(\mathfrak{t}^1)}$ of $\tau_{\mathfrak{N}^5}$ is not positively weighted whenever $\mathbf{n}_1(\mathfrak{t}^1)<0$; see Definition 4.4. Nonetheless, if we contract the minimal edge $\mathsf{e}^1_{\mathbf{n}_1(\mathfrak{t}^1)}$ of $\tau_{\mathfrak{N}^5}$, it is a direct check that there exists a unique list

$$\tilde{\gamma}\in\Gamma^1_{\mathrm{vic}},\quad \tilde{\mathfrak{N}}^1\in\pi_0(\mathfrak{M}^1_{\tilde{\gamma}}),\quad [\tilde{\mathfrak{t}}^1]\in\overline{\Lambda}_{\tilde{\mathfrak{N}}^1},\quad \ldots,\quad \tilde{\gamma}^5\in\Gamma^5_{\mathrm{vi}3},\quad \tilde{\mathfrak{N}}^5\in\pi_0(\mathfrak{M}^5_{\tilde{\gamma}^5}) \tag{4.34}$$

such that $\tilde{\gamma}^5 = \gamma^5_{(\mathsf{e}^1_{\mathbf{n}_1(\mathfrak{t}^1)})}$, $\mathrm{Cl}_{\mathfrak{M}^5}\tilde{\mathfrak{N}}^5\cap\mathfrak{N}^5\neq\varnothing$, and $\mathbf{n}_1(\tilde{\mathfrak{t}}^1) = (\mathbf{n}_1(\mathfrak{t}^1))^\sharp$; see (2.3) for notation. Thus, $\mathsf{v}^1_{(\mathbf{n}_1(\mathfrak{t}^1))^\sharp}$ is a minimal vertex of $\tau_{\tilde{\mathfrak{N}}^5}$ in this case. Similarly, if we contract a non-exceptional minimal edge e of $\tau_{\mathfrak{N}^5}$ that is directly attached to an exceptional vertex v^1_i, $i\in[\![\mathbf{n}_1(\mathfrak{t}^1),0)\!)$, then it is a direct check that there exists a unique list as in (4.34) such that $\tilde{\gamma}^5=\gamma^5_{(e)}$, $\mathrm{Cl}_{\mathfrak{M}^5}\tilde{\mathfrak{N}}^5\cap\mathfrak{N}^5\neq\varnothing$, and $\mathbf{n}_1(\tilde{\mathfrak{t}}^1)=i$. Thus, v^1_i is a minimal vertex of $\tau_{\tilde{\mathfrak{N}}^5}$ in this case. The verification of the rest of the conditions in Definition 2.16 is analogous to Lemma 4.9 and Example 2.17, hence is omitted. $\square$

Lemmas 4.21 and 4.22, along with Theorem 2.18, give rise to the following statement.

Corollary 4.23. *Let $(\mathfrak{M}^5)^{\mathrm{tf}}_{\Lambda^5}$ be constructed as in Theorem 2.18 (p_1). Then, $(\mathfrak{M}^5)^{\mathrm{tf}}_{\Lambda^5}$ is a smooth algebraic stack and the composite forgetful morphism $(\mathfrak{M}^5)^{\mathrm{tf}}_{\Lambda^5}\longrightarrow\mathfrak{M}^1=\mathfrak{P}_2$ is proper and birational.*

4.7. **The sixth step of the recursive construction.** In this step, we take

$$\mathfrak{M}^6 = (\mathfrak{M}^5)^{\mathrm{tf}}_{\Lambda^5}, \qquad \Gamma^6_{\mathrm{vi}3} = (\Gamma^5_{\mathrm{vi}3})^{\mathrm{der}}_{\Lambda^5}.$$

Similar to Lemma 4.11, we have the following statement.

Lemma 4.24. *The stack $\mathfrak{M}^6$ comes equipped with a $\Gamma^6_{\mathrm{vi}3}$-stratification that is the derived stratification of $\mathfrak{M}^6$ with respect to $(\mathfrak{M}^5, \Gamma^5_{\mathrm{vi}3}, \Lambda^5)$.*

Hereafter, the recursive construction becomes a bit different, which is due to the expressions of the structural homomorphism in [9, §2] that are related to the conjugate and Weierstrass points. In Step 6, the strategy is as follows.

(1) We first describe a substack K of the boundary Δ^6 of $\mathfrak{M}^6$ that is related to the conjugate and Weierstrass points and satisfies the first assumption of Corollary 3.3.
(2) We then construct a treelike structure Λ^6 satisfying the second assumption and (3.6) of Corollary 3.3.
(3) Finally, we apply Corollary 3.3 to obtain the grafted stratification Γ^6_{gft} with respect to K and the induced treelike structure Λ^6_{gft}.

We start with the substack K. The graphs of $\Gamma^6_{\mathrm{vi}3}$ are in the form

$$\gamma^6 := \rho_{\gamma^5,[\mathfrak{t}^5]} \in \Gamma^6_{\mathrm{vi}3}, \qquad \text{where } \gamma^5 \in \Gamma^5_{\mathrm{vi}3}, \quad [\mathfrak{t}^5] \in \overline{\Lambda}^5_{\mathfrak{N}^5}, \quad \mathfrak{N}^5 \in \pi_0\big(\mathfrak{M}^5_{\gamma^5}\big); \tag{4.35}$$

see (4.28) for γ^5 and (4.29) for $\mathfrak{N}^5$. The vertex that is the image of o_{γ^5} in γ^6 is denoted by o_{γ^6}. The decoration $(\gamma^5)^\star$ of γ^5 induces the decoration $(\gamma^6)^\star$ of γ^6. Notice that $\mathbf{w}\big((\gamma^6)^\star_{\mathrm{cor}}\big) \geqslant 2$.

By Corollary 3.2, for every $\gamma^6 \in \Gamma^6_{\mathrm{vi}3}$, the connected components of $\mathfrak{M}^6_{\gamma^6}$ are in the form

$$\mathfrak{N}^6 := (\mathfrak{N}^5)^{\mathrm{tf}}_{[\mathfrak{t}^5]} \in \pi_0\big(\mathfrak{M}^6_{\gamma^6}\big). \tag{4.36}$$

Recall that $\mathbf{n}_j(\mathfrak{t}^k)$ denotes the lowest level of $\mathfrak{t}^k$ on which there are at most j vertices that are contained in the paths from the root to the minimal vertices on the $\mathbf{m}(\mathfrak{t}^k)$-th level; see (2.5). Consider the following sets of the connected components of the strata of $\mathfrak{M}^6$:

