A VANISHING EXT-BRANCHING THEOREM FOR $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$

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Abstract
We prove a conjecture of Dipendra Prasad on Ext-branching from $\mathrm{GL}_{n+1}(F)$ *to* $\mathrm{GL}_n(F)$*, where* F *is a* p*-adic field, and we give a projectivity criterion, resulting in some interesting consequences.*

1. Introduction

Decomposing a smooth representation of $\mathrm{GL}_{n+1}(F)$, when restricted to $\mathrm{GL}_n(F)$, is a well-known and well-studied problem introduced by Prasad in [19]. Today, this problem is one of a large family of Gan–Gross–Prasad restriction problems (see [15]) at the center of much research in representation theory and number theory. In order to describe what is known and what is new in our research here, we let $G_n = \mathrm{GL}_n(F)$, with $\mathrm{Alg}(G_n)$ the category of smooth representations of G_n. For every $\pi \in \mathrm{Alg}(G_n)$, let $\mathrm{Wh}(\pi)$ be the space of Whittaker functionals on π. If π is irreducible, then $\mathrm{Wh}(\pi)$ is one- or zero-dimensional. We say that π is *generic* or *degenerate*, respectively. Let π_1 be an irreducible representation of G_{n+1}. One of the most significant results in the subject is that the restriction of π_1 to G_n is multiplicity-free (see [1], [2], [23]); that is, for every irreducible representation π_2 of G_n,

$$\dim \mathrm{Hom}_{G_n}(\pi_1, \pi_2) \leq 1,$$

and it is 1 if both representations are generic. On the other hand, Prasad proved in [20] the following beautiful formula:

$$\mathrm{EP}(\pi_1, \pi_2) := \sum (-1)^i \dim \mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) = \dim \mathrm{Wh}(\pi_1) \cdot \dim \mathrm{Wh}(\pi_2).$$

In particular, the formula implies that $\mathrm{EP}(\pi_1, \pi_2) = 1$ if both representations are generic. Since $\dim \mathrm{Hom}_{G_n}(\pi_1, \pi_2) = 1$, Prasad had conjectured that $\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2)$ will vanish for $i > 0$ if both representations are generic.

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The first main result in this article is a proof of this conjecture. In [13], we will generalize the result to other Bessel and Fourier–Jacobi models (in the sense of [15]). The proof is based on the theory of Bernstein–Zelevinsky derivatives (see [5], [6]) with the following, additional ingredient. The theory of derivatives describes how a smooth representation of G_{n+1} restricts to the mirabolic subgroup M_{n+1}. However, instead of M_{n+1}, one can consider the transpose $M_{n+1}^{\top}$ of M_{n+1} and develop a theory of derivatives with respect to $M_{n+1}^{\top}$. Thus, we have two notions of derivatives: those with respect to M_{n+1} are called *right derivatives*, and those with respect to $M_{n+1}^{\top}$ are called *left derivatives*. These two derivatives are related by the outer automorphism of G_{n+1} defined by $\theta_{n+1}(g) = (g^{-1})^{\top}$. Since $M_{n+1}^{\top}$ is not conjugated to M_{n+1} in G_{n+1}, the information provided by left and right derivatives taken together is stronger and is essential to our combinatorial arguments. Let us illustrate the argument when π_1 is the Steinberg representation of $\mathrm{GL}_2(F)$. Let $\nu(g) = |g|$ be a character of GL_1. The theory of derivatives implies that the restriction of π_1 to $\mathrm{GL}_1(F)$ is given by the following Bernstein–Zelevinsky filtration:

$$0 \to C_c(F^{\times}) \to \pi_1 \to \mathbb{C} \to 0,$$

where $C_c(F^{\times})$ is the space of locally constant, compactly supported functions on $F^{\times}$, and $\mathrm{GL}_1(F)$ acts on $\mathbb{C}$ by the character ν or ν^{-1}, depending on whether we use right or left derivatives, respectively. Thus, for a given character π_2 of $\mathrm{GL}_1(F)$, one can clearly arrange that the character on the quotient $\mathbb{C}$ in the above sequence is different from π_2. Now higher extension spaces vanish since $C_c(F^{\times})$ is projective. Even the multiplicity 1 statement is clear since it holds for $C_c(F^{\times})$. The general case, restricting from G_{n+1} to G_n, follows this strategy. The bottom piece of the Bernstein–Zelevinsky filtration of π_1 is the Gelfand–Graev representation of G_n, and thus the vanishing of higher extensions and multiplicity 1 for generic representations follow from projectivity (see [14]) and multiplicity 1 for the Gelfand–Graev representation of G_n, respectively.

The theory of left and right derivatives is expected to have more applications on restriction problems. In [12], we further prove that there are no isomorphic irreducible quotients (and submodules) for the ith left- and ith right-shifted derivatives of an irreducible representation of G_n unless the derivatives are the highest one. This result has consequences on the indecomposability of a restricted representation, as well as to the submodule restriction problem.

Let K_r be the rth principal congruence subgroup in G_n. Let $\pi \in \mathrm{Alg}(G_n)$ be generated by the subspace π^{K_r} of K_r-fixed vectors (so π is contained in finitely many Bernstein components). Then the left ${}^{(i)}\pi$ and the right $\pi^{(i)}$ derivatives are related by the isomorphism $(\pi^{(i)})^{\vee} \cong {}^{(i)}(\pi^{\vee})$. We establish this as a consequence of a "second adjointness isomorphism" for Bernstein–Zelevinsky derivatives, which naturally involves the left derivative, proved in the appendix. This result is of independent interest.

The second main result is a projectivity criterion for the representation π_1 of G_{n+1}, when restricted to G_n, formulated in [12]. In [12], we use the criteria to classify all irreducible representations which are projective when restricted from G_{n+1} to G_n. Assume that π_1 is projective as a G_n-module. Then higher extension spaces vanish without assuming that π_2 is generic. Now assume that π_1 or π_2 is degenerate. Then $\mathrm{EP}(\pi_1, \pi_2) = 0$ by Prasad's formula. If π_2 is a quotient of π_1, then $\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) \neq 0$ for some $i > 0$, and this contradicts the projectivity of π_1. Thus, a necessary condition for π_1 to be G_n-projective is that it is generic and all its irreducible quotients are generic. In this paper, we show that this is also a sufficient condition. The proof relies heavily on the Hecke algebra methods from our earlier paper [14]. Moreover, if π_1 is projective, we identify each Bernstein component of π_1 with an explicit projective Hecke algebra module, independent of π_1. We also show that the necessary condition is satisfied if π_1 is an essentially square-integrable representation. Therefore, essentially square-integrable representations of G_{n+1} are projective G_n-modules, and any two such representations are isomorphic as G_n-modules. This result generalizes the classical result of Bernstein and Zelevinsky which says that any two cuspidal representations of G_{n+1} are isomorphic when restricted to the mirabolic subgroup M_{n+1} of G_{n+1}.

Finally, we would like to point out the following consequence of our results to the submodule restriction problem: $\mathrm{Hom}_{G_n}(\pi_2, \pi_1)$ (note that π_1 and π_2 have switched the places). The two restriction problems are related by a cohomological duality, due to Nori and Prasad [18] (also see [11]),

$$\mathrm{Ext}^i_{G_n}(\pi_2, \pi_1)^\vee \cong \mathrm{Ext}^{d(\pi_2)-i}_{G_n}\big(\pi_1, D(\pi_2)\big),$$

where $d(\pi_2)$ is the cohomological dimension of π_2 and $D(\pi_2)$ is the Aubert involute of π_2. This duality gives an additional importance to the cohomological restriction problem that we study here. Since $d(\pi_2) > 0$, due to the presence of the one-dimensional center in G_n, it follows that $\mathrm{Hom}_{G_n}(\pi_2, \pi_1) = 0$ for all irreducible π_2 if π_1 is projective; in particular, this is true if π_1 is an essentially square-integrable representation.

2. Bernstein–Zelevinsky derivatives

In this section we study *Bernstein–Zelevinsky derivatives* (or simply "derivatives") as functors from $\mathrm{Alg}(G_n)$ to $\mathrm{Alg}(G_{n-i})$. We state a *second adjointness isomorphism* for these functors, as well as an Ext version of the formula. The mirabolic group will appear in the next section.

2.1. Notation

Let $G_n = \mathrm{GL}_n(F)$, where F is a p-adic field. Let $\nu(g) = |\det(g)|$ be the character of G_n, where $|\cdot|$ is the absolute value on F. Let B_n be the Borel subgroup of G_n consisting of upper triangular matrices, and let U_n be the unipotent radical of B_n. Let

$$R_{n-i} = \left\{ \begin{pmatrix} g & x \\ 0 & u \end{pmatrix} : g \in G_{n-i}, u \in U_i, x \in \mathrm{Mat}_{n-i,i}(F) \right\}.$$

We have an obvious Levi decomposition $R_{n-i} = G_{n-i} E_{n-i}$, where E_{n-i} is the unipotent radical of R_{n-i}. Moreover, $E_{n-i} = N_{n-i} U_i$, where N_{n-i} is the unipotent radical of the maximal parabolic subgroup P_{n-i} consisting of block upper-triangular matrices and Levi factor $G_{n-i} \times G_i$. Fix a nonzero additive character ψ of F. Let ψ_i be the character of E_{n-i} defined by

$$\psi_i \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} = \psi(u_{1,2} + \cdots + u_{i-1,i}),$$

where $u_{1,2}, \ldots, u_{i-1,i}$ are the entries of u above the diagonal. Let δ_{R_i} be the modular character of R_{n-i}. The modular character is trivial on the unipotent radical E_{n-i}, and it is equal to ν^i on the Levi factor G_{n-i}. Let π be a smooth representation of G_n on a vector space V. The right ith Bernstein–Zelevinsky derivative of π is a smooth representation $\pi^{(i)}$ of G_{n-i} on the vector space $V^{(i)}$ defined by

$$V^{(i)} = V / \langle \pi(e)v - \psi_i(e)v : e \in E_{n-i}, v \in V \rangle.$$

The representation $\pi^{(i)}$ is the natural action of the Levi factor G_{n-i} on $V^{(i)}$ twisted by $\delta_{R_{n-i}}^{-1/2}$; that is, Bernstein–Zelevinsky derivatives in this paper are normalized, as is parabolic induction and corresponding Jacquet functors.

