Estimating Tessellation Parameter Intervals for Rational Curves and Surfaces

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Abstract

This paper presents a method for determining *a priori* a constant parameter interval with which a rational curve or surface can be tessellated such that the deviation of the curve or surface from its piecewise linear approximation is within a specified tolerance. The parameter interval is estimated based on information about the second order derivatives in the homogeneous coordinates, instead of using affine coordinates directly. This new step size can be found with roughly the same amount of computation as the step size presented in [Cheng 1992], though it can be proven to always be larger than Cheng's step size. In fact, numerical experiments show the new step is typically orders of magnitude larger than the step size in [Cheng 1992]. Furthermore, for rational cubic and quartic curves, the new step size is generally twice as large as the step size found by computing bounds on the Bernstein polynomial coefficients of the second derivatives function.

Keywords: Rational curves and surfaces, tessellation, flatness, derivative bounds, step size, projection distance.

1   Introduction

In computer graphics and modeling, parametric curves and surfaces are often tessellated into piecewise linear segments for rendering [Lane and Carpenter, 1979; Rappoport, 1991; Abi-Ezzi and Shirman, 1991; Klassen, 1994], mesh generation [Sheng and Hirsch, 1992; Piegl and Richard, 1995] and surface/surface intersection [Wang, 1984; Filip et al, 1986]. In such applications, the approximation error is typically taken as the maximal distance between the original and the approximating segments.

Many criteria and methods have been developed for the task of assuring that a piecewise linear tessellation satisfies an error bound [Lane and Riesenfeld, 1980; Schaback, 1993; Elber, 1996; Tookey and Cripps, 1997]. One popular approach is to determine a *global parameter interval* or *step size* δ that is valid over the entire domain [Wang, 1984; Filip et al, 1986; Cheng, 1992; Sheng and Hirsch, 1992; Klassen, 1994]. Then the tessellation can be generated either by recursively splitting the curve or surface in half to a depth computed from δ, or by

just sampling the curve or surface at points separated by a parameter interval δ. Such an approach lends itself well to forward differencing. A step size should ideally be as large as possible without violating the prescribed tolerance. The calculation of such a step size is usually not easy for rational curves and surfaces.

The general approach to obtaining such a step size is to establish a relatively simple relationship between the step size and a bound on the maximal deviation. Once the bound is found, the step size follows immediately. Based on theorems from approximation theory [deBoor, 1978], Wang proposed using bounds on second order derivatives of the curve or surface to estimate deviation from the linear approximation [Wang, 1984]. This approach was later enhanced by Filip et al [Filip et al, 1986]. For a C^2 continuous parametric curve $\mathbf{r}(t), t \in [\alpha, \beta]$, a bound on the displacement of $\mathbf{r}(t)$ from its linear approximation $\mathbf{L}(t)$ which interpolates $\mathbf{r}(\alpha)$ and $\mathbf{r}(\beta)$ is given by

$$\sup_{t \in [\alpha,\beta]} \|\mathbf{r}(t) - \mathbf{L}(t)\| \leq \frac{1}{8}\delta^2 \sup_{t \in [\alpha,\beta]} \|\mathbf{r}''(t)\|, \quad \delta = \beta - \alpha \tag{1}$$

For a C^2 continuous parametric surface $\mathbf{r}(u, v)$ defined over a right triangle T with two vertical edge lengths of δ_u and δ_v, a bound on the deviation of $\mathbf{r}(u, v)$ from the base triangle $\mathbf{L}(u, v)$ which linearly interpolates $\mathbf{r}(u, v)$ at three vertices of T is

$$\sup_{(u,v) \in T} \|\mathbf{r}(u, v) - \mathbf{L}(u, v)\| \leq \frac{1}{8} \left(D_{uu}\delta_u^2 + 2D_{uv}\delta_u\delta_v + D_{vv}\delta_v^2 \right) \tag{2}$$

where

$$D_{uu} = \sup_{(u,v) \in T} \left\| \frac{\partial^2 \mathbf{r}(u, v)}{\partial u^2} \right\|, \quad D_{uv} = \sup_{(u,v) \in T} \left\| \frac{\partial^2 \mathbf{r}(u, v)}{\partial u \partial v} \right\|, \quad D_{vv} = \sup_{(u,v) \in T} \left\| \frac{\partial^2 \mathbf{r}(u, v)}{\partial v^2} \right\| \tag{3}$$

In these formulas, we need to compute the *sup*'s of the second order (partial) derivatives over the domain of the curve or surface. For polynomial curves and surfaces, the convex hull property of the control points of the Bezier representation offers a straightforward way to compute upper bounds. For rational curves and surfaces, estimating the *sup*'s of their second order (partial) derivatives is much more expensive. The rational case is important not only because it is more powerful in modeling than the polynomial scheme and because NURBS have become an industry standard in CAD/CAM systems, but also because perspective transformation changes a polynomial curve or surface to a rational curve or surface.

To date, not many papers have been published that address the rational case. Several papers have been written that study efficient ways to obtain bounds on first derivatives for rational curves and surfaces [Float, 1992; Hermann, 1992; Wang et al, 1997], but these techniques are still somewhat costly. Experience with the first derivative suggests that global bounds on second derivatives are either very expensive to compute, or very

conservative. Cheng proposed a new method that does not require bounds on the second derivatives [Cheng, 1992]. He instead treats a rational surface as a polynomial surface in projective $4D$ space and shows that, given a tolerance ϵ and a rational surface $\mathbf{r}(u,v) = \mathbf{R}(u,v)/w(u,v)$, we only need to compute a bound for its associated homogeneous surface $\mathbf{S}(u,v) = (\mathbf{R}(u,v), w(u,v))$ with a new tolerance ϵ_H, which guarantees the ϵ-flatness of $\mathbf{r}(u,v)$. The new tolerance is

$$
\epsilon_H = \begin{cases} \dfrac{\inf |w(u,v)|\epsilon}{(1+(\sup||\mathbf{r}(u,v)||-\epsilon)^2)^{1/2}}, & 0 \le \epsilon \le \sup||\mathbf{r}(u,v)|| \\[2ex] \inf|w(u,v)|\epsilon, & \sup||\mathbf{r}(u,v)|| < \epsilon \le 2\sup||\mathbf{r}(u,v)|| \\[2ex] +\infty, & 2\sup||\mathbf{r}(u,v)|| < \epsilon \end{cases}
\tag{4}
$$

Thus the problem of evaluating the bounds on $\mathbf{r}(u,v)$'s second order derivatives turns to estimating $\mathbf{S}(u,v)$'s, which is much easier. However, numerical experiments show that this estimation is often too loose.

In this paper, we adhere to Cheng's idea that using homogeneous coordinates for rational curves and surfaces makes derivatives simple, and hence we attempt to bound the maximum deviation based on the associated homogeneous coordinates. Section 2 derives a simple bound expressed by the homogeneous coordinates for the distance of two projection points. Then the new step size formulas are developed for rational curves and surfaces in Sections 3 and 4 respectively. It can be found that the new bound is tighter than Cheng's. When all the weights of the rational curves or surfaces are the same, the bound reduces to that of polynomial Bézier curves or surfaces. It is expected that the improved bound will lead to more efficient algorithms in graphics and modeling applications.

2 Projection distance estimation

Our basic strategy is to perform evaluation in homogeneous space. This requires us to approximate the effect of the perspective projection efficiently. The following theorems provide relationships between homogeneous and Euclidean spaces for this purpose. Throughout the paper, E^n stands for n-dimensional Euclidean space.