$$\begin{aligned}
N^6_{a;1} &= \big\{\mathfrak{N}^6 \in \pi_0\big(\mathfrak{M}^6_{\gamma^6}\big) : \gamma^6 \in \Gamma^6_{\mathrm{vi}3};\ \mathbf{n}_2(\mathfrak{t}^1) < \mathbf{n}_1(\mathfrak{t}^1) = 0;\ \mathbf{w}\big((\gamma^2)^\star_{\mathrm{cor}}\big) = \mathbf{w}\big((\gamma^6)^\star_{\mathrm{cor}}\big)\big\},\\
N^6_{a;2} &= \big\{\mathfrak{N}^6 \in \pi_0\big(\mathfrak{M}^6_{\gamma^6}\big) : \gamma^6 \in \Gamma^6_{\mathrm{vi}3};\ \mathbf{n}_1(\mathfrak{t}^1), \mathbf{n}_1(\mathfrak{t}^2) < 0;\ \mathbf{w}\big((\gamma^2)^\star_{\mathrm{cor}}\big) < \mathbf{w}\big((\gamma^3)^\star_{\mathrm{cor}}\big) = \mathbf{w}\big((\gamma^6)^\star_{\mathrm{cor}}\big)\big\},\\
N^6_{a;3} &= \big\{\mathfrak{N}^6 \in \pi_0\big(\mathfrak{M}^6_{\gamma^6}\big) : \gamma^6 \in \Gamma^6_{\mathrm{vi}3};\ \mathbf{n}_1(\mathfrak{t}^1), \mathbf{n}_1(\mathfrak{t}^3) < 0;\\
&\qquad\qquad \mathbf{w}\big((\gamma^2)^\star_{\mathrm{cor}}\big) = \mathbf{w}\big((\gamma^3)^\star_{\mathrm{cor}}\big) < \mathbf{w}\big((\gamma^4)^\star_{\mathrm{cor}}\big) = \mathbf{w}\big((\gamma^6)^\star_{\mathrm{cor}}\big)\big\},\\
N^6_b &= \big\{\mathfrak{N}^6 \in \pi_0\big(\mathfrak{M}^6_{\gamma^6}\big) : \gamma^6 \in \Gamma^6_{\mathrm{vi}3};\ \mathbf{n}_1(\mathfrak{t}^5) < 0\big\},\\
N^6_c &= \big\{\mathfrak{N}^6 \in \pi_0\big(\mathfrak{M}^6_{\gamma^6}\big) : \gamma^6 \in \Gamma^6_{\mathrm{vi}3};\ \mathbf{w}\big((\gamma^6)^\star_{\mathrm{cor}}\big) = 2;\ \gamma_{\mathrm{vic}} \neq \tau_\bullet\big\},\\
N^6_d &= \big\{\mathfrak{N}^6 \in \pi_0\big(\mathfrak{M}^6_{\gamma^6}\big) : \gamma^6 \in \Gamma^6_{\mathrm{vi}3};\ \mathbf{n}_1(\mathfrak{t}^4) < 0;\ \mathbf{w}\big((\gamma^6)^\star_{\mathrm{cor}}\big) = \mathbf{w}\big((\gamma^5)^\star_{\mathrm{cor}}\big)\big\}.
\end{aligned}$$

We take

$$N^\dagger = \bigcup_{1 \leqslant s \leqslant 3} N^6_{a;s} \cup N^6_b \cup N^6_c \cup N^6_d\,; \qquad \mathfrak{M}^\dagger = \bigsqcup_{\mathfrak{N}^6 \in N^\dagger} \mathfrak{N}^6 \subset \mathfrak{M}^6\,.$$

The condition $\gamma_{\mathrm{vic}} \neq \tau_\bullet$ in N^6_c excludes the stratum consisting of smooth genus two curves and degree 2 line bundles, which is a connected component of $\mathfrak{M}^6_{\tau_\bullet}$. Thus, $\mathfrak{M}^\dagger$ is a substack of the boundary Δ^6 of $\mathfrak{M}^6$. In addition, we have the following:

Lemma 4.25. *The substack $\mathfrak{M}^\dagger$ is closed.*

Proof. Fix $\mathfrak{N}^6 \in N^\dagger$. By Lemma 2.15, for every $\tilde{\gamma}^6 \in \Gamma^6_{\mathrm{vi}3}$ and $\widetilde{\mathfrak{N}}^6 \in \pi_0(\mathfrak{M}^6_{\tilde{\gamma}^6})$, if $\mathrm{Cl}_{\mathfrak{M}^6}\mathfrak{N}^6 \cap \widetilde{\mathfrak{N}}^6 \neq \varnothing$, then $\tilde{\gamma}^6 \preceq \gamma^6$. Here the notation $\tilde{}$ is to distinguish $\tilde{\gamma}^6$ from γ^6. Analyzing the possible graphs $\tilde{\gamma}^6$ with $\tilde{\gamma}^6 \preceq \gamma^6$ and keeping track of the weights $\mathbf{w}\big((\tilde{\gamma}^s)^\star_{\mathrm{cor}}\big)$, $2 \leqslant s \leqslant 6$, we observe that

$$\mathrm{Cl}_{\mathfrak{M}^6}\mathfrak{N}^6 \subset \begin{cases} \bigsqcup_{\mathfrak{N}' \in N^6_{a;1} \cup N^6_{a;2} \cup N^6_{a;3} \cup N^6_b \cup N^6_d} \mathfrak{N}' & \text{if } \mathfrak{N}^6 \in N^6_{a;1};\\ \bigsqcup_{\mathfrak{N}' \in N^6_{a;s} \cup N^6_b \cup N^6_d} \mathfrak{N}' & \text{if } \mathfrak{N}^6 \in N^6_{a;s},\ s = 2, 3;\\ \bigsqcup_{\mathfrak{N}' \in N^6_i} \mathfrak{N}' & \text{if } \mathfrak{N}^6 \in N^6_i,\ i = b, c, d. \end{cases} \tag{4.37}$$

Therefore, $\mathfrak{M}^\dagger$ is closed. $\square$

To construct $K \subset \mathfrak{M}^\dagger \subset \Delta^6$, we consider the algebraic stack $\mathfrak{D}_2$ of the stable pairs (C, D), where C are genus 2 nodal curves and D are effective divisors on C. A pair (C, D) in $\mathfrak{D}_2$ is said to be stable if every rational irreducible component without any divisoral marking contains at least three nodal points. It is known that $\mathfrak{D}_2$ is smooth and there exists a smooth morphism

$$\mathfrak{D}_2 \longrightarrow \mathfrak{P}_2, \qquad (C, D) \mapsto (C, \mathscr{O}_C(D)).$$

Thus, every smooth chart $\mathcal{V} \longrightarrow \mathfrak{D}_2$ gives a smooth chart $\mathcal{V} \longrightarrow \mathfrak{P}_2$.

Fix $\widetilde{x}\in\mathfrak{M}^6_{\gamma^6}$. Let

$$x=(C,L)\in\mathfrak{M}^1=\mathfrak{P}_2$$

be the image of $\widetilde{x}$ under the forgetful morphism $\varpi\colon\mathfrak{M}^6\longrightarrow\mathfrak{M}^1$ and let $(C,D)\in\mathfrak{D}_2$ be such that $\mathscr{O}_C(D)=L$. We take $\mathcal{V}\longrightarrow\mathfrak{D}_2(\longrightarrow\mathfrak{M}^1)$ to be a smooth chart containing (C,D) (and hence x). Let $(\mathcal{C},\mathcal{D})/\mathcal{V}$ be the universal family. W.l.o.g. we assume that

$$\mathcal{D}=\mathcal{D}_1+\cdots+\mathcal{D}_m,$$

where the sections $\mathcal{D}_i\in\Gamma(\mathcal{C}/\mathcal{V})$ are disjoint.

Assume that $\widetilde{x}\in\mathfrak{N}^6\in N^\dagger$. We shall make the following choices on $\mathcal{D}_1$ and $\mathcal{D}_2$. If $\mathfrak{N}^6\in N^6_{a;1}$, the normalization at the two nodes corresponding to $\mathfrak{E}_{\mathbf{n}_2(\mathfrak{t}^1)}(\mathfrak{t}^1)\cap(\mathfrak{E}^\perp_{\mathbf{m};\min})^{\succeq}(\mathfrak{t}^1)$ yields two (possibly nodal) rational sub-curves of C; w.l.o.g. we assume that $\mathcal{D}_1(x)$ and $\mathcal{D}_2(x)$ are on each of them, respectively. Similarly, if $\mathfrak{N}^6\in N^6_{a;2}\sqcup N^6_{a;3}\sqcup N^6_b$, we take $\mathcal{D}_1(x)$ and $\mathcal{D}_2(x)$ to be on each of the two rational sub-curves of C obtained from the normalization at the two nodes corresponding to $\mathfrak{E}_{\mathbf{n}_1(\mathfrak{t}^1)}(\mathfrak{t}^1)\cap(\mathfrak{E}^\perp_{\mathbf{m};\min})^{\succeq}(\mathfrak{t}^1)$ and $\mathfrak{E}_{\mathbf{n}_1(\mathfrak{t}^s)}(\mathfrak{t}^s)\cap(\mathfrak{E}^\perp_{\mathbf{m};\min})^{\succeq}(\mathfrak{t}^s)$ $(s=2,3,5)$, respectively. If $\mathfrak{N}^6\in N^6_c$, then $\mathbf{w}((\gamma^6)^\star_{\rm cor})=2$, which we assume to come from $\mathcal{D}_1(x)$ and $\mathcal{D}_2(x)$. If $\mathfrak{N}^6\in N^6_c$, then $\mathbf{w}\big((\gamma^4_{\rm vi3})^\star_{\rm cor}\big)=1$, which we assume to come from $\mathcal{D}_1(x)$; we then take $\mathcal{D}_2(x)$ to be on the rational sub-curve of C obtained from the normalization at the node corresponding to $\mathfrak{E}_{\mathbf{n}_1(\mathfrak{t}^4)}(\mathfrak{t}^4)\cap(\mathfrak{E}^\perp_{\mathbf{m};\min})^{\succeq}(\mathfrak{t}^4)$.

