Available online at www.sciencedirect.com

SCIENCE DIRECT®

Computer Aided Geometric Design 22 (2005) 275–280

COMPUTER AIDED GEOMETRIC DESIGN

www.elsevier.com/locate/cagd

Minimizing the maximal ratio of weights of a rational Bézier curve

Jianmin Zheng

School of Computer Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798

Received 15 May 2004; received in revised form 8 December 2004; accepted 16 December 2004

Available online 29 December 2004

Abstract

This paper presents a solution to the problem of reparameterizing a rational curve by a Möbius transformation such that the maximal ratio of weights in the reparameterized representation is minimized. The problem is reduced to solving a linear programming problem, which can be solved directly and simply. The result can be used to reparameterize rational curves so as to yield tight bounds on derivatives.

Keywords: Rational Bézier curves; Möbius transformation; Weights; LP problem

1. Introduction

In geometric modeling, bounds on derivatives are used in many curve and surface algorithms (Filip et al., 1986). For rational Bézier curves or surfaces, the estimates of such bounds are often related to the ratios of weights (Floater, 1992; Saito et al., 1995). For example, consider a rational Bézier curve

$$r(t)=\frac{\sum_{i=0}^{n} w_i P_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}, \quad t\in[0,1], \tag{1}$$

where the weights w_i are assumed to be positive. Floater (1992) derived two bounds on the first order derivative:

$$\left|r'(t)\right| \leqslant n\frac{\max_{0\leqslant i\leqslant n} w_i}{\min_{0\leqslant i\leqslant n} w_i}\max_{0\leqslant i,j\leqslant n}|P_i-P_j|$$

E-mail address: asjmzheng@ntu.edu.sg (J. Zheng).

0167-8396/$ – see front matter
doi:10.1016/j.cagd.2004.12.003

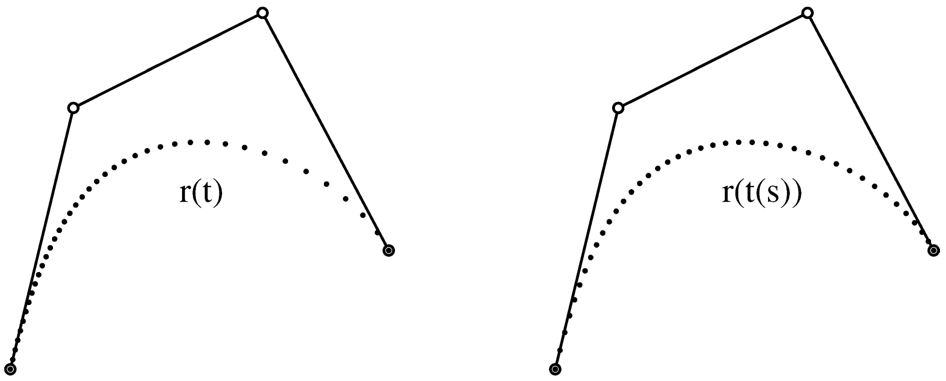

Fig. 1. Different parametric flows.

and

$$\left|r'(t)\right| \leqslant n\left(\frac{\max_{0\leqslant i\leqslant n} w_i}{\min_{0\leqslant i\leqslant n} w_i}\right)^2 \max_{1\leqslant i\leqslant n} |P_i - P_{i-1}|.$$

The uneven distribution of the weights leads to loose bounds. If these bounds on the derivative are used to determine a global step size for tessellating or rendering rational curves/surfaces such as in (Abi-Ezzi and Shirman, 1991; Rockwood, 1987), it will result in excessive segments and defect further computations. We are thus motivated to find a rational Bézier representation with a small ratio of the largest weight over the smallest weight.