Theorem 1 *Let $\mathbf{Q}_1 = (\mathbf{R}_1, w_1)$ and $\mathbf{Q}_2 = (\mathbf{R}_2, w_2)$ with $w_1, w_2 > 0$ be two points in E^{n+1}, whose corresponding projections are $\mathbf{r}_1 = \mathbf{R}_1/w_1$, $\mathbf{r}_2 = \mathbf{R}_2/w_2 \in E^n$, and let r be any number not smaller than $\max(||\mathbf{r}_1||, ||\mathbf{r}_2||)$. Then*

$$
||\mathbf{r}_1 - \mathbf{r}_2|| \le \frac{||\mathbf{R}_1 - \mathbf{R}_2|| + (r - ||\mathbf{r}_1 - \mathbf{r}_2||)|w_1 - w_2|}{\min(w_1, w_2)}
\tag{5}
$$

Proof: Without loss of generality, we assume that $w_1 \leq w_2$. Then

$$\|\mathbf{r}_1 - \mathbf{r}_2\| \quad = \quad \|\tfrac{\mathbf{R}_1}{w_1} - \tfrac{\mathbf{R}_2}{w_2}\| \leq \|\tfrac{\mathbf{R}_1}{w_1} - \tfrac{\mathbf{R}_1}{w_2}\| + \|\tfrac{\mathbf{R}_1}{w_2} - \tfrac{\mathbf{R}_2}{w_2}\|$$

$$\leq \quad \|\tfrac{\mathbf{R}_1}{w_1}\|\tfrac{|w_1 - w_2|}{w_2} + \tfrac{\|\mathbf{R}_1 - \mathbf{R}_2\|}{w_2} \leq r\tfrac{|w_1 - w_2|}{w_2} + \tfrac{\|\mathbf{R}_1 - \mathbf{R}_2\|}{w_2}$$

$$= \quad \tfrac{r|w_1 - w_2| + \|\mathbf{R}_1 - \mathbf{R}_2\|}{w_1 + w_2 - w_1} = \tfrac{r|w_1 - w_2| + \|\mathbf{R}_1 - \mathbf{R}_2\|}{\min(w_1, w_2) + |w_1 - w_2|}$$

Thus we obtain

$$(\min(w_1, w_2) + |w_1 - w_2|)\|\mathbf{r}_1 - \mathbf{r}_2\| \leq r|w_1 - w_2| + \|\mathbf{R}_1 - \mathbf{R}_2\|$$

i.e.

$$\min(w_1, w_2)\|\mathbf{r}_1 - \mathbf{r}_2\| \leq \|\mathbf{R}_1 - \mathbf{R}_2\| + (r - \|\mathbf{r}_1 - \mathbf{r}_2\|)|w_1 - w_2|$$

Therefore

$$\|\mathbf{r}_1 - \mathbf{r}_2\| \leq \frac{\|\mathbf{R}_1 - \mathbf{R}_2\| + (r - \|\mathbf{r}_1 - \mathbf{r}_2\|)|w_1 - w_2|}{\min(w_1, w_2)}$$

∎

Figure 1: Geometric interpretation for the case of n=1.

Theorem 1 has an intuitive geometric interpretation. See Figure 1 for an example in the case of $n = 1$. The multiplication of w_1 and the Euclidean distance of $\mathbf{r}_1$ and $\mathbf{r}_2$ equals the length of $\mathbf{Q}_1\mathbf{I}$, which is the sum of $\|\mathbf{Q}_1\mathbf{H}\|$ and $\|\mathbf{H}\mathbf{I}\|$. Note that $\|\mathbf{H}\mathbf{I}\| = \|\mathbf{Q}_2\mathbf{H}\|/\tan \alpha$, $\|\mathbf{Q}_1\mathbf{H}\|$ is $\|\mathbf{R}_1 - \mathbf{R}_2\|$ and $\|\mathbf{Q}_2\mathbf{H}\|$ is just $|w_1 - w_2|$. Therefore the inequality (5) is equivalent to $\tan \alpha = \|\mathbf{OC}\|/\|\mathbf{r}_2\mathbf{C}\| \geq 1/(r - \|\mathbf{r}_1 - \mathbf{r}_2\|)$.

Theorem 2 *Let* $\mathbf{Q}_1 = (\mathbf{R}_1, w_1)$ *and* $\mathbf{Q}_2 = (\mathbf{R}_2, w_2)$ *be two points in* E^{n+1}, *and let* ϵ *and* r *be two positive numbers. If* $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *satisfy*

$$w_1, w_2 \geq 0, \quad \max(\|\frac{\mathbf{R}_1}{w_1}\|, \|\frac{\mathbf{R}_2}{w_2}\|) \leq r \tag{6}$$

and

$$\frac{\|\mathbf{R}_1 - \mathbf{R}_2\| + (r - \epsilon)|w_1 - w_2|}{\min(w_1, w_2)} \leq \epsilon \tag{7}$$

then the distance of their projections $\mathbf{r}_1 = \mathbf{R}_1/w_1$ *and* $\mathbf{r}_2 = \mathbf{R}_2/w_2$ *is less than or equal to* ϵ:

$$\|\mathbf{r}_1 - \mathbf{r}_2\| \leq \epsilon \tag{8}$$

Proof: Let $\eta = \|\mathbf{r}_1 - \mathbf{r}_2\|$. Then

$$\eta \leq \frac{\|\mathbf{R}_1 - \mathbf{R}_2\| + (r - \eta)|w_1 - w_2|}{\min(w_1, w_2)} = \frac{\|\mathbf{R}_1 - \mathbf{R}_2\| + (r - \epsilon)|w_1 - w_2|}{\min(w_1, w_2)} + \frac{(\epsilon - \eta)|w_1 - w_2|}{\min(w_1, w_2)}$$

$$\leq \epsilon + (\epsilon - \eta)\frac{|w_1 - w_2|}{\min(w_1, w_2)}$$

Thus $(\epsilon - \eta)\left(1 + \frac{|w_1 - w_2|}{\min(w_1, w_2)}\right) \geq 0$, which arrives at the conclusion: $\epsilon \geq \eta = \|\mathbf{r}_1 - \mathbf{r}_2\|$. ∎

This theorem shows that the closeness requirement of two points can be imposed on a simple expression represented by the homogeneous coordinates.

3 Step size for rational curves

3.1 General theory

Since curves have simpler equations, we discuss the curve case first. For a rational curve $\mathbf{r}(t) = \frac{\mathbf{R}(t)}{w(t)}, t \in [0, 1]$, the tessellation problem can be formulated: given a tolerance ϵ, find a maximum parameter step δ such that

$$\sup_{t \in [\alpha, \beta]} \|\mathbf{r}(t) - \mathbf{L}(t)\| \leq \epsilon \tag{9}$$

where $\alpha, \beta \in [0, 1]$ and $\delta = \beta - \alpha$, $\mathbf{L}(t)$ is the line segment connecting $\mathbf{r}(\alpha)$ and $\mathbf{r}(\beta)$ parameterized as follows:

$$\mathbf{L}(t) = \frac{\mathbf{L}_n(t)}{L_d(t)} = \frac{[(\beta - t)\mathbf{R}(\alpha) + (t - \alpha)\mathbf{R}(\beta)]/(\beta - \alpha)}{[(\beta - t)w(\alpha) + (t - \alpha)w(\beta)]/(\beta - \alpha)} \tag{10}$$

In practice, we typically content ourselves with finding a step size for which (9) is an upper bound. Surprisingly, situations exist for which a step size $\delta_1 < \delta$ can actually violate the ϵ-error, even though a step size of δ does not. Figure 2 illustrates such a case.
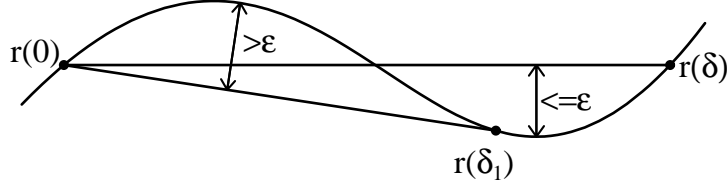
Figure 2: Line $\mathbf{r}(0)\mathbf{r}(\delta_1)$ is not within ϵ-flatness while $\mathbf{r}(0)\mathbf{r}(\delta)$ is.

