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Restriction for general linear groups: The local non-tempered Gan–Gross–Prasad conjecture (non-Archimedean case)

By *Kei Yuen Chan* at Shanghai

Abstract. We prove a local Gan–Gross–Prasad conjecture on predicting the branching law for the non-tempered representations of general linear groups in the case of non-Archimedean fields. We also generalize to Bessel and Fourier–Jacobi models and study a possible generalization to Ext-branching laws.

1. Introduction

In 1990s, Gross and Prasad [22] formulated conjectures which determine when an irreducible generic representation of $\mathrm{SO}_{n-1}(F)$ appears in a quotient of an irreducible generic representation of $\mathrm{SO}_n(F)$, where F is a local field. The conjectural answer is in terms of symplectic root numbers, providing deep connections with number theory. About ten years ago, Gan, Gross and Prasad [18] generalized the conjectures to other classical groups. For p-adic groups, the local generic conjectures in orthogonal, unitary and symplectic-metaplectic cases have been respectively settled by Waldspurger [43], Mœglin and Waldspurger [32], and by Beuzart-Plessis [8], Gan and Ichino [20], and by Atobe [4]; and for real groups, the unitary cases for tempered representations and independently for discrete series are settled by Beuzart-Plessis [9] and H. He [25], respectively. We remark that the generic case for general linear groups has been known long from the work of Jacquet, Piateski-Shapiro and Shalika [27].

Recently, Gan, Gross and Prasad [19] have formulated new conjectures for certain non-tempered representations arising from a local component of an automorphic representation. The main goal of this paper is to prove one of those conjectures for general linear groups over a non-Archimedean local field and study related generalizations.

1.1. Local non-tempered Gan–Gross–Prasad conjecture. We begin with a precise formulation of the non-tempered conjecture. Let $G_n = \mathrm{GL}_n(F)$, the general linear group over a local field F. Let W_F be the Weil group of F. The Weil–Deligne group WD_F of F is defined as

$$\mathrm{WD}_F = \begin{cases} W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is non-Archimedean,} \\ W_F & \text{if } F \text{ is Archimedean.} \end{cases}$$

The set of Langlands parameters of G_n is the set of equivalence classes of homomorphisms

$$\phi : \mathrm{WD}_F \to {}^L G = \mathrm{GL}_n(\mathbb{C}),$$

under conjugation by elements in $\mathrm{GL}_n(\mathbb{C})$, and the restriction to the factor of $\mathrm{SL}_2(\mathbb{C})$ in W_F is algebraic. The local Langlands correspondence for $\mathrm{GL}_n(F)$ is now known by [24,26,31,37,44].

Define the Arthur parameters [3] as the set of ${}^L G$-orbits of maps

$$\psi : \mathrm{WD}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

such that $\psi|_{\mathrm{WD}_F}$ has bounded image, i.e. has tempered Langlands parameter, and the restriction to the $\mathrm{SL}_2(\mathbb{C})$ factor is algebraic. For each Arthur parameter ψ, one assigns an L-parameter given by

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right).$$

Let $\mathrm{Sym}^k(\mathbb{C}^2)$ be the unique $(k+1)$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. The Arthur parameter, as a finite $\mathrm{WD}_F \times \mathrm{SL}_2(\mathbb{C})$-representation ψ, takes the form

$$M_A = \sum_d M_d \otimes \mathrm{Sym}^d(\mathbb{C}^2), \tag{1.1}$$

where each M_d is a representation of WD_F such that $\psi|_{\mathrm{WD}_F}$ has bounded image, i.e. each M_i corresponds to a tempered representation. It gives rise to a Langlands parameter M as described above, and gives a G_n-representation denoted by π_M. Any irreducible smooth representation of G_n associated to the Langlands parameter ϕ_ψ coming from an Arthur parameter is called a representation of Arthur type.

A key notion in [19] is the relevant pair which governs the branching law of representations of Arthur type:

Definition 1.1 ([19]). For any $n, m \in \mathbb{Z}_{\geq 0}$, two Arthur parameters M_A and N_A for respective G_n and G_m are said to form a *relevant* pair if there exists WD_F-representations $M_0^+, \ldots, M_r^+, M_0^-, \ldots, M_s^-$ (possibly zero) corresponding to tempered representations such that

$$M_A = \sum_{d=0}^{r} M_d^+ \otimes \mathrm{Sym}^d(\mathbb{C}^2) \oplus \sum_{d=1}^{s} M_d^- \otimes \mathrm{Sym}^{d-1}(\mathbb{C}^2) \tag{1.2}$$

and

$$N_A = \sum_{d=1}^{r} M_d^+ \otimes \mathrm{Sym}^{d-1}(\mathbb{C}^2) \oplus \sum_{d=0}^{s} M_d^- \otimes \mathrm{Sym}^d(\mathbb{C}^2). \tag{1.3}$$

We remark that in the above definition, the dimensions of the Arthur parameters M_A and N_A are not required to be of corank 1.

We regard G_n as a subgroup of G_{n+1} via the embedding $g \mapsto \mathrm{diag}(g, 1)$. A non-tempered Gan–Gross–Prasad conjecture predicts which Arthur-type representations of G_n appear in the quotient of an Arthur-type representation of G_{n+1}, in terms of relevant pairs.

Conjecture 1.2 ([19, Conjecture 5.1]). Let F be a local field, and let π_M and π_N be Arthur-type representations of $\mathrm{GL}_{n+1}(F)$ and $\mathrm{GL}_n(F)$, respectively. Then $\mathrm{Hom}_{G_n}(\pi_M, \pi_N)$ is non-zero if and only if their respective associated Arthur parameters M_A and N_A are relevant.

The main result of the paper is to prove the conjecture for non-Archimedean field F. Previously, for non-Archimedean F, certain cases including when the Deligne $\mathrm{SL}_2(\mathbb{C})$ in WD_F acts trivially are proved in [19], and M. Gurevich [23] proves the only if direction. We shall give another proof for the only if direction in this paper. Recently, Gourevitch and Sayag [21] have results towards the Archimedean case. The unitary restriction problem is studied in [42] by Venkatesh.

Theorem 1.3. *If F is non-Archimedean, then Conjecture* 1.2 *holds.*

1.2. Representation-theoretic reformulation. From now on, we assume F is non-Archimedean. Let $\mathrm{Alg}(G_n)$ be the category of smooth G_n-representations. We first reformulate the problem into a representation theory setup.

For representations π_i in $\mathrm{Alg}(G_{n_i})$ $(i = 1, \dots, k)$ and $n = n_1 + \cdots + n_k$, we define the product

$$\pi_1 \times \cdots \times \pi_k \in \mathrm{Alg}(G_n)$$

to be the normalized parabolic parabolically induced module from $\pi_1 \boxtimes \cdots \boxtimes \pi_k$. For more detailed notions of Zelevinsky segments and product, see Section 2.4. For an irreducible unitarizable cuspidal representation ρ of G_l, let

$$\Delta_\rho(m) = [\nu^{-(m-1)/2}\rho, \nu^{(m-1)/2}\rho]$$

be a Zelevinsky segment. Any square integrable representation is known to be isomorphic to $\mathrm{St}(\Delta_\rho(m))$ for some such Zelevinsky segment $\Delta_\rho(m)$ (see [44]). Any tempered representation is isomorphic to a product of some square-integrable representations, and corresponds to a WD_F-representation ψ with bounded image $\psi(\mathrm{WD}_F)$.

Let $u_\rho(m, d)$ be the unique irreducible quotient of the product

$$\mathrm{St}(\nu^{(d-1)/2}\Delta_\rho(m)) \times \cdots \times \mathrm{St}(\nu^{-(d-1)/2}\Delta_\rho(m)),$$

which is so-called a Speh representation and is unitarizable. Each factor $M_d \otimes \mathrm{Sym}^d(\mathbb{C})$ in (1.1) corresponds to a product of Speh representations of the form

$$u_{\rho_1}(m_1, d+1) \times \cdots \times u_{\rho_r}(m_r, d+1). \tag{1.4}$$

Any Arthur-type representation is a product of some Speh representations. It follows from [5, 41] that such a product is irreducible, and is independent of the ordering of Speh representations.

The notion of a derivative is defined in [44] (see Section 2.1 for the detail). For an irreducible $\pi \in \mathrm{Alg}(G_r)$, let $\widetilde{\pi}$ be the highest derivative of π and let $\pi^- = \nu^{1/2}\widetilde{\pi}$, where $\nu(g) = |\det g|_F$. A key observation in [19] is that

$$u_\rho(m, d+1)^- \cong u_\rho(m, d) \tag{1.5}$$

(when $d = 0$, we regard $u_\rho(m, 0) = 1$) and so

$$(u_{\rho_1}(m_1, d+1) \times \cdots \times u_{\rho_r}(m_r, d+1))^- \cong u_{\rho_1}(m_1, d) \times \cdots \times u_{\rho_r}(m_r, d), \tag{1.6}$$

which is also a motivation for the notion of relevant pairs in [19]. The isomorphism (1.5) follows from the well-known highest derivative of Zelevinsky [44] (and its translation to the Zelevinsky classification via [41]).

Thus combining Definition 1.1, (1.4) and (1.6), we have the following reformulation.

Reformulation of Conjecture 1.2 in the non-Archimedean case. Let F be a non-Archimedean local field. Let π_M and π_N be Arthur-type representations of $\mathrm{GL}_{n+1}(F)$ and $\mathrm{GL}_n(F)$, respectively. Then $\mathrm{Hom}_{G_n}(\pi_M, \pi_N) \neq 0$ if and only if there exist Speh representations $\pi_{p,1}, \dots, \pi_{p,r}$ and $\pi_{q,1}, \dots, \pi_{q,s}$ such that

$$\pi_M \cong \pi_{p,1} \times \cdots \times \pi_{p,r} \times \pi_{q,1}^- \times \cdots \times \pi_{q,s}^-$$

and

$$\pi_N \cong \pi_{p,1}^- \times \cdots \times \pi_{p,r}^- \times \pi_{q,1} \times \cdots \times \pi_{q,s}.$$

1.3. Generalizations. The first generalization is on Bessel and Fourier–Jacobi models (Theorem 5.12). Such a generalization is also expected in [19]. The strategy for proving general cases is connecting those models functorially via Bernstein–Zelevinsky theory (Corollary 6.3) and then using the reduction to basic case similar to [18]. The functorial connection is a key difference of our study from the one in [18]. We remark that we also deduce the equal rank Fourier–Jacobi case from the basic case of restricting G_{n+1} to G_n representations, which differs from that some results (such as multiplicity one theorems, e.g., [39]) are proved separately for equal rank Fourier–Jacobi models.

In more detail, let

$$H_r^R = \left\{ \begin{pmatrix} g & & x \\ & 1 & v^t \\ & & u \end{pmatrix} : g \in G_r,\ x \in \mathrm{Mat}_{r\times(n-r)},\ v \in F^{n-r},\ u \in U_{n-r} \right\} \subset G_{n+1},$$

where U_{n-r} is the subgroup of unipotent upper triangular matrices. It is sometimes referred to a Rankin–Selberg subgroup. Let ψ be a non-degenerate character on a subgroup $U_{n-r} \ltimes F^{n-r}$, extending trivially to H_r^R (also see Section 5.2). We show that the restriction problem for a Bessel model or a Fourier–Jacobi model is equivalent to the problem of determining the corresponding Rankin–Selberg model (Corollary 6.3), i.e. determining if

$$\mathrm{Hom}_{H_r^R}(\pi_1 \otimes \psi \otimes \nu^{-(n-r)/2}, \pi_2) \neq 0,$$

where π_1 and π_2 are respective irreducible G_{n+1} and G_r representations.

The second generalization is on Ext-branching laws. The generic case for Ext-branching law is simpler: for respective generic irreducible representations π_1 and π_2 of G_{n+1} and G_n,

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \cong \mathbb{C}$$

and

$$\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) = 0 \quad \text{for } i \geq 1.$$

The Ext-vanishing part is conjectured by D. Prasad [36] and proved in [16], and the Ext-result also extends to standard modules in [12]. One may consider an analogous problem of Ext-branching laws for Arthur representations. However, there is no such general Ext-vanishing result for Arthur representations, and we do not have a way predicting non-vanishing Ext at the moment.

Nevertheless, we formulate a conjecture in Section 7.1, which reduces computations of Ext-groups for branching laws to computation of Ext-groups of derivatives. The conjecture is partly based on the derivative approach in [19], as well as some examples computed in this paper.

1.4. Outline of the proof of non-tempered GGP. We shall consider the reformulated problem in Section 1.2. Let

$$\pi_M = \pi_{p,1} \times \cdots \times \pi_{p,r} \in \mathrm{Alg}(G_{n+1}), \tag{1.7}$$

and

$$\pi_N = \pi_{q,1} \times \cdots \times \pi_{q,s} \in \mathrm{Alg}(G_n),$$

where each $\pi_{p,i}$ and $\pi_{q,j}$ is an (irreducible) Speh representation.

The proof is on the induction of the total number of factors $\pi_{p,i}$ and $\pi_{q,j}$ which are not cuspidal representations. The basic case is that all factors are cuspidal representations. Then the associated Arthur parameters M_A and N_A are automatically relevant. Since the representations π_M and π_N are generic in this case, we always have $\mathrm{Hom}_{G_n}(\pi_M, \pi_N) \neq 0$.

The strategy of the general case is to find a suitable filtration on $\pi_M|_{G_n}$

$$0 \to \lambda \to \pi_M|_{G_n} \to \omega \to 0$$

such that

$$\mathrm{Hom}_{G_n}(\omega, \pi_N) = \mathrm{Ext}^1_{G_n}(\omega, \pi_N) = 0 \tag{1.8}$$

and $\mathrm{Hom}_{G_n}(\lambda, \pi_N)$ can be transferred to another Hom space computable from the inductive case. Now a long exact sequence argument gives

$$\mathrm{Hom}_{G_n}(\pi_M|_{G_n}, \pi_N) \cong \mathrm{Hom}_{G_n}(\lambda, \pi_N)$$

and so one concludes the former from the latter one. The way to find such filtration is based on a combination of Bernstein–Zelevinsky filtration and Mackey theory, and (1.8) would follow from comparing cuspidal supports on ω and π_N. A more systematic filtration is given in Proposition 5.13.

In more detail, an Arthur-type representation π_M is written as a product of Speh representations in (1.7). Now we write $\pi_{p,k} = v_{\rho_k}(m_k, d_k)$ for all k. As shown in Proposition 4.1, there is a duality between the original restriction problem $\mathrm{Hom}_{G_n}(\pi_M, \pi_N) \neq 0$ and the dual restriction problem

$$\mathrm{Hom}_{G_{n+1}}(\sigma' \times \pi_N, \pi_M) \neq 0,$$

where σ' is a certain unitarizable cuspidal representation of G_2. With the commutation of the Speh representations in the product, we may assume that $m_1 + d_1$ is the largest among all Speh representation factors in π_M and π_N. Such arrangement allows one (easily) finds a suitable filtration to obtain the vanishing (1.8).

Now some cuspidal support consideration in the filtration reduces to the study of the bottom layer (of the filtration):

$$\mathrm{Hom}_{G_{n+1}}(v^- \times ((\Pi \,\bar{\times}\, \pi')|_{G_k}), \pi_N), \tag{1.9}$$

for some G_k, where $v = v_{\rho_1}(m_1, d_1)$, $\pi' = \pi_{p,2} \times \cdots \times \pi_{p,r}$, Π is the Gelfand–Graev representation and $\bar{\times}$ indicates the mirabolic induction considered in Section 3. A key here is that a Gan–Gross–Prasad-type reduction can be used to transfer the study of $(\Pi \,\bar{\times}\, \pi')$ to $(\sigma \times \pi')|_{G_k}$ for some suitable choice of a unitarizable cuspidal representation σ. (Here $\sigma \times \pi'$ is an irreducible G_{k+1}-representation.)

Now $\sigma \times \pi'$ is still Arthur type and so induction can be applied. It is clear that if λ is a quotient of $(\sigma \times \pi')|_{G_k}$, then $\upsilon^- \times \lambda$ is still a quotient of $\upsilon^- \times ((\sigma \times \pi')|_{G_k})$, which basically deals with the if direction. The converse of the statement is not true in general, but holds under suitable assumption that fulfills our purpose. For which, we have to study the product with $\pi^-_{p,1}$ preserves extensions in some situations (Corollary 9.4), which handles the only if direction. We also need some product preserving irreducibility results from [30].

1.5. Remarks. For irreducible generic quotients of G_n appearing in an irreducible generic representation of G_{n+1} (also known as generic GGP conjecture for GL-case), it is shown by Rankin–Selberg integrals [27,35]. In [16], G. Savin and the author give another proof for the generic case using variations of Bernstein–Zelevinsky filtrations. We also remark that in the above outline, one may replace the mirabolic induction $\Pi \bar{\times} \pi$ with certain Rankin–Selberg model discussed in Section 5. Such interpretation is later motivated by the approach in the generic case of orthogonal groups by Mœglin and Waldspurger [32].

Our method for Arthur-type representations is again a variation of Bernstein–Zelevinsky filtration method which exploits the product structure of Arthur representations. To illustrate how the refinement gives more information, we consider respective representations in $\mathrm{GL}_5(F)$ and $\mathrm{GL}_4(F)$ in [19, Remark 5.6] with A-parameters:

$$M_A = 1 \otimes \mathrm{Sym}^0(\mathbb{C}^2) \otimes \mathrm{Sym}^2(\mathbb{C}^2) \oplus 1 \otimes \mathrm{Sym}^0(\mathbb{C}^2) \otimes \mathrm{Sym}^0(\mathbb{C}^2) \oplus 1 \otimes \mathrm{Sym}^0(\mathbb{C}^2) \otimes \mathrm{Sym}^0(\mathbb{C}^2)$$

and

$$N_A = 1 \otimes \mathrm{Sym}^0(\mathbb{C}^2) \otimes \mathrm{Sym}^1(\mathbb{C}^2) \oplus 1 \otimes \mathrm{Sym}^1(\mathbb{C}^2) \otimes \mathrm{Sym}^0(\mathbb{C}^2).$$

(Here 1 is the trivial representation of the Weil group and the first Sym^k factor is the irreducible $(k+1)$-dimensional representation of the $\mathrm{SL}_2(\mathbb{C})$ in the Weil–Deligne group.) Their respective representations take the form

$$\pi_1 = \langle [\nu^{-1}, \nu] \rangle \times 1 \times 1 \quad \text{and} \quad \pi_2 = \langle [\nu^{-1/2}, \nu^{1/2}] \rangle \times \mathrm{St}([\nu^{-1/2}, \nu^{1/2}]).$$

(Here 1 is the trivial character of $F^\times$.) Now the Mackey theory gives two layers on $\pi_1|_{G_4}$:

$$\langle [\nu^{-1/2}, \nu^{3/2}] \rangle \times ((1 \times 1)|_{G_1}) \quad \text{and} \quad \langle [\nu^{-1/2}, \nu^{1/2}] \rangle \times ((1|_{M_1} \times 1 \times 1)|_{G_2}).$$

Set $\tau = \langle [\nu^{-1/2}, \nu^{1/2}] \rangle$. A key difference of our method from the one in [19] is to use transfer in (1.9) to deduce that $\tau \times ((1|_{M_1} \times 1 \times 1)|_{G_2})$ has a quotient of π_2, as G_4 representations. The above filtration is coarser than the full Bernstein–Zelevinsky filtration, but has the advantage of using transfer and induction as mentioned before. It can also deal with an obstruction in [19, Remark 5.6].

An irreducible cuspidal representation of G_n restricted to the mirabolic subgroup is isomorphic to the Gelfand–Graev representation, which is an essential step in our proof. This classical fact plays crucial roles, and is generalized to essentially square-integrable representations when restricted to G_{n-1} via Hecke algebra realization [14–16] (also see [13] for further generalization to representations restricted to be projective), but we do not critically need any Hecke algebra technique in this paper. We also remark that such fact also plays important roles, for example, in the reductions in [18] and in proving the Ext-vanishing theorem in [16].

Our approach in only if direction only requires a study of producting with one Speh representations (rather than several generalized Speh representations), compared with the study of multiple products of generalized Speh representations in [23]. We also work directly in the p-adic group without passing to other categories, and so the method is different from [23]. We show under some conditions on cuspidal supports that producting with a Speh representation preserves extensions and is a fully-faithful functor. This improves one of results of Lapid and Mínguez [30] which shows producting with Speh representations preserves irreducibility under a related condition.

1.6. Summary of results and structure of the paper. We summarize the key results of this paper below:

(1) A proof of Conjecture 1.2 in the case of a non-Archimedean field (Theorem 4.5).

(2) Generalize Conjecture 1.2 to Bessel, Fourier–Jacobi and other mixed models (Theorem 5.12).

(3) A filtration as a tool to study restriction problem for parabolically induced modules (Proposition 5.13).

(4) Product with a Speh representation preserves indecomposability and is a fully-faith functor under some conditions (Theorems 8.1 and 9.1).

In Section 2, we set up notations and recall some results such as properties of Speh representations. In Section 3, we study parabolically induced modules restricted to mirabolic subgroups. In Section 4, we prove Conjecture 1.2 for non-Archimedean case. In Section 5, we generalize results to general cases including Bessel models and Fourier–Jacobi models. In Section 6, we establish connections between models. In Section 7, we study Ext-branching laws for Arthur-type representations. In Sections 8 and 9, we prove Theorems 8.1 and 9.1.

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2. Notations and Preliminaries

2.1. Bernstein–Zelevinsky functors. For a connected reductive group G, let $\mathrm{Alg}(G)$ be the category of smooth (complex) representations of G. Let $G_n = \mathrm{GL}_n(F)$. All representations in this paper are smooth and we usually drop the term "smooth". For a representation π of G_n, set $n_\pi = n$.

