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ESTIMATION OF THE HEAT CONDUCTED BY A CLUSTER OF SMALL CAVITIES AND CHARACTERIZATION OF THE EQUIVALENT HEAT CONDUCTION*

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Abstract. We estimate the heat conducted by a cluster of many small cavities. We show that the dominating heat is a sum, over the number of cavities, of the heat generated by each cavity after interacting with each other. This interaction is described through densities computable as solutions of a closed, and invertible, system of time domain integral equations of second kind. As an application of these expansions, we derive the effective heat conductivity which generates approximately the same heat as the cluster of cavities, distributed in a three-dimensional bounded domain, with explicit error estimates in terms of that cluster. At the analysis level, we use time domain integral equations. Doing that, we have two choices. First, we can favor the space variable by reducing the heat potentials to the ones related to the Laplace operator (avoiding Laplace transform). Second, we can favor the time variable by reducing the representation to the Abel integral operator. As the model under investigation has time-independent parameters, we follow the first approach here.

Key words. asymptotic analysis, heat equation, effective medium, layer potentials

AMS subject classifications. 35C20, 35K05

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1. Introduction.

Let D be a bounded domain in $\mathbb{R}^3$ and consider the following initial boundary value problem:

$$
\begin{cases}
(\partial_t - \Delta)u = 0 & \text{in } (\mathbb{R}^3 \setminus \overline{D})_T, \\
u = f & \text{on } (\partial D)_T, \\
u = 0 & \text{at } t = 0.
\end{cases}
\tag{1.1}
$$

For simplicity of notations, here and throughout this paper, we denote $X \times (0, T)$ and $\partial X \times (0, T)$ by X_T and $(\partial X)_T$, respectively, where X is a domain in $\mathbb{R}^3$ and ∂X denotes its boundary. We also set $T_\varepsilon := T/\varepsilon^2$, where ε is a positive parameter. The model (1.1) has many important applications in science and engineering [39, 40, 41]. For example, in active thermography, D is regarded as a cavity embedded in the background medium.

We assume in (1.1) that the compatibility condition $f(x, 0) = u(x, 0)$ holds on ∂D. To ensure the uniqueness of solutions to (1.1), we require that the solution $u(x, t)$ satisfies the growth condition

$$
|u(x, t)| \le C_0 \exp(b|x|^2) \qquad \text{as } |x| \to +\infty
\tag{1.2}
$$

for some positive constants C_0 and $b < (4T)^{-1}$. Under certain regularity assumptions

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on D and f, the unique solvability of the problem (1.1) can be proved; see, for instance, [44].

In this work, we consider the case where D is given by a union of small cavities, i.e., $D := \cup_{j=1}^{M} D_j$, with the maximum radius of D_j of small order ε. Our goal is to estimate the heat generated by such a cluster in terms of the size and the number of cavities. To give sense to the boundary conditions on the surfaces of the small cavities, we take a source defined, at least, in a domain containing the whole cluster and then the boundary conditions on each cavity's surface is the trace of this source on it.

We could consider more general sources in the model (1.1), namely,

$$\begin{cases} (\partial_t - \Delta) w = F & \text{in } (\mathbb{R}^3 \setminus \overline{D})_T, \\ w = G & \text{on } (\partial D)_T, \\ w = H & \text{at } t = 0 \end{cases} \tag{1.3}$$

with the needed compatibility conditions for the sources G and H, i.e., $G(\cdot, 0) = H(\cdot)$ on ∂D. However, setting v to be the solution in the absence of the cavities, by extending first F and G to the whole space $\mathbb{R}^3$, we see that $u := w - v$ solves (1.1) with the boundary source $f := G - v|_{(\partial D)_T}$. A particular source of heat that is used in practice is given by

$$f(x, t) := f(x, t; z^*) = \Phi(x, t; z^*, 0), \tag{1.4}$$

where

$$\Phi(x, t; y, \tau) := \begin{cases} \dfrac{1}{[4\pi(t-\tau)]^{3/2}} \exp\left(-\dfrac{|x-y|^2}{4(t-\tau)} \right), & t > \tau, \\ 0, & t \le \tau, \end{cases} \tag{1.5}$$

is the fundamental solution of the heat operator $\partial_t - \Delta$ for the three-dimensional space and z^* is the source point which is located away from the cluster $\cup_{j=1}^{M} D_j$. This source is initially a Dirac source supported on the source point z^*, i.e., $f(x, 0) = \delta(z^*)$. Indeed, with such sources, we see from (1.1), that $u_f := f - u$ satisfies the problem

$$\begin{cases} (\partial_t - \Delta) u_f = 0 & \text{in } (\mathbb{R}^3 \setminus \overline{D})_T, \\ u_f = 0 & \text{on } (\partial D)_T, \\ u_f = \delta(z^*) & \text{at } t = 0 \end{cases} \tag{1.6}$$

with the growth condition as in (1.2).

Let $B_1, B_2, \ldots, B_M$ be M open, bounded, and simply connected domains in $\mathbb{R}^3$ with C^2-boundaries containing the origin. Assume that the Lipschitz constants of B_j, $j = 1, 2, \ldots, M$ are uniformly bounded. Set $D_j := \varepsilon B_j + z_j$ to be small cavities characterized by the parameter $\varepsilon > 0$ and the locations $z_j \in \mathbb{R}^3$, $j = 1, 2, \ldots, M$.

Definition 1.1. *Let D be a union of the small cavities, i.e., $D := \cup_{j=1}^{j=M} D_j$. We set*

1. *a as the maximum among the diameters of the small cavities, i.e.,*

$$a := \max_{1 \le j \le M} \operatorname{diam}(D_j) = \varepsilon \max_{1 \le j \le M} \operatorname{diam}(B_j); \tag{1.7}$$

2. *d as the minimum distance between the small cavities, i.e.,*

$$d := \min_{\substack{1 \le i,j \le M \\ i \ne j}} d_{ij}, \quad d_{ij} := \operatorname{dist}(D_i, D_j). \tag{1.8}$$

We are interested in regimes where

$$M \sim a^{-s} \quad \text{and} \quad d \sim a^{\beta} \tag{1.9}$$

with nonnegative real numbers s and β.

Our first result is the following approximation property.

THEOREM 1.2. *Let $D_i, i = 1, \ldots, M$, and the source f be as described above. Under the condition*

$$a \max_{1\le i\le M} \sum_{\substack{j=1\\ j\neq i}}^{M} d_{ij}^{-2} < 1, \tag{1.10}$$

which means that $1 - 2\beta - \frac{s}{3} \geq 0$, the heat conducted by this cluster is approximately estimated for x away from $\cup_{j=1}^{j=M} D_j$ and $t > 0$, as

$$u(x, t) = \sum_{i=1}^{M} C_i \int_0^t \Phi(x, t; z_i, \tau)\, \alpha_i(\tau)\, d\tau + O(a^{3-3\beta-s} |\ln a|) + O(a^{2-s}) \tag{1.11}$$

as $a \to 0$, where each constant C_i is the capacitance of the cavity D_i, $i = 1, \ldots, M$, defined as $C_i := \int_{\partial D_i} \sigma_i(x)\, ds(x)$ with σ_i as the unique solution of the integral equation $\int_{\partial D_i} \frac{\sigma_i(y)}{4\pi|x-y|}\, ds(y) = 1,\ x \in \partial D_i$, and $\{\alpha_i(t)\}_{i=1}^{M}$ is the unique solution of the linear system

$$\alpha_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{M} \int_0^t C_j\, \Phi(z_i, t; z_j, \tau)\, \alpha_j(\tau)\, d\tau = \Phi(z_i, t; z^*, 0), \quad t \in (0, T), \tag{1.12}$$

for $i = 1, 2, \ldots, M$, which is invertible from $L^2(0, T)$ to itself.

We can show that $\max_{1\le i\le M} \sum_{j\neq i} d_{ij}^{-2} \leq d^{-2} M^{\frac{1}{3}}$ (see [13] for instance), then (1.10) is satisfied if $a\, d^{-2} M^{\frac{1}{3}} < 1$ and, from (1.9), it implies that $1 - 2\beta - \frac{s}{3} \geq 0$.

As an application of such results, we derive the effective heat conductivity distribution that can produce the same heat as the cluster above. To show this, let Ω be a bounded domain containing the cavities D_j, $j = 1, 2, \ldots, M$. We divide Ω into $[a^{-1}]$ subdomains Ω_j, $j = 1, 2, \ldots, [a^{-1}]$, periodically arranged for instance,[1] such that the Ω_j's are disjoint and of volume a. Here, we denote by $[x]$ the unique integer n such that $n \leq x < n + 1$, i.e., n is the floor number. Each subdomain Ω_j contains one single hole. Such a distribution obeys the condition (1.10). Indeed, from the estimate $\max_{1\le i\le M} \sum_{j\neq i} d_{ij}^{-2} \leq d^{-2} M^{\frac{1}{3}}$ and as $M = [a^{-1}] \leq a^{-1}$, i.e., $s = 1$, we have $a \max_{1\le i\le M} \sum_{j\neq i} d_{ij}^{-2} \leq a^{\frac{2}{3}} d^{-2} = (a^{\frac{1}{3}} d^{-1})^2$. Taking $d = d_0 a^{\frac{1}{3}}$, i.e., $\beta = \frac{1}{3}$, we see $a \max_{1\le i\le M} \sum_{j\neq i} d_{ij}^{-2} \leq \frac{1}{d_0^2} < 1$ if $d_0 > 1$. This last condition is obviously satisfied according to the distribution described above.

Now, we state our second main result.

THEOREM 1.3. *Let Ω be a bounded and Lipschitz domain in $\mathbb{R}^3$. We distribute the cavities as described above. Then for any $t > 0$ and x away from Ω, we have the approximation*

$$u(x, t) - W(x, t) = O(a^{\frac{1}{3}}) \quad \text{as } a \ll 1, \tag{1.13}$$

[1] The periodicity is actually not needed. We assume it only for simplicity of exposition.

where W is the unique solution of the problem

$$\begin{cases} (\partial_t - \Delta + \overline{C}\chi_\Omega)W = \overline{C}\chi_\Omega\Phi(x,\, t;\, z^*,\, 0) & \text{in } \mathbb{R}^3 \times (0,\, T), \\ W(x,\, 0) = 0 & \text{in } \mathbb{R}^3, \\ |W(x,\, t)| \leq C_0 \exp(b|x|^2) & \text{as } |x| \to +\infty \end{cases} \tag{1.14}$$

for some positive constants C_0 and $b < (4T)^{-1}$. Here $\overline{C}$ is the capacitance of the unscaled domains B_m, $1 \leq m \leq M$ (that are assumed to be the same). Finally χ_Ω is the characteristic function of the domain Ω.

As a corollary, we deduce the following result. Let σ be the unique solution of the problem

$$\begin{cases} -\Delta\sigma + \overline{C}\sigma = 0 & \text{in } \Omega, \\ \sigma = 1 & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

As $\overline{C}$ is positive in Ω, by the maximum principle for a general second-order uniformly elliptic operator [21, section 6.4], the unique solution of (1.15) is positive in Ω. We extend σ from Ω to $\mathbb{R}^3$ simply by 1 in $\mathbb{R}^3 \setminus \overline{\Omega}$. By a simple change of variable $\tilde{W} := \sigma^{-1}\, W$, we see that $\tilde{W}$ satisfies the problem

$$\begin{cases} \sigma\, \partial_t\tilde{W} - \sigma^{-1}\nabla \cdot \sigma^2\nabla\tilde{W} = \overline{C}\chi_\Omega\Phi(x,\, t;\, z^*,\, 0) & \text{in } \mathbb{R}^3 \times (0,\, T), \\ \tilde{W}(x,\, 0) = 0 & \text{in } \mathbb{R}^3, \\ |\tilde{W}(x,\, t)| \leq C_0 \exp(b|x|^2) & \text{as } |x| \to +\infty. \end{cases} \tag{1.16}$$

Observe that the first equation in (1.16) can be written as $\rho\, c\, \partial_t\tilde{W} - \nabla \cdot \gamma\nabla\tilde{W} = F$, where the density ρ and the heat capacity c are such that $\rho\, c = \sigma^2$ and the heat conductivity γ is give by $\gamma := \sigma^2$.

As $\tilde{W} = W$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, from Theorem 1.3 we deduce the following result.

COROLLARY 1.4. *Let Ω be as described in Theorem* 1.3. *Then for any $t > 0$ and x away from Ω, we have the approximation*

$$u(x,\, t) - \tilde{W}(x,\, t) = O(a^{\frac{1}{3}}) \quad \text{as } a \ll 1, \tag{1.17}$$

*where $\tilde{W}$ is the unique solution of the problem (*1.16*) with σ being the unique solution of the problem* (1.15)*.*

The results provided in Theorems 1.2 and 1.3 and Corollary 1.4 are given for boundary sources $f(x,\, t)$ as point sources initially located at z^*. Actually, these results are valid for any source f in the Sobolev space $H_0^1((0,\, T);\, W^{1,\infty}(\mathbb{R}^3)) := \{f|\, f,\, \partial_t f \in L^2((0,\, T);\, W^{1,\infty}(\mathbb{R}^3)) \text{ and } f \equiv 0 \text{ for } t \leq 0\}$.

Next, observe that $G_C(x,\, t;\, z^*,\, 0) := \Phi(x,\, t;\, z^*,\, 0) - W(x,\, t)$ satisfies

$$\begin{cases} (\partial_t - \Delta + \overline{C}\chi_\Omega)G_C = \delta_x(z^*)\delta_t(0) & \text{in } \mathbb{R}^3 \times (0,\, T), \\ G_C(x,\, 0) = 0 & \text{in } \mathbb{R}^3, \\ |G_C(x,\, t)| \leq C_0 \exp(b|x|^2) & \text{as } |x| \to +\infty. \end{cases} \tag{1.18}$$

Now, using the form $f(x,\, t) := \Phi(x,\, t;\, z^*,\, 0)$ in (1.1), we see that $G_D(x,\, t;\, z^*,\, 0) := \Phi(x,\, t;\, z^*,\, 0) - u(x,\, t)$ satisfies

$$\begin{cases} (\partial_t - \Delta)G_D = \delta_x(z^*)\delta_t(0) & \text{in } (\mathbb{R}^3 \setminus \overline{D})_T, \\ G_D = 0 & \text{on } (\partial D)_T, \\ G_D = 0 & \text{at } t = 0. \end{cases} \tag{1.19}$$

From Theorem 1.3, we see that, for $t > 0$ and $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, we have

$$G_D(x, t; z^*, 0) - G_C(x, t; z^*, 0) = O(a^{\frac{1}{3}}) \text{ as } a \ll 1.$$

As z^* is arbitrary in $\mathbb{R}^3 \setminus \overline{\Omega}$, this approximation implies that for any given initial source function $H := H(x)$ compactly supported in $\mathbb{R}^3 \setminus \overline{\Omega}$, then for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$ and $t > 0$, we have

$$(G_D * H)(x, t) - (G_C * H)(x, t) = O(a^{\frac{1}{3}}) \text{ as } a \ll 1,$$

where $G_D * H$ and $G_C * H$ stand for the convolutions in the space variable of the Green kernels G_D and G_C, respectively, with H. Hence, $G_D * H$ is the solution of (1.3) with H as the initial data, with $F = 0$ and $G = 0$, i.e.,

$$\begin{cases} (\partial_t - \Delta)(G_D * H) = 0 & \text{in } (\mathbb{R}^3 \setminus \overline{D})_T, \\ (G_D * H) = 0 & \text{on } (\partial D)_T, \\ (G_D * H) = H & \text{at } t = 0 \end{cases} \tag{1.20}$$

and $G_C * H$ satisfies the problem

$$\begin{cases} (\partial_t - \Delta + \overline{C}\chi_\Omega)(G_C * H) = 0 & \text{in } \mathbb{R}^3 \times (0, T), \\ (G_C * H)(x, 0) = H & \text{in } \mathbb{R}^3, \\ |(G_C * H)(x, t)| \leq C_0 \exp(b|x|^2) & \text{as } |x| \to +\infty. \end{cases} \tag{1.21}$$

A few remarks are in order:

1. The observation described above is not a surprise. That means, estimating the heat, generated by the cluster $\cup_{i=1}^M D_i$, starting from an initial heat supported on the source points z^* arbitrarily located outside Ω is enough to estimate the heat starting from any source supported in $\mathbb{R}^3 \setminus \overline{\Omega}$.
2. The condition that the initial source H is supported outside Ω is not a restriction. Indeed (see (1.20)) as $H = 0$ on $\cup_{i=1}^M \partial D_i$ and the set of cavities is densely distributed in Ω, then necessarily we should assume that $H = 0$ in Ω.
3. The condition (1.10) can be relaxed as follows. Recall that this condition translates as $1 - 2\beta - \frac{s}{3} \geq 0$. This condition is enough to derive the effective conductivity where we need $s = 1$ and $\beta = \frac{1}{3}$. However, performing the analysis done in this work more carefully, we can handle larger regimes where $s > 1$ and $\beta > \frac{1}{3}$.
4. In Theorem 1.3, we divided Ω into a family of Ω_m's which contain a single cavity each. Actually, we can put an arbitrary number of cavities in each Ω_m. This would be translated by the appearance of the local distribution density, which we denote by $K := K(z)$, in (1.14), replacing $\overline{C}$ by $\overline{C}K$.
5. In Corollary 1.4, we have seen that the cluster behaves as a conductive heat medium modeled by a heat conductivity $\gamma := \sigma^2$. This is possible because such cavities, modeled by Dirichlet boundary conditions, are actually resonating and hence enhance the generated heats. This is translated by the fact that the additive potential $\overline{C}\chi_\Omega$ appears with a positive sign and hence the generated conductivity σ via the boundary value problem (1.15) is positive. This phenomenon can also occur for other resonating particles as the electromagnetic nanoparticles enjoying balanced contrast/size ratios. However, this is not always the case for other boundary conditions, as the impedance boundary conditions, for which the sign of the corresponding coefficient $\overline{C}$ can

be negative. In this case, the corresponding boundary value problem (1.15) does not enjoy positivity for its solutions and hence the heat conductivity σ might not be generated.