In light of this problem, we modify our definition of δ as follows:

$$\delta = \sup\{\rho|\ \sup_{t\in[\alpha,\alpha+\eta]}||\mathbf{r}(t)-\mathbf{L}(t)|| \leq \epsilon \text{ for any positive } \eta \leq \rho, \ \alpha, \alpha+\rho \in [0,1]\}$$

With Theorem 2, we need to compute a bound for $\|\mathbf{R}(t)-\mathbf{L}_n(t)\| + (r-\epsilon)|w(t)-L_d(t)|$. The tighter the bound, the larger (and the more economical) the step size δ. The formula (1) can be simply used here by finding the bounds on $\|\mathbf{R}''(t)\|$ and $|w''(t)|$, and combining them. However, this is different from taking $\|\mathbf{R}''(t)\| + (r-\epsilon)|w''(t)|$ as a whole. In general, the latter is smaller than the former since the maxima for $\|\mathbf{R}''(t)\|$ and $|w''(t)|$ may occur at different parameter values. In the following we show that the proof of Theorem 2 in [Filip et al, 1986] can be generalized for this.

Lemma 1 *Let* $\mathbf{Q}(t) = (\mathbf{R}(t), w(t)) \in E^{n+1}$ *be a* C^2 *vector-valued function defined over* $[\alpha,\beta]$ *with* $\mathbf{Q}(\alpha) = \mathbf{Q}(\beta) = 0$. *Let* k *be any nonnegative number. Then*

$$\sup_{t\in[\alpha,\beta]}(\|\mathbf{R}(t)\| + k|w(t)|) \leq \frac{(\beta-\alpha)^2}{8}\sup_{t\in[\alpha,\beta]}(\|\mathbf{R}''(t)\| + k|w''(t)|) \tag{11}$$

Proof: Let $d(t) = \|\mathbf{R}(t)\| + k|w(t)|$. Then $d(t)$ is a continuous function in $[\alpha,\beta]$ with $d(\alpha) = d(\beta) = 0$ and $d(t) \geq 0$, which means $d(t)$ can attain its maximum value at some $\xi \in (\alpha,\beta)$. It is obvious that $d(t)$ is C^2 continuous in $[\alpha,\beta]$ possibly except those points where $w(t) = 0$ or $\mathbf{R}(t) = 0$. Now we discuss the problem in three cases:

case 1 $\quad w(\xi) = 0$.

In this case, $d(\xi) = \|\mathbf{R}(\xi)\|$. From (1),

$$\|\mathbf{R}(\xi)\| \leq \frac{(\beta-\alpha)^2}{8}\sup_{t\in[\alpha,\beta]}\|\mathbf{R}''(t)\|$$

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Let η be so that $\|\mathbf{R}''(\eta)\| = \sup\limits_{t\in[\alpha,\beta]} \|\mathbf{R}''(t)\|$. We have

$$
\begin{aligned}
d(\xi) &= \|\mathbf{R}(\xi)\| \leq \frac{(\beta-\alpha)^2}{8}\|\mathbf{R}''(\eta)\| \\
&\leq \frac{(\beta-\alpha)^2}{8}(\|\mathbf{R}''(\eta)\| + k|w''(\eta)|) \leq \frac{(\beta-\alpha)^2}{8}\sup\limits_{t\in[\alpha,\beta]}(\|\mathbf{R}''(\eta)\| + k|w''(\eta)|)
\end{aligned}
$$

case 2 $\mathbf{R}(\xi) = 0$.

Then $d(\xi) = k|w(\xi)|$. By a like argument as in case 1, we obtain

$$
d(\xi) = \frac{(\beta-\alpha)^2}{8}k\sup\limits_{t\in[\alpha,\beta]}|w''(t)| \leq \frac{(\beta-\alpha)^2}{8}\sup\limits_{t\in[\alpha,\beta]}(\|\mathbf{R}''(t)\| + k|w''(t)|)
$$

case 3 $w(\xi) \neq 0$, $\mathbf{R}(\xi) \neq 0$.

Thus $d(t)$ is C^2 at $t = \xi$ and $d'(\xi) = 0$, i.e.

$$
d'(\xi) = \frac{\mathbf{R}}{\|\mathbf{R}\|}\mathbf{R}'(\xi) + k\frac{w}{|w|}w'(\xi) = 0 \tag{12}
$$

Now we express $\mathbf{Q}(t)$ in its Taylor expansion about ξ:

$$
(\mathbf{R}(t), w(t)) = (\mathbf{R}(\xi), w(\xi)) + (\mathbf{R}'(\xi), w'(\xi))(t - \xi) + \int_\xi^t (\mathbf{R}''(s), w''(s))(t - s)ds \tag{13}
$$

Without loss of generality, we assume that $\xi \in [\frac{\alpha+\beta}{2}, \beta)$. Then substituting $t = \beta$ into the equation (13) yields

$$
0 = (\mathbf{R}(\beta), w(\beta)) = (\mathbf{R}(\xi), w(\xi)) + (\mathbf{R}'(\xi), w'(\xi))(\beta - \xi) + \int_\xi^\beta (\mathbf{R}''(s), w''(s))(\beta - s)ds \tag{14}
$$

Taking the dot product of equation (14) with $(\frac{\mathbf{R}(\xi)}{\|\mathbf{R}(\xi)\|}, k\frac{w(\xi)}{|w(\xi)|})$, we obtain

$$
0 = \|\mathbf{R}(\xi)\| + k|w(\xi)| + 0 + \int_\xi^\beta \left(\frac{\mathbf{R}(\xi)\mathbf{R}''(s)}{\|\mathbf{R}(\xi)\|} + k\frac{w(\xi)w''(s)}{|w(\xi)|}\right)(\beta - s)ds
$$

Thus

$$
\begin{aligned}
d(\xi) &= \|\mathbf{R}(\xi)\| + k|w(\xi)| = \left|\int_\xi^\beta (\frac{\mathbf{R}(\xi)\mathbf{R}''(s)}{\|\mathbf{R}(\xi)\|} + k\frac{w(\xi)w''(s)}{|w(\xi)|})(\beta - s)ds\right| \\[2mm]
&\leq \int_\xi^\beta (|\frac{\mathbf{R}(\xi)\mathbf{R}''(s)}{\|\mathbf{R}(\xi)\|}| + k|\frac{w(\xi)w''(s)}{|w(\xi)|}|)(\beta - s)ds \leq \int_\xi^\beta (\|\mathbf{R}''(s)\| + k|w''(s)|)(\beta - s)ds \\[2mm]
&\leq \sup\limits_{t\in[\alpha,\beta]}(\|\mathbf{R}''(t)\| + k|w''(t)|)\int_\xi^\beta (\beta - s)ds = \sup\limits_{t\in[\alpha,\beta]}(\|\mathbf{R}''(t)\| + k|w''(t)|)\frac{(\beta-\xi)^2}{2} \\[2mm]
&\leq \frac{(\beta-\alpha)^2}{8}\sup\limits_{t\in[\alpha,\beta]}(\|\mathbf{R}''(t)\| + k|w''(t)|)
\end{aligned}
$$

∎

Now we are ready to derive the formula for the step size.