Let $K^{\mathcal{V}}_{12}\subset\mathcal{V}\longrightarrow\mathfrak{D}_2$ be the locus on which $\mathcal{D}_1$ and $\mathcal{D}_2$ are conjugate. Shrinking $\mathcal{V}$ if necessary, we can further assume that $K^{\mathcal{V}}_{12}=\varnothing$ if $\mathcal{D}_1(x)$ and $\mathcal{D}_2(x)$ are not conjugate. By [9, Lemma 2.8.2], $K^{\mathcal{V}}_{12}$ is a Cartier divisor of $\mathcal{V}$, hence there exists $\kappa^{\mathcal{V}}_{12}\in\Gamma(\mathscr{O}_{\mathcal{V}})$ such that

$$K^{\mathcal{V}}_{12}=\{\kappa^{\mathcal{V}}_{12}=0\}. \tag{4.38}$$

Let $\mathfrak{U}=\mathfrak{U}_{\widetilde{x}}\longrightarrow\mathfrak{M}^6$ be a twisted chart that is centered at $\widetilde{x}$ and satisfies $\varpi(\mathfrak{U})\subset\mathcal{V}$, and $\xi^{\mathfrak{U}}_s$, $s\in\widetilde{\mathbb{I}}(\mathfrak{t}^5)$, be the twisted parameters on $\mathfrak{U}$; see Theorem 2.18 for notation. We denote by $\breve{K}^{\mathfrak{U}}_{12}$ the proper transform of $K^{\mathcal{V}}_{12}$ in $\mathfrak{U}$, which is still a Cartier divisor, hence there exists $\kappa^{\mathfrak{U}}_{12}\in\Gamma(\mathscr{O}_{\mathfrak{U}})$ such that

$$\breve{K}^{\mathfrak{U}}_{12}=\{\kappa^{\mathfrak{U}}_{12}=0\}. \tag{4.39}$$

Lemma 4.26. *With notation as above, the function $\kappa^{\mathfrak{U}}_{12}$ can be taken so that $\{\xi^{\mathfrak{U}}_s\}_{s\in\widetilde{\mathbb{I}}(\mathfrak{t}^5)}\sqcup\{\kappa^{\mathfrak{U}}_{12}\}$ is a subset of local parameters on $\mathfrak{U}$. Moreover, for any $\widetilde{y}\in\mathfrak{U}$ and any twisted chart $\mathfrak{U}'=\mathfrak{U}_{\widetilde{y}}$ centered at $\widetilde{y}$, we have*

$$\breve{K}^{\mathfrak{U}}_{12}\cap\mathfrak{M}^\dagger\cap\mathfrak{U}'=\breve{K}^{\mathfrak{U}'}_{12}\cap\mathfrak{M}^\dagger\cap\mathfrak{U},$$

hence the local substacks $\breve{K}^{\mathfrak{U}}_{12}\cap\mathfrak{M}^\dagger$ of $\mathfrak{U}\cap\mathfrak{M}^\dagger$ can be glued together to form a substack $K\subset\mathfrak{M}^\dagger$.

Proof. Assume that $\widetilde{x}\in\mathfrak{N}^6\,(\in N^\dagger)$. Let $K^{\mathcal{V}}_{12}$ be as in (4.38) and $x=(C,L)\in\mathfrak{P}_2$ be the image of $\widetilde{x}$. We denote by $C_{\rm cor}$ the smallest connected genus 2 sub-curve of C and by $\langle\delta_1\rangle$ and $\langle\delta_2\rangle$ respectively the images of $\mathcal{D}_1(x)$ and $\mathcal{D}_2(x)$ on $C_{\rm cor}$ after contracting all the irreducible components of C that are away from $C_{\rm cor}$. It is possible that $\langle\delta_1\rangle=\langle\delta_2\rangle$.

According to [9, §2.2 & Lemma 2.8.2], there are two possible situations for $K^{\mathcal{V}}_{12}$: either $K^{\mathcal{V}}_{12}$ is smooth and transverse to every local divisor $\{\zeta_e=0\}$ as in (4.2), or $\langle\delta_1\rangle$ and $\langle\delta_2\rangle$ belong to a non-separating bridge B, i.e. a chain of rational irreducible components whose complement in $C_{\rm cor}$ is connected.

In the former situation, $\kappa^{\mathfrak{U}}_{12}$ can simply be taken as the pullback of $\kappa^{\mathcal{V}}_{12}$ to $\mathfrak{U}$ so that the first statement of Lemma 4.26 holds; see Theorem 2.18(p_3).

In the latter situation, w.l.o.g. we assume that B is the largest non-separating bridge containing $\langle\delta_1\rangle$ and $\langle\delta_2\rangle$. The nodes where B is attached to a connected genus 1 sub-curve C_1 of $C_{\rm cor}$ are denoted by q_{ϵ_1} and q_{ϵ_2}. For $i=1,2$, let $E_i\subset{\rm Edg}(\gamma_{C_{\rm cor}})$ index the nodes between C_1 and $\langle\delta_i\rangle$, satisfying that E_1 and E_2 are disjoint. The positions of $\langle\delta_1\rangle$ and $\langle\delta_2\rangle$ imply that

$$\mathfrak{N}^6\in N^6_{a;1}\cup N^6_{a;3}\cup N^6_b\cup N^6_d,\qquad E_1\sqcup E_2\subset{\rm E}_{\mathfrak{N}^3}={\rm Edg}(\tau_{\mathfrak{N}^3}),\qquad \ell_{\mathfrak{t}^3}(\hbar^+_{\epsilon_1})=\ell_{\mathfrak{t}^3}(\hbar^+_{\epsilon_2})=:l_0.$$

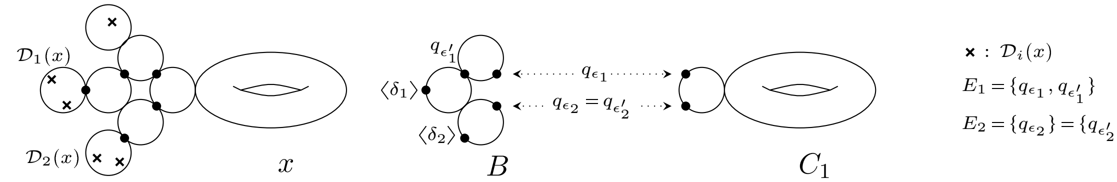


FIGURE 5. A point x of $K_{12}^{\mathcal{V}}$

The edges ϵ_1 and ϵ_2 labeling the nodes q_{ϵ_1} and q_{ϵ_2} are respectively the maximal elements of E_1 and E_2 (relative to the tree order on $\tau_{\mathfrak{N}^3}$); their minimal elements are respectively denoted by ϵ_1' and ϵ_2'; see Figure 5 for illustration.

By [9, §2.2 & Lemma 2.8.2], the parameters ζ_e on $\mathcal{V}$ can be chosen such that

$$\kappa_{12}^{\mathcal{V}}=\prod_{e\in E_1}\zeta_e+\prod_{e\in E_2}\zeta_e.$$

Let $\varpi_{4;6}:\mathfrak{M}^6\longrightarrow\mathfrak{M}^4$ be the composite forgetful morphism, $\widetilde{x}^4=\varpi_{4;6}(\widetilde{x})$, and $\mathfrak{U}^3$ be a twisted chart of $\mathfrak{M}^4$ centered at $\widetilde{x}^4$.