6. In Theorem 1.3, we distributed the cavities in three-dimensional domains. It is natural, and interesting in applications, to consider distributions in two-dimensional surfaces or one-dimensional curves and their different superpositions. The corresponding results for these situations will be reported elsewhere.

The estimation of the fields generated by a cluster of small particles (of different kinds) is well developed in the literature for elliptic models, both stationary and nonstationary cases. When the small particles are periodically distributed or statistically arranged, then homogenization techniques are applicable; see, for instance, [12, 18, 25, 27, 29]. There are also extensive results on asymptotic expansions for various models of mathematical physics without periodicity assumptions on the distributions of the clusters. These clusters are distributed in bounded domains however. These results are obtained using different ideas. Next, we describe some of them as proposed in the literature.

In [33], mesoscale asymptotic approximations to solutions of elliptic boundary value problems in regions containing many perforations are presented on the basis of asymptotic analysis of the Green's functions. Under geometrical constraints on the small perforations, the solutions to the boundary value problems are written as a linear combination of model fields with remainder estimates. The method has been extended to the Dirichlet problem for elasticity in a perforated elastic body [34] and to solid problems containing clouds of voids [35]. A common tool to these works is the use of the maximum principle which is applicable to stationary models. Recently, an eigenvalue problem in a solid for a bounded domain with a large cluster of small inclusions is also studied in [36]. Using the integral equation methods, asymptotic approximations are developed in [13, 17, 30, 45] for acoustic, electromagnetic, and elastic scattering problems from many small particles of different types. Under sufficient conditions on M, ε, and d (see above for their definitions), asymptotic expansions of the scattered fields are described as a linear combination of weighted point sources, i.e., called Foldy–Lax fields, where the weights are solutions of the Foldy–Lax algebraic system; see [30]. Moreover, the precise estimates of the reminders are established in terms of M, ε, and d. The analysis in achieving these approximations is based on the integral representations of the scattered fields and the precise scalings of the boundary integral operators between the corresponding Sobolev spaces. This avoids, in particular, the need to use the maximum principle and hence allows us to deal with nonstationary models with arbitrary frequencies. As in [32, 33, 34, 35, 37, 38, 43], the particles can have arbitrary shapes and their distributions are not necessarily periodic. Another approach is based on the matching asymptotic expansion method (see [11]), which is, so far, limited to a fixed number of and well separated small particles. But it can handle approximations of any order in a unified way.

It is of particular interest to study the case when the number of small defects becomes large (and hence can be close to each other). Based on the above described asymptotic approximations, in the limit case, some effective materials are derived and they generate almost the same fields as the clusters of defects [1, 6, 22, 23, 24, 31, 42, 45, 46]. Motivations of such studies come from different areas such as mathematical imaging and material sciences in particular.

However, the situation is much less clear for time domain models, as those related to parabolic, Schrödinger, or hyperbolic equations, unless for periodic media; see, for instance, [18, 29]. Nevertheless, for the particular situations of pointlike particles, there are several works published in the framework of singular perturbations; see, for instance, [2, 19, 20]. Motivated by applications in mathematical imaging, asymptotic expansions of the heat generated by single, or well separated, small particles are derived in [5, 7] based on energy methods or Laplace transform. Here, we avoid using the Laplace transform and, rather, we use time domain integral equations. One of the basic arguments we used here can be explained as follows. Based on integral representations, via the single-layer (or double-layer) heat potentials, we have two choices. Either, we favor the space variable, i.e., reduce the single-layer heat potential to the one related to the Laplace operator, or the time variable, i.e., reduce the representation to the Abel integral. In this paper, since the cavities are fixed and not moving in time, we follow the first approach as it allows us to extract in a straightforward way the dominant part of the heat generated by the cluster. However, the other approach is also interesting as it might allow moving cavities (for instance, allowing their centers to be time dependent).

The rest of the paper is organized as follows. In section 2, we recall some properties of the integral equation method for the heat equation, mainly, the single-layer heat potential. In section 3, we provide the analysis for the case of a single cavity to describe the main steps of our approach. In section 4, we provide the proof of Theorem 1.2 and in section 5, the one of Theorem 1.3. In section 6, we give some conclusions of our work and discuss some potential applications. The proofs of several technical lemmas are given in the appendices.

2. Preliminaries. Throughout the paper, we use the notation "$\lesssim$" to denote "$\leq$" with its right-hand side multiplied by a generic positive constant. We recall some known properties of the single-layer heat operator. Using the change of variables $\zeta := |x-y|^2/(4(t-\tau))$ and the estimate $\zeta^\beta\, e^{-\zeta} \leq \beta^\beta\, e^{-\beta}$ for $0 < \zeta,\, \beta < +\infty$ (see, for instance, [28, Chapter 9]), we can easily derive the following classical singularity estimates for the fundamental solution $\Phi(x,\, t;\, y,\, \tau)$:

$$|\Phi(x,\, t;\, y,\, \tau)| \lesssim \frac{1}{(t-\tau)^\mu}\,\frac{1}{|x-y|^{3-2\mu}}, \quad \mu < \frac{3}{2}, \tag{2.1}$$

$$|\partial_{x_j}\Phi(x,\, t;\, y,\, \tau)| \lesssim \frac{1}{(t-\tau)^\gamma}\,\frac{1}{|x-y|^{4-2\gamma}}, \quad \gamma < \frac{5}{2},\ j = 1,\, 2,\, 3, \tag{2.2}$$

$$|D_t^{1/2}\Phi(x,\, t;\, y,\, \tau)| \lesssim \frac{1}{(t-\tau)^\gamma}\,\frac{1}{|x-y|^{4-2\gamma}}, \quad \gamma < \frac{5}{2}, \tag{2.3}$$

for $0 \leq \tau < t \leq T$ and $x,\, y \in \mathbb{R}^3$ with $x \neq y$. Here $D_t^{1/2}$ denotes the time derivative of order 1/2 defined by

$$D_t^{1/2} h(t) := \frac{1}{\sqrt{\pi}} \int_0^t \frac{h'(s)}{\sqrt{t-s}}\, ds.$$

Define the single-layer heat potential $\mathcal{S}_{\Gamma_T}$ by

$$\mathcal{S}_{\Gamma_T}[\varphi](x,\, t) := \int_0^t \int_\Gamma \Phi(x,\, t;\, y,\, \tau)\, \varphi(y,\, \tau)\, ds(y) d\tau, \quad (x,\, t) \in \Gamma \times (0,\, T), \tag{2.4}$$

where Γ is the boundary of a bounded domain in $\mathbb{R}^3$. To look for the weak solution of (1.1) by the single-layer potential, we introduce the anisotropic Sobolev space $H^{1,1/2}(\Gamma_T)$. It is a closure of the space

$$\left\{w :\ w = v|_{\Gamma_T},\, v \in C_0^\infty\left(\mathbb{R}^3 \times (0,\, +\infty)\right)\right\}$$

with respect to the norm
(2.5)

$$\|w\|^2_{H^{1,1/2}(\Gamma_T)} := \int_0^T \int_\Gamma \left[|\nabla_{\tan} w(x, t)|^2 + w^2(x, t) + \left(D_t^{1/2} w(x, t) \right)^2 \right] ds(x) dt,$$

where $\nabla_{\tan} w$ is the tangential gradient of w on Γ defined by $\nabla_{\tan} w := \nabla w - (\partial_\nu w)\nu$ with ν being the outward unit normal vector to Γ. Then the invertibility of $\mathcal{S}_{\Gamma_T}$ can be stated as follows [14, 15].

LEMMA 2.1. *Assume Γ to be Lipschitz regular. The operator $\mathcal{S}_{\Gamma_T} : L^2(\Gamma_T) \to H^{1,1/2}(\Gamma_T)$ is invertible with a bounded inverse.*

To investigate the classical solution of (1.1) and consider its regularity, we define the anisotropic Hölder spaces of functions

$$C_\lambda^{k,\alpha}(\Gamma_T) = \left\{ u \middle| \, \|u\|_\lambda^{(k,\,\alpha)}(\Gamma_T) < +\infty \right\}, \quad k = 0,\, 1,\ \alpha > 0,\ \lambda > 0,$$

where

$$\|u\|_\lambda^{(0,\,\alpha)}(\Gamma_T) := \sup_{(x,\,t)\in\Gamma_T} \frac{|u(x,\,t)|}{\exp \lambda t} + \sup_{\substack{(x,\,t),\,(x+\Delta x,\,t+\Delta t)\in\Gamma_T \\ |(\Delta x,\,\Delta t)|\neq 0}} \frac{|u(x+\Delta x,\,t) - u(x,\,t)|}{|\Delta x|^\alpha [\exp \lambda t + \exp \lambda (t+\Delta t)]},$$

$$\|u\|_\lambda^{(1,\,\alpha)}(\Gamma_T) := \sup_{(x,\,t)\in\Gamma_T} \frac{|u(x,\,t)|}{\exp \lambda t} + \sup_{\substack{(x,\,t),\,(x,\,t+\Delta t)\in\Gamma_T \\ |\Delta t|\neq 0}} \frac{|u(x,\,t+\Delta t) - u(x,\,t)|}{|\Delta t|^{(1+\alpha)/2} [\exp \lambda t + \exp \lambda (t+\Delta t)]} + \sum_{i=1}^3 \|\partial_{x_i} u\|_\lambda^{(0,\,\alpha)}(\Gamma_T).$$

Define $\overset{\circ}{C}{}_\lambda^{k,\alpha}(\Gamma_T)$ by

$$\overset{\circ}{C}{}_\lambda^{k,\alpha}(\Gamma_T) := \left\{ u \in C_\lambda^{k,\alpha}(\Gamma_T) \,\middle|\, u(x,\,0) = 0 \right\}.$$

Then we have the following result [9, 10].

LEMMA 2.2. *Assume Γ to be of class C^2. The operator $\mathcal{S}_{\Gamma_T} : \overset{\circ}{C}{}_\lambda^{0,\alpha}(\Gamma_T) \to \overset{\circ}{C}{}_\lambda^{1,\alpha}(\Gamma_T)$ is invertible with a bounded inverse.*

Observe that by Sobolev embedding, if $f \in H_0^1((0,\,T);\, W^{1,\,\infty}(\mathbb{R}^3))$ then $f|_{\Gamma_T} \in \overset{\circ}{C}{}_\lambda^{0,\frac{1}{2}}(\Gamma_T)$. Based on Lemma 2.1, we see that the problem (1.1) is well-posed for any source f belonging to $L^2(\Gamma_T)$. In addition, based on Lemma 2.2, the problem enjoys regularity properties. A particular property that we need is that if $f(\cdot,\,0) = 0$, then the density φ also satisfies the same property, i.e., $\varphi(\cdot,\,0) = 0$. Actually, in our analysis we need this property and this is the only place where we need the C^2 regularity of Γ. Otherwise, only the Lipschitz smoothness of Γ is needed.

Let $D = \varepsilon B + z$, where B is a bounded and simply connected domain in $\mathbb{R}^3$ with Lipschitz boundary containing the origin. For any functions φ and ψ defined on $(\partial D)_T$ and $(\partial B)_{T_\varepsilon}$, respectively, we use the notations

$$\hat{\varphi}(\eta,\,\tilde{\tau}) = \varphi^\wedge(\eta,\,\tilde{\tau}) := \varphi(\varepsilon\eta + z,\, \varepsilon^2\tilde{\tau}), \quad \check{\psi}(x,\,t) = \psi^\vee(x,\,t) := \psi\left(\frac{x-z}{\varepsilon},\, \frac{t}{\varepsilon^2}\right)$$

for $(x,\,t) \in (\partial D)_T$ and $(\eta,\,\tilde{\tau}) \in (\partial B)_{T_\varepsilon}$. Then we have the following lemmas.

LEMMA 2.3. *Suppose* $0 < \varepsilon \leq 1$ *and* $D = \varepsilon B + z \subset \mathbb{R}^3$. *Then, for* $\varphi \in H^{1,1/2}((\partial D)_T)$ *and* $\psi \in L^2((\partial D)_T)$, *we have*

$$\|\psi\|_{L^2((\partial D)_T)} = \varepsilon^2 \|\hat{\psi}\|_{L^2((\partial B)_{T_\varepsilon})} \tag{2.6}$$

and

$$\varepsilon^2 \|\hat{\varphi}\|_{H^{1,1/2}((\partial B)_{T_\varepsilon})} \leq \|\varphi\|_{H^{1,1/2}((\partial D)_T)} \leq \varepsilon \|\hat{\varphi}\|_{H^{1,1/2}((\partial B)_{T_\varepsilon})}. \tag{2.7}$$

Proof. Let $x = \varepsilon\xi + z$ and $t = \varepsilon^2 \tilde{t}$. Then for $\psi \in L^2((\partial D)_T)$ we have

$$\begin{aligned}\|\psi\|^2_{L^2((\partial D)_T)} &= \int_0^T \int_{\partial D} |\psi(x, t)|^2 \, ds(x)dt = \varepsilon^4 \int_0^{T_\varepsilon} \int_{\partial B} |\psi(\varepsilon\xi + z, \varepsilon^2 \tilde{t})|^2 \, ds(\xi)d\tilde{t} \\ &= \varepsilon^4 \|\hat{\psi}\|_{L^2((\partial B)_{T_\varepsilon})},\end{aligned}$$

which leads to (2.6).

For $\varphi \in H^{1,1/2}((\partial D)_T)$, we have

$$\begin{aligned}\|\varphi\|^2_{H^{1,1/2}((\partial D)_T)} &= \int_0^T \int_{\partial D} \left[|\nabla_{\tan}\varphi(x, t)|^2 + \varphi^2(x, t) + \left(D_t^{1/2} \varphi(x, t) \right)^2 \right] ds(x)dt \\ &= \varepsilon^4 \int_0^{T_\varepsilon} \int_{\partial B} \left[\varepsilon^{-2} |\nabla_{\tan}\varphi(\varepsilon\xi + z, \varepsilon^2\tilde{t})|^2 + \varphi^2(\varepsilon\xi + z, \varepsilon^2\tilde{t}) \right] ds(\xi)d\tilde{t} \\ &\quad + \varepsilon^2 \int_0^{T_\varepsilon} \int_{\partial B} \left(D_{\tilde{t}}^{1/2} \varphi(\varepsilon\xi + z, \varepsilon^2\tilde{t}) \right)^2 ds(\xi)d\tilde{t},\end{aligned}$$

which implies (2.7) by noticing that $\varepsilon^4 \leq \varepsilon^2$. ∎

LEMMA 2.4. *For* $\varphi \in H^{1,1/2}((\partial D)_T)$ *and* $\psi \in L^2((\partial D)_T)$, *we have*

$$\mathcal{S}_{(\partial D)_T}[\psi] = \varepsilon (\mathcal{S}_{(\partial B)_{T_\varepsilon}}[\hat{\psi}])^\vee, \tag{2.8}$$

$$\mathcal{S}^{-1}_{(\partial D)_T}[\varphi] = \varepsilon^{-1} (\mathcal{S}^{-1}_{(\partial B)_{T_\varepsilon}}[\hat{\varphi}])^\vee, \tag{2.9}$$

and

$$\left\| \mathcal{S}^{-1}_{(\partial D)_T} \right\|_{\mathcal{L}\left(H^{1,1/2}((\partial D)_T),\, L^2((\partial D)_T)\right)} \leq \varepsilon^{-1} \left\| \mathcal{S}^{-1}_{(\partial B)_{T_\varepsilon}} \right\|_{\mathcal{L}\left(H^{1,1/2}((\partial B)_{T_\varepsilon}),\, L^2((\partial B)_{T_\varepsilon})\right)}. \tag{2.10}$$