From the definition of derivatives, the factorization $R_{n-i} = G_{n-i} E_{n-i}$, and the Frobenius reciprocity, one can easily prove the *first adjointness isomorphism* for right Bernstein–Zelevinsky derivatives. For any smooth representation π of G_n and smooth representation σ of G_{n-i},

$$\mathrm{Hom}_{G_n}\big(\pi, \mathrm{Ind}_{R_{n-i}}^{G_n}(\sigma \otimes \psi_i)\big) \cong \mathrm{Hom}_{G_{n-i}}(\pi^{(i)}, \sigma).$$

We termed the derivative "right" because there is also a left derivative, which is taken with respect to the transpose of the groups used to define right derivatives. More precisely, the underlying vector space for the left derivative ${}^{(i)}\pi$ is

$${}^{(i)}V = V / \langle \pi(e)v - \psi_i^{\top}(e)v : e \in E_{n-i}^{\top}, v \in V \rangle,$$

where $\psi_i^{\top}$ is the character of $E_{n-i}^{\top}$ defined by

$$\psi_i^{\top} \begin{pmatrix} 1 & 0 \\ x & u \end{pmatrix} = \bar{\psi}(u_{2,1} + \cdots + u_{i,i-1}).$$

Let $\theta_n(g) = (g^{-1})^\top$ be the outer automorphism of G_n, where $g^\top$ is the transpose of g. Then the left derivative of π is related to the right derivative by the identity

$$^{(i)}\pi = \theta_{n-i}\big(\theta_n(\pi)^{(i)}\big).$$

Let K_r be the rth principal congruence subgroup in G_n. Let π be a representation of G_n generated by π^{K_r}, the space of K_r-fixed vectors in π. By Theorem 4.2 in [5], any submodule of π is also generated by its subspace of K_r-fixed vectors. Thus, representations of G_n generated by K_r-fixed vectors form a categorical direct summand. The following is the *second adjointness isomorphism* for left Bernstein–Zelevinsky derivatives, proved in the appendix.

LEMMA 2.1
Let K_r be the rth principal congruence subgroup in G_n. For any representation π of G_n generated by π^{K_r} and any smooth representation σ of G_{n-i},

$$\mathrm{Hom}_{G_n}\big(\mathrm{ind}_{R_{n-i}}^{G_n}(\sigma \otimes \bar{\psi}_i), \pi\big) \cong \mathrm{Hom}_{G_{n-i}}(\sigma, {}^{(i)}\pi).$$

This isomorphism is functorial in both σ and π.

We now derive some consequences of the two adjointness isomorphisms. The first consequence is a relationship between right and left derivatives via the contragredient.

LEMMA 2.2
Let K_r be the rth principal congruence subgroup in G_n. Let π be a representation of G_n generated by π^{K_r}. Then $(\pi^{(i)})^\vee \cong {}^{(i)}(\pi^\vee)$.

Proof
If we insert $\pi^\vee$ in Lemma 2.1, then for every smooth representation σ of G_{n-i},

$$\mathrm{Hom}_{G_n}\big(\mathrm{ind}_{R_{n-i}}^{G_n}(\sigma \otimes \bar{\psi}_i), \pi^\vee\big) \cong \mathrm{Hom}_{G_{n-i}}\big(\sigma, {}^{(i)}(\pi^\vee)\big).$$

On the other hand, by Proposition 4.2 in [20],

$$\mathrm{Hom}_{G_n}\big(\mathrm{ind}_{R_{n-i}}^{G_n}(\sigma \otimes \bar{\psi}_i), \pi^\vee\big) \cong \mathrm{Hom}_{G_{n-i}}\big(\sigma, (\pi^{(i)})^\vee\big).$$

Thus we have an isomorphism

$$\mathrm{Hom}_{G_{n-i}}\big(\sigma, (\pi^{(i)})^\vee\big) \cong \mathrm{Hom}_{G_{n-i}}\big(\sigma, {}^{(i)}(\pi^\vee)\big)$$

functorial in σ. Now we regard the functors

$$h_1 := \mathrm{Hom}_{G_{n-i}}\big(\cdot, (\pi^{(i)})^\vee\big), h_2 := \mathrm{Hom}_{G_{n-i}}\big(\cdot, {}^{(i)}(\pi^\vee)\big)$$

as objects in the functor category $\mathscr{F}$ which contains contravariant functors from the category of representations of G_{n-i} to the category of Abelian groups. The Yoneda lemma (see, e.g., Lemma 4.3.5 in [22]) asserts that there are natural isomorphisms

$$\mathrm{Hom}_{\mathscr{F}}(h_1, h_2) \cong \mathrm{Hom}_{G_{n-i}}\big((\pi^{(i)})^\vee, {}^{(i)}(\pi^\vee)\big)$$

and

$$\mathrm{Hom}_{\mathscr{F}}(h_2, h_1) \cong \mathrm{Hom}_{G_{n-i}}\big({}^{(i)}(\pi^\vee), (\pi^{(i)})^\vee\big).$$

The naturality is in the sense of Lemma 4.3.5 in [22], and so $h_1 \cong h_2$ in $\mathscr{F}$ implies that ${}^{(i)}(\pi^\vee) \cong (\pi^{(i)})^\vee$. □

Remark

The statement of Lemma 2.1 is optimal in the sense that it cannot be extended to all smooth representations π. To that end, observe that Lemma 2.2, in the case $i = n$, says that we have an isomorphism of vector spaces $(\pi^\vee)_{U_n,\psi_n} \cong (\pi_{U_n,\psi_n})^\vee$ for every G_n-module π generated by π^{K_r}. Let π be any smooth G_n-module. It can be written as a direct sum

$$\pi \cong \bigoplus_{r=1}^{\infty} \pi_r,$$

where π_r is generated by $\pi_r^{K_r}$ and $\pi_r^{K_{r-1}} = 0$. Then

$$\pi^\vee \cong \bigoplus_{r=1}^{\infty} \pi_r^\vee;$$

hence, $\pi^\vee_{U_n,\psi_n}$ is a direct sum of $(\pi_r^\vee)_{U_n,\psi_n} \cong ((\pi_r)_{U_n,\psi_n})^\vee$. But $(\pi_{U_n,\psi_n})^\vee$ is a product of $((\pi_r)_{U_n,\psi_n})^\vee$ and, hence, much larger unless the sum over r is finite.

The following lemma is not needed in this work; however, it is used in the sequel to this paper [12] to prove that both the socle and cosocle of derivatives of irreducible representations are multiplicity-free (see Proposition 2.5 in [12]).

LEMMA 2.3

Let π be an irreducible representation of G_n. The socle of ${}^{(i)}\pi$ is isomorphic to the cosocle of ${}^{(i)}\pi$. In particular, if the irreducible subquotients of ${}^{(i)}\pi$ are multiplicity-free, then ${}^{(i)}\pi$ is a direct sum of its irreducible subquotients.

Proof

The key observation is that, in view of Lemma 2.2, we have two ways to compute ${}^{(i)}\pi$:

$$ {}^{(i)}\pi = \theta_{n-i}\big(\theta_n(\pi)^{(i)}\big) = \big((\pi^\vee)^{(i)}\big)^\vee. $$

Since π is irreducible, we have $\theta_n(\pi) \cong \pi^\vee$, and if we denote by σ either of two isomorphic representations $\theta_n(\pi)^{(i)}$ and $(\pi^\vee)^{(i)}$, we see that ${}^{(i)}\pi$ is obtained from σ on one hand by applying the covariant functor θ and, on the other hand, by applying the contravariant functor taking the contragradient. Since these two functors coincide on irreducible representations, the corollary follows. ☐

LEMMA 2.4

For any representation π of G_n generated by π^{K_r} and smooth representation σ of G_{n-i},

$$ \mathrm{Ext}^j_{G_n}\big(\mathrm{ind}^{G_n}_{R_{n-i}}(\sigma \otimes \bar{\psi}_i), \pi\big) \cong \mathrm{Ext}^j_{G_{n-i}}(\sigma, {}^{(i)}\pi). $$

Proof

In order to compute the right-hand side, we need to use a projective resolution of σ. By using the induction in stages,

$$ \mathrm{ind}^{G_n}_{R_{n-i}}(\sigma \otimes \bar{\psi}_i) \cong \mathrm{Ind}^{G_n}_{P_{n-i}}\big(\sigma \boxtimes \mathrm{ind}^{G_i}_{U_i}(\bar{\psi}_i)\big). $$

The Gelfand–Graev representation $\mathrm{ind}^{G_i}_{U_i}(\bar{\psi}_i)$ is projective by Corollary A.6 of [14]. Thus, if σ is projective, then it follows that $\mathrm{ind}^{G_n}_{R_{n-i}}(\sigma \otimes \bar{\psi}_i)$ is projective, since the parabolic induction takes projective modules into projective modules. So we have shown that taking a projective resolution of σ also gives a projective resolution of $\mathrm{ind}^{G_n}_{R_{n-i}}(\sigma \otimes \bar{\psi}_i)$. Hence, the lemma follows from Lemma 2.1. ☐

2.2. *Zelevinsky segments*

Here we follow [24]. Let ρ be a cuspidal representation of G_r. For any $a, b \in \mathbb{C}$ with $b - a \in \mathbb{Z}_{\geq 0}$, we have a Zelevinsky segment $\Delta = [\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^b\rho]$. The absolute length of Δ is defined to be $r(b-a+1)$, and the relative length is defined to be $b-a+1$. We can truncate Δ from each side to obtain two segments of absolute length $r(b-a)$:

$$ {}^{-}\Delta = [\nu^{a+1}\rho, \ldots, \nu^b\rho] \qquad \text{and} \qquad \Delta^{-} = [\nu^a\rho, \ldots, \nu^{b-1}\rho]. $$

Moreover, if we perform the truncation k times, then the resulting segments will be denoted by ${}^{(k)}\Delta$ and $\Delta^{(k)}$. The induced representation $\nu^a\rho \times \nu^{a+1}\rho \times \cdots \times \nu^b\rho$ contains a unique irreducible submodule denoted by $\langle\Delta\rangle$.