Let $G = G_n$. For a closed subgroup H of G and a representation π in $\mathrm{Alg}(H)$, let $\mathrm{Ind}_H^G \pi$ be the space of smooth functions $f : G \to \pi$ satisfying

$$f(hg) = \delta(h)^{1/2} h.f(g),$$

where δ is the modulus function of H. The G-action on $\mathrm{Ind}_H^G \pi$ is given by

$$(g.f)(g_0) = f(g_0 g) \quad \text{for any } g, g_0 \in G.$$

Let $\mathrm{ind}_H^G \pi$ be the subrepresentation of $\mathrm{Ind}_H^G \pi$ containing all functions with compact support modulo H. We shall use ${}^u\mathrm{ind}$ and ${}^u\mathrm{Ind}$ for corresponding unnormalized inductions of ind and Ind, respectively. Those functors Ind, ind, ${}^u\mathrm{Ind}$, ${}^u\mathrm{Ind}$ are exact [7, Proposition 2.25 (a)].

Let M_n be the mirabolic subgroup of G_n, i.e. M_n is the subgroup of G_n with all the matrices with the last row $(0, \dots, 0, 1)$. We shall also regard G_{n-1} as a subgroup of M_n via the embedding

$$g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

Thus we have a chain of subgroups:

$$1 = G_0 = M_1 \subset \cdots \subset M_{n-1} \subset G_{n-1} \subset M_n \subset G_n.$$

For $\pi \in \mathrm{Alg}(G_n)$, we may simply write $\pi|_M$ for the restriction $\pi|_{M_n}$

Let $V = V_{n-1}$ be the unipotent radical of M_n. Let $\bar{\psi} : F \to \mathbb{C}$ be a non-degenerate character. Let $\psi : V \to \mathbb{C}$ by $\psi(v) = \bar{\psi}(v_{n-1})$, where v_{n-1} is the last entry in v. Note the action of M_{n-1} stabilizes $\psi : V \to \mathbb{C}$. For a character λ of V and a representation π of M_n, define

$$\pi_{V,\lambda} = \delta^{-1/2}\pi/\langle v.x - \lambda(v)x : v \in V,\ x \in \pi\rangle,$$

where δ is the modulus function of M_n. When $\lambda = 1$ (respectively, $\lambda = \psi$), we regard $\pi_{V,\lambda}$ as G_{n-1}-representation (respectively, M_{n-1}-representation).

Define $\theta = \theta_n : G_n \to G_n$ by $\theta(g) = g^{-t}$, the Gelfand–Kazhdan involution [7, Section 7]. For any irreducible representation π of G_n, $\theta(\pi) \cong \pi^\vee$.

Define

$$\Phi^+ : \mathrm{Alg}(M_n) \to \mathrm{Alg}(M_{n+1}), \qquad \Psi^+ : \mathrm{Alg}(G_n) \to \mathrm{Alg}(M_{n+1}),$$
$$\Phi^- : \mathrm{Alg}(M_{n+1}) \to \mathrm{Alg}(M_n), \qquad \Psi^- : \mathrm{Alg}(M_{n+1}) \to \mathrm{Alg}(G_n).$$

by

$$\Phi^+(\pi) = \mathrm{ind}_{M_n V_n}^{M_{n+1}} \pi \boxtimes \psi, \quad \Psi^+(\pi) = \mathrm{ind}_{G_n V_n}^{M_{n+1}} \pi \boxtimes 1,$$
$$\Phi^-(\pi) = \pi_{V_n,\psi}, \qquad \Psi^-(\pi) = \pi_{V_n,1}.$$

In particular, Ψ^+ is just an inflation of representations. Some major properties of the functors [6, Proposition 3.2]:

(1) All the above functors are exact.

(2) Φ^- is left-adjoint to Φ^+ and Ψ^- is left-adjoint to Φ^+.

(3) $\Phi^-\Psi^+ = 0$ and $\Psi^-\Phi^+ = 0$

(4) There is an exact sequence:

$$0 \to \Phi^+\Phi^- \to \mathrm{Id} \to \Psi^+\Psi^- \to 0.$$

(5) All the irreducible representations of M_n are isomorphic to $(\Phi^+)^{k-1}\Psi^+(\pi)$ for some k and some irreducible smooth G_{n-k}-representation.

(6) [7, 5.18] For any cuspidal representation σ of G_n, $\sigma|_{M_n} \cong (\Phi^+)^{n-1}(1)$. Here 1 is the 1-dimensional representation of M_1.

Denote, the Gelfand–Graev representation,

$$\Pi_n := (\Phi^+)^{n-1}(1) \in \mathrm{Alg}(M_n).$$

Let $\nu = \nu_n : G_n \to \mathbb{C}$ be a character given by $\nu(g) = |\det(g)|_F$. For $\pi \in \mathrm{Alg}(G_n)$, the k-th right and left derivatives of π are respectively defined as

$$\pi^{(k)} = \Psi^-(\Phi^-)^{k-1}(\pi|_{M_n}), \quad {}^{(k)}\pi = \theta(\theta(\pi)^{(k)}).$$

and the k-th shifted right and left derivatives of π are defined as

$$\pi^{[k]} = \nu^{1/2} \cdot \pi^{(k)}, \quad {}^{[k]}\pi = \nu^{-1/2} \cdot {}^{(k)}\pi.$$

Let k^* be the largest integer such that $\pi^{(k^*)} \neq 0$. We shall call $\pi^{(k^*)}$ to be the highest derivative of π, and k^* to be the level of π. We also set $\pi^- = \pi^{[k^*]}$.

2.2. Parabolic induction and Jacquet functors. Let U_n be the subgroup of G_n containing all unipotent upper triangular matrices. Let N_i be the unipotent subgroup of G_n containing matrices of the form

$$\begin{pmatrix} I_{n-i} & u \\ & I_i \end{pmatrix}$$

for any $(n-i) \times i$ matrices u over F. We regard $G_{n-i} \times G_i$ as a subgroup of G_n via the embedding $(g_1, g_2) \mapsto \mathrm{diag}(g_1, g_2)$. Let P_i be the parabolic subgroup $(G_{n-i} \times G_i)N_i$.

For $\pi_1 \in \mathrm{Alg}(G_{n-i})$ and $\pi_2 \in \mathrm{Alg}(G_i)$, define the product of π_1 and π_2 as

$$\pi_1 \times \pi_2 = \mathrm{Ind}_{(G_{n-i} \times G_i) \ltimes N_i}^{G_n} \pi_1 \boxtimes \pi_2 \boxtimes 1.$$

For a family of representations $\pi_i \in \mathrm{Alg}(G_{n_i})$ $(i = 1, \dots, k)$, define

$$\pi_1 \times \dots \times \pi_k := \pi_1 \times (\dots \times (\pi_{k-1} \times \pi_k) \cdots).$$

The parabolic induction is an exact functor [7]. For more properties for parabolic inductions, see [30].

Let $N_i^- = N_i^t$ be the opposite unipotent subgroup. For $\pi \in \mathrm{Alg}(G_n)$, we shall denote by π_{N_i} and $\pi_{N_i^-}$ the corresponding normalized Jacquet modules, as $G_{n-i} \times G_i$-representations. They are also exact functors. Since the parabolic induction has usual and opposite Jacquet functors as left and right adjoint functors respectively, parabolic induction also preserves injective and projective objects.

For an irreducible representation π of G_k, there is a unique set (with multiplicities) of cuspidal representations $\rho_1, \dots, \rho_r$ such that π is a composition factor of $\rho_1 \times \dots \times \rho_r$, and we denote the multiset

$$\mathrm{cupp}(\pi) = \{\rho_1, \dots, \rho_r\},$$

and denote the set

$$\mathrm{cupp}_{\mathbb{Z}}(\pi) = \{\nu^i \rho_j\}_{i \in \mathbb{Z}, j=1,\dots,r}.$$

2.3. Bernstein–Zelevinsky filtrations. Since Bernstein–Zelevinsky filtration [6, Section 3.5] (and its variations) is a main tool in this article, we recall in this section.

Let π be in $\mathrm{Alg}(G_{n+1})$. Then $\pi|_{G_n}$ admits a filtration

$$0 = \pi_{n+1} \subset \pi_n \subset \dots \subset \pi_1 \subset \pi_0 = \pi|_{G_n}$$

such that

$$\pi_{k-1}/\pi_k \cong \pi^{[k]} \times \mathrm{ind}_{U_{k-1}}^{G_{k-1}} \psi_k,$$

where ψ_k is a non-degenerate character on U_{k-1}. Note that $\mathrm{ind}_{U_{k-1}}^{G_{k-1}} \psi_k \cong \Pi_{k+1}|_{G_k}$. There is a "left" version of the filtration (see [16] and [12]), while we do not need this in this article.

2.4. Speh representations and Zelevinsky segments. Let ρ be an irreducible cuspidal representation of G_m. For any $a, b \in \mathbb{C}$ with $b - a \in \mathbb{Z}_{\geq 0}$, a Zelevinsky segment

$$\Delta = [\nu^a \rho, \nu^b \rho]$$

is the set $\{\nu^a \rho, \nu^{a+1} \rho, \dots, \nu^b \rho\}$, and we denote $a(\Delta) = \nu^a \rho$ and $b(\Delta) = \nu^b \rho$. Denote by $\langle \Delta \rangle$ (respectively, $\mathrm{St}(\Delta)$) the unique submodule (respectively, quotient) of $\nu^a \rho \times \cdots \times \nu^b \rho$.

A Zelevinsky multisegment is a multiset of Zelevinsky segments. For a Zelevinsky multisegment

$$\mathfrak{m} = \{\Delta_1, \dots, \Delta_r\},$$

denote by $\langle \mathfrak{m} \rangle$ the unique irreducible subrepresentation of $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_r \rangle$, and denote by $\mathrm{St}(\mathfrak{m})$ the unique irreducible quotient of $\mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_r)$, where $\Delta_1, \dots, \Delta_r$ are ordered in the way as in [44, Theorem 6.1]. We also denote the parabolic induction $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_r \rangle$ by $\zeta(\mathfrak{m})$.

Let $\mathrm{Irr}^c(G_k)$ be the set of all (isomorphism classes of) irreducible cuspidal representations of G_k, and let $\mathrm{Irr}^c = \bigsqcup_{k \geq 0} \mathrm{Irr}^c(G_k)$. Let $\mathrm{Irr}^{u,c}(G_k)$ be the set of irreducible unitarizable cuspidal representations of G_k. Let $\mathrm{Irr}^{u,c} = \bigsqcup_{k \geq 0} \mathrm{Irr}^{u,c}(G_k)$.

Let $\rho \in \mathrm{Irr}^c(G_m)$. For a positive integer d, define

$$\Delta_\rho(d) = [\nu^{-(d-1)/2} \rho, \nu^{(d-1)/2} \rho].$$

For a positive integer m, define

$$u_\rho(m, d) = \langle \{\nu^{-(m-1)/2} \Delta_\rho(d), \dots, \nu^{(m-1)/2} \Delta_\rho(d)\} \rangle.$$

When ρ is unitarizable, we shall call those representations to be Speh representations, and they are unitarizable [5, Section 8] (see [41]).

In Section 1.2, we also introduce the notion $v_\rho(m, d)$. The two notions coincide (and here we do not assume ρ to be unitarizable):

Lemma 2.1 ([41, Theorem A10]). *For any $\rho \in \mathrm{Irr}^c(G_n)$, any $d, m \geq 1$,*

$$v_\rho(m, d) \cong u_\rho(m, d).$$

The above result can also be deduced from Mœglin–Waldspurger algorithm.

Explicit derivatives of a Speh representation are particularly simple to describe, and one refers to [29] (also see [15, Section 7]). We collect some useful information for our study:

Lemma 2.2 ([29, Theorem 14]). *Let $\pi = u_\rho(m, d)$ be a Speh representation.*

(1) *The level of π is $n_\rho m$.*

(2) *If k is not the level of π and $\pi^{[k]} \neq 0$, then the cuspidal support of $\pi^{[k]}$ contains $\nu^{(d+m-2)/2+1/2} \rho$.*

(3) *If k is the level of π, then $\pi^- = \pi^{[k]} \cong u_\rho(m, d-1)$ and $\pi^{(k)} \cong \nu^{-1/2} u_\rho(m, d-1)$.*

2.5. Weakly relevant condition for general branching law. For a segment Δ, set ${}^{[0]}\Delta = \nu^{-1/2} \Delta$ and $\Delta^{[0]} = \nu^{1/2} \Delta$. For a segment $\Delta = [\nu^a \rho, \nu^b \rho]$, set $\Delta^- = [\nu^a \rho, \nu^{b-1} \rho]$ and ${}^-\Delta = [\nu^{a+1} \rho, \nu^b \rho]$; and set $\Delta^{[-]} = \nu^{1/2} \Delta^-$ and ${}^{[-]}\Delta = \nu^{-1/2} \cdot {}^-\Delta$.

Let $\mathfrak{m}$ and $\mathfrak{n}$ be two Zelevinsky multisegments. We say that $\mathfrak{m}$ and $\mathfrak{n}$ are *weakly relevant* if there exists segments

$$\Delta_{p,1},\dots,\Delta_{p,r},\quad \Delta_{q,1},\dots,\Delta_{q,s}$$

and

$$\Delta_{a,1},\dots,\Delta_{a,k},\quad \Delta_{b,1},\dots,\Delta_{b,l}$$

such that

$$\mathfrak{m}=\{\Delta_{p,1},\dots,\Delta_{p,r},\Delta_{q,1}^{[-]},\dots,\Delta_{q,s}^{[-]}\}\cup\{\Delta_{a,1},\dots,\Delta_{a,k},{}^{[0]}\Delta_{b,1},\dots{}^{[0]}\Delta_{b,l}\}$$

and

$$\mathfrak{n}=\{{}^{[-]}\Delta_{p,1},\dots,{}^{[-]}\Delta_{p,r},\Delta_{q,1},\dots,\Delta_{q,s}\}\cup\{{}^{[0]}\Delta_{a,1},\dots,{}^{[0]}\Delta_{a,k},\Delta_{b,1},\dots,\Delta_{b,l}\}.$$

While we do not need the following result, it gives one guiding principle in general smooth branching law. One may also compare with an Archimedean result [21] in terms of wavefront sets. We remark that the converse is not true in general (for example, see the quotient branching law for the Steinberg representation in [16]).

Proposition 2.3. *Let π_1,π_2 be irreducible smooth representations of G_{n+1} and G_n, respectively. Let $\mathfrak{m}$ and $\mathfrak{n}$ be their associated Zelevinsky multisegments. If* $\mathrm{Hom}_{G_n}(\pi_1,\pi_2)\neq 0$*, then $\mathfrak{m}$ and $\mathfrak{n}$ are weakly relevant.*

Proof. Since $\mathrm{Hom}_{G_n}(\pi_1,\pi_2)\neq 0$, we have that $\mathrm{Hom}_{G_n}(\theta(\zeta(\mathfrak{m}))^\vee,\zeta(\mathfrak{n}))\neq 0$. (We remark that $\theta(\zeta(\mathfrak{m}))^\vee$ has a quotient of $\langle\mathfrak{m}\rangle$ as $\theta(\langle\mathfrak{m}\rangle)^\vee\cong\langle\mathfrak{m}\rangle$.) Let $\zeta_1=\theta(\zeta(\mathfrak{m}))^\vee$ and $\zeta_2=\zeta(\mathfrak{n})$. Now using Bernstein–Zelevinsky filtration (Section 2.3, also see [16, Lemma 2.4] and [13, Lemma 2.1]), we obtain that, for some i,

$$\mathrm{Hom}_{G_{n+1-i}}(\zeta_1^{[i+1]},{}^{(i)}\zeta_2)\neq 0.$$

Write $\mathfrak{m}=\{\Delta_1,\dots,\Delta_k\}$ and $\mathfrak{n}=\{\widetilde{\Delta}_1,\dots,\widetilde{\Delta}_l\}$. For convenience, we also set the notion $\Delta^0=\Delta$ and ${}^0\Delta=\Delta$ (i.e. no effect on Δ). Using the geometric lemma on ζ_1 (with suitable arrangement of segments, see for example the proof of [13, Lemma 6.3]), we obtain a filtration on $\zeta_1^{[i+1]}$ whose successive quotients are $\theta(\zeta(\mathfrak{p})^\vee)$ with $\mathfrak{p}$ taking the form

$$\mathfrak{p}=\{\nu^{1/2}\Delta_1^{\#},\dots,\nu^{1/2}\Delta_k^{\#}\}$$

and similarly a filtration on ${}^{(i)}\zeta_2$ whose successive quotients are $\zeta(\mathfrak{q})$ with $\mathfrak{q}$ taking the form

$$\mathfrak{q}=\{{}^{\#}\widetilde{\Delta}_1,\dots,{}^{\#}\widetilde{\Delta}_l\},$$

where each $\#=-$ or $=0$. Now the previous non-vanishing Hom implies that

$$\mathrm{Hom}_{G_n}(\theta(\zeta(\mathfrak{p})^\vee),\zeta(\mathfrak{q}))\neq 0$$

for some $\mathfrak{p}$ and $\mathfrak{q}$ of the form. The ordering [44, Theorem 7.1] of Zelevinsky classification implies that a non-zero map in $\mathrm{Hom}_{G_n}(\theta(\zeta(\mathfrak{p})^\vee),\zeta(\mathfrak{q}))$ factors through the Zelevinsky submodule of $\zeta(\mathfrak{q})$ and hence $\mathfrak{p}=\mathfrak{q}$ for some $\mathfrak{p}$ and $\mathfrak{q}$ taking the above form. In other words, we have for each j, $\nu^{1/2}\Delta_j^-=\widetilde{\Delta}_{i_j}$ or $\nu^{1/2}\Delta_j^-=1$ (as G_0-representation), or $\Delta_j=\nu^{-1/2}\cdot{}^-\widetilde{\Delta}_{i_j}$, or $\nu^{1/2}\Delta_j=\widetilde{\Delta}_{i_j}$, or $\nu^{1/2}\Delta_j^-={}^-\widetilde{\Delta}_{i_j}$ for some i_j. These conditions give the weakly relevant condition on the pair $(\mathfrak{m},\mathfrak{n})$. □

2.6. Ext-vanishing on cuspidal supports. The following result is standard. Since we shall frequently use the following result, we give a proof on it.

Lemma 2.4. *Let π_1 be an irreducible G_{n-i}-representation and let $\pi_2 \in \mathrm{Alg}(G_i)$ (not necessarily admissible). Let π be an admissible G_n-representation. Suppose that for any simple composition factor τ of π, $\mathrm{cupp}(\tau) \cap \mathrm{cupp}(\pi_1) = \emptyset$. Then, for all i,*

$$\mathrm{Ext}^i_{G_n}(\pi_1 \times \pi_2, \pi) = 0$$

Proof. It suffices to prove for π to be irreducible. One first applies Frobenius reciprocity, for any i,

$$\mathrm{Ext}^i_{G_n}(\pi_1 \times \pi_2, \pi) \cong \mathrm{Ext}^i_{G_{n-i} \times G_i}(\pi_1 \boxtimes \pi_2, \pi_{N_i^-}).$$

Since $\pi_{N_i^-}$ is admissible, it suffices to check the Ext-vanishing for each simple composition factor τ' of $\pi_{N_i^-}$. Now we write $\tau' = \tau_a \boxtimes \tau_b$ for simple G_{n-i} and G_i representations τ_a and τ_b, respectively. Now we have that $\mathrm{Ext}^i_{G_i}(\pi_1, \tau_a) = 0$ since, by using the cuspidal support condition, one can find an element in the Bernstein center which acts by a different scalar on π_1 and τ_a. Now we conclude $\mathrm{Ext}^i_{G_n}(\pi_1 \times \pi_2, \tau') = 0$ by Künneth formula. □

3. Mirabolic induction

In this section, we discuss inductions involving mirabolic subgroups, which will be used in Sections 4, 5 and 6.

3.1. Mirabolic induction. Let $\tau \in \mathrm{Alg}(M_m)$ and let $\pi \in \mathrm{Alg}(G_n)$. Define two types of mirabolic inductions, similar to [6, Section 4.12].

(1) Type 1: Let $Q = P_m \cap M_{n+m} \subset G_{n+m}$, i.e.

$$Q = \left\{ \begin{pmatrix} g & u \\ & m \end{pmatrix} : g \in G_n,\ m \in M_m,\ u \in \mathrm{Mat}_{n \times m} \right\}.$$

Let $\epsilon : Q \to \mathbb{C}$ be the identity.

(2) Type 2: Let $Q = P^t_m \cap M_{n+m} \subset G_{n+m}$, i.e.

$$Q = \left\{ \begin{pmatrix} g & & \\ u & h & v \\ & & 1 \end{pmatrix} : g \in G_n,\ u \in \mathrm{Mat}_{m-1,n},\ h \in G_{m-1},\ v \in F^{m-1} \right\}.$$

Let $\epsilon : Q :\to \mathbb{C}$ given by $\epsilon = \nu^{-1/2}$.

For type 1 (respectively, type 2), extend $\pi \boxtimes \tau$ trivially to Q. Define the M_{n+m}-representation $\pi \bar{\times} \tau$ (respectively, $\tau \bar{\times} \pi$) to be the space of smooth functions $f : M_{n+m} \to \pi \boxtimes \tau$ satisfying $f(qg) = \epsilon(q)\delta(q)^{1/2} q.f(g)$ for any $q \in Q$ and $g \in M_{n+m}$, and f is compactly-supported modulo Q, where δ is the modulus function of Q.

In type 1, when restricting to G_{n+m-1}, we have

$$(\pi \bar{\times} \tau)|_{G_{n+m-1}} \cong (\nu^{1/2}\pi) \times (\tau|_{G_{m-1}}), \tag{3.1}$$

where the isomorphism is given by $f \mapsto (g \mapsto f(\mathrm{diag}(g, 1)))$. Here we naturally identify $\pi \boxtimes \tau$ and $(\nu^{1/2}\pi) \boxtimes (\tau|_{G_{m-1}})$. We may also sometimes simply write $\times$ for $\bar{\times}$.

