A POLYHEDRON COMPARISON THEOREM FOR 3-MANIFOLDS WITH POSITIVE SCALAR CURVATURE

CHAO LI

ABSTRACT. The study of comparison theorems in geometry has a rich history. In this paper, we establish a comparison theorem for polyhedra in 3-manifolds with nonnegative scalar curvature, answering affirmatively a dihedral rigidity conjecture by Gromov. For a large collections of polyhedra with interior non-negative scalar curvature and mean convex faces, we prove the dihedral angles along its edges cannot be everywhere less or equal than those of the corresponding Euclidean model, unless it is a isometric to a flat polyhedron.

1. INTRODUCTION

A fundamental question in differential geometry is to understand metric/measure properties of Riemannian manifolds under global curvature conditions, and study notions of curvature lower bounds in spaces with low regularity. Such goals are usually achieved via geometric comparison theorems. The quest started with Alexandrov [Ale51], who introduced the notion of *sectional* curvature lower bounds for metric spaces via geometric comparison theorems for geodesic triangles. Similar questions for *Ricci* curvature have also attracted a wide wealth of research recently (Cheeger-Colding-Naber theory; see, e.g., [CC97, CC00a, CC00b, CN12, CN13]; for an optimal transport approach, see, e.g., [LV09] [Stu06a, Stu06b, Stu06c]).

The case of scalar curvature lower bounds, however, is not as well established, possibly due to a lack of satisfactory geometric comparison theory. The first progress in this direction was made by Shi-Tam [ST02], who proved a total boundary mean curvature comparison theorem for regions in manifolds with nonnegative scalar curvature. However, it requires a presumption of the existence of boundary isometric embedding into Euclidean spaces, which is not satisfied for general domains.

As triangles play an essential role in the comparison theorems for sectional curvature, Gromov [Gro14] suggested that Riemannian **polyhedra** should be of particular importance for the study of scalar curvature. In this paper, we place our focus specifically in three dimensions, and make the following

Definition 1.1. Let P be a flat polyhedron in \mathbb{R}^3 . A closed Riemannian manifold M^3 with non-empty boundary is called a P-type polyhedron, if it admits a Lipschitz diffeomorphism $\phi : M \to P$, such that ϕ^{-1} is smooth when restricted to the interior, the faces and the edges of P. We thus define

the faces, edges and vertices of M as the image of ϕ^{-1} when restricted to the corresponding objects of P.

The first case that Gromov investigated was cube-type polyhedra in threemanifolds with nonnegative scalar curvature $(P = [0, 1]^3 \subset \mathbf{R}^3)$. Let (M^n, g) be a cube-type polyhedron with faces F_j . Let $\angle_{ij}(M, g)$ denote the (possibly nonconstant) dihedral angle between two adjacent faces F_i and F_j . Then Gromov proposed that (M, g) cannot simultaneously satisfy:

- (1) the scalar curvature $R(g) \ge 0$;
- (2) each F_i is mean convex;
- (3) for all pairs (i, j), $\measuredangle_{ij}(M, g) < \frac{\pi}{2}$.

Notice that conditions (2) and (3) above may be interpreted as C^0 properties of g. In fact, a face F is strictly mean convex if and only if it is locally one-sided area minimizing: for any outward compactly support small perturbation F' of F, we have |F| < |F|'; the dihedral angle can be measured with the metric g.

The crucial and elegant observation from Gromov is that, if such a cube exists, then by "doubling" M three times across the front, the right and the bottom faces, the new cube \tilde{M} has isometric opposite faces. Then we identify the opposite faces of \tilde{M} and obtain a torus T^3 with a singular metric \tilde{g} . Due to the geometric assumptions, the metric \tilde{g} has positive scalar curvature away from a stratified singular set $S = F^2 \cup L^1 \cup V^0$, where:

- (1) \tilde{g} is smooth on both sides from F^2 . The mean curvatures of F^2 from two sides satisfy a positive jump;
- (2) \tilde{g} is an edge metric along L with angle less than 2π ;
- (3) \tilde{g} is bounded measurable across isolated vertices V^0 .

It is known that condition (1) above implies that \tilde{g} has positive scalar curvature on F^2 in a weak sense: a Yamabe nonpositive manifold cannot support any metric which is singular along a hypersurface satisfying the "positive jump of mean curvature" assumption, and has positive scalar curvature on its regular part [Mia02][ST16]. The affect of condition (2) and (3) above on the Yamabe type of a manifold was investigated by C. Mantoulidis and the author in a recent paper [LM17]. We proved that in dimension 3, skeleton singularities with cone angle less than 2π do not effect the Yamabe type. We refer the readers to these papers and the references therein for more details.

This idea of Gromov relies on the fact that cubes are the fundamental domains of the \mathbb{Z}^3 actions on \mathbb{R}^3 , hence is not applicable to general polyhedra. An interesting question is then: which types of polyhedra share properties like those observed by Gromov for cube-type polyhedra in manifolds with nonnegative scalar curvature?

Another related question concerns the rigidity: what types of polyhedra are "mean convexly extremetal"? Surprisingly, this question is unsettled even in the Euclidean spaces: Conjecture 1.2 (Dihedral rigidity conjecture, section 2.2 of [Gro14]). Let $P \in \mathbf{R}^n$ be a convex polyhedron with faces F_i . Let $P' \subset \mathbf{R}^n$ be a P-type polyhedron with faces F'_i . If

(1) each F'_i is mean convex, and (2) the dihedral angles satisfy $\measuredangle'_{ij}(P') \leq \measuredangle_{ij}(P)$,

then P' is flat.