If $\mathfrak{N}^6\in N_{a;1}^6\cup N_{a;3}^6$, then $\ell_{\mathfrak{t}^3}(\hbar^-_{\epsilon_1'})\leqslant\mathbf{m}(\mathfrak{t}^3)$ and $\ell_{\mathfrak{t}^3}(\hbar^-_{\epsilon_2'})=\mathbf{m}(\mathfrak{t}^3)$. By Theorem 2.18 ($\mathsf{p}_4$), the products $\prod_{e\in E_1}\zeta_e$ and $\prod_{e\in E_2}\zeta_e$ respectively pull back to

$$\prod_{i\in[\![\mathbf{m},l_0)\!)_{\mathfrak{t}^3}}\varpi_{4;6}^*\,\xi_i^{\mathfrak{U}^3}\cdot\prod_{e\in E_1}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3}\qquad\text{and}\qquad\prod_{i\in[\![\mathbf{m},l_0)\!)_{\mathfrak{t}^3}}\varpi_{4;6}^*\,\xi_i^{\mathfrak{U}^3}\cdot\prod_{e\in E_2}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3},$$

where $\xi_e^{\mathfrak{U}^3}\equiv 1$ whenever e is one of the special edges e_i of $\mathfrak{t}^3$ as in Theorem 2.18 (p_2). We observe that $\prod_{e\in E_1}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3}$ and $\prod_{e\in E_2}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3}$ are products of *pairwise distinct* local parameters on $\mathfrak{U}$. Moreover, $\prod_{e\in E_2}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3}$ is a unit on $\mathfrak{U}$. Thus,

$$\kappa_{12}^{\mathfrak{U}}=\prod_{e\in E_1}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3}+\prod_{e\in E_2}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3}$$

is a local parameter on $\mathfrak{U}$ that defines $\breve{K}_{12}^{\mathfrak{U}}$. The construction of the rooted tree $\tau_{\mathfrak{N}^5}$ that determines the twisted parameters on $\xi_s^{\mathfrak{U}}$ implies that $\{\xi_s^{\mathfrak{U}}\}_{s\in\widetilde{\mathbb{I}}(\mathfrak{t}^5)}\sqcup\{\kappa_{12}^{\mathfrak{U}}\}$ is a subset of local parameters on $\mathfrak{U}$.

If $\mathfrak{N}^6\in N_b^6\cup N_d^6$, the argument is analogous to the last paragraph, possibly with a new situation when $\ell_{\mathfrak{t}^3}(\hbar^-_{\epsilon_1'})\leqslant\mathbf{m}(\mathfrak{t}^3)$ but $\ell_{\mathfrak{t}^3}(\hbar^-_{\epsilon_2'})>\mathbf{m}(\mathfrak{t}^3)$. In that case, the products $\prod_{e\in E_1}\zeta_e$ and $\prod_{e\in E_2}\zeta_e$ respectively pull back to

$$\prod_{i\in[\![\mathbf{m},l_0)\!)_{\mathfrak{t}^3}}\varpi_{4;6}^*\,\xi_i^{\mathfrak{U}^3}\cdot\prod_{e\in E_1}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3}\qquad\text{and}\qquad\prod_{i\in[\![\ell_{\mathfrak{t}^3}(\hbar^-_{\epsilon_2'}),\,l_0)\!)_{\mathfrak{t}^3}}\varpi_{4;6}^*\,\xi_i^{\mathfrak{U}^3}\cdot\prod_{e\in E_2}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3},$$

and $\prod_{e\in E_2}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3}$ is a unit on $\mathfrak{U}$. Thus,

$$\kappa_{12}^{\mathfrak{U}}=\prod_{i\in[\![\mathbf{m}\,,\,\ell_{\mathfrak{t}^3}(\hbar^-_{\epsilon_2'}))\!)_{\mathfrak{t}^3}}\varpi_{4;6}^*\,\xi_i^{\mathfrak{U}^3}\cdot\prod_{e\in E_1}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3}+\prod_{e\in E_2}\varpi_{4;6}^*\,\xi_e^{\mathfrak{U}^3}$$

is a local parameter on $\mathfrak{U}$ that defines $\breve{K}_{12}^{\mathfrak{U}}$, which together with $\{\xi_s^{\mathfrak{U}}\}_{s\in\widetilde{\mathbb{I}}(\mathfrak{t}^5)}$ still form a subset of local parameters on $\mathfrak{U}$. This establishes the first statement of Lemma 4.26.

The second statement of Lemma 4.26 follows from a direct check. The key fact is that for every $\widetilde{x}\in\mathfrak{N}^6\in N^\dagger$, the choices of $\mathcal{D}_1,\mathcal{D}_2\in\Gamma(\mathcal{C}/\mathcal{V})$ described in the paragraph above (4.38) may not be unique. For instance, in Figure 5, we may choose the sections $\mathcal{D}_i$ such that $\mathcal{D}_1(x)$ (resp. $\mathcal{D}_2(x)$)

is the other marked point on the same irreducible component. Nonetheless, we observe that $\breve{K}_{12}^{\mathfrak{U}}\cap\mathfrak{M}^{\dagger}$ is independent of such choices (even though $\breve{K}_{12}^{\mathfrak{U}}$ is). This, along with (4.37), gives rise to the second statement of Lemma 4.26. $\square$

Next, we construct a treelike structure Λ^6 on $(\mathfrak{M}^6,\Gamma^6_{\mathrm{vi}_3})$ satisfying the second assumption and (3.6) of Corollary 3.3. For each $\mathfrak{N}^6$ as in (4.36), we describe the tuple $(\tau_{\mathfrak{N}^6},\mathrm{E}_{\mathfrak{N}^6},\beta_{\mathfrak{N}^6})$ as follows.

Let $\widehat{\tau}^5$ be as above (4.32). Then by (4.33),

$$\mathrm{Edg}(\tau_{\mathfrak{N}^5}) = \mathrm{E}_{\mathfrak{N}^5} \subset \mathrm{Edg}(\widehat{\tau}^5) \subset \mathrm{Edg}(\gamma^5)\backslash\{\, \mathsf{e}_i^k : i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k},\ k=2,3,4\} \subset \mathrm{Edg}(\gamma^5). \tag{4.40}$$

We can thus take

$$\widehat{\tau}^6 := \rho_{\widehat{\tau}^5;[\mathfrak{t}^5]},$$

and let $(\widehat{\tau}^6)^\star$ be the induced decoration of $\widehat{\tau}^6$. Since $\widehat{\tau}^5$ is a rooted tree, so is $\widehat{\tau}^6$. Analogous to (4.32),

$$\mathrm{Edg}(\widehat{\tau}^6) \subset \mathrm{Edg}(\gamma^6)\backslash\{\mathsf{e}_i^k : k=2,3,4;\ i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k}\} \subset \mathrm{Edg}(\gamma^6), \qquad o_{\widehat{\tau}^6} \subset o_{\gamma^6}. \tag{4.41}$$

We are ready to construct $\tau_{\mathfrak{N}^6}$.