It is noted that the representation of a rational Bézier curve is not unique. It is well known that a rational Bézier curve can be reparameterized by a Möbius transformation such that the parameter domain and the control points remain unchanged (Lucian, 1991). What changes is the way of tracing the curve. For example, a cubic rational Bézier curve $r(t)$, $t \in [0,1]$ with control points $(0,0)$, $(0.5, 2.2)$, $(2,3)$, $(3,1)$ and weights 8, 4, 1 and 1 is transformed by the Möbius transformation $t = 2s/(1+s)$ into another cubic rational Bézier curve $r(t(s))$, $s \in [0,1]$, with the same control points and new weights 8, 8, 4 and 8. The sequences of discrete points on the curves corresponding to a fixed parametric increment are shown in Fig. 1. This naturally raises a question: what is the good Möbius transformation for a given rational curve? Of course, it relies on given measure of "goodness". Farin and Worsey (1991) suggested reparameterizing a rational curve so as to make the first and last weights be 1. This representation is called the *standard form*. The standard form is likely to give a more symmetric parametric flow for a rational curve. Optimal reparameterization was also studied by Farouki (1997) who introduced an optimality criterion describing the deviation of the curve from the unit speed parameterization. The identification of optimal parameterizations for a polynomial curve was analytically reducible to determining the unique real root on (0, 1) of a quadratic equation. In this paper, we seek a rational reparameterization that leads to minimizing the maximal ratio of weights. We first show that finding such a parameter transformation can be reduced to a linear programming problem. Then we show that no "programming" is in fact required and a direct solution is presented.

2. The optimization problem

The general form of the Möbius transformation $t = t(s)$ that maps the interval $[0, 1]$ into itself with $t(0) = 0$ and $t(1) = 1$ is

$$t = t(s) = \frac{\gamma s}{\gamma s + (1 - s)}, \tag{2}$$

where $\gamma > 0$ is a free parameter. Substituting (2) into (1), we obtain a rational Bézier curve of degree n:

$$r(t(s)) = \frac{\sum_{i=0}^{n} \gamma^i w_i P_i B_i^n(s)}{\sum_{i=0}^{n} \gamma^i w_i B_i^n(s)} \tag{3}$$

with the same control points P_i, but the weights becoming $\gamma^i w_i$.

The goal of this paper is to find the "best" parameter transformation among the family of Möbius transformation (2). That is to determine a value for γ so that the curve $r(t(s))$ has such a property that $\max_i\{\gamma^i w_i\}/\min_i\{\gamma^i w_i\}$ is minimized for all possible $\gamma > 0$.

Note that $\max_{0\leqslant i\leqslant n}\{\gamma^i w_i\}/\min_{0\leqslant i\leqslant n}\{\gamma^i w_i\} = \max_{0\leqslant i,j\leqslant n}\{\gamma^{i-j} w_i/w_j\}$. So if we let $\rho = \max_{0\leqslant i,j\leqslant n}\{\gamma^{i-j} w_i/w_j\}$, then $\rho \geqslant \gamma^{i-j} w_i/w_j$ for all i and j. Based on this observation, we can convert our problem to finding the smallest ρ and the corresponding γ satisfying

$$\gamma^{i-j} \frac{w_i}{w_j} \leqslant \rho \tag{4}$$

for all i and j. Due to the exchangeability of indices i and j, it is obvious that $\rho \geqslant 1$.

Taking logarithm of both sides of inequality (4) and letting $H = \log\gamma$, $G = \log\rho$, gives

$$(i - j)H + \log w_i - \log w_j \leqslant G, \quad 0 \leqslant i, j \leqslant n,$$

and we arrive at the following linear programming (LP) problem:

$$\begin{aligned} &\text{minimize} && G \\ &\text{subject to} && -G \leqslant (i - j)H + \log w_i - \log w_j \leqslant G, \quad 0 \leqslant j < i \leqslant n. \end{aligned}$$

Observe that the constraints in the above LP problem contain $n(n+1)$ inequalities. However, it is possible that the constraints can be specified by only $2n$ inequalities. To show this, we divide the inequalities into n groups, each of which is labelled by an integer k that ranges from 1 to n. The kth group has the index restriction $i - j = k$ and is described by

$$-G \leqslant kH + \log w_i - \log w_{i-k} \leqslant G, \quad i = k, \ldots, n. \tag{5}$$