The primary scope of this paper is to answer affirmatively this conjecture for a large collection of polyhedral types in three-manifolds with nonnegative scalar curvature. We also obtain a comparison theorem for Riemannian polyhedra.

Let us define two general polyhedron types.

(1) Let $k \geq 3$ be an integer. In \mathbb{R}^3 , let $B \subset \{x^3 = 0\}$ Definition 1.3. be a convex k-polygon, and $p \in \{x_3 = 1\}$ be a point. Call the set

$$\{tp + (1-t)x : t \in [0,1], x \in B\}$$

a(B,p)-cone. Call B the base face and all the other faces side faces.

(2) Let $k \ge 3$ be an integer. In \mathbb{R}^3 , let $B_1 \subset \{x^3 = 0\}, B_2 \subset \{x_3 = 1\}$ be two similar convex k-polygons whose corresponding edges are parallel *(i.e. the polygons are congruent up to scaling but not rotation). Call* the set

 $\{tp + (1-t)q : t \in [0,1], p \in B_1, q \in B_2\}$

a (B_1, B_2) -prism. Call B_1, B_2 the base faces and all the other faces side faces.

If (M, q) is a Riemannian polyhedron of P-type, where P is a (B, p)-cone (or a (B_1, B_2) -prism), we call (M, g) is of cone type (prism type, respectively).



FIGURE 1. A (B, p)-cone and a (B_1, B_2) -prism.

The major objects we consider are Riemannian polyhedra (M^3, q) of cone type or prism type, as in Definition 1.3. Let us fix some notations that will be used throughout the paper. We use F_1, \dots, F_k to denote the side faces of M; if M is of cone type, we use p to denote the cone vertex, and Bto denote its base face; if M is of prism type, we use B_1, B_2 to denote its two bases. Let $F = \bigcup_{j=1}^{k} F_j$ be the union of all side faces. Our first theorem makes a comparison between Riemannian polyhedra with nonnegative scalar curvature and their Euclidean models:

Theorem 1.4. Let (M^3, g) be a Riemannian polyhedron of P-type with side faces F_1, \dots, F_k , where $P \subset \mathbf{R}^3$ is a cone or prism with side faces F'_1, \dots, F'_k . Denote γ_j the angle between F'_j and the base face of P (if P is a prism, fix one base face). Assume that everywhere along $F_j \cap F_{j+1}$,

$$|\pi - (\gamma_j + \gamma_{j+1})| < \measuredangle(F_j, F_{j+1}).$$
(1.1)

Then the strict comparison statement holds for (M, g). Namely, if $R(g) \ge 0$, and each F_j is mean convex, then the dihedral angles of M cannot be everywhere less than those of P.

Our theorem should be contextualized in the rich history of the study of comparison theorems in differential geometry. In fact, it is not hard to argue as in [Gro14] that the converse is also true: on a three-manifold with negative scalar curvature, one may construct a polyhedron which entirely invalidates the conclusions of Theorem 1.4. Thus the metric properties introduced by Theorem 1.4 characterize $R(g) \ge 0$ faithfully, and may very well serve as a definition of $R(g) \ge 0$ for a metric g that is only continuous.

A more refined analysis enables us to characterize the rigidity behavior for Theorem 1.4, thus answering Conjecture 1.2 for cone type and prism type polyhedra, with the very mild a priori angle assumptions (1.1). In fact, we obtain:

Theorem 1.5. Under the same assumptions of Theorem 1.4 and the extra assumption that

$$\gamma_j \le \pi/2, j = 1, 2, \cdots, k, \quad or \quad \gamma_j \ge \pi/2, j = 1, 2, \cdots, k,$$
 (1.2)

we have the rigidity statement. Namely, if $R(g) \ge 0$, each F_j is mean convex, and $\measuredangle_{ij}(M,g) \le \measuredangle_{ij}(P,g_{Euclid})$, then (M,g) is isometric to a flat polyhedron in \mathbb{R}^3 .

The angle assumption (1.1) may be regarded as a mild regularity assumption on (M, g). It is satisfied, for instance, by any small C^0 perturbation of the Euclidean polyhedron P. Moreover, assumption (1.1) is vacuous, if all the angles γ_j are $\pi/2$. In this case, the Euclidean model is a prism with orthogonal base and side faces, and we are able to obtain the prism inequality in section 5.4, [Gro14] as a special case.

Motivated by the Schoen-Yau dimension reduction argument [SY79], we have also been able to generalize Theorem 1.4 and Theorem 1.5 in higher dimensions. They will appear in a forthcoming paper.

Now we indicate the strategy of the proof for Theorem 1.4 and Theorem 1.5 and the organization of the paper. Consider the following energy functional:

$$\mathcal{F}(E) = \mathcal{H}^2(\partial E \cap \mathring{M}) - \sum_{j=1}^k (\cos \gamma_j) \mathcal{H}^2(\partial E \cap F_j), \qquad (1.3)$$

and the variational problem

$$\mathcal{I} = \inf\{\mathcal{F}(E) : E \in \mathscr{E}\},\tag{1.4}$$

here \mathscr{E} is the collection of contractible open subset E' such that: if M is of cone type, then $p \in E'$ and $E' \cap B = \emptyset$; if M is of prism type, then $B_2 \subset E'$ and $E' \cap B_1 = \emptyset$. If the solution to (1.4) is regular, its boundary $\Sigma^2 = \partial E \cap \mathring{M}$ is called a capillary minimal surface. That is, Σ is a minimal surface that contacts each side face F_j at constant angle γ_j . The existence, regularity and geometric properties of capillary surfaces have attracted a wealth of research throughout the rich history of geometric variational problems. We refer the readers to the book of Finn [Fin86] for a beautiful and thorough introduction.