- If $\mathfrak{N}^6\in N^6_{a;1}$, we dissolve the image of $\mathsf{v}^1_{\mathbf{n}_2(\mathfrak{t}^1)}$ in $\widehat{\tau}^6$ as well as all the non-root vertices v of $\widehat{\tau}^6$ with $\mathbf{w}(v)>0$, and take $\tau_{\mathfrak{N}^6}$ to be the unique connected component that contains $o_{\widehat{\tau}^6}$.
- If $\mathfrak{N}^6\in N^6_b$ (which implies $\mathbf{n}_2(\mathfrak{t}^1)\leqslant\mathbf{n}_1(\mathfrak{t}^1)<0$), consider the following two sub-cases:
  - if $(\mathfrak{E}^{\perp}_{\mathbf{m};\min}(\mathfrak{t}^5))^{\succeq}=\{\mathsf{e}_i^1 : i\in[\![\mathbf{n}_1(\mathfrak{t}^1),0)\!)_{\mathfrak{t}^1}\}$, then we dissolve the image of $\mathsf{v}^1_{\mathbf{n}_2(\mathfrak{t}^1)}$ in $\widehat{\tau}^6$ as well as all the non-root vertices v of $\widehat{\tau}^6$ with $\mathbf{w}(v)>0$ and take $\tau_{\mathfrak{N}^6}$ to be the unique connected component that contains $o_{\widehat{\tau}^6}$;
  - otherwise, we first dissolve the image of $\mathsf{v}^1_{\mathbf{n}_1(\mathfrak{t}^1)}$ in $\widehat{\tau}^6$ as well as all the non-root vertices v of $\widehat{\tau}^6$ with $\mathbf{w}(v)>0$, then contract the edges in
  $$\big\{\,\mathsf{e}_i^5 : i\in[\![\mathbf{m}(\mathfrak{t}^5),\lambda)\!)_{\mathfrak{t}^5}\big\} \quad\text{with}\quad \lambda:=\min\big(\{\ell_{\mathfrak{t}^5}(\mathsf{v}_j^1) : \mathsf{e}_j^1\in(\mathfrak{E}^{\perp}_{\mathbf{m};\min}(\mathfrak{t}^5))^{\succeq}\}\sqcup\{0\}\big),$$
  and finally take $\tau_{\mathfrak{N}^6}$ to be the unique connected component that contains $o_{\widehat{\tau}^6}$.
- If $\mathfrak{N}^6\in N^6_{a;2}\cup N^6_{a;3}\cup N^6_d$, we dissolve the image of $\mathsf{v}^1_{\mathbf{n}_1(\mathfrak{t}^1)}$ in $\widehat{\tau}^6$ as well as all the non-root vertices v of $\widehat{\tau}^6$ with $\mathbf{w}(v)>0$, and take $\tau_{\mathfrak{N}^6}$ to be the unique connected component that contains $o_{\widehat{\tau}^6}$.
- For other $\mathfrak{N}^6$, we set $\tau_{\mathfrak{N}^6}=\tau_\bullet$ and $o_{\mathfrak{N}^6}=o_{\gamma^6}$.

By (4.41), we have

$$\mathrm{Edg}(\tau_{\mathfrak{N}^6}) \subset \mathrm{Edg}(\widehat{\tau}^6) \subset \mathrm{Edg}(\gamma^6)\backslash\{\mathsf{e}_i^k : k=2,\cdots,5;\ i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k}\} \subset \mathrm{Edg}(\gamma^6), \tag{4.42}$$

and moreover, $\mathrm{Edg}(\tau_{\mathfrak{N}^6})$ is the set of the edges of a *connected* subgraph $\tau'_{\mathfrak{N}^6}$ of γ^6 that satisfies

$$o_{\mathfrak{N}^6} \subset o_{\gamma^6} \in \mathrm{Ver}(\tau'_{\mathfrak{N}^6}).$$

We set

$$\Lambda^6 = \big(\,(\tau_{\mathfrak{N}^6},\ \mathrm{E}_{\mathfrak{N}^6} := \mathrm{Edg}(\tau_{\mathfrak{N}^6}),\ \mathrm{Id})\,\big)_{\gamma^6\in\Gamma^6_{\mathrm{vi}_3},\,\mathfrak{N}^6\in\pi_0(\mathfrak{M}^6_{\gamma^6})}.$$

Mimicking the proof of Lemma 4.22 and taking Lemma 4.25 into consideration, we obtain the following statement.

Lemma 4.27. *The set Λ^6 gives a treelike structure on $(\mathfrak{M}^6,\Gamma^6_{\mathrm{vi}_3})$ as in Definition 2.16.*

Notice that $\mathfrak{N}^6\in N^\dagger$ whenever $\tau_{\mathfrak{N}^6}\neq\tau_\bullet$. Corollary 3.3, Lemmas 4.26 and 4.27, and Theorem 2.18 together lead to the following statement.

Corollary 4.28. *The stack* $\mathfrak{M}^6$ *has a* Γ^6_{gft}*-stratification*

$$\mathfrak{M}^6 = \bigsqcup_{\gamma' \in \Gamma^6_{\mathrm{gft}}} (\mathfrak{M}^6_{\mathrm{gft}})_{\gamma'}, \qquad \text{where} \quad \Gamma^6_{\mathrm{gft}} = (\Gamma^6_{\mathrm{vi3}})_{\mathrm{gft}},$$

which is the grafted stratification with respect to $(\mathfrak{M}^6, \Gamma^6_{\mathrm{vi3}}, \Lambda^6)$*, along with the induced treelike structure* Λ^6_{gft} *as in Corollary 3.3. Furthermore, the stack* $(\mathfrak{M}^6)^{\mathrm{tf}}_{\Lambda^6_{\mathrm{gft}}}$ *as in Theorem 2.18* (p_1) *is a smooth algebraic stack and the composite forgetful morphism* $(\mathfrak{M}^6)^{\mathrm{tf}}_{\Lambda^6_{\mathrm{gft}}} \longrightarrow \mathfrak{M}^1 = \mathfrak{P}_2$ *is proper and birational.*

4.8. **The seventh step of the recursive construction.** In this step, we take

$$\mathfrak{M}^7 = (\mathfrak{M}^6)^{\mathrm{tf}}_{\Lambda^6_{\mathrm{gft}}}, \qquad \Gamma^7 = (\Gamma^6_{\mathrm{gft}})^{\mathrm{der}}_{\Lambda^6_{\mathrm{gft}}}.$$

Analogous to Lemma 4.11, we have the following statement.

Lemma 4.29. *The stack* $\mathfrak{M}^7$ *comes equipped with a* Γ^7*-stratification that is the derived stratification of* $\mathfrak{M}^7$ *with respect to* $(\mathfrak{M}^6, \Gamma^6_{\mathrm{gft}}, \Lambda^6_{\mathrm{gft}})$*.*

The graphs of Γ^7 are in the form

$$\begin{aligned} &\gamma^7 := \rho_{\gamma^6_{\mathrm{gft}}, [\mathfrak{t}^6]} \in \Gamma^7, \\ &\text{where} \quad \gamma^6_{\mathrm{gft}} := (\gamma^6, o_{\gamma^6})_{\mathrm{gft}} \in \Gamma^6_{\mathrm{gft}}, \quad [\mathfrak{t}^6] \in \overline{\Lambda}^6_{\mathfrak{N}^6_{\mathrm{gft}}}, \quad \mathfrak{N}^6_{\mathrm{gft}} := \mathfrak{N}^6 \cap K \in \pi_0\big((\mathfrak{M}^6_{\mathrm{gft}})_{\gamma^6_{\mathrm{gft}}}\big); \end{aligned} \tag{4.43}$$

see (4.35) for γ^6, (4.36) for $\mathfrak{N}^6$, and Lemma 4.26 for K. Let o_{γ^7} be the image of o_{γ^6} in γ^5 and $(\gamma^7)^\star$ be the derived decoration. By Corollary 3.2, for every $\gamma^7 \in \Gamma^7$ as in (4.43), the connected components of $\mathfrak{M}^7_{\gamma^7}$ are in the form

$$\mathfrak{N}^7 := (\mathfrak{N}^6_{\mathrm{gft}})^{\mathrm{tf}}_{[\mathfrak{t}^6]} \in \pi_0\big(\mathfrak{M}^7_{\gamma^7}\big). \tag{4.44}$$

For each $\mathfrak{N}^7$, we will describe the tuple $(\tau_{\mathfrak{N}^7}, \mathrm{E}_{\mathfrak{N}^7}, \beta_{\mathfrak{N}^7})$ to provide Λ^7 on $(\mathfrak{M}^7, \Gamma^7)$.

