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HARMONIC MAPS FOR HITCHIN REPRESENTATIONS

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Abstract. Let (S, g_0) be a hyperbolic surface, ρ be a Hitchin representation for $PSL(n, \mathbb{R})$, and f be the unique ρ-equivariant harmonic map from $(\widetilde{S}, \widetilde{g}_0)$ to the corresponding symmetric space. We show its energy density satisfies $e(f) \geq 1$ and equality holds at one point only if $e(f) \equiv 1$ and ρ is the base n-Fuchsian representation of (S, g_0). In particular, we show given a Hitchin representation ρ for $PSL(n, \mathbb{R})$, every ρ-equivariant minimal immersion f from the hyperbolic plane $\mathbb{H}^2$ into the corresponding symmetric space X is distance-increasing, i.e. $f^* g_X \geq g_{\mathbb{H}^2}$. Equality holds at one point only if it holds everywhere and ρ is an n-Fuchsian representation.

1 Introduction

In this paper, we study equivariant harmonic maps into the noncompact symmetric space for Hitchin representations. Consider a closed orientable surface S of genus $g \geq 2$. Let Σ denote a Riemann surface structure on S and g_0 be the unique conformal hyperbolic metric on Σ of constant Gaussian curvature -1. By the uniformization theorem, the Riemann surface structure Σ, or the hyperbolic metric g_0 on S, gives rise to a Fuchsian holonomy "representation" j_Σ of $\pi_1(S)$ into $PSL(2, \mathbb{R})$. Composing j_Σ with the unique irreducible representation of $PSL(2, \mathbb{R})$ into $PSL(n, \mathbb{R})$, we obtain a representation of $\pi_1(S)$ into $PSL(n, \mathbb{R})$ and call it *the base n-Fuchsian representation of Σ or (S, g_0)*. *An n-Fuchsian representation* is defined to be a base n-Fuchsian representation of some Riemann surface structure Σ. The Hitchin representations are exactly deformations of the n-Fuchsian representations in the space of all representations of $\pi_1(S)$ into $PSL(n, \mathbb{R})$. These form a connected component inside the representation variety for $PSL(n, \mathbb{R})$, called the Hitchin component. Hitchin representations possess many nice properties, for example: Hitchin [Hit92] showed that they are irreducible; Labourie [Lab06] showed that they are discrete, faithful quasi-isometric embeddings of $\pi_1(S)$ into $PSL(n, \mathbb{R})$.

Fix M, N two Riemannian manifolds and consider an equivariant map $f : \widetilde{M} \to N$ for a representation $\rho : \pi_1(M) \to Isom(N)$. The *energy density* of f is $e(f) = \frac{1}{2}||df||^2$ with respect to the metrics on $\widetilde{M}$ and N. By equivariance, $e(f)$ is invariant under $\pi_1(M)$, so it gives a well-defined function on M, also called the energy density.

The *energy* $E(f)$ is the integral of the energy density $e(f)$ over M. Equivariant harmonic maps are defined to be critical points of the energy functional. When M is of dimension 2, the energy and harmonicity are invariant under conformal change of the metric on M and thus we can talk about harmonic maps from a Riemann surface instead of a Riemannian surface.

Fix a semisimple subgroup G of $SL(n,\mathbb{C})$. A representation ρ of $\pi_1(S)$ into G is said to be *irreducible* (or *completely reducible*) if the induced representation on $\mathbb{C}^n$ is irreducible (or completely reducible). Fix a Riemann surface structure Σ on S. Following the work of Donaldson [Don87] and Corlette [Cor88], for any irreducible representation ρ into G, there exists a unique ρ-equivariant harmonic map f from $\widetilde{\Sigma}$ to the corresponding symmetric space of G. For the base n-Fuchsian representation of Σ, the harmonic map f is a totally geodesic embedding of $\mathbb{H}^2$. The equivariant harmonic map further gives rise to a Higgs bundle, a pair (E,ϕ) consisting of a holomorphic vector bundle over Σ and a holomorphic section of $End(E)\otimes K$, the Higgs field. Conversely, by the work of Hitchin [Hit92] and Simpson [Sim88], a stable Higgs bundle admits a unique metric on the bundle which solves the Hitchin equation, this metric will be referred to as the harmonic metric. The harmonic metric further gives rise to an irreducible representation ρ into G and a ρ-equivariant harmonic map into the corresponding symmetric space. These two directions between representations and Higgs bundles together give the celebrated non-abelian Hodge correspondence. In particular, Hitchin in [Hit92] used Higgs bundles to give a nice description of the Hitchin component.

We study the energy density of equivariant harmonic maps for Hitchin representations. Let X denote the symmetric space $SL(n,\mathbb{R})/SO(n)$. Renormalize the metric on X so that the totally geodesic hyperbolic plane associated to any n-Fuchsian representation has curvature -1. Here we prove

Theorem 1.1 (Theorem 4.2). *Let ρ be a Hitchin representation for $PSL(n,\mathbb{R})$, g_0 be a hyperbolic metric on S, and f be the unique ρ-equivariant harmonic map from $(\widetilde{S},\widetilde{g}_0)$ to the symmetric space X. Then its energy density $e(f)$ satisfies*

$$e(f)\geq 1.$$

Moreover, equality holds at one point only if $e(f)\equiv 1$ in which case ρ is the base n-Fuchsian representation of (S,g_0).

Remark 1.2. Consider the $(1,1)$-part of a Riemannian metric g, that is, $g^{1,1} = 2Re(g(\partial_z,\partial_{\bar{z}})dz\otimes d\bar{z})$. Equivalently, Theorem 1.1 says the pullback metric of f satisfies $(f^*g_X)^{1,1}\geq\widetilde{g}_0$ as the domination of two 2-tensors, meaning their difference is a definite 2-tensor.

Remark 1.3. In the case $n=2$, Theorem 1.1 concerns harmonic diffeomorphism between surfaces and is proven in Sampson [Sam78].

Understanding the harmonic maps involves estimating the harmonic metric solving the Hitchin equation for Higgs bundles. The Hitchin equation is a second-order nonlinear elliptic system, which is highly non-trivial in general. The property of cyclic Higgs bundles used in [DL] is that the solution metric is diagonal with respect to a certain holomorphic splitting of the bundle E, which is also essential in Labourie's proof [Lab17] showing the uniqueness of an equivariant minimal surface for Hitchin representations into split real rank 2 Lie group.

However, the diagonal property no longer holds for general cases and it seems hard to directly analyze the Hitchin equation for a particular Higgs bundle. Here we are able to analyze the Hitchin equation without requiring cyclic condition. This is done by considering a holomorphic filtration of the bundle E and choosing another orthogonal splitting of E. In this way, considering the holomorphic filtration and the new splitting is an effective tool to study Hitchin representations and related harmonic maps.

Recall the energy of f is the integral of the energy density on S with respect to the hyperbolic volume form. As a corollary of Theorem 1.1, we reprove the following result of Hitchin [Hit92].

Corollary 1.4. *Under the same assumptions of Theorem 1.1, the energy $E(f)$ of every ρ-equivariant harmonic map f satisfies*

$$E(f) \geq -2\pi \cdot \chi(S).$$

Equality holds if and only if ρ is the base n-Fuchsian representation of (S, g_0).

Remark 1.5. The energy is also the L^2-norm of the Higgs field, which is a Morse-Bott function on the moduli space of Higgs bundles. It was first considered by Hitchin [Hit87,Hit92] to study the topology of the moduli space of G-Higgs bundles for $G = SL(2,\mathbb{C})$, $PSL(n,\mathbb{R})$ and hence the corresponding representation variety. Hitchin's original proof of Corollary 1.4 in the Higgs bundle language relies on the Kähler structure on the moduli space and that the energy function is a moment map of an S^1-action on the moduli space.

When the harmonic map is conformal, it is a minimal immersion. Labourie [Lab08] showed that for any Hitchin representation ρ, there exists a ρ-equivariant immersed minimal surface inside the symmetric space X and conjectured that the minimal surface is unique.

As a special case of Theorem 1.1, we obtain the following corollary immediately, which verifies the metric domination conjecture in Dai-Li [DL].

Corollary 1.6. *Let ρ be a Hitchin representation for $PSL(n,\mathbb{R})$ which is not an n-Fuchsian representation. Then the pullback metric of every ρ-equivariant minimal immersion f of $\widetilde{S}$ into the symmetric space X satisfies*

$$f^*g_X \geq \lambda \cdot g_{\mathbb{H}^2} \quad \textit{for some positive constant } \lambda > 1,$$

where $g_{\mathbb{H}^2}$ denotes the unique hyperbolic metric on $\widetilde{S}$ in the conformal class of f^*g_X.