Proof. Let $x = \varepsilon\xi + z$, $y = \varepsilon\eta + z$, $t = \varepsilon^2\tilde{t}$, $\tau = \varepsilon^2\tilde{\tau}$. By direct calculations, we obtain

$$\begin{aligned}\mathcal{S}_{(\partial D)_T}[\psi](x, t) &= \int_0^t \int_{\partial D} \frac{1}{[4\pi(t-\tau)]^{3/2}} \exp\left(-\frac{|x-y|^2}{4(t-\tau)} \right) \psi(y, \tau) \, ds(y)d\tau \\ &= \int_0^{\tilde{t}} \int_{\partial B} \frac{1}{[4\pi\varepsilon^2(\tilde{t}-\tilde{\tau})]^{3/2}} \exp\left(-\frac{|\xi-\eta|^2}{4(\tilde{t}-\tilde{\tau})} \right) \psi(\varepsilon\eta + z, \varepsilon^2\tilde{\tau}) \, \varepsilon^4 \, ds(\eta)d\tilde{\tau} \\ &= \varepsilon \int_0^{\tilde{t}} \int_{\partial B} \frac{1}{[4\pi(\tilde{t}-\tilde{\tau})]^{3/2}} \exp\left(-\frac{|\xi-\eta|^2}{4(\tilde{t}-\tilde{\tau})} \right) \hat{\psi}(\eta, \tilde{\tau}) \, ds(\eta)d\tilde{\tau} \\ &= \varepsilon \mathcal{S}_{(\partial B)_{T_\varepsilon}}[\hat{\psi}](\xi, \tilde{t}),\end{aligned}$$

which gives (2.8). Further, the identity (2.9) follows from the following derivation:

$$\mathcal{S}_{(\partial D)_T}[(\mathcal{S}^{-1}_{(\partial B)_{T_\varepsilon}}[\hat{\varphi}])^\vee] = \varepsilon\left(\mathcal{S}_{(\partial B)_{T_\varepsilon}}\left[\mathcal{S}^{-1}_{(\partial B)_{T_\varepsilon}}[\hat{\varphi}]\right]\right)^\vee = \varepsilon(\hat{\varphi})^\vee = \varepsilon\varphi.$$

To show the estimate (2.10), we use the definition of the operator norm

$$\left\|\mathcal{S}^{-1}_{(\partial D)_T}\right\|_{\mathcal{L}\left(H^{1,1/2}((\partial D)_T),\, L^2((\partial D)_T)\right)} := \sup_{0\neq\varphi\in H^{1,1/2}((\partial D)_T)} \frac{\|\mathcal{S}^{-1}_{(\partial D)_T}[\varphi]\|_{L^2((\partial D)_T)}}{\|\varphi\|_{H^{1,1/2}((\partial D)_T)}},$$

and then obtain from (2.6) and (2.7) that

$$\begin{aligned}\left\|\mathcal{S}^{-1}_{(\partial D)_T}\right\|_{\mathcal{L}\left(H^{1,1/2}((\partial D)_T),\, L^2((\partial D)_T)\right)} &\leq \sup_{0\neq\varphi\in H^{1,1/2}((\partial D)_T)} \frac{\varepsilon^2\|(\mathcal{S}^{-1}_{(\partial D)_T}[\varphi])^\wedge\|_{L^2((\partial B)_{T_\varepsilon})}}{\varepsilon^2\|\hat{\varphi}\|_{H^{1,1/2}((\partial B)_{T_\varepsilon})}} \\ &= \sup_{0\neq\varphi\in H^{1,1/2}((\partial D)_T)} \frac{\|(\mathcal{S}^{-1}_{(\partial D)_T}[\varphi])^\wedge\|_{L^2((\partial B)_{T_\varepsilon})}}{\|\hat{\varphi}\|_{H^{1,1/2}((\partial B)_{T_\varepsilon})}}.\end{aligned}$$

Using (2.9), we further have

$$\begin{aligned}&\left\|\mathcal{S}^{-1}_{(\partial D)_T}\right\|_{\mathcal{L}\left(H^{1,1/2}((\partial D)_T),\, L^2((\partial D)_T)\right)} \\ &\quad\leq \sup_{0\neq\hat{\varphi}\in H^{1,1/2}((\partial B)_{T_\varepsilon})} \frac{\|(\varepsilon^{-1}(\mathcal{S}^{-1}_{(\partial B)_{T_\varepsilon}}[\hat{\varphi}])^\vee)^\wedge\|_{L^2((\partial B)_{T_\varepsilon})}}{\|\hat{\varphi}\|_{H^{1,1/2}((\partial B)_{T_\varepsilon})}} \\ &\quad= \sup_{0\neq\hat{\varphi}\in H^{1,1/2}((\partial B)_{T_\varepsilon})} \frac{\varepsilon^{-1}\|\mathcal{S}^{-1}_{(\partial B)_{T_\varepsilon}}[\hat{\varphi}]\|_{L^2((\partial B)_{T_\varepsilon})}}{\|\hat{\varphi}\|_{H^{1,1/2}((\partial B)_{T_\varepsilon})}} \\ &\quad= \varepsilon^{-1}\left\|\mathcal{S}^{-1}_{(\partial B)_{T_\varepsilon}}\right\|_{\mathcal{L}\left(H^{1,1/2}((\partial B)_{T_\varepsilon}),\, L^2((\partial B)_{T_\varepsilon})\right)}.\end{aligned}$$

Thus, the proof is complete. □

Finally, we need the following result [8, 14].

Lemma 2.5. *The single-layer potential operator*

$$\mathcal{S}_{(\partial B)_{T_\varepsilon}} : L^2((\partial B)_{T_\varepsilon}) \to H^{1,1/2}((\partial B)_{T_\varepsilon})$$

is invertible, and its inverse can be bounded by a constant independent of T and ε.

3. Proof of Theorem 1.2: The single cavity case. In this section, we develop the asymptotic approximation of the solution to (1.1) as $\varepsilon \to 0$ for the single cavity case, i.e., $D = \varepsilon B + z$. We recall that f in (1.1) is a heat point source, namely, $f(x, t) = \Phi(x, t; z^*, 0)$ with $z^* \in \mathbb{R}^3\backslash\overline{D}$. Express the solution of (1.1) as a single-layer heat potential

$$u(x, t) = \int_0^t \int_{\partial D} \Phi(x, t; y, \tau)\, \sigma(y, \tau)\, ds(y) d\tau, \quad (x, t) \in (\mathbb{R}^3 \setminus \overline{D})_T. \tag{3.1}$$

In terms of the boundary condition in (1.1), the density function σ should satisfy

$$\mathcal{S}_{(\partial D)_T}[\sigma](x, t) = \Phi(x, t; z^*, 0), \quad (x, t) \in (\partial D)_T. \tag{3.2}$$

Since $\mathcal{S}_{(\partial D)_T} : L^2((\partial D)_T) \to H^{1,1/2}((\partial D)_T)$ is invertible, it follows that

$$\sigma(x, t) = \mathcal{S}^{-1}_{(\partial D)_T}[\Phi_{(z^*, 0)}](x, t), \quad (x, t) \in (\partial D)_T,$$

where we use the notation $\Phi_{(z^*, 0)}(x, t) := \Phi(x, t; z^*, 0)$. By Lemma 2.5 and the estimate (2.10), we derive that

$$\begin{aligned}
\|\sigma\|_{L^2((\partial D)_T)} &\le \left\|\mathcal{S}^{-1}_{(\partial D)_T}\right\|_{\mathcal{L}\left(H^{1,1/2}((\partial D)_T),\, L^2((\partial D)_T)\right)} \|\Phi_{(z^*, 0)}\|_{H^{1,1/2}((\partial D)_T)} \\
&\le \varepsilon^{-1} \left\|\mathcal{S}^{-1}_{(\partial B)_{T_\varepsilon}}\right\|_{\mathcal{L}\left(H^{1,1/2}((\partial B)_{T_\varepsilon}),\, L^2((\partial B)_{T_\varepsilon})\right)} \|\Phi_{(z^*, 0)}\|_{H^{1,1/2}((\partial D)_T)} \\
&\lesssim \varepsilon^{-1}|\partial D|^{1/2} \le C. \qquad (3.3)
\end{aligned}$$

This leads to

$$\int_0^t \int_{\partial D} |\sigma(y, \tau)|\, ds(y) d\tau \le \|1\|_{L^2((\partial D)_T)} \cdot \|\sigma\|_{L^2((\partial D)_T)} \lesssim \varepsilon. \qquad (3.4)$$

In our argument, we also need to estimate $\partial_t \sigma$. Since $\Phi(x, t; z^*, 0)$ is infinitely smooth with respect to $(x, t) \in (\partial D)_T$ and goes to zero identically as $t \to 0$, we derive from Lemma 2.2 that $\sigma(\cdot, 0) = 0$ identically. Then, by taking the derivative of (3.1) with respect to t and using integration by parts, we get

$$\begin{aligned}
\partial_t u(x, t) &= \int_0^t \int_{\partial D} \partial_t \Phi(x, t; y, \tau)\, \sigma(y, \tau)\, ds(y) d\tau \\
&= -\int_0^t \int_{\partial D} \partial_\tau \Phi(x, t; y, \tau)\, \sigma(y, \tau)\, ds(y) d\tau \\
&= \int_0^t \int_{\partial D} \Phi(x, t; y, \tau)\, \partial_\tau \sigma(y, \tau)\, ds(y) d\tau, \quad (x, t) \in (\mathbb{R}^3 \setminus \overline{D})_T.
\end{aligned}$$

Again, we obtain from the boundary condition in (1.1) that

$$\int_0^t \int_{\partial D} \Phi(x, t; y, \tau)\, \partial_\tau \sigma(y, \tau)\, ds(y) d\tau = \partial_t \Phi(x, t; z^*, 0), \quad (x, t) \in (\partial D)_T. \qquad (3.5)$$

By the same derivations as for (3.3) and (3.4), we have

$$\|\partial_t \sigma\|_{L^2((\partial D)_T)} \le C \qquad (3.6)$$

and then

$$\int_0^t \int_{\partial D} |\partial_\tau \sigma(y, \tau)|\, ds(y) d\tau \lesssim \varepsilon. \qquad (3.7)$$

We now construct an asymptotic expansion of the solution to (1.1) as $\varepsilon \to 0$, based on the integral representation (3.1). First, we observe that, for any fixed $(x, t) \in (\mathbb{R}^3 \setminus \overline{D})_T$, the function $\Phi(x, t; y, \tau)$ is sufficiently smooth with respect to $(y, \tau) \in \partial D \times (0, t)$. It follows from Taylor's expansion that

$$|\Phi(x, t; y, \tau) - \Phi(x, t; z, \tau)| \lesssim \varepsilon$$

for $z \in D$, $y \in \partial D$, $x \in \mathbb{R}^3 \setminus \overline{D}$. Then we obtain

$$\left| \int_0^t \int_{\partial D} [\Phi(x,\, t;\, y,\, \tau) - \Phi(x,\, t;\, z,\, \tau)]\, \sigma(y,\, \tau)\, ds(y) d\tau \right| \lesssim \varepsilon^2, \quad (x,\, t) \in (\mathbb{R}^3 \setminus \overline{D})_T,$$

and hence

$$\begin{aligned} u(x,\, t) &= \int_0^t \int_{\partial D} \Phi(x,\, t;\, z,\, \tau)\, \sigma(y,\, \tau)\, ds(y) d\tau \\ &\quad + \int_0^t \int_{\partial D} [\Phi(x,\, t;\, y,\, \tau) - \Phi(x,\, t;\, z,\, \tau)]\, \sigma(y,\, \tau)\, ds(y) d\tau \\ &= \int_0^t \Phi(x,\, t;\, z,\, \tau) \left(\int_{\partial D} \sigma(y,\, \tau)\, ds(y) \right) d\tau + O(\varepsilon^2), \quad (x,\, t) \in (\mathbb{R}^3 \setminus \overline{D})_T. \end{aligned} \tag{3.8}$$

To derive the asymptotic expansion for the integral $\int_{\partial D} \sigma(y,\, t)\, ds(y)$, we rewrite the single-layer heat potential (3.1) as

$$\begin{aligned} &\int_0^t \int_{\partial D} \Phi(x,\, t;\, y,\, \tau)\, \sigma(y,\, \tau)\, ds(y) d\tau \\ &= \int_{\partial D} \frac{1}{4\pi |x-y|} \left[\int_0^t \frac{|x-y|}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{|x-y|^2}{4(t-\tau)} \right) \sigma(y,\, \tau)\, d\tau \right] ds(y) \end{aligned} \tag{3.9}$$

for $x \in \mathbb{R}^3 \setminus \partial D$ and $t \in (0,\, T]$. Therefore, we can view the single-layer heat potential (3.1) as a single-layer harmonic potential with the density

$$\begin{aligned} \varphi(x,\, y,\, t) := \int_0^t \frac{|x-y|}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{|x-y|^2}{4(t-\tau)} \right) \sigma(y,\, \tau)\, d\tau, \\ x \in \mathbb{R}^3,\, y \in \partial D,\, t \in (0,\, T]. \end{aligned} \tag{3.10}$$

By Lemma 2.2, the density $\sigma(x,\, t)$ in (3.2) is continuous on $(\partial D)_T$, since $\Phi(x,\, t;\, z^*,\, 0)$ is sufficiently smooth with respect to $(x,\, t) \in (\partial D)_T$. Based on this property, we state the following two lemmas, which are crucial in the next steps. Their proofs can be found in Appendices A and B, respectively.

LEMMA 3.1. *If $\sigma(y,\, t)$ is continuous on $\partial D \times (0,\, T]$, the function $\varphi(x,\, y,\, t)$ is continuous on $\mathbb{R}^3 \times \partial D \times (0,\, T]$ with*

$$\lim_{x \to y} \varphi(x,\, y,\, t) = \sigma(y,\, t) \tag{3.11}$$

for all $y \in \partial D$ and $t \in (0,\, T]$.

LEMMA 3.2. *Let $\sigma(y,\, t)$ be the solution to* (3.2). *Then we have*

$$\varphi(x,\, y,\, t) - \sigma(y,\, t) = O\left(|x-y|\, \|\partial_t \sigma(y,\, \cdot)\|_{L^2(0,\, t)} \right) \tag{3.12}$$

for x, y such that $|x-y| \ll 1$ and $t \in (0,\, T]$ uniformly with respect to D.

We are in a position to show the asymptotic expansion of the solution to (1.1) for the single cavity case.

THEOREM 3.3. *For $x \in \mathbb{R}^3 \setminus \overline{D}$ and $t \in (0,\, T]$, the solution $u(x,\, t)$ to* (1.1) *has the following asymptotic expansion:*

$$u(x,\, t) = C_0 \int_0^t \Phi(x,\, t;\, z,\, \tau)\, \Phi(z,\, \tau;\, z^*,\, 0)\, d\tau + O(\varepsilon^2) \int_0^t \Phi(x,\, t;\, z,\, \tau)\, d\tau + O(\varepsilon^2),\ \varepsilon \to 0, \tag{3.13}$$

with the constant C_0 defined by

$$C_0 := \int_{\partial D} S_{\partial D}^{-1}[1](y)\, ds(y),$$

where $S_{\partial D}^{-1}$ is the inverse of the single-layer potential operator $S_{\partial D}$ corresponding to the Laplace equation, namely,

$$S_{\partial D}[\varphi](x) := \int_{\partial D} \frac{1}{4\pi|x-y|}\, \varphi(y)\, ds(y), \quad x \in \partial D. \tag{3.14}$$

Proof. From (3.2), (3.9), and (3.10), we obtain that

$$\int_{\partial D} \frac{1}{4\pi|x-y|}\, \sigma(y, t)\, ds(y) = \Phi(x, t; z^*, 0) + \int_{\partial D} \frac{1}{4\pi|x-y|}\, [\sigma(y, t) - \varphi(x, y, t)]\, ds(y) \tag{3.15}$$

for $(x, t) \in (\partial D)_T$. Note that

$$\Phi(x, t; z^*, 0) = \Phi(z, t; z^*, 0) + O(\varepsilon), \quad (x, t) \in (\partial D)_T,\ z^* \in \mathbb{R}^3 \setminus \overline{D},$$

and, by Lemma 3.2, we have

$$\int_{\partial D} \frac{1}{4\pi|x-y|}\, [\sigma(y, t) - \varphi(x, y, t)]\, ds(y) = O\left(\varepsilon^2\right), \quad (x, t) \in (\partial D)_T.$$

Let σ_z be the unique solution of

$$\int_{\partial D} \frac{1}{4\pi|x-y|}\, \sigma_z(y, t)\, ds(y) = \Phi(z, t; z^*, 0), \quad (x, t) \in (\partial D)_T. \tag{3.16}$$

Then we have

$$\int_{\partial D} \frac{1}{4\pi|x-y|}\, [\sigma(y, t) - \sigma_z(y, t)]\, ds(y) = O(\varepsilon) + O\left(\varepsilon^2\right) = O(\varepsilon) \text{ uniformly for } x \in \partial D. \tag{3.17}$$

We recall that $S_{\partial D}$ from $H^{-1}(\partial D) := \left(H^1(\partial D)\right)^*$ to $L^2(\partial D)$ is an isomorphism and $\|S_{\partial D}^{-1}\|_{\mathcal{L}(L^2(\partial D), H^{-1}(\partial D))} = O(\varepsilon^{-1})$ by the scaling argument; see for instance [17]. This implies that

$$\begin{aligned} \|\sigma - \sigma_z\|_{H^{-1}(\partial D)} &= \left\|S_{\partial D}^{-1}\right\|_{\mathcal{L}(L^2(\partial D),\, H^{-1}(\partial D))} \|O(\varepsilon)\|_{L^2(\partial D)} \\ &= \left\|S_{\partial D}^{-1}\right\|_{\mathcal{L}(L^2(\partial D),\, H^{-1}(\partial D))} O(\varepsilon^2) = O(\varepsilon), \end{aligned} \tag{3.18}$$

and then

$$\left|\int_{\partial D} [\sigma(y, t) - \sigma_z(y, t)]\, ds(y)\right| = |\langle 1, \sigma - \sigma_z\rangle| \leq \|1\|_{H^1(\partial D)}\, \|\sigma - \sigma_z\|_{H^{-1}(\partial D)} = O(\varepsilon^2),$$

where $\langle\cdot, \cdot\rangle$ is the duality paring between $H^1(\partial D)$ and $H^{-1}(\partial D)$. So we have

$$\int_{\partial D} \sigma(y, t)\, ds(y) = \int_{\partial D} \sigma_z(y, t)\, ds(y) + O(\varepsilon^2), \quad t \in (0, T). \tag{3.19}$$

From (3.16), we know that

$$\sigma_z(x, t) = S_{\partial D}^{-1}[1](x)\, \Phi(z, t; z^*, 0), \quad (x, t) \in (\partial D)_T,$$

and therefore

$$\int_{\partial D} \sigma_z(y,\, t)\, ds(y) = C_0\, \Phi(z,\, t;\, z^*,\, 0), \quad t \in (0,\, T). \tag{3.20}$$

That is,

$$\int_{\partial D} \sigma(y,\, t)\, ds(y) = C_0\, \Phi(z,\, t;\, z^*,\, 0) + O(\varepsilon^2), \quad t \in (0,\, T). \tag{3.21}$$

Then, by inserting it into (3.8), we have, for $x \in \mathbb{R}^3 \setminus \overline{D}$ and $t \in (0,\, T)$, that

$$u(x,\, t) = C_0 \int_0^t \Phi(x,\, t;\, z,\, \tau)\, \Phi(z,\, \tau;\, z^*,\, 0)\, d\tau + O(\varepsilon^2) \int_0^t \Phi(x,\, t;\, z,\, \tau)\, d\tau + O(\varepsilon^2). \tag{3.22}$$

The proof is now complete. □

Remark 3.4. Observe that when the source of the heat z^* and the location of the receiver x are away from the cavity D then for $t \sim \varepsilon$, the dominating term $C_0 \int_0^t \Phi(x,\, t;\, z,\, \tau)\, \Phi(z,\, \tau;\, z^*,\, 0)\, d\tau$ behaves as ε^2 and hence it is lost in the error term. But this is not a surprise. However, when z^* and/or x are close to the cavity D, then the first term stays a dominating term even for very short time t, i.e., $t \sim \varepsilon$.