PROPOSITION 2.5
Let $i > 0$ be an integer. The ith left and right derivatives of $\langle \Delta \rangle$ vanish unless $i = r$ when

$$^{(r)}\langle \Delta \rangle = \langle {}^{-}\Delta \rangle \qquad \text{and} \qquad \langle \Delta \rangle^{(r)} = \langle \Delta^{-} \rangle.$$

COROLLARY 2.6
Let $\Delta_1, \ldots, \Delta_k$ be segments. Let π be an irreducible subquotient of $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle$. If a right derivative of π is generic, then every Δ_j is of the relative length 1 or 2, and if the relative length is 2, then Δ_j^- contributes to the cuspidal support of the right derivative of π. Similarly, if a left derivative of π is generic, then ${}^{-}\Delta_j$ contributes to the cuspidal support of the left derivative.

Proof
Observe that $\langle \Delta \rangle$ is generic if and only if the relative length of Δ is 1. By the Leibniz rule, a right derivative of $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle$ has a filtration whose subquotients are $\langle \Delta_1' \rangle \times \cdots \times \langle \Delta_k' \rangle$, where Δ_j' is Δ_j or Δ_j^-. This representation is generic if and only if the relative length of every Δ_j is 1 or 2, and if it is 2, then $\Delta_j' = \Delta_j^-$. $\square$

We summarize some other results from [24] that we will need. The induced representation $\nu^a \rho \times \nu^{a+1} \rho \times \cdots \times \nu^b \rho$ also contains a unique irreducible quotient denoted by $\mathrm{St}(\Delta)$. This representation is an essentially square-integrable representation; that is, its matrix coefficients are square-integrable when restricted to the derived subgroup. Every essentially square-integrable representation is isomorphic to $\mathrm{St}(\Delta)$ for some segment Δ.

PROPOSITION 2.7
Let $i > 0$ be an integer. The ith left and right derivatives of $\mathrm{St}(\Delta)$ vanish unless $i = jr$, for some integer j, when

$$^{(i)}\mathrm{St}(\Delta) = \mathrm{St}(\Delta^{(j)}) \qquad \text{and} \qquad \mathrm{St}(\Delta)^{(i)} = \mathrm{St}({}^{(j)}\Delta).$$

Finally, let $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$ be a multisegment, that is, a multiset of segments. Let

$$\mathrm{St}(\mathfrak{m}) = \mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_k).$$

We observe that this representation depends on the ordering of the segments, but its semisimplification does not. One can say that $\mathfrak{m}$ is generic if no two segments are linked (see [24, p. 184]). Then, by Theorems 4.2 and 9.7 in [24], $\mathrm{St}(\mathfrak{m})$ is an irreducible generic representation, and every such representation arises in this way.

3. Bernstein–Zelevinsky filtration

In this section, we begin our study of the restriction problem from G_{n+1} to G_n. Using the second adjointness formula, for both left and right derivatives, we prove that degenerate representations of G_n cannot be quotients of essentially square-integrable representations of G_{n+1}.

3.1. Bernstein–Zelevinsky functors

Let $M_{n+1} \subseteq G_{n+1}$ be the mirabolic subgroup

$$M_{n+1} = \left\{ \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} : g \in G_n, u \in \mathrm{Mat}_{n,1}(F) \right\}.$$

We have an obvious Levi decomposition $M_{n+1} = G_n E_n$. Abusing notation, let ψ be the character of E_n defined by $\psi(u) = \psi(u_n)$, where u_n is the bottom entry of the column vector u. Note that the stabilizer of ψ in G_n is M_n. We have a pair of functors

$$\Phi^- : \mathrm{Alg}(M_{n+1}) \to \mathrm{Alg}(M_n) \qquad \text{and} \qquad \Phi^+ : \mathrm{Alg}(M_n) \to \mathrm{Alg}(M_{n+1})$$

defined by $\Phi^-(\tau) = \tau_{E_n,\psi}$ and $\Phi^+(\tau) = \mathrm{ind}_{M_n E_n}^{M_{n+1}}(\tau \boxtimes \psi)$. We also have a pair of functors

$$\Psi^- : \mathrm{Alg}(M_{n+1}) \to \mathrm{Alg}(G_n) \qquad \text{and} \qquad \Psi^+ : \mathrm{Alg}(G_n) \to \mathrm{Alg}(M_{n+1}),$$

where $\Psi^-(\tau) = \tau_{E_n}$ and Ψ^+ is simply the inflation. All functors are normalized as in [6]. Any $\tau \in \mathrm{Alg}(M_{n+1})$ has an M_{n+1}-filtration

$$\tau_n \subset \cdots \subset \tau_0 = \tau,$$

where $\tau_i = (\Phi^+)^i (\Phi^-)^i (\tau)$ and

$$\tau_i / \tau_{i+1} = (\Phi^+)^i \Psi^+ \Psi^- (\Phi^-)^i (\tau).$$

Observe that $\Psi^-(\Phi^-)^i(\tau) = \tau^{(i+1)}$ is the $(i+1)$th derivative, and the subquotients of the filtration, considered as G_n-modules, are

$$\tau_i / \tau_{i+1} \cong \mathrm{ind}_{R_{n-i}}^{G_n} (\nu^{1/2} \cdot \tau^{(i+1)} \boxtimes \psi_i),$$

where we have used notation from the preceding section. In particular, τ_n is a multiple of the Gelfand–Graev representation. We derive some consequences of this filtration that we will need later.

LEMMA 3.1
Let $\tau \in \mathrm{Alg}(M_{n+1})$ be such that its derivatives are all finitely generated. When τ is considered as a G_n-module, its Bernstein components are finitely generated.

Proof
Recall that $P_{n-i} \supseteq R_{n-i}$ is the maximal parabolic subgroup of G_n with the Levi factor $G_i \times G_{n-i}$. By using induction in stages, the ith subquotient in the Bernstein–Zelevinsky filtration of τ can be written as

$$\mathrm{Ind}_{P_{n-i}}^{G_n}\big(\nu^{1/2} \cdot \tau^{(i+1)} \boxtimes \mathrm{ind}_{U_i}^{G_i}(\psi_i)\big).$$

By assumption, $\tau^{(i+1)}$ is a finitely generated G_{n-i}-module and the Bernstein components of the Gelfand–Graev representation $\mathrm{ind}_{U_i}^{G_i}(\psi_i)$ are finitely generated (see [7]). The lemma follows since parabolic induction sends finitely generated modules to finitely generated modules by Variante 3.11 in [4]. □

LEMMA 3.2
Let $\pi_1 \in \mathrm{Alg}(G_{n+1})$, and let π_2 be an admissible representation of G_n. If π_2 is a quotient of π_1, then for some $i, j \geq 0$,

$$\mathrm{Hom}_{G_{n-i}}(\nu^{1/2} \cdot \pi_1^{(i+1)}, {}^{(i)}\pi_2) \neq 0 \qquad \textit{and}$$
$$\mathrm{Hom}_{G_{n-j}}(\nu^{-1/2} \cdot {}^{(j+1)}\pi_1, \pi_2^{(j)}) \neq 0.$$

Proof
In order to prove the first isomorphism, we restrict π_1 to G_n, by way of M_{n+1}, and we use the second adjointness formula. For the second, we restrict to G_n, by way of $M_{n+1}^{\top}$; that is, we reverse the roles of left and right derivatives. □

3.2. Essentially square-integrable representations

THEOREM 3.3
Let $\Delta = [\nu^a \rho, \ldots, \nu^b \rho]$ be a segment of absolute length $n+1$, where ρ is a cuspidal representation of G_r. Let π be an irreducible G_n-module. If π is a quotient of $\mathrm{St}(\Delta)$, then π is generic.

Proof
Let $l = b - a + 1$; in particular, $n + 1 = lr$. Assume that π is degenerate. Let $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$ be a multisegment, from the Zelevinsky classification, such that π is the unique submodule of $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_k \rangle$. Since π is degenerate, by Theorem 8.1 in [24] one segment in $\mathfrak{m}$ has relative length at least 2. If π is a quotient of $\mathrm{St}(\Delta)$, then by Lemma 3.2, ${}^{(i)}\pi$ contains $\nu^{1/2} \cdot \mathrm{St}(\Delta)^{(i+1)}$ as a generic submodule for some i. Now we can apply Corollary 2.6: the relative length of each segment in $\mathfrak{m}$ is 1 or 2, and one of them is $[\nu^{c-1/2}\rho, \nu^{c+1/2}\rho]$, where $\nu^{c+1/2}\rho$ contributes to the cuspidal support of $\nu^{1/2} \cdot \mathrm{St}(\Delta)^{(i+1)}$. It follows that $\nu^{1/2} \cdot \mathrm{St}(\Delta)^{(i+1)}$ is a generalized Steinberg representation corresponding to a segment ending in $\nu^{b+1/2}$ and containing $\nu^{c+1/2}\rho$.