3.2. Associative property. The following lemma follows from an inspection. We omit the details.

Lemma 3.1 ([44]). *Let* $\pi_1 \in \mathrm{Alg}(G_{n_1})$. *Let* $\pi_2 \in \mathrm{Alg}(G_{n_2})$. *Let* $\tau \in \mathrm{Alg}(M_r)$. *Then:*

(1) $(\pi_1 \bar{\times} \tau) \bar{\times} \pi_2 \cong \pi_1 \bar{\times} (\tau \bar{\times} \pi_2)$,

(2) $(\pi_1 \times \pi_2) \bar{\times} \tau \cong \pi_1 \bar{\times} (\pi_2 \bar{\times} \tau)$,

(3) $(\tau \bar{\times} \pi_1) \bar{\times} \pi_2 \cong \tau \bar{\times} (\pi_1 \times \pi_2)$.

3.3. From parabolic to mirabolic induction. The appearance of mirabolic inductions comes from the study of parabolic inductions when restricting to the mirabolic subgroup via Mackey theory. The following lemma will be used several times.

Lemma 3.2 ([6, Proposition 4.13]). *Let* π_1 *and* π_2 *be* G_{n_1} *and* G_{n_2}*-representations. Then* $(\pi_1 \times \pi_2)|_M$ *admits a short exact sequence*

$$0 \to \pi_1|_M \bar{\times} \pi_2 \to (\pi_1 \times \pi_2)|_M \to \pi_1 \bar{\times} (\pi_2|_M) \to 0.$$

3.4. Connection to Bernstein–Zelevinsky functors.

Lemma 3.3 ([6, Proposition 4.13]). *Let* $\pi \in \mathrm{Alg}(G_n)$. *Let* $\tau \in \mathrm{Alg}(M_k)$. *Then:*

(1) $\Psi^-(\tau \bar{\times} \pi) \cong \Psi^-(\tau) \times \pi$,

(2) $0 \to \Phi^-(\tau) \bar{\times} \pi \to \Phi^-(\tau \bar{\times} \pi) \to \Psi^-(\tau) \bar{\times} (\pi|_M) \to 0$.

The following result is standard. We omit the details.

Lemma 3.4. *For* $\pi \in \mathrm{Alg}(G_r)$,

$$(\Phi^+)^k \Psi^+(\pi) \cong \pi \bar{\times} \Pi_{k+1}.$$

It is also convenient to define another functor:

$$\Lambda : \mathrm{Alg}(G_n) \to \mathrm{Alg}(M_{n+1})$$

by

$$\Lambda(\pi) = {}^u\mathrm{Ind}_{G_n}^{M_{n+1}} \nu^{-1/2}\pi.$$

By definitions, $\Lambda(\pi) \cong 1|_{M_1} \bar{\times} \pi$. When $n = 0$, then Λ defines an isomorphism between vector spaces.

Proposition 3.5. *Let* $r \geq 0$. *Let* $\pi \in \mathrm{Alg}(G_r)$. *For* $s \geq 0$,

$$\Pi_{s+1} \bar{\times} \pi \cong (\Phi^+)^s(\Lambda(\pi)).$$

Proof. Recall that $\Pi_{s+1} = (\Phi^+)^s(1)$ (and 1 is the trivial representation of M_1) and so, by $\Psi^- \circ \Phi^+ = 0$, $\Psi^-(\Pi_{s+1-k}) = 0$ for $k = 0, \ldots, s-1$. This with Lemma 3.3(2) implies, for $k = 0, \ldots, s-1$,

$$(*) \qquad \Phi^-(\Pi_{s+1-k} \bar{\times} \pi) \cong \Phi^-(\Pi_{s+1-k}) \bar{\times} \pi \cong \Pi_{s-k} \bar{\times} \pi.$$

Here in the last isomorphism, we use $\Pi_{s+1-k} = (\Phi^+)^{s-k}(1)$ for any k and $\Phi^- \circ \Phi^+ \cong \mathrm{Id}$.

Now, for $0 \leq k \leq s-1$,

$$(**) \qquad (\Psi^-)(\Pi_{s+1-k} \bar{\times} \pi) = 0,$$

where the equality follows from Lemma 3.3 (1) and above discussions.

Now repeatedly using Bernstein–Zelevinsky theory [6, Proposition 3.2] (see property (4) of the functors in Section 2.1) on $\Pi_{s+1-k} \bar{\times} \pi$ ($k = 0, 1, \dots, s-1$) with $(*)$ and $(**)$, we have

$$\Pi_{s+1} \bar{\times} \pi \cong \Phi^+(\Pi_s \bar{\times} \pi) \cong \cdots \cong (\Phi^+)^s(\Pi_1 \bar{\times} \pi).$$

The last isomorphism simply yields

$$\Pi_{s+1} \bar{\times} \pi \cong (\Phi^+)^s(\Lambda(\pi)). \qquad \square$$

3.5. A transfer lemma. We shall need the following transfer or reduction:

Lemma 3.6. *Let $\pi_1 \in \mathrm{Alg}(G_k)$ and $\pi_2 \in \mathrm{Alg}(G_l)$. Let $\pi_3 \in \mathrm{Alg}(G_n)$ with $n \geq l+k$. Let $a = n+1-(k+l)$. Then, for any σ in $\mathrm{Irr}^c(G_{n+1-(k+l)})$ such that $\sigma \notin \mathrm{csupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_3)$, and for any i,*

$$\mathrm{Ext}^i_{G_n}(\pi_1 \times ((\sigma \times \pi_2)|_{G_{n-k}}), \pi_3) \cong \mathrm{Ext}^i_{G_n}(\pi_1 \times ((\Pi_a \bar{\times} \pi_2)|_{G_{n-k}}), \pi_3).$$

Proof. Again Lemma 3.2 gives a filtration on $(\sigma \times \pi_2)|_{M_{n+1-k}}$ as

$$0 \to \sigma|_M \bar{\times} \pi_2 \to (\sigma \times \pi_2)|_M \to \sigma \bar{\times} (\pi_2|_M) \to 0.$$

Restricting to G_{n-k}, this gives the filtration

$$0 \to (\sigma|_M \bar{\times} \pi_2)|_{G_{n-k}} \to (\sigma \times \pi_2)|_{G_{n-k}} \to (\nu^{1/2}\sigma) \times (\pi_2|_{G_{l-1}}) \to 0.$$

With $\Pi_a = \sigma|_M$, producting with π_1 gives the exact sequence

$$(3.2) \qquad 0 \to \pi_1 \times ((\Pi_a \bar{\times} \pi_2)|_{G_{n-k}}) \to \pi_1 \times ((\sigma \times \pi_2)|_{G_{n-k}}) \to \pi_1 \times (\nu^{1/2}\sigma) \times (\pi_2|_{G_{l-1}}) \to 0.$$

The standard argument using second adjointness of Frobenius reciprocity and comparing cuspidal support at $\nu^{1/2}\sigma$ gives that, for all i,

$$\mathrm{Ext}^i_{G_n}(\pi_1 \times (\nu^{1/2}\sigma) \times (\pi_2|_{G_{l-1}}), \pi_3) = 0.$$

Thus long exact sequence from (3.2) gives that, for all i,

$$\mathrm{Ext}^i_{G_n}(\pi_1 \times ((\Pi_a \bar{\times} \pi_2)|_{G_{n-k}}), \pi_3) \cong \mathrm{Ext}^i_{G_n}(\pi_1 \times ((\sigma \times \pi_2)|_{G_{n-k}}), \pi_3). \qquad \square$$

3.6. A lemma on Speh representation.

Lemma 3.7. *Let $\pi = u_\rho(m, d)$ be a Speh representation. Let π' be in $\mathrm{Alg}(G_k)$. Let $n+1 = n_\pi + k$. Let π'' be an irreducible representation of G_n such that $\nu^{1/2}(\nu^{(m+d-2)/2}\rho)$ is not in $\mathrm{cupp}(\pi'')$. Then there exists a short exact sequence, as G_n-representations,*

$$0 \to K \to (\pi|_M \bar{\times} \pi')|_{G_n} \to Q \to 0$$

such that, for all i,

$$\mathrm{Ext}^i_{G_n}(Q,\pi'')=0$$

and

$$K\cong((\nu^{-1/2}u_\rho(m,d-1))\bar{\times}(\Pi_p\bar{\times}\pi'))|_{G_n}\cong u_\rho(m,d-1)\times((\Pi_p\bar{\times}\pi')|_{G_{k+p-1}}),$$

where $p=n_\rho m$, and

$$\mathrm{Ext}^i_{G_n}(K,\pi'')\cong\mathrm{Ext}^i_{G_n}((\pi|_M\bar{\times}\pi')|_{G_n},\pi'').$$

Proof. From the bottom piece of Bernstein–Zelevinsky filtration (Lemma 2.2), $\pi|_M$ has the submodule (see Section 2.1 and Lemma 3.4)

$$K':=\nu^{-1/2}u_\rho(m,d-1)\bar{\times}\Pi_p$$

and $(\pi|_M)/K'$ admits a M-filtration whose successive quotients isomorphic to $\pi^{(j)}\bar{\times}\Pi_j$ for $j<p$ (see similar discussions in Section 2.3). Let $G=G_n$. Now taking mirabolic product is exact and so one would have, by a long exact sequence argument,

$$\mathrm{Ext}^i_G((\pi|_M\bar{\times}\pi')|_G,\pi'')\cong\mathrm{Ext}^i_G((K'\bar{\times}\pi')|_G,\pi'')$$

if we can show that, for all i,

$$\mathrm{Ext}^i_G(((\pi|_M)/K'\bar{\times}\pi'|_G,\pi'')=0$$

To show the last Ext vanishing, it suffices to show that for each piece of Bernstein–Zelevinsky layer $\tau=\pi^{(j)}\times\Pi_j$ $(j<p)$ appearing in $(\pi|_M)/K'$,

$$\mathrm{Ext}^i_G((\tau\bar{\times}\pi')|_G,\pi'')=0$$

for any i, which indeed follows from:

$$\begin{aligned}&\mathrm{Ext}^i_G(((\pi^{(j)}\bar{\times}\Pi_j)\bar{\times}\pi')|_G,\pi'')\\&\quad\cong\mathrm{Ext}^i_G((\nu^{1/2}\pi^{(j)})\times((\Pi_j\bar{\times}\pi')|_{G_{j+k-1}}),\pi'')\\&\quad\cong\mathrm{Ext}^i_{G_{n_\pi-j}\times G_{j+k-1}}((\nu^{1/2}\pi^{(j)})\boxtimes(\Pi_j\bar{\times}\pi'),(\pi'')_{N^-_{j+k-1}})\\&\quad\cong 0,\end{aligned}$$

where the first isomorphism follows from Lemma 3.1 (1) and (3.1), the second isomorphism follows from Frobenius reciprocity, and the last isomorphism follows from Lemma 2.2 (2) with comparing cuspidal supports. The last isomorphism follows from Lemma 2.4. □

4. Proof of Conjecture 1.2 (non-Archimedean)

The main goal of this section is to prove Conjecture 1.2 (non-Archimedean) modulo Proposition 4.1 and Proposition 4.2. Roughly speaking, Lemmas 3.6 and 3.7 reduce to a bottom layer in a filtration and then Lemma 4.4 reduces the computation of the bottom layer to an inductive case. One also needs a Gan–Gross–Prasad-type reduction (Lemma 4.3) to transfer the study to the inductive case.

4.1. Dual restriction.

Proposition 4.1. *Let π_1 and π_2 be irreducible representations of G_{n+1} and G_n, respectively. For $\sigma \in \mathrm{Irr}^c(G_2)$ such that σ is not in $\mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_1^\vee) \cup \mathrm{cupp}_{\mathbb{Z}}(\pi_2)$, and for all i,*

$$\mathrm{Ext}^i_{G_n}(\pi_1|_{G_n}, \pi_2^\vee) \cong \mathrm{Ext}^i_{G_{n+1}}((\pi_2 \times \sigma)|_{G_{n+1}}, \pi_1^\vee).$$

The proof of Proposition 4.1 will be postponed to Proposition 5.5, where we will prove a more general statement. Note that the additional cuspidal support condition $\sigma \notin \mathrm{cupp}_{\mathbb{Z}}(\pi_2)$ (cf. Proposition 5.5) guarantees that $\sigma \times \pi_2 \cong \pi_2 \times \sigma$, while it is not critical in the proof of the GGP conjecture.

4.2. Product preserving quotients .

Proposition 4.2. *Let $\rho \in \mathrm{Irr}^{u,c}$. Fix m, d. Let π_1 be a (not necessarily admissible) representation of G_n. Let $p = n_\rho md$. Let π_2 be an irreducible representation of G_{n+p} such that any cuspidal representation in $\mathrm{cupp}(\pi_2)$ is either*

1. *lying in $\mathrm{cupp}(u_\rho(m,d)) = \{\nu^{-(m+d-2)/2}\rho, \dots, \nu^{(m+d-2)/2}\rho\}$, or*
2. *not lying in $\{\nu^n\nu^{(m+d)/2}\rho\}_{n\in\mathbb{Z}}$.*

Then if

$$\mathrm{Hom}_{G_{n+p}}(u_\rho(m,d) \times \pi_1, \pi_2) \neq 0,$$

then there exists a non-zero irreducible quotient ω of π_1 such that $\pi_2 \cong u_\rho(m,d) \times \omega$, moreover, if π_2 is an irreducible Arthur-type representation, then such ω is also an irreducible Arthur-type representation.

Proposition 4.2 will be proved as a special case of Corollary 9.4. Proposition 4.2 is only needed for the only if direction.

4.3. Proof of non-tempered GGP.

Recall that $\mathrm{Irr}^{u,c}(G_k)$ is the set of irreducible unitarizable cuspidal representations of G_k.

The following two lemmas are the keys for reductions to an inductive case.

Lemma 4.3. *Let π_p and π_q be Arthur-type representations of G_{n+1} and G_n, respectively. Write*

$$\pi_p = \pi_{p,1} \times \dots \times \pi_{p,r}, \quad \pi_q = \pi_{q,1} \times \dots \times \pi_{q,s}$$

for some Speh representations $\pi_{p,i}, \pi_{q,j}$. Write $\pi_{p,i} = u_{\rho_i}(m_i, d_i)$ and $\pi_{q,j} = u_{\sigma_j}(l_j, e_j)$. Suppose $m_1 + d_1 \geq m_i + d_i$ and $m_1 + d_1 \geq l_j + e_j$ for all i, j. Then

$$\mathrm{Hom}_{G_n}(\pi_p, \pi_q) \neq 0$$

if and only if for any $\widetilde{\sigma} \in \mathrm{Irr}^{u,c}(G_{n_{\rho_1}m_1})$ such that $\widetilde{\sigma} \notin \mathrm{cupp}_{\mathbb{Z}}(\pi_p) \cup \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_q)$,

$$\mathrm{Hom}_{G_n}(u_{\rho_1}(m_1, d_1 - 1) \times ((\widetilde{\sigma} \times \pi'_p)|_{G_a}), \pi_q) \neq 0,$$

where $\pi'_p = \pi_{p,2} \times \dots \times \pi_{p,r}$ and $a = n - n_{\rho_1}m_1(d_1 - 1)$.

Proof. By Lemma 3.2,

$$0 \to \pi_{p,1}|_M \bar{\times} \pi'_p \to \pi_p|_M \to \pi_{p,1} \bar{\times} (\pi'_p|_M) \to 0. \tag{4.1}$$

Let $n_1 = n_{\rho_1} d_1 m_1$ and $n' = n - n_1$. Now

$$\begin{aligned} \mathrm{Ext}^i_{G_n}((\pi_{p,1} \bar{\times} (\pi'_p|_M))|_{G_n}, \pi_q) &\cong \mathrm{Ext}^i_{G_n}((\nu^{1/2}\pi_{p,1}) \times (\pi'_p|_{G_{n'}}), \pi_q) \\ &\cong \mathrm{Ext}^i_{G_{n_1} \times G_{n-n_1}}((\nu^{1/2}\pi_{p,1}) \boxtimes (\pi'_p|_{G_{n'}}), (\pi_q)_{N^-_{n-n_1}}) \\ &= 0, \end{aligned}$$

where the first isomorphism follows from (3.1) and Lemma 3.1 (1) and the second isomorphism follows from second adjointness of Frobenius reciprocity and the third isomorphism follows by comparing cuspidal support at $\nu^{1/2}\nu^{(d_1+m_1-2)/2}\rho_1$.

Thus long exact sequence argument on (4.1) gives that, for all i,

$$\mathrm{Ext}^i_{G_n}((\pi_{p,1}|_M \bar{\times} \pi'_p)|_{G_n}, \pi_q) \cong \mathrm{Ext}^i_{G_n}(\pi_p|_{G_n}, \pi_q). \tag{4.2}$$

Set $u' = \pi^-_{p,1} \cong u_{\rho_1}(m_1, d_1 - 1)$ and $u'' = \nu^{-1/2}u'$. Now Lemma 3.7 gives that

$$\mathrm{Ext}^i_{G_n}(((u'' \bar{\times} \Pi) \times \pi'_p)|_{G_n}, \pi_q) \cong \mathrm{Ext}^i_{G_n}((\pi_{p,1}|_M \bar{\times} \pi'_p)|_{G_n}, \pi_q), \tag{4.3}$$

where $\Pi = \Pi_{n_{\rho_1} m_1}$.

For any $\widetilde{\sigma} \in \mathrm{Irr}^{u,c}(G_{n_{\rho_1} m_1})$ not appearing in $\mathrm{cupp}_{\mathbb{Z}}(\pi_p) \cup \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_q)$,

$$\begin{aligned} \mathrm{Ext}^i_{G_n}(u' \times ((\widetilde{\sigma} \times \pi'_p)|_{G_t}), \pi_q) &\cong \mathrm{Ext}^i_{G_n}(u' \times ((\Pi \bar{\times} \pi'_p)|_{G_t}), \pi_q) \\ &\cong \mathrm{Ext}^i_{G_n}(((u'' \bar{\times} \Pi) \times \pi'_p)|_{G_n}, \pi_q), \end{aligned} \tag{4.4}$$

where $t = n' + n_{\rho_1} m_1$. Here the first isomorphism follows from Lemma 3.6 and the second isomorphism follows from Lemma 3.1 (1) and (3.1),

By equations (4.2), (4.3) and (4.4) at the case that $i = 0$, we obtain the following equivalent statements:

(1) one has $\mathrm{Hom}_{G_n}(\pi_p|_{G_n}, \pi_q) \neq 0$,

(2) one has $\mathrm{Hom}_{G_n}(u_{\rho_1}(m_1, d_1 - 1) \times ((\widetilde{\sigma} \times \pi'_p)|_{G_t}), \pi_q) \neq 0$ for any $\widetilde{\sigma} \in \mathrm{Irr}^{u,c}(G_{n_{\rho_1} m_1})$ not appearing in $\mathrm{cupp}_{\mathbb{Z}}(\pi_p) \cup \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_q)$. □

Lemma 4.4. *We keep using notations in the previous lemma. We still assume that* $m_1 + d_1 \geq m_i + d_i$ *and* $m_1 + d_1 \geq l_j + e_j$ *for all* i, j. *Then*

$$\mathrm{Hom}_{G_n}(\pi_p|_{G_n}, \pi_q) \neq 0$$

if and only if there exists k *such that*

$$\pi_{q,k} \cong u_{\rho_1}(m_1, d_1 - 1),$$

and for any $\widetilde{\sigma} \in \mathrm{Irr}^{u,c}(G_{n_{\rho_1} m_1})$ *with* $\widetilde{\sigma} \notin \mathrm{cupp}_{\mathbb{Z}}(\pi_p) \cup \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_q)$,

$$\mathrm{Hom}_{G_{n'}}((\widetilde{\sigma} \times \pi'_p)|_{G_{n'}}, \pi'_q) \neq 0,$$

where $n' = n - n_{\rho_1} m_1(d_1 - 1)$ *and* $\pi'_q = \pi_{q,1} \times \cdots \times \pi_{q,k-1} \times \pi_{q,k+1} \times \cdots \times \pi_{q,s}$.

Proof. We first consider the "if" direction. To this end, let $\widetilde{\sigma} \in \mathrm{Irr}^{u,c}(G_{n_{\rho_1} m_1})$ not appear in $\mathrm{cupp}_{\mathbb{Z}}(\pi_p) \cup \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_q)$. By the hypothesis of the "if" direction, $\widetilde{\sigma} \times \pi'_p$ has a quotient π'_q (where π'_q is defined as in the lemma). Hence, by exactness of parabolic induction,

$$u_{\rho_1}(m_1, d_1 - 1) \times (\widetilde{\sigma} \times \pi'_p)$$

has a quotient

$$\pi_{q,k} \times \pi'_q \cong u_{\rho_1}(m_1, d_1 - 1) \times \pi'_q \cong \pi_q.$$

Thus, by the "if" part of Lemma 4.3, we obtain

$$\mathrm{Hom}_{G_n}(\pi_p|_{G_n}, \pi_q) \neq 0.$$

We now consider the "only if" direction. Suppose $\mathrm{Hom}_{G_n}(\pi_p|_{G_n}, \pi_q) \neq 0$. By using the "only if" part of Lemma 4.3, we have

$$\mathrm{Hom}_{G_n}(u_{\rho_1}(m_1, d_1 - 1) \times ((\widetilde{\sigma} \times \pi'_p)|_{G_t}), \pi_q) \neq 0$$

for some $\widetilde{\sigma} \in \mathrm{Irr}^{u,c}(G_{n_{\rho_1} m_1})$ not in $\mathrm{cupp}_{\mathbb{Z}}(\pi_p) \cup \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_q)$. Here

$$t = n - n_{\rho_1} m_1 (d_1 - 1).$$

From the condition on $m_1 + d_1$, one checks the conditions in Proposition 4.2 and so we can apply it to obtain that

$$\pi_2 \cong u_{\rho_1}(m_1, d_1 - 1) \times \omega$$

for some irreducible Arthur-type quotient ω of $(\widetilde{\sigma} \times \pi'_p)|_{G_t}$. Now by uniqueness of factorization of Arthur-type representations in terms of Speh representations, there exists some k^* such that

$$\pi_{q,k^*} \cong u_{\rho_1}(m_1, d_1 - 1), \quad \pi_{q,1} \times \cdots \times \pi_{q,k^*-1} \times \pi_{q,k^*+1} \times \cdots \times \pi_{q,s} \cong \omega.$$

This proves the only if direction. □

Theorem 4.5. *Conjecture* 1.2 *holds for non-Archimedean field F.*

Proof. We shall prove the reformulated problem in Section 1.2. Let π_p and π_q be Arthur-type representations of G_{n+1} and G_n, respectively. We can write as the product of Speh representations, i.e.

$$\pi_p = \pi_{p,1} \times \cdots \times \pi_{p,r} \quad \text{and} \quad \pi_q = \pi_{q,1} \times \cdots \times \pi_{q,s}$$

such that each $\pi_{p,i}$ (respectively, $\pi_{q,j}$) is an (irreducible unitarizable) Speh representation $u_{\rho_i}(m_i, d_i)$ (respectively, $u_{\sigma_j}(l_j, e_j)$). Let $N(\pi_p, \pi_q)$ be the total number of factors $\pi_{p,i}$ and $\pi_{p,j}$ which are not cuspidal representation. The basic case is that all $\pi_{p,i}$ and $\pi_{q,j}$ are cuspidal representations, i.e. $N(\pi_p, \pi_q) = 0$, and so π_p and π_q are generic. In that case, it is well known from [18, 27].