Our first observation is that \mathcal{I} is always finite: since M is compact, we deduce that

$$\mathcal{I} \ge -\sum_{j=1}^{k} (\cos \gamma_j) \mathcal{H}^2(F_j) > -\infty.$$

Thus a minimizing sequence exists. The existence and boundary regularity of the solution to (1.4) was treated by Taylor [Tay77] (see page 328-(6); see also the discussion for more general anisotropic capillary problems by De Philippis-Maggi [DPM15]). Using the language of integral currents, Taylor proved the existence of the minimizer Σ , and that Σ is C^{∞} regular up to its boundary, where ∂M is smooth. However, the variational problem (1.4) has obstacles: the base face(s) of M. To overcome this difficulty, we apply the interior varifold maximum principle [SW89] and a new boundary maximum principle, and reduce (1.4) to a variational problem without obstacles. We then adapt ideas from Simon [Sim80] and Lieberman [Lie88], and obtain a $C^{1,\alpha}$ regularity property of Σ at its corners. This is the only place we need to use the angle assumption (1.1). The existence and regularity of Σ is established in section 2. In section 3, we unveil the connection between interior scalar curvature, the boundary mean curvature and the dihedral angle captured by the variational problem (1.4), and derive various geometric consequences with Σ . We prove Theorem 1.4 with the second variational inequality and the Gauss-Bonnet formula. We then proceed to section 4 for the proof of Theorem 1.5, where an analysis for the "infinitesimally rigid" minimal capillary surface Σ is carried out, with the idea pioneered by Bray-Brendle-Neves [BBN10]. The new challenge here is to deal with the case when $\mathcal{I} = 0$. We develop a new general existence result of constant mean curvature capillary foliations near the vertex p, and establish the dynamical behavior of such foliations in nonnegative scalar curvature.

Acknowledgement: The author wishes to thank Rick Schoen, Brian White, Leon Simon, Rafe Mazzeo, Or Hershkovits and Christos Mantoulidis for stimulating conversations. He also wishes to thanks the referee for greatly improving the exposition. Part of this work was carried out when the author was visiting the University of California, Irvine. He wants to thank Department of Mathematics, UCI, for their hospitality.

2. EXISTENCE AND REGULARITY

We discuss the existence and regularity of the minimizer for the variational problem (1.4). The goal of this section is:

Theorem 2.1. Consider the variational problem (1.4) in a Riemannian polyhedron (M^3, g) of cone or prism type. Assume $\mathcal{I} < 0$ if M is of cone type. Then \mathcal{I} is achieved by an open subset E. Moreover, $\Sigma = E \cap \mathring{M}$ is an area minimizing surface, $C^{1,\alpha}$ to its corners for some $\alpha > 0$, and meets F_j at constant angle γ_j .

We first introduce some notations and basic geometric facts on capillary surfaces. Then we reduce the obstacle problem (1.4) equivalently to a variational problem without any obstacle. This is done via a varifold maximum principle. Hence the regularity theory developed in [Tay77] is applicable, and we get regularity in $\mathring{\Sigma}$, and in $\partial \Sigma$ in \mathring{F}_j . The regularity at the corners of Σ is then studied with an idea of Simon [Sim80]. At the corner, we prove that the surface is graphical over its planar tangent cone. Then we invoke the result of Lieberman [Lie88], which showed that the unit normal vector field is Hölder continuous up to the corners.

2.1. **Preliminaries.** We start by discussing some geometric properties of capillary surfaces. In particular, we deduce the first and second variation formulas for the energy functional (1.3). Let us fix some notation.

Let P be an orientable Riemannian manifold of dimension p and M a closed compact polyhedron of cone or prism types in N. Let Σ^{n-1} be an orientable n-1 dimensional compact manifold with non-empty boundary $\partial \Sigma$ and $\partial \Sigma \subset \partial M$. We denote the topological interior of a set U by \mathring{U} . Assume Σ separates \mathring{M} into two connected components. Fix one component and call it E. Denote X the outward pointing unit normal vector field of ∂M in M, N the unit normal vector field of Σ in E pointing into E, ν the outward pointing unit normal vector field of $\partial \Sigma$ in Σ , $\overline{\nu}$ the unit normal vector field of $\partial \Sigma$ in ∂M pointing outward E. Let A denote the second fundamental form of $\Sigma \subset E$, II denote the second fundamental form of $\partial M \subset M$. We take the convention that $A(X_1, X_2) = \langle \nabla_{X_1} X_2, N \rangle$. Denote H, \overline{H} the mean curvature of $\Sigma \subset E$, $\partial M \subset M$, respectively. Note that in our convention, the unit sphere in \mathbb{R}^3 has mean curvature 2.