Thus, for every $d = c, \dots, b$, $\nu^{d+1/2}\rho$ contributes to the cuspidal support of ${}^{(i)}\pi$ as well as to the cuspidal support of π. Similarly, if we use the second identity in Lemma 3.2, then for every $d = a, \dots, c$, $\nu^{d-1/2}\rho$ contributes to the cuspidal support of π. We see that $\mathfrak{m}$ contains segments of total relative length at least l and absolute length $(l+1)r = n+1+r > n$. This is a contradiction. ☐

4. Vanishing of Ext's

The purpose of this section is to prove the following result.

THEOREM 4.1

Let π_1 be an irreducible generic representation of G_{n+1}, and let π_2 be an irreducible generic representation of G_n. Then

$$\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) = 0 \quad \text{if } i > 0 \qquad \text{and} \qquad \mathrm{Hom}_{G_n}(\pi_1, \pi_2) = \mathbb{C}.$$

Let us explain the strategy of the proof. Fix π_2, and assume that π_2 is a subquotient of $\rho_1 \times \cdots \times \rho_k$, where ρ_i are cuspidal representations. Let $m(\pi_1)$ be the integer that counts the number of cuspidal representations ρ in the support of π_1 such that ρ is an unramified twist of a ρ_i, for some $1 \le i \le k$. The proof is by induction on $m(\pi_1)$. The base case $m(\pi_1) = 0$ is easy. It is deduced from the Bernstein–Zelevinsky filtration of π_1, where the bottom piece is the Gelfand–Graev representation of G_n. Assume now that $\pi_1 = \mathrm{St}(\mathfrak{m}_1)$ and $\pi_2 = \mathrm{St}(\mathfrak{m}_2)$ for a pair of generic multisegments $\mathfrak{m}_1$ and $\mathfrak{m}_2$, that is, no two segments in $\mathfrak{m}_i$ are linked. Let $\Delta = [\nu^a\rho, \dots, \nu^b\rho]$ be a segment in $\mathfrak{m}_1$ such that ρ contributes to $m(\pi_2)$. Assume that Δ is also a shortest such segment. Write $\pi_1 = \mathrm{St}(\Delta) \times \pi$, where $\pi = \mathrm{St}(\mathfrak{m})$ and $\mathfrak{m} = \mathfrak{m}_1 \setminus \Delta$. Let r be the integer such that $\rho \in \mathrm{Alg}(G_r)$. Let $\rho' \in \mathrm{Alg}(G_r)$ be another cuspidal representation such that no unramified twist of ρ' appears in the cuspidal supports of π_1 and π_2. Now both $\rho' \times \mathrm{St}({}^-\Delta) \times \pi$ and $\rho' \times \mathrm{St}(\Delta^-) \times \pi \in \mathrm{Alg}(G_{n+1})$ are irreducible and satisfy the induction assumption. We will use this information to prove the theorem for π_1.

4.1. *Transfer*

Let $l = s + r$. Recall that P_s is the maximal parabolic of G_l with the Levi $G_s \times G_r$. Starting with $\sigma \in \mathrm{Alg}(G_s)$ and $\tau \in \mathrm{Alg}(M_r)$, we can manufacture two representations of M_l. The first one is obtained by the (normalized) induction from $P_s \cap M_l$ and, by abusing notation, is denoted by $\sigma \times \tau$. The second is obtained by the normalized induction from $P_s^\top \cap M_l$ but only after σ is multiplied by $\nu^{-1/2}$ (see [6, p. 457]), where the definition uses a different subgroup but is conjugated in M_l. This representation is denoted by $\tau \times \sigma$.

Our interest in these representations comes from the following.

PROPOSITION 4.2 ([6, Proposition 4.13])
Let $\rho \in \mathrm{Alg}(G_r)$, let $\sigma \in \mathrm{Alg}(G_s)$, and let $\tau \in \mathrm{Alg}(M_r)$. Let $\rho|_M$ and $\sigma|_M$ denote restrictions to M_r and M_s, respectively.

(1) *There exists an exact sequence in $\mathrm{Alg}(M_l)$,*

$$0 \to (\rho|_M) \times \sigma \to \rho \times \sigma \to \rho \times (\sigma|_M) \to 0.$$

(2) *If Ω is any of the four functors $\Phi^{\pm}$ and $\Psi^{\pm}$, then*

$$\Omega(\sigma \times \tau) = \sigma \times \Omega(\tau).$$

(3) $\Psi^{-}(\tau \times \sigma) = \Psi^{-}(\tau) \times \sigma$*, and there exists an exact sequence in $\mathrm{Alg}(M_{l-1})$,*

$$0 \to \Phi^{-}(\tau) \times \sigma \to \Phi^{-}(\tau \times \sigma) \to \Psi^{-}(\tau) \times (\sigma|_M) \to 0.$$

PROPOSITION 4.3
Let $\Delta = [\nu^a\rho, \ldots, \nu^b\rho]$ be a segment where $\rho \in \mathrm{Alg}(G_r)$. Let $\tau_r = (\Phi^{+})^{r-1}(1) \in \mathrm{Alg}(M_r)$, the Gelfand–Graev module. Then $\mathrm{St}(\Delta)|_M$ is isomorphic to $\tau_r \times \mathrm{St}({}^{-}\Delta)$.

Proof
Recall that $\rho|_M \cong \tau_r$. (This is true for every cuspidal representation.) Note that $\mathrm{St}(\Delta)$ is a quotient of $\nu^a\rho \times \mathrm{St}({}^{-}\Delta)$. By Proposition 4.2(1), we have an exact sequence of mirabolic subgroup modules

$$0 \to \tau_r \times \mathrm{St}({}^{-}\Delta) \to \nu^a\rho \times \mathrm{St}({}^{-}\Delta) \to \nu^a\rho \times \big(\mathrm{St}({}^{-}\Delta)|_M\big) \to 0.$$

By Proposition 4.2(2), any derivative of the quotient in the above sequence is equal to $\nu^a\rho \times \mathrm{St}({}^{(k)}\Delta)$ with $k > 1$. Since $\nu^a\rho$ and ${}^{(k)}\Delta$ are not linked, the corresponding subquotients in the Bernstein–Zelevinsky filtration are irreducible as mirabolic subgroup modules. Observe that they are nonisomorphic to the subquotients of the Bernstein–Zelevinsky filtration of $\mathrm{St}(\Delta)$. Hence, the projection from $\nu^a\rho \times \mathrm{St}({}^{-}\Delta)$ onto $\mathrm{St}(\Delta)$ restricted to $\tau_r \times \mathrm{St}({}^{-}\Delta)$ gives the desired isomorphism. □

Now we arrive at a key result.

COROLLARY 4.4
Let $\rho, \rho' \in \mathrm{Alg}(G_r)$ be any two irreducible cuspidal representations. Let $\Delta = [\nu^a\rho, \ldots, \nu^b\rho]$, and let $\pi \in \mathrm{Alg}(G_s)$. Then we have an isomorphism of mirabolic modules

$$\mathrm{St}(\Delta)|_M \times \pi \cong \rho'|_M \times \big(\mathrm{St}({}^{-}\Delta) \times \pi\big).$$

Proof
By Proposition 4.3, we can substitute $\mathrm{St}(\Delta)|_M = \tau_r \times \mathrm{St}({}^-\Delta)$. Furthermore, we have a natural isomorphism

$$\big(\tau_r \times \mathrm{St}({}^-\Delta)\big) \times \pi \cong \tau_r \times \big(\mathrm{St}({}^-\Delta) \times \pi\big)$$

given by the induction in stages in two different orders. We finish by observing that $\tau_r = \rho'|_M$. □

Now we continue with the proof of vanishing for $\pi_1 = \mathrm{St}(\Delta) \times \pi$, with notation as in the start of the section. By Proposition 4.2(1), there is an exact sequence in $\mathrm{Alg}(M_{n+1})$,

$$0 \to \big(\mathrm{St}(\Delta)|_M\big) \times \pi \to \mathrm{St}(\Delta) \times \pi \to \mathrm{St}(\Delta) \times (\pi|_M) \to 0.$$

Likewise, there is an exact sequence in $\mathrm{Alg}(M_{n+1})$,

$$0 \to \rho'|_M \times \big(\mathrm{St}({}^-\Delta) \times \pi\big) \to \rho' \times \big(\mathrm{St}({}^-\Delta) \times \pi\big) \to \rho' \times \big(\mathrm{St}({}^-\Delta) \times \pi\big)|_M \to 0.$$

Note that the submodules in the two sequences are isomorphic by Corollary 4.4. Furthermore, by the choice of ρ',

$$\mathrm{Ext}^i_{G_n}\big(\rho' \times \big(\mathrm{St}({}^-\Delta) \times \pi\big)|_M, \pi_2\big) = 0 \quad \text{if } i \geq 0.$$

Now we can apply the induction assumption to $\rho' \times \mathrm{St}({}^-\Delta) \times \pi$ and conclude that

$$\mathrm{Ext}^i_{G_n}\big(\big(\mathrm{St}(\Delta)|_M\big) \times \pi, \pi_2\big) = 0 \quad \text{if } i > 0 \qquad \text{and} \qquad \cong \mathbb{C} \quad \text{if } i = 0.$$

Hence, in order to establish the conjecture for the pair (π_1, π_2), it suffices to show that

$$\mathrm{Ext}^i_{G_n}\big(\mathrm{St}(\Delta) \times (\pi|_M), \pi_2\big) = 0 \text{ if } i \geq 0,$$

and, to do this, it suffices to show vanishing for each subquotient in the Bernstein–Zelevinsky filtration of $\mathrm{St}(\Delta) \times (\pi|_M)$. By Proposition 4.2(2), the derivatives of $\mathrm{St}(\Delta) \times (\pi|_M)$ are computed on the second factor. Therefore, by combining this with the second adjointness formula, it suffices to show that

- $\mathrm{Ext}^j_{G_n}(\nu^{1/2}\,\mathrm{St}(\Delta) \times \pi^{(i+1)}, {}^{(i)}\pi_2) = 0$ for $i, j \geq 0$.