By [41, Theorem 7.1], we may and shall assume that for $1 \leq i \leq r$, $1 \leq j \leq s$,

$$m_1 + d_1 \geq m_i + d_i \quad \text{and} \quad l_1 + e_1 \geq l_j + e_j.$$

We may also assume that $m_1 + d_1 > 2$ or $l_1 + e_1 > 2$, and so either $\pi_{p,1}$ or $\pi_{q,1}$ is not cuspidal. Otherwise, it is the basic case.

We now consider two cases:

Case 1: $m_1 + d_1 \geq l_1 + e_1$. Then

$$\frac{m_1 + d_1 - 2}{2} + \frac{1}{2} > \frac{l_i + e_i - 2}{2}$$

for all i, and so $\nu^{1/2}\nu^{(d_1+m_1-2)/2}\rho_1$ is in $\mathrm{cupp}(\nu^{1/2}\pi_{p,1})$, but is not in the cuspidal support of any $\pi_{q,i}$. Let

$$\pi'_p = \pi_{p,2} \times \cdots \times \pi_{p,r}.$$

Let $u = \pi_{p,1} \cong u_{\rho_1}(m_1, d_1)$.

We first prove the "only if" direction and assume that $\mathrm{Hom}_{G_n}(\pi_p, \pi_q) \neq 0$. Using Lemma 4.4, there exists $\sigma \in \mathrm{Irr}^{u,c}(G_{n_{\rho_1} m_1})$ with $\sigma \notin \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_q)$ and k^* such that

$$\pi_{q,k^*} = u^- \quad \text{and} \quad \mathrm{Hom}_{G_t}(\sigma \times \pi'_p, \pi'_q) \neq 0,$$

where $\pi'_q = \pi_{q,1} \times \cdots \times \pi_{q,k^*-1} \times \pi_{q,k^*+1} \times \cdots \times \pi_{q,s}$ and $t = n - n_{\rho_1} m_1 (d_1 - 1)$. Since $\sigma \times \pi'_p$ is also an Arthur-type representation with

$$N(\sigma \times \pi'_p, \pi'_q) = N(\pi'_p, \pi'_q) < N(\pi_p, \pi_q),$$

we can apply inductive hypothesis to obtain

$$\sigma \times \pi'_p \cong \tau_{p,1} \times \cdots \times \tau_{p,k} \times \tau^-_{q,1} \times \cdots \times \tau^-_{q,l}$$

and

$$\pi'_q \cong \tau^-_{p,1} \times \cdots \times \tau^-_{p,k} \times \tau_{q,1} \times \cdots \times \tau_{q,l}$$

for some Speh representations $\tau_{p,1}, \ldots, \tau_{p,k}, \tau_{q,1}, \ldots, \tau_{q,l}$. Since the product is uniquely determined by the factors of those Speh representations [41] and $\sigma \notin \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi'_q)$, we must have $\tau_{p,i^*} \cong \sigma$ for some i^*. Since the products between Speh representations commute, we may simply set $i^* = 1$. With $\tau^-_{p,1} = 1$, now we have

$$\pi_p \cong u \times \pi'_p \cong u \times \tau_{p,2} \times \cdots \times \tau_{p,k} \times \tau^-_{q,1} \times \cdots \times \tau^-_{q,l}$$

and

$$\pi_q \cong u^- \times \pi'_q \cong u^- \times \tau^-_{p,2} \times \cdots \times \tau^-_{p,k} \times \tau_{q,1} \times \cdots \times \tau_{q,l},$$

as desired.

Now we prove the "if" direction and so we consider

$$\pi_p \cong \tau_{p,1} \times \cdots \times \tau_{p,k} \times \tau^-_{q,1} \times \cdots \times \tau^-_{q,l}$$

and

$$\pi_q \cong \tau^-_{p,1} \times \cdots \times \tau^-_{p,k} \times \tau_{q,1} \times \cdots \times \tau_{q,l}$$

for some Speh representations $\tau_{p,1}, \ldots, \tau_{p,k}, \tau_{q,1}, \ldots, \tau_{q,l}$. From our choice of $\pi_{p,1}$ and the assumption for Case 1, we must have that, by reindexing if necessary,

$$\tau_{p,1} \cong \pi_{p,1}.$$

Then $\tau^-_{p,1} \cong u_{\rho_1}(m_1, d_1 - 1)$. This implies that

$$\tau_{p,2} \times \cdots \times \tau_{p,k} \times \tau^-_{q,1} \times \cdots \times \tau^-_{q,l} \cong \pi_{p,2} \times \cdots \times \pi_{p,r} = \pi'_p,$$

by unique factorization of Speh representations [41]. Since

$$N(\sigma \times \pi'_p, \pi''_q) = N(\pi'_p, \pi''_q) < N(\pi_p, \pi''_q) \leq N(\pi_p, \pi_q),$$

induction gives that for any σ of $\mathrm{Irr}^{u,c}(G_{n_{\rho_1} m_1})$,

$$\mathrm{Hom}_{G_{n'}}(\sigma \times \pi'_p, \pi''_q) \neq 0,$$

where

$$\pi''_q \cong \sigma^- \times \tau^-_{p,2} \times \cdots \times \tau^-_{p,k} \times \tau_{q,1} \times \cdots \times \tau_{q,l}.$$

Lemma 4.4 implies that $\mathrm{Hom}_{G_n}(\pi_p|_{G_n}, \pi_q) \neq 0$, as desired.

Case 2: $l_1 + e_1 > m_1 + d_1$. Then

$$\frac{l_1 + e_1 - 2}{2} + \frac{1}{2} > \frac{m_1 + d_1 - 2}{2}.$$

There are infinitely many unitarizable cuspidal representations of G_2, and we can find one satisfying the hypothesis in Proposition 4.1 so that

$$\begin{aligned} \mathrm{Hom}_{G_{n+1}}(\pi_q \times \sigma|_{G_{n+1}}, \pi_p) \neq 0 &\iff \mathrm{Hom}_{G_n}(\pi_p^\vee|_{G_n}, \pi_q^\vee) \neq 0 \\ &\iff \mathrm{Hom}_{G_{n+1}}(\overline{\pi}_p|_{G_n}, \overline{\pi}_q) \neq 0 \\ &\iff \mathrm{Hom}_{G_{n+1}}(\pi_p|_{G_n}, \pi_q) \neq 0. \end{aligned}$$

Here $\overline{\pi}_p, \overline{\pi}_q$ are complex conjugate representations of π_p, π_q, respectively, and so the last "if and only if" implication is immediate. The first "if and only if" implication follows from Proposition 4.1 and the second one follows from that π_p, π_q are unitarizable and so Hermitian self-dual.

We also have that $\pi_q \times \sigma$ is still an Arthur-type representation. Note that

$$N(\pi_q \times \sigma, \pi_p) = N(\pi_p, \pi_q).$$

We now use the argument in Case 1 and inductive hypothesis to prove this case, where the role of $\pi_{q,1}$ replaces the one of $\pi_{p,1}$. □

5. General cases: Bessel, Fourier–Jacobi and Rankin–Selberg models

In this section, we shall generalize the non-tempered GGP to other models of general linear groups. We study some connections between models, which will be continued in Section 6. We also improve some previous multiplicity results for Bessel and Fourier–Jacobi models to the Ext versions.

5.1. Equal rank Fourier–Jacobi models. Let $S(F^n)$ be the space of Bruhat–Schwartz functions on F^n. For a character μ of G_n, let $\omega_{\mu,0}$ (respectively, $\widehat{\omega}_{\mu,0}$) be a G_n-representation with underlying space $S(F^n)$ and the G_n-action given by

$$(g.f)(v) = \mu(g) f(g^{-1}v) \quad \text{(respectively, } (g.f)(v) = \mu(g) f(g^t v)).$$

Let $\pi \in \mathrm{Alg}(G_n)$. Since $G_n \setminus M_{n+1} \cong F^n$ as topological spaces, and $\omega_{\mu\nu^{-1/2},0} \otimes \pi$ can be viewed as the space of smooth compactly-supported functions $f : F^n \to \mu\nu^{-1/2}\pi$ with G_n acting by $(g.f)(v) = g.f(g^{-1}v)$, we have

$$\mu \otimes \Lambda(\pi)|_{G_n} \cong \omega_{\mu\nu^{-1/2},0} \otimes \pi$$

via the natural map for $f \in \Lambda(\pi)$,

$$f \mapsto \left(v \mapsto f\left(\begin{pmatrix} I_n & v \\ 0 & 1 \end{pmatrix} \right) \right).$$

Set $\zeta^F = \omega_{\nu^{-1/2},0}$ and set $\widehat{\zeta}^F = \widehat{\omega}_{\nu^{1/2},0}$.

Proposition 5.1. *Let $\pi, \pi' \in \mathrm{Alg}(G_n)$. Then there exists a character χ of $F^\times$ such that $\chi \notin \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi')$ and, for all i,*

$$\mathrm{Ext}^i_{G_n}((\chi \times \pi)|_{G_n}, \pi') \cong \mathrm{Ext}^i_{G_n}(\pi \otimes \zeta^F, \pi').$$

The assertion also holds if we replace for ζ^F by $\widehat{\zeta}^F$.

Proof. By Lemma 3.2,

$$0 \to \chi|_{M_1} \bar{\times} \pi \to (\chi \times \pi)|_M \to \chi \bar{\times} (\pi|_M) \to 0. \tag{5.1}$$

Then $\chi|_{M_1} \bar{\times} \pi \cong \Lambda(\pi)$ by the definition of mirabolic induction. By using the above identification, we have

$$\chi|_{M_1} \bar{\times} \pi \cong \pi \otimes \zeta^F. \tag{5.2}$$

On the other hand, via Frobenius reciprocity, due to the condition that $\chi \notin \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi')$, Lemma 2.4 implies that for all i,

$$\mathrm{Ext}^i_{G_n}((\chi \bar{\times} (\pi|_M))|_{G_n}, \pi') \cong \mathrm{Ext}^i_{G_n}((\nu^{1/2}\chi) \times (\pi|_{G_{n-1}}), \pi') = 0. \tag{5.3}$$

Now standard long exact sequence argument on (5.1) with (5.2) and (5.3) gives, for all i,

$$\mathrm{Ext}^i_{G_n}(\chi \times \pi, \pi') \cong \mathrm{Ext}^i_{G_n}((\chi|_{M_1} \bar{\times} \pi)|_{G_n}, \pi') \cong \mathrm{Ext}^i_{G_n}(\pi \otimes \zeta^F, \pi').$$

The proof for $\widehat{\zeta}^F$ is similar. □

Remark 5.2. From Proposition 5.1, one can deduce explicit restriction for equal rank Fourier–Jacobi model from the basic restriction from G_{n+1} to G_n. One may also compare with the method using theta correspondence to deduce Fourier–Jacobi models from Bessel models in [20] and [4].

5.2. Bessel, Rankin–Selberg and mixed models. Let $m_1, m_2, r \geq 0$. Recall that $\bar{\psi}$ is a choice of a non-degenerate character on F. Let

$$H = \left\{ \begin{pmatrix} u_1 & x & y \\ & h & z \\ & & u_2 \end{pmatrix} : u_1 \in U_{m_1},\ u_2 \in U_{m_2},\ h \in \widetilde{G}_{r+1},\ x \in \mathrm{Mat}_{m_1 \times (r+1)},\right.$$
$$\left. z \in \mathrm{Mat}_{(r+1)\times m_2},\ y \in \mathrm{Mat}_{m_1 \times m_2} \right\} \subset G_{m_1+m_2+r+1}$$

and

$$\widetilde{G}_{r+1} = \{\mathrm{diag}(1, g) : g \in G_r\}.$$

We shall also write H^B or $H^B_{m_1,m_2,r}$ for H.

Let $\varphi_n : U_n \to \mathbb{C}$ be a non-degenerate character on U_n. For example, one may take

$$\varphi_n(u) = \bar{\psi}(u_{1,2} + \cdots + u_{n-1,n}).$$

Let $\zeta : H \to \mathbb{C}$ such that

$$\zeta\left(\begin{pmatrix} u_1 & x & y \\ & g & z \\ & & u_2 \end{pmatrix}\right) = \varphi_{m_1}(u_1)\varphi_{m_2}(u_2)\bar{\psi}(x_{m_1,1})\bar{\psi}(z_{1,1})\nu(g)^{-(m_2-m_1)/2},$$

where $x_{m_1,1}$ (respectively, $z_{1,1}$) is the $(m_1, 1)$-(respectively, $(1, 1)$-)coordinate of x (respectively, z). We shall also sometimes write ζ^B for ζ. Note that $\nu^{m_2-m_1}$ is the modulus function of H (i.e. a normalizing factor).

Let U' be the unipotent radical of the group H. The orbit by the conjugation action of $(T_{m_1+1} \times G_r \times T_{m_2})U'$ on ϕ is the unique dense orbit on the character space of U', where T_{m_1+1} (respectively, T_{m_2}) be the subgroup of diagonal matrices of G_{m_1+1} (respectively, G_{m_2}), and as subgroup of H via embedding to the upper (respectively, lower) corner.

Remark 5.3. The Bessel subgroup defined in [18, Sections 12 and 13] is conjugate to $H^B_{m,m,r}$, where $r = n - 2m$, for some m. When $m_1 = 0$ or $m_2 = 0$, the model is sometimes called a Rankin–Selberg model [17, 21]. We shall also write

$$H^R_{m,r} = H^B_{0,m,r} \quad \text{and} \quad \zeta^R = \zeta^B.$$

(The matrix $H^R_{m,r}$ is conjugate to the one in Section 1.3.) When $r = 0$, the model is Whittaker [38], and when $m_1 = m_2 = 0$, it is related to the restriction from G_{n+1} to G_n in [1].

There is another formulation of Bessel models, using Bernstein–Zelevinsky functors.

Proposition 5.4. *Let π be a G_r-representation, which extends to an H-representation trivially. Let $n = m_1 + m_2 + r + 1$. Then there exist natural isomorphisms:*

$$\begin{aligned} {}^u\mathrm{ind}_{H^B_{m_1,m_2,r}}^{G_n} \pi \otimes \zeta^B \otimes \nu^{(m_2-m_1)} &\cong (\Phi^+)^{m_2+1}(\Pi_{m_1+1} \bar{\times} \pi)|_{G_n} \\ &\cong (\Phi^+)^{(m_1+m_2+1)}(\Lambda(\pi))|_{G_n} \\ &\cong {}^u\mathrm{ind}_{H^R_{m_1+m_2,r}}^{G_n} \pi \otimes \zeta^R \otimes \nu^{m_1+m_2}. \end{aligned}$$

Proof. The second isomorphism follows from Proposition 3.5. Note that the last isomorphism is a special case of the first isomorphism. It remains to prove the first isomorphism. Let

$$w = \mathrm{diag}\left(\begin{pmatrix} 0 & I_r \\ I_{m_1+1} & 0 \end{pmatrix}, I_{m_2+1}\right).$$

Using induction in stages, the subgroup from which

$$(\Phi^+)^{m_2+1}(\Pi_{m_1+1} \bar{\times} \pi))|_{G_n}$$

is induced, takes the form

$$Q' = \begin{pmatrix} g & & & * \\ * & m & * & * \\ & & 1 & * \\ & & & u \end{pmatrix},$$

where $g \in G_r$, $m \in G_{m_1}$ and $u \in U_{m_2}$, and so

$$w^{-1} Q' w = H^B_{m_1,m_2,r}.$$

The conjugation by the element w then defines a map Γ from ${}^u\mathrm{ind}_H^{G_n} \pi \otimes \zeta^B \otimes \nu^{m_2-m_1}$ to $(\Phi^+)^{m_2+1}(\Pi_{m_1+1} \bar{\times} \pi))|_{G_n}$, as vector spaces, given by

$$f \mapsto \left(g \mapsto f \left(w \begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) \right).$$

Restricted to the unipotent subgroup U' of H^B, $\Gamma(f)$ is copies of character ζ^B, while a function in $(\Phi^+)^{m_2+1}(\Pi_{m_1+1} \bar{\times} \pi)$ restricted to U' is copies of another character in the same B'-orbit as ζ^B, where B' contains matrices of the form $\mathrm{diag}(I_r, T_l U_l)$, where $l = m_1+m_2+1$. Hence there exists $b \in B'$ such that the map

$$f \mapsto \left(g \mapsto f \left(bw \begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) \right)$$

is a G_n-isomorphism.

We also remark that the character $\nu^{1/2}$ arisen when restricted to G_n cancels with the character $\nu^{-1/2}$ arisen from the mirabolic induction in $\Pi_{m_1+1} \bar{\times} \pi$. □

The following result is proved by a similar method as in [18], also see [17].

Proposition 5.5. *Let π_1, π_2 be representations of G_n and G_r, respectively. Let*

$$m_1 + m_2 + r + 1 = n.$$

For any irreducible cuspidal representation σ of $G_{m_1+m_2+2}$ such that $\sigma \notin \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_1^\vee)$, and for all i,

$$\mathrm{Ext}^i_{H^B_{m_1,m_2,r}}(\pi_1 \otimes \zeta^B, \pi_2^\vee) \cong \mathrm{Ext}^i_{G_n}(\sigma \times \pi_2, \pi_1^\vee).$$

Remark 5.6. Proposition 4.1 is a particular case of Proposition 5.5 for $m_1 = 0, m_2 = 0$ and $r = n-1$.

Proof. By Lemma 3.2 again,

$$0 \to \sigma|_M \bar{\times} \pi_2 \to (\sigma \times \pi_2)|_M \to \sigma \bar{\times} (\pi_2|_M) \to 0.$$

Since σ is cuspidal, we have $\sigma|_M \cong \Pi_{m_1+m_2+2}$. Now with Propositions 3.5 and 5.4,

$${}^u\mathrm{ind}_H^G \pi_2 \otimes \zeta^B = (\sigma|_M \bar{\times} \pi_2)|_{G_n}.$$

Again the cuspidal condition guarantees that, for all i,

$$\mathrm{Ext}^i_{G_n}((\nu^{1/2}\sigma)\times(\pi_2|_{G_{r-1}}),\pi_1^\vee)=0.$$

Now similar argument with the proof of Proposition 5.1, one reduces to, for all i,

$$\begin{aligned}\mathrm{Ext}^i_{G_n}(\sigma\times\pi_2,\pi_1^\vee) &\cong \mathrm{Ext}^i_{G_n}((\sigma|_M\bar{\times}\pi_2)|_{G_n},\pi_1^\vee)\\ &\cong \mathrm{Ext}^i_{G_n}({}^u\mathrm{ind}_H^{G_n}\pi_2\otimes\zeta^B\otimes\nu^{m_2-m_1},\pi_1^\vee)\\ &\cong \mathrm{Ext}^i_{G_n}(\pi_1,{}^u\mathrm{Ind}_H^{G_n}(\pi_2\otimes\zeta^B)^\vee) && \text{(taking duals)}\\ &\cong \mathrm{Ext}^i_H(\pi_1,(\pi_2\otimes\zeta^B)^\vee) && \text{(Frobenius reciprocity)}\\ &\cong \mathrm{Ext}^i_H(\pi_1\otimes\zeta^B,\pi_2^\vee) && \text{(taking duals).}\end{aligned}$$

For the last three isomorphism, also see [36]. □

5.3. Fourier–Jacobi models. Let $S(F^r)$ be the space of Bruhat–Schwartz functions on F^r. Let $W=F^r$ and let K_r be the Heisenberg group, i.e. K_r is the group isomorphic to $F\oplus W\oplus W^\vee$ with the multiplication

$$(a,v,w)\cdot(a',v',w')=(a+a'+w^tv',v+v',w+w').$$

Define

$$H'_r=\left\{\begin{pmatrix}1 & w^t & a\\ & g & v\\ & & 1\end{pmatrix} : v,w\in F^r,\ a\in F,\ g\in G_r\right\}$$

and so $H'_r\cong G_r\ltimes K_r$. Here we identify W and $W^\vee$ with F^r so that $y(x)=y^tx$ for $x\in W$ and $y\in W^\vee$.