Theorem 3 *Given a C^2 rational curve $\mathbf{r}(t) = \frac{\mathbf{R}(t)}{w(t)}, t \in [\alpha, \beta]$ with $w(t) > 0$, and r is a number satisfying*

$r \geq \sup\limits_{t \in [\alpha,\beta]} \|\mathbf{r}(t)\| = \sup\limits_{t \in [\alpha,\beta]} \|\frac{\mathbf{R}(t)}{w(t)}\|$. *Let* $\mathbf{L}(t) = \frac{\mathbf{L}_n(t)}{L_d(t)}$ *be defined as in (10), then* $\sup\limits_{t \in [\alpha,\beta]} \|\mathbf{r}(t) - \mathbf{L}(t)\| \leq \epsilon$ *if*

$$
\delta = \beta - \alpha \leq
\begin{cases}
\sqrt{\frac{8 \inf_t\{w(t)\}\epsilon}{\sup_t(\|\mathbf{R}''(t)\| + (r - \epsilon)|w''(t)|)}}, & \epsilon < r \\[3ex]
\sqrt{\frac{8 \inf_t\{w(t)\}\epsilon}{\sup_t \|\mathbf{R}''(t)\|}}, & r \leq \epsilon < 2r \\[3ex]
1, & 2r \leq \epsilon
\end{cases}
\tag{15}
$$

Proof: First let us consider the case of $2r \leq \epsilon$. Since

$$
\mathbf{L}(t) = \frac{\mathbf{R}(\alpha)(\beta - t) + \mathbf{R}(\beta)(t - \alpha)}{w(\alpha)(\beta - t) + w(\beta)(t - \alpha)} = \frac{\mathbf{R}(\alpha)}{w(\alpha)} \frac{(\beta - t)w(\alpha)}{(\beta - t)w(\alpha) + (t - \alpha)w(\beta)} + \frac{\mathbf{R}(\beta)}{w(\beta)} \frac{(t - \alpha)w(\beta)}{(\beta - t)w(\alpha) + (t - \alpha)w(\beta)},
$$

$\|\mathbf{L}(t)\| \leq \max(\|\frac{\mathbf{R}(\alpha)}{w(\alpha)}\|, \|\frac{\mathbf{R}(\beta)}{w(\beta)}\|) \leq r$. Then for any $t \in [\alpha, \beta]$, $\|\mathbf{r}(t) - \mathbf{L}(t)\| \leq \|\mathbf{r}(t)\| + \|\mathbf{L}(t)\| \leq r + r = 2r \leq \epsilon$.

Second, in the case of $r \leq \epsilon < 2r$, applying (1) to $\mathbf{R}(t)$, we get

$$
\sup\limits_{t \in [\alpha,\beta]} \|\mathbf{R}(t) - \mathbf{L}_n(t)\| \leq \frac{(\beta - \alpha)^2}{8} \sup\limits_{t \in [\alpha,\beta]} \|\mathbf{R}''(t)\|
$$

As $L_d(t)$ is a convex combination of $w(\alpha)$ and $w(\beta)$, the value of $L_d(t)$ is between $w(\alpha)$ and $w(\beta)$. Hence

$$
\sup\limits_{t \in [\alpha,\beta]} \frac{\|\mathbf{R}(t) - \mathbf{L}_n(t)\| + (r - \epsilon)|w(t) - L_d(t)|}{\min(w(t), L_d(t))} \leq \sup\limits_{t \in [\alpha,\beta]} \frac{\|\mathbf{R}(t) - \mathbf{L}_n(t)\|}{\min(w(t), L_d(t))} \leq \frac{\delta^2}{8} \frac{\sup\limits_{t \in [\alpha,\beta]} \|\mathbf{R}''(t)\|}{\inf\limits_{t \in [\alpha,\beta]} w(t)} \leq \epsilon
$$

By Theorem 2, $\sup\limits_{t \in [\alpha,\beta]} \|\mathbf{r}(t) - \mathbf{L}(t)\| \leq \epsilon$.

Finally, for the case of $\epsilon < r$, $r - \epsilon > 0$. Note that $\mathbf{R}(\alpha) - \mathbf{L}_n(\alpha) = \mathbf{R}(\beta) - \mathbf{L}_n(\beta) = 0$ and $w(\alpha) - L_d(\alpha) = w(\beta) - L_d(\beta) = 0$. From Lemma 1,

$$
\sup\limits_{t \in [\alpha,\beta]} (\|\mathbf{R}(t) - \mathbf{L}_n(t)\| + (r - \epsilon)|w(t) - L_d(t)|) \leq \frac{(\beta - \alpha)^2}{8} \sup\limits_{t \in [\alpha,\beta]} (\|\mathbf{R}''(t)\| + (r - \epsilon)|w''(t)|)
$$

So

$$
\sup\limits_{t \in [\alpha,\beta]} \frac{\|\mathbf{R}(t) - \mathbf{L}_n(t)\| + (r - \epsilon)|w(t) - L_d(t)|}{\min(w(t), L_d(t))} \leq \frac{\sup\limits_{t \in [\alpha,\beta]} (\|\mathbf{R}''(t)\| + (r - \epsilon)|w''(t)|)}{\inf_t w(t)} \frac{(\beta - \alpha)^2}{8} \leq \epsilon
$$

Thus the proof is completed by Theorem 2.

∎

3.2 Rational Bezier curves

For a rational Bézier curve $\mathbf{r}(t)$ defined by

$$
\mathbf{r}(t) = \frac{\mathbf{R}(t)}{w(t)} = \frac{\sum\limits_{i=0}^{n} \mathbf{P}_i w_i B_i^n(t)}{\sum\limits_{i=0}^{n} w_i B_i^n(t)}, \quad t \in [0, 1]
\tag{16}
$$

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$\mathbf{R}(t)$ and $w(t)$ are in Bézier form. The estimation for the *sup*'s of their second order derivatives is straightforward due to the convex-hull property. Define Δ to be a forward difference operator so that $\Delta P_i = P_{i+1} - P_i$. We have

Corollary 1 *Let $\mathbf{r}(t)$ be a rational Bézier curve (16), $r = \max_i \|\mathbf{P}_i\|$ and $w = \min_i\{w_i\} > 0$. For any $\epsilon > 0$, the deviation of the curve segment over $[\alpha, \beta] \subseteq [0, 1]$ from its fractional linear approximation is within ϵ, i.e. $\sup_{t \in [\alpha, \beta]} \|\mathbf{r}(t) - \mathbf{L}(t)\| \leq \epsilon$ if $\delta = \beta - \alpha$ satisfies*

$$\delta \leq \begin{cases} \sqrt{\dfrac{8w\epsilon}{n(n-1)\max_i(\|\Delta^2(w_i\mathbf{P}_i)\| + (r-\epsilon)|\Delta^2 w_i|)}}, & \epsilon < r \\[3ex] \sqrt{\dfrac{8w\epsilon}{n(n-1)\max_i\|\Delta^2(w_i\mathbf{P}_i)\|}}, & r \leq \epsilon < 2r \\[3ex] 1, & 2r \leq \epsilon \end{cases} \tag{17}$$

Proof: The second order derivative of a Bézier curve is also a Bézier curve with degree lower by 2, e.g.

$$(\mathbf{R}''(t), w''(t)) = n(n-1)\sum_{i=0}^{n-2} \Delta^2(w_i\mathbf{P}_i, w_i)B_i^{n-2}(t) \tag{18}$$

The conclusion follows immediately from the inequalities

$$\|\mathbf{R}''(t)\| \leq n(n-1)\sum_{i=0}^{n-2}\|\Delta^2(w_i\mathbf{P}_i)\|B_i^{n-2}(t) \leq n(n-1)\max_i\|\Delta^2(w_i\mathbf{P}_i)\| \tag{19}$$

and

$$\begin{aligned} \|\mathbf{R}''(t)\| + (r-\epsilon)|w''(t)| &\leq n(n-1)\sum_{i=0}^{n-2}(\|\Delta^2(w_i\mathbf{P}_i)\| + (r-\epsilon)|\Delta^2 w_i|)B_i^{n-2}(t) \\ &\leq n(n-1)\max_i(\|\Delta^2(w_i\mathbf{P}_i)\| + (r-\epsilon)|\Delta^2 w_i|) \end{aligned} \tag{20}$$

∎

Examples.