Note that when ρ is an n-Fuchsian representation, there is a unique ρ-equivariant minimal immersion f of $\widetilde{S}$ into the symmetric space X whose image is a totally geodesic hyperbolic plane of curvature -1.

REMARK 1.7. For a general representation which is not Hitchin, one should not expect that a similar phenomenon happens. For example, for any completely reducible $SL(2,\mathbb{C})$-representation ρ, every ρ-equivariant minimal mapping from $\mathbb{H}^2$ into $\mathbb{H}^3$ is distance-decreasing by Ahlfors' lemma (see [DT]). Even for split real groups, there exist many maximal $Sp(4,\mathbb{R})$-representations and equivariant minimal immersions from $\mathbb{H}^2$ into $Sp(4,\mathbb{R})/U(2)$ which are distance-decreasing (see [DL18]).

We equip ρ with a translation length spectrum $\bar{l}_\rho : \gamma \in \pi_1(S) \to \mathbb{R}^+$ given by

$$\bar{l}_\rho(\gamma) := \inf_M \inf_{x\in M} d_M(x, \rho(\gamma)(x)),$$

where M goes through all ρ-equivariant minimal surfaces in the symmetric space X and d_M is induced by the metric on the minimal surface M. If Labourie's conjecture on the uniqueness of minimal surface holds, then the first "inf" in the definition of $\bar{l}_\rho$ is not needed. One may compare $\bar{l}_\rho$ with the classical translation length spectrum $l_\rho : \gamma \in \pi_1(S) \to \mathbb{R}^+$ given by

$$l_\rho(\gamma) := \inf_{x\in X} d_X(x, \rho(\gamma)(x)),$$

where d_X is induced by the unique G-invariant Riemannian metric on X. Notice that for an n-Fuchsian representation j, $\bar{l}_j = l_j$ and the fact $d_X \le d_M$ implies that $l_\rho(\gamma) \le \bar{l}_\rho(\gamma)$, for any element $\gamma \in \pi_1(S)$. As a direct corollary of Theorem 1.6, we have:

COROLLARY 1.8. *For any Hitchin representation ρ for $PSL(n,\mathbb{R})$ which is not an n-Fuchsian representation, there exists an n-Fuchsian representation j and a positive constant $\lambda > 1$ such that for any non-identity element $\gamma \in \pi_1(S)$,*

$$\bar{l}_\rho(\gamma) > \lambda \cdot \bar{l}_j(\gamma) = \lambda \cdot l_j(\gamma). \tag{1}$$

Organization of the paper. In Section 2, we review the theory of Higgs bundles and the non-abelian Hodge correspondence. In Section 3, we first explain the Higgs bundle description of Hitchin representations in terms of holomorphic differentials, then introduce a new expression of the Higgs bundles in the Hitchin section, and in the end deduce the Hitchin equation using the new expression. In Section 4, we show the main theorem. In Section 5, we discuss some further questions.

2 Preliminaries

In this section, we briefly recall the theory of Higgs bundles and the non-abelian Hodge correspondence. One may refer [Bar10, DL, Lab17] for more details. For $p \in S$, let $\pi_1 = \pi_1(S,p)$ be the fundamental group of S. Let Σ be a Riemann surface structure on S, $\widetilde{\Sigma}$ be the universal cover, and $K_\Sigma, K_{\widetilde{\Sigma}}$ be the canonical line bundle over $\Sigma, \widetilde{\Sigma}$ respectively. Let g_0 be the unique conformal hyperbolic metric on Σ of constant curvature -1. Through this whole section, G denotes $SL(n,\mathbb{C})$ and K denotes $SU(n)$. There is a G-invariant Riemannian metric $g_{G/K}$ on the symmetric space G/K which is unique up to scalar. We will see the explicit definition of $g_{G/K}$ later in Equation (2).

Let ρ be a representation of π_1 into G and we consider a ρ-equivariant smooth map $f : (\widetilde{S}, \widetilde{g}_0) \to (G/K, g_{G/K})$. We first recall the definition of f being harmonic. The *energy density* of f is $e(f) = \frac{1}{2}||df||^2$ with respect to the metrics $\widetilde{g}_0$ and $g_{G/K}$ on $\widetilde{S}$ and G/K respectively. By equivariance, $e(f)$ is invariant under π_1, so it gives a well-defined function on S, also called the energy density. The *energy* $E(f)$ is the integral of the energy density $e(f)$ over S. A ρ-equivariant smooth map f is called *harmonic* if it is a critical point of the energy. Note that the energy $E(f)$ is invariant under the conformal change of the metric on S, and so is the harmonicity of f. Hence we can talk about equivariant harmonic maps from the universal cover of a Riemann surface. However, the energy density still varies under the conformal change of the metric on S and when we talk about the energy density, we will usually choose g_0 as the conformal metric on Σ.

2.1 From Higgs bundles to harmonic maps and representations.

Definition 2.1. *A G-Higgs bundle over Σ is a pair (E,ϕ) consisting of a holomorphic rank n vector bundle E of trivial determinant and a trace-free holomorphic bundle map from E to $E \otimes K_\Sigma$.*

We call (E,ϕ) is stable if any proper ϕ-invariant holomorphic subbundle F has a negative degree, and is polystable if it is a direct sum of stable Higgs bundles of degree 0.

The moduli space of G-Higgs bundles consists of gauge equivalent classes of polystable G-Higgs bundles, denoted as $\mathcal{M}_{Higgs}(G)$.

Theorem 2.2 (Hitchin [Hit87] and Simpson [Sim88]). *Let (E,ϕ) be a polystable G-Higgs bundle. Then there exists a Hermitian metric H on E such that the induced metric on* $\det E \cong \mathcal{O}$ *is* 1*, solving the Hitchin equation*

$$F^{\nabla^H} + [\phi, \phi^{*_H}] = 0,$$

where ∇^H is the Chern connection of H, F^{∇^H} is the curvature of the connection ∇^H, and ϕ^{*_H} is the adjoint of ϕ with respect to H.

The Hermitian metric H is called the harmonic metric. And the harmonic metric is unique if (E,ϕ) is stable.

The Hitchin equation is equivalent to the G-connection $D = \nabla^H + \phi + \phi^{*_H}$ being flat. The holonomy of D gives a completely reducible representation $\rho : \pi_1 \to G$ and the pair (E,D) is isomorphic to $\widetilde{\Sigma} \times_\rho \mathbb{C}^n$ equipped with the natural flat connection.

Let P_G denote the unimodule frame bundle of the vector bundle $E \cong \widetilde{\Sigma} \times_\rho \mathbb{C}^n$, which is isomorphic to $\widetilde{\Sigma} \times_\rho G$. A choice of a Hermitian metric H on E is equivalent to a reduction of P_G to K by considering the unitary unimodule frame bundle of (E, H). A Hermitian metric H then descends to be a section s_H of the fiber bundle $P_G/K = \widetilde{\Sigma} \times_\rho G/K$ over Σ. Such a section s_H over Σ corresponds to a ρ-equivariant map $f : \widetilde{\Sigma} \to G/K$. Moreover, the map arising from a harmonic metric is harmonic. If the image of the representation lies in $SL(n, \mathbb{R})$, the harmonic map lies in a totally geodesic copy of $SL(n, \mathbb{R})/SO(n)$ inside $G/K = SL(n, \mathbb{C})/SU(n)$.

2.2 From representations and harmonic maps to Higgs bundles. Given a completely reducible representation ρ, by the work of Corlette [Cor88] and Donaldson [Don87], there exists a ρ-equivariant harmonic map $f : \widetilde{\Sigma} \to G/K$, which is unique up to the centralizer of $\rho(\pi_1)$. The set of conjugate classes of completely reducible representations of $\pi_1(S)$ into G is called *the representation variety* and will be denoted by $Rep(\pi_1, G)$.