Fix a character μ of G_r. Let λ be a non-trivial character on F. The Weil representation $\omega_{\mu,\lambda}$ of K_r associated to λ is the representation with underlying space as $S(W)$ with the action of K_r given by: for $f\in S(W)\cong S(F^r)$,

$$((a,v,w).f)(x)=\lambda(a-w^tx-w^tv)f(x+v).$$

and for $f\in S(W^\vee)\cong S(F^r)$,

$$((a,v,w).f)(y)=\lambda(a+y^tv)f(y+w).$$

This extends $\omega_{\mu,\lambda}$ to an H'_r-representation $\widetilde{\omega}_{\mu,\lambda}$ (respectively, $\widehat{\omega}_{\mu,\lambda}$) given by: for $g\in G_r$ and $f\in S(W)$ (respectively, $f\in S(W^\vee)$),

$$(g.f)(x)=\mu(g)\cdot f(g^{-1}.x)\quad\text{(respectively, }(g.f)(y)=\mu(g)\cdot f(g^t.y)).$$

Lemma 5.7. *Let $\pi\in\mathrm{Alg}(G_r)$, extend trivially to H'_r. Then*

$$\pi\otimes\widehat{\omega}_{\mu,\bar{\psi}}\cong {}^u\mathrm{ind}_{H^B_{0,1,r}}^{H'_r}\mu\pi\otimes(\zeta^B\otimes\nu^{1/2}).$$

Proof. We can identify $\nu^{-1/2}\pi\otimes\widehat{\omega}_{\mu,\bar{\psi}}$ with the space of smooth compactly supported functions $f:F^r\to\nu^{-1/2}\pi$ with the action given by

$$(g.f)(y)=g.f(g^ty).$$

Since $H^B_{0,1,r} \setminus H'_r \cong F^r$ as topological spaces, the identification gives a map

$$\mathcal{F} : \pi \otimes \widehat{\omega}_{\mu,\bar{\psi}} \to {}^u\mathrm{ind}_{H^B_{0,1,r}}^{H'_r} \mu\pi \otimes (\zeta^B \nu^{1/2})$$

given by

$$\mathcal{F}(f)(y) = f\left(\begin{pmatrix} 1 & y^t & \\ & I_r & \\ & & 1 \end{pmatrix}\right). \qquad \square$$

Now we consider general Fourier–Jacobi models. Let $m_1, m_2 \geq 1$. Let H (respectively, U_H) be the subgroups of $G_{m_1+m_2+r}$ containing all elements of the form

$$\begin{pmatrix} u_1 & x & y \\ & h & z \\ & & u_2 \end{pmatrix} \quad \left(\text{respectively, } \begin{pmatrix} u_1 & x & y \\ & I_{r+2} & z \\ & & u_2 \end{pmatrix}\right)$$

with entries $u_1 \in U_{m_1-1}$, $u_2 \in U_{m_2-1}$, $h \in H'_r$, $x \in \mathrm{Mat}_{m_1-1,r+2}$, $y \in \mathrm{Mat}_{m_1-1,m_2-1}$ and $z \in \mathrm{Mat}_{r+2,m_2-1}$. We shall also write $H^F_{m_1,m_2,r}$ or H^F for H. Note that $H \cong H'_r \ltimes U_H$. In the case that $m_1 = m_2 = 1$, it recovers the notion for H'_r.

We now extend the representations $\omega_{\mu,\lambda}$ of H'_r to be a representation of H, still denoted $\omega_{\mu,\lambda}$ by abuse of notation, whose underlying space is $S(F^r)$ with the action, for $f \in S(F^r)$,

$$\begin{pmatrix} u_1 & x & y \\ & h & z \\ & & u_2 \end{pmatrix}.f = \varphi_{m_1}(u_1)\varphi_{m_2}(u_2)(h.f).$$

We similarly define the representation $\widehat{\omega}_{\mu,\lambda}$.

Set

$$\zeta = \zeta^F_{m_1,m_2,r,\lambda} = \zeta^F = \nu^{(m_1-m_2)/2}\widetilde{\omega}_{\nu^{-1/2},\lambda}$$

and

$$\widehat{\zeta} = \widehat{\zeta}^F_{m_1,m_2,r,\lambda} = \widehat{\zeta}^F = \nu^{(m_1-m_2)/2}\widehat{\omega}_{\nu^{1/2},\lambda}.$$

Again when $m_1 = m_2$, it is the original notion of Fourier–Jacobi model in [18, Section 15]. The restriction problems involving ζ^F (and $\widehat{\zeta}^F$) (i.e. $\mathrm{Hom}_H(\pi_1 \otimes \zeta^F, \pi_2)$) do not depend on a choice of λ.

Proposition 5.8. *Let $n = m_1 + m_2 + r$ with $m_1, m_2, r \geq 1$. Let $\pi \in \mathrm{Alg}(G_r)$. Then*

$${}^u\mathrm{ind}_{H^B_{m_1-1,m_2,r}}^{G_n} \pi \otimes \zeta^B \otimes \nu^{m_2-m_1+1} \cong {}^u\mathrm{ind}_{H^F_{m_1,m_2,r}}^{G_n} \pi \otimes \widehat{\zeta}^F \otimes \nu^{m_2-m_1}.$$

Proof. From constructions, $\zeta^B|_{U_H} \cong \widehat{\zeta}^F|_{U_H}$. Note that H'_r normalizes U_H and the conjugation action of H'_r on $\widehat{\zeta}^F|_{U_H}$ is trivial. One can extend the identification in Lemma 5.7 to, as H^F-representations,

$$\pi \otimes (\widehat{\zeta}^F \otimes \nu^{-1/2}\nu^{(m_2-m_1)/2}) \cong {}^u\mathrm{ind}_{H^B}^{H^F} \pi \otimes (\zeta^B \otimes \nu^{(m_2-m_1+1)/2}).$$

Now applying induction from H^F to G, an induction by stages gives the lemma. $\square$

In view of Propositions 3.5, 5.4 and 5.8, we can prove in a similar way as in the proof of Proposition 5.5 (also similar to the proof of Proposition 5.1). We omit the details.

Proposition 5.9. *Let $m_1, m_2, r \geq 1$. Let $n = m_1 + m_2 + r$. Let $\pi_1 \in \mathrm{Alg}(G_n)$ and let $\pi_2 \in \mathrm{Alg}(G_r)$. Then, for any cuspidal representation σ of G_{n+1-r} that does not appear in $\mathrm{cupp}_{\mathbb{Z}}(\pi_2) \cup \mathrm{cupp}_{\mathbb{Z}}(\nu^{-1/2}\pi_1^{\vee})$, and for any i,*

$$\mathrm{Ext}^i_{H^F_{m_1,m_2,r}}(\pi_1 \otimes \widehat{\zeta}^F, \pi_2^{\vee}) \cong \mathrm{Ext}^i_{G_n}(\sigma \times \pi_2, \pi_1^{\vee}).$$

Now we give a connection of the two notions ζ^F and $\widehat{\zeta}^F$.

Proposition 5.10. *Let $m_1, m_2, r \geq 1$ and let $n = m_1 + m_2 + r$. Let $\pi_1 \in \mathrm{Alg}(G_n)$ and let $\pi_2 \in \mathrm{Alg}(G_r)$. For all i,*

$$\mathrm{Ext}^i_H(\pi_1 \otimes \zeta^F, \pi_2^{\vee}) \cong \mathrm{Ext}^i_{\widetilde{H}}(\theta(\pi_1) \otimes \widehat{\zeta}^F, \theta(\pi_2)^{\vee}),$$

where $\widetilde{H} = H^F_{m_2,m_1,r} = w\theta(H)w^{-1}$. Here w is the matrix with all 1 in the antidiagonal and 0 elsewhere.

Proof. Let θ^w be the action of θ followed by the conjugation of w. We use the same θ^w for the induced map on representations. Note that $\theta^w(\pi_1) \cong \theta(\pi_1)$ as G_n-representations, $\theta^w(\pi_2^{\vee}) \cong \theta(\pi_2^{\vee}) \cong \theta(\pi_2)^{\vee}$ as G_r-representation, and $\theta^w(\zeta^F_{\lambda}) \cong \widehat{\zeta}^F_{\lambda^{-1}}$. □

5.4. Restrictions. We state the multiplicity one and finiteness for the general cases (cf. [18]):

Corollary 5.11. *Let (H, ζ) be any pair described in Sections 5.1, 5.2 and 5.3. Let π_1 and π_2 be irreducible representations of G_n and G_r, respectively. Then*

$$\dim \mathrm{Hom}_H(\pi_1 \otimes \zeta, \pi_2) \leq 1$$

and for all i,

$$\dim \mathrm{Ext}^i_H(\pi_1 \otimes \zeta, \pi_2) < \infty$$

Proof. Proposition 5.5 reduces to the case that restricting from G_{n+1} to G_n, which is proved in [1] for Hom and follows from [2, 36] for higher Ext. □

Theorem 5.12. *Let (H, ζ) be any pair described in Sections 5.1, 5.2 and 5.3. Let π_M and π_N be Arthur-type representations of G_n and G_r, respectively. Then*

$$\mathrm{Hom}_H(\pi_M \otimes \zeta, \pi_N) \neq 0$$

if and only if their associated Arthur parameters M_A and N_A are relevant.

Proof. When $r = 0$, the model is Whittaker and it is well known. Assume $r \geq 1$. For the Bessel models, this follows from Proposition 5.5 (in which we choose σ to be a unitarizable cuspidal representation) and Theorem 4.5. For the Fourier–Jacobi models, using Propositions 5.9 and 5.10, it is equivalent to show that $\theta(\pi_M)$ and $\theta(\pi_N)$ have relevant Arthur parameters. By the Gelfand–Kazhdan isomorphism [7], $\theta(\pi_M) \cong \pi_M^\vee$ and $\theta(\pi_N) \cong \pi_N^\vee$. Thus now the statement follows from that π_M, π_N have relevant Arthur parameter if and only if $\pi_M^\vee, \pi_N^\vee$ have relevant Arthur parameter. □

5.5. A filtration on parabolically induced modules. The notion of those models also provide a convenient way to state the following filtration, which can be regarded as a systematic tool for studying restriction of parabolically induced representations (e.g. [12]). For example, one may use it to replace some arguments in Lemmas 3.6 and 3.7.

Proposition 5.13. *Let $\pi_1 \in \mathrm{Alg}(G_{n_1})$ and let $\pi_2 \in \mathrm{Alg}(G_{n_2})$. Let $n_1 + n_2 = n + 1$. Then there exists a filtration on $(\pi_1 \times \pi_2)|_{G_n}$*

$$0 \subset \tau_n \subset \tau_{n-1} \subset \cdots \subset \tau_1 \subset \tau_0 = \pi_1 \times \pi_2$$

such that

$$\tau_0/\tau_1 \cong (\nu^{1/2}\pi_1) \times (\pi_2|_{G_{n_2-1}})$$

and

$$\tau_1/\tau_2 \cong \pi_1^{[1]} \times (\pi_2 \otimes \zeta^F),$$

and for $k \geq 2$,

$$\tau_k/\tau_{k+1} \cong \pi_1^{[k]} \times {}^u\mathrm{ind}_{H^R_{k-2,n_2}}^{G_{n_2+k-1}} \pi_2 \otimes \zeta^R \otimes \nu^{k-2}.$$

Proof. This is a consequence of Lemma 3.2, Bernstein–Zelevinsky filtrations (for some details, see Lemma 3.7), and Proposition 3.5. □

5.6. Consequence on Ext-branching law. We also deduce the Ext-analog result in [16] for Bessel and Fourier–Jacobi models.

Corollary 5.14. *Let (H, ζ) be any pair described in Sections 5.1, 5.2 and 5.3. Let π_1 and π_2 be irreducible generic representations of G_n and G_r, respectively. Then, for all $i \geq 1$,*

$$\mathrm{Ext}^i_H(\pi_1 \otimes \zeta, \pi_2) = 0.$$

Proof. The case of the Bessel model for $r = n - 1$ is proved in [16]. The general case now follows from the case in [16] and Propositions 5.1, 5.5, 5.9 and 5.10. (We remark that for a suitable choice of $\sigma \in \mathrm{Irr}^{u,c}$, $\sigma \times \pi_1$ is still generic.) □

6. Fourier–Jacobi models and Bernstein–Zelevinsky theory

In Section 5, we apply Bernstein–Zelevinsky theory to obtain isomorphisms of models. In this section, we further investigate the isomorphisms, and a goal is to obtain Corollary 6.3.

6.1. Fourier–Jacobi model and its dual. Recall that ζ_F and $\widehat{\zeta}_F$ are defined in Sections 5.1 and 5.3. We first consider the equal rank case.

Proposition 6.1. *In the equal rank case, $\zeta^F \cong \widehat{\zeta}^F$ as G_n-representations.*

Proof. Let $a \in F^{\times}$. For $f \in S(F^r)$, define the Fourier transform

$$\widehat{f}(y) = \int_{F^r} \bar{\psi}(ay^t x) f(x)\, dx, \tag{6.1}$$

which is still smooth and compactly supported, and so in $S(F^r)$, and we regard it as a map from ζ_F to $\widehat{\zeta}_F$. It is straightforward to check well-definedness of the map. One can define the inverse similarly. □

The above proposition can also be proved by considering the Hecke algebra realization at each Bernstein component, and deduced from left and right filtrations in [13, 16].

Proposition 6.2. *We use the Fourier–Jacobi models in Section* 5.3 *and the Fourier transform defined in* (6.1). *The map* $\Omega : S(F^r) \to S(F^r)$ *by* $f \mapsto (y \mapsto \widehat{f}(-a^{-1}y))$ *defines an* H_r'*-map from* $\zeta^F_{1,1,r,\bar{\psi}}$ *to* $\widehat{\zeta}^F_{1,1,r,\bar{\psi}}$.

Proof. It follows from a straightforward computation as in the previous proposition. We omit the details. □

We summarize the identifications as follow:

Corollary 6.3. *Let* $\pi \in \mathrm{Alg}(G_r)$. *For* $m_1, m_2, r \geq 1$,

$$\begin{aligned} {}^u\mathrm{ind}_{H^F_{m_1,m_2,r}}^{G_n} \pi \otimes \zeta^F \otimes \nu^{m_2-m_1} &\cong {}^u\mathrm{ind}_{H^F_{m_1,m_2,r}}^{G_n} \pi \otimes \widehat{\zeta}^F \otimes \nu^{m_2-m_1} \\ &\cong {}^u\mathrm{ind}_{H^B_{m_1-1,m_2,r}}^{G_n} \pi \otimes \zeta^B \otimes \nu^{m_2-m_1+1} \\ &\cong {}^u\mathrm{ind}_{H^R_{m_1-1+m_2,r}}^{G_n} \pi \otimes \zeta^R \otimes \nu^{m_1+m_2-1}. \end{aligned}$$

Proof. Proposition 6.2 implies that, as $H^F_{m_1,m_2,r}$-representations,

$$\pi \otimes \zeta^F \cong \pi \otimes \widehat{\zeta}^F$$

and hence we obtain the isomorphism. Now the remaining isomorphisms follow from Proposition 5.4. □

Remark 6.4. As we have seen, there is a more direct connection via (5.8) and the first isomorphism of Proposition 5.4

$${}^u\mathrm{ind}_{H^F_{m_1,m_2,r}}^{G_n} \pi \otimes \widehat{\zeta}^F \otimes \nu^{m_2-m_1} \cong (\Phi^+)^{m_2+1}(\Pi_{m_1} \bar{\times} \pi))|_{G_n}, \tag{6.2}$$

and similarly, we can obtain

$${}^u\mathrm{ind}_{H^F_{m_1,m_2,r}}^{G_n} \pi \otimes \zeta^F \otimes \nu^{m_2-m_1} \cong (\Phi^+)^{m_2}(\Pi_{m_1+1} \bar{\times} \pi))|_{G_n}. \tag{6.3}$$

The left-hand side of (6.2) and (6.3) are connected via Fourier transform in Proposition 6.2, while the right-hand side of (6.2) and (6.3) can be directly connected via Bernstein–Zelevinsky theory (Proposition 3.5).

Corollary 6.5. *Let $m_1, m_2, r \geq 1$. Let $\pi_1 \in \mathrm{Alg}(G_{m_1+m_2+r})$ and let $\pi_2 \in \mathrm{Alg}(G_r)$. There are natural isomorphisms*

$$\begin{aligned}\mathrm{Ext}^i_{H^F_{m_1,m_2,r}}(\pi_1 \otimes \zeta^F, \pi_2^\vee) &\cong \mathrm{Ext}^i_{H^F_{m_1,m_2,r}}(\pi_1 \otimes \widehat{\zeta}^F, \pi_2^\vee)\\ &\cong \mathrm{Ext}^i_{H^B_{m_1-1,m_2,r}}(\pi_1 \otimes \zeta^B, \pi_2^\vee)\\ &\cong \mathrm{Ext}^i_{H^R_{m_1+m_2-1,r}}(\pi_1 \otimes \zeta^R, \pi_2^\vee).\end{aligned}$$

Example 6.6. We consider the equal rank Fourier–Jacobi model. For a generalized Steinberg representation $\mathrm{St}(\Delta)$ of G_n, we expect that $\mathrm{St}(\Delta) \otimes \zeta^F$ is projective and is isomorphic to the Gelfand–Graev representation of G_n (cf. [13–16]).

7. Ext-branching laws

7.1. Conjecture on Ext-branching laws. We formulate the following question about Ext-branching laws stated in the form of a conjecture, which gives a possible generalization of some observations in [19].

Conjecture 7.1. Let π_M and π_N be Arthur-type representations of G_{n+1} and G_n, respectively. Then, for any i,

$$\mathrm{Ext}^i_{G_n}(\pi_M, \pi_N) \cong \bigoplus_k \mathrm{Ext}^i_{G_{n+1-k}}(\pi_M^{[k]}, {}^{(k-1)}\pi_N).$$

It would be an interesting question to give a more precise formulation on predicting non-vanishing Ext-groups of Arthur-type representations (see [19, Proposition 5.7, Remark 5.8]).

We remark that the appearance of left derivatives in the second spot comes from the second adjointness property of an induction in the Bernstein–Zelevinsky filtration (see e.g. [16, Lemma 2.4]). We shall give few examples of the above conjecture below.

7.2. Hom-branching.

Example 7.2. Let π_M and π_N be generic Arthur-type representations of G_{n+1} and G_n, respectively. Then $\pi_M = \mathrm{St}(\mathfrak{m})$ and $\pi_N = \mathrm{St}(\mathfrak{n})$ for some multisegments $\mathfrak{m}$ and $\mathfrak{n}$. A computation via comparing cuspidal support gives that, for $i \neq 0$ or $k \neq n$,

$$\mathrm{Ext}^i_{G_n}(\pi_M^{[k+1]}, {}^{(k)}\pi_N) = 0.$$

Then

$$\mathrm{Hom}_{G_n}(\pi_M, \pi_N) \cong \mathrm{Hom}_{G_0}(\pi_M^{[n+1]}, {}^{(n)}\pi_N) \cong \mathbb{C}.$$

This recovers the Ext-vanishing theorem [16, 36] and the multiplicity one theorem [1, 40] in this special case.

We remark that the same formulation of Conjecture 7.1 for arbitrary respective generic representations π_M and π_N of G_{n+1} and G_n is not true.

Example 7.3. Let π_M and π_N be Arthur-type representations of G_{n+1} and G_n, respectively. Suppose their associated Arthur parameters are relevant. Write those Arthur parameters

M_A and N_A as (1.2) and (1.3), respectively. Then Conjecture 7.1 for Hom-case follows from (Theorem 4.5 and) the following:

$$\text{(7.1)}\quad \mathrm{Hom}_{G_{n+1-k}}(\pi_M^{[k+1]}, {}^{(k)}\pi_N) \neq 0 \iff k = \sum_{d=0}^{r} \dim M_d^{+} - 1 = \sum_{d=0}^{s} \dim M_d^{-}.$$

The direction "⇐" is easy. For the "⇒" direction, one may hope to compute the Hom of those derivatives directly while it seems it have not been done so far. We shall sketch how to modify the proof of Theorem 4.5 to see (7.1). We use all the notations in the proof of Theorem 4.5, and in particular, write

$$\pi_M = \pi_p = \pi_{p,1} \times \cdots \times \pi_{p,r}, \text{ and } \quad \pi_N = \pi_q = \pi_{q,1} \times \cdots \times \pi_{q,s}.$$

The basic case is again all $\pi_{p,i}, \pi_{q,j}$ are cuspidal, which is included in Example 7.2. Since taking duals behaves well with derivatives, Case 2 (in Theorem 4.5) follows from Case 1.

We only consider Case 1. Again, we use the short exact sequence

$$0 \to \pi_{p,1}|_M \bar{\times} \pi_p' \to \pi_p|_M \to \pi_{p,1} \bar{\times} (\pi_p'|_M) \to 0.$$

Note that any Bernstein–Zelevinsky layer of $\pi_{p,1} \times (\pi_p'|_M)$ cannot contribute a non-zero Hom with π_2, by comparing cuspidal support. With similar consideration as in Theorem 4.5, the only Bernstein–Zelevinsky layer that can contribute non-zero Hom with π_1 takes the form

$$(\nu^{-1/2}\pi_{p,1}^{-}) \bar{\times} (\Pi \times \pi_p'),$$

which can then be transferred to study the layers in $(\nu^{-1/2}\pi_{p,1}^{-}) \bar{\times} ((\sigma \times \pi_p')|_M)$. Now one applies induction on the unique layer in $(\sigma \times \pi_p')|_M$ that can contribute non-zero Hom with π_q, which gives the required integer in (7.1).

7.3. Generic representations. An irreducible representation π of G_n is generic if it admits a Whittaker model or equivalently $\pi^{(n)} \neq 0$. The classification of generic representations of G_n in terms of segments is obtained in [44, Section 9]. We now treat the case that when one of Arthur-type representations is tempered and hence is generic. Compared to the Hom-case (also see [23, Theorem 5.1] and [13, Corollary 2.8]), a wider class of Arthur-type representations can be paired to obtain non-vanishing higher Ext-groups.

Theorem 7.4. *Let π_p and π_q be Arthur-type representations of G_{n+1} and G_n, respectively. Suppose at least one of π_p or π_q is generic.*

(1) *Then there exists at most one integer j^* such that*

$$\mathrm{Ext}^i_{G_n}(\pi_p^{[j^*]}, {}^{(j^*-1)}\pi_q) \neq 0$$

for some i and furthermore if π_p (respectively, π_q) is not generic, then j^* (respectively, j^*-1) is the level of π_p (respectively, π_q); and if both π_p and π_q are generic, then $j^* = n+1$.

(2) *Suppose π_p is generic. Then such j^* in* (1) *exists if and only if $\pi_p \cong \pi_q^{\mathrm{gen}} \times \pi'$, where π_q^{gen} is the generic representation with same cuspidal support as π_q^{-}, and π' is some irreducible generic (tempered) representation.*

(3) *Suppose π_q is generic. An analogous statement holds by switching the role of π_p and π_q in* (2).