Alternatively, by reversing the roles of left and right derivatives, it suffices to show that

- $\mathrm{Ext}^j_{G_n}(\nu^{-1/2}\,\mathrm{St}(\Delta) \times {}^{(i+1)}\pi, \pi_2^{(i)}) = 0$ for $i, j \geq 0$.

Hence, it suffices to show that the cuspidal support of $\nu^{1/2}(\mathrm{St}(\Delta) \times \pi^{(i+1)})$ and of ${}^{(i)}\pi_2$ are different for all i, or they are different for $\nu^{-1/2}(\mathrm{St}(\Delta) \times {}^{(i+1)}\pi)$ and $\pi_2^{(i)}$ for all i. The strategy is to show that if both statements fail, then $\mathfrak{m}_2$ contains linked segments.

4.2. Combinatorics

Let $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$ be a multisegment. Then $\mathrm{St}(\mathfrak{m})$ is generic but reducible if some segments are linked. However, if Δ_i and Δ_j are linked, then they can be replaced by $\Delta_i \cap \Delta_j$ and $\Delta_i \cup \Delta_j$. This process (called *recombination* henceforth) eventually leads to a generic segment such that the corresponding irreducible generic representation is the unique generic subquotient in $\mathrm{St}(\mathfrak{m})$. An important observation is that the recombination does not change the cuspidal support. The following is a key lemma.

LEMMA 4.5

Let $\mathfrak{m}$ be a generic multisegment, and let $\mathfrak{m}'$ be a multisegment obtained by truncating $\mathfrak{m}$ from the right. Then the generic segment corresponding to $\mathfrak{m}'$ by recombination is also obtained from $\mathfrak{m}$ by truncating from the right.

Proof

This is proved by induction on the number of steps in the recombination process. If that number is 0, then there is nothing to prove. Otherwise, there is a pair of linked segments Δ' and Δ'' in $\mathfrak{m}'$ such that the first step in the recombination is replacing Δ' and Δ'' by $\Delta' \cap \Delta''$ and $\Delta' \cup \Delta''$, respectively. It is trivial to see that the resulting multisegment is also obtained by right truncation from $\mathfrak{m}$. The proof follows by induction. □

4.3. Finishing the proof

Let $l = b - a + 1$ be the relative length of Δ. We note that ${}^{(i)}\pi_2$ is glued from $\mathrm{St}(\mathfrak{m}_2')$, where $\mathfrak{m}_2'$ runs over all multisegments obtained from $\mathfrak{m}_2$ by truncating from the right i times (in the sense of absolute length). By Lemma 4.5, the cuspidal support of ${}^{(i)}\pi_2$ is given by such generic multisegments. Likewise, $\mathrm{St}(\Delta) \times \pi^{(i+1)}$ is glued from $\mathrm{St}(\Delta) \times \mathrm{St}({}'\mathfrak{m})$, where ${}'\mathfrak{m}$ runs over all multisegments obtained from $\mathfrak{m}$ by truncating from the left $i+1$ times; to determine the cuspidal support, we need to consider only generic ${}'\mathfrak{m}$. However, $\{\Delta\} \cup {}'\mathfrak{m}$ need not be generic. There could be segments in ${}'\mathfrak{m}$ linked to Δ. Since Δ is not linked to any segment in $\mathfrak{m}$ and ${}'\mathfrak{m}$ is obtained from $\mathfrak{m}$ by left truncation, it follows that linking occurs over the right endpoint of Δ. Let Δ_0 be the longest segment in ${}'\mathfrak{m}$ linked to Δ. It is easy to see that $\Delta \cup \Delta_0$ is a segment in the generic multisegment corresponding to $\{\Delta\} \cup {}'\mathfrak{m}$ by the recombination process. Note that $\Delta \cup \Delta_0$ starts with $\nu^a \rho$ and is of relative length at least l. Thus, the cuspidal supports of $\nu^{1/2}(\mathrm{St}(\Delta) \times \pi^{(i+1)})$ and ${}^{(i)}\pi_2$ can have a point in common only if $\mathfrak{m}_2$ contains a segment starting with $\nu^{a+1/2}\rho$ and of relative length at least l. Similarly, the cuspidal supports of $\nu^{-1/2}(\mathrm{St}(\Delta) \times {}^{(i+1)}\pi)$ and $\pi_2^{(i)}$ can have a point in common only if $\mathfrak{m}_2$ contains a segment ending with $\nu^{b-1/2}\rho$ and of length at least l. In other words, we have constructed a pair of linked segments in $\mathfrak{m}_2$, which is a contradiction. This completes the proof of the Ext-vanishing theorem, Theorem 4.1. □

5. Hecke algebra methods

The main goal of this section is to prove that an irreducible representation π_1 of G_{n+1} when restricted to G_n is projective if π_1 is generic and all its irreducible G_n-quotients are generic. The proof uses Hecke algebras and identifies all Bernstein components of a projective π_1 with the sign-projective module of the Hecke algebra corresponding to the Bushnell–Kutzko type (see [8]–[10]). As a consequence, any two projective representations of G_{n+1} are isomorphic when restricted to G_n.

5.1. Hecke algebras

Let $\Delta = [\nu^a\rho, \ldots, \nu^b\rho]$ be a Zelevinsky segment. Let $m = b - a + 1$. The Bernstein component of $\mathrm{St}(\Delta)$ is equivalent to the category of representations of a Hecke algebra $\mathcal{H}_m$ arising from a simple Bushnell–Kutzko type τ_Δ; that is, if π is a smooth representation in the Bernstein component, then $\mathrm{Hom}(\tau_\Delta, \pi)$ is the corresponding $\mathcal{H}_m$-module. The algebra $\mathcal{H}_m$ is isomorphic to the Iwahori Hecke algebra of $\mathrm{GL}_m(E)$, for some field E. Thus, as an abstract algebra, $\mathcal{H}_m$ is generated by $\theta_1, \ldots, \theta_m$ and T_w ($w \in S_m$) satisfying the following relations (see, e.g., [16, (50) and (57)]):

(1) $\theta_k\theta_l = \theta_l\theta_k$ for any $k, l = 1, \ldots, m$;

(2) $T_{s_k}\theta_k - \theta_{k+1}T_{s_k} = (q-1)\theta_k$, where q is a prime power depending on τ_Δ and s_k is the transposition of numbers k and $k+1$;

(3) $T_{s_k}\theta_l = \theta_l T_{s_k}$ if $l \neq k, k+1$;

(4) $(T_{s_k} - q)(T_{s_k} + 1) = 0$, where s_k is as in (2), and T_w satisfies a braid relation.

Let $\mathcal{A}_m = \mathbb{C}[\theta_1^{\pm 1}, \ldots, \theta_m^{\pm 1}]$, and let $\mathcal{H}_{S_m}$ be the span of T_w, $w \in S_m$. Then $\mathcal{H}_m \cong \mathcal{A}_m \otimes \mathcal{H}_{S_m}$. The finite-dimensional algebra $\mathcal{H}_{S_m}$ has a one-dimensional sign representation $\mathrm{sgn}(T_w) = (-1)^{\ell(w)}$, where ℓ is the length function on S_m. An irreducible representation π in the component is Whittaker-generic if and only if $\mathrm{Hom}(\tau_\Delta, \pi)$ contains the sign type as an $\mathcal{H}_{S_m}$-module (see [14]).

Let $\Delta_1, \ldots, \Delta_r$ be segments such that, for $i \neq j$, the cuspidal representations ρ_i and ρ_j are not unramified twists of each other. The Bernstein component of $\mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_r)$ is equivalent to the category of representations of a Hecke algebra $\mathcal{H}$ arising from a semisimple Bushnell–Kutzko type τ. We have $\mathcal{H} \cong \mathcal{H}_{m_1} \otimes \cdots \otimes \mathcal{H}_{m_r}$ and $\mathcal{H} \cong \mathcal{A} \otimes \mathcal{H}_S$, where $\mathcal{A} \cong \mathcal{A}_{m_1} \otimes \cdots \otimes \mathcal{A}_{m_r}$ and $\mathcal{H}_S \cong \mathcal{H}_{S_{m_1}} \otimes \cdots \otimes \mathcal{H}_{S_{m_r}}$. The subalgebra $\mathcal{A}$ is isomorphic to the ring of Laurent polynomials in $m = m_1 + \cdots + m_r$ variables, while $\mathcal{H}_S$ is spanned by T_w, $w \in S = S_{m_1} \times \cdots \times S_{m_r}$. An irreducible representation π in the component can be written as $\pi_1 \times \cdots \times \pi_r$, where π_i is in the component of $\mathrm{St}(\Delta_i)$. Thus, it is clear that π is Whittaker-generic if and only if $\mathrm{Hom}(\tau, \pi)$ contains the sign type of $\mathcal{H}_S$.

5.2. Some projective modules

Let χ be a character of $\mathcal{A}$. The $\mathcal{H}$-module $\mathcal{H} \otimes_{\mathcal{A}} \chi$ is called a *principal series representation* of $\mathcal{H}$. A twisted Steinberg representation of $\mathcal{H}$ is any one-dimensional $\mathcal{H}$-module such that the restriction to $\mathcal{H}_S$ is the sign type. For example, if $\pi = \mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_r)$, then $\mathrm{Hom}(\tau, \pi)$ is a twisted Steinberg representation.

The following is from [14], where it is stated for $\mathcal{H}$ arising from the singleton partition (m), but the proof is applicable to a general partition $(m_1, \ldots, m_r)$.