Let's explain how to extract a Higgs bundle from an equivariant harmonic map. Denote

$$\mathfrak{g} = Lie(G) = sl(n, \mathbb{C}), \quad \mathfrak{k} = Lie(K) = su(n), \quad \text{and} \quad \mathfrak{p} = i \cdot su(n),$$

On a Lie group G, there is a natural 1-form ω valued in $\mathfrak{g}$, called the Maurer-Cartan form, which is defined as follows: at a point $g \in G$, ω_g is the tangential map of the left multiplication by g^{-1}, $\omega_g := (L_{g^{-1}})_* : T_g G \to T_e G \cong \mathfrak{g}$. With respect to the $Ad(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we can decompose the Maurer-Cartan form of G, $\omega = \omega^{\mathfrak{k}} + \omega^{\mathfrak{p}}$, where $\omega^{\mathfrak{k}} \in \Omega^1(G, \mathfrak{k})$ and $\omega^{\mathfrak{p}} \in \Omega^1(G, \mathfrak{p})$. The form $\omega^{\mathfrak{k}}$ is a connection on the K-bundle $G \to G/K$ and $\omega^{\mathfrak{p}}$ descends to be an element in $\Omega^1(G/K, G \times_{Ad_K} \mathfrak{p})$, giving an isomorphism $T(G/K) \cong G \times_{Ad_K} \mathfrak{p}$. One can refer to Chapter 1 in [BR90] for details on the Maurer-Cartan form calculus. Pulling back the principal K-bundle $G \to G/K$ to $\widetilde{\Sigma}$ by the map f, we get a principal K-bundle P_K on $\widetilde{\Sigma}$. Then

(1) $f^*\omega^{\mathfrak{k}}$ is a connection form on P_K, hence a unitary connection form A on the complexified bundle P_G. Denote the covariant derivative of A as d_A.

(2) $f^*\omega^{\mathfrak{p}}$ is a section of $T^*\widetilde{\Sigma} \otimes (P_K \times_{Ad_K} \mathfrak{p})$ over $\widetilde{\Sigma}$, whose complexification is

$$\begin{aligned}(T^*\widetilde{\Sigma} \otimes \mathbb{C}) \otimes (P_K \times_{Ad_K} \mathfrak{p} \otimes \mathbb{C}) &\cong (K_{\widetilde{\Sigma}} \oplus \bar{K}_{\widetilde{\Sigma}}) \otimes (P_G \times_{Ad_G} \mathfrak{g}) \\ &\cong (K_{\widetilde{\Sigma}} \oplus \bar{K}_{\widetilde{\Sigma}}) \otimes End_0(E),\end{aligned}$$

where $End_0(E)$ is the trace-free endomorphism bundle of E.

The harmonicity of the map f from a Riemann surface can be interpreted as a holomorphic condition and hence assures that the pair $(d_A^{0,1}, (f^*\omega^{\mathfrak{p}})^{1,0})$ is a G-Higgs bundles over $\widetilde{\Sigma}$. By equivariance of f, this Higgs bundle descends to Σ. Since the Maurer-Cartan form ω induces the flat connection on P_G, by comparing these two decompositions, the harmonic map associated to this Higgs bundle coincides with the original one.

In this way, we obtain a homeomorphism between the moduli space of G-Higgs bundles and the representation variety:

$$\mathcal{M}_{Higgs}(G) \cong Rep(\pi_1, G),$$

i.e. the non-abelian Hodge correspondence.

2.3 Harmonic maps in terms of Higgs field. By the above discussion, the pullback of the form $\omega^{\mathfrak{p}} \in \Omega^1(G/K, G \times_{Ad_K} \mathfrak{p})$ by the map $f : \widetilde{\Sigma} \to G/K$ is given by:

$$(f^*\omega^{\mathfrak{p}})^{1,0} = \pi^*\phi \in \Omega^{1,0}(\widetilde{\Sigma}, End_0(E)),$$
$$(f^*\omega^{\mathfrak{p}})^{0,1} = \pi^*\phi^{*_H} \in \Omega^{0,1}(\widetilde{\Sigma}, End_0(E)).$$

The rescaled Killing form $B(X,Y) = \frac{12}{n(n^2-1)} \cdot \mathrm{tr}(XY)$ on $\mathfrak{g}$ induces a Riemannian metric $g_{G/K}$ on G/K: for two vectors $Y_1, Y_2 \in T_p(G/K)$,

$$g_{G/K}(Y_1, Y_2) = B(\omega^{\mathfrak{p}}(Y_1), \omega^{\mathfrak{p}}(Y_2)). \tag{2}$$

Remark 2.3. We will later see in Equation (19) that this rescaled Killing form is such that for the base n-Fuchsian representation of (S, g_0), the associated equivariant harmonic map is a totally geodesic isometric embedding of $(\widetilde{S}, \widetilde{g}_0)$ inside G/K.

Pulling back the metric on G/K to $\widetilde{\Sigma}$ by f: for any two vectors $\widetilde{X}, \widetilde{Y} \in T\widetilde{\Sigma}$,

$$f^*g_{G/K}(\widetilde{X}, \widetilde{Y}) = g_{G/K}(f_*(\widetilde{X}), f_*(\widetilde{Y})) = B(\omega^{\mathfrak{p}}(f_*(\widetilde{X})), \omega^{\mathfrak{p}}(f_*(\widetilde{Y})).$$

Since f is ρ-equivariant and $g_{G/K}$ is G-invariant, $f^*g_{G/K}$ also descends to Σ. From now on, we won't distinguish notations on Σ and $\widetilde{\Sigma}$. So the pullback metric $f^*g_{G/K}$ on Σ is given by

$$f^*g_{G/K} = \frac{12}{n(n^2-1)}(\mathrm{tr}(\phi^2) + 2Re(\mathrm{tr}(\phi\phi^{*_H})) + \mathrm{tr}(\phi^{*_H}\phi^{*_H})). \tag{3}$$

The Hopf differential of f is defined to be the $(2,0)$-part of the pull back metric $f^*g_{G/K}$, which is denoted by Hopf(f). So it is given by

$$\mathrm{Hopf}(f) = (f^*g_{G/K})^{2,0} = \frac{12}{n(n^2-1)} \cdot \mathrm{tr}(\phi^2). \tag{4}$$

The conformal hyperbolic metric g_0 on Σ induces a Hermitian metric h on K_Σ^{-1} satisfying $g_0 = 2Re(h)$. In a local coordinate chart of coordinate z, we denote by $\hat{g}_0$ the local function $g_0(\partial_z, \partial_{\bar{z}}) = h(\partial_z, \partial_z)$. The metric g_0 is locally given by

$$g_0 = 2Re(\hat{g}_0 dz \otimes d\bar{z}) = \hat{g}_0(dz \otimes d\bar{z} + d\bar{z} \otimes dz) = 2\hat{g}_0(dx^2 + dy^2).$$

Since the Gaussian curvature formula of g_0 is $K_{g_0} = -\frac{1}{\hat{g}_0}\partial_{\bar{z}}\partial_z \log \hat{g}_0$ and the Gaussian curvature of g_0 is -1, the local function $\hat{g}_0$ satisfies

$$\partial_{\bar{z}}\partial_z \log \hat{g}_0 = \hat{g}_0. \tag{5}$$

Now let us derive the exact formula of the energy density $e(f) = \frac{1}{2}||df||^2$, the square norm of df as a section of $T^*\widetilde{S} \otimes f^*T(G/K)$. Locally, set $\Phi = \phi + \phi^{*_H}$ and denote $\phi = \hat{\phi}dz$ and $\phi^{*_H} = \hat{\phi}^{*_H}d\bar{z}$, the energy density is given by

$$\begin{aligned} e(f) = \frac{1}{2}||df||^2 &= \frac{1}{4\hat{g}_0}(g_{G/K}(f_*(\partial_x), f_*(\partial_x)) + g_{G/K}(f_*(\partial_y), f_*(\partial_y))) \\ &= \frac{1}{4\hat{g}_0}(B(\Phi(\partial_x), \Phi(\partial_x)) + B(\Phi(\partial_y), \Phi(\partial_y))) \\ &= \frac{1}{4\hat{g}_0}(B(\hat{\phi} + \hat{\phi}^{*_H}, \hat{\phi} + \hat{\phi}^{*_H}) + B(i(\hat{\phi} - \hat{\phi}^{*_H}), i(\hat{\phi} - \hat{\phi}^{*_H})) \\ &= \frac{1}{\hat{g}_0}B(\hat{\phi}, \hat{\phi}^{*_H}) = \frac{12}{n(n^2-1)} \cdot \frac{1}{\hat{g}_0}\mathrm{tr}(\hat{\phi}\hat{\phi}^{*_H}). \end{aligned}$$

Hence globally,

$$e(f) = \frac{12}{n(n^2-1)} \cdot 2Re(\mathrm{tr}(\phi\phi^{*_H}))/g_0. \tag{6}$$

3 Higgs Bundles in the Hitchin Section

In the first part of this section, we explain the Higgs bundle description of Hitchin representations in terms of holomorphic differentials $\bigoplus_{i=2}^n H^0(\Sigma, K^i)$. In the second part, we introduce a new expression of the Higgs bundles in the Hitchin section which behave nicely with respect to the harmonic metric. Lastly, we express the Hitchin equation in our new Higgs bundle expression.