FIGURE 2. Capillary surfaces

By an admissible deformation we mean a diffeomorphism $\Psi : (-\varepsilon, \varepsilon) \times \Sigma \to M$ such that $\Psi_t : \Sigma \to M$, $t \in (-\varepsilon, \varepsilon)$, defined by $\Psi_t(q) = \Psi(t, q), q \in \Sigma$, is an embedding satisfying $\Psi_t(\Sigma) \subset \mathring{M}$ and $\Psi_t(\partial \Sigma) \subset \partial M$, and $\Psi_0(x) = x$ for all $x \in \Sigma$. Denote $\Sigma_t = \Psi_t(\Sigma)$. Let E_t be the corresponding component separated by Σ_t . Denote $Y = \frac{\partial \Psi(t, \cdot)}{\partial t}|_{t=0}$ the vector field generating Ψ . Then Y is tangential to ∂M along $\partial \Sigma$. Fix the angles $\gamma_1, \cdots, \gamma_k \in (0, \pi)$ on the faces F_1, \cdots, F_k of M. Consider the energy functional

$$F(t) = \mathcal{H}^{n-1}(\Sigma_t) - \sum_{j=1}^k (\cos \gamma_j) \mathcal{H}^{n-1}(\partial E_t \cap F_j)$$

We now deduce the first variation formula of F(t). Let $f = \langle Y, N \rangle$ be the normal component of the vector field Y. By the usual first variation formula on volume function and integration by parts,

$$\frac{d}{dt}\Big|_{t=0}\mathcal{H}^{n-1}(\Sigma_t) = \int_{\Sigma} \operatorname{div}_{\Sigma} Y d\mathcal{H}^{n-1} = -\int_{\Sigma} H f d\mathcal{H}^{n-1} + \int_{\partial \Sigma} \langle Y, \nu \rangle \, d\mathcal{H}^{n-2}.$$

On the other hand, for each $j, 1 \le j \le k$,

$$\left. \frac{d}{dt} \right|_{t=0} - (\cos \gamma_j) \mathcal{H}^{n-1}(\partial E_t \cap F_j) = -\cos \gamma_j \int_{\partial \Sigma \cap F_j} \langle Y, \overline{\nu} \rangle \, d\mathcal{H}^{n-2}$$

Adding the above two equations, the first variation of F(t) is given by

$$\frac{d}{dt}\Big|_{t=0}F(t) = -\int_{\Sigma} Hf d\mathcal{H}^{n-1} + \sum_{j=1}^{k} \int_{\partial\Sigma\cap F_j} \langle Y, \nu - (\cos\gamma_j)\overline{\nu}\rangle \, d\mathcal{H}^{n-2}, \quad (2.1)$$

We note that (2.1) holds more generally in the context of varifolds, see (2.21). Also, in the first variation (2.1), there is no contribution from the corners of Σ . The surface Σ is said to be minimal capillary if F'(t) = 0 for any admissible deformations. If follows from (2.1) that Σ is minimal capillary if and only if $H \equiv 0$ and $\nu - (\cos \gamma_j)\overline{\nu}$ is normal to F_j ; that is, along F_j the angle between the normal vectors N and X, or equivalently, between ν and $\overline{\nu}$, is everywhere equal to γ_j .

Assume Σ is minimal capillary. We then have the second variational formula:

$$\frac{d^2}{dt^2}\Big|_{t=0}F(0) = -\int_{\Sigma} (f\Delta f + (|A|^2 + \operatorname{Ric}(N, N))f^2)d\mathcal{H}^{n-1} + \sum_{j=1}^k \int_{\partial\Sigma\cap F_j} f\left(\frac{\partial f}{\partial\nu} - Qf\right)d\mathcal{H}^{n-2}, \quad (2.2)$$

where on $\partial \Sigma \cap F_j$,

$$Q = \frac{1}{\sin \gamma_j} \operatorname{II}(\overline{\nu}, \overline{\nu}) + \cot \gamma_j A(\nu, \nu).$$

Here Δ is the Laplace operator of the induced metric on Σ , and Ric is the Ricci curvature of M. For a proof of the second variation formula, we refer the readers to the appendix of [RS97].

2.2. Maximum principles. We first observe that (1.4) is a variational problem with obstacles: $E \cap B_1 = \emptyset$ if M is of cone type, and $B_2 \subset E$, $E \cap B_1 = \emptyset$ if M is of prism type. To apply the existence and the regularity theories of Taylor [Tay77], we first prove that it suffices to consider a variational problem without any obstacles. Such reduction is usually achieved via varifold maximum principles, see e.g. [SW89, Whi10][LZ17]. In our case, the maximum principles hinge upon the special structure of the obstacle: that B (or B_1, B_2) is mean convex, and that the dihedral angles along $\partial F_j \cap B$ are nowhere larger than γ_j . In fact, if $\Sigma = \partial E \cap \mathring{M}$ is a C^1 surface with piecewise smooth boundary, then it is not hard to see from the first variational formula (2.1) that

- Σ and B do not touch in the interior.
- $\partial \Sigma$ does not contain any point on $F_j \cap B$ where the dihedral angle is strictly less than γ_j .

Thus Σ is a minimal surface that meets each F_i at constant angle γ_i .

The interior maximum principle has been investigated in different scenarios [Sim87][SW89][Whi10][Wic14]. Notice that the energy minimizer of (1.4) is necessarily area minimizing in the interior. We apply the strong maximum principle by Solomon-White [SW89] and conclude that the surface $\Sigma = \partial E \cap \mathring{M}$ cannot touch the base face B, unless lies entirely in B.