THEOREM 5.1 ([14, Theorem 2.1])

Let Π be an $\mathcal{H}$-module. Assume that

(1) Π *is projective and finitely generated;*

(2) $\dim \mathrm{Hom}_{\mathcal{H}}(\Pi, \pi) \leq 1$ *for an irreducible principal series representation π;*

(3) *a twisted Steinberg representation is a quotient of* Π.

Then $\Pi \cong \mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$. *Conversely,* $\mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$ *satisfies the above properties.*

As in [14], we have the following corollary.

COROLLARY 5.2 ([14, Theorem 3.4])

Let Γ be the summand of the Gelfand–Graev representation corresponding to the Bernstein component of $\mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_r)$. *Then we have an isomorphism* $\mathrm{Hom}(\tau, \Gamma) \cong \mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$ *of $\mathcal{H}$-modules.*

5.3. Projectivity for Hecke algebras

Let $\mathcal{Z}$ be the center of $\mathcal{H}$. Recall that $\mathcal{Z} = \mathcal{A}^S$; in particular, $\mathcal{H}$ is a finitely generated $\mathcal{Z}$-module. Let $\mathcal{J}$ be a maximal ideal in $\mathcal{Z}$. Let $\hat{\mathcal{H}}$ denote the $\mathcal{J}$-adic completion of $\mathcal{H}$ (see [3]). For every $\mathcal{H}$-module Π, let $\hat{\Pi}$ denote the $\mathcal{J}$-adic completion of Π. If Π is finitely generated, then $\hat{\Pi} \cong \hat{\mathcal{H}} \otimes_{\mathcal{H}} \Pi$.

THEOREM 5.3

Let Π be a finitely generated $\mathcal{H}$-module, and let $\mathcal{J}$ be a maximal ideal in $\mathcal{Z}$. Let π be the unique irreducible $\mathcal{H}$-module annihilated by $\mathcal{J}$ and containing the sign type. Assume that

(1) $\dim \mathrm{Hom}_{\mathcal{H}}(\Pi, \pi) = 1$;

(2) Π *has no other irreducible quotients annihilated by $\mathcal{J}$;*

(3) Π *contains a torsion-free element for $\mathcal{A}$.*

Then $\hat{\Pi} \cong \hat{\mathcal{H}} \otimes_{\mathcal{H}_S} \mathrm{sgn}$.

Proof
In order to simplify notation, write $\Sigma = \mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$. Since Π is finitely generated, $\hat{\Pi}/\mathcal{J}\hat{\Pi} \cong \Pi/\mathcal{J}\Pi$ is a finite-dimensional $\mathcal{H}$-module, annihilated by $\mathcal{J}$. By (2), it must be generated by the sign-type subspace. Let r be the dimension of the sign type in $\Pi/\mathcal{J}\Pi$. By Frobenius reciprocity, we have a surjection $f : \Sigma^{\oplus r} \to \Pi/\mathcal{J}\Pi$ which descends to a surjection $\bar{f} : (\Sigma/\mathcal{J}\Sigma)^{\oplus r} \to \Pi/\mathcal{J}\Pi$. Observe that $\bar{f}$ is bijective on the sign type, since the sign type in $\Sigma/\mathcal{J}\Sigma$ is one-dimensional. Since π is the unique irreducible quotient of $\Sigma/\mathcal{J}\Sigma$ and $\bar{f}$ is bijective on the sign type, it follows that π^r is a quotient of $\Pi/\mathcal{J}\Pi$. This forces $r = 1$ by (1), and by the Nakayama lemma, we have a surjection $\hat{f} : \hat{\Sigma} \to \hat{\Pi}$. Since $\hat{\Sigma} \cong \hat{\mathcal{A}}$, as $\hat{\mathcal{A}}$-modules, (3) implies that the surjection is in fact an isomorphism. □

COROLLARY 5.4
Let Π be a finitely generated $\mathcal{H}$-module, and let $\mathcal{J}$ be a maximal ideal in $\mathcal{Z}$. Assume that the conditions of Theorem 5.3 are satisfied. Then, for all $\mathcal{H}$-modules σ annihilated by $\mathcal{J}$ and for all $i > 0$,

$$\mathrm{Ext}^i_{\mathcal{H}}(\Pi, \sigma) = 0.$$

Proof
To compute $\mathrm{Ext}^i_{\mathcal{H}}(\Pi, \sigma)$, we take a sufficiently long free resolution of Π,

$$\cdots \to \mathcal{H}^r \to \mathcal{H}^s \to \Pi \to 0.$$

Let $\hat{\mathcal{Z}}$ be the $\mathcal{J}$-adic completion of $\mathcal{Z}$. By Proposition 10.13 in [3], the completion of finitely generated $\mathcal{Z}$-modules is isomorphic to tensoring by $\hat{\mathcal{Z}}$. Since $\hat{\mathcal{Z}}$ is a flat $\mathcal{Z}$-module, by Proposition 10.14 in [3], it follows that

$$\cdots \to \hat{\mathcal{H}}^r \to \hat{\mathcal{H}}^s \to \hat{\Pi} \to 0$$

is also exact. Now, since σ is annihilated by $\mathcal{J}$, it is easy to check that

$$\mathrm{Ext}^i_{\mathcal{H}}(\Pi, \sigma) \cong \mathrm{Ext}^i_{\hat{\mathcal{H}}}(\hat{\Pi}, \sigma).$$

The latter spaces are trivial for $i > 0$ by the projectivity of $\hat{\mathcal{H}} \otimes_{\mathcal{H}_S} \mathrm{sgn}$. □

COROLLARY 5.5
Let Π be a finitely generated $\mathcal{H}$-module. Assume that the conditions of Theorem 5.3 are satisfied for every maximal ideal in $\mathcal{Z}$. Then $\Pi \cong \mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$.

Proof
Corollary 5.4 implies that $\operatorname{Ext}^i_{\mathcal{H}}(\Pi,\sigma)=0$, $i>0$, for all finite length modules σ. Since Π is also finitely generated, it is projective by [14, Theorem A.1]. Now we can apply Theorem 5.1. □

5.4. *Projectivity for groups*
Now we can apply the Hecke module results to the restriction problem, one Bernstein component at a time. Let π_1 be an irreducible generic representation of G_{n+1}, and fix a Bushnell–Kutzko type τ for G_n. Let $\Pi=\operatorname{Hom}(\tau,\pi_1)$ be the corresponding $\mathcal{H}$-module. Note that the conditions (1) and (3) in Theorem 5.3 are satisfied for every maximal ideal $\mathcal{J}$. Indeed, condition (1) is satisfied because all irreducible generic G_n-representations are quotients of π_1 with multiplicity one, and (3) is satisfied because π_1, restricted to G_n, contains the Gelfand–Graev representation whose τ-component is a free $\mathcal{A}$-module. Theorem 5.3 implies the following local Ext-vanishing result for groups.

THEOREM 5.6
Let π_1 be an irreducible generic representation of G_{n+1}. Let $\mathcal{J}$ be a maximal ideal of the Bernstein center of G_n. Assume that no degenerate irreducible representation of G_n annihilated by $\mathcal{J}$ is a quotient of π_1. Then $\operatorname{Ext}^i_{G_n}(\pi_1,\pi_2)=0$, $i>0$, for all irreducible representations π_2 of G_n annihilated by $\mathcal{J}$.

Finally, we have the following result (see [12]).

THEOREM 5.7
Let π_1 be an irreducible generic representation of G_{n+1} whose irreducible G_n-quotients are all generic. Then π_1, considered as a G_n-module, is projective. Moreover, any two such representations of G_{n+1} are isomorphic as G_n-modules. This holds for all essentially square-integrable representations of G_{n+1}.

Proof
Indeed, by Corollary 5.5, $\operatorname{Hom}(\tau,\pi_1)\cong\mathcal{H}\otimes_{\mathcal{H}_S}\operatorname{sgn}$ for any Bernstein component of π_1. Thus, every component of π_1 is a projective G_n-module independent of π_1, as long as π_1 has no degenerate quotients. And these conditions are satisfied for essentially square-integrable representations by Theorem 3.3. □

Appendix

In the following, we prove Lemma 2.1, that is, the second adjointness isomorphism for Bernstein–Zelevisky derivatives. The key ingredient is Rodier's approximation of the Whittaker character by characters of compact pro-p groups (see [21]).

A.1. Groups

Let F be a p-adic field, let R be its ring of integers, and let P be the maximal ideal generated by a prime ϖ. Let ψ be the character of F of conductor R. Let $G = \mathrm{GL}_n(F)$, and let U be the group of unipotent upper triangular matrices in G. Let $\psi_U : U \to \mathbb{C}$ be a Whittaker character defined by

$$\psi_U(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}),$$

where $u_{i,j}$ denote the entries of the matrix u.

For every natural number r, let L_r be the lattice in $M_n(F)$ consisting of all matrices whose entries are in P^r. Then

$$K_r = 1 + L_r$$

is a principal congruence subgroup of G. Let $t = (t_i) \in G$ be a diagonal matrix such that $t_i/t_{i+1} = \varpi^2$ for $i = 1, \ldots, n-1$. Let $H_r = t^{-r} K_r t^r$. Let $B^\top$ be the Borel subgroup of lower triangular matrices. Then we have a parahoric decomposition

$$H_r = (H_r \cap B^\top)(H_r \cap U).$$

The sequence of groups $H_r \cap B^\top$ is decreasing with trivial intersection, while the sequence of groups $H_r \cap U$ is increasing with union U. Let ψ_r be a character of H_r defined by

$$\psi_r(g) = \psi(g_{1,2} + \cdots + g_{n-1,n}).$$

Observe that

$$\psi_r|_{H_r \cap U} = \psi_U|_{H_r \cap U}.$$

A.2. Representations

Let π be a smooth G-module. For every nonnegative integer r, we have a projection map $P_r : \pi \to \pi^{H_r, \psi_r}$ defined by

$$P_r(v) = \mathrm{vol}(H_r)^{-1} \int_{H_r} \bar{\psi}_r(u) \pi(g) v \, dg.$$

For $r \leq s$, we have maps $i_r^s : \pi^{H_r,\psi_r} \to \pi^{H_s,\psi_s}$ defined by restricting P_s to π^{H_r,ψ_r}. From the parahoric decomposition of H_r, it is easy to see that

$$i_r^s(v) = \mathrm{vol}(H_s \cap U)^{-1} \int_{H_s \cap U} \bar{\psi}_s(u)\pi(u)v\,du.$$

This formula, in turn, implies that these maps form a direct system, that is, $i_s^t \circ i_r^s = i_r^t$, for $r \leq s \leq t$. We have natural maps $i_r : \pi^{H_r,\psi_r} \to \pi_{U,\psi_U}$. Observe that $i_s \circ i_r^s = i_r$. Hence, we have a map from a direct limit

$$i_\pi : \lim_r \pi^{H_r,\psi_r} \to \pi_{U,\psi_U}.$$

PROPOSITION A.1
For every smooth G-module π, the map i_π is an isomorphism of vector spaces.