Let

$$\begin{aligned} \mathrm{E}^6 &:= \mathfrak{E}^{\perp}_{\geqslant \mathbf{m}}(\mathfrak{t}^6) \subset \mathrm{Edg}(\tau_{\mathfrak{N}^6_{\mathrm{gft}}}) = \mathrm{E}_{\mathfrak{N}^6_{\mathrm{gft}}} \\ &\subset \mathrm{Edg}(\hat{\tau}^6_{\mathrm{gft}}) \subset \mathrm{Edg}(\gamma^6_{\mathrm{gft}}) \backslash \{\mathsf{e}^k_i : i \in [\![\mathbf{m}, 0)\!)_{\mathfrak{t}^k}, k = 2, 3, 4, 6\} \subset \mathrm{Edg}(\gamma^6_{\mathrm{gft}}), \end{aligned} \tag{4.45}$$

which follows from (4.42). Consider the rooted tree

$$\hat{\tau}^7 := (\hat{\tau}^6_{\mathrm{gft}})_{(\mathrm{E}^6 \sqcup \{\mathsf{e}^k_i : i \in [\![\mathbf{m}, 0)\!)_{\mathfrak{t}^k}, k = 1, 5\})}, \tag{4.46}$$

whose induced decoration is denoted by $(\hat{\tau}^7)^\star$. By (4.45) and (4.46), we have

$$\mathrm{Edg}(\hat{\tau}^7) \subset \mathrm{Edg}(\gamma^7) \backslash \{\mathsf{e}^k_i : i \in [\![\mathbf{m}, 0)\!)_{\mathfrak{t}^k}, k = 1, \cdots, 6\} \subset \mathrm{Edg}(\gamma^7). \tag{4.47}$$

We are ready to construct $\tau_{\mathfrak{N}^7}$.

- If either $\mathbf{w}(o_{\hat{\tau}^7}) > 2$, or $\mathbf{w}(o_{\hat{\tau}^7}) = 2$ but $e_{\mathrm{gft}} \in \mathrm{E}^6$, we set $\tau_{\mathfrak{N}^7} = \tau_\bullet$ and $o_{\mathfrak{N}^7} = o_{\gamma^7}$.
- If $\mathbf{w}(o_{\hat{\tau}^7}) = 2$ and $e_{\mathrm{gft}} \notin \mathrm{E}^6$, we dissolve all the non-root vertices v in $\hat{\tau}^7$ satisfying $\mathbf{w}(v) > 0$ and take $\tau_{\mathfrak{N}^7}$ to be the unique connected component that contains $o_{\hat{\tau}^7}$. Thus, $\tau_{\mathfrak{N}^7}$ is also a rooted tree with $o_{\mathfrak{N}^7} = o_{\hat{\tau}^7}$.

By (4.47) and the above construction of $\tau_{\mathfrak{N}^7}$, we observe that

$$\mathrm{Edg}(\tau_{\mathfrak{N}^7}) \subset \mathrm{Edg}(\hat{\tau}^7) \subset \mathrm{Edg}(\gamma^7) \backslash \{\mathsf{e}^k_i : i \in [\![\mathbf{m}, 0)\!)_{\mathfrak{t}^k}, k = 1, \cdots, 6\} \subset \mathrm{Edg}(\gamma^7), \tag{4.48}$$

and moreover, $\mathrm{Edg}(\tau_{\mathfrak{N}^7})$ is the set of the edges of a *connected* subgraph $\tau'_{\mathfrak{N}^7}$ of γ^7 that satisfies

$$o_{\tau_{\mathfrak{N}^7}} \subset o_{\gamma^7} \in \mathrm{Ver}(\tau'_{\mathfrak{N}^7}).$$

We set

$$\Lambda^7 = \big(\, (\tau_{\mathfrak{N}^7},\ \mathrm{E}_{\mathfrak{N}^7} := \mathrm{Edg}(\tau_{\mathfrak{N}^7}),\ \mathrm{Id})\,\big)_{\gamma^7\in\Gamma^7,\,\mathfrak{N}^7\in\pi_0(\mathfrak{M}^7_{\gamma^7})}.$$

The following statement is the analogue of Lemmas 4.9 and 4.19 in this step.

Lemma 4.30. *The set Λ^7 gives a treelike structure on $(\mathfrak{M}^7,\Gamma^7)$ as in Definition 2.16.*

Lemma 4.30, along with Theorem 2.18, gives rise to the following statement.

Corollary 4.31. *Let $(\mathfrak{M}^7)^{\mathrm{tf}}_{\Lambda^7}$ be constructed as in Theorem 2.18 (p_1). Then, $(\mathfrak{M}^7)^{\mathrm{tf}}_{\Lambda^7}$ is a smooth algebraic stack and the composite forgetful morphism $(\mathfrak{M}^7)^{\mathrm{tf}}_{\Lambda^7}\longrightarrow\mathfrak{M}^1=\mathfrak{P}_2$ is proper and birational.*

4.9. **The eighth step of the recursive construction.** In this step, we take

$$\mathfrak{M}^8 = (\mathfrak{M}^7)^{\mathrm{tf}}_{\Lambda^7}, \qquad \Gamma^8 = (\Gamma^7)^{\mathrm{der}}_{\Lambda^7}.$$

Analogous to Lemma 4.11, we have the following statement.

Lemma 4.32. *The stack $\mathfrak{M}^8$ comes equipped with a Γ^8-stratification that is the derived stratification of $\mathfrak{M}^8$ with respect to $(\mathfrak{M}^7,\Gamma^7,\Lambda^7)$.*

The graphs of Γ^8 are in the form

$$\gamma^8 := \rho_{\gamma^7,[\mathfrak{t}^7]}\in\Gamma^8, \qquad \text{where} \quad \gamma^7\in\Gamma^7,\ \ [\mathfrak{t}^7]\in(\overline{\Lambda}^7)_{\mathfrak{N}^7},\ \ \mathfrak{N}^7\in\pi_0(\mathfrak{M}^7_{\gamma^7}); \tag{4.49}$$

see (4.43) for γ^7 and (4.44) for $\mathfrak{N}^7$. Let o_{γ^8} be the image of o_{γ^7} in γ^7 and $(\gamma^8)^\star$ be the derived decoration. By Corollary 3.2, for every $\gamma^8\in\Gamma^8$, the connected components of $\mathfrak{M}^8_{\gamma^8}$ are in the form

$$\mathfrak{N}^8 := (\mathfrak{N}^7)^{\mathrm{tf}}_{[\mathfrak{t}^7]}\ \in\pi_0\big(\mathfrak{M}^8_{\gamma^8}\big). \tag{4.50}$$

For each $\mathfrak{N}^8$, we will describe the tuple $(\tau_{\mathfrak{N}^8},\mathrm{E}_{\mathfrak{N}^8},\beta_{\mathfrak{N}^8})$ to provide Λ^8 on $(\mathfrak{M}^8,\Gamma^8)$.

Let

$$\begin{aligned}\mathrm{E}^7 &= \mathfrak{E}^{\perp}_{\geqslant\mathbf{m}}(\mathfrak{t}^7)\subset\mathrm{Edg}(\tau_{\mathfrak{N}^7})=\mathrm{E}_{\mathfrak{N}^7}\\ &\subset\mathrm{Edg}(\widehat{\tau}^7)=\mathrm{Edg}(\gamma^7)\backslash\{\mathsf{e}^k_i : i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k},\ k=1,\cdots,6\}\subset\mathrm{Edg}(\gamma^7),\end{aligned}\tag{4.51}$$

which follows from (4.48). By (4.31), the derived graph $\widetilde{\gamma}^2_{(\mathrm{E}^3)}$ is well-defined and satisfies

$$\mathrm{E}^4\subset\mathrm{Edg}(\widetilde{\gamma}^2_{(\mathrm{E}^3)}),$$

so we can consider the derived graph

$$\widetilde{\gamma}^4 := \rho_{\widetilde{\gamma}^2_{(\mathrm{E}^3)};[\mathfrak{t}^4]}$$

and the derived decoration $(\widetilde{\gamma}^4)^\star$. Let

$$\widetilde{\tau}^5 \qquad\text{and}\qquad (\widetilde{\tau}^5)^\star$$

be the rooted tree obtained from $\widetilde{\gamma}^4$ by contracting the edges of $(\widetilde{\gamma}^4)^\star_{\mathrm{cor}}$ into the root $o_{\widetilde{\tau}^5}$ and the induced decoration, respectively.