3.1 Hitchin fibration and Hitchin section. Let us explain Hitchin's parametrization in [Hit92] and refer to Section 2.2 in [Bar10] for details and [Kos59] for the basics on principal 3-dimensional subalgebras. Recall that a principal 3-dimensional subalgebra $\mathfrak{s}$ in $\mathfrak{g} = sl(n, \mathbb{C})$ is spanned by a triple $\{\tilde{e}_1, x, e_1\}$ consisting of a semisimple element x, regular nilpotent elements $\tilde{e}_1$ and e_1 satisfying

$$[x, e_1] = e_1, \quad [x, \tilde{e}_1] = -\tilde{e}_1, \quad [e_1, \tilde{e}_1] = x.$$

Principal 3-dimensional subalgebras are unique up to conjugacy.

Moreover, we require $\mathfrak{s}$ to be real with respect to a compact real form ρ of $\mathfrak{g}$, that is,

$$\rho(x) = -x, \quad \rho(e_1) = -\tilde{e}_1.$$

The adjoint representation of $\mathfrak{s}$ decomposes $\mathfrak{g}$ into a direct sum of irreducible representations V_i of dimension $2i+1$ for $1 \leq i \leq n-1$ and V_1 is just $\mathfrak{s}$ itself. We choose a vector $e_i \in V_i$ as an eigenvector of $ad(x)$ associated with eigenvalue i.

For elements $f = \tilde{e}_1 + \alpha_2 e_1 + \cdots + \alpha_n e_{n-1} \in \mathfrak{g}$, there exist invariant homogeneous polynomials $p_i, i = 2, \cdots, n$ of degree i on $\mathfrak{g}$ such that $p_i(f) = \alpha_i$. The Hitchin fibration is then defined as a map from the moduli space of $SL(n,\mathbb{C})$-Higgs bundles over Σ to the direct sum of the holomorphic differentials

$$\begin{aligned} h : M_{Higgs}(SL(n,\mathbb{C})) &\longrightarrow \bigoplus_{j=2}^{n} H^0(\Sigma, K^j) \\ (E,\phi) &\longmapsto (p_2(\phi), p_3(\phi), \cdots, p_n(\phi)). \end{aligned}$$

Remark 3.1. The polynomial p_2 of degree 2 is $p_2(X) = c \cdot tr(X^2), X \in \mathfrak{g}$ for some fixed constant c. Recall that the Hopf differential of the associated harmonic map is $\frac{12}{n(n^2-1)} \cdot \mathrm{tr}(\phi^2)$, see Equation (4). Thus the first term $p_2(\phi) \in H^0(\Sigma, K^2)$ of the Hitchin fibration $h(E,\phi)$ coincides with the Hopf differential of associated harmonic map up to a constant scalar.

Hitchin in [Hit92] defines a section s of the Hitchin fibration as follows: at $(q_2, \cdots, q_n) \in \bigoplus_{j=2}^{n} H^0(\Sigma, K^j)$, $s(q_2, \cdots, q_n)$ is defined to be

$$E = K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{\frac{1-n}{2}}, \quad \phi = \tilde{e}_1 + q_2 e_1 + q_3 e_2 + \cdots + q_n e_n.$$

This section s will be referred to as the Hitchin section. Hitchin in [Hit92] showed that the Higgs bundles in the Hitchin section are stable and have holonomy inside a copy of $SL(n,\mathbb{R})$ inside $SL(n,\mathbb{C})$. Under the non-abelian Hodge correspondence, the image of the Hitchin section corresponds to the Hitchin component in the representation variety for $PSL(n,\mathbb{R})$. Different choices of principal 3-dimensional subalgebras give gauge equivalent Higgs bundles.

Now we will write the Higgs bundles in the Hitchin section in matrix form explicitly.

Choose the compact real form on $sl(n,\mathbb{C})$ as $\rho(X) = -\overline{X}^T$. Choose the Cartan subalgebra as the trace-free diagonal matrices $\mathfrak{h} \ni H = diag(t_1, \cdots, t_n)$ and a positive Weyl chamber in $\mathfrak{h}$ consisting of H satisfying $t_i > t_{i+1}$. The root system $\triangle$, the system $\triangle^+$ of positive roots, and the system Π of corresponding simple roots are

$$\begin{aligned} &\triangle = \{\alpha_{ij} \in \mathfrak{h}^*, i \neq j | \alpha_{ij}(H) = t_i - t_j, \forall = diag(t_1, \cdots, t_n) \in \mathfrak{h}\}, \\ &\triangle^+ = \{\alpha_{ij} \in \triangle, i < j\}, \quad \Pi = \{\alpha_{i,i+1} \in \triangle\}. \end{aligned}$$

Let E_{ij} be a $n \times n$ matrix such that its (i,j)-entry is 1 and 0 elsewhere. Denote by $<,>$ the Killing form of $\mathfrak{g}$. We use the same symbol $<,>$ to denote its restriction to $\mathfrak{h}$ and its dual extension to $\mathfrak{h}^*$. For each root $\alpha \in \triangle$, the coroot $h_\alpha \in \mathfrak{h}$ is defined by $< h_\alpha, u > = \frac{2}{<\alpha,\alpha>}\alpha(u)$ for $u \in \mathfrak{h}$. So the coroot of α_{ij} is $h_{\alpha_{ij}} = E_{ii} - E_{jj}$, and its root space is spanned by $x_{\alpha_{ij}} := E_{ij}$. We then choose $\tilde{e}_1, x, e_1$ as follows:

$$x = \frac{1}{2}\sum_{\alpha\in\triangle^+} h_\alpha = \sum_{\alpha_{i,i+1}\in\Pi} r_i h_{\alpha_{i,i+1}} = \begin{pmatrix} \frac{n-1}{2} & & & & \\ & \frac{n-3}{2} & & & \\ & & \ddots & & \\ & & & \frac{3-n}{2} & \\ & & & & \frac{1-n}{2} \end{pmatrix},$$

$$e_1 = \sum_{\alpha_{i,i+1}\in\Pi} \sqrt{r_i} x_{\alpha_{i,i+1}} = \begin{pmatrix} 0 & \sqrt{r_1} & & & \\ & 0 & \sqrt{r_2} & & \\ & & \ddots & \ddots & \\ & & & 0 & \sqrt{r_{n-1}} \\ & & & & 0 \end{pmatrix},$$

$$\tilde{e}_1 = \sum_{\alpha_{i,i+1}\in\Pi} \sqrt{r_i} x_{-\alpha_{i,i+1}} = \begin{pmatrix} 0 & & & & \\ \sqrt{r_1} & 0 & & & \\ & \sqrt{r_2} & 0 & & \\ & & \ddots & \ddots & \\ & & & \sqrt{r_{n-1}} & 0 \end{pmatrix},$$

where $r_i = \frac{i(n-i)}{2}$ for $1 \le i \le n-1$.

One can check that $\mathfrak{s} = span\{\tilde{e}_1, x, e_1\}$ is a principal 3-dimensional subalgebra which is real with respect to the compact real form ρ. As a generalization of e_1, the vector $e_i = \sum_{k=1}^{n-i}(\prod_{j=k}^{k+i-1}\sqrt{r_j})\cdot E_{k,k+i}$ is an eigenvector of $ad(x)$ associated with eigenvalue i and satisfies $[e_1, e_i] = 0$. The vector space V_i generated by applying $ad_{\mathfrak{s}}$ to e_i is of dimension $2i+1$, and the vector $e_i \in V_i$ is in the highest weightspace for the Lie algebra representation of $\mathfrak{s}$ on V_i.