4. Proof of Theorem 1.2: The multiple cavities case. In this section, we show the asymptotic analysis of the solution to (1.1) as $\varepsilon \to 0$ for the multiple cavities case. We recall that $D_j := \varepsilon B_j + z_j$ are small cavities characterized by the parameter $\varepsilon > 0$ and the locations $z_j \in \mathbb{R}^3$, $j = 1,\, 2,\, \ldots,\, M$.

4.1. Integral representation of the solution. We express the solution to (1.1) as a single-layer heat potential

$$u(x,\, t) = \sum_{j=1}^{M} \int_0^t \int_{\partial D_j} \Phi(x,\, t;\, y,\, \tau)\, \sigma_j(y,\, \tau)\, ds(y) d\tau, \quad (x,\, t) \in (\mathbb{R}^3 \setminus \overline{D})_T, \tag{4.1}$$

where σ_j, $j = 1,\, 2,\, \ldots,\, M$ are density functions to be determined. In terms of the boundary condition, the density functions should satisfy

$$\sum_{j=1}^{M} \mathcal{S}_{(\partial D_j)_T}[\sigma_j](x,\, t) = \Phi(x,\, t;\, z^*,\, 0), \quad (x,\, t) \in (\partial D)_T. \tag{4.2}$$

Define the operator $\mathcal{S}_{ji}$ by

$$\mathcal{S}_{ji}[\sigma_j](x,\, t) := \int_0^t \int_{\partial D_j} \Phi(x,\, t;\, y,\, \tau)\, \sigma_j(y,\, \tau)\, ds(y) d\tau, \quad (x,\, t) \in (\partial D_i)_T. \tag{4.3}$$

Then (4.2) can be rewritten as

$$\mathcal{S}_{ii}[\sigma_i] + \sum_{\substack{j=1 \\ j \neq i}}^{M} \mathcal{S}_{ji}[\sigma_j] = \Phi(x,\, t;\, z^*,\, 0), \quad (x,\, t) \in (\partial D_i)_T, \quad i = 1,\, 2,\, \ldots,\, M, \tag{4.4}$$

or

$$\sigma_i + \mathcal{S}_{ii}^{-1} \sum_{\substack{j=1 \\ j \neq i}}^{M} \mathcal{S}_{ji}[\sigma_j] = \mathcal{S}_{ii}^{-1} \left[\Phi_{(z^*,\, 0)}\right], \quad (x,\, t) \in (\partial D_i)_T, \quad i = 1,\, 2,\, \ldots,\, M. \tag{4.5}$$

The unique solvability of the system (4.4), or (4.5), can be justified using the standard Fredholm alternative. Indeed, the operators $\mathcal{S}_{ii} : L^2((\partial D_i)_T) \to H^{1,1/2}((\partial D_i)_T)$ are invertible, while $\mathcal{S}_{ji}, j \neq i$, are compact ones. To show the uniqueness of solutions to (4.4), we use the uniqueness result of the problem (1.1) and the jump relations of the adjoint of the double-layer potential operator. The details of this argument can be found in [17, section 2.1.1], where the Helmholtz equation case is considered.

The next step is to derive an a priori estimate of the densities σ_i, $i = 1, 2, \dots, M$.

4.2. A priori estimate of the densities $\boldsymbol{\sigma_i}$'s. We start with the following singularity properties related to the heat fundamental solution. The proofs will be given in Appendices C and D, respectively.

LEMMA 4.1. *For $x \neq y$ and $|x-y| \to 0$, we have*

$$\left(\int_0^T \int_0^T |\Phi(x,\, t;\, y,\, \tau)|^2 \, d\tau dt\right)^{1/2} = O(|x-y|^{-2}), \tag{4.6}$$

$$\left(\int_0^T \int_0^T |\partial_{x_i}\Phi(x,\, t;\, y,\, \tau)|^2 \, d\tau dt\right)^{1/2} = O(|x-y|^{-3}), \quad i = 1,\, 2,\, 3, \tag{4.7}$$

$$\left(\int_0^T \int_0^T |D_t^{1/2}\Phi(x,\, t;\, y,\, \tau)|^2 \, d\tau dt\right)^{1/2} = O(|x-y|^{-3}). \tag{4.8}$$

LEMMA 4.2. *For $x \neq y$ and $|x-y| \to 0$, we have*

$$\left(\int_0^t |\nabla_x\Phi(x,\, t;\, y,\, \tau)|^2 \, d\tau\right)^{1/2} = O(|x-y|^{-3}), \quad t \in (0,\, T]. \tag{4.9}$$

To proceed, we recall the single-layer operator $S_{\partial D_i}$ as defined in (3.14):

$$S_{\partial D_i}[\sigma_i](x,\, t) := \int_{\partial D_i} \frac{1}{4\pi|x-y|}\, \sigma_i(y,\, t)\, ds(y), \quad (x,\, t) \in (\partial D_i)_T, \tag{4.10}$$

and write

$$\mathcal{S}_{ii}[\sigma_i] = S_{\partial D_i}[\sigma_i] + S_{\partial D_i}[\varphi_i - \sigma_i],$$

where

$$\varphi_i(x,\, y,\, t) := \int_0^t \frac{|x-y|}{2\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{|x-y|^2}{4(t-\tau)}\right) \sigma_i(y,\, \tau)\, d\tau, \tag{4.11}$$
$$x \in \mathbb{R}^3,\; y \in \partial D_i,\; t \in (0,\, T).$$

In addition, we define

$$q_j(t) := \int_{\partial D_j} \sigma_j(y,\, t)\, ds(y), \quad t \in (0,\, T), \tag{4.12}$$

and write

$$\mathcal{S}_{ji}[\sigma_j](x,\, t) = \int_0^t \Phi(z_i,\, t;\, z_j,\, \tau)\, q_j(\tau)\, d\tau + A_{ji} + B_{ji}, \quad (x,\, t) \in (\partial D_i)_T,$$

with

$$A_{ji} := \int_0^t [\Phi(x, t; z_j, \tau) - \Phi(z_i, t; z_j, \tau)]\, q_j(\tau)\, d\tau,$$
$$B_{ji} := \int_0^t \int_{\partial D_j} [\Phi(x, t; y, \tau) - \Phi(x, t; z_j, \tau)]\, \sigma_j(y, \tau)\, ds(y) d\tau.$$

Hence, (4.4) becomes

(4.13)
$$S_{\partial D_i}[\sigma_i] = -\sum_{\substack{j=1 \\ j\neq i}}^{M} \int_0^t \Phi(z_i, t; z_j, \tau)\, q_j(\tau)\, d\tau + \Phi_{(z^*, 0)} - S_{\partial D_i}[\varphi_i - \sigma_i] - \sum_{\substack{j=1 \\ j\neq i}}^{M} (A_{ji} - B_{ji})\,.$$

To deduce the estimate of σ_i's, using the mean value theorem, we decompose A_{ji} as

$$A_{ji} = A_{ji}^{(1)} + A_{ji}^{(2)}$$

with

$$A_{ji}^{(1)} := \int_0^t (x - z_i) \cdot \nabla_x \Phi(z_i, t; z_j, \tau)\, q_j(\tau)\, d\tau \tag{4.14}$$

and

$$A_{ji}^{(2)} := \frac{1}{2} \sum_{k, l=1}^{3} \int_0^t (x_k - z_{ik})\, (x_l - z_{il})\, \partial_{x_k} \partial_{x_l} \Phi(z_i^*, t; z_j, \tau)\, q_j(\tau)\, d\tau, \tag{4.15}$$

where $z_i^* = z_i + \theta(x - z_i)$, $0 < \theta < 1$. Then we decompose σ_i as $\sigma_i = \sigma_i^{(1)} + \sigma_i^{(2)} + \sigma_i^{(3)}$, where $\sigma_i^{(1)}$, $\sigma_i^{(2)}$, and $\sigma_i^{(3)}$ are defined by

$$S_{\partial D_i}[\sigma_i^{(1)}](x, t) = -\sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ji}^{(1)}, \quad (x, t) \in (\partial D_i)_T, \tag{4.16}$$

$$S_{\partial D_i}[\sigma_i^{(2)}](x, t) = -S_{\partial D_i}[\varphi_i - \sigma_i], \quad (x, t) \in (\partial D_i)_T, \tag{4.17}$$

and

$$\begin{aligned} S_{\partial D_i}[\sigma_i^{(3)}](x, t) = &-\sum_{\substack{j=1 \\ j\neq i}}^{M} \int_0^t \Phi(z_i, t; z_j, \tau)\, q_j(\tau)\, d\tau \\ &+ \Phi_{(z^*, 0)} - \sum_{\substack{j=1 \\ j\neq i}}^{M} \left(A_{ji}^{(2)} - B_{ji}\right), \quad (x, t) \in (\partial D_i)_T, \end{aligned} \tag{4.18}$$

respectively.

First, we estimate $\sigma_i^{(1)}$. Plugging (4.14) into (4.16) and taking the inverse of $S_{\partial D_i}$, we get

$$\|\sigma_i^{(1)}\|_{L^2(\partial D_i)} \lesssim \sum_{\substack{j=1\\ j\neq i}}^{M} \|S_{\partial D_i}^{-1}[x-z_i]\|_{L^2(\partial D_i)} \int_0^t |\nabla_x \Phi(z_i,\, t;\, z_j,\, \tau)|\, |q_j(\tau)|\, d\tau.$$

By the Cauchy–Schwartz inequality and Lemma 4.2, we have

$$\begin{aligned}\|\sigma_i^{(1)}\|_{L^2(\partial D_i)} &\lesssim \sum_{\substack{j=1\\ j\neq i}}^{M} \varepsilon \left(\int_0^t |\nabla_x \Phi(z_i,\, t;\, z_j,\, \tau)|^2\, d\tau\right)^{1/2} \left(\int_0^t |q_j(\tau)|^2\, d\tau\right)^{1/2}\\ &\lesssim \sum_{\substack{j=1\\ j\neq i}}^{M} \varepsilon\, d_{ji}^{-3}\, \|1\|_{L^2(\partial D_j)}\, \|\sigma_j\|_{L^2((\partial D_j)_T)}\\ &\lesssim \sum_{\substack{j=1\\ j\neq i}}^{M} \varepsilon^2\, d_{ji}^{-3}\, \|\sigma_j\|_{L^2((\partial D_j)_T)}. \qquad (4.19)\end{aligned}$$

Second, let us estimate $\sigma_i^{(2)}$. To this end, we need the following result, which will be proved in Appendix E.

LEMMA 4.3. *Let $\sigma_i(y,\, t)$, $i = 1,\, 2,\, \ldots,\, M$, be the solution to* (4.2). *Then we have*
(4.20)

$$\varphi_i(x,\, y,\, t)-\sigma_i(y,\, t) = \sum_{n=1}^{+\infty} \alpha_n(t)\, |x-y|^n\, \sigma_i(y,\, t)+\sum_{n=1}^{+\infty}\left(\sum_{m=n}^{+\infty} \beta_{m,n}(t)\, |x-y|^m\right)\partial_t^n \sigma_i(y,\, t)$$

for x, y such that $|x-y| \ll 1$ and $t \in (0,\, T]$ uniformly with respect to D_i, where $\alpha_n(t)$ and $\beta_{m,n}(t)$ are smooth in $(0,\, T]$.

In view of Lemma 4.3, let us consider the following two equations:

$$S_{\partial D_i}[\sigma_n^{(2)}](x,\, t) = S_{\partial D_i}[|x-y|^n\, \sigma_i(y,\, t)], \quad (x,\, t) \in (\partial D_i)_T, \qquad (4.21)$$

$$S_{\partial D_i}[\sigma_{m,n}^{(2)}](x,\, t) = S_{\partial D_i}[|x-y|^m\, \partial_t^n \sigma_i(y,\, t)], \quad (x,\, t) \in (\partial D_i)_T, \qquad (4.22)$$

where the operator $S_{\partial D_i}$ is defined by (4.10). By scaling on the space variable, we get from (4.21) that

$$\varepsilon\, S_{\partial B_i}[\tilde{\sigma}_n^{(2)}] = \varepsilon^{n+1}\, S_{\partial B_i}[|\xi-\eta|^n\, \tilde{\sigma}_i(\eta,\, t)],$$

where $x = \varepsilon\, \xi$, $y = \varepsilon\, \eta$, $\tilde{\sigma}_n^{(2)}(\eta,\, t) = \sigma_n^{(2)}(\varepsilon\eta,\, t)$, and $\tilde{\sigma}_i(\eta,\, t) = \sigma_i(\varepsilon\, \eta,\, t)$. Then we have

$$\|\tilde{\sigma}_n^{(2)}\|_{L^2(\partial B_i)} \lesssim \varepsilon^n \|\tilde{\sigma}_i\|_{L^2(\partial B_i)}$$

and also

$$\|\sigma_n^{(2)}\|_{L^2(\partial D_i)} \lesssim \varepsilon^n \|\sigma_i\|_{L^2(\partial D_i)}. \qquad (4.23)$$

Similarly, we obtain from (4.22) that

$$\|\sigma_{n,n}^{(2)}\|_{L^2(\partial D_i)} \lesssim \varepsilon^n \|\partial_t^n \sigma_i\|_{L^2(\partial D_i)}. \qquad (4.24)$$

Based on Lemma 4.3, we derive from (4.17), (4.21), and (4.22) that

$$\sigma_i^{(2)} = -\sum_{n=1}^{+\infty} \left[\alpha_n(t)\sigma_n^{(2)} + O(|x-y|^n)\sigma_{n,n}^{(2)}\right].$$

Then, using (4.23) and (4.24), we have

$$\begin{aligned} \|\sigma_i^{(2)}\|_{L^2(\partial D_i)} &\lesssim \sum_{n=1}^{+\infty} \varepsilon^n \left[\|\sigma_i\|_{L^2(\partial D_i)} + \|\partial_t^n \sigma_i\|_{L^2(\partial D_i)}\right] \\ &= O(\varepsilon\|\sigma_i\|_{L^2(\partial D_i)}) + \sum_{n=1}^{+\infty} \varepsilon^n \|\partial_t^n \sigma_i\|_{L^2(\partial D_i)}. \end{aligned} \tag{4.25}$$

Finally, let us estimate $\sigma_i^{(3)}$. Using the estimate of the gradient of the heat kernel (see (2.2)), we see that if z^* is away from $\cup_{i=1}^M D_i$, then

$$\|\Phi_{(z^*,0)}\|_{H^1(\partial D_i)} = O(\varepsilon). \tag{4.26}$$

To proceed, we show the following estimates.