1. Quadratic rational Bézier curve: the step size for the case of $\epsilon < r$ is

$$\delta = \sqrt{\frac{4\min\{w_0, w_1, w_2\}\epsilon}{\|w_2\mathbf{P}_2 - 2w_1\mathbf{P}_1 + w_0\mathbf{P}_0\| + (r-\epsilon)|w_2 - 2w_1 + w_0|}}$$

2. Cubic rational Bézier curve: the step size for the case of $\epsilon < r$ is

$$\delta = \sqrt{\frac{4\min\{w_0, w_1, w_2, w_3\}\epsilon}{3\max\{\|\Delta^2(w_0\mathbf{P}_0)\| + (r-\epsilon)|\Delta^2 w_0|, \|\Delta^2(w_1\mathbf{P}_1)\| + (r-\epsilon)|\Delta^2 w_1|\}}}$$

Remarks.

1. Here we show that the new step size δ_n given by (17) is larger than Cheng's. For a rational Bézier curve, Cheng's step size is

$$\delta_c = \sqrt{\frac{8\epsilon_H}{n(n-1)\max_i \|\Delta^2(w_i\mathbf{P}_i, w_i)\|}}$$

and ϵ_H is defined in (4). When $\epsilon < r$,

$$
\begin{aligned}
(8w\epsilon/n(n-1)\delta_c^2)^2 &= (1+(r-\epsilon)^2)\max_i(\|\Delta^2(w_i\mathbf{P}_i)\|^2 + |\Delta^2 w_i|^2) \\[2mm]
&= \max_i(\|\Delta^2(w_i\mathbf{P}_i)\|^2 + (r-\epsilon)^2|\Delta^2 w_i|^2 + (r-\epsilon)^2\|\Delta^2(w_i\mathbf{P}_i)\|^2 + |\Delta^2 w_i|^2) \\[2mm]
&\geq \max_i(\|\Delta^2(w_i\mathbf{P}_i)\|^2 + (r-\epsilon)^2|\Delta^2 w_i|^2 + 2(r-\epsilon)\|\Delta^2(w_i\mathbf{P}_i)\||\Delta^2 w_i|) \\[2mm]
&= \max_i(\|\Delta^2(w_i\mathbf{P}_i)\| + (r-\epsilon)|\Delta^2 w_i|)^2 = (8w\epsilon/n(n-1)\delta_n^2)^2
\end{aligned}
$$

so $\delta_n \geq \delta_c$. The other cases are trivial.

2. When all weights are identical, the rational Bézier curve reduces to a polynomial curve

$$\mathbf{r}(t) = \sum_{i=0}^{n} \mathbf{P}_i B_i^n(t), \quad t \in [0,1].$$

In this case $\Delta^2 w_i = 0$, so the new step size is the same as that for a Bézier curve, which was given in [Filip et al, 1986], i.e.

$$\delta \leq \sqrt{\frac{8\epsilon}{n(n-1)\max_i \|\Delta^2\mathbf{P}_i\|}} \tag{21}$$

However Cheng's estimation, in general, does not reduce to — and is more conservative than — (21).

3.3 Further improvements

Further improvements without much additional computation are possible. The first observation is that the bound for the second derivative curve (18) might be tightened. Farin points out that a rational curve can be bounded in a tighter convex hull generated by the end points and so-called "weight points" [Farin, 1993]. The weight points are obtained after an additional subdivision without doing the full work of subdividing. In our case, the weight points are just the average of the adjacent homogeneous points. That is

$$\mathbf{W}_i = (\mathbf{WR}_i, Ww_i) = n(n-1)\frac{\Delta^2(w_{i-1}\mathbf{P}_{i-1}, w_{i-1}) + \Delta^2(w_i\mathbf{P}_i, w_i)}{2}, \quad i = 1 \cdots n-2 \tag{22}$$

If we also denote the end points by $\mathbf{W}_0 = n(n-1)\Delta^2(w_0\mathbf{P}_0, w_0)$ and $\mathbf{W}_{n-1} = n(n-1)\Delta^2(w_{n-2}\mathbf{P}_{n-2}, w_{n-2})$, then the curve (18) lies in the convex hull that includes only $\mathbf{W}_i$ $(i = 0, \cdots, n-1)$. Therefore for any $t \in [0,1]$, there exist nonnegative coefficients $\lambda_i(t)$ such that

$$(\mathbf{R}''(t), w''(t)) = \sum_{i=0}^{n-1} \lambda_i(t)\mathbf{W}_i, \qquad \lambda_0(t) + \lambda_1(t) + \cdots + \lambda_{n-1}(t) = 1$$

Therefore

$$
\begin{aligned}
\|\mathbf{R}''(t)\| &\leq \sum_{i=0}^{n-1} \|\mathbf{WR}_i\|\lambda_i(t) \leq \max_i \|\mathbf{WR}_i\| \\
&= n(n-1)\max_i\{\|\Delta^2(w_0\mathbf{P}_0)\|, \tfrac{\|\Delta^2(w_{i-1}\mathbf{P}_{i-1})+\Delta^2(w_i\mathbf{P}_i)\|}{2}, \|\Delta^2(w_{n-2}\mathbf{P}_{n-2})\|\}
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
\|\mathbf{R}''(t)\| + (r-\epsilon)|w''(t)| &\leq \sum_{i=0}^{n-1} (\|\mathbf{WR}_i\| + (r-\epsilon)|Ww_i|)\lambda_i(t) \\
&\leq \max_i(\|\mathbf{WR}_i\| + (r-\epsilon)|Ww_i|) \\
&= n(n-1)\max_i\{\|\Delta^2(w_0\mathbf{P}_0)\| + (r-\epsilon)|\Delta^2 w_0|, \\
&\qquad \tfrac{\|\Delta^2(w_{i-1}\mathbf{P}_{i-1})+\Delta^2(w_i\mathbf{P}_i)\|}{2} + (r-\epsilon)\tfrac{|\Delta^2 w_{i-1}+\Delta^2 w_i|}{2}, \\
&\qquad \|\Delta^2(w_{n-2}\mathbf{P}_{n-2})\| + (r-\epsilon)|\Delta^2 w_{n-2}|\}
\end{aligned}
\tag{24}
$$

Usually this gives a tighter bound. The improvement depends on the distribution of the control points of curve (18). If the adjacent control points are close, the averages formed will not be much less in magnitude than the original ones. For a cubic rational curve, (18) is a linear polynomial, in which case there is no improvement in using the weight-point method.