By (3.1), we have

$$\mathrm{Edg}(\widetilde{\gamma}^2_{(\underline{\mathrm{E}})^4})\subset\mathrm{Edg}(\widetilde{\gamma}^4) \qquad\text{and}\qquad \mathrm{Edg}\big((\widetilde{\gamma}^2_{(\underline{\mathrm{E}})^4})^\star_{\mathrm{cor}}\big)=\mathrm{Edg}\big((\widetilde{\gamma}^4)^\star_{\mathrm{cor}}\big).$$

Therefore,

$$\mathrm{Edg}(\widehat{\tau}^5)\subset\mathrm{Edg}(\widetilde{\tau}^5).$$

Parallel to the constructions of $\widehat{\tau}^6$ above (4.41) and $\widehat{\tau}^7$ in (4.46), we define

$$\widetilde{\tau}^7 = \big((\rho_{\widetilde{\tau}^5;[\mathfrak{t}^5]})_{\mathrm{gft}}\big)_{(\mathrm{E}^6\sqcup\{\mathsf{e}^k_i:\, i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k},\, k=1,5\})}$$

and observe that

$$\mathrm{E}^7\subset\mathrm{Edg}(\widehat{\tau}^7)\subset\mathrm{Edg}(\widetilde{\tau}^7)\subset\mathrm{Edg}(\gamma^7).$$

Let $\hat{\tau}^8 := \tilde{\tau}^7_{(\mathrm{E}^8)}$, along with the induced decoration $(\hat{\tau}^8)^\star$. Then,

$$\mathrm{Edg}(\hat{\tau}^8) \subset \mathrm{Edg}(\gamma^8)\backslash\{\mathsf{e}^k_i : i \in [\![\mathbf{m}, 0)\!)_{\mathfrak{t}^k}, k = 1, 2, 3, 5, 6, 7\} \subset \mathrm{Edg}(\gamma^8). \tag{4.52}$$

We are ready to construct $\tau_{\mathfrak{N}^8}$.

- If either $\mathfrak{N}^6 \notin N^6_d$, or $\mathfrak{N}^6 \in N^6_d$ but either $e_{\mathrm{gft}} \in \mathrm{E}^6 \sqcup \mathrm{E}^7$ or $\mathbf{w}\big((\gamma^6)^\star_{\mathrm{cor}}\big) < \mathbf{w}\big((\gamma^8)^\star_{\mathrm{cor}}\big)$, we set $\tau_{\mathfrak{N}^8} = \tau_\bullet$ and $o_{\mathfrak{N}^8} = o_{\gamma^8}$.
- If $\mathfrak{N}^6 \in N^6_d$, $e_{\mathrm{gft}} \notin \mathrm{E}^6 \sqcup \mathrm{E}^7$, and $\mathbf{w}\big((\gamma^6)^\star_{\mathrm{cor}}\big) = \mathbf{w}\big((\gamma^8)^\star_{\mathrm{cor}}\big)$, then we dissolve $\mathsf{v}^4_{\mathbf{n}_1(\mathfrak{t}^4)}$ as well as all the non-root vertices v in $\hat{\tau}^8$ satisfying $\mathbf{w}(v) > 0$, and take $\tau_{\mathfrak{N}^8}$ to be the unique connected component that contains $o_{\hat{\tau}^8}$. Thus, $\tau_{\mathfrak{N}^8}$ is a rooted tree with $o_{\mathfrak{N}^8} = o_{\hat{\tau}^8}$.

By (4.52) and the above construction of $\tau_{\mathfrak{N}^8}$, we observe that

$$\mathrm{Edg}(\tau_{\mathfrak{N}^8}) \subset \mathrm{Edg}(\hat{\tau}^8) \subset \mathrm{Edg}(\gamma^8)\backslash\{\mathsf{e}^k_i : i \in [\![\mathbf{m}, 0)\!)_{\mathfrak{t}^k}, k = 1, 2, 3, 5, 6, 7\} \subset \mathrm{Edg}(\gamma^8), \tag{4.53}$$

and moreover, $\mathrm{Edg}(\tau_{\mathfrak{N}^8})$ is the set of the edges of a *connected* subgraph $\tau'_{\mathfrak{N}^8}$ of γ^8 that satisfies

$$o_{\tau_{\mathfrak{N}^8}} \subset o_{\gamma^8} \in \mathrm{Ver}(\tau'_{\mathfrak{N}^8}).$$

We set

$$\Lambda^8 = \big(\,(\tau_{\mathfrak{N}^8},\, \mathrm{E}_{\mathfrak{N}^8} := \mathrm{Edg}(\tau_{\mathfrak{N}^8}),\, \mathrm{Id})\,\big)_{\gamma^8 \in \Gamma^8,\, \mathfrak{N}^8 \in \pi_0(\mathfrak{M}^8_{\gamma^8})}.$$

The following statement is the analogue of Lemma 4.22 in this step.

Lemma 4.33. *The set Λ^8 gives a treelike structure on $(\mathfrak{M}^8, \Gamma^8)$ as in Definition 2.16.*

Lemma 4.33, along with Theorem 2.18, gives rise to the following statement.

Corollary 4.34. *Let $(\mathfrak{M}^8)^{\mathrm{tf}}_{\Lambda^8}$ be constructed as in Theorem 2.18 (p_1). Then,*

$$\widetilde{\mathfrak{P}}^{\mathrm{tf}}_2 := (\mathfrak{M}^8)^{\mathrm{tf}}_{\Lambda^8}$$

is a smooth algebraic stack and the composite forgetful morphism $\widetilde{\mathfrak{P}}^{\mathrm{tf}}_2 = (\mathfrak{M}^8)^{\mathrm{tf}}_{\Lambda^8} \longrightarrow \mathfrak{M}^1 = \mathfrak{P}_2$ is proper and birational.

4.10. **Proof of Theorem 1.2.** The properness of the forgetful morphism $\varpi : \widetilde{\mathfrak{P}}^{\mathrm{tf}}_2 \longrightarrow \mathfrak{P}_2$ established in Corollary 4.31 implies Theorem 1.2 (2). Moreover, $\varpi : \widetilde{\mathfrak{P}}^{\mathrm{tf}}_2 \longrightarrow \mathfrak{P}_2$ restricts to the identity map on the open subset $\varpi^{-1}(\mathfrak{M}_{\tau_\bullet})$, which gives rise to Theorem 1.2 (3).

It remains to prove Theorem 1.2 (1) and (4). By [9, Theorem 6.1.1 & Lemma 2.4.1], on a smooth chart $\mathcal{V} \longrightarrow \mathfrak{P}_2$, the stack $\overline{M}_2(\mathbb{P}^n, d)$ and the direct image sheaf $\pi_* \mathfrak{f}^* \mathscr{O}_{\mathbb{P}^n}(k)$ can respectively be identified with the kernel of homomorphisms:

$$(0 \oplus \varphi)^{\oplus n} : \big(\mathscr{O}_{\mathcal{V}}^{\oplus(m+1)}\big)^{\oplus n} \longrightarrow \big(\mathscr{O}_{\mathcal{V}}^{\oplus 2}\big)^{\oplus n},$$

where $m = d$ for $\overline{M}_2(\mathbb{P}^n, d)$ and $m = kd$ for $\pi_* \mathfrak{f}^* \mathscr{O}_{\mathbb{P}^n}(k)$. The homomorphisms φ for $\overline{M}_2(\mathbb{P}^n, d)$ coincides with that for $\pi_* \mathfrak{f}^* \mathscr{O}_{\mathbb{P}^n}(1)$. For each $k \geqslant 1$, φ can be considered as a $2 \times m$ matrix whose explicit expressions (for all possible chart $\mathcal{V} \longrightarrow \mathfrak{P}_2$) are given in [9, Proposition 2.6.1, Lemma 2.7.1, & Corollaries 2.7.2 & 2.7.3]. In particular, the entries of the matrix φ after suitable trivialization are monomials in the parameters corresponding to the smoothing of the nodes as in (4.2) and/or the function $\kappa^{\mathcal{V}}_{12}$ as in (4.38). Applying Theorem 2.18 (p_4) repeatedly to the entries of φ and taking suitable trivialization again if necessary, we observe that the pullback $\tilde{\varphi}^{\mathrm{tf}}$ of φ to any twisted chart $\mathfrak{U}_x \longrightarrow \widetilde{\mathfrak{P}}^{\mathrm{tf}}_2$ centered at any $\tilde{x} \in \widetilde{\mathfrak{P}}^{\mathrm{tf}}_2$ must take the form

$$\tilde{\varphi}^{\mathrm{tf}}|_{\mathfrak{U}_{\tilde{x}}} = \begin{pmatrix} z_1 & 0 & 0 & \cdots \\ 0 & z_2 & 0 & \cdots \end{pmatrix}, \tag{4.54}$$

where z_1 and z_2 are monomials in the pullbacks of the twisted parameters of $\widetilde{\mathfrak{P}}_2^{\rm tf}$ such that z_1 is a factor of z_2. Since this approach is parallel to the argument in [9, §5], we only provide the key steps below and omit further details in this paper.