With respect to $\mathfrak{s}$, the Higgs bundle in the Hitchin section corresponding to $(q_2, \cdots, q_n)$ is of the form:

$$\begin{aligned} E &= K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{\frac{1-n}{2}}, \\ \phi &= \tilde{e}_1 + q_2 e_1 + q_3 e_2 + \cdots + q_n e_{n-1} \\ &= \begin{pmatrix} 0 & \sqrt{r_1}q_2 & \sqrt{r_1 r_2}q_3 & \cdots & \prod_{i=1}^{n-2}\sqrt{r_i}q_{n-1} & \prod_{i=1}^{n-1}\sqrt{r_i}q_n \\ \sqrt{r_1} & 0 & \sqrt{r_2}q_2 & \cdots & \cdots & \prod_{i=2}^{n-1}\sqrt{r_i}q_{n-1} \\ & \sqrt{r_2} & 0 & \sqrt{r_3}q_2 & \cdots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 0 & \sqrt{r_{n-2}}q_2 & \sqrt{r_{n-2}r_{n-1}}q_3 \\ & & & \sqrt{r_{n-2}} & 0 & \sqrt{r_{n-1}}q_2 \\ & & & & \sqrt{r_{n-1}} & 0 \end{pmatrix}. \end{aligned} \tag{7}$$

Remark 3.2. When $n = 2$, the Higgs bundles in Form (7) form the Higgs bundle parametrization of Fuchsian representations. As noted in the beginning of the Introduction, the base n-Fuchsian representation of Σ is the composition of the uniformization representation j_Σ with the unique irreducible representation of

$PSL(2,\mathbb{R})$ into $PSL(n,\mathbb{R})$ and *an n-Fuchsian representation* is a base n-Fuchsian representation of some Riemann surface Σ. In the case $(q_2,\cdots,q_n)=(0,\cdots,0)$, the corresponding representation is the base n-Fuchsian representation of Σ. In the case $(q_2,\cdots,q_n)=(q_2,0,\cdots,0)$, the corresponding representation is an n-Fuchsian representation.

For our convenience later, we will work with the Higgs bundle $g\cdot(E,\phi)$ under the following gauge transformation

$$g=\begin{pmatrix}1 & & & & \\ & \sqrt{r_1} & & & \\ & & \sqrt{r_1r_2} & & \\ & & & \ddots & \\ & & & & \prod_{i=1}^{n-1}\sqrt{r_i}\end{pmatrix}.$$

In the Higgs bundle $g\cdot(E,\phi)$, the holomorphic structure on E does not change and the Higgs field ϕ becomes $g^{-1}\phi g$.

So a Higgs bundle in the Hitchin section is of the following form:

$$E=K^{\frac{n-1}{2}}\oplus K^{\frac{n-3}{2}}\oplus\cdots\oplus K^{\frac{1-n}{2}},$$
$$\phi=\begin{pmatrix}0 & r_1q_2 & r_1r_2q_3 & r_1r_2r_3q_4 & \cdots & \prod_{i=1}^{n-2}r_iq_{n-1} & \prod_{i=1}^{n-1}r_iq_n\\ 1 & 0 & r_2q_2 & r_2r_3q_3 & \cdots & \cdots & \prod_{i=2}^{n-1}r_iq_{n-1}\\ & 1 & 0 & r_3q_2 & r_3r_4q_3 & \cdots & \vdots\\ & & \ddots & \ddots & \ddots & \ddots & \vdots\\ & & & 1 & 0 & r_{n-2}q_2 & r_{n-2}r_{n-1}q_3\\ & & & & 1 & 0 & r_{n-1}q_2\\ & & & & & 1 & 0\end{pmatrix}, \quad (8)$$

where $r_i=\frac{i(n-i)}{2}$ for $1\le i\le n-1$.

3.2 A smooth orthogonal decomposition of the bundle. Since (E,ϕ) is stable, there is a unique harmonic metric H. The above Higgs bundle expression (8) has a disadvantage that the holomorphic decomposition of E is not an orthogonal decomposition with respect to H. To achieve an orthogonal splitting of the bundle, we sacrifice the holomorphic splitting of the bundle together with the explicit expression of the Higgs field in terms of holomorphic differentials. However, we retain enough control on the holomorphic structure and the Higgs field by carefully choosing a nice orthogonal splitting of the bundle E.

The new decomposition goes as follows. We start with (E,ϕ) in the expression (8). The bundle E is a direct sum of holomorphic line bundles $K^{\frac{n-1}{2}}\oplus K^{\frac{n-3}{2}}\oplus\cdots\oplus K^{\frac{1-n}{2}}$. Define $F_k=\bigoplus_{i=1}^{k}K^{\frac{n+1-2i}{2}}$ to be the direct sum of the first k line bundles. Therefore E admits a holomorphic filtration

$$0=F_0\subset F_1\subset\cdots F_{n-1}\subset F_n=E.$$

We can equip each line bundle F_k/F_{k-1} with the quotient holomorphic structure. Then

(1) the natural inclusion $K^{\frac{n+1-2k}{2}} \subset F_k$ induces an isomorphism of holomorphic line bundles between $K^{\frac{n+1-2k}{2}}$ with F_k/F_{k-1}.;

(2) the Higgs field ϕ takes F_k to $F_{k+1}\otimes K$ and the induced map from $F_k/F_{k-1} \to F_{k+1}/F_k \otimes K$ is the constant map $1 : K^{\frac{n+1-2k}{2}} \to K^{\frac{n+1-2(k+1)}{2}} \otimes K$ under the isomorphism between F_k/F_{k-1} with $K^{\frac{n+1-2k}{2}}$.

Remark 3.3. In summary, a point in the Hitchin section gives the data consisting of an $SL(n,\mathbb{C})$-Higgs bundle (E,ϕ) together with a filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_n = E$ of holomorphic subbundles of E such that for each $1 \leq k \leq n-1$, ϕ takes F_k to $F_{k+1}\otimes K$ and the induced linear map $\bar{\phi} : F_k/F_{k-1} \to F_{k+1}/F_k \otimes K$ is an isomorphism of line bundles. Conversely, following from Example 3.8 in [CW], such data is enough to determine a point in the Hitchin section.

This equivalent definition of a point in the Hitchin section using a holomorphic filtration is similar to an $SL(n,\mathbb{C})$-oper. If we replace the Higgs field with a holomorphic flat connection in the above definition of a point in the Hitchin section, we can obtain an oper. Precisely, an $SL(n,\mathbb{C})$-oper on Σ is a holomorphic vector bundle E over Σ such that $\det E = \mathcal{O}$, a holomorphic flat connection ∇ inducing the trivial connection d on $\det E$, and a filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_n = E$ of holomorphic subbundles of E such that for each $1 \leq k \leq n-1$, ∇ takes F_k to $F_{k+1}\otimes K$ and the induced linear map $\bar{\nabla} : F_k/F_{k-1} \to F_{k+1}/F_k \otimes K$ is an isomorphism of line bundles. One may refer to [BD] and Chapter 4 in [Dal08] for details on opers.

Define L_k as the H-orthogonal line bundle inside F_k with respect to F_{k-1} for $1 \leq k \leq n$. So the inclusion map $L_k \subset F_k$ induces an isomorphism between L_k with the quotient line bundle F_k/F_{k-1}. Then L_k is equipped with the pullback quotient holomorphic structure of F_k/F_{k-1}. In this way, E admits a C^∞-decomposition as

$$L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

where L_k is a holomorphic line bundle which is isomorphic to $K^{\frac{n+1-2k}{2}}$. Note that L_k is not necessarily a holomorphic subbundle of E except for when $k = 1$.

Let us record some important information about the decomposition from the construction:

(i) the decomposition is orthogonal with respect to H;
(ii) For each k, $\bar{\partial}_E$ preserves $F_k = \bigoplus_{i=1}^k L_i$, since $F_k = \bigoplus_{i=1}^k L_i$ is a holomorphic subbundle of E;
(iii) For each k, ϕ takes $F_k = \bigoplus_{i=1}^k L_i$ to $F_{k+1}\otimes K = (\bigoplus_{i=1}^{k+1} L_i)\otimes K$ and the induced map from

$$L_k \xrightarrow{\phi} F_{k+1}\otimes K \xrightarrow{\mathrm{pr}_{k+1}} L_{k+1}\otimes K$$

is the constant map $1 : K^{\frac{n+1-2k}{2}} \to K^{\frac{n+1-2(k+1)}{2}} \otimes K$ under the isomorphism between L_k with $K^{\frac{n+1-2k}{2}}$.