LEMMA 4.4. *For $t \in (0, T]$ and $i, j = 1, 2, \ldots, M$ with $i \neq j$, we have*

$$\|A_{ji}^{(1)}\|_{H^1(\partial D_i)}^2 \lesssim \left(\varepsilon^6 d_{ji}^{-6} + \varepsilon^4 d_{ji}^{-6}\right) \|\sigma_j\|_{L^2((\partial D_j)_T)}^2, \tag{4.27}$$

$$\|A_{ji}^{(2)}\|_{H^1(\partial D_i)}^2 \lesssim \left(\varepsilon^8 d_{ji}^{-8} + \varepsilon^6 d_{ji}^{-8}\right) \|\sigma_j\|_{L^2((\partial D_j)_T)}^2, \tag{4.28}$$

$$\|B_{ji}\|_{H^1(\partial D_i)}^2 \lesssim \left(\varepsilon^6 d_{ji}^{-6} + \varepsilon^6 d_{ji}^{-8}\right) \|\sigma_j\|_{L^2((\partial D_j)_T)}^2. \tag{4.29}$$

Proof. We only prove the estimate (4.27), since the others can be shown in the same way.

Recalling the definition of q_j given by (4.12) and using the Cauchy–Schwartz inequality, we derive

$$\begin{aligned} &\int_{\partial D_i} \left|A_{ji}^{(1)}\right|^2 ds(x) \\ &= \int_{\partial D_i} \left|\int_0^t (x-z_i)\cdot\nabla_x\Phi(z_i, t; z_j, \tau)\, q_j(\tau)\, d\tau\right|^2 ds(x) \\ &\lesssim \|\sigma_j\|_{L^2((\partial D_j)_T)}^2 \int_{\partial D_i}\int_{\partial D_j}\left(\int_0^t |x-z_i|^2\, |\nabla_x\Phi(z_i, t; z_j, \tau)|^2\, d\tau\right) ds(x)ds(y). \end{aligned}$$

Then, by Definition 1.1 and Lemma 4.2, we have

$$\int_{\partial D_i} \left|A_{ji}^{(1)}\right|^2 ds(x) \lesssim \|\sigma_j\|_{L^2((\partial D_j)_T)}^2\, \varepsilon^2 d_{ji}^{-6} |\partial D_i|\, |\partial D_j| \lesssim \varepsilon^6 d_{ji}^{-6} \|\sigma_j\|_{L^2((\partial D_j)_T)}^2.$$

Similarly, using Lemma 4.2, we know

$$\begin{aligned} &\int_{\partial D_i} \left|\nabla_{\tan} A_{ji}^{(1)}\right|^2 ds(x) \\ &\lesssim \int_{\partial D_i} \left(\int_0^t \left|\nabla_x\Phi(z_i, t; z_j, \tau)\, q_j(\tau)\right|^2\, d\tau\right) ds(x) \end{aligned}$$

$$\lesssim \|\sigma_j\|^2_{L^2((\partial D_j)_T)} \int_{\partial D_i} \int_{\partial D_j} \left(\int_0^t |\nabla_x \Phi(z_i,\, t;\, z_j,\, \tau)|^2 \, d\tau \right) ds(x) ds(y)$$
$$\lesssim \varepsilon^4 d_{ji}^{-6} \|\sigma_j\|^2_{L^2((\partial D_j)_T)}.$$

So, the estimate (4.27) is justified. ∎

Note that the term $\sum_{j\neq i} \int_0^t \Phi(z_i,\, t;\, z_j,\, \tau)\, q_j(\tau)\, d\tau$ is independent of the space variable, hence its $H^1(\partial D_i)$-norm reduces to its $L^2(\partial D_i)$-norm. Then we have

$$\int_{\partial D_i} \left| \int_0^t \Phi(z_i,\, t;\, z_j,\, \tau)\, q_j(\tau)\, d\tau \right|^2 ds(x) \lesssim \varepsilon^2 \|q_j\|^2_{L^2(0,\,T)} \int_0^t |\Phi(z_i,\, t;\, z_j,\, \tau)|^2 \, d\tau$$
$$\lesssim \varepsilon^2 d_{ji}^{-4} \, \|q_j\|^2_{L^2(0,\,T)} \lesssim \varepsilon^4 d_{ji}^{-4} \|\sigma_j\|^2_{L^2((\partial D_j)_T)}.$$

By (4.18), we get

$$\|S_{\partial D_i}[\sigma_i^{(3)}]\|_{H^1(\partial D_i)} \lesssim \sum_{\substack{j=1 \\ j\neq i}}^{M} \varepsilon^2 d_{ji}^{-2} \|\sigma_j\|_{L^2((\partial D_j)_T)} + O(\varepsilon) + \sum_{\substack{j=1 \\ j\neq i}}^{M} O\left(\varepsilon^3 \, d_{ji}^{-4} \, \|\sigma_j\|_{L^2((\partial D_j)_T)}\right).$$

Using the property $\|S^{-1}_{\partial D_i}\|_{\mathcal{L}(H^1(\partial D_i),\, L^2(\partial D_i))} = O(\varepsilon^{-1})$ and integrating over t, we deduce that

$$\|\sigma_i^{(3)}\|_{L^2((\partial D_i)_T)} \lesssim \sum_{\substack{j=1 \\ j\neq i}}^{M} \varepsilon d_{ji}^{-2} \|\sigma_j\|_{L^2((\partial D_j)_T)} + O(1) + \sum_{\substack{j=1 \\ j\neq i}}^{M} O\left(\varepsilon^2 \, d_{ji}^{-4} \, \|\sigma_j\|_{L^2((\partial D_j)_T)}\right). \tag{4.30}$$

Combining (4.19), (4.25), and (4.30), we have

$$\|\sigma_i\|_{L^2((\partial D_i)_T)} \lesssim \sum_{\substack{j=1 \\ j\neq i}}^{M} \varepsilon d_{ji}^{-2} \|\sigma_j\|_{L^2((\partial D_j)_T)} + O(1) + \sum_{\substack{j=1 \\ j\neq i}}^{M} O\left(\varepsilon^2 \, d_{ji}^{-4} \, \|\sigma_j\|_{L^2((\partial D_j)_T)}\right)$$
$$+O(\varepsilon\|\sigma_i\|_{L^2(\partial D_i)}) + \sum_{n=1}^{+\infty} \varepsilon^n \, \|\partial_t^n \sigma_i\|_{L^2(\partial D_i)}. \tag{4.31}$$

To estimate $\|\partial_t^n \sigma_i\|_{L^2((\partial D_i)_T)}$, we perform the time derivative for (4.4), and use integration by parts for the first term and the fact that $\sigma_i(\cdot,\, 0) = 0$. Then we get

$$\mathcal{S}_{ii}[\partial_t \sigma_i] + \sum_{\substack{j=1 \\ j\neq i}}^{M} \partial_t \left(\mathcal{S}_{ji}[\sigma_j]\right) = \partial_t \Phi(x,\, t;\, z^*,\, 0), \quad (x,\, t) \in (\partial D_i)_T,\ i = 1,\, 2,\, \ldots,\, M, \tag{4.32}$$

or equivalently,

$$\partial_t \sigma_i = \mathcal{S}_{ii}^{-1} \left[\partial_t \Phi(x,\, t;\, z^*,\, 0) - \sum_{\substack{j=1 \\ j\neq i}}^{M} \partial_t \left(\mathcal{S}_{ji}[\sigma_j]\right) \right], \quad (x,\, t) \in (\partial D_i)_T,\ i = 1,\, 2,\, \ldots,\, M. \tag{4.33}$$

Since we have

$$\left\|\mathcal{S}_{ii}^{-1}\right\|_{\mathcal{L}\left(H^{1,1/2}((\partial D_i)_T),\, L^2((\partial D_i)_T)\right)} \lesssim \varepsilon^{-1}, \tag{4.34}$$

it suffices to estimate

$$\left\| \partial_t \Phi(x,\, t;\, z^*,\, 0) - \sum_{\substack{j=1 \\ j \neq i}}^{M} \partial_t \left(\mathcal{S}_{ji}[\sigma_j] \right) \right\|_{H^{1,1/2}((\partial D_i)_T)}.$$

First, observe that for $z^* \in \mathbb{R}^3 \backslash \overline{D}$, the function $\Phi(x,\, t;\, z^*,\, 0)$ is sufficiently smooth with respect to $x \in \partial D_i$ and $t \in (0,\, T)$. Then, by the definition of the $H^{1,1/2}$-norm (2.5), there exists a positive constant $\tilde{c}$ such that

$$\begin{aligned} &\left\| \partial_t \Phi(x,\, t;\, z^*,\, 0) \right\|^2_{H^{1,1/2}((\partial D_i)_T)} \\ &= \int_0^T \int_{\partial D_i} \left[|\nabla_{\tan} \Phi_{(z^*,\, 0)}(x,\, t)|^2 + |\Phi_{(z^*,\, 0)}(x,\, t)|^2 + |D_t^{1/2} \Phi_{(z^*,\, 0)}(x,\, t)|^2 \right] ds(x) dt \\ &\leq \tilde{c} \int_0^T \int_{\partial D_i} 1 \, ds(x) dt \\ &\leq \tilde{c}\, T\, |\partial D_i|, \end{aligned}$$

which implies

$$\left\| \partial_t \Phi(x,\, t;\, z^*,\, 0) \right\|_{H^{1,1/2}((\partial D_i)_T)} \lesssim \varepsilon. \tag{4.35}$$

Second, let us define $\mathcal{S}'_{ji} := \partial_t(\mathcal{S}_{ji})$ and estimate $\left\| \mathcal{S}'_{ji} \right\|_{\mathcal{L}\left(L^2((\partial D_j)_T),\, H^{1,1/2}((\partial D_i)_T)\right)}$ for $i \neq j$. Note that

$$\begin{aligned} &\left\| \mathcal{S}'_{ji} \right\|_{\mathcal{L}\left(L^2((\partial D_j)_T),\, H^{1,1/2}((\partial D_i)_T)\right)} \\ &:= \sup_{0 \neq \varphi \in L^2((\partial D_j)_T)} \frac{\left\| \mathcal{S}'_{ji}[\varphi] \right\|_{H^{1,1/2}((\partial D_i)_T)}}{\|\varphi\|_{L^2((\partial D_j)_T)}} \\ &\leq \sup_{0 \neq \varphi \in L^2((\partial D_j)_T)} \left(\frac{\left\| \mathcal{S}'_{ji}[\varphi] \right\|_{L^2((\partial D_i)_T)} + \left\| \nabla_{\tan} \mathcal{S}'_{ji}[\varphi] \right\|_{L^2((\partial D_i)_T)}}{\|\varphi\|_{L^2((\partial D_j)_T)} + \frac{\left\| D_t^{1/2} \mathcal{S}'_{ji}[\varphi] \right\|_{L^2((\partial D_i)_T)}}{\|\varphi\|_{L^2((\partial D_j)_T)}} \right). \end{aligned} \tag{4.36}$$

We deduce that

$$\begin{aligned} &\left\| \mathcal{S}'_{ji}[\varphi] \right\|^2_{L^2((\partial D_i)_T)} \\ &= \int_0^T \int_{\partial D_i} \left| \int_0^t \int_{\partial D_j} \partial_t \Phi(x,\, t;\, y,\, \tau)\, \varphi(y,\, \tau)\, ds(y) d\tau \right|^2 ds(x) dt \\ &\lesssim \|\varphi\|^2_{L^2((\partial D_j)_T)} \int_{\partial D_i} \int_{\partial D_j} \left(\int_0^T \int_0^t |\partial_t \Phi(x,\, t;\, y,\, \tau)|^2 \, d\tau dt \right) ds(x) ds(y) \\ &\lesssim d_{ij}^{-8}\, \|\varphi\|^2_{L^2((\partial D_j)_T)}\, |\partial D_j|\, |\partial D_i|, \end{aligned} \tag{4.37}$$

which implies that

$$\left\|\mathcal{S}'_{ji}[\varphi]\right\|_{L^2((\partial D_i)_T)} \lesssim \varepsilon^2\, d_{ij}^{-4}\, \|\varphi\|_{L^2((\partial D_j)_T)}. \tag{4.38}$$

Analogously, we can also prove that

$$\left\|\nabla_{\tan}\mathcal{S}'_{ji}[\varphi]\right\|_{L^2((\partial D_i)_T)} \lesssim \varepsilon^2\, d_{ij}^{-5}\, \|\varphi\|_{L^2((\partial D_j)_T)} \tag{4.39}$$

and

$$\left\|D_t^{1/2}\mathcal{S}'_{ji}[\varphi]\right\|_{L^2((\partial D_i)_T)} \lesssim \varepsilon^2\, d_{ij}^{-5}\, \|\varphi\|_{L^2((\partial D_j)_T)}. \tag{4.40}$$

So we obtain from (4.36) that

$$\left\|\mathcal{S}'_{ji}\right\|_{\mathcal{L}\left(L^2((\partial D_j)_T),\, H^{1,1/2}((\partial D_i)_T)\right)} \lesssim \varepsilon^2\, d_{ij}^{-5}. \tag{4.41}$$

This leads to

$$\begin{aligned} \left\|\sum_{\substack{j=1\\ j\neq i}}^{M} \partial_t\left(\mathcal{S}_{ji}[\sigma_j]\right)\right\|_{H^{1,1/2}((\partial D_i)_T)} &\lesssim \varepsilon^2 \sum_{\substack{j=1\\ j\neq i}}^{M} d_{ij}^{-5}\|\sigma_j\|_{L^2((\partial D_j)_T)} \\ &\lesssim \varepsilon^2\, d^{-5} \sup_j \|\sigma_j\|_{L^2((\partial D_j)_T)}, \end{aligned} \tag{4.42}$$

where we used the following estimate shown in [4]:

$$\sum_{\substack{j=1\\ j\neq i}}^{M} d_{ij}^{-5} = O(d^{-5}).$$

As a consequence, we have proved that

$$\|\partial_t\sigma_i\|_{L^2((\partial D_i)_T)} \lesssim O(1) + \varepsilon\, d^{-5} \sup_j \|\sigma_j\|_{L^2((\partial D_j)_T)}. \tag{4.43}$$

By repeatedly using this argument, we have

$$\|\partial_t^n\sigma_i\|_{L^2((\partial D_i)_T)} \lesssim O(1) + \varepsilon\, d^{-3-2n} \sup_j \|\sigma_j\|_{L^2((\partial D_j)_T)}. \tag{4.44}$$

Inserting (4.44) into (4.31), we get

$$\begin{aligned} \|\sigma_i\|_{L^2((\partial D_i)_T)} \lesssim{}& \sum_{\substack{j=1\\ j\neq i}}^{M} \varepsilon d_{ji}^{-2}\|\sigma_j\|_{L^2((\partial D_j)_T)} + O(1) + \sum_{\substack{j=1\\ j\neq i}}^{M} O\left(\varepsilon^2\, d_{ji}^{-4}\, \|\sigma_j\|_{L^2((\partial D_j)_T)}\right) \\ &+ O(\varepsilon\|\sigma_i\|_{L^2(\partial D_i)}) + O(\varepsilon) + O\left(\varepsilon^2\, d^{-5}\, \sup_j \|\sigma_j\|_{L^2((\partial D_j)_T)}\right). \end{aligned} \tag{4.45}$$

Hence, we conclude that if $\varepsilon\sum_j\, d_{ij}^{-2} < 1$ or, equivalently, $1-2\beta-s/3 \geq 0$, then

$$\|\sigma_i\|_{L^2((\partial D_i)_T)} = O(1) \quad \text{as } \varepsilon \ll 1.$$

As we derived before, the functions $\partial_t\sigma_i$, $i = 1, 2, \ldots, M$ satisfy the system (4.32). Using the same steps for σ_i, $i = 1, 2, \ldots, M$, we also have

$$\|\partial_t\sigma_i\|_{L^2((\partial D_i)_T)} = O(1) \quad \text{as } \varepsilon \ll 1.$$

We state these results in the following proposition.

PROPOSITION 4.5. *Under the following condition on the distribution of the small cavities,*

$$\varepsilon \max_{1\le i\le M} \sum_{j\ne i} d_{ij}^{-2} < 1, \tag{4.46}$$

the solution of the system of integral equation (4.4) *has the following estimates:*

$$\|\sigma_i\|_{L^2((\partial D_i)_T)} = O(1), \quad \|\partial_t \sigma_i\|_{L^2((\partial D_i)_T)} = O(1) \quad \text{as } \varepsilon \ll 1. \tag{4.47}$$

4.3. Invertibility of the algebraic system. For $j = 1, 2, \ldots, M$, we define

$$C_j := \int_{\partial D_j} S_{\partial D_j}^{-1}[1](y)\, ds(y). \tag{4.48}$$

We state the following system of integral equations

$$q_i(t) + \sum_{\substack{j=1\\ j\ne i}}^{M} \int_0^t C_j\, \Phi(z_i, t; z_j, \tau)\, q_j(\tau)\, d\tau = f_i(t),\ t \in (0, T),\ i = 1, 2, \ldots, M. \tag{4.49}$$

This system is naturally linked to the proof of our main results in section 4.4. Here, we show the invertibility of the system (4.49) and estimate $\sum_{i=1}^{M} \|q_i\|_{L^2(0,T)}^2$.