Here we develop a new boundary maximum principle. For the purpose of this paper, we only consider energy minimizing currents of codimension 1 associated to (1.4). However, we conjecture that a similar statement should hold for varifolds with boundary in general codimension. (See, for instance, the boundary maximum principle of Li-Zhou [LZ17].)

Proposition 2.2. Let M be a polyhedron of cone type. Let $T \in \mathcal{D}^2(M), E \in \mathcal{D}^3(U)$ be rectifiable currents with $T = \partial E \cap \mathring{M}$ and $\operatorname{spt}(\partial T) \subset F$. Assume E is an energy minimizer of (1.4). Then $\operatorname{spt}(T)$ does not contain any point on the edge $F_i \cap B$ where the dihedral angle is less than γ_i .

By a similar argument, in the case that M is of prism type, $\operatorname{spt}(T)$ does not contain any point on $F_j \cap B_1$ where the dihedral angle is less than $\pi - \gamma_j$, or any point on $F_j \cap B_2$ where the dihedral angle is less than γ_j . Combine this with the interior maximum principle, we conclude that the minimizer to (1.4) lies in the interior of M, and hence an energy-minimizer for the \mathcal{F} without any barriers. Thus the existence and regularity theory developed in [Tay77][DPM15] concludes that the minimizer $T = \partial E \cap \mathring{M}$ exists, and is regular away from the corners.

Proof. Assume, for the sake of contradiction, that a point $q \in F_j \cap B$ is also in spt T, and that the dihedral angle between F_j and B at q is less than

 γ_j . In the rest of proof we use ||T|| to denote the associated varifold. Fix a vector field Y tangential to ∂M , such that Y is transversal along B and points into M at B. Since $T = \partial E \cap \mathring{M}$, it is a rectifiable current with multiplicity one, the first variational formula for the energy functional \mathcal{F} applies:

$$\frac{d}{dt}\Big|_{t=0}\mathcal{F}(\Psi_t(E)) = -\int Hfd\|T\| + \sum_j \int \langle Y, \nu - (\cos\gamma_j)\overline{\nu} \rangle \, d\|\partial T\|, \quad (2.3)$$

where $f, \overline{\nu}$ are the geometric quantities defined as before, H is the generalized mean curvature of T, and ν is the generalized outward unit normal of $\|\partial T\|$. Since the dihedral angle between F_j and B at q is strictly less than γ_j , we have

$$\langle Y, \nu' - \cos \gamma_j \overline{\nu} \rangle > 0,$$
 (2.4)

for any ν' at q which is the outward unit normal vector of some two-plane in $T_q M$. By the interior maximum principle, $H \equiv 0.^1$ Hence

$$\|\partial T\|(\operatorname{spt}(T) \cap \mathcal{B}) = 0, \qquad (2.5)$$

where

$$\mathcal{B} = \bigcup_{j} \{ q \in F_j \cap B : \text{the dihedral angle at } q \text{ is less than } \gamma_j. \}$$

The boundary regularity theorem of Taylor [Tay77] implies that for any point $q' \in \partial T \setminus \mathcal{B}$, the current T is smooth up to q'. In particular, the density of T at q' is given by $\Theta^2(||T||, q') = \frac{1}{2}$. Denote W the two dimensional varifold $v(\partial E \cap F_j)$ associated with $E \cap F_j$, $Z = ||T|| - \cos \gamma_j W$. Since the faces F_j and B intersects smoothly at q, we have the following monotonicity formula (we delay the derivation of a more general monotonicity formula in the next section, see (2.22)):

$$\exp(cr^{\alpha})\frac{\|Z\|(B_r(q))}{r^2} \text{ is increasing in } r, \qquad (2.6)$$

for r sufficiently small, where c and $\alpha > 0$ depends only on the geometry of F_j and B. It is then straightforward to check as in Theorem 3.5-(1) in [All75] that the $\theta^2(||T||, \cdot)$ is an uppersemicontinuous function on $\operatorname{spt}(T) \cap \partial T$. By (2.5) we then conclude

$$\Theta^2(||T||, \cdot) \ge \frac{1}{2} > 0$$
 (2.7)

everywhere on $\operatorname{spt}(T) \cap \partial T$.

Consider a tangent cone T_{∞} of T at q. Let E_{∞} be the associated three dimensional current with $T_{\infty} = \partial E_{\infty}$. By the monotonicity (2.6) and the lower density bound (2.7), T_{∞} is a nonempty cone in C through q_{∞} , where C is the region in \mathbf{R}^3 enclosed by the two planes $F_{i,\infty}$ and B_{∞} intersecting

¹The same argument here applies to the general case where the barrier B has bounded mean curvature, see Remark 2.3.

at an angle $\gamma' < \gamma_j$, and where $q_{\infty} \in F_{\infty} \cap B_{\infty}$. By scaling, for any open set $U \subset \mathbb{R}^3$, E_{∞} minimizes the energy functional

$$\mathcal{F}_{\infty}(E') = \mathcal{H}^2(\partial E' \cap \mathring{C} \cap U) - (\cos \gamma_j) \mathcal{H}^2(\partial E' \cap F_{\infty} \cap U)$$
(2.8)

among open sets E' with $\partial E' \subset \overline{F_{\infty}}$. Since two-planes are the only minimal cones in \mathbb{R}^3 , T_{∞} is a two-plane through q_{∞} . However, since $\measuredangle(F_{\infty}, \mathring{B_{\infty}}) < \gamma_j$, no two-plane through q_{∞} can be the minimizer of (2.8). Contradiction.