More explicitly, with respect to the smooth decomposition

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_n, \quad L_k = K^{\frac{n+1-2k}{2}},$$

we have:

I. the Hermitian metric H solving the Hitchin equation is given by

$$H = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix} \tag{9}$$

where h_i is the Hermitian metric on L_i and under the isomorphism $det(E) \cong \mathcal{O}$, $det(H)$ is a constant metric 1 on the line bundle $det(E)$;

II. the holomorphic structure on E is given by the $\bar{\partial}$-operator

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1n} \\ & \bar{\partial}_2 & \beta_{23} & \cdots & \beta_{2n} \\ & & \bar{\partial}_3 & \cdots & \beta_{3n} \\ & & & \ddots & \vdots \\ & & & & \bar{\partial}_n \end{pmatrix} \tag{10}$$

where $\bar{\partial}_k$ are $\bar{\partial}$-operators defining the holomorphic structures on L_k, and $\beta_{ij} \in \Omega^{0,1}(Hom(L_j, L_i))$;

III . the Higgs field is of the form

$$\phi = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 1_1 & a_{22} & a_{23} & \cdots & a_{2n} \\ & 1_2 & a_{33} & \cdots & a_{3n} \\ & & \ddots & \ddots & \vdots \\ & & & 1_{n-1} & a_{nn} \end{pmatrix} \tag{11}$$

where $a_{ij} \in \Omega^{1,0}(Hom(L_j, L_i))$ and 1_l is the constant map $1_l : K^{\frac{n+1-2l}{2}} \to K^{\frac{n+1-2(l+1)}{2}} \otimes K$ (the subscript l will be useful later).

Remark 3.4. If the holomorphic decomposition in the expression (8) is already orthogonal with respect to the harmonic metric H, then the new decomposition coincides with the old one. By the work of Baraglia [Bar10] and Collier [Col16], there are several families of Higgs bundles in the Hitchin section such that H splits on the holomorphic splitting:

(a) cyclic case: $(0, \cdots, 0, q_n)$;

(b) sub-cyclic case: $(0, \cdots, 0, q_{n-1}, 0)$; and
(c) Fuchsian case: $(q_2, 0, \cdots, 0)$.

In particular, if (E, ϕ) corresponds to the base n-Fuchsian representation of Σ, all the β_{ij}, a_{ij} vanish.

3.3 Hitchin equation in terms of orthogonal decomposition. We are going to describe the Chern connection ∇^H and the Hermitian adjoint ϕ^{*_H} of the Higgs field. The Chern connection is given by

$$\nabla^H = \begin{pmatrix} \nabla^{h_1} & \beta_{12} & \beta_{13} & \cdots & \beta_{1n} \\ -\beta_{12}^* & \nabla^{h_2} & \beta_{23} & \cdots & \beta_{2n} \\ -\beta_{13}^* & -\beta_{23}^* & \nabla^{h_3} & \cdots & \beta_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\beta_{1n}^* & -\beta_{2n}^* & -\beta_{2n}^* & \cdots & \nabla^{h_n} \end{pmatrix}, \tag{12}$$

where ∇^{h_l} is Chern connection on each L_l, β_{ij}^* is the Hermitian adjoint of β_{ij}, that is, $h_i(\beta_{ij}(e_j), e_i) = h_j(e_j, \beta_{ij}^*(e_i))$ for local sections e_i, e_j of L_i, L_j respectively. In a local frame, $\beta_{ij}^* = \bar{\beta}_{ij} \cdot h_i h_j^{-1}$.

Since the metric H is diagonal, the Hermitian adjoint of the Higgs field with respect to H is

$$\phi^{*_H} = \begin{pmatrix} a_{11}^* & 1_1^* & & & \\ a_{12}^* & a_{22}^* & 1_2^* & & \\ a_{13}^* & a_{23}^* & a_{33}^* & \ddots & \\ \vdots & \vdots & \vdots & \ddots & 1_{n-1}^* \\ a_{1n}^* & a_{2n}^* & a_{3n}^* & \cdots & a_{nn}^* \end{pmatrix}, \tag{13}$$

where a_{ij}^* is the Hermitian adjoint of a_{ij}, that is, $h_i(a_{ij}(e_j), e_i) = h_j(e_j, a_{ij}^*(e_i))$ for local sections e_i, e_j of L_i, L_j respectively. Similarly, 1_l^* is the Hermitian adjoint of $1_l \in H^0(Hom(L_{l+1}, L_l) \otimes K)$. In a local frame, $a_{ij}^* = \bar{a}_{ij} \cdot h_i h_j^{-1}$, and $1_l^* = h_l^{-1} h_{l+1}$.

For $1 < l \leq n$, the (l, l)-entry of the Hitchin equation

$$\nabla^H \circ \nabla^H + [\phi, \phi^{*_H}] = 0$$

is

$$\begin{aligned} F^{\nabla^{h_l}} &- \sum_{j=l+1}^{n} \beta_{lj} \wedge \beta_{lj}^* - \sum_{j=1}^{l-1} \beta_{jl}^* \wedge \beta_{jl} \\ &+ \sum_{j=l+1}^{n} a_{lj} \wedge a_{lj}^* + \sum_{j=1}^{l-1} a_{lj}^* \wedge a_{lj} + 1_{l-1} \wedge 1_{l-1}^* + 1_l^* \wedge 1_l = 0. \end{aligned} \tag{14}$$

In the case $l = 1$, $1_{l-1} \wedge 1_{l-1}^*$ does not appear.

In a local coordinate z, choose a local holomorphic frame $s_l = dz^{\frac{n+1-2l}{2}}$ of L_l. Denote the local function $h_l(s_l, s_l)$ as $\hat{h}_l$. Then locally, $F^{\nabla^{h_l}} = (\partial_{\bar{z}}\partial_z \log \hat{h}_l) d\bar{z} \wedge dz$. Getting rid of the 2-form $d\bar{z} \wedge dz$, Equation (14) is locally

$$\begin{aligned} &\partial_{\bar{z}}\partial_z \log \hat{h}_l - \sum_{j=l+1}^{n} (|\beta_{lj}|^2 + |a_{lj}|^2)\hat{h}_l \hat{h}_j^{-1} \\ &\quad + \sum_{j=1}^{l-1} (|\beta_{jl}|^2 + |a_{jl}|^2)\hat{h}_j \hat{h}_l^{-1} - \hat{h}_{l-1}^{-1}\hat{h}_l + \hat{h}_l^{-1}\hat{h}_{l+1} = 0. \end{aligned} \tag{15}$$

In the case $l = 1$, $\hat{h}_{l-1}^{-1}\hat{h}_l$ does not appear.

Recall from Section 2.3 that g_0 is the conformal hyperbolic metric on Σ of constant curvature -1, h is the induced Hermitian metric on K^{-1} and in local coordinate z, $g_0 = 2\hat{g}_0(dx^2 + dy^2)$ and $h = \hat{g}_0 dz \otimes d\bar{z}$. Since h_l and $h^{-\frac{n+1-2l}{2}}$ are both two Hermitian metric on L_l, they differ by a positive function on the surface, i.e. $h_l = h^{-\frac{n+1-2l}{2}} \cdot e^{u_l}$, for some smooth function u_l on Σ. Using Equation (15) and Equation (5), u_l locally satisfies

$$\begin{aligned} &\frac{1}{\hat{g}_0}\partial_{\bar{z}}\partial_z u_l - \sum_{j=l+1}^{n} (|\beta_{lj}|^2 + |a_{lj}|^2) e^{u_l - u_j} \hat{g}_0^{l-j-1} + \sum_{j=1}^{l-1} (|\beta_{jl}|^2 + |a_{jl}|^2) e^{u_j - u_l} \hat{g}_0^{j-l-1} \\ &\quad - \hat{h}_{l-1}^{-1}\hat{h}_l + \hat{h}_l^{-1}\hat{h}_{l+1} - \frac{n+1-2l}{2} = 0. \end{aligned} \tag{16}$$

As noted in Remark 3.4, when (E, ϕ) corresponds to the base n-Fuchsian representation of Σ, all the β_{jl}, a_{lj} vanish. Denote by $\tilde{h}_l$ the Hermitian metric on L_l. Using the fact $\sum_{l=1}^{k} \frac{n+1-2l}{2} = \frac{(n-k)k}{2}$, one can check that $\tilde{h}_l = h^{-\frac{n+1-2l}{2}} \cdot e^{\tilde{u}_l}$, where $\tilde{u}_l$'s are constants satisfying

$$e^{-\tilde{u}_l + \tilde{u}_{l+1}} = \frac{(n-l)l}{2}. \tag{17}$$

This is equivalent to saying that

$$\tilde{h}_l^{-1}\tilde{h}_{l+1} = \frac{l(n-l)}{2} h. \tag{18}$$

Using Equation (3), the pullback metric of corresponding harmonic map $\tilde{f}$ is given by

$$\begin{aligned} \tilde{f}^* g_{G/K} &= 2Re\left(\frac{12}{n(n^2-1)} \cdot tr(\phi\phi^{*_H})\right) = 2Re\left(\frac{12}{n(n^2-1)} \cdot \sum_{l=1}^{n-1} \tilde{h}_l^{-1}\tilde{h}_{l+1}\right) \\ &= 2Re\left(\frac{12}{n(n^2-1)} \cdot \sum_{l=1}^{n-1} \frac{(n-l)l}{2} \cdot h\right) = 2Re(h) = g_0, \end{aligned} \tag{19}$$

where the fourth equality uses the fact $\sum_{l=1}^{n-1}\frac{(n-l)l}{2}=\frac{n(n^2-1)}{12}$. Using Equation (6), the energy density $e(\tilde{f})\equiv 1$.