Proof. We first consider (1). Assume that π_p is not a generic representation and π_q is a generic representation. Let

$$\pi_p = \pi_{p,1} \times \cdots \times \pi_{p,r}, \quad \pi_q = \pi_{q,1} \times \cdots \times \pi_{q,s},$$

where each $\pi_{p,i}$ is a Speh representation and each $\pi_{q,j}$ is isomorphic to $\mathrm{St}(\Delta_{q,j})$ for some segment $\Delta_{q,j}$. Then the i-th derivative $\pi_p^{[i]}$ takes the form, for $i_1 + \cdots + i_r = i$,

$$\nu^{1/2}(\pi_{p,1}^{(i_1)} \times \cdots \times \pi_{p,r}^{(i_r)}).$$

For each representation ω, we call the cuspidal support $\mathrm{cupp}(\omega)$ is:

(1) G-positive (respectively, G-negative) if for each irreducible unitarizable cuspidal representation σ and for all positive (respectively, negative) integer a, the multiplicity of $\nu^a\sigma$ in $\mathrm{cupp}(\omega)$ is at least that of $\nu^{-a}\sigma$,

(2) balanced if $\mathrm{cupp}(\omega)$ is both G-positive and G-negative.

Write $\pi_{p,j} = u_\rho(m,d)$. Note that for any i such that $\pi_{p,j}^{(i)}$ is non-zero,

$$\mathrm{cupp}(\nu^{1/2}\pi_{p,j}^{(i)}) = \mathrm{cupp}(\pi_{p,j}^-) + \mathrm{cupp}(\mathrm{St}(\Delta))$$

for $\Delta = [\nu^{(m-d)/2+k+1/2}\rho, \nu^{(m+d-2)/2+1/2}\rho]$, where $k = i/n_\rho$. Since $\mathrm{cupp}(\pi_{p,j}^-)$ is balanced and $\mathrm{cupp}(\mathrm{St}(\Delta))$ is G-positive, it follows that $\nu^{1/2}\pi_{p,j}^{(i)}$ is G-positive for any i and is balanced only if i is the level of $\pi_{p,j}$.

On the other hand, since $\pi_{q,j}$ is a generalized Steinberg representation, it follows that ${}^{(i-1)}\pi_{q,j}$ is G-negative for all i and is balanced only if $i = 0$ or i is the level of $\pi_{q,j}$. Thus we have $\mathrm{cupp}(\pi_1^{[i]}) = \mathrm{cupp}({}^{(i-1)}\pi_2)$ only if i is the level of π_1 as desired.

Other cases are similar, or one may use Lemma 4.1.

We now consider (2). The above discussion proves the only if direction by the cuspidal support consideration. It remains to prove the if direction. From above discussion, it suffices to show that $\mathrm{Ext}^i_{G_{n+1-j^*}}(\pi_q^{\mathrm{gen}}, \pi_q^-) \neq 0$ for some i. Since π_q^{gen} is generic, we can write π_q^{gen} as

$$\pi_q^{\mathrm{gen}} \cong \mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_k),$$

where each $\Delta_i = [\nu^{-a}\rho, \nu^a\rho]$ for some a and some unitarizable representation ρ. For simplicity, set $\pi' = \pi_q^-$. Thus via Frobenius reciprocity, it suffices to show

$$(*) \qquad E_i := \mathrm{Ext}^i(\mathrm{St}(\Delta_1) \boxtimes \cdots \boxtimes \mathrm{St}(\Delta_k), \pi'_{N^-}) \neq 0,$$

where N^- is the opposite unipotent radical associated to the parabolic subgroup in the product $\mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_k)$. Now the Jacquet module of π'_{N^-} is computed in [28]. In order to describe the composition factors of π'_{N^-} which contribute non-zero Ext take the form, we need some more notations: For a Speh representation $\tau = u_\rho(m,d)$, we associate with a collection $\mathcal{S}_{\rho,m,d}$ of "hook-shaped multisegments":

$$\{[\nu^{-(m+d-2)/2}\rho], \dots, [\nu^{(m-d)/2-1}\rho], [\nu^{(m-d)/2}\rho, \nu^{(m+d-2)/2}\rho]\},$$
$$\{[\nu^{-(m+d-2)/2+1}\rho], \dots, [\nu^{(m-d)/2-1}\rho], [\nu^{(m-d)/2}\rho, \nu^{(m+d-2)/2-1}\rho]\}, \ \dots \ .$$

It ends in a segment depending on m, d: if $m > d$, the last multisegment takes the form

$$\{[\nu^{-(m-d)/2}\rho], \dots, [\nu^{(m-d)/2}\rho]\}$$

and, if $m < d$, the last multisegment takes the form

$$\{[\nu^{-(m-d)/2}\rho, \nu^{(m-d)/2}\rho]\}$$

and if $m = d$, the last multisegment takes the form $[\rho]$.

Now we arrange the segments $\Delta_1, \dots, \Delta_k$ such that if $\Delta_i \cap \Delta_j \neq \emptyset$ and $i < j$, then $\Delta_i \subset \Delta_j$. For a Speh representation $u_\rho(m, d)$, define $X(u_\rho(m, d)) = (m - d)/2$. We shall arrange the Speh representations in

$$\pi' = \pi_q^- = \pi_{q,1}^- \times \cdots \times \pi_{q,s}^-$$

such that $X(\pi_{q,1}^-) \leq \cdots \leq X(\pi_{q,s}^-)$.

Using the Kret–Lapid description of Jacquet modules of Speh representations [28], we have the following key properties of $u_\rho(m, d)_{N_r^-}$ (for some r):

- $u_\rho(m, d)_{N_r^-}$ is semisimple,
- for any irreducible composition factor $\omega_1 \boxtimes \omega_2$ of $u_\rho(m, d)_{N_r^-}$, $\nu^{(m-d)/2}\rho$ is in $\mathrm{cupp}(\omega_2)$.

In order to compute $(*)$, we first consider Ext of the form

$$(**) \qquad \mathrm{Ext}^i((\mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_{k-1}) \boxtimes \mathrm{St}(\Delta_k), \pi'_{N^-})$$

and so we have to compute π'_{N^-}. By the geometric lemma, a composition factor takes the form

$$\tau \boxtimes (\omega_1 \times \cdots \times \omega_s),$$

where each ω_l comes from an irreducible composition factor $\delta \boxtimes \omega_l$ in some Jacquet functor $(\pi_{q,l}^-)_{N^-}$ for some opposite unipotent subgroup N^-. We claim the following.

Claim. *Let $\sigma \in \Delta_k$. Suppose*

$$(***) \qquad \mathrm{Ext}^i((\mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_{k-1}) \boxtimes \mathrm{St}(\Delta_k), \tau \boxtimes (\omega_1 \times \cdots \times \omega_s)) \neq 0.$$

Then the sequence $\mathrm{cupp}(\omega_1), \dots, \mathrm{cupp}(\omega_s)$ *satisfies a descending pattern, which means that, for any $\nu^c\rho \in \mathrm{cupp}(\omega_x)$ and $\nu^d\rho \in \mathrm{cupp}(\omega_y)$ with $x < y$, we have $c > d$.*

Proof of Claim. In order to have $(***)$ to be non-zero, by Künneth formula, we must have that $\mathrm{Ext}^{i'}(\mathrm{St}(\Delta_k), \omega_1 \times \cdots \times \omega_s) \neq 0$ for some i'. Now one applies Frobenius reciprocity, and the corresponding Jacquet functor on $\mathrm{St}(\Delta_k)$ is known (see Section 8.2 below). The claim then follows by comparing cuspidal support. □

Recall that the Speh representations in π' is specially arranged, and so the claim with the second bullet of the key properties above implies that there is *exactly one* ω_l is not the trivial representation of G_0.

Now we also recall that the Steinberg representations in π_q^{gen} is specially arranged with Δ_k satisfying certain maximality condition. Such arrangement actually forces that the underlying multisegment of ω_l is a hook-shaped multisegment, which comes from $\pi_{q,l}$ and the corresponding composition factor in π'_{N^-} takes the form $\kappa \boxtimes \omega_l$, where

$$\kappa = \pi_{q,1} \times \cdots \times \pi_{q,l-1} \times \widetilde{\pi}_{q,l} \times \pi_{q,l+1} \times \cdots \times \pi_{q,s},$$

and $\widetilde{\pi}_{q,l}$ is a Speh representation such that $\widetilde{\pi}_{q,l} \boxtimes \omega_l$ is a compaction factor of π'_{N^-}.

Now, rearranging the Speh representations in κ if necessary, one proceeds similarly and inductively for $\Delta_{k-1}, \dots, \Delta_1$ to find a composition factor in π'_{N-} contributing a non-zero E_i.

Hence, the composition factor of π'_{N-} that could contribute a non-zero Ext is isomorphic to objects taking the form

$$\boxtimes_{\rho,m,d} \boxtimes_{\mathfrak{m}\in\cup\mathcal{S}_{\rho,m,d}} \langle \mathfrak{m} \rangle,$$

where ρ, m, d runs through all the data that $u_\rho(m, d)$ is a factor in the Arthur-type representation π_N^-, counting multiplicities. Now using Künneth formula, the computation of Ext^i follows from

$$(\bullet) \qquad \bigoplus_{\sum k_{\rho,m,d}=i} \bigotimes_{\rho,m,d} \bigotimes_{\mathfrak{m}\in\mathcal{S}_{\rho,m,d}} \mathrm{Ext}^{k_{\rho,m,d}}(\mathrm{St}(\Delta^{\mathrm{gen}}(\mathfrak{m})), \langle \mathfrak{m} \rangle),$$

where $\Delta^{\mathrm{gen}}(\mathfrak{m})$ is the segment with the same cuspidal representations as $\langle \mathfrak{m} \rangle$, and ρ, m, d run over all data as above. Then when $\rho = 1$, it follows from [34] that for each set of data ρ, m, d, there is at least one k such that

$$\mathrm{Ext}^k(\mathrm{St}(\Delta^{\mathrm{gen}}(\mathfrak{m})), \langle \mathfrak{m} \rangle) \neq 0$$

and one can deduce the general case from a transfer argument of Hecke algebra. We pick the smallest such integers k and denote the sum of those integers k by k^*. Such a k^* is the smallest integer such that Ext of the form

$$\mathrm{Ext}^{k^*}\Bigg(\bigotimes_{\rho,m,d} \bigotimes_{\mathfrak{m}\in\mathcal{S}_{\rho,m,d}} \mathrm{St}(\Delta^{\mathrm{gen}}(\mathfrak{m})), \bigotimes_{\rho,m,d} \bigotimes_{\mathfrak{m}\in\mathcal{S}_{\rho,m,d}} \langle \mathfrak{m} \rangle \Bigg)$$

is non-zero. The hook-shaped multisegments obtained above (see $(**)$ and $(***)$) come from all Speh representations $\{\pi_{q,a}\}_{a=1}^{s}$ possibly in different orders, but any simple composition factors in π'_{N-} obtained above will still give the same $(\bullet)$ after Künneth formula. Thus, a long exact sequence argument can conclude that $E_{k^*} \neq 0$.

Assertion (3) is similar to (2). We omit the details. □

Remark 7.5. For some other related computations of Arthur-type representations, e.g., see Ext-groups of tempered representations [33] and Speh representations from Koszul resolution [10].

7.4. Another example. One can obtain different information from various filtrations on restricted representations [13, 16, 35] such as left and right Bernstein–Zelevinsky filtrations [13, 16]. We shall see another example below using combinations of filtrations:

Example 7.6. Let $\Delta[d] = [\nu^{-(d-1)/2}, \nu^{(d-1)/2}]$. For $e \geq 3$, let

$$\pi_1 = \langle \Delta[e] \rangle \times \mathrm{St}(\Delta[e-2]) \times \sigma,$$

and let

$$\pi_2 = \mathrm{St}(\Delta[e-1]) \times \langle \Delta[e-1] \rangle,$$

where σ is a ramified character. We first investigate possible Bernstein–Zelevinsky layers contributing non-zero Ext-groups. Consider the derivatives

$$^{(i_1)}\langle \Delta[e] \rangle \times {}^{(i_2)}\mathrm{St}(\Delta[e-2]) \times {}^{(i_3)}\sigma \quad \text{and} \quad \mathrm{St}(\Delta[e-1])^{(j_1)} \times \langle \Delta[e-1] \rangle^{(j_2)}$$

and, by comparing cuspidal supports, we must have $i_1 = 1$. Then we have the following two possibilities: either

(1) $j_1 = e - 1$, or

(2) $j_2 = 1$.

In the case $j_1 = e - 1$, by comparing cuspidal support, we have $j_2 = 0$, and then $i_2 = e - 2$. In the case $j_2 = 1$, we have two possibilities:

(1) $j_1 = 0, i_2 = 0$.

(2) $j_1 = e - 2, i_2 = e - 2$

Now we find a cuspidal representation σ' as in Proposition 4.1 to consider the representation $\pi_2 \times \sigma'$. Now we observe that there is two layers $(\pi_2 \times \sigma')|_M$ that contribute non-zero Ext-groups (after restricting to G): Now $(j_1, j_2) = (e - 1, 0)$, it contributes one layer

$$\lambda_1 := \langle \Delta[e-1] \rangle \times \Pi_{e+1}$$

and $(j_1, j_2) = (0, 1)$, it contributes one layer

$$\lambda_2 := \mathrm{St}(\Delta[e-1]) \times \langle \nu^{-1/2} \Delta[e-2] \rangle \times \Pi_3$$

and $(j_1, j_2) = (e - 2, 1)$, it contributes one (reducible) layer

$$\lambda_3 := \lambda = \langle \nu^{-1/2} \Delta[e-2] \rangle \times \nu^{(e-1)/2} \times \Pi_{e+1}.$$

We remark that λ_3 is indecomposable as $\langle \nu^{1/2} \Delta[e-2] \rangle \times \nu^{-(e-1)/2}$ is indecomposable.

We now consider the dual restriction problem in Proposition 4.1, and so we consider the restriction for $\pi_2 \times \sigma'$ for some cuspidal representation σ' of G_2.

Using the following short exact sequence (Lemma 3.2):

$$\begin{aligned} 0 \to \langle \Delta[e-1] \rangle|_M \mathbin{\bar{\times}} (\mathrm{St}(\Delta[e-1]) \times \sigma') \to (\pi_2 \times \sigma')|_M \\ \to \langle \Delta[e-1] \rangle \mathbin{\bar{\times}} ((\mathrm{St}(\Delta[e-1]) \mathbin{\bar{\times}} \sigma')|_M) \to 0, \end{aligned}$$

and letting

$$X^* = \langle \Delta[e-1] \rangle|_M \mathbin{\bar{\times}} (\mathrm{St}(\Delta[e-1]) \times \sigma'),$$

X^* admits a filtration, in which there is one successive quotient isomorphic to λ_2 and another successive quotient isomorphic to λ_3.

Using Bernstein–Zelevinsky filtration, we obtain a filtration on $(\pi_2 \times \sigma')|_M$ of the form

$$0 = Y_{2e} \subset Y_{2e-1} \subset \cdots \subset Y_0 = (\pi_2 \times \sigma')|_M$$

so that

(1) $Y_e / Y_{e+1} \cong (\pi_2 \times \sigma')^{(e+1)} \mathbin{\bar{\times}} \Pi_{e+1}$, and

(2) Y_{e+1} is a simple module which is not isomorphic to any simple composition factor of $\lambda_1, \lambda_2, \lambda_3$, and

(3) Y_e / Y_{e+1} admits a filtration with one quotient isomorphic to λ_1 and another quotient isomorphic to λ_3.

The key of two filtrations is to obtain the following filtration, as M_{n+2}, and the direct sum in the quotient roughly contributes the direct sum of Ext-groups in Conjecture 7.1:

$$0 \to I \to X^* + Y_e \to X^*/I \oplus Y_e/I \to 0,$$

where $I = X^* \cap Y_e$. Let

$$\beta := \langle \{\nu^{-1/2}\Delta[e-2], \nu^{(e-1)/2}\}\rangle \times \Pi_{e+1},$$

which has multiplicity one in $\pi_2 \times \sigma'|_M$. With the above information on X^* and Y_e, we can obtain further structure on I. The multiplicity forces that I contains the unique composition factor β, but the indecomposability of λ_3 also forces I contains the composition factor β, and a count on multiplicities gives that other composition factor of I is not isomorphic to λ_1, λ_2 or β (those are all the possible factors contributing non-zero Ext). Thus, we have that, for all k,

$$\mathrm{Ext}^k_{G_{n+1}}(I|_{G_{n+1}}, \pi_1) = \mathrm{Ext}^k_{G_{n+1}}(\lambda_3|_{G_{n+1}}, \pi_1) = 0.$$

Then we have

$$\begin{aligned}\mathrm{Ext}^k_{G_{n+1}}(\pi_2 \times \sigma', \pi_1) &\cong \mathrm{Ext}^k_{G_{n+1}}((X^* + Y_e)|_{G_{n+1}}, \pi_1)\\ &\cong \mathrm{Ext}^k_{G_{n+1}}(X^*/I, \pi_1) \oplus \mathrm{Ext}^k_{G_{n+1}}(Y_e/I, \pi_1)\\ &\cong \mathrm{Ext}^k_{G_{n+1}}(\lambda_2, \pi_1) \oplus \mathrm{Ext}^k_{G_{n+1}}(\lambda_1, \pi_1)\\ &\cong \mathrm{Ext}^k_{G_{n-1}}((\pi_2^{[1]}, {}^{(2)}\pi_1) \oplus \mathrm{Ext}^k_{G_{n+1-e}}(\pi_2^{[e-1]}, {}^{(e)}\pi_1)\end{aligned}$$

The first isomorphism follows from that the quotients by $X^* + Y_e$ has zero Ext by looking at the possible composition factors and some computations on comparing cuspidal supports. The fourth isomorphism follows from the adjointness of the functors (see [16, Lemma 2.1] for more discussions).

Since $\pi_1^\vee \cong \pi_1$ and $\pi_2^\vee \cong \pi_2$, taking duals and using Proposition 4.1 gives that

$$\mathrm{Ext}^k_{G_n}(\pi_1, \pi_2) \cong \mathrm{Ext}^k_{G_{n-1}}(\pi_1^{[2]}, {}^{(1)}\pi_2) \oplus \mathrm{Ext}^k_{G_{n+1-e}}(\pi_1^{[e]}, {}^{(e-1)}\pi_2).$$

The last isomorphism follows from [16, Lemma 2.2].

8. Product preserving extensions

A motivating example in this section is the following. Let σ be an irreducible cuspidal representation of G_n. Let π_1 and π_2 be two admissible representations of G_k such that the cuspidal supports of irreducible composition factors of π_1 and π_2 do not contain σ. Then a simple application of Frobenius reciprocity and geometric lemma gives

$$\mathrm{Hom}_{G_{n+k}}(\sigma \times \pi_1, \sigma \times \pi_2) \cong \mathrm{Hom}_{G_n}(\sigma, \sigma) \boxtimes \mathrm{Hom}_{G_k}(\pi_1, \pi_2) \cong \mathrm{Hom}_{G_k}(\pi_1, \pi_2).$$

Our goal is to generalize the above isomorphism to a larger class of examples in a functorial way, which is Theorem 9.1.

8.1. Preserving extensions. Let $\mathcal{C} \subset \mathrm{Irr}^c$. Define $\mathrm{Alg}_{\mathcal{C}}(G_m)$ to be the full subcategory of $\mathrm{Alg}(G_m)$ whose objects π have finite lengths and satisfy the property that for any simple composition factor π' of π, and for any $\sigma \in \mathrm{cupp}(\pi')$, σ lies in $\mathcal{C}$.

Theorem 8.1. *Let $\rho \in \mathrm{Irr}^c(G_k)$. Let $\mathcal{C} = \mathcal{C}_{u_\rho(d,m)}$. Let $\pi \in \mathrm{Alg}_{\mathcal{C}}(G_n)$ with length 2. Then π is indecomposable if and only if $u_\rho(m,d) \times \pi$ is indecomposable.*

We will prove Theorem 8.1 in Section 8.5. We expect to prove a more general result elsewhere using some ideas from [11] (see Section 9.2) as well as the case mentioned here.

The idea of the proof is to first prove for a large Speh representation (in the sense of Section 8.3). In such case, one can compute via some simpler computations of Jacquet modules and standard modules. The general case is deduced from "truncating" large Speh representations to the desired one.

Remark 8.2. In general, a product does not preserve extensions even if it preserves irreducibility of composition factors. The standard example is that $\nu \times (1 \times \nu)$, which is of length 2. In this case, $\nu \times \langle [1, \nu] \rangle$ and $\nu \times \mathrm{St}([1, \nu])$ are both irreducible, but $1 \times \nu$ is indecomposable and $\nu \times (1 \times \nu)$ is semisimple.

8.2. Jacquet functors. Recall that N_p is the subgroup of G_n containing all matrices

$$\begin{pmatrix} I_{n-p} & u \\ & I_p \end{pmatrix},$$

where $u \in \mathrm{Mat}_{n-p,p}$. Let $\Delta = [\nu^a \rho, \nu^b \rho]$ be a Zelevinsky segment. Let $m = n_\rho$. Then, by [44, Propositions 3.4 and 9.5], the Jacquet modules are

$$\begin{aligned} \langle \Delta \rangle_{N_{mi}} &= [\nu^a \rho, \nu^{b-i} \rho] \boxtimes [\nu^{b-i+1} \rho, \nu^b \rho], \\ \langle \Delta \rangle_{N_{mi}^-} &= \langle [\nu^{a+i} \rho, \nu^b \rho] \rangle \boxtimes \langle [\nu^a \rho, \nu^{a+i-1} \rho] \rangle, \\ \mathrm{St}(\Delta)_{N_{mi}} &= \mathrm{St}([\nu^{a+i} \rho, \nu^b \rho]) \boxtimes \mathrm{St}([\nu^a \rho, \nu^{a+i-1} \rho]), \\ \mathrm{St}(\Delta)_{N_{mi}^-} &= \mathrm{St}([\nu^a \rho, \nu^{b-i} \rho] \boxtimes \mathrm{St}([\nu^{b-i+1} \rho, \nu^b \rho]). \end{aligned}$$

Note that computing $\pi_{N_i^-}$ is equivalent to first computing $\pi_{N_{n-i}}$ to obtain a $G_i \times G_{n-i}$-representation, then twisting by the action by the element

$$\begin{pmatrix} 0 & I_i \\ I_{n-i} & 0 \end{pmatrix}$$

to obtain a $G_{n-i} \times G_i$-representation.