Proof
To prove surjectivity, let $v \in \pi$. Since $H_r \cap B^\top \to \{1\}$ there exists r such that v is $H_r \cap B^\top$-invariant. Let

$$w = \mathrm{vol}(H_r \cap U)^{-1} \int_{H_r \cap U} \bar{\psi}_r(u)\pi(u)v\,du \in \pi^{H_r,\psi_r}.$$

Then v and w have the same projection on π_{U,ψ_U}. To prove injectivity, let $v \in \pi^{H_r,\psi_r}$ that projects to 0 in π_{U,ψ_U}. Then there exists an open compact subgroup $U_c \subset U$ such that

$$\int_{U_c} \bar{\psi}_s(u)\pi(u)v\,du = 0.$$

Since $H_s \cap U \to U$ there exists $s \geq r$ such that $H_s \cap U \supset U_c$. Then the above integral, with U_c substituted by $H_s \cap U$, vanishes. In other words, $i_r^s(v) = 0$, and hence $v = 0$, viewed as an element of the direct limit. □

For $r \leq s$ we have maps $p_r^s : \pi^{H_s,\psi_s} \to \pi^{H_r,\psi_r}$, going in the opposite direction, defined by restricting P_r to $\pi^{H_s\psi_s}$. From the parahoric decomposition of H_r, it is easy to see that

$$p_r^s(v) = \mathrm{vol}(H_r \cap B^\top)^{-1} \int_{H_r \cap B^\top} \pi(g)v dg,$$

and this implies that these maps form an inverse system; that is, $p_r^s \circ p_s^t = p_r^t$, for $r \leq s \leq t$.

By Proposition 4 in [21] (see also [21, Section VI, p. 169]), there exists an integer r_0, independent of π, such that $p_r^s \circ i_r^s$ is a nonzero multiple of the identity on

π^{H_r,ψ_r} if $r_0 \le r \le s$. Thus, i_r^s is an injection, and p_s^r is a surjection. It follows, from Proposition A.1, that the maps $i_r : \pi^{H_r,\psi_r} \to \pi_{U,\psi_U}$ are injections, for all $r \ge r_0$.

We will use the surjectivity of the maps p_r^s to construct a natural complement of π^{H_r,ψ_r} in π_{U,ψ_U}. So fix $r \ge r_0$, and for every $s \ge r$, let τ_s be the kernel of p_r^s. Observe that τ_s is a complement of π^{H_r,ψ_r} in π^{H_s,ψ_s}, where we have identified π^{H_r,ψ_r} with its image in π^{H_s,ψ_s}. We claim that τ_s, for $s \ge r$, will form an injective subsystem. To that end, let $t \ge s$. We need to prove that if $v \in \tau_s$, then $i_s^t(v) \in \tau_t$; that is, $p_r^t(i_s^t(v)) = 0$. Write $p_r^t = p_r^s \circ p_s^t$. Then

$$p_r^t\big(i_s^t(v)\big) = p_r^s \circ p_s^t\big(\big(i_s^t(v)\big) = p_r^s\big(p_s^t \circ i_s^t(v)\big) = 0,$$

where for the last equality we used the fact that $p_s^t \circ i_s^t(v)$ is a multiple of v. Hence, the direct limit

$$\pi_c^{H_r,\psi_r} := \lim_{s \ge r} \tau_s$$

is a complement of π^{H_r,ψ_r} in $\lim_{s\ge r} \pi^{H_r,\psi_r} \cong \pi_{U,\psi_U}$.

We apply the above considerations to $\pi = S(G)$, the space of locally constant, compactly supported functions on G, considered as a G-module with respect to the action by left translations. In this case, the vector spaces π^{H_r,ψ_r} and π_{U,ψ_U} are naturally G-modules, coming from the right translation action of G on $S(G)$, and the maps i_r^s, p_r^s, and i_r are G-morphisms. Observe that $S(G)^{H_r,\psi_r} = \mathrm{ind}_{H_r}^G(\psi_r)$ and that $S(G)_{U,\psi_U} \cong \mathrm{ind}_U^G(\psi)$, the Gelfand–Graev representation. Hence, $\lim_r \mathrm{ind}_{H_r}^G(\psi_r) \cong \mathrm{ind}_U^G(\psi)$, as G-modules. Moreover, if $r \ge r_0$, then $\mathrm{ind}_{H_r}^G(\psi_r)$ is a direct summand of $\mathrm{ind}_U^G(\psi)$. We record this in the following.

PROPOSITION A.2
For every $r \ge r_0$, $\mathrm{ind}_{H_r}^G(\psi_r)$ is a direct G-invariant summand of $\mathrm{ind}_U^G(\psi)$:

$$\mathrm{ind}_U^G(\psi) \cong \mathrm{ind}_{H_r}^G(\psi_r) \oplus \mathrm{ind}_{H_r}^G(\psi_r)_c.$$

PROPOSITION A.3
Fix $r \ge r_0$. For almost all $s \ge r$, $(\mathrm{ind}_{H_s}^G(\psi_s)_c)^{K_r}$ is trivial.

Proof
The key is the following lemma.

LEMMA A.4
Let $r \ge r_0$. Let π be an irreducible Whittaker generic G-module such that $\pi^{K_r} \ne 0$. There exists a positive integer m, independent of π, such that $\pi^{H_{mr}\psi_{mr}} \ne 0$.

Proof
The first step in the proof is a reduction to supercuspidal representations. Let $P = MN$ be a standard parabolic subgroup of block upper triangular matrices. Assume that π is a Whittaker generic subquotient of $\mathrm{Ind}_{P^\top}^G \sigma$, where $P^\top$ is the transpose of P. Let $K = \mathrm{GL}_n(R)$. By using $G = P^\top K$ and the normality of K_r in K, it is easy to see that $\pi^{K_r} \neq 0$ implies that $\sigma^{K_r^M} \neq 0$, where $K_r^M = K_r \cap M$. Now assume that $\sigma^{H_s^M, \psi_s^M} \neq 0$, where $H_s^M = H_s \cap M$ and ψ_s^M is the restriction of ψ_s to H_s^M. Let $v \in \sigma^{H_s^M, \psi_s^M}$, and define $f \in \mathrm{Ind}_{P^\top}^G \sigma$, supported on $P^\top(H_s \cap N)$, such that $f(1) = v$ and such that it is right $(\psi_s)|_{H_s \cap N}$-invariant. Then $f \in (\mathrm{Ind}_{P^\top}^G \sigma)^{H_s, \psi_s}$. This type must belong to the Whittaker generic subquotient of the induced representation by injectivity of the map i_s.

It remains to deal with supercuspidal π. Let ℓ be a Whittaker functional on π, and for every $v \in \pi$ we have a Whittaker function $f_v(g) = \ell(\pi(g)v)$. Let $T(r) \subset T$ be the subset of $t = (t_1, \dots, t_n)$ such that $1/q^{(2m-2)r} \leq |t_i/t_{i+1}| \leq q^{(2m-2)r}$, for all i. By Theorem 2.1 in [17], there exists m, independent of π, such that f_v is supported on $UT(r)K$ for all $v \in \pi^{K_r}$. Since f_v is nonzero, for a nonzero v, there exist $t \in T(r)$ and $k \in K$ such that $\ell(\pi(tk)v) \neq 0$. Since K normalizes K_r, $\pi(k)v \in \pi^{K_r}$. It follows that $\pi(tk)v$ is fixed by $tK_r t^{-1}$. Observe that this group contains $H_{mr} \cap B^\top$, by the definition of $T(r)$; hence,

$$w = \mathrm{vol}(H_{mr} \cap U)^{-1} \int_{H_{mr} \cap U} \bar{\psi}_U(u) \pi(u) \pi(tk) v \in \pi^{H_{mr}, \psi_{mr}},$$

and it is nonzero since $\ell(w) = \ell(\pi(tk)v) \neq 0$. The lemma is proved. □

Take $s \geq mr$, where m is as in the lemma. Recall that, by [7], Bernstein's components of $\mathrm{ind}_U^G \psi_U$ are finitely generated and, hence, admit irreducible quotients. Thus, if $(\mathrm{ind}_{H_s}^G(\psi_s)_c)^{K_r} \neq 0$, then $\mathrm{ind}_{H_s}^G(\psi_s)_c$ has an irreducible quotient π such that $\pi^{K_r} \neq 0$. Then $\pi^{H_s, \psi_s} \neq 0$ by the lemma, and hence $\dim_G(\mathrm{ind}_U^G \psi_U, \pi) \geq 2$, by Proposition A.2, which is a contradiction. □