Remark 2.3. The above proof only uses the fact that T is minimal in a very weak manner. In fact, the same argument holds under the assumption that the generalized mean curvature H is bounded measurable. This is implied, for instance, by that the barrier \overline{B} has bounded mean curvature (instead of being mean convex).

Remark 2.4. The fact that T is energy minimizing is only used to guarantee the existence of an area minimizing tangent cone. Motivated by [SW89], we speculate that a similar statement should hold for varifolds with boundary that are stationary for the energy functional (1.4).

2.3. Regularity at the corners. We proceed to study the regularity of the minimizer $T = \partial E \cap \mathring{M}$ at the corners. Since T is regular away from the corners, our idea is to adapt the argument of Simon [Sim80], and prove $\operatorname{spt}(T)$ is graphical near a corner. We refer the readers to [Sim80] for full details. Then we apply the theorem of Lieberman [Lie88] to conclude that $\operatorname{spt}(T)$ has a Hölder continuous unit normal vector field to its corners.

Consider any two adjacent side faces F_j, F_{j+1} and let $L = F_j \cap F_{j+1}$. Without loss of generality let j = 1. Fix a point $q \in \operatorname{spt}(T) \cap L$. Let θ be the angle between F_1 and F_2 at q. Recall that we assume

$$|\pi - (\gamma_1 + \gamma_2)| < \theta \le \theta',$$

where θ' is the (constant) dihedral angle between corresponding faces in the Euclidean model. To start the regularity discussion, we first make the following simple calculation in Euclidean space.

Lemma 2.5. Let Γ_1, Γ_2 be two half-planes in \mathbb{R}^3 , enclosing a wedge region W with opening angle $\theta' \in (0, \pi)$. Suppose Γ is a plane in \mathbb{R}^3 , such that the dihedral angle between Γ and Γ_i , i = 1, 2, is $\gamma_i \in (0, \pi)$. Then we have

$$|\pi - (\gamma_1 + \gamma_2)| < \theta' < \pi - |\gamma_1 - \gamma_2|.$$

Proof. Let ν_i be the unit normal vector of Γ_i pointing out of W, ν is the unit normal vector of Γ . Then by assumption, we have that

$$\nu_1 \cdot \nu_2 = -\cos \theta', \quad \nu_i \cdot \nu = \cos \gamma_i, \quad i = 1, 2$$

We calculate

$$(\nu_1 \times \nu) \cdot (\nu_2 \times \nu) = \nu_1 \cdot \nu_2 - (\nu_1 \cdot \nu)(\nu_2 \cdot \nu) = -\cos\theta' - \cos\gamma_1 \cos\gamma_2.$$

On the other hand, we have that $|\nu_1 \times \nu| = \sin \gamma_i$, i = 1, 2. Hence $|(\nu_1 \times \nu) \cdot (\nu_2 \times \nu)| \leq \sin \gamma_1 \sin \gamma_2$. Since Γ intersects Γ_1, Γ_2 transversely, we actually have $|(\nu_1 \times \nu) \cdot (\nu_2 \times \nu)| < \sin \gamma_1 \sin \gamma_2$. By a simple calculation, this implies that

$$\cos(\gamma_1 + \gamma_2) < \cos(\pi - \theta') < \cos(\gamma_1 - \gamma_2).$$

Thus $|\pi - (\gamma_1 + \gamma_2)| < \theta' < \pi - |\gamma_1 - \gamma_2|$, as desired.

We therefore may assume that

$$|\pi - (\gamma_1 + \gamma_2)| < \theta < \pi - |\gamma_1 - \gamma_2|.$$
(2.9)

As an immediate observation following Lemma 2.5, (2.9) is a necessary condition for the regularity statement in Theorem 2.1². Precisely, if the capillary surface Σ is $C^{1,\alpha}$ regular up to the corners, its tangent plane at the corner satisfies the assumption of Lemma 2.5, imposing the range for θ as in (2.9). To prove Theorem 2.1, we verify that condition (2.9) is also sufficient to guarantee the regularity of Σ .

For $\rho > 0$, denote $C_{\rho} = \{x \in M : \operatorname{dist}_{M}(x, L) < \rho\}$, $B_{\rho} = \{x \in M : \operatorname{dist}_{M}(x,q) < \rho\}$. In this section, and subsequently, let c be a constant that may change from line to line, but only depend on the geometry of the polyhedron M. Our argument is parallel to that of [Sim80]: we prove a uniform lower density bound around q, and consequently analyze the tangent cone at q.

2.3.1. Lower density bound. Our first task is to establish an upper bound for the area of T. Precisely, we prove:

Lemma 2.6. For ρ small enough, $\mathcal{H}^2(T \cap C_{\rho}) \leq c\rho$.

Proof. This is straightforward consequence of the fact that $T = \partial E \cap \tilde{M}$ minimizes the energy \mathcal{F} . In fact, for any open subset $U \subset M$, E minimizes the functional

$$\mathcal{H}^2(E' \cap \mathring{M} \cap U) - \sum_{j=1}^k (\cos \gamma_j) \mathcal{H}^2(\partial E' \cap \partial M \cap U)$$

among all sets $E' \subset M$ with finite perimeter, p (or $B_2 \subset E'$, $E' \cap B = \emptyset$. In particular, choose E' to be a small open neighborhood of p when M is a

²When condition (2.9) is not satisfied, we conjecture that there will be a "cusp" singularity forming at the corner. For instance, see (0.4) and (0.5) in [Sim80], and the discussion therein.