8.3. Fully-faith product for large Speh. For $\rho \in \mathrm{Irr}^c(G_k)$, $d, m \in \mathbb{Z}_{\geq 1}$, let

$$\widetilde{\Delta}_\rho(d, k) = [\nu^{-(d-1)/2} \rho, \nu^{(d-1)/2+k} \rho].$$

We first consider

$$\widetilde{\mathfrak{m}}_\rho(m, d, k) = \{\nu^{-(m-1)/2} \widetilde{\Delta}(d, k), \ldots, \nu^{(m-1)/2} \widetilde{\Delta}(d, k)\}.$$

Let $\widetilde{u}_\rho(m, d, k) = \langle \widetilde{\mathfrak{m}}_\rho(m, d, k) \rangle$, which is sometimes called essentially Speh representation as it is a Speh representation twisted by a character. In particular, $\widetilde{u}_\rho(m, d, 0) = u_\rho(m, d)$ if ρ is unitarizable.

Lemma 8.3. *Let π_1, π_2 be admissible representations of G_n. Fix $\rho \in \mathrm{Irr}^c$ and integers $d, m \in \mathbb{Z}_{\geq 1}$. For any $k \geq 0$, set $\widetilde{u}_k = \widetilde{u}_\rho(m, d, k)$. For k large enough, we have a natural isomorphism*

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \cong \mathrm{Hom}_{G_{n+p}}(\widetilde{u}_k \times \pi_1, \widetilde{u}_k \times \pi_2),$$

where $p = n_\rho m(d + k)$. Here naturality holds when the isomorphism holds, for both π_1 and π_2 for the same k.

Proof. We set k large enough such that $\nu^{(d-m)/2+k}\rho$ is not in the cuspidal supports of any irreducible representation of π_1 and π_2.

Let $\mathfrak{m} = \widetilde{\mathfrak{m}}_\rho(m,d,k)$ and let $\widetilde{u} = \widetilde{u}_\rho(m,d,k)$. By using the injection

$$\widetilde{u} \times \pi_2 = \langle \mathfrak{m} \rangle \times \pi_2 \hookrightarrow \zeta(\mathfrak{m}) \times \pi_2,$$

the left exactness of $\mathrm{Hom}_{G_{n+p}}(\widetilde{u} \times \pi_1, \cdot)$ gives

$$\mathrm{Hom}_{G_{n+p}}(\widetilde{u} \times \pi_1, \zeta(\mathfrak{m}) \times \pi_2) \hookleftarrow \mathrm{Hom}_{G_{n+p}}(\widetilde{u} \times \pi_1, \widetilde{u} \times \pi_2). \tag{8.1}$$

Let $\Delta = [\nu^{(-d+m)/2}\rho, \nu^{(d+m-2)/2+k}\rho]$. Since $\zeta(\mathfrak{m}) = \langle \Delta \rangle \times \zeta(\mathfrak{m} \setminus \{\Delta\})$,

$$\mathrm{Hom}_{G_{n+p}}(\langle \mathfrak{m} \rangle \times \pi_1, \zeta(\mathfrak{m}) \times \pi_2) \cong \mathrm{Hom}_{G_{n+p}}(\langle \mathfrak{m} \rangle \times \pi_1, \langle \Delta \rangle \times \pi'),$$

where $\pi' = \zeta(\mathfrak{m} \setminus \{\Delta\}) \times \pi_2$.

Let $q = n_\rho m$. Now Frobenius reciprocity gives that

$$\mathrm{Hom}_{G_{n+p}}(\langle \mathfrak{m} \rangle \times \pi_1, \langle \Delta \rangle \times \pi') \cong \mathrm{Hom}_{G_q \times G_{n+p-q}}((\langle \mathfrak{m} \rangle \times \pi_1)_{N_{n+p-q}}, \langle \Delta \rangle \boxtimes \pi').$$

Note that $\nu^{(d+m-2)/2+k}\rho$ does not appear in the cuspidal support of irreducible factors of π_1. With some analysis on Jacquet module from the geometric lemma (see, for example the proof of Lemma 8.6 below for more details), the only composition factor in $(\langle \mathfrak{m} \rangle \times \pi_1)_{N_{n+p-q}}$ that has the same cuspidal support as $\langle \Delta \rangle \boxtimes \pi'$ is

$$\langle \Delta \rangle \boxtimes \langle \mathfrak{m} \setminus \{\Delta\} \rangle \times \pi_1.$$

Thus we have

$$\begin{aligned} \mathrm{Hom}(\langle \mathfrak{m} \rangle \times \pi_1, \langle \Delta \rangle \times \pi') &\cong \mathrm{Hom}(\langle \mathfrak{m} \setminus \{\Delta\} \rangle \times \pi_1, \pi') \\ &= \mathrm{Hom}(\langle \mathfrak{m}' \rangle \times \pi_1, \zeta(\mathfrak{m}') \times \pi_2), \end{aligned}$$

where $\mathfrak{m}' = \mathfrak{m} \setminus \{\Delta\}$, and so

$$\mathrm{Hom}(\langle \mathfrak{m} \rangle \times \pi_1, \zeta(\mathfrak{m}) \times \pi_2) = \mathrm{Hom}(\langle \mathfrak{m}' \rangle \times \pi_1, \zeta(\mathfrak{m}') \times \pi_2).$$

Since $\nu^{(d+m-2)/2+k-1}\rho$ does not appear in the cuspidal support of π' (when $k \geq 2$, otherwise we are done), we can repeat the similar process by replacing $\mathfrak{m} \setminus \{\Delta\}$ with $\mathfrak{m}$. Inductively (which works by our choice of large k), we obtain

$$\mathrm{Hom}_{G_{n+p}}(\langle \mathfrak{m} \rangle \times \pi_1, \zeta(\mathfrak{m}) \times \pi_2) \cong \mathrm{Hom}_{G_n}(\pi_1, \pi_2).$$

With (8.1),

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \hookleftarrow \mathrm{Hom}_{G_{n+p}}(\widetilde{u} \times \pi_1, \widetilde{u} \times \pi_2). \tag{8.2}$$

Viewing $\widetilde{u}\times$ as a functor and using the faithfulness of $\widetilde{u}\times$ (see Section 9.1 below), we have

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \hookrightarrow \mathrm{Hom}_{G_{n+p}}(\widetilde{u} \times \pi_1, \widetilde{u} \times \pi_2) \tag{8.3}$$

Since we are dealing with admissible representations, the injections in (8.2) and (8.3) must be isomorphisms. Hence, we have

$$\mathrm{Hom}_{G_n}(\pi_1, \pi_2) \cong \mathrm{Hom}_{G_{n+p}}(\widetilde{u} \times \pi_1, \widetilde{u} \times \pi_2). \qquad \square$$

Remark 8.4. We remark that the above lemma does not require π_1 and π_2 to be in $\mathrm{Alg}_{\mathcal{C}}(G_n)$. In such case, $\widetilde{u}_\rho(m,d,k) \times \pi_1$ may have more complicated structure. For example, when π_1 has unique quotient, the cosocle of $\widetilde{u}_\rho(m,d,k) \times \pi$ may not be irreducible. We give an example here.

Let $\Delta = [\nu^{1/2}, \nu^k]$ for sufficiently large k. Let $\pi = \nu^{-1/2} \times \nu^{1/2}$, which is reducible with length 2. Then

$$\langle \Delta \rangle \times \pi$$

has the quotient $\langle [\nu^{-1/2}, \nu^k] \rangle \times \nu^{1/2}$ since $\langle \Delta \rangle \times \nu^{-1/2}$ has quotient $\langle [\nu^{-1/2}, \nu^k] \rangle$, and has the quotient $\langle \Delta \rangle \times \mathrm{St}([\nu^{-1/2}, \nu^{1/2}])$, which is irreducible (deduced from similar way as in [13, Appendix]), since π has the quotient $\mathrm{St}([\nu^{-1/2}, \nu^{1/2}])$.

8.4. Product for irreducibility. We use the notations introduced in the previous subsection.

Lemma 8.5 ([30]). *Fix m, d and $\rho \in \mathrm{Irr}^c$. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be multisegments with each segment Δ satisfying that any cuspidal representation in Δ is in $\mathcal{C}_{u_\rho(m,d)}$. Then, for any $k \geq 0$:*

(1) *$\widetilde{u}_\rho(m,d,k) \times \langle \mathfrak{m}_i \rangle$ $(i = 1, 2)$ is irreducible.*

(2) *$\widetilde{u}_\rho(m,d,k) \times \langle \mathfrak{m}_1 \rangle \cong \widetilde{u}_\rho(m,d,k) \times \langle \mathfrak{m}_2 \rangle$ if and only if $\mathfrak{m}_1 = \mathfrak{m}_2$.*

(3) *$\widetilde{u}_\rho(m,d,k) \times \langle \mathfrak{m}_i \rangle \cong \langle \mathfrak{m}_i \rangle \times \widetilde{u}_\rho(m,d,k)$, for $i = 1, 2$.*

(4) *Let ω be an irreducible representation of G_{a+p}. If $\widetilde{u}_\rho(m,d,k) \boxtimes \pi$ is an irreducible quotient of ω_N, then $\omega \cong \widetilde{u}_\rho(m,d,k) \times \pi$. The statement also holds if we replace ω_N by ω_{N^-} as well as replace quotient by submodule.*

Proof. Assertions (1) and (2) follow from [30, Corollary 6.7]. We only sketch how to deduce from [13, Appendix]. Using a modified version of a lemma in [13, Appendix], we have

$$\theta(\zeta(\widetilde{\mathfrak{m}}_\rho(m,d,k) + \mathfrak{m}_i))^\vee \twoheadrightarrow u_\rho(m,d,k) \times \langle \mathfrak{m}_i \rangle \hookrightarrow \zeta(\widetilde{\mathfrak{m}}_\rho(m,d,k) + \mathfrak{m}_i),$$

which forces that $\widetilde{u}_\rho(m,d,k) \times \langle \mathfrak{m}_i \rangle$ is the unique submodule of $\zeta(\widetilde{\mathfrak{m}}_\rho(m,d,k) + \mathfrak{m}_i)$. Assertion (4) follows from Frobenius reciprocity and (1). Finally, assertion (3) follows from the Gelfand–Kazhdan involution. □

8.5. Proof of Theorem 8.1. We fix ρ, d, m. For simplicity, set $\widetilde{u}_k = u_\rho(m,d,k)$ for $k \geq 0$. Let $\Delta_{k+1} = [\nu^{(m-d)/2+k+1}\rho, \nu^{(m+d-2)/2+k+1}\rho]$. Let $\mathcal{C}$ be as in Theorem 8.1 for such ρ, d and m.

Lemma 8.6. *Let $p = n_\rho m$. Let π' be an irreducible representation in $\mathrm{Alg}_{\mathcal{C}}(G_{n'})$. Let $n = n' + (d+k+1)mn_\rho$. There is a unique irreducible composition factor ω in*

$$(\mathrm{St}(\Delta_{k+1}) \times \widetilde{u}_k \times \pi')_{N^-_{n-p}}$$

which is isomorphic to $\mathrm{St}(\Delta_{k+1}) \boxtimes \tau$ for some irreducible τ of G_{n-p}, and moreover,

$$\omega \cong \mathrm{St}(\Delta_{k+1}) \boxtimes (\widetilde{u}_k \times \pi').$$

Proof. For simplicity, set $\lambda = \widetilde{u}_k \times \pi'$, which is irreducible by Lemma 8.5. Note that $\nu^{(m+d-2)/2+k+1}\rho$ is not in the cuspidal support of $\widetilde{u}_k \times \pi'$. To compute $(\mathrm{St}(\Delta_{k+1}) \times \lambda)_{N^-_{n-p}}$, we first compute

$$(\mathrm{St}(\Delta_{k+1}) \times \lambda)_{N_p}$$

(see discussions in Section 8.2), and then twisting the action by an element. Then geometric lemma on $(\mathrm{St}(\Delta_{k+1}) \times \lambda)_{N_p}$ yields a filtration successive quotients of the form

$$\mathrm{St}([\nu^{l+1}\rho, \nu^b\rho]) \times \omega \boxtimes \mathrm{St}([\nu^a\rho, \nu^l]) \times \omega',$$

and this gives a filtration on $(\widetilde{u}_k \times \mathrm{St}(\Delta_{k+1}))_{N^-_{n-p}}$ with successive quotients taking the form

$$\mathrm{St}([\nu^a\rho, \nu^l\rho]) \times \omega' \boxtimes \mathrm{St}(\nu^{l+1}\rho, \nu^b\rho]) \times \omega. \tag{8.4}$$

Here ω and ω' are representations whose cuspidal supports do not contain $\nu^{(m+d-2)/2+k+1}\rho$. It follows that an irreducible composition factor γ of $(\mathrm{St}(\Delta_{k+1}) \times \lambda)_{N^-_{n-p}}$ can take the form $\mathrm{St}(\Delta_{k+1}) \boxtimes \tau$ only if $l = b$ in (8.4). In such case, the successive quotient from geometric lemma is irreducible and is isomorphic to $\gamma \cong \mathrm{St}(\Delta_{k+1}) \boxtimes \lambda$. □

Lemma 8.7. *There exists a surjection from* $\mathrm{St}(\Delta_{k+1}) \times \widetilde{u}_k$ *to* $\widetilde{u}_{k+1}$.

Proof. Let $\Delta = \Delta_{k+1}$. It follows from Lemma 2.1 that there is a surjection

$$\tau := \mathrm{St}(\Delta) \times \mathrm{St}(\nu^{-1}\Delta) \times \cdots \times \mathrm{St}(\nu^{-(d+k)}\Delta) \to \widetilde{u}_{k+1},$$

and similarly, $\tau' := \mathrm{St}(\nu^{-1}\Delta) \times \cdots \times \mathrm{St}(\nu^{-(d+k)}\Delta) \to \widetilde{u}_k$. By uniqueness of the irreducible quotient for τ, we then also have that $\mathrm{St}(\Delta) \times \widetilde{u}_k$ has the same unique irreducible quotient as τ. This gives surjections

$$\tau = \mathrm{St}(\Delta) \times \tau' \twoheadrightarrow \mathrm{St}(\Delta) \times \widetilde{u}_k \twoheadrightarrow \widetilde{u}_{k+1}.$$

□

Lemma 8.8. *Let K be the kernel of the surjection in Lemma* 8.7*. Then, for any π in* $\mathrm{Alg}_{\mathcal{C}}(G_{n'})$ *and any π' in* $\mathrm{Alg}_{\mathcal{C}}(G_{n'})$*,*

$$\mathrm{Hom}(K \times \pi, \widetilde{u}_{k+1} \times \pi') = 0.$$

Proof. Let $\Delta = \Delta_{k+1}$. We have the short exact sequence

$$0 \to K \to \mathrm{St}(\Delta) \times \widetilde{u}_k \to \widetilde{u}_{k+1} \to 0,$$

which gives the short exact sequence

$$0 \to K \times \pi \to \mathrm{St}(\Delta) \times \widetilde{u}_k \times \pi \to \widetilde{u}_{k+1} \times \pi \to 0.$$

Let $N^- = N^-_{n'+n_\rho m(d+k)}$. The Jacquet functor is exact and so we have another short exact sequence

$$0 \to (K \times \pi)_{N^-} \to (\mathrm{St}(\Delta) \times \widetilde{u}_k \times \pi)_{N^-} \to (\widetilde{u}_{k+1} \times \pi)_{N^-} \to 0. \tag{8.5}$$

Now, by second adjointness of Frobenius reciprocity, we have a map

$$\mathrm{St}(\Delta) \boxtimes (\widetilde{u}_k \times \pi) \to (\widetilde{u}_{k+1} \times \pi)_{N^-}.$$

The map is indeed injective. This follows first from the case that π is irreducible by using irreducibility of $\widetilde{u}_k \times \pi$ (Lemma 8.5), and then lift to the general case by an inductive argument using functoriality of Frobenius reciprocity. (One can also prove the map is injective by directly computing the composition factors of $(\widetilde{u}_{k+1} \times \pi)_{N^-}$ taking the form $\mathrm{St}(\Delta) \boxtimes \tau$, see the proof of Lemma 8.6.)

Now by Lemma 8.6 and counting on composition factors, all irreducible composition factors of the form $\mathrm{St}(\Delta) \boxtimes \tau$ in $(\mathrm{St}(\Delta) \times \widetilde{u}_k \times \pi)_{N^-}$ are mapped onto $(\widetilde{u}_{k+1} \times \pi)_{N^-}$ under the surjection map in (8.5).

Thus there is no irreducible composition factor of $(K \times \pi)_{N^-}$ taking the form $\mathrm{St}(\Delta) \boxtimes \tau$. On the other hand, for any irreducible π', $(\widetilde{u}_{k+1} \times \pi')_{N^-}$ has irreducible composition factor of the form $\mathrm{St}(\Delta) \boxtimes \tau$, which can be deduced by an argument using Frobenius reciprocity. Hence, following from the exactness of Jacquet functor (and Lemma 8.5 (1)), we must have

$$\mathrm{Hom}(K \times \pi, \widetilde{u}_{k+1} \times \pi') = 0. \qquad \square$$

Proof of Theorem 8.1. We keep using the above notations. Let $\pi \in \mathrm{Alg}_{\mathcal{C}}(G_n)$ be of length 2. The if direction is easy and so we now consider the only if direction. Suppose π is indecomposable. We shall use backward induction to prove that, for any $k \geq 0$, $\widetilde{u}_k \times \pi$ is indecomposable, and moreover $\widetilde{u}_k \times \pi$ has unique irreducible quotient. When k is sufficiently large, Lemma 8.5 implies that $\widetilde{u}_k \times \pi$ has length 2, and Lemma 8.3 (and Lemma 8.5 (2)) imply the uniqueness of the quotient, which also then implies the indecomposability.

Let π_1 and π_2 be the two irreducible composition factors of π. Let $\lambda_i = \widetilde{u}_k \times \pi_i$ for $i = 1, 2$. Note that λ_1 and λ_2 are irreducible, and $\pi_1 \cong \pi_2$ if and only if $\lambda_1 \cong \lambda_2$ by Lemma 8.5.

Suppose $\widetilde{u}_k \times \pi$ is not indecomposable. Let $\Delta = \Delta_{k+1}$. This gives an isomorphism

$$\widetilde{u}_k \times \pi \cong \lambda_1 \oplus \lambda_2,$$

and so there exists surjections, by Lemma 8.7,

$$\mathrm{St}(\Delta) \times \widetilde{u}_k \times \pi \cong \mathrm{St}(\Delta) \times \lambda_1 \oplus \mathrm{St}(\Delta) \times \lambda_2 \to \widetilde{u}_{k+1} \times \pi_1 \oplus \widetilde{u}_{k+1} \times \pi_2.$$

This implies that:

(1) If $\lambda_1 \ncong \lambda_2$, then for *both* $i = 1, 2$,

$$\mathrm{Hom}_G(\mathrm{St}(\Delta) \times \widetilde{u}_k \times \pi, \widetilde{u}_{k+1} \times \pi_i) \neq 0.$$

(2) If $\lambda_1 \cong \lambda_2$, then

$$\dim \mathrm{Hom}_G(\mathrm{St}(\Delta) \times \widetilde{u}_k \times \pi, \widetilde{u}_{k+1} \times \pi_1) \geq 2.$$

On the other hand, we have the following short exact sequence from Lemma 8.7:

$$0 \to K \times \pi \to \mathrm{St}(\Delta) \times \widetilde{u}_k \times \pi \to \widetilde{u}_{k+1} \times \pi \to 0.$$

By Lemma 8.8, $\mathrm{Hom}(K \times \pi, \widetilde{u}_{k+1} \times \pi_i) = 0$ for $i = 1, 2$. Hence we have

$$\mathrm{Hom}(\widetilde{u}_{k+1} \times \pi, \widetilde{u}_{k+1} \times \pi_i) \cong \mathrm{Hom}(\mathrm{St}(\Delta) \times \widetilde{u}_k \times \pi, \widetilde{u}_{k+1} \times \pi_i).$$

However, by induction hypothesis and the irreducibility of $\widetilde{u}_{k+1} \times \pi_i$, the former Hom has dimension one for both $i = 1$ or 2 if $\lambda_1 \cong \lambda_2$, and has dimension one for precisely one of $i = 1, 2$ if $\lambda_1 \ncong \lambda_2$. This gives a contradiction to (1) or (2) above. Thus $\widetilde{u}_k \times \pi$ is indecomposable as desired, and since $\widetilde{u}_k \times \pi$ has length 2, it also has unique irreducible quotient. This completes the proof. $\square$

9. Product functor of a Speh representation

9.1. Fully-faithful product. For an irreducible cuspidal representation ρ of some G_k, define $\mathrm{cupp}_{\mathbb{Z}}(\rho) = \{\nu^n \rho\}_{n\in\mathbb{Z}}$.

Let $\pi \in \mathrm{Alg}_{\mathcal{C}}(G_p)$. Define the functor

$$\times_{\pi,\mathcal{C}} = \times_{\pi,\mathcal{C},n} : \mathrm{Alg}_{\mathcal{C}}(G_n) \to \mathrm{Alg}_{\mathcal{C}}(G_{n+p})$$

as

$$\times_{\pi,\mathcal{C}}(\omega) = \pi \times \omega,$$

and, for a map $\Omega : \omega_1 \to \omega_2$ in $\mathrm{Alg}_{\mathcal{C}}(G_n)$,

$$\times_{\pi,\mathcal{C}}(\Omega)(f)(g) = (\mathrm{Id}_\pi \boxtimes \Omega)(f(g)),$$

where $f \in u_\rho(m,d) \times \omega_1$ is a smooth function $f : G_{n+p} \to u_\rho(m,d) \boxtimes \omega_1$ (Section 2.2). Note that since $\times_{\pi,\mathcal{C}}$ is exact and sends a non-zero object to a non-zero object, it follows that $\times_{\pi,\mathcal{C}}$ is faithful. We may sometimes simply write $\times_\pi$ for $\times_{\pi,\mathcal{C}}$.