The second observation is that translation does not change the shape of the curve and the translated curve should have the same step size as the original curve. However the formula (15) or (17) is dependent on the affine coordinate system. This suggests that we move the original point of the affine coordinate system to make the value of $r = \max_i \|\mathbf{P}_i\|$ as small as possible. One intuitive way is to find the min-max bounding box of the rational curve and to choose the center of the box as the new original point, or we can find a bounding circle or sphere and then use the center as the new original point. Thus each control point should be transformed to a new position. The method proposed above can be applied to these new control points. It should be pointed out that this is just a heuristic approach. When the control points are moved, the values of $\|\Delta^2(w_i\mathbf{P}_i)\|$ are also changed. So sometimes the result might be worse. But the numerical examples indicate that in most cases doing a certain translation gives a better step size, especially when the control points are located very asymmetrically around the original point .

Summarizing, we have the following algorithm:

Input: the control points P_i and weights w_i, the error tolerance ϵ.

Output: a global step size

Procedure:

Step 1. Find a min-max bounding box of the curve, and compute its center point, denoted by C.

Step 2. Translate the control points by C: $\quad P_i \leftarrow P_i - C$.

Step 3. Compute the Bézier representation (18) for the second derivative of the homogeneous Bézier curve.

Step 4. Compute $\sup(\|\mathbf{R}''(t)\| + (r - \epsilon)|w''(t)|)$ or $\sup \|\mathbf{R}''(t)\|$ by formula (24) or (23).

Step 5. compute the step size δ by (15).

3.4 Numerical experiments

We present a few planar curve examples. The data defining the rational Bézier curves are generated randomly. $C_i(t)$ represents the curve of degree i.

$$
\begin{array}{ll}
C_1(t): & (2, 5, 5.6), (1, 8, 0.7) \\
C_2(t): & (-3, -10, 0.96), (6, 8, 2.3), (2, 4, 0.63) \\
C_3(t): & (19, 61, 0.08), (-61, 52, 0.5), (17, 55, 1), (49, -20, 0.4) \\
C_4(t): & (1, 5, 6.1), (7, 7, 0.39), (-8, -10, 18.4), (-1, -10, 1.1), (-6, -3, 0.03) \\
C_5(t): & (53, -6, 0.7), (-7, 66, 1.8), (-64, -46, 147), (-71, 43, 6.6), (97, -68, 4), (-66, 57, 0.7) \\
C_6(t): & (36, -23, 1.7), (48, 54, 0.8), (14, -13, 0.2), (64, 13, 1), (-68, 54, 1.4), (43, -1, 0.4), (34, 92, 0.2) \\
C_7(t): & (9, 9, 1.5), (-4, 0, 3.1), (5, 0, 3.3), (-7, 0, 2.7), (10, -10, 2.6), (2, -3, 0.7), \\
& (-4, 9, 1.1), (1, -5, 1.3) \\
C_8(t): & (3, 2, 0.2), (-5, 8, 1.6), (1, 4, 0.8), (7, 10, 0.8), (1, -8, 1.1), (-7, 5, 1.3), (0, 10, 0.4), \\
& (4, 1, 0.7), (6, -1, 2.3)
\end{array}
$$

In each tuple, the first two numbers stand for the control point's x, y-components, and the third for the weight.

Different methods are used to compute the step size δ with the tolerance $\epsilon = 0.1$. The results are listed in Table 1. Here, we test the new methods with just using formula (17), or using weight-point method, or translating the control points first and then using (17), or translating the control points first and then using weight-point method. They are respectively denoted by "new", "new-w", "new-t" and "new-tw". Cheng's method is labelled "cheng". A well-known method for finding a step in a parametric curve is

$$
\delta = \sqrt{\frac{8\epsilon}{D}} \tag{25}
$$

where D is an upper bound on the magnitude of the second derivative vector of the curve. Variations of this step size are discussed in [Wang 1984, Filip et. al. 1986]. For a rational curve of degree n, the second derivative vector is degree $3n$. For comparison with the new method, we computed in two different ways a step size using the method in (25). First, we expressed the second derivative in rational Bernstein form and took D to be the largest distance from the origin to any control point. This method is labelled "approx-D" in the table. We also went to the expense of computing the tight upper bound on the second derivative curve. This amounted

to finding the global maximum of a degree $6n$ polynomial over the $[0, 1]$ interval, a computation that is far too costly for practical use, but which is of interest because it yields the largest step size of which (25) is capable. The step size produced in this way is labelled "exact-D" in the table.

δ	$C_1(t)$	$C_2(t)$	$C_3(t)$	$C_4(t)$	$C_5(t)$	$C_6(t)$	$C_7(t)$	$C_8(t)$
cheng	1.0	0.017	0.0003	0.0001	0.00008	0.00018	0.0027	0.0012
new	1.0	0.0549	0.0075	0.0015	0.0007	0.0035	0.0108	0.0072
new-w	1.0	0.0549	0.0075	0.0019	0.00098	0.0048	0.0132	0.0072
new-t	1.0	0.0551	0.0081	0.0016	0.0007	0.0037	0.0111	0.0075
new-tw	1.0	0.0551	0.0081	0.0021	0.001	0.0052	0.0132	0.0075
approx-D	0.048	0.0335	0.0039	0.0015	0.001	0.0052	0.0121	0.0032
exact-D	0.085	0.0457	0.0039	0.0019	0.0013	0.0058	0.0130	0.0036

Table 1: The step sizes computed by different methods with $\epsilon = 0.1$.

We also ran several sets of numerical comparisons on several hundred randomly generated test cases of degree three and four. In one batch of tests, the (x, y) coordinates of the control points were generated as random numbers in the interval $([-1000, 1000], [-1000, 1000])$. We also ran tests in the intervals $([-100, 100], [-100, 100])$ and $([-10, 10], [-10, 10])$. Control point weights were generated as the ratio of two random numbers in the intervals $[1, 10000]$, and ϵ was fixed at 0.1. In each batch, we ran several hundred test cases, computed the ratio between the "new-tw" step size, and those produced by Cheng's method and by the "approx-D" method. The results are displayed in Table 2.

Interval	Degree	$\frac{new-tw}{cheng}$	$\frac{new-tw}{approx-D}$
$([-1000, 1000], [-1000, 1000])$	3	66.4	2.03
$([-1000, 1000], [-1000, 1000])$	4	72.6	1.95
$([-100, 100], [-100, 100])$	3	20.2	1.92
$([-10, 10], [-10, 10])$	3	6.8	2.56

Table 2: Average ratio of step sizes with $\epsilon = 0.1$.

Two things are noteworthy in this table. First, it is somewhat surprising that "new-tw" gives on average twice as large of a step size as "approx-D". This can be explained. In the case of a polynomial Bézier curve, it is easily shown that the two methods will give exactly the same step size. However, if the weights vary significantly, "new-tw" is often able to provide a larger step size than "approx-D" because it is finding the distance between a rational curve segment and a *fractional*-linearly parametrized line segment. By contrast, "approx-D" finds the distance between the rational curve segment and a linearly-parametrized line segment.

The second curiosity in Table 2 is that Cheng's method appears to improve as the size of the control point coordinates decreases! Inspection of Cheng's formulae confirms that this should indeed occur. Furthermore, it

can be seen that Cheng's method will produce a different step size *even if the control points and ϵ are scaled the same amount*! This suggests that Cheng's method might be able to produce a larger step size if we first scale the control points and ϵ by some constant, and that there might be an optimal scale factor that will produce an optimally large step size. Closer study reveals that, in fact, our step size is exactly the one produced by Cheng's method after applying the optimal scaling!

4   Step sizes for rational surfaces

The development for rational surfaces parallels that for rational curves. In this section we just list the results without proofs except the following lemma.