(B, p)-cone, and a small tubular neighborhood of B_2 when M is a (B_1, B_2) -prism. Let $T' = \partial E' \cap \mathring{M}$. Choose $U = C_{\rho}$. We conclude that

$$\mathcal{H}^{2}(T \cap C_{\rho}) - \sum_{j=1}^{2} (\cos \gamma_{j}) \mathcal{H}^{2}(\partial E \cap C_{\rho} \cap F_{j})$$
$$\leq \mathcal{H}^{2}(T' \cap C_{\rho}) - \sum_{j=1}^{2} (\cos \gamma_{j}) \mathcal{H}^{2}(\partial E' \cap C_{\rho} \cap F_{j}). \quad (2.10)$$

By the rough estimate that

$$\mathcal{H}^2(C_{\rho} \cap F_j) \le c\rho \quad \text{and} \quad \mathcal{H}^2(C_{\rho} \cap B) \le c\rho^2,$$

we conclude the proof.

Denote $\Sigma = \operatorname{spt}(T) \setminus L$. Since the mean curvature of T is zero in its interior, from the first variation formula for varifolds, we have that, for any C^1 vector field ϕ compactly supported in $M \setminus L$,

$$\int_{\Sigma} \operatorname{div}_{\Sigma} \phi d\mathcal{H}^2 = \int_{\partial \Sigma} \phi \cdot \nu d\mathcal{H}^1.$$
(2.11)

We first bound the length of $\partial \Sigma$. Precisely, let r be the radial distance function $r = \operatorname{dist}(\cdot, L)$, let ϕ be any vector field, supported in M with $\sup r|D\phi| < \infty$ and C^1 in \mathring{M} . (Note that we allow ϕ to have support on L.) By a standard approximation argument as in [Sim80], we deduce that

$$\rho^{-1} \int_{\Sigma \cap C_{\rho}} \phi \cdot \nabla_{\Sigma} r d\mathcal{H}^{2} - \int_{\partial \Sigma} \min\left\{\frac{r}{\rho}, 1\right\} \phi \cdot \nu d\mathcal{H}^{1}$$
$$= -\int_{\Sigma} \min\left\{\frac{r}{\rho}, 1\right\} \operatorname{div}_{\Sigma} \phi d\mathcal{H}^{2}. \quad (2.12)$$

We are going to use (2.12) in two different ways. By the angle assumption (2.9), $|\pi - (\gamma_1 + \gamma_2)| < \theta$. Therefore in the 2-plane $(T_qL)^{\perp} \subset T_qM$, there is a unit vector τ such that

$$(-X)|_{F_j} \cdot \tau > \cos \gamma_j, \quad j = 1, 2,$$
 (2.13)

where $(-X)|_{F_j}$ is the inward pointing unit normal vector of $\partial M \subset M$, restricted to the face F_j . Extend τ in a neighborhood of $q \in M$ as a constant vector field, and with slight abuse of notation, denote this constant vector field also by τ . In (2.12), replace ϕ to be the constant vector τ . (2.13) then implies that

$$(-\nu) \cdot \tau \ge c > 0, \tag{2.14}$$

in $\Sigma \cap C_{\rho_0}$, for sufficiently small ρ_0 . Taking $\rho \to 0$ in (2.12) and using Lemma 2.6, we deduce that

$$\mathcal{H}^1(\partial \Sigma \cap C_\rho) < \infty. \tag{2.15}$$

A POLYHEDRON COMPARISON THEOREM IN POSITIVE SCALAR CURVATURE 13

The angle assumption $\theta < \pi$ then guarantees that the vector $\tau \in T_q M$ defined above also satisfies

$$\tau \cdot \nabla_M r \ge c > 0, \tag{2.16}$$

where r is the radial distance function. See Figure 3 for an illustration of the choice of τ .



FIGURE 3. The choice of the vector τ .

Now we use (2.12) in a second way. We replace ϕ by $\psi\tau$, where τ is the constant vector field defined in a neighborhood of $q \in M$ as above. Then by the same argument as (1.8)-(1.10) in [Sim80],

$$\rho^{-1} \int_{\Sigma \cap C_{\rho}} \psi \tau \cdot \nabla_{M} r d\mathcal{H}^{2} - \int_{\partial \Sigma} \psi \tau \cdot \nu d\mathcal{H}^{1} \\ \leq c \int_{\Sigma} (\psi + |\nabla_{M} \psi|) d\mathcal{H}^{2} + o(1). \quad (2.17)$$

As a consequence of (2.14) and (2.16),

$$\|\delta T\|(\psi) \le c \int (\psi + |\nabla_M \psi|) d\|T\|, \qquad (2.18)$$

where here we view T as the associated varifold. Then we apply the isoperimetric inequality (7.1 in [All72]) and the Moser type iteration (7.5(6) in [All72]) as in [Sim80], and conclude that

$$\mathcal{H}^2(T \cap B_\rho(q)) \ge c\rho^2. \tag{2.19}$$

Remark 2.7. The argument above does not use the fact that Σ is a two dimensional surface in an essential way. The same argument should work for capillary surfaces in general dimensions.