We will use the following notations:

$$\triangle_{g_0}=\frac{1}{\hat{g}_0}\partial_{\bar{z}}\partial_z,\quad ||\beta_{lj}||^2=|\beta_{lj}|^2e^{u_l-u_j}\hat{g}_0^{l-j-1},\quad ||a_{lj}||^2=|a_{lj}|^2e^{u_l-u_j}\hat{g}_0^{l-j-1},$$
$$\text{and}\quad z_l=u_l-\tilde{u}_l,$$

each of these objects is globally well-defined on the surface. So the following equation of u_l holds globally,

$$\triangle_{g_0}u_l-\sum_{j=l+1}^{n}(||\beta_{lj}||^2+||a_{lj}||^2)$$
$$+\sum_{j=1}^{l-1}(||\beta_{jl}||^2+||a_{jl}||^2)-e^{u_l-u_{l-1}}+e^{u_{l+1}-u_l}-\frac{n+1-2l}{2}=0. \tag{20}$$

In the case $l=1$, $e^{u_l-u_{l-1}}$ does not appear.

Note that the function z_l measures how far h_l is from $\tilde{h}_l$, the base n-Fuchsian case. Using Equation (20) and the fact that $\tilde{u}_l$'s are constants satisfying Equation (17), the function z_l satisfies the following equation:

$$\triangle_{g_0}z_l-\sum_{j=l+1}^{n}(||\beta_{lj}||^2+||a_{lj}||^2)+\sum_{j=1}^{l-1}(||\beta_{jl}||^2+||a_{jl}||^2)-e^{z_l-z_{l-1}}\cdot\frac{(l-1)(n+1-l)}{2}$$
$$+e^{z_{l+1}-z_l}\cdot\frac{l(n-l)}{2}-\frac{n+1-2l}{2}=0. \tag{21}$$

In the case $l=1$, $e^{z_l-z_{l-1}}$ does not appear.

4 Proof of Main Theorem

We proceed to prove the main theorem in this section.

The function z_l in Equation (21) is difficult to estimate directly since the unknown terms $||\beta_{ij}||^2, ||a_{ij}||^2$ do not appear with the same sign. The main observation is that the sum of the first k equations turns out to have the same sign surprisingly. So we prove in the following Lemma 4.1 that for $1\le k\le n-1$, the harmonic metric on the determinant line bundle of the holomorphic bundle $F_k=\bigoplus_{i=1}^k L_i$ is always smaller than the one in the base n-Fuchsian case.

LEMMA 4.1. *Let $v_k=\sum_{l=1}^k z_l$, if ρ is not the base n-Fuchsian representation of Σ, then*

$$v_k<0,\quad \textit{for } 1\le k\le n-1, \textit{ and}\quad v_n\equiv 0.$$

Proof. The statement $v_n \equiv 0$ follows directly from both H and $\tilde{H}$ are of unit determinant.

Summing up Equation (21) over $1 \leq l \leq k$, we obtain

$$\triangle_{g_0}\left(\sum_{l=1}^{k} z_l\right) - \sum_{l=1}^{k}\sum_{j=l+1}^{n}(||\beta_{lj}||^2 + ||a_{lj}||^2) + \sum_{l+1}^{k}\sum_{j=1}^{l-1}(||\beta_{jl}||^2 + ||a_{jl}||^2)$$
$$+e^{-z_k+z_{k+1}} \cdot \frac{k(n-k)}{2} - \sum_{l=1}^{k}\frac{n+1-2l}{2} = 0. \tag{22}$$

Step 1: We first simplify the above Equation (22).
The following formula is key to us.

$$-\sum_{l=1}^{k}\sum_{j=l+1}^{n}(||\beta_{lj}||^2 + ||a_{lj}||^2) + \sum_{l=1}^{k}\sum_{j=1}^{l-1}(||\beta_{jl}||^2 + ||a_{jl}||^2)$$
$$= -\sum_{l=1}^{k}\sum_{j=k+1}^{n}(||\beta_{lj}||^2 + ||a_{jl}||^2). \tag{23}$$

It holds because

$$\begin{aligned} LHS &= -\sum_{l=1}^{k}\sum_{j=l+1}^{n}(||\beta_{lj}||^2 + ||a_{lj}||^2) + \sum_{j=1}^{k-1}\sum_{l=j+1}^{k}(||\beta_{jl}||^2 + ||a_{jl}||^2) \\ &= -\sum_{l=1}^{k}\sum_{j=l+1}^{n}(||\beta_{lj}||^2 + ||a_{lj}||^2) + \sum_{l=1}^{k-1}\sum_{j=l+1}^{k}(||\beta_{lj}||^2 + ||a_{lj}||^2) \\ &= -\sum_{l=1}^{k}\sum_{j=k+1}^{n}(||\beta_{lj}||^2 + ||a_{lj}||^2) = RHS. \end{aligned}$$

Using Formula (23) and the fact $\sum_{l=1}^{k}\frac{n+1-2l}{2} = \frac{(n-k)k}{2}$, Equation (22) becomes

$$\triangle_{g_0}\left(\sum_{l=1}^{k} z_l\right) - \sum_{l=1}^{k}\sum_{j=k+1}^{n}(||\beta_{lj}||^2 + ||a_{lj}||^2) + (e^{-z_k+z_{k+1}} - 1) \cdot \frac{(n-k)k}{2} = 0. \tag{24}$$

Set $v_k = \sum_{l=1}^{k} z_k$ for $1 \leq k \leq n$ and $v_0 = 0$. Noting that $e^{-z_k+z_{k+1}} = e^{v_{k-1}+v_{k+1}-2v_k}$ for $1 \leq k \leq n-1$, we rewrite the above equation (24) as

$$\triangle_{g_0} v_k - \sum_{l=1}^{k}\sum_{j=k+1}^{n}(||\beta_{lj}||^2 + ||a_{lj}||^2) + (e^{v_{k-1}+v_{k+1}-2v_k} - 1) \cdot \frac{(n-k)k}{2} = 0. \tag{25}$$

Therefore we obtain

$$\triangle_{g_0} v_k + (e^{v_{k-1}+v_{k+1}-2v_k} - 1) \cdot \frac{(n-k)k}{2} \geq 0. \tag{26}$$

Step 2: We show $v_k \leq 0$ for all $1 \leq k \leq n$.

Set $M := \max_{1 \leq k \leq n} \max_{p \in \Sigma} v_k(p)$. Since $v_n \equiv 0$, $M \geq 0$. It suffices to show $M = 0$. Suppose not, i.e. $M > 0$. Assume k_0 is the largest integer such that v_{k_0} achieves M at some point. Since $v_n \equiv 0$, we have $k_0 \leq n-1$. Then at some maximum point p of v_{k_0}, $\triangle_{g_0} v_{k_0}(p) \leq 0$, using Equation (26), we obtain

$$e^{(v_{k_0-1}+v_{k_0+1}-2v_{k_0})(p)} - 1 \geq 0 \Longrightarrow v_{k_0-1}(p) + v_{k_0+1}(p) \geq 2v_{k_0}(p) = \max_{\Sigma} 2v_{k_0} = 2M.$$

By the definition of M being maximum, $v_{k_0-1} + v_{k_0+1} \leq 2M$, hence $v_{k_0-1}(p) = v_{k_0+1}(p) = M$. This contradicts the choice of k_0 being the largest integer such that v_{k_0} achieves M at some point.