To obtain (4.54), let $\varpi^k : \widetilde{\mathfrak{P}}_2^{\rm tf} \longrightarrow (\mathfrak{M}^k)^{\rm tf}_{\Lambda^k}$, $1 \leqslant k \leqslant 8$, be the corresponding composite forgetful morphism, $\mathfrak{U}^k := \mathfrak{U}_{\varpi^k(\widetilde{x})}$ be a twisted chart on $(\mathfrak{M}^k)^{\rm tf}_{\Lambda^k}$ centered at $\varpi^k(\widetilde{x})$, and $\xi_i^{\mathfrak{U}^k}$ be the twisted parameters on $\mathfrak{U}^k$; see Theorem 2.18 (p_2) for notation.

Let $\mathfrak{N}^9$ be the connected component of the stratum of the $(\Gamma^8)^{\rm der}_{\Lambda^8}$-stratification of $\widetilde{\mathfrak{P}}_2^{\rm tf}$ that contains $\widetilde{x}$. Following the recursive construction in §4.2-4.9, we see that $\mathfrak{N}^9$ is determined by a unique list:

$$\gamma \in \Gamma^1_{\rm vic}, \quad \mathfrak{N}^1 \in \pi_0(\mathfrak{M}^1_\gamma), \quad [\mathfrak{t}^1] \in \overline{\Lambda}_{\mathfrak{N}^1}, \quad \ldots \quad , \; [\mathfrak{t}^8] \in \overline{\Lambda}_{\mathfrak{N}^8}, \;\; \gamma^9 \in (\Gamma^8)^{\rm der}_{\Lambda^8}, \quad \mathfrak{N}^9 \in \pi_0\big((\widetilde{\mathfrak{P}}_2^{\rm tf})_{\gamma^9}\big).$$

Some of these rooted level trees $\mathfrak{t}^s$ are possibly trivial (i.e. the underlying rooted tree is edgeless).

If $\mathfrak{t}^2$ is trivial, we can find an entry in the first row of φ corresponding to a vertex on the $\mathbf{m}(\mathfrak{t}^1)$-th level in $\mathfrak{t}^1$, such that the pullback z_1 of this entry to $\widetilde{\mathfrak{P}}_2^{\rm tf}$, according to Theorem 2.18 (p_4), is in the form $\prod_{i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^1}}(\varpi^1)^*\xi_i^{\mathfrak{U}^1}$ up to multiplication by a unit. If $\mathfrak{t}^2$ is not trivial, we can analogously find an entry in the first row of φ corresponding to a vertex on the $\mathbf{m}(\mathfrak{t}^2)$-th level in $\mathfrak{t}^2$, such that the pullback z_1 of this entry to $\widetilde{\mathfrak{P}}_2^{\rm tf}$ is in the form $\prod_{k=1,2;\; i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k}}(\varpi^k)^*\xi_i^{\mathfrak{U}^k}$ up to multiplication by a unit. In either situation, we observe that z_1 is a factor of any other entries in the first row of $\widetilde{\varphi}^{\rm tf}$, thus after taking suitable trivialization, we obtain the first row as well as the leftmost 0 of the second row of (4.54).

Following the same idea, we can find an entry in the second row (excluding the first column) of φ corresponding to a vertex on the $\mathbf{m}(\mathfrak{t}^j)$-th level in $\mathfrak{t}^j$, where j is the last step of the recursive construction in which $\mathfrak{t}^j$ is non-trivial (i.e. $\mathbf{m}(\mathfrak{t}^j) \neq 0$). By considering all the possible treelike structures and $[\mathfrak{t}^i]$'s prior to the j-th step, we see that the pullback z_2 of this entry to $\widetilde{\mathfrak{P}}_2^{\rm tf}$ is in the form $\prod_{1\leqslant k\leqslant j,\; i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k}}(\varpi^k)^*\xi_i^{\mathfrak{U}^k}$ up to multiplication by a unit and z_2 is a factor of any entries in the second row of $\widetilde{\varphi}^{\rm tf}$. In sum, we obtain (4.54) with

$$z_1 = \prod_{k=1,2;\; i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k}} (\varpi^k)^*\xi_i^{\mathfrak{U}^k}\;, \qquad z_2 = \prod_{1\leqslant k\leqslant 8,\; i\in[\![\mathbf{m},0)\!)_{\mathfrak{t}^k}} (\varpi^k)^*\xi_i^{\mathfrak{U}^k}$$

up to multiplication by units.

The local expression (4.54) of $\widetilde{\varphi}^{\rm tf}$ implies that its kernel admits at worst normal crossing singularities, which justifies the properties (1) and (4) of Theorem 1.2 simultaneously.

References

[1] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *Compactification of strata of Abelian differentials*, Duke Math. J. 167 (2018), no. 12, 2347–2416.

[2] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *The moduli space of multi-scale differentials*, math/1910.13492

[3] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Holomorphic anomalies in topological field theories*, Nucl. Phys. B405 (1993), 279–304.

[4] H.-L. Chang and J. Li, *An algebraic proof of the hyperplane property of the genus oneGW-invariants of quintics,* JDG 100 (2015), no. 2, 251–299.

[5] T. Coates and C. Manolache, *A splitting of the virtual class for genus one stablemaps,* math/1809.04162

[6] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I*, Ann. of Math., 2, 79 (1), (1964) 109–203.

[7] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. II*, Ann. of Math., 2, 79 (1), (1964) 205–326.

[8] Y. Hu and J. Li, *Genus-one stable maps, local equations and Vakil-Zinger's desingularization*, Math. Ann. 348 (2010), no. 4, 929–963.
[9] Y. Hu, J. Li, and J. Niu, *Genus two stable maps, local equations and modular resolutions*, math/1201.2427v3
[10] Y. Hu and J. Niu, *Moduli of curves of genus one with twisted fields*, math/1906.10527
[11] A. de Jong, *Smoothness, semi-stability and alterations,* Publ. Math. IHES 83, (1996) 51-93.
[12] J. Kollár, *Lectures on resolution of singularities,* Annals of Mathematics Studies 166, (2007).
[13] S. Lee and J. Oh, *Algebraic reduced genus one Gromov-Witten invariants for complete intersections in projective spaces,* math/1809.10995
[14] S. Lee and J. Oh, *Algebraic reduced genus one Gromov-Witten invariants for complete intersections in projective spaces, Part 2,* math/2004.07436
[15] J. Li and A. Zinger, *On the genus-one Gromov-Witten invariants of complete inter-sections*, JDG 82 (2009), no. 3, 641–690.
[16] R. Vakil, *Murphy's law in algebraic geometry: Badly-behaved deformation spaces*, Invent. Math. 164 (2006), no. 3, 569–590.
[17] R. Vakil and A. Zinger, *A desingularization of the main component of the moduli space of genus-one stable maps into* $\mathbb{P}^n$, Geom. Topol. 12 (2008), no. 1, 1–95.
[18] K. Yeats, *A combinatorial perspective on quantum field theory,* SpringerBriefs in Mathematical Physics 15, (2017)
[19] A. Zinger, *Standard versus reduced genus-one Gromov–Witten invariants,* Geom. Top. (2008), no. 2, 1203–1241.
[20] A. Zinger, *Reduced genus-one Gromov-Witten invariants*, JDG 83 (2009), no. 2, 407–460.
[21] A. Zinger, *The reduced genus-one Gromov-Witten invariants of Calabi-Yau hypersurfaces*, JAMS 22 (2009), no. 3, 691–737.

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