For an irreducible representation π, we define a stable cuspidal set $\mathcal{C}_\pi$ of π as

$$\mathcal{C}_\pi = \mathrm{cupp}(\pi) \cup (\mathrm{Irr}^c \setminus \mathrm{cupp}_{\mathbb{Z}}(\pi)).$$

(Here we regard $\mathrm{cupp}(\pi)$ as a set.) A motivation for the term stable cuspidal set is in the case that for $\pi = u_\rho(d,m)$, and for any $\rho \in \mathcal{C}_\pi$, $\rho \times \pi$ is irreducible. (However, this is not true for general π. We avoid some complications for the generality in our study for branching laws.)

Theorem 9.1. *Let d, m be positive integers, and let $\rho \in \mathrm{Irr}^{u,c}(G_k)$. Let*

$$\mathcal{C} = \mathcal{C}_{u_\rho(m,d)}.$$

Then the functor $\times_{u_\rho(m,d),\mathcal{C}}$ is fully-faithful.

Proof. It suffices to check the conditions in Lemma A.1 in Appendix A. It follows from definition that $\mathrm{Alg}_{\mathcal{C}}(G_k)$ is Serre. Condition (1) is automatic. Condition (2) follows from Theorem 8.1. Conditions (3) and (4) follow from Lemma 8.5 (see [30]). □

As mentioned in introduction, a key input for the above result is the irreducibility of parabolic induction due to Lapid-Mínguz [30]. It is possible to modify the proof of Theorem 8.1 to give another proof of Theorem 9.1 without deducing from the length 2 case while the length 2 case is simpler.

Let $p = n_\rho md$. For $\pi \in \mathrm{Alg}_{\mathcal{C}}(G_{n+p})$, define $R_{u_\rho(m,d)}(\pi) = \mathrm{Hom}_{G_p}(u_\rho(m,d), \pi_{N_n^-})$, which is regarded as a G_n-representation by $(g.f)(u) = \mathrm{diag}(1,g).(f(u))$, and is an object in $\mathrm{Alg}_{\mathcal{C}}(G_n)$. Using adjointness, one checks that $\times_{u_\rho(m,d)}$ is left adjoint to $R_{u_\rho(m,d)}$.

Corollary 9.2. *Let $u = u_\rho(m,d)$. Let π be in $\mathrm{Alg}_{\mathcal{C}}(G_n)$. Then*

$$\pi \cong R_{u_\rho(m,d)}(u_\rho(m,d) \times \pi).$$

Proof. Since $R_{u_\rho(m,d)}$ is right adjoint to $\times_{u_\rho(m,d)}$, it follows from Theorem 9.1 that $R_{u_\rho(m,d)} \circ \times_{u_\rho(m,d)}$ is isomorphic to the identity functor (see e.g. [45, Lemma 4.24.3]). □

Corollary 9.2 also gives the following:

Corollary 9.3. *Let π' be in $\mathrm{Alg}_{\mathcal{C}}(G_n)$. Suppose that π is an irreducible quotient of $u_\rho(m,d)\times\pi'$. Then $\pi \cong u_\rho(m,d)\times\omega$ for an irreducible quotient ω of π'.*

Proof. By Frobenius reciprocity, $\omega' \hookrightarrow R_{u_\rho(m,d)}(\pi)$ for some irreducible composition factor ω' of π'. Since ω' is also in $\mathrm{Alg}_{\mathcal{C}}(G_n)$, we have

$$\pi \cong u_\rho(m,d)\times\omega'$$

(see Lemma 8.5). Applying the Frobenius reciprocity on the quotient map from $u_\rho(m,d)\times\pi'$ to $\pi \cong u_\rho(m,d)\times\omega'$ and Corollary 9.2, we have a non-zero map from π' to ω', as desired. □

We need a stronger variation for Corollary 9.3:

Corollary 9.4. *Let $\mathcal{C}$ be as in Theorem 9.1. Let π_1 be a (not necessarily admissible) representation of G_n. Let π_2 be in $\mathrm{Alg}_{\mathcal{C}}(G_{n+p})$, where $p = n_\rho md$. Then if π_2 is a quotient of $u_\rho(m,d)\times\pi_1$, there exists a non-zero quotient ω of π_1 such that*

$$\pi_2 \cong u_\rho(m,d)\times\omega.$$

In particular, if π_2 is irreducible, then

$$\pi_2 \cong u_\rho(m,d)\times\omega$$

for an irreducible quotient ω of π_1. If π_2 is an irreducible Arthur-type (respectively, unitarizable) representation, then

$$\pi_2 \cong u_\rho(m,d)\times\omega$$

for some irreducible Arthur-type (respectively, unitarizable) representation ω.

Proof. Let $u = u_\rho(m,d)$. By adjointness, we have

$$0 \neq \mathrm{Hom}_{G_{n+p}}(u\times\pi_1,\pi_2) \cong \mathrm{Hom}_{G_n}(\pi_1, R_u(\pi_2)).$$

Let f be the map in $\mathrm{Hom}_{G_n}(\pi_1, R_u(\pi_2))$ corresponding to the surjection from $u_\rho(m,d)\times\pi_1$ to π_2.

Now using adjointness, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{G_{n+p}}(u\times\omega,\pi_2) & \longleftarrow & \mathrm{Hom}_{G_{n+p}}(u\times\pi_1,\pi_2) & \longleftarrow & \mathrm{Hom}_{G_{n+p}}(u\times\tau,\pi_2) & \longleftarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
\mathrm{Hom}_{G_n}(\omega, R_u(\pi_2)) & \longleftarrow & \mathrm{Hom}_{G_n}(\pi_1, R_u(\pi_2)) & \longleftarrow & \mathrm{Hom}_{G_n}(\tau, R_u(\pi_2)) & \longleftarrow & 0,
\end{array}$$

where the two horizontal rows are exact from the short exact sequence

$$0 \to \omega = \ker f \to \pi_1 \to \tau = \mathrm{im}\, f \to 0.$$

In particular, we have $\mathrm{im}\, f$ embeds to $R_u(\pi_2)$.

The image of the embedding under the leftmost bottom horizontal map is zero by definition and by the commutative diagram, it comes from an element in $\mathrm{Hom}_{G_{n+p}}(u\times\pi_1,\pi_2)$ with zero image by the leftmost top horizontal map. Thus when adjointness back, we get back the surjective map

$$u\times\tau \to \pi_2,$$

and the injection

$$\operatorname{im} f \cong \tau \hookrightarrow R_u(\pi_2).$$

Since π_2 is in $\mathrm{Alg}_{\mathcal{C}}(G_{n+p})$, it follows that τ is also in $\mathrm{Alg}_{\mathcal{C}}(G_n)$. Thus the first surjection implies that the number of composition factors in π_2 is at most that of τ by Lemma 8.5. By Corollary 9.2, for each irreducible $\pi' \in \mathrm{Alg}_{\mathcal{C}}(G_{n+p})$, $R_u(\pi')$ is either irreducible or zero. Thus with the fact that R_u is a left exact functor, the number of composition factors of π_2 is at least that of composition factors of $R_u(\pi_2)$. Hence, the second injection implies that the number of composition factors in π_2 is at least that of τ. This implies the coincidence on the numbers and so the surjection must be an isomorphism, i.e. $u \times \tau \cong \pi_2$.

It remains to prove the last statement. Suppose $\pi_2 \cong u_\rho(m,d) \times \omega$ is an Arthur-type representation. Then π_2 and $u_\rho(m,d)$ being Hermitian self-dual implies that

$$\bar{\omega}^\vee \times u_\rho(m,d) \cong \bar{\pi}_2^\vee \cong \pi_2 \cong u_\rho(m,d) \times \omega \cong \omega \times u_\rho(m,d).$$

This implies that $\bar{\omega}^\vee \cong \omega$ by Lemma 8.5 and so it is Hermitian self-dual. Thus ω is unitarizable by a result of Bernstein [5, Corollary 8.2]. Now the classification [41] of unitarizable representations and unique factorization give that ω is an Arthur-type representation. The proof for the assertion for unitarizable representation is similar. □

9.2. Generalizations. While our result of Theorem 9.1 is for a special class of representations, one can generalize to a larger class of examples as long as an analogue of Theorem 8.1 is established.

In [30, Proposition 5.1], it describes a criteria which $\pi \times \langle \Delta \rangle$ is irreducible for an irreducible representation π and a segment Δ. For a fixed irreducible representation π, let $\mathrm{Alg}_\pi(G_n)$ be the full subcategory of $\mathrm{Alg}(G_n)$ which contains objects of finite length with simple composition factors τ satisfying the property that $\pi \times \langle \Delta \rangle$ for any segment Δ in the associated multisegment $\mathfrak{m}$ of π. We expect that if τ is $\mathrm{Alg}_\pi(G_n)$ is an indecomposable representation of length 2, then $\pi \times \tau$ is also indecomposable of length 2.

A. Some homological algebra

Let $\mathcal{A} = \mathrm{Alg}(G_l)$. Let $\mathcal{B} = \mathrm{Alg}(G_n)$. Via Yoneda extension, any element in $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)$ corresponds to a short exact sequence in $\mathcal{A}$, and zero element corresponds to the split sequence. Then, for an additive exact functor $\mathcal{F}$, $\mathcal{F}$ sends a short exact sequence to a short exact sequence, and this defines a map from $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)$ to $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y))$.

Lemma A.1. *Let $\mathcal{C}$ be a full Serre subcategory of $\mathcal{A} = \mathrm{Alg}(G_l)$. Let $\mathcal{B} = \mathrm{Alg}(G_n)$ and let $\mathcal{D}$ be a Serre full subcategory of $\mathcal{B}$. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be an exact additive functor. We also regard objects in $\mathcal{C}$ as objects in $\mathcal{A}$ via the inclusion. Assume that:*

(1) *any object in $\mathcal{C}$ is of finite length,*

(2) *for any simple objects X, Y in the subcategory $\mathcal{C}$, the induced map of $\mathcal{F}$, from $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)$ to $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y))$ is an injection,*

(3) *$\mathcal{F}(X)$ is a simple object in $\mathcal{D}$ if X is simple in $\mathcal{C}$,*

(4) *for any simple objects X and Y in $\mathcal{C}$, $\mathcal{F}(X) \cong \mathcal{F}(Y)$ if and only if $X \cong Y$.*

Then for any objects X, Y in $\mathcal{C}$, the induced map from $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)$ to $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y))$ is also injective, and $\mathcal{F}:\mathcal{C}\to\mathcal{D}$ is fully-faithful, i.e.

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y)) \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X),\mathcal{F}(Y)) \cong \mathrm{Hom}_{\mathcal{C}}(X,Y) \cong \mathrm{Hom}_{\mathcal{A}}(X,Y)$$

for any objects X, Y in $\mathcal{C}$.

Proof. Let X and Y be objects in $\mathcal{C}$. When both lengths of X and Y are 1 in $\mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X),\mathcal{F}(Y)) \cong \mathrm{Hom}_{\mathcal{C}}(X,Y), \quad \mathrm{Ext}^1_{\mathcal{A}}(X,Y) \hookrightarrow \mathrm{Ext}^1_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y))$$

are guaranteed by (2), (3) and (4). We first fix the length of X to be at most some n. We shall prove the statement for arbitrary Y by induction on the length of Y.

For an object Y in $\mathcal{C}$, let Y_1 be an irreducible quotient of Y. Then we have a short exact sequence

$$0 \to Y_2 \to Y \to Y_1 \to 0.$$

Since $\mathcal{C}$ is Serre, it follows that Y_1 and Y_2 are in $\mathcal{C}$.

Note that we have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{A}}(X,Y_1) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{A}}(X,Y_2) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{A}}(X,Y) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{A}}(X,Y_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y_1)) & \to & \mathrm{Ext}^1_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y_2)) & \to & \mathrm{Ext}^1_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y)) & \to & \mathrm{Ext}^1_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y_1)),
\end{array}$$

where the horizontal maps come from long exact sequences, in which the connecting homomorphism is the Yoneda product, and vertical maps for Ext^1 are described in the beginning of this section, and the vertical map for Hom is the map induced from the functor.

We have the first vertical arrow is isomorphism and the second and forth vertical arrows are injections by induction hypothesis. Then it is direct to check that the third vertical arrow is also an injection.

Now we consider another commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(X,Y_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(X,Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(X,Y_2) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{A}}(X,Y_1) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y_1)) & \to & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y)) & \to & \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y_2)) & \to & \mathrm{Ext}^1_{\mathcal{B}}(\mathcal{F}(X),\mathcal{F}(Y_1)).
\end{array}$$

The first and third vertical arrows are isomorphisms by induction and the last vertical arrow is an injection by induction again. Thus we have that the second vertical arrow is an isomorphism.

Now we switch the role of X and Y, and use similar argument to prove that the assertion is true for X and Y of arbitrary finite length. □

Remark A.2. The above lemma is also valid for arbitrary abelian categories $\mathcal{A}$ and $\mathcal{B}$ which are Schurian k-categories, where k is a field, i.e.

$$\mathrm{Hom}_{\mathcal{A}}(X,X) \cong k \quad \text{and} \quad \mathrm{Hom}_{\mathcal{B}}(Y,Y) \cong k$$

for any simple objects X and Y in $\mathcal{A}$ and $\mathcal{B}$, respectively.

References

[1] *A. Aizenbud*, *D. Gourevitch*, *S. Rallis* and *G. Schiffmann*, Multiplicity one theorems, Ann. of Math. (2) **172** (2010), no. 2, 1407–1434.

[2] *A. Aizenbud* and *E. Sayag*, Homological multiplicities in representation theory of p-adic groups, Math. Z. **294** (2020), no. 1–2, 451–469.

[3] *J. Arthur*, Unipotent automorphic representations: conjectures, in: Orbites unipotentes et représentations, II, Asterisque **171–172**, Société Mathématique de France, Paris (1989), 13–71.

[4] *H. Atobe*, The local theta correspondence and the local Gan–Gross–Prasad conjecture for the symplectic-metaplectic case, Math. Ann. **371** (2018), no. 1–2, 225–295.

[5] *I. N. Bernstein*, P-invariant distributions on $\mathrm{GL}(N)$ and the classification of unitary representations of $\mathrm{GL}(N)$ (non-Archimedean case), in: Lie group representations, II (College Park 1982/1983), Lecture Notes in Math. **1041**, Springer, Berlin (1984), 50–102.

[6] *I. N. Bernstein* and *A. V̇. Zelevinsky*, Induced representations of reductive $\mathfrak{p}$-adic groups. I, Ann. Sci. Éc. Norm. Supér. (4) **10** (1977), no. 4, 441–472.

[7] *I. N. Bernstein* and *A. V. Zelevinsky*, Representations of the group $\mathrm{GL}(n, F)$ where F is a local non-Archimedean field, Russian Math. Surveys **31** (1976), no. 3, 1–68.

[8] *R. Beuzart-Plessis*, Endoscopie et conjecture locale raffinée de Gan–Gross–Prasad pour les groupes unitaires, Compos. Math. **151** (2015), no. 7, 1309–1371.

[9] *R. Beuzart-Plessis*, A local trace formula for the Gan–Gross–Prasad conjecture for unitary groups: The archimedean case, preprint 2015, `https://arxiv.org/abs/1506.01452`.

[10] *K. Y. Chan*, Duality for Ext-groups and extensions of discrete series for graded Hecke algebras, Adv. Math. **294** (2016), 410–453.

[11] *K. Y. Chan*, Some methods of computing first extensions between modules of graded Hecke algebras, Algebr. Represent. Theory **21** (2018), no. 4, 859–895.

[12] *K. Y. Chan*, Ext-multiplicity theorem for standard representations of $(\mathrm{GL}_{n+1}, \mathrm{GL}_n)$, preprint 2021, `https://arxiv.org/abs/2104.11528`.

[13] *K. Y. Chan*, Homological branching law for $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$: Projectivity and indecomposability, Invent. Math. **225** (2021), no. 1, 299–345.

[14] *K. Y. Chan* and *G. Savin*, Iwahori component of the Gelfand–Graev representation, Math. Z. **288** (2018), no. 1–2, 125–133.

[15] *K. Y. Chan* and *G. Savin*, Bernstein–Zelevinsky derivatives: A Hecke algebra approach, Int. Math. Res. Not. IMRN **2019** (2019), no. 3, 731–760.

[16] *K. Y. Chan* and *G. Savin*, A vanishing Ext-branching theorem for $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$, Duke Math. J. **170** (2021), no. 10, 2237–2261.

[17] *F. Chen* and *B. Sun*, Uniqueness of Rankin–Selberg periods, Int. Math. Res. Not. IMRN **2015** (2015), no. 14, 5849–5873.

[18] *W. T. Gan*, *B. H. Gross* and *D. Prasad*, Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups, in: Sur les conjectures de Gross et Prasad. I, Asterisque **346**, Société Mathématique de France, Paris (2012), 1–109.

[19] *W. T. Gan*, *B. H. Gross* and *D. Prasad*, Branching laws for classical groups: the non-tempered case, Compos. Math. **156** (2020), no. 11, 2298–2367.

[20] *W. T. Gan* and *A. Ichino*, The Gross–Prasad conjecture and local theta correspondence, Invent. Math. **206** (2016), no. 3, 705–799.

[21] *D. Gourevitch* and *E. Sayag*, Annihilator varieties of distinguished modules of reductive lie algebras, preprint 2020, `https://arxiv.org/abs/2001.11746v2`.

[22] *B. H. Gross* and *D. Prasad*, On the decomposition of a representation of SO_n when restricted to SO_{n-1}, Canad. J. Math. **44** (1992), no. 5, 974–1002.

[23] *M. Gurevich*, On restriction of unitarizable representations of general linear groups and the non-generic local Gan–Gross–Prasad conjecture, preprint 2020, `https://arxiv.org/abs/1808.02640v3`; J. Eur. Math. Soc. (JEMS), to appear.

[24] *M. Harris* and *R. Taylor*, The geometry and cohomology of some simple Shimura varieties. With an appendix by Vladimir G. Berkovich, Ann. of Math. Stud. **151**, Princeton University Press, Princeton 2001.

[25] *H. He*, On the Gan–Gross–Prasad conjecture for $U(p, q)$, Invent. Math. **209** (2017), no. 3, 837–884.

[26] *G. Henniart*, Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique, Invent. Math. **139** (2000), no. 2, 439–455.

[27] *H. Jacquet*, *I. I. Piatetskii-Shapiro* and *J. A. Shalika*, Rankin–Selberg convolutions, Amer. J. Math. **105** (1983), no. 2, 367–464.

[28] *A. Kret* and *E. Lapid*, Jacquet modules of ladder representations, C. R. Math. Acad. Sci. Paris **350** (2012), no. 21–22, 937–940.
[29] *E. Lapid* and *A. Mínguez*, On a determinantal formula of Tadić, Amer. J. Math. **136** (2014), no. 1, 111–142.
[30] *E. Lapid* and *A. Mínguez*, On parabolic induction on inner forms of the general linear group over a non-archimedean local field, Selecta Math. (N.S.) **22** (2016), no. 4, 2347–2400.
[31] *G. Laumon*, *M. Rapoport* and *U. Stuhler*, D-elliptic sheaves and the Langlands correspondence, Invent. Math. **113** (1993), no. 2, 217–338.
[32] *C. Mœglin* and *J.-L. Waldspurger*, La conjecture locale de Gross–Prasad pour les groupes spéciaux orthogonaux: le cas général, in: Sur les conjectures de Gross et Prasad. II, Asterisque **347**, Société Mathématique de France, Paris (2012), 167–216.
[33] *E. Opdam* and *M. Solleveld*, Extensions of tempered representations, Geom. Funct. Anal. **23** (2013), no. 2, 664–714.
[34] *S. Orlik*, On extensions of generalized Steinberg representations, J. Algebra **293** (2005), no. 2, 611–630.
[35] *D. Prasad*, On the decomposition of a representation of GL(3) restricted to GL(2) over a p-adic field, Duke Math. J. **69** (1993), no. 1, 167–177.
[36] *D. Prasad*, Ext-analogues of branching laws, in: Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, World Scientific Publishing, Hackensack (2018), 1367–1392.
[37] *P. Scholze*, The local Langlands correspondence for GL_n over p-adic fields, Invent. Math. **192** (2013), no. 3, 663–715.
[38] *J. A. Shalika*, The multiplicity one theorem for GL_n, Ann. of Math. (2) **100** (1974), 171–193.
[39] *B. Sun*, Multiplicity one theorems for Fourier–Jacobi models, Amer. J. Math. **134** (2012), no. 6, 1655–1678.
[40] *B. Sun* and *C.-B. Zhu*, Multiplicity one theorems: the Archimedean case, Ann. of Math. (2) **175** (2012), no. 1, 23–44.
[41] *M. Tadić*, Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case), Ann. Sci. Éc. Norm. Supér. (4) **19** (1986), no. 3, 335–382.
[42] *A. Venkatesh*, The Burger–Sarnak method and operations on the unitary dual of $\mathrm{GL}(n)$, Represent. Theory **9** (2005), 268–286.
[43] *J.-L. Waldspurger*, La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes spéciaux orthogonaux, in: Sur les conjectures de Gross et Prasad. II, Asterisque **347**, Société Mathématique de France, Paris (2012), 103–165.
[44] *A. V. Zelevinsky*, Induced representations of reductive p-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$, Ann. Sci. Éc. Norm. Supér. (4) **13** (1980), no. 2, 165–210.
[45] The Stacks Project Authors, Stacks Project, `https://stacks.math.columbia.edu`, 2020.

Kei Yuen Chan, Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, P. R. China
https://orcid.org/0000-0001-9051-183X
e-mail: kychan@fudan.edu.cn

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