Step 3: We show that $v_k < 0$ for all $1 \leq k \leq n-1$.

For each k, applying $v_{k-1}, v_{k+1} \leq 0$ to Equation (26), we obtain

$$\triangle_{g_0} v_k + (e^{-2v_k} - 1) \cdot \frac{(n-k)k}{2} \geq 0.$$

Observe that the constant function 0 satisfies

$$\triangle_{g_0}(0) + (e^{-2\times 0} - 1)\frac{(n-k)k}{2} = 0.$$

By the strong maximum principle (see [Jos07], page 43), $v_k < 0$ or $v_k \equiv 0$. In fact, the latter case cannot happen for any $1 \leq k \leq n-1$. Suppose $v_{k_0} \equiv 0$, Equation (26) for v_{k_0} implies $v_{k_0-1} = v_{k_0+1} \equiv 0$. Repeating this process, we obtain that for all k, $v_k \equiv 0$. Plugging in $v_k \equiv 0$ to Equation (25) for all k, we have that every β_{lj}, a_{lj} must vanish. Then the Higgs bundle corresponds to the base n-Fuchsian representation of Σ, which cannot happen by assumption. $\square$

Now we are ready to show the main theorem.

Theorem 4.2 *Let ρ be a Hitchin representation for $PSL(n, \mathbb{R})$, g_0 be a hyperbolic metric on S, and f be the unique ρ-equivariant harmonic map from $(\widetilde{S}, \widetilde{g}_0)$ to $SL(n, \mathbb{R})/SO(n)$. Then its energy density $e(f)$ satisfies*

$$e(f) \geq 1.$$

Moreover, equality holds at one point only if $e(f) \equiv 1$ in which case ρ is the base n-Fuchsian representation of (S, g_0).

Proof. Firstly, if ρ is the base n-Fuchsian representation of Σ, we have $e(f) \equiv 1$ from Equation (19). Now suppose ρ is not the base n-Fuchsian representation of Σ, from the expression of ϕ and ϕ^* in Equations (11) and (13) respectively, we have

$$\begin{aligned}\mathrm{tr}(\phi\phi^{*_H}) &= \left(\sum_{k=1}^{n-1} h_k^{-1}h_{k+1}\right) + \sum_{l=1}^{n}\sum_{j=1}^{l} \|a_{jl}\|^2 \geq \sum_{k=1}^{n-1} h_k^{-1}h_{k+1} \\ &= \sum_{k=1}^{n-1} \tilde{h}_k^{-1}\tilde{h}_{k+1} \cdot e^{-z_k+z_{k+1}},\end{aligned}$$

where $\tilde{h}_k, z_k$ are defined in Section 3.3. From Equation (18), $\tilde{h}_k^{-1}\tilde{h}_{k+1} = \frac{k(n-k)}{2}h$. So

$$\mathrm{tr}(\phi\phi^{*_H})/h \geq \sum_{k=1}^{n-1}\left[\frac{k(n-k)}{2} \cdot e^{-z_k+z_{k+1}}\right] \tag{27}$$

$$\geq \left[\sum_{k=1}^{n-1}\frac{k(n-k)}{2}\right] \cdot \left\{e^{\sum_{k=1}^{n-1}\frac{k(n-k)}{2}(-z_k+z_{k+1})}\right\}^{\frac{1}{\sum_{k=1}^{n-1}\frac{k(n-k)}{2}}} \tag{28}$$

$$= \frac{n(n^2-1)}{12} \cdot \left(e^{\sum_{k=1}^{n-1}\frac{k(n-k)}{2}(-z_k+z_{k+1})}\right)^{\frac{12}{n(n^2-1)}} \tag{29}$$

$$= \frac{n(n^2-1)}{12} \cdot \left(e^{-\sum_{k=1}^{n}\frac{n+1-2k}{2}z_k}\right)^{\frac{12}{n(n^2-1)}} \tag{30}$$

$$= \frac{n(n^2-1)}{12} \cdot \left(e^{-[(n-1)z_1+(n-2)z_2+\cdots+2z_{n-2}+z_{n-1}]}\right)^{\frac{12}{n(n^2-1)}} \tag{31}$$

$$= \frac{n(n^2-1)}{12} \cdot \left(e^{-\sum_{k=1}^{n-1}z_k} \cdot e^{-\sum_{k=1}^{n-2}z_k} \cdot \cdots \cdot e^{-\sum_{k=1}^{2}z_k} \cdot e^{-z_1}\right)^{\frac{12}{n(n^2-1)}} \tag{32}$$

$$> \frac{n(n^2-1)}{12}. \tag{33}$$

Here Equality (29) uses the fact $\sum_{k=1}^{n-1}\frac{k(n-k)}{2} = \frac{n(n^2-1)}{12}$; Equality (31) uses $z_n = -z_1 - z_2 - \cdots - z_{n-1}$; Inequality (33) uses the fact $v_l = \sum_{k=1}^{l} z_k < 0$ in Lemma 4.1; and the other equations follow from direct calculations.

Using Formula (6) of the energy density and Equation (33), we have

$$e(f) = 2Re\left(\frac{12}{n(n^2-1)} \cdot \mathrm{tr}(\phi\phi^{*_H})\right)/g_0 > 2Re(h)/g_0 = 1. \qquad \square$$

5 Further Questions

There is a natural $\mathbb{C}^*$-action on the moduli space M_{Higgs} of $SL(n,\mathbb{C})$-Higgs bundles given by

$$\mathbb{C}^* \times \mathcal{M}_{Higgs}(SL(n,\mathbb{C})) \to \mathcal{M}_{Higgs}(SL(n,\mathbb{C}))$$
$$t \cdot [(E,\phi)] = [(E,t\phi)].$$

Hitchin [Hit92] showed that along the $\mathbb{C}^*$-flow, the energy is monotonically increasing as $|t|$ increases. From integral monotonicity to pointwise monotonicity, we make the following conjecture.

Conjecture 5.1. *Along the $\mathbb{C}^*$-flow on the moduli space $\mathcal{M}_{Higgs}(SL(n,\mathbb{C}))$, the energy density of the corresponding harmonic maps is monotonically increasing as $|t|$ increases.*

For any Higgs bundle (E,ϕ) in the Hitchin section, as $t \to 0$, the limit $\lim_{t\to 0} t \cdot (E,\phi)$ corresponds to the base n-Fuchsian representation of Σ. Therefore, Conjecture 5.1 is a natural generalization of Theorem 4.2. For cyclic Higgs bundles, Conjecture 5.1 is shown by the author and Dai in [DL18].

We recall the definition of the Hitchin fibration in Section 3.1 and call each fiber of the Hitchin fibration a Hitchin fiber. Then the $\mathbb{C}^*$ action always takes elements in one Hitchin fiber to elements in another distinct Hitchin fiber unless the element is in the Hitchin fiber based at the origin. If instead, we stay at in a single Hitchin fiber, then we expect the maximum of the energy density to occur exactly at the image of the Hitchin section.

Conjecture 5.2 ([DL18]). *Let $(\tilde{E},\tilde{\phi})$ be a Higgs bundle in the Hitchin section and (E,ϕ) be a distinct polystable $SL(n,\mathbb{C})$-Higgs bundle in the same Hitchin fiber. Then the corresponding harmonic maps $f,\tilde{f}$ satisfy $e(f) < e(\tilde{f})$ and hence $f^*g_{G/K} < \tilde{f}^*g_{G/K}$.*

As a result, the energy satisfies $E(f) < E(\tilde{f})$.

In the $SL(2,\mathbb{C})$ case, Conjecture 5.2 is shown by Deroin and Tholozan in [DT]. We remark that in the work of Deroin and Tholozan, they use purely the language of harmonic maps instead of Higgs bundles. Given two distinct completely reducible representations $\rho_1,\rho_2 : \pi_1(S) \to SL(2,\mathbb{C})$ where ρ_1 is Fuchsian and suppose there exists a Riemann surface structure Σ on S such that the ρ_i-equivariant harmonic maps $f_i : \widetilde{\Sigma} \to \mathbb{H}^2$ $(i = 1,2)$ have the same Hopf differential. Then Deroin and Tholozan in [DT] show the energy density of f_1 is strictly larger than the one of f_2. Their result can be easily translated into verifying Conjecture 5.2 for $SL(2,\mathbb{C})$ case because following from Remark 3.1, the two associated Higgs bundles over Σ for ρ_1 and ρ_2 are in the same Hitchin fiber.

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