Introducing the new variables

$$\overline{W}_k = \max_{k\leqslant i\leqslant n}\{\log w_i - \log w_{i-k}\}, \qquad \underline{W}_k = \min_{k\leqslant i\leqslant n}\{\log w_i - \log w_{i-k}\}, \tag{6}$$

it is easy to check that the region defined by $2(n - k + 1)$ inequalities (5) in (H, G)-plane is the same as defined by two inequalities $G - kH \geqslant \overline{W}_k$ and $G + kH \geqslant -\underline{W}_k$. Thus the total number of the inequalities can be reduced to $2n$ and the LP problem can be simplified to

$$\begin{aligned} &\text{minimize} && G \\ &\text{subject to} && G - kH \geqslant \overline{W}_k, \quad k = 1, \ldots, n \\ & && G + kH \geqslant -\underline{W}_k, \quad k = 1, \ldots, n. \end{aligned} \tag{7}$$

To summarize, we have reduced the problem of finding the best Möbius transformation to a linear programming problem (7). Though this LP problem can be solved using the standard solver, such as the simplex method (Murty, 1976), we provide a more straightforward method in the next section.

3. The solution

Since (7) is a special LP problem, we expect a conceptually or geometrically simple solution, without resort to classical LP solvers. A trivial case is that all weights of the curve are the same and thus the curve is a polynomial one. Then the constraints become $-G \leqslant kH \leqslant G$ for $k = 1, \dots, n$. The optimal solution is $G = H = 0$. In general, we have

Theorem 1. *Let integers j_0, k_0 be such that*

$$\frac{j_0\overline{W}_{k_0} - k_0\underline{W}_{j_0}}{j_0 + k_0} = \max_{1\leqslant j,k\leqslant n}\left\{\frac{j\overline{W}_k - k\underline{W}_j}{j+k}\right\}. \tag{8}$$

Then

$$(H_0, G_0) = \left\{-\frac{\overline{W}_{j_0} + \underline{W}_{k_0}}{j_0 + k_0}, \frac{j_0\overline{W}_{k_0} - k_0\underline{W}_{j_0}}{j_0 + k_0}\right\}$$

is the unique optimal solution to LP problem (7).

Proof. We first show that in the feasible region in (H, G)-plane, which is defined by the inequalities of (7), G_0 given in the theorem is not greater than all possible values of G. In fact, for each j and k, the H and G values of any point (H, G) in the feasible region must satisfy $G - kH \geqslant \overline{W}_k$ and $G + jH \geqslant -\underline{W}_j$. Solving these two inequalities yields

$$G \geqslant \frac{j\overline{W}_k - k\underline{W}_j}{j+k}.$$

Since this inequality should hold for all j and k, $G \geqslant G_0$.

Next we show that (H_0, G_0) is a point in the feasible region. Note that (H_0, G_0) is the intersection point of lines $G - k_0H = \overline{W}_{k_0}$ and $G + j_0H = -\underline{W}_{j_0}$. For each line $G - kH = \overline{W}_k$, it intersects line $G + j_0H = -\underline{W}_{j_0}$ at a certain point, say (H_1, G_1). By definition of G_0 and (8), we have $G_0 \geqslant G_1$. Since both (H_0, G_0) and (H_1, G_1) are on line $G + j_0H = -\underline{W}_{j_0}$, $H_0 \leqslant H_1$. Thus for line $G - kH = \overline{W}_k$, $kH_0 + \overline{W}_k \leqslant kH_1 + \overline{W}_k = G_1 \leqslant G_0$. Similarly, we can prove that for each line $G + jH = -\underline{W}_j$, $G_0 \geqslant -jH_0 - \underline{W}_j$. This means point (H_0, G_0) is in the feasible region. Therefore (H_0, G_0) is an optimal solution.

Last we prove the uniqueness. Assume there exists another optimal solution (H_1, G_1). G_1 must equal to G_0. If $H_1 > H_0$, then $k_0H_1 + \overline{W}_{k_0} > k_0H_0 + \overline{W}_{k_0} = G_0 = G_1$. If $H_1 < H_0$, then $-j_0H_1 - \underline{W}_{j_0} > -j_0H_0 - \underline{W}_{j_0} = G_0 = G_1$. So in both cases, (H_1, G_1) is not a feasible solution, which contradicts the assumption. Therefore the only possibility is $H_1 = H_0$. □

The geometric meaning of Theorem 1 is that we only need to consider the intersection point of each pair of lines: one line from group $\{G - kH = \overline{W}_k\colon\ k = 1, \dots, n\}$ with positive slopes and the other from group $\{G + jH = -\underline{W}_j\colon\ j = 1, \ldots, n\}$ with negative slopes. The optimal solution is one of these intersection points, which has maximal G coordinate.