THEOREM 4.6. *If*

$$C \max_{1\le i\le M} \sum_{j\ne i} |z_i - z_j|^{-2} < 1 \tag{4.50}$$

with $C := \max_{1\le j\le M} C_j$, *then the system* (4.49) *is uniquely solvable. Moreover, we have the estimate*

$$\left(\sum_{i=1}^{M} \|q_i\|_{L^2(0,T)}^2\right)^{1/2} \le \left(1 - C \max_{1\le i\le M} \sum_{j\ne i} |z_i - z_j|^{-2}\right)^{-1} \left\{\sum_{i=1}^{M} \|f_i\|_{L^2(0,T)}^2\right\}^{1/2}. \tag{4.51}$$

Proof. Observe from (4.49) that

$$\sum_{i=1}^{M} \int_0^T q_i^2(t)\, dt + \sum_{i=1}^{M} \sum_{\substack{j=1\\ j\ne i}}^{M} C_j \int_0^T \int_0^t \Phi(z_i, t; z_j, \tau)\, q_j(\tau)\, q_i(t)\, d\tau dt = \sum_{i=1}^{M} \int_0^T f_i(t)\, q_i(t)\, dt. \tag{4.52}$$

Note that

$$\begin{aligned} &\int_0^T \int_0^t |\Phi(z_i, t; z_j, \tau)\, q_j(\tau)\, q_i(t)|\, d\tau dt \\ &\le \int_0^T \left(\int_0^t |\Phi(z_i, t; z_j, \tau)|^2\, d\tau\right)^{1/2} \|q_j\|_{L^2(0,T)}\, |q_i(t)|\, dt \\ &\le \left(\int_0^T \int_0^t |\Phi(z_i, t; z_j, \tau)|^2\, d\tau dt\right)^{1/2} \|q_j\|_{L^2(0,T)}\, \|q_i\|_{L^2(0,T)}. \end{aligned} \tag{4.53}$$

Using Lemma 4.1, we deduce from (4.52) that

$$\sum_{i=1}^{M} \|q_i\|^2_{L^2(0,\,T)} - C \sum_{i=1}^{M} \sum_{\substack{j=1\\ j\neq i}}^{M} |z_i - z_j|^{-2} \, \|q_i\|^2_{L^2(0,\,T)}$$

$$\leq \left(\sum_{i=1}^{M} \|f_i\|^2_{L^2(0,\,T)} \right)^{1/2} \left(\sum_{i=1}^{M} \|q_i\|^2_{L^2(0,\,T)} \right)^{1/2} . \tag{4.54}$$

Then we have

$$\left(1 - C \max_i \sum_{\substack{j=1\\ j\neq i}}^{M} |z_i - z_j|^{-2} \right) \sum_{i=1}^{M} \|q_i\|^2_{L^2(0,\,T)}$$

$$\leq \left(\sum_{i=1}^{M} \|f_i\|^2_{L^2(0,\,T)} \right)^{1/2} \left(\sum_{i=1}^{M} \|q_i\|^2_{L^2(0,\,T)} \right)^{1/2}$$

and, therefore,

$$\left(1 - C \max_i \sum_{\substack{j=1\\ j\neq i}}^{M} |z_i - z_j|^{-2} \right) \left(\sum_{i=1}^{M} \|q_i\|^2_{L^2(0,\,T)} \right)^{1/2} \leq \left(\sum_{i=1}^{M} \|f_i\|^2_{L^2(0,\,T)} \right)^{1/2} . \tag{4.55}$$

Thus, the unique solvability of (4.49) and the estimate (4.51) follow from the condition (4.50). The proof is now complete. □

4.4. End of the proof of Theorem 1.2. In the following, we derive the asymptotic formula for the solution to (1.1) in the case of multiple cavities. For $x \in \partial D_i$ with $i \neq j$, using Taylor's expansion and (4.9), we have

$$\left| \int_0^t \int_{\partial D_j} \left[\Phi(x,\, t;\, y,\, \tau) - \Phi(x,\, t;\, z_j,\, \tau) \right] \, \sigma_j(y,\, \tau) \, ds(y) d\tau \right|$$

$$\leq \left(\int_0^t \int_{\partial D_j} |\nabla_y \Phi(x,\, t;\, z_j^*,\, \tau)|^2 \, |y - z_j|^2 \, ds(y) d\tau \right)^{1/2}$$

$$\times \left(\int_0^t \int_{\partial D_j} \sigma_j^2(y,\, \tau) \, ds(y) d\tau \right)^{1/2}$$

$$\lesssim \varepsilon^2 \, d_{ij}^{-3} \, \|\sigma_j\|_{L^2((\partial D_j)_T)} \tag{4.56}$$

and

$$\left| \int_0^t \left[\Phi(x,\, t;\, z_j,\, \tau) - \Phi(z_i,\, t;\, z_j,\, \tau) \right] \left(\int_{\partial D_j} \sigma_j(y,\, \tau) \, ds(y) \right) d\tau \right|$$

$$\leq \left(\int_0^t |\nabla_x \Phi(z_i^*,\, t;\, z_j,\, \tau)|^2 \, |x - z_i|^2 \, d\tau \right)^{1/2}$$

$$\times \left[\int_0^t \left(\int_{\partial D_j} \sigma_j(y,\, \tau) \, ds(y) \right)^2 d\tau \right]^{1/2}$$

$$\lesssim \varepsilon^2 \, d_{ij}^{-3} \, \|\sigma_j\|_{L^2((\partial D_j)_T)}, \tag{4.57}$$

where $z_j^* = z_j + \theta_1 (y - z_j)$, $y \in D_j$, $0 < \theta_1 < 1$ and $z_i^* = z_i + \theta_2 (x - z_i)$, $x \in D_i, 0 < \theta_2 < 1$. Then we derive that

$$\begin{aligned}
&\int_0^t \int_{\partial D_j} \Phi(x, t; y, \tau) \, \sigma_j(y, \tau) \, ds(y) d\tau \\
&= \int_0^t \Phi(x, t; z_j, \tau) \left(\int_{\partial D_j} \sigma_j(y, \tau) \, ds(y) \right) d\tau \\
&\quad + \int_0^t \int_{\partial D_j} [\Phi(x, t; y, \tau) - \Phi(x, t; z_j, \tau)] \; \sigma_j(y, \tau) \, ds(y) d\tau \\
&= \int_0^t \Phi(x, t; z_j, \tau) \left(\int_{\partial D_j} \sigma_j(y, \tau) \, ds(y) \right) d\tau + O(d_{ij}^{-3} \, \varepsilon^2) \|\sigma_j\|_{L^2((\partial D_j)_T)} \\
&= \int_0^t \Phi(z_i, t; z_j, \tau) \left(\int_{\partial D_j} \sigma_j(y, \tau) \, ds(y) \right) d\tau \\
&\quad + \int_0^t [\Phi(x, t; z_j, \tau) - \Phi(z_i, t; z_j, \tau)] \left(\int_{\partial D_j} \sigma_j(y, \tau) \, ds(y) \right) d\tau \\
&\quad + O(d_{ij}^{-3} \varepsilon^2) \, \|\sigma_j\|_{L^2((\partial D_j)_T)} \\
(4.58) &= \int_0^t \Phi(z_i, t; z_j, \tau) \left(\int_{\partial D_j} \sigma_j(y, \tau) \, ds(y) \right) d\tau + O(d_{ij}^{-3} \, \varepsilon^2) \, \|\sigma_j\|_{L^2((\partial D_j)_T)}
\end{aligned}$$

for $(x, t) \in (\partial D_i)_T$. Hence, the boundary integral system (4.2) can be rewritten as

$$\begin{aligned}
&\int_0^t \int_{\partial D_i} \Phi(x, t; y, \tau) \, \sigma_i(y, \tau) \, ds(y) d\tau \\
&\quad + \sum_{\substack{j=1 \\ j \neq i}}^{M} \int_0^t \Phi(z_i, t; z_j, \tau) \left(\int_{\partial D_j} \sigma_j(y, \tau) \, ds(y) \right) d\tau \\
&= \Phi(z_i, t; z^*, 0) + O\left(\sum_{\substack{j=1 \\ j \neq i}}^{M} d_{ij}^{-3} \, \varepsilon^2 \right) + O(\varepsilon), \quad (x, t) \in (\partial D_i)_T \qquad (4.59)
\end{aligned}$$

for $i = 1, 2, \ldots, M$.

Recall that

$$\varphi_i(x, y, t) = \int_0^t \frac{|x - y|}{2\sqrt{\pi}(t - \tau)^{3/2}} \, \exp\left(-\frac{|x - y|^2}{4(t - \tau)} \right) \, \sigma_i(y, \tau) \, d\tau$$

for $x \in \mathbb{R}^3$, $y \in \partial D_i$, $t \in (0, T)$. We note that the result of Lemma 3.2 is true for any

D_i in the multiple cavities case. Then, we have

$$\begin{aligned}
&\int_0^t \int_{\partial D_i} \Phi(x,\,t;\,y,\,\tau)\,\sigma_i(y,\,\tau)\,ds(y)d\tau \\
&= \int_{\partial D_i} \frac{1}{4\pi|x-y|}\,\sigma_i(y,\,t)\,ds(y) \\
&\qquad + \int_{\partial D_i} \frac{1}{4\pi|x-y|}\,[\varphi_i(x,\,y,\,t) - \sigma_i(y,\,t)]\,ds(y) \\
&= \int_{\partial D_i} \frac{1}{4\pi|x-y|}\,\sigma_i(y,\,t)\,ds(y) + O\left(\int_\partial D_i \|\partial_t \sigma_i\|_{L^2(0,\,t)}\right) \\
&= \int_{\partial D_i} \frac{1}{4\pi|x-y|}\,\sigma_i(y,\,t)\,ds(y) + O\left(\varepsilon\right), \quad (x,\,t) \in (\partial D_i)_T.
\end{aligned} \tag{4.60}$$

It follows from (4.59) and (4.60) that

$$\begin{aligned}
&\int_{\partial D_i} \frac{1}{4\pi|x-y|}\sigma_i(y,\,t)\,ds(y) \\
&\qquad + \sum_{\substack{j=1\\ j\neq i}}^M \int_0^t \Phi(z_i,\,t;\,z_j,\,\tau)\,\left(\int_{\partial D_j} \sigma_j(y,\,\tau)\,ds(y)\right) d\tau \\
&= \Phi(z_i,\,t;\,z^*,\,0) + O\left(\sum_{\substack{j=1\\ j\neq i}}^M d_{ij}^{-3}\,\varepsilon^2\right) + O(\varepsilon), \quad (x,\,t) \in (\partial D_i)_T.
\end{aligned} \tag{4.61}$$

Hence, we deduce that

$$\begin{aligned}
&\int_{\partial D_i} \sigma_i(y,\,t)\,ds(y) + \left(\int_{\partial D_i} S^{-1}_{\partial D_i}[1](y)\,ds(y)\right) \\
&\qquad \times \sum_{\substack{j=1\\ j\neq i}}^M \int_0^t \Phi(z_i,\,t;\,z_j,\,\tau)\left(\int_{\partial D_j} \sigma_j(y,\,\tau)\,ds(y)\right) d\tau \\
&= \Phi(z_i,\,t;\,z^*,\,0) \int_{\partial D_i} S^{-1}_{\partial D_i}[1](y)\,ds(y) \\
&\qquad + \tilde{E}_i \int_{\partial D_i} S^{-1}_{\partial D_i}[1](y)\,ds(y), \quad t \in (0,\,T),
\end{aligned} \tag{4.62}$$

where

$$\tilde{E}_i := O\left(\sum_{\substack{j=1\\ j\neq i}}^M d_{ij}^{-3}\,\varepsilon^2\right) + O(\varepsilon) = O\left(\frac{\varepsilon^2\,|\ln\varepsilon|}{d^3}\right) + O(\varepsilon) = O\left(\varepsilon^{2-3\beta}\,|\ln\varepsilon|\right) + O(\varepsilon).$$

Then we have the following system:

$$\begin{aligned}
&\frac{q_i(t)}{C_i} + \sum_{\substack{j=1\\ j\neq i}}^M \int_0^t C_j\,\Phi(z_i,\,t;\,z_j,\,\tau)\,\left(\frac{q_j(\tau)}{C_j}\right) d\tau \\
&= \Phi(z_i,\,t;\,z^*,\,0) + O\left(\varepsilon^{2-3\beta}\,|\ln\varepsilon|\right) + O(\varepsilon),\ t \in (0,\,T),\ i = 1,\,2,\,\ldots,\,M.
\end{aligned} \tag{4.63}$$

We are now in a position to state our main result of this section.

THEOREM 4.7. *For $x \in \mathbb{R}^3 \setminus \overline{D}$ and $t \in (0, T)$, the solution to (1.1) has the following asymptotic expansion:*

(4.64)
$$u(x, t) = \sum_{i=1}^{M} C_i \int_0^t \Phi(x, t; z_i, \tau)\, \alpha_i(\tau)\, d\tau + O(\varepsilon^{3-3\beta-s}|\ln \varepsilon|) + O(\varepsilon^{2-s}) \quad \text{as } \varepsilon \to 0$$

under the following condition on the distribution of the cavities:

$$\varepsilon \max_{1\le i\le M} \sum_{j\neq i} d_{ij}^{-2} < 1 \tag{4.65}$$

or $1 - 2\beta - \frac{s}{3} \geq 0$, where the constant C_i's are defined by (4.48) and $\{\alpha_i(t)\}_{i=1}^M$ is the unique solution of the linear system

(4.66)
$$\alpha_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{M} \int_0^t C_j\, \Phi(z_i, t; z_j, \tau)\, \alpha_j(\tau)\, d\tau = \Phi(z_i, t; z^*, 0),\ t \in (0, T),\ i = 1, 2, \ldots, M.$$

Proof. Define

$$w_i(t) := q_i(t)/C_i - \alpha_i(t), \quad t \in (0, T),\ i = 1, 2, \ldots, M.$$

Then, by (4.63) and (4.66), we have

(4.67)
$$w_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{M} \int_0^t C_j\, \Phi(z_i, t; z_j, \tau)\, w_j(\tau)\, d\tau = O\left(\varepsilon^{2-3\beta}\, |\ln \varepsilon|\right) + O(\varepsilon) := \overline{E}_i,\ i = 1, 2, \ldots, M.$$

From Theorem 4.6, we obtain that

$$\sum_{i=1}^{M} \|w_i\|^2_{L^2(0,T)} \lesssim \sum_{i=1}^{M} \|\overline{E}_i\|^2_{L^2(0,T)} =: E_1 \sim M(\varepsilon^{2-3\beta}\, |\ln \varepsilon| + \varepsilon)^2. \tag{4.68}$$

Thus, we derive for $x \in \mathbb{R}^3 \setminus \overline{D}$ and $t \in (0, T)$ that

$$\begin{aligned}
u(x, t) &= \sum_{i=1}^{M} \int_0^t \int_{\partial D_i} \Phi(x, t; y, \tau)\, \sigma_i(y, \tau)\, ds(y) d\tau \\
&= \sum_{i=1}^{M} \int_0^t \int_{\partial D_i} \Phi(x, t; z_i, \tau)\, \sigma_i(y, \tau)\, ds(y) d\tau \\
&\quad + \sum_{i=1}^{M} \int_0^t \int_{\partial D_i} [\Phi(x, t; y, \tau) - \Phi(x, t; z_i, \tau)]\ \sigma_i(y, \tau)\, ds(y) d\tau \\
&= \sum_{i=1}^{M} \int_0^t \Phi(x, t; z_i, \tau)\, q_i(\tau)\, d\tau + O(\varepsilon^2) \sum_{i=1}^{M} \|\sigma_i\|_{L^2(\partial D_i)_T} \\
&= \sum_{i=1}^{M} C_i \int_0^t \Phi(x, t; z_i, \tau)\, \alpha_i(\tau)\, d\tau + O(C(ME_1)^{\frac{1}{2}}) + O(M\varepsilon^2)
\end{aligned}$$

$$(4.69)\qquad = \sum_{i=1}^{M} C_i \int_0^t \Phi(x,\, t;\, z_i,\, \tau)\, \alpha_i(\tau)\, d\tau + O(\varepsilon^{3-3\beta-s}|\ln\varepsilon|) + O(\varepsilon^{2-s})$$

for $M \sim \varepsilon^{-s},\ d \sim \varepsilon^{\beta}$. This completes the proof. ◻

5. Proof of Theorem 1.3. Let Ω be a bounded domain containing the cavities D_j, $j = 1, 2, \ldots, M$. We divide Ω into $[a^{-1}]$ periodical subdomains Ω_j, $j = 1, 2, \ldots, [a^{-1}]$ such that Ω_j's are disjoint and each Ω_j contains one single cavity D_j and has a volume a. We also assume that the cavities D_j, $j = 1, 2, \ldots, M$, have the same shape. This means that $C_i = C_j$ for $i, j = 1, 2, \ldots, M$. Define

$$C := C_j = \overline{C}\, a,$$

where $\overline{C}$ is the scaled value of the heat capacitance C_j defined by (4.48) for each cavity D_j.

As Ω can have an arbitrary shape, the set of the cubes intersecting $\partial\Omega$ is not empty (unless if Ω has a simple shape as a cube). Later in our analysis, we will need the estimate of the volume of this set. Since each Ω_j has volume of the order a, and then its maximum radius is of the order $a^{\frac{1}{3}}$, then the intersecting surfaces with $\partial\Omega$ have an area of the order $a^{\frac{2}{3}}$. As the area of $\partial\Omega$ is of the order one, we conclude that the number of such cubes will not exceed the order $a^{-\frac{2}{3}}$. Hence the volume of this set will not exceed the order $a^{-\frac{2}{3}}a = a^{\frac{1}{3}}$, as $a \to 0$.