Lemma 2 *Let $\mathbf{Q}(u,v) = (\mathbf{R}(u,v), w(u,v))$ be a C^2 vector-valued function defined over a right triangle T with vertices A, B and C of the form $B = A + (\delta_u, 0)$ and $C = A + (0, \delta_v)$, and vanish at these three vertices $\mathbf{Q}(A) = \mathbf{Q}(B) = \mathbf{Q}(C) = 0$. Then for any nonnegative number k,*

$$\sup_{(u,v)\in T} (\|\mathbf{R}(u,v)\| + k|w(u,v)|) \leq (D_{uu}\delta_u^2 + 2D_{uv}\delta_u\delta_v + D_{vv}\delta_v^2)/8 \tag{26}$$

where

$$
\begin{aligned}
D_{uu} &= \sup_{(u,v)\in T} (\|\mathbf{R}_{uu}''(u,v)\| + k|w_{uu}''(u,v)|) \\
D_{uv} &= \sup_{(u,v)\in T} (\|\mathbf{R}_{uv}''(u,v)\| + k|w_{uv}''(u,v)|) \\
D_{vv} &= \sup_{(u,v)\in T} (\|\mathbf{R}_{vv}''(u,v)\| + k|w_{vv}''(u,v)|)
\end{aligned}
\tag{27}
$$

Proof: This lemma can be proven in a way similar to that in [Filip et al, 1986]. Let $D(u,v) = \|\mathbf{R}(u,v)\| + k|w(u,v)|$. $D(u,v)$ is a continuous function and $D(u,v) \geq 0$, $D(A) = D(B) = D(C) = 0$. Then $D(u,v)$ must get its maximum value at some point $P_0 = (u_0, v_0)$ in T. Suppose P_0 lies in region T_1 (see Figure 3). The other cases can be proven similarly.

Let $V = P_0 - A = (l\cos\theta, l\sin\theta)$ with $l = \|P_0 - A\|$ and θ the angle between AB and V. Now consider the curve $\mathbf{g}(t)$ from A to P_0 on $\mathbf{Q}(u,v)$. Let $\mathbf{g}(t) = \mathbf{Q}(A+tV)$ and $d(t) = D(A+tV) = \|\mathbf{R}(A+tV)\| + k|w(A+tV)|$. Then $\mathbf{g}(0) = \mathbf{Q}(A)$ and $\mathbf{g}(1) = \mathbf{Q}(P_0)$. So $d(0) = D(A) = 0$ and $d(1) = D(P_0) = \sup_{(u,v)\in T}\{D(u,v)\}$. Like the argument in Lemma 1, here we also consider three cases:

case 1 $w(P_0) = 0$.   Applying (2) gives

$$
\begin{aligned}
D(u,v) &\leq d(P_0) = \|\mathbf{R}(P_0)\| \leq \tfrac{1}{8}\left(\sup\|\mathbf{R}_{uu}''\|\delta_u^2 + 2\sup\|\mathbf{R}_{uv}''\|\delta_u\delta_v + \sup\|\mathbf{R}_{vv}''\|\delta_v^2\right) \\
&\leq \tfrac{1}{8}\left(D_{uu}\delta_u^2 + 2D_{uv}\delta_u\delta_v + D_{vv}\delta_v^2\right)
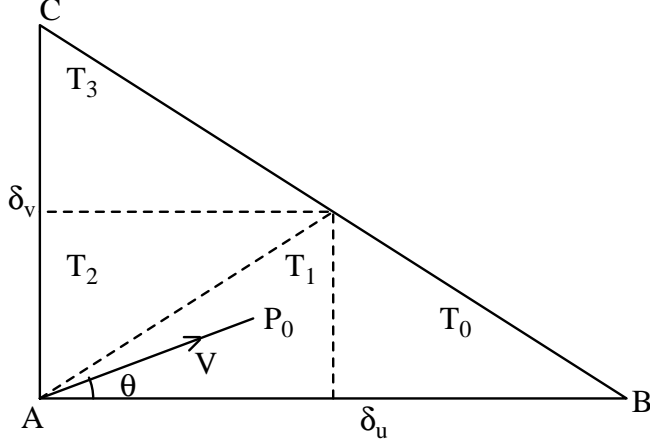\end{aligned}
$$

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Figure 3: The region of parametric triangle T.

case 2 $\mathbf{R}(P_0) = 0$.   Just as in case 1,

$$
\begin{aligned}
D(u,v) &\leq d(P_0) = k|w(P_0)| \leq \tfrac{1}{8}\left(\sup|w''_{uu}|\delta_u^2 + 2\sup|w''_{uv}|\delta_u\delta_v + \sup|w''_{vv}|\delta_v^2\right) \\
&\leq \tfrac{1}{8}\left(D_{uu}\delta_u^2 + 2D_{uv}\delta_u\delta_v + D_{vv}\delta_v^2\right)
\end{aligned}
$$

case 3 $w(P_0) \neq 0$, $\mathbf{R}(P_0) \neq 0$.

Thus $d(t)$ is C^2 at $t = 1$ and $d'(1) = 0$, i.e.

$$
d'(1) = \frac{\mathbf{R}(P_0)}{\|\mathbf{R}(P_0)\|}(\mathbf{R}'_u(P_0)l\cos\theta + \mathbf{R}'_v(P_0)l\sin\theta) + k\frac{w(P_0)}{|w(P_0)|}(w'_u(P_0)l\cos\theta + w'_v(P_0)l\sin\theta) = 0 \qquad (28)
$$

Writing $\mathbf{g}(t)$ in its Taylor expansion at $t = 1$:

$$
\mathbf{g}(t) = \mathbf{g}(1) + \mathbf{g}'(1)(t-1) + \int_1^t \mathbf{g}''(s)(t-s)ds \qquad (29)
$$

Letting $t = 0$ gives

$$
\mathbf{g}(0) = \mathbf{g}(1) - \mathbf{g}'(1) + \int_0^1 \mathbf{g}''(s)s\,ds
$$

or

$$
0 = (\mathbf{R}(A), w(A)) = (\mathbf{R}(P_0), w(P_0)) - (\mathbf{R}'_u(P_0), w'_u(P_0))l\cos\theta - (\mathbf{R}'_v(P_0), w'_v(P_0))l\sin\theta + I \qquad (30)
$$

where

$$
I = \int_0^1 \left((\frac{\partial^2\mathbf{R}(s)}{\partial u^2}, \frac{\partial^2 w(s)}{\partial u^2})l^2\cos^2\theta + 2(\frac{\partial^2\mathbf{R}(s)}{\partial u\partial v}, \frac{\partial^2 w(s)}{\partial u\partial v})l^2\cos\theta\sin\theta + (\frac{\partial^2\mathbf{R}(s)}{\partial v^2}, \frac{\partial^2 w(s)}{\partial v^2})l^2\sin^2\theta\right) s\,ds
$$

Taking the dot product of equation (30) with $(\frac{\mathbf{R}(P_0)}{\|\mathbf{R}(P_0)\|}, k\frac{w(P_0)}{|w(P_0)|})$ yields

$$
0 = \|\mathbf{R}(P_0)\| + k|w(P_0)| - 0 - 0 + I \cdot (\frac{\mathbf{R}(P_0)}{\|\mathbf{R}(P_0)\|}, k\frac{w(P_0)}{|w(P_0)|})
$$

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Therefore we get

$$
\begin{aligned}
d(P_0) \ &\leq \ |I \cdot (\tfrac{\mathbf{R}(P_0)}{\|\mathbf{R}(P_0)\|}, k\tfrac{w(P_0)}{|w(P_0)|})| \\[2mm]
&\leq \ \int_0^1 \left[(\|\mathbf{R}''_{uu}(s)\| + k|w''_{uu}(s)|)\delta_u^2/4 + 2(\|\mathbf{R}''_{uv}\| + k|w''_{uv}|)\delta_u\delta_v/4 + (\|\mathbf{R}''_{vv}\| + k|w''_{vv}|)\delta_v^2/4 \right] s\,ds \\[2mm]
&\leq \ \tfrac{1}{4}(D_{uu}\delta_u^2 + 2D_{uv}\delta_u\delta_v + D_{vv}\delta_v^2)\int_0^1 s\,ds \leq \tfrac{1}{8}(D_{uu}\delta_u^2 + 2D_{uv}\delta_u\delta_v + D_{vv}\delta_v^2)
\end{aligned}
$$