Now we are ready to describe the procedure for finding the required Möbius parameter transformation: (i) compute $\overline{W}_k$ and $\underline{W}_k$ by (6); (ii) find j_0 and k_0 by (8); (iii) the Möbius transformation is (2) with

$$\gamma = \exp\left(-\frac{\overline{W}_{j_0} + \underline{W}_{k_0}}{j_0 + k_0}\right).$$

As an example, we examine a quadratic rational Bézier curve

$$r_2(t) = \frac{w_0 P_0 B_0^2(t) + w_1 P_1 B_1^2(t) + w_2 P_2 B_2^2(t)}{w_0 B_0^2(t) + w_1 B_1^2(t) + w_2 B_2^2(t)}.$$

The corresponding LP problem is

$$\begin{aligned} &\text{minimize} && G \\ &\text{subject to} && G - H \geqslant \overline{W}_1 \\ &&& G + H \geqslant -\underline{W}_1 \\ &&& G - 2H \geqslant \overline{W}_2 \\ &&& G + 2H \geqslant -\underline{W}_2, \end{aligned}$$

where $\overline{W}_2 = \underline{W}_2 = \log w_2 - \log w_0$ and $\overline{W}_1 = \max\{\log w_1 - \log w_0, \log w_2 - \log w_1\}$, $\underline{W}_1 = \min\{\log w_1 - \log w_0, \log w_2 - \log w_1\}$. Therefore $\overline{W}_2 = \underline{W}_2 = \overline{W}_1 + \underline{W}_1$. By Theorem 1, $j_0 = k_0 = 1$, the optimal G is thus equal to $(\overline{W}_1 - \underline{W}_1)/2$, and $H = -(\overline{W}_1 + \underline{W}_1)/2 = \log\sqrt{w_0/w_2}$ or $\gamma = \sqrt{w_0/w_2}$. In particular, if the initial weights w_0 and w_2 are the same, then $\gamma = 1$, i.e., the required Möbius transformation is the identity map $t = s$. This also concludes that for a quadratic rational Bézier curve, the standard form is the "best" form.

Acknowledgements

The author would like to thank the reviewers for the constructive comments. This work is supported by the URC-SUG 8/04 of Nanyang Technological University.

References

Abi-Ezzi, A.S., Shirmann, L.A., 1991. Tessellation of curved surfaces under highly varying transformations. In: Proc. EUROGRAPHICS 91, pp. 385–397.

Farin, G., Worsey, A.J., 1991. Reparametrization and degree elevation for rational Bézier curves. In: Farin, G. (Ed.), NURBS for Curve and Surface Design. SIAM, ISBN 0-89871-286-6.

Farouki, R., 1997. Optimal parameterizations. Computer Aided Geometric Design 14 (2), 153–168.

Filip, D., Magedson, R., Markot, R., 1986. Surface algorithms using bounds on derivatives. Computer Aided Geometric Design 3, 295–311.

Floater, M.S., 1992. Derivatives of rational Bézier curves. Computer Aided Geometric Design 9 (2), 161–174.

Lucian, M., 1991. Linear fractional transformations of rational Bézier curves. In: Farin, G. (Ed.), NURBS for Curve and Surface Design. SIAM, ISBN 0-89871-286-6.

Murty, K., 1976. Linear and Combinatorial Programming. Wiley, Toronto, ISBN 0-471-57370-1.
Rockwood, A., 1987. A generalized scanning technique for display of parametrically defined surfaces. IEEE Computer Graphics Appl., 15–26.
Saito, T., Wang, G., Sederberg, T., 1995. Hodographs and normals of rational curves and surfaces. Computer Aided Geometric Design 12 (4), 417–430.