We consider the integral equation

$$(5.1)\qquad v(x,\, t) + \int_0^t \int_\Omega \overline{C}\, \Phi(x,\, t;\, z,\, \tau)\, v(z,\, \tau)\, dz d\tau = \Phi(x,\, t;\, z^*,\, 0), \quad (x,\, t) \in \Omega_T,\ z^* \notin \overline{\Omega}.$$

The unique solvability can be proved as follows.

Lemma 5.1. *The integral equation* (5.1) *is uniquely solvable in* $L^2(\Omega_T)$.

Proof. Due to the estimate (2.1), the volume potential operator $\mathcal{V}$ defined by

$$\mathcal{V}[\varphi](x,\, t) := \int_0^t \int_\Omega \Phi(x,\, t;\, z,\, \tau)\, \varphi(z,\, \tau)\, dz d\tau, \quad (x,\, t) \in \Omega_T,$$

has a weakly singular kernel and is bounded from $L^2(\Omega_T)$ into $L^2(\Omega_T)$. We note that the operator norm of $\mathcal{V}$ goes to zero as $T \to 0$. Thus the inverse of $I + \mathcal{V}$ can be written as the series $\sum_{j=0}^{\infty}(-\mathcal{V})^j$ for small T. The result for large T follows by an iteration argument. ◻

Define

$$(5.2)\quad V(x,\, t) := \begin{cases} v(x,\, t) & \text{in } \Omega_T, \\ \Phi(x,\, t;\, z^*,\, 0) - \int_0^t \int_\Omega \overline{C}\, \Phi(x,\, t;\, z,\, \tau)\, v(z,\, \tau)\, dz d\tau & \text{in } (\mathbb{R}^3 \setminus \overline{\Omega})_T. \end{cases}$$

Then we have the following result.

Lemma 5.2. *The function* v *is the solution of* (5.1) *if and only if* V *is a solution of* $(\partial_t - \Delta + \overline{C}\chi_\Omega)V = \delta(x - z^*)\, \delta(t)$ *in* $\mathbb{R}^3$.

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Set

$$W(x,\, t) := \Phi(x,\, t;\, z^*,\, 0) - V(x,\, t), \quad x \in \mathbb{R}^3,\, t \in (0,\, T).$$

We obtain that

$$\begin{cases} (\partial_t - \Delta + \overline{C}\chi_\Omega)W = \overline{C}\chi_\Omega \Phi(x,\, t;\, z^*,\, 0) & \text{in } \mathbb{R}^3 \times (0,\, T), \\ W(x,\, 0) = 0 & \text{in } \mathbb{R}^3, \\ |W(x,\, t)| \leq C_0 \exp(b|x|^2) & \text{as } |x| \to +\infty. \end{cases} \tag{5.3}$$

As z^* is outside Ω, the right-hand side of the first equation in (5.3) is smooth. Then the solution W is in $C\left([0,\, T];\, H^1_{loc}(\mathbb{R}^3)\right)$; see [26] for instance. By Sobolev embedding, we deduce that $W \in C\left([0,\, T];\, L^p(\Omega)\right)$ for any $p < 3$.

From (5.1), we see

$$|v(x,\, t)| \lesssim \int_0^t \int_\Omega |\Phi(x,\, t;\, z,\, \tau)||v(z, \tau)|\, dzd\tau + |\Phi(x,\, t;\, z^*,\, 0)|, \quad (x,\, t) \in \Omega_T,$$

and hence

$$|v(x,\, t)| \lesssim \left(\int_0^t \int_\Omega |\Phi(x,\, t;\, z,\, \tau)|^q\, dzd\tau\right)^{\frac{1}{q}} \|v\|_{C([0,\, T];\, L^p(\Omega))} + O(1) \quad \text{ with } \frac{1}{p} + \frac{1}{q} = 1.$$

By the singularity estimate (2.1), we have

$$|\Phi(x,\, t;\, z,\, \tau)|^q \lesssim \frac{1}{(t-\tau)^{\mu q}} \frac{1}{|x-z|^{(3-2\mu)q}}, \quad 0 \leq \tau < t \leq T,\ x,\, z \in \mathbb{R}^3, \text{ with } x \neq z,$$

and this function is integrable in $\Omega \times (0,\, T)$ if $\mu q < 1$ and $(3-2\mu)q < 3$. As $p < 3$, then $q > \frac{3}{2}$. Choosing μ smaller but near $\frac{2}{3}$, then these two conditions on q are satisfied. Hence v is in $C\left([0,\, T];\, L^\infty(\Omega)\right)$. In addition, from (5.1), we get

$$\partial_{x_j} v(x,\, t) = -\int_0^t \int_\Omega \overline{C}\, \partial_{x_j}\Phi(x,\, t;\, z,\, \tau)\, v(z,\, \tau)\, dzd\tau + \partial_{x_j}\Phi(x,\, t;\, z^*,\, 0) \tag{5.4}$$

for $(x,\, t) \in \Omega_T,\ z^* \notin \Omega$ and, then,

$$|\partial_{x_j} v(x,\, t)| \lesssim \left(\int_0^t \int_\Omega |\partial_{x_j}\Phi(x,\, t;\, z,\, \tau)|\, dzd\tau\right) \|v\|_{L^\infty((0,\, T);\, L^\infty(\Omega))} + |\partial_{x_j}\Phi(x,\, t;\, z^*,\, 0)|. \tag{5.5}$$

By the singularity estimate (2.2) with $\mu > 1/2$, we see

$$\int_0^t \int_\Omega |\partial_{x_j}\Phi(x,\, t;\, z,\, \tau)|\, dz = O(1), \quad (x,\, t) \in \Omega_T.$$

This means that $\partial_{x_j} v \in C\left([0,\, T];\, L^\infty(\Omega)\right)$. So we obtain that $v \in C\left([0,\, T];\, W^{1,\infty}(\Omega)\right)$.

We now rewrite the integral equation (5.1) at $x = z_l$ for $1 \leq l \leq M$ as

$$v(z_l,\, t) + \int_0^t \sum_{\substack{j=1 \\ j \neq l}}^{[a^{-1}]} \int_{\Omega_j} \overline{C}\, \Phi(z_l,\, t;\, z,\, \tau)\, v(z,\, \tau)\, dzd\tau = \Phi(z_l,\, t;\, z^*,\, 0) + \mathcal{A} + \mathcal{A}_l, \quad t \in (0,\, T), \tag{5.6}$$

or

$$v(z_l,\,t)+\sum_{\substack{j=1\\ j\neq l}}^{M}\overline{C}\,a\int_0^t\Phi(z_l,\,t;\,z_j,\,\tau)\,v(z_j,\,\tau)\,dzd\tau \tag{5.7}$$
$$=\Phi(z_l,\,t;\,z^*,\,0)+\mathcal{A}+\mathcal{A}_l+\mathcal{B}_l,\quad t\in(0,\,T),$$

with

$$\mathcal{A}:=-\int_0^t\int_{\Omega\setminus(\cup_{j=1}^{[a^{-1}]}\Omega_j)}\overline{C}\,\Phi(z_l,\,t;\,z,\,\tau)\,v(z,\,\tau)\,dzd\tau,$$
$$\mathcal{A}_l:=-\int_0^t\int_{\Omega_l}\overline{C}\,\Phi(z_l,\,t;\,z,\,\tau)\,v(z,\,\tau)\,dzd\tau,$$
$$\mathcal{B}_l:=-\sum_{\substack{j=1\\ j\neq l}}^{[a^{-1}]}\overline{C}\int_0^t\int_{\Omega_j}\Phi(z_l,\,t;\,z,\,\tau)\,v(z,\,\tau)\,dzd\tau+\sum_{\substack{j=1\\ j\neq l}}^{M}\overline{C}\,a\int_0^t\Phi(z_l,\,t;\,z_j,\,\tau)\,v(z_j,\,\tau)\,d\tau.$$

As $v\in L^\infty(\Omega_T)$, then by (2.1), we have

$$\mathcal{A}_l\sim\int_0^t\int_{\Omega_l}|\Phi(z_l,\,t;\,z,\,\tau)|\,dzd\tau=O\left(\int_{\Omega_l}|z-z_l|^{2\mu-3}\,dz\right)$$

for $0<\mu<1$ and, hence, by a scaling, we derive the estimate

$$\mathcal{A}_l=O\left(\varepsilon^{\frac{2\mu}{3}}\right)\quad\text{as }\varepsilon\ll 1. \tag{5.8}$$

Let us estimate $\mathcal{B}_l$. As $|\Omega_l|=a$, we have

$$\mathcal{B}_l=-\sum_{\substack{j=1\\ j\neq l}}^{[a^{-1}]}\overline{C}\int_0^t\int_{\Omega_j}\left[\Phi(z_l,\,t;\,z,\,\tau)\,v(z,\,\tau)-\Phi(z_l,\,t;\,z_j,\,\tau)\,v(z_j,\,\tau)\right]\,dzd\tau.$$

We write the above integrand as

$$\Phi(z_l,\,t;\,z,\,\tau)\,v(z,\,\tau)-\Phi(z_l,\,t;\,z_j,\,\tau)\,v(z_j,\,\tau)$$
$$=[\Phi(z_l,\,t;\,z,\,\tau)-\Phi(z_l,\,t;\,z_j,\,\tau)]v(z,\,\tau)+\Phi(z_l,\,t;\,z_j,\,\tau)[v(z,\,\tau)-v(z_j,\,\tau)].$$

Then we see

$$\mathcal{B}_l=O\left(\sum_{\substack{j=1\\ j\neq l}}^{[a^{-1}]}\int_0^t\int_{\Omega_j}\left[|\nabla_z\Phi(z_l,\,t;\,z_j,\,\tau)|\,|z-z_j|+|\Phi(z_l,\,t;\,z_j,\,\tau)|\,|z-z_j|\right]\,dzd\tau\right).$$

But

$$\int_{\Omega_j}|\Phi(z_l,\,t;\,z_j,\,\tau)|\,|z-z_j|\,dz=O\left((t-\tau)^{-\mu}\,|z_l-z_j|^{-3+2\mu}\right)\int_{\Omega_j}|z-z_j|\,dz$$
$$=O\left((t-\tau)^{-\mu}\,|z_l-z_j|^{-3+2\mu}\right)a^{\frac{4}{3}}$$

with $0 < \mu < 1$ and, similarly,

$$\int_{\Omega_j} |\nabla_z \Phi(z_l,\, t;\, z_j,\, \tau)|\, |z - z_j|\, dz = O\left((t-\tau)^{-\gamma}\, |z_l - z_j|^{-4+2\gamma}\right)\, a^{\frac{4}{3}}$$

with $0 < \gamma < 1$. Hence, choosing $\mu = \gamma \in [1/2,\, 1)$, we get

$$\mathcal{B}_l = O\left(\sum_{\substack{j=1\\ j\neq l}}^{[a^{-1}]} |z_l - z_j|^{-4+2\gamma}\right)\, a^{\frac{4}{3}} = O\left(d^{-3}\, a^{\frac{4}{3}}\right) = O\left(a^{\frac{1}{3}}\right). \tag{5.9}$$

To estimate the term $\mathcal{A}$, following [3, 16], we distinguish between the following two cases:

(a) The point z_m is away from the boundary $\partial\Omega$ and so $\Phi(z_m,\, t;\, z,\, \tau)$ is bounded in z near the boundary.

(b) The point z_m is located near one of the Ω_j's touching the boundary $\partial\Omega$. In this case, we split the estimate into two parts. By N_m we denote the part that involves Ω_j's close to z_m, and we denote the remaining part by F_m. The integral over F_m can be estimated in a manner similar to the case (a) discussed above. Also note that $F_m \subset \Omega \setminus \cup_{j=1}^{[a^{-1}]}\Omega_j$ and so $\mathrm{Vol}\,(F_m)$ is of the order $a^{\frac{1}{3}}$ as $a \to 0$.
To estimate the integral over N_m, we observe that owing to the fact a is small, the Ω_j's close to z_m are located near a small region of the boundary $\partial\Omega$. Since we assume that the boundary is smooth enough, this region can be assumed to be flat. We now divide this layer into concentric layers as in the estimate of $\mathcal{B}_l$. In this case, we have at most $(2n+1)^2$ cubes intersecting the surface for $n = 0,\, \ldots,\, [a^{-\frac{1}{3}}]$. So the number of cavities in the nth layer $(n \neq 0)$ will be at most $[(2n+1)^2 - (2n-1)^2]$ and their distance from Ω_m is at least $n(a^{\frac{1}{3}} - \frac{a}{2})$.

Similarly to Lemma 4.2, we have $\int_0^t \Phi(x,\, t;\, z,\, \tau)\, d\tau = O(|x-z|^{-1})$. Therefore we can write

$$\begin{aligned}
|\mathcal{A}| &= \left|\int_0^t \int_{\Omega\setminus(\cup_{j=1}^{[a^{-1}]}\Omega_j)} \overline{C}\, \Phi(z_m,\, t;\, z,\, \tau)\, v(z,\, \tau)\, dz d\tau\right| \\
&= \left|\int_0^t \int_{N_m} \overline{C}\, \Phi(z_m,\, t;\, z,\, \tau)\, v(z,\, \tau)\, dz d\tau\right| + \left|\int_0^t \int_{F_m} \overline{C}\, \Phi(z_m,\, t;\, z,\, \tau)\, v(z,\, \tau)\, dz d\tau\right| \\
&\leq \sum_{l=1}^{[a^{-\frac{2}{3}}]} \overline{C}\, \|v\|_{L^\infty(\Omega_T)}\, \mathrm{Vol}\,(\Omega_l)\, \frac{1}{d_{ml}} + \overline{C}\, \|\Phi(z_m,\, t;\, \cdot,\, \tau)\,\|_{L^\infty((F_m)_T)}\, \|v\|_{L^\infty(\Omega_T)}\, \mathrm{Vol}\,(F_m) \\
&\leq O\left(a \sum_{l=1}^{[a^{-\frac{1}{3}}]} \frac{1}{d_{ml}} + \tilde{C}\, a^{\frac{1}{3}}\right) \\
&\leq O\left(a \sum_{l=1}^{[a^{-\frac{1}{3}}]} \left[(2n+1)^2 - (2n-1)^2\right] \frac{1}{n\left(a^{\frac{1}{3}} - \frac{a}{2}\right)} + \tilde{C}\, a^{\frac{1}{3}}\right) \\
&= O\Big(a\, O(a^{-\frac{2}{3}}) + O(a^{\frac{1}{3}})\Big)
\end{aligned}$$

and, hence,

$$|\mathcal{A}| = O\left(a^{\frac{1}{3}}\right). \tag{5.10}$$

Gathering the estimates (5.8), (5.9), and (5.10), and taking $\mu = 1/2$, we have

$$\sum_l \left(|\mathcal{A}|^2 + |\mathcal{A}_l|^2 + |\mathcal{B}_l|^2\right) = O\left(Ma^{\frac{2}{3}} + Ma^{\frac{4\mu}{3}}\right) = O\left(a^{-\frac{1}{3}}\right).$$

Using the invertibility property and the estimate (4.51) for the algebraic system (4.49), we deduce the following estimate:

$$\sum_{j=1}^{M} \|\alpha_j(t) - v(z_j,\, t)\|^2_{L^2(0,\,T)} = O(a^{-\frac{1}{3}}) \quad \text{as } a \to 0. \tag{5.11}$$

Let x be away from $\Omega \cup \{z^*\}$. Then we recall that

$$W(x,\, t) := \Phi(x,\, t;\, z^*,\, 0) - V(x,\, t) = \int_0^t \int_\Omega \overline{C}\, \Phi(x,\, t;\, z,\, \tau)\, v(z,\, \tau)\, dz d\tau,$$

and rewrite it as

$$W(x,\, t) = \sum_{j=1}^{[a^{-1}]} |\Omega_j| \int_0^t \overline{C}\, \Phi(x,\, t;\, z_j,\, \tau)\, v(z_j,\, \tau)\, d\tau + \mathcal{C}$$

with

$$\mathcal{C} = \sum_{j=1}^{[a^{-1}]} \int_0^t \int_{\Omega_j} \overline{C}\, \left[\Phi(x,\, t;\, z,\, \tau)\, v(z,\, \tau) - |\Omega_j|\, \Phi(x,\, t;\, z_j,\, \tau)\, v(z_j,\, \tau)\right]\, dz d\tau.$$

Following similar steps as for estimating $\mathcal{B}_l$, and as the integrands are smooth here, it can be proved that $\mathcal{C} = o(\varepsilon^{\frac{1}{3}})$ as $\varepsilon \to 0$. Then we have

$$W(x,\, t) = \sum_{j=1}^{[a^{-1}]} \overline{C}\, |\Omega_j| \int_0^t \Phi(x,\, t;\, z_j,\, \tau)\, \alpha_j(\tau)\, d\tau + \mathcal{D} + o(\varepsilon^{\frac{1}{3}})$$

with

$$\mathcal{D} := -\sum_{j=1}^{[a^{-1}]} \overline{C}\, |\Omega_j| \int_0^t \Phi(x,\, t;\, z_j,\, \tau)\, [\alpha_j(\tau) - v(z_j,\, \tau)]\, d\tau.$$

The term $\mathcal{D}$ can be estimated as

$$\begin{aligned}\mathcal{D} &= O\left(\sum_{j=1}^{[a^{-1}]} |\Omega_j| \int_0^t |\Phi(x,\, t;\, z_j,\, \tau)|\, |\alpha_j(\tau) - v(z_j,\, \tau)|\, d\tau\right)\\ &= O\left(a \sum_{j=1}^{[a^{-1}]} \left(\int_0^t |\Phi(x,\, t;\, z_j,\, \tau)|^2\, d\tau\right)^{1/2} \left(\int_0^t |\alpha_j(\tau) - v(z_j,\, \tau)|^2\, d\tau\right)^{1/2}\right),\end{aligned}$$

and then

$$\mathcal{D} = O\left(a\, M^{1/2} a^{-1/6}\right) = O\left(a^{\frac{1}{3}}\right).$$

Hence, we conclude that

$$W(x,\, t) = u(x,\, t) + O\left(a^{\frac{1}{3}}\right), \quad x \in \mathbb{R}^3 \setminus \left\{\overline{\Omega} \cup \{z^*\}\right\} \text{ as } a \to 0. \tag{5.12}$$

∎

6. Conclusions. In this work, we studied a transient heat conduction problem in an infinite domain containing a cluster of small cavities. Under the specified geometrical constraints on the size a and the distance d_{ij} between the cavities, we developed an asymptotic expansion of the temperature field as $a \to 0$. Explicitly, the temperature field is approximately described as a linear combination of fields involving the heat capacitance of each cavity and the convolution of the fundamental solution with functions that are determined from a linear algebraic system. Based on the boundary integral equation approach, we gave a rigorous justification of the asymptotic expansion. As an application of the expansion, in the limit case when the cavities are densely distributed and occupy a bounded domain, we derived an effective conductivity medium that generates approximately the same temperature field as the cluster.