PROPOSITION A.5
For every G-module π generated by π^{K_r} and every vector space σ,

$$\mathrm{Hom}_G(\sigma \otimes \mathrm{ind}_U^G \psi_U, \pi) \cong \mathrm{Hom}(\sigma, \pi_{U,\psi_U}).$$

Proof
By Propositions A.2 and A.3, for almost all $s \geq r$,

$$\mathrm{Hom}_G(\sigma \otimes \mathrm{ind}_U^G \psi_U, \pi) \cong \mathrm{Hom}_G\big(\sigma \otimes \mathrm{ind}_{H_s}^G(\psi_s), \pi\big).$$

Let $\mathbb{C}[G]$ denote the group algebra of G. Then we can write

$$\sigma \otimes \mathrm{ind}_{H_s}^G(\psi_s) \cong \mathrm{ind}_{H_s}^G(\sigma \otimes \psi_s) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H_s]} (\sigma \otimes \psi_s).$$

Hence, by the Frobenius reciprocity,

$$\mathrm{Hom}_G\big(\sigma \otimes \mathrm{ind}_{H_s}^G(\psi_s), \pi\big) \cong \mathrm{Hom}(\sigma, \pi^{H_s,\psi_s}).$$

Now observe that the starting space $\mathrm{Hom}_G(\sigma \otimes \mathrm{ind}_U^G \psi_U, \pi)$ does not depend on s. It follows that the spaces π^{H_s,ψ_s} are isomorphic for almost all s. In particular, $\pi^{H_s,\psi_s} \cong \pi_{U,\psi_U}$ for such s. Hence,

$$\mathrm{Hom}_G(\sigma \otimes \mathrm{ind}_U^G \psi_U, \pi) \cong \mathrm{Hom}\big(\sigma, (\pi)_{U,\psi_U}\big). \qquad \square$$

A.3. Second adjointness

Now we are ready to prove Lemma 2.1. We resume using the notation from the main body of the paper; in particular, $G_n = \mathrm{GL}_n(F)$, U_n is the group of upper triangular unipotent matrices, and $P_{n-i} = M_{n-i}N_{n-i}$ is the standard maximal parabolic subgroup of block upper triangular matrices with the Levi $G_{n-i} \times G_i$. Let π be a smooth representation of G_n generated by vectors fixed by the rth principal congruence subgroup in G_n, and let σ be a smooth representation of G_{n-i}, as in the statement of the lemma. By using induction in stages,

$$\mathrm{ind}_{R_{n-i}}^{G_n}(\sigma \otimes \bar{\psi}_i) \cong \mathrm{Ind}_{P_{n-i}}^{G_n}\big(\sigma \boxtimes \mathrm{ind}_{U_i}^{G_i}(\bar{\psi}_i)\big).$$

By the second adjointness isomorphism for parabolic induction, due to Bernstein,

$$\mathrm{Hom}_{G_n}\big(\mathrm{Ind}_{P_{n-i}}^{G_n}\big(\sigma \boxtimes \mathrm{ind}_{U_i}^{G_i}(\bar{\psi}_i)\big), \pi\big) \cong \mathrm{Hom}_{G_{n-i}\times G_i}\big(\sigma \boxtimes \mathrm{ind}_{U_i}^{G_i}(\bar{\psi}_i), \pi_{N_{n-i}^{\top}}\big).$$

It is easy to see that $\pi_{N_{n-i}^{\top}}$, as a G_i-module, is also generated by vectors fixed by the rth principal congruence subgroup in G_i. Thus, we can apply Proposition A.5 to G_i to derive that

$$\mathrm{Hom}_{G_{n-i}\times G_i}\big(\sigma \boxtimes \mathrm{ind}_{U_i}^{G_i}(\bar{\psi}_i), \pi_{N_{n-i}^{\top}}\big) \cong \mathrm{Hom}_{G_{n-i}}(\sigma, {}^{(i)}\pi).$$

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References

[1] A. AIZENBUD and D. GOUREVITCH, *Multiplicity one theorem for* $(\mathrm{GL}_{n+1}(\mathbb{R}), \mathrm{GL}_n(\mathbb{R}))$, Selecta Math. (N.S.) **15** (2009), no. 2, 271–294. MR 2529937. DOI 10.1007/s00029-009-0544-7. *(2237)*

[2] A. AIZENBUD, D. GOUREVITCH, S. RALLIS, and G. SCHIFFMANN, *Multiplicity one theorems*, Ann. of Math. (2) **172** (2010), no. 2, 1407–1434. MR 2680495. DOI 10.4007/annals.2010.172.1413. *(2237)*

[3] M. F. ATIYAH and I. G. MACDONALD, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, 1969. MR 0242802. *(2252, 2253)*

[4] J. N. BERNSTEIN, "Le centre de Bernstein" in *Representations of Reductive Groups over a Local Field*, Travaux en Cours, Hermann, Paris, 1984, 1–32. MR 0771671. *(2246)*

[5] I. N. BERNSTEIN and A. V. ZELEVINSKY, *Representations of the group* $\mathrm{GL}(n, F)$ *where* F *is a local non-Archimedean field*, Uspekhi Mat. Nauk **31** (1976), no. 3, 5–70; English translation in Russian Math. Surveys **31** (1976), no. 3, 1–68. MR 0425030. *(2238, 2241)*

[6] ———, *Induced representations of reductive p-adic groups*, I, Ann. Sci. Éc. Norm. Supér. (4) **10** (1977), 441–472. MR 0579172. *(2238, 2245, 2247, 2248)*

[7] C. J. BUSHNELL and G. HENNIART, *Generalized Whittaker models and the Bernstein center*, Amer. J. Math. **125** (2003), no. 3, 513–547. MR 1981032. *(2246, 2258)*

[8] C. J. BUSHNELL and P. C. KUTZKO, *The Admissible Dual of* $\mathrm{GL}(N)$ *via Compact Open Subgroups*, Ann. of Math. Stud. **129**, Princeton Univ. Press, Princeton, 1993. MR 1204652. DOI 10.1515/9781400882496. *(2251)*

[9] ———, *Smooth representations of reductive p-adic groups: Structure theory via types*, Proc. Lond. Math. Soc. (3) **77** (1998), no. 3, 582–634. MR 1643417. DOI 10.1112/S0024611598000574. *(2251)*

[10] ———, *Semisimple types in* GL_n, Compos. Math. **119** (1999), no. 1, 53–97. MR 1711578. DOI 10.1023/A:1001773929735. *(2251)*

[11] K. Y. CHAN, *Duality for Ext-groups and extensions of discrete series for graded Hecke algebras*, Adv. Math. **294** (2016), 410–453. MR 3479568. DOI 10.1016/j.aim.2016.03.002. *(2239)*

[12] ———, *Homological branching law for* $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$*: Projectivity and indecomposability*, Invent. Math., published online 15 February 2021. DOI 10.1007/s00222-021-01033-5. *(2238, 2239, 2242, 2254)*

[13] ———, *Restriction for general linear groups: The local non-tempered Gan-Gross-Prasad conjecture*, preprint, arXiv:2006.02623v1 [math.RT]. *(2238)*

[14] K. Y. CHAN and G. SAVIN, *Bernstein-Zelevinsky derivatives: A Hecke algebra approach*, Int. Math. Res. Not. IMRN **2019**, no. 3, 731–760. MR 3910471. DOI 10.1093/imrn/rnx138. *(2238, 2239, 2243, 2251, 2252, 2254)*

[15] W. T. GAN, B. H. GROSS, and D. PRASAD, *Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups*, Astérisque **346** (2012), 1–109. MR 3202556. *(2237, 2238)*

[16] R. HOWE, "Hecke algebras and p-adic GL_n" in *Representation Theory and Analysis on Homogeneous Spaces (New Brunswick, 1993)*, Contemp. Math. **177**, Amer. Math. Soc., Providence, 1994, 65–100. MR 1303601. DOI 10.1090/conm/177/01922. *(2251)*

[17] E. LAPID, "On the support of matrix coefficients of supercuspidal representations of the general linear group over a local non-archimedean field" in *Representations of Reductive Groups*, Proc. Sympos. Pure Math. **101**, Amer. Math. Soc., Providence, 2019, 211–218. MR 3930019. *(2258)*

[18] M. NORI and D. PRASAD, *On a duality theorem of Schneider-Stuhler*, J. Reine Angew. Math. **762** (2020), 261–280. MR 4195661. DOI 10.1515/crelle-2018-0028. *(2239)*

[19] D. PRASAD, *On the decomposition of a representation of* GL(3) *restricted to* GL(2) *over a p-adic field*, Duke Math. J. **69** (1993), no. 1, 167–177. MR 1201696. DOI 10.1215/S0012-7094-93-06908-6. *(2237)*

[20] ———, "Ext-analogues of branching laws" in *Proceedings of the International Congress of Mathematicians, Vol. II: Invited Lectures (Rio de Janeiro, 2018)*, World Sci., Hackensack, 2018, 1367–1392. MR 3966813. *(2237, 2241)*

[21] F. RODIER, "Modéle de Whittaker et caractéres de représentations" in *Non-commutative Harmonic Analysis (Marseille-Luminy, 1974)*, Lecture Notes in Math. **466**, Springer, Berlin, 1975, 151–171. MR 0393355. *(2255, 2256)*

[22] THE STACKS PROJECT AUTHORS, *Stacks Project*, https://stacks.math.columbia.edu (accessed 19 April 2021). *(2242)*

[23] B. SUN and C. B. ZHU, *Multiplicity one theorems: The Archimedean case*, Ann. of Math. (2) **175** (2012), no. 1, 23–44. MR 2874638. DOI 10.4007/annals.2012.175.1.2. *(2237)*

[24] A. V. ZELEVINSKY, *Induced representations of reductive p-adic groups, II: On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. Éc. Norm. Supér. **13** (1980), no. 2, 165–210. MR 0584084. *(2243, 2244, 2246)*

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