∎

From Lemma 2 and Theorem 2, we can derive the required step sizes for rational surfaces:

Theorem 4 *Let $T \subset E^2$ be a right triangle with vertices A, B and $C : B = A + (\delta_u, 0), C = A + (0, \delta_v)$. $\mathbf{r}(u,v) = \frac{\mathbf{R}(u,v)}{w(u,v)}, (w(u,v) > 0) : T-> E^3$ is a C^2 rational surface. $\mathbf{L}(u,v)$ is a fractional linearly parameterized triangle with $\mathbf{r}(A) = \mathbf{L}(A)$, $\mathbf{r}(B) = \mathbf{L}(B)$ and $\mathbf{r}(C) = \mathbf{L}(C)$. Then $\sup\limits_{(u,v)\in T} \|\mathbf{r}(u,v) - \mathbf{L}(u,v)\| \leq \epsilon$ if*

$$
D_{uu}\delta_u^2 + 2D_{uv}\delta_u\delta_v + D_{vv}\delta_v^2 \leq 8\epsilon \inf_{(u,v)\in T} w(u,v) \tag{31}
$$

where

$$
\begin{aligned}
r \ &\geq \ \sup_T \|\mathbf{r}(u,v)\| \\[2mm]
D_{uu} \ &= \ \begin{cases}
\sup\limits_T(\|\mathbf{R}''_{uu}(u,v)\| + (r-\epsilon)|w''_{uu}(u,v)|), & \epsilon < r \\
\sup\limits_T \|\mathbf{R}''_{uu}(u,v)\|, & r \leq \epsilon < 2r \\
0, & 2r \leq \epsilon
\end{cases} \\[2mm]
D_{uv} \ &= \ \begin{cases}
\sup\limits_T(\|\mathbf{R}''_{uv}(u,v)\| + (r-\epsilon)|w''_{uv}(u,v)|), & \epsilon < r \\
\sup\limits_T \|\mathbf{R}''_{uv}(u,v)\|, & r \leq \epsilon < 2r \\
0, & 2r \leq \epsilon
\end{cases} \\[2mm]
D_{vv} \ &= \ \begin{cases}
\sup\limits_T(\|\mathbf{R}''_{vv}(u,v)\| + (r-\epsilon)|w''_{vv}(u,v)|), & \epsilon < r \\
\sup\limits_T \|\mathbf{R}''_{vv}(u,v)\|, & r \leq \epsilon < 2r \\
0, & 2r \leq \epsilon
\end{cases}
\end{aligned} \tag{32}
$$

Note that formula (31) has two unknowns δ_u and δ_v. It requires a second condition to determine them. In case $D_{uu} = 0$ or $D_{vv} = 0$, which implies the surface is fractional linear in the corresponding direction, we let $\delta_u = 1$ and get

$$
\delta_v = \frac{\sqrt{D_{uv}^2 + 8D_{vv}\epsilon\inf\{w(u,v)\}} - D_{uv}}{D_{vv}}
$$

or let $\delta_v = 1$ and get

$$
\delta_u = \frac{\sqrt{D_{uv}^2 + 8D_{uu}\epsilon\inf\{w(u,v)\}} - D_{uv}}{D_{uu}}
$$

Otherwise, set $\delta_u/\delta_v = \sqrt{D_{vv}/D_{uu}}$ and we have

$$
\delta_u = \sqrt{\frac{4D_{vv}\epsilon\inf\{w(u,v)\}}{D_{uu}D_{vv} + D_{uv}\sqrt{D_{uu}D_{vv}}}}, \quad \delta_v = \sqrt{\frac{4D_{uu}\epsilon\inf\{w(u,v)\}}{D_{uu}D_{vv} + D_{uv}\sqrt{D_{uu}D_{vv}}}}
$$

This setting will minimize the number of triangles for tessellating the surface, as indicated in [Abi-Ezzi and Shirman, 1991]. A special case is $D_{uu} = D_{vv} = 0$ and $D_{uv} \neq 0$. We simply let $\delta_u = \delta_v$, and thus

$$\delta_u = \delta_v = \sqrt{\frac{4\epsilon \inf\{w(u,v)\}}{D_{uv}}}$$

For a rational Bézier surface expressed by

$$\mathbf{r}(u,v) = \frac{\mathbf{R}(u,v)}{w(u,v)} = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} w_{ij}\mathbf{P}_{ij}B_i^n(u)B_j^m(v)}{\sum_{i=0}^{n}\sum_{j=0}^{m} w_{ij}B_i^n(u)B_j^m(v)}, \quad u,v \in [0,1] \tag{33}$$

$\mathbf{R}(u,v)$ and $w(u,v)$ are in Bézier form, and the evaluation of D_{uu}, D_{uv} and D_{vv} is easy.

$$D_{uu} = n(n-1) \max_{\substack{0 \leq i \leq n-2 \\ 0 \leq j \leq m}} \{\|w_{i+2,j}\mathbf{P}_{i+2,j} - 2w_{i+1,j}\mathbf{P}_{i+1,j} + w_{i,j}\mathbf{P}_{i,j}\| + (r-\epsilon)|w_{i+2,j} - 2w_{i+1,j} + w_{i,j}|\}$$

$$D_{uv} = nm \max_{\substack{0 \leq i \leq n-1 \\ 0 \leq j \leq m-1}} \{\|w_{i+1,j+1}\mathbf{P}_{i+1,j+1} - w_{i+1,j}\mathbf{P}_{i+1,j} - w_{i,j+1}\mathbf{P}_{i,j+1} + w_{i,j}\mathbf{P}_{i,j}\|$$
$$+ (r-\epsilon)|w_{i+1,j+1} - w_{i+1,j} - w_{i,j+1} + w_{i,j}|\}$$

$$D_{vv} = m(m-1) \max_{\substack{0 \leq j \leq m-2 \\ 0 \leq i \leq n}} \{\|w_{i,j+2}\mathbf{P}_{i,j+2} - 2w_{i,j+1}\mathbf{P}_{i,j+1} + w_{i,j}\mathbf{P}_{i,j}\| + (r-\epsilon)|w_{i,j+2} - 2w_{i,j+1} + w_{i,j}|\}$$

where r can be chosen to be $r = \max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} \|\mathbf{P}_{ij}\|$.

It can be shown that the above estimation improves Cheng's result for surfaces. Also further improvements may be possible by bounding the second order derivatives with the "weight points", or by transforming the control points with a certain translation.

5   Conclusion

We have presented an approach to computing the tessellation step sizes for rational curves and surfaces. The method is derived in the homogeneous coordinates, so it is numerically accessible. The new formulas make a substantial improvement over previous known results. This is achieved by:

· efficiently estimating the effect of perspective transformation;

· treating $\|\mathbf{R}(t) - \mathbf{L}_n(t)\| + k|w(t) - L_d(t)|$ as a whole;

· performing more precise estimations.

Our approach is developed based on the parametric displacement of a curve or surface $\mathbf{r}$ from its fractional linearly parameterized interpolant $\mathbf{L}$. Future work could consider the geometric displacement, i.e., the perpendicular distance of $\mathbf{r}$ from $\mathbf{L}$, for the global step size estimation.

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