Our work provides an insight into tackling time-dependent models with many small defects. We can handle more general initial boundary value problems of second-order parabolic operators. For instance, we can handle the initial boundary value problem for the operator $\partial_t - \nabla \cdot \gamma(x)\nabla + q(x)$, since under certain conditions the fundamental solution of this operator exists and has the same singularity estimates as that for $\partial_t - \Delta$. In addition, we frequently encounter the heat conduction problem in a bounded thermal conductor containing small inclusions, which is modeled by a transmission problem for the heat equation. By the argument developed in this paper, we expect to derive an asymptotic expansion for this transmission problem, which is already proved in [5], for fixed and well separated small inclusions, by using a variational method. Based on the asymptotic expansion, we can design a direct imaging method for locating small inclusions in the thermal conductors from boundary measurements allowing them to be very close (and large in number). Moreover, and most importantly, the expected asymptotic approximation can be used for heat generation (for the purpose of heat therapy) if we choose the small particles to be resonating. These issues will be reported in forthcoming works.

Appendix A. Proof of Lemma 3.1.

Proof. Let

$$\delta = \frac{|x-y|}{2\sqrt{t-\tau}}. \tag{A.1}$$

The direct calculations give

$$\begin{aligned}
\varphi(x,\, y,\, t) \;&=\; \int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty} \frac{2}{\sqrt{\pi}}\, \exp\left(-\delta^2\right)\, \sigma\left(y,\, t - \frac{|x-y|^2}{4\delta^2}\right) d\delta \\
&=\; \frac{2}{\sqrt{\pi}} \int_{\frac{|x-y|}{2\sqrt{t}}}^{\sqrt{|x-y|}} \exp\left(-\delta^2\right)\, \sigma\left(y,\, t - \frac{|x-y|^2}{4\delta^2}\right) d\delta \\
&\quad+\; \frac{2}{\sqrt{\pi}} \int_{\sqrt{|x-y|}}^{+\infty} \exp\left(-\delta^2\right) \left[\sigma\left(y,\, t - \frac{|x-y|^2}{4\delta^2}\right) - \sigma(y,\, t)\right] d\delta \\
&\quad+\; \frac{2\sigma(y,\, t)}{\sqrt{\pi}} \int_{\sqrt{|x-y|}}^{+\infty} \exp(-\delta^2)\, d\delta \\
&=:\; I_1 + I_2 + I_3.
\end{aligned} \tag{A.2}$$

Clearly, we have

$$\lim_{x \to y} I_1(x,\, y,\, t) = 0 \tag{A.3}$$

uniformly on ∂D and on compact subintervals of $(0,\, T]$. And also, we can easily see that

$$\lim_{x \to y} I_3(x,\, y,\, t) = \sigma(y,\, t) \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \exp\left(-\delta^2\right)\, d\delta = \sigma(y,\, t) \tag{A.4}$$

uniformly on $\partial D \times (0,\, T]$.

Let us consider $I_2(x,\, y,\, t)$. Since $\sigma(y,\, s)$ is continuous on $\partial D \times (0,\, T]$ for any $\epsilon > 0$ there exists a positive constant δ_0 such that $|\sigma(y,\, t_1) - \sigma(y,\, t_2)| \le \epsilon$ for all t_1 and t_2 with $|t_1 - t_2| < \delta_0$. Then, for all $|x - y| < 4\delta_0$ and all $\delta \ge \sqrt{|x-y|}$, we have

$$\frac{|x-y|^2}{4\delta^2} \le \frac{|x-y|}{4} < \delta_0$$

and, hence,

$$\left|\sigma\left(y,\, t - \frac{|x-y|^2}{4\delta^2}\right) - \sigma(y,\, t)\right| \le \epsilon.$$

Consequently, we obtain

$$|I_2(x,\, y,\, t)| \le \epsilon \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \exp\left(-\delta^2\right)\, d\delta \le \epsilon,$$

which implies that

$$\lim_{x \to y} I_2(x,\, y,\, t) = 0. \tag{A.5}$$

Combining (A.3), (A.4), and (A.5) yields (3.11). The proof is complete. ∎

Appendix B. Proof of Lemma 3.2.

Proof. We start from the formula

$$\varphi(x,\, y,\, t) = \int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty} \frac{2}{\sqrt{\pi}} \exp(-\delta^2)\, \sigma\left(y,\, t - \frac{|x-y|^2}{4\delta^2}\right) d\delta,$$

which we rewrite as

$$\varphi(x,\, y,\, t) = \int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty} \frac{2}{\sqrt{\pi}} \exp(-\delta^2) \left[\sigma\left(y,\, t - \frac{|x-y|^2}{4\delta^2}\right) - \sigma(y,\, t)\right] d\delta + \sigma(y,\, t) \int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty} \frac{2}{\sqrt{\pi}} \exp(-\delta^2)\, d\delta.$$

Thus, we obtain

$$\varphi(x,\, y,\, t) - \sigma(y,\, t) = \int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty} \frac{2}{\sqrt{\pi}} \exp(-\delta^2) \left[\sigma\left(y,\, t - \frac{|x-y|^2}{4\delta^2}\right) - \sigma(y,\, t)\right] d\delta - \sigma(y,\, t) \int_0^{\frac{|x-y|}{2\sqrt{t}}} \frac{2}{\sqrt{\pi}} \exp(-\delta^2)\, d\delta.$$

As $\sigma(y,\,0) = 0$, then $\sigma(y,\,t) = \int_0^t \partial_t\sigma(y,\,s)\,ds$ and by the Cauchy–Schwartz inequality, we derive the estimate

$$\sigma(y,\,t) = O\left(t^{1/2}\,\|\sigma_t(y,\,\cdot)\|_{L^2(0,\,t)}\right).$$

Similarly, we have

$$\sigma\left(y,\,t - \frac{|x-y|^2}{4\delta^2}\right) - \sigma(y,\,t) = \int_t^{t-\frac{|x-y|^2}{4\delta^2}} \partial_t\sigma(y,\,s)\,ds.$$

Recalling (A.1), we easily see that $\delta \geq \frac{|x-y|}{2\sqrt{t}}$, then $t - \frac{|x-y|^2}{4\delta^2} \geq 0$ and, hence,

$$\int_t^{t-\frac{|x-y|^2}{4\delta^2}} \partial_t\sigma(y,\,s)\,ds = O\left(\frac{|x-y|^2}{4\delta^2}\,\|\partial_t\sigma(y,\,\cdot)\|_{L^2(0,\,t)}\right).$$

This means

$$\begin{aligned}\varphi(x,\,y,\,t) - \sigma(y,\,t) = O\left(\int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty} \frac{\exp(-\delta^2)}{\delta^2}\,d\delta\,\|\partial_t\sigma(y,\,\cdot)\|_{L^2(0,\,t)}\,|x-y|^2\right)\\ + O\left(\int_0^{\frac{|x-y|}{2\sqrt{t}}} \exp(-\delta^2)\,d\delta\,\|\partial_t\sigma(y,\,\cdot)\|_{L^2(0,\,t)}\,t^{1/2}\right).\end{aligned}$$

For any fixed $t \in (0,\,T]$ and $|x-y| \ll 1$, we see that

$$\int_0^{\frac{|x-y|}{2\sqrt{t}}} \exp(-\delta^2)\,d\delta = O\left(\frac{|x-y|}{\sqrt{t}}\right)$$

and

$$\int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty} \frac{\exp(-\delta^2)}{\delta^2}\,d\delta = -\frac{\exp(-\delta^2)}{\delta}\Bigg|_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty} - 2\int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty} \exp(-\delta^2)\,d\delta = O\left(\frac{\sqrt{t}}{|x-y|}\right).$$

Hence, we have

$$\varphi(x,\,y,\,t) - \sigma(y,\,t) = O\left(|x-y|\,\|\partial_t\sigma(y,\,\cdot)\|_{L^2(0,\,t)}\right).$$

The proof is complete. ∎

Appendix C. Proof of Lemma 4.1.

Proof. Let

$$\zeta = |x-y|, \quad \delta = \frac{\zeta}{2\sqrt{t-\tau}}. \tag{C.1}$$

By direct calculations, we have

$$\int_0^T \int_0^T |\Phi(x,\, t;\, y,\, \tau)|^2 \, d\tau dt = \int_0^T \int_0^t \frac{1}{[4\pi(t-\tau)]^3} \exp\left(-\frac{|x-y|^2}{2(t-\tau)}\right) d\tau dt$$

$$= \frac{\zeta^{-4}}{2\pi^3} \int_0^T dt \int_{\frac{\zeta}{2\sqrt{t}}}^{+\infty} \delta^3 \exp(-2\delta^2) \, d\delta$$

$$= \frac{\zeta^{-4}}{16\pi^3} \int_0^T \left[\frac{\zeta^2}{2t} \exp(-\frac{\zeta^2}{2t}) + \exp(-\frac{\zeta^2}{2t}) \right] dt. \tag{C.2}$$

Since

$$\int_0^T \exp(-\frac{\zeta^2}{2t}) \, dt = \left[t \exp(-\frac{\zeta^2}{2t}) \right]_0^T - \int_0^T \frac{\zeta^2}{2t} \exp(-\frac{\zeta^2}{2t}) \, dt$$

$$= T \exp(-\frac{\zeta^2}{2T}) - \int_0^T \frac{\zeta^2}{2t} \exp(-\frac{\zeta^2}{2t}) \, dt, \tag{C.3}$$

we obtain from (C.2) that

$$\int_0^T \int_0^T |\Phi(x,\, t;\, y,\, \tau)|^2 \, dt d\tau = \frac{T\zeta^{-4}}{16\pi^3} \exp\left(-\frac{\zeta^2}{2T}\right) \sim \frac{T\zeta^{-4}}{16\pi^3} \quad \text{as } \zeta \to 0. \tag{C.4}$$

This completes the proof of (4.6).

Next, let us prove (4.7). The direct calculations give

$$\int_0^T \int_0^T |\partial_{x_i} \Phi(x,\, t;\, y,\, \tau)|^2 \, d\tau dt \le \int_0^T \int_0^t \frac{|x-y|^2}{[4\pi(t-\tau)]^5} \exp\left(-\frac{|x-y|^2}{2(t-\tau)}\right) d\tau dt$$

$$= \frac{\zeta^{-6}}{2\pi^5} \int_0^T dt \int_{\frac{\zeta}{2\sqrt{t}}}^{+\infty} \delta^7 \exp(-2\delta^2) \, d\delta. \tag{C.5}$$

By repeatedly using integration by parts, we have

$$\int_{\frac{\zeta}{2\sqrt{t}}}^{+\infty} \delta^7 \exp(-2\delta^2) \, d\delta$$

$$= \frac{\zeta^6}{2^8 t^3} \exp(-\frac{\zeta^2}{2t}) + \frac{3\zeta^4}{2^7 t^2} \exp(-\frac{\zeta^2}{2t}) + \frac{3}{16} \left[\frac{\zeta^2}{2t} \exp(-\frac{\zeta^2}{2t}) + \exp(-\frac{\zeta^2}{2t}) \right]. \tag{C.6}$$

It follows that

$$\int_0^T dt \int_{\frac{\zeta}{2\sqrt{t}}}^{+\infty} \delta^7 \exp(-2\delta^2) \, d\delta = O(1), \quad \zeta \to 0. \tag{C.7}$$

Then the estimate (4.7) comes from (C.5) and (C.7).

Finally, let us show (4.8). Due to the estimate

$$|D_t^{1/2} \Phi(x,\, t;\, y,\, \tau)| \lesssim \frac{1}{(t-\tau)^{3/2}} \, \frac{1}{|x-y|} \, \exp\left(-c_0 \frac{|x-y|^2}{t-\tau}\right) \tag{C.8}$$

with a positive constant c_0 (see [14, Lemma A.1]), we get

$$\begin{aligned}
&\int_0^T \int_0^T |D_t^{1/2}\Phi(x,\,t;\,y,\,\tau)|^2\,d\tau dt \\
&\lesssim \int_0^T \int_0^t \frac{1}{(t-\tau)^3}\,\frac{1}{|x-y|^2}\exp\left(-2c_0\frac{|x-y|^2}{t-\tau}\right)d\tau dt \\
&= 2^5\zeta^{-6}\int_0^T dt\int_{\frac{\zeta}{2\sqrt{t}}}^{+\infty}\delta^3\exp(-8c_0\delta^2)\,d\delta \\
&= O(\zeta^{-6}). \qquad \text{(C.9)}
\end{aligned}$$

The proof is now complete. ∎

Appendix D. Proof of Lemma 4.2.

Proof. Using the change of variables (C.1), we have

$$\begin{aligned}
&\int_0^t \left|\nabla_x\Phi(x,\,t;\,y,\,\tau)\right|^2\,d\tau \\
&\leq \int_0^t \frac{\zeta^2}{[4\pi(t-\tau)]^5}\exp\left(-\frac{\zeta^2}{2(t-\tau)}\right)d\tau = \frac{\zeta^{-6}}{2\pi^5}\int_{\frac{\zeta}{2\sqrt{t}}}^{+\infty}\delta^7\exp(-2\delta^2)\,d\delta, \qquad \text{(D.1)}
\end{aligned}$$

which gives (4.9) by using (C.6). The proof is complete. ∎

Appendix E. Proof of Lemma 4.3.

Proof. Recall that

$$\begin{aligned}
\varphi_i(x,\,y,\,t)-\sigma_i(y,\,t) &= \int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty}\frac{2}{\sqrt{\pi}}\exp(-\delta^2)\left[\sigma_i\left(y,\,t-\frac{|x-y|^2}{4\delta^2}\right)-\sigma_i(y,\,t)\right]d\delta \\
&\quad -\sigma_i(y,\,t)\int_0^{\frac{|x-y|}{2\sqrt{t}}}\frac{2}{\sqrt{\pi}}\exp(-\delta^2)\,d\delta, \qquad \text{(E.1)}
\end{aligned}$$

where δ is defined by (A.1). Note that

$$\int_0^{\frac{|x-y|}{2\sqrt{t}}}\frac{2}{\sqrt{\pi}}\exp(-\delta^2)\,d\delta = \sum_{n=1}^{+\infty}\tilde{\alpha}_n\left(\frac{|x-y|}{2\sqrt{t}}\right)^n =: \sum_{n=1}^{+\infty}\alpha_n(t)\,|x-y|^n. \qquad \text{(E.2)}$$

In addition, for fixed y, we have

$$\sigma_i\left(y,\,t-\frac{|x-y|^2}{4\delta^2}\right)-\sigma_i(y,\,t) = \sum_{n=1}^{+\infty}\frac{1}{n!}\,\partial_t^n\sigma_i(y,\,t)\left(-\frac{|x-y|^2}{4\delta^2}\right)^n,$$

and then

$$\begin{aligned}
&\int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty}\frac{2}{\sqrt{\pi}}\exp(-\delta^2)\left[\sigma_i\left(y,\,t-\frac{|x-y|^2}{4\delta^2}\right)-\sigma_i(y,\,t)\right]d\delta \\
&= \sum_{n=1}^{+\infty}\frac{1}{n!}\,(-|x-y|^2)^n\,\partial_t^n\sigma_i(y,\,t)\int_{\frac{|x-y|}{2\sqrt{t}}}^{+\infty}\frac{2}{\sqrt{\pi}}\,\frac{\exp(-\delta^2)}{(4\delta^2)^n}\,d\delta \\
&= \sum_{n=1}^{+\infty}\left(\sum_{m=n}^{+\infty}\beta_{m,n}(t)\,|x-y|^m\right)\partial_t^n\sigma_i(y,\,t). \qquad \text{(E.3)}
\end{aligned}$$

The proof is completed by inserting (E.2) and (E.3) into (E.1). ∎

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