Remark 2.8. Notice that we only require the weaker angle assumption $|\pi - (\gamma_1 + \gamma_2)| < \theta < \pi$ for the lower density bound. We are going to see that the assumption $\theta < \pi - |\gamma_1 - \gamma_2|$ is used to classify the tangent cone.

2.3.2. Monotonicity and tangent plane. We proceed to derive the monotonicity formula and study the tangent cone at a point $q \in \operatorname{spt}(T) \cap L$. For j = 1, 2, denote $W_j = E_j \cap (F_j \setminus L)$. We also use W_j to denote the associated two dimensional varifold. The divergence theorem implies that

$$\delta W_j(\psi\phi) = \int_{\partial\Sigma} \psi\phi \cdot \overline{\nu}, \qquad (2.20)$$

where recall $\overline{\nu}$ is the unit normal vector of $\partial \Sigma$ which is tangent to ∂M and points outward E.

Since ϕ is tangential on $F_j \setminus L$, $\nu - (\cos \gamma_j)\overline{\nu}$ is normal to ϕ . We then multiply $-\cos \gamma_j$ to (2.20) and add the result to (2.11), thus obtaining

$$\left[\delta \|T\| - \sum_{j=1}^{2} (\cos \gamma_j) \delta W_j\right] (\psi \phi) = 0.$$
(2.21)

Denote $Z = ||T|| - \sum_{j=1}^{2} (\cos \gamma_j) W_j$. Now since F_1, F_2 are smooth surfaces intersecting transversely on L, we may choose a vector field ϕ that is $C^{1,\alpha}$ close to the radial vector field $\nabla_M \operatorname{dist}(\cdot, q)$. For a complete argument, see (2.4) of [Sim80]. Then by a minor modification of the argument of 5.1 in [All72], we conclude that

$$\exp(c\rho^{\alpha})\frac{\|Z\|(B_{\rho}(0))}{\rho^2} \text{ is increasing in } \rho, \text{ for } \rho < \rho_0.$$
(2.22)

We thus deduce from (2.19) and (2.22) that there is a nontrivial tangent cone Z_{∞} of Z at q. Precisely, under the homothetic transformations μ_r defined by $x \mapsto r(x-q)$ (r > 0), $(\mu_{r_k \#} T, \mu_{r_k \#} W_1, \mu_{r_k \#} W_2, \mu_{r_k \#} Z)$ subsequentially converges to $Z_{\infty} = ||T_{\infty}|| - \sum_{j=1}^{2} W_{j,\infty}$ in \mathbf{R}^3 . Let $F_{j,\infty}$, j = 1, 2, denote the corresponding limit planes in \mathbf{R}^3 of F_j , $L_{\infty} = F_{1,\infty} \cap F_{2,\infty}$. Denote $P_{\infty} = L^{\perp}$ the two-plane through 0 that is perpendicular to L. Denote D_r the open disk of radius r centered at 0 on the plane P_{∞} .

Proposition 2.9. The tangent cone $T_{\infty} \subset \mathbf{R}^3$ is a single-sheeted planar domain that intersects $F_{j,\infty}$, j = 1, 2, at angle γ_j . Moreover, it is unique. Namely, T_{∞} does not depend on the choice of subsequence for its construction.

Proof. We first notice that the tangent cone $||T_{\infty}||$ is nontrivial by virtue of (2.19). Moreover, since (T, E) solves the variational problem (1.4), (T_{∞}, E_{∞}) minimizes the corresponding energy in \mathbf{R}^3 . Precisely, let C be the open set in \mathbf{R}^3 enclosed by $F_{1,\infty}$ and $F_{2,\infty}$. Then for any open subset $U \subset \subset \mathbf{R}^3$, E_{∞} minimizes the energy

$$\mathcal{F}(E'_{\infty}) = \mathcal{H}^2(\partial E'_{\infty} \cap \mathring{C} \cap U) - \sum_{j=1}^2 (\cos \gamma_j) \mathcal{H}^2(\partial E'_{\infty} \cap \partial C \cap U).$$
(2.23)

It follows immediately that T_{∞} is minimal in \hat{C} . Therefore each connected component of T_{∞} is part of a two-plane. We conclude that

$$T_{\infty} = \bigcup_{j=1}^{N} \pi_j \cap C, \qquad (2.24)$$

where π_j are planes through the origin and $\pi_i \cap \pi_j \cap C = \emptyset$ for $i \neq j$. Therefore we conclude either Case 1 N = 1 and $T_{\infty} = \pi_1 \cap C$ for some plane π_1 such that $\pi_1 \cap L_{\infty} = \{0\}$, or

Case 2 $N < \infty$ and $T_{\infty} = \bigcup_{j=1}^{N} \pi_j \cap C$, where π_1, \cdots, π_N are planes with the line L_{∞} in common.

Now we rule out case 2 by constructing proper competitors. Notice that in case 2, $E_{\infty} = E_{\infty}^{(1)} \times \mathbf{R}$ for some open $E_{\infty}^{(1)} \subset P_{\infty}$, here P_{∞} is a wedge region in \mathbf{R}^2 such that $C = P_{\infty} \times \mathbf{R}$, and $\partial E_{\infty}^{(1)}$ a finite union of rays emanating from the origin. Define the functional

$$\mathcal{F}_{\infty}^{(1)}(E') = \mathcal{H}^1(\partial E' \cap C \cap D_1) - \sum_{j=1}^2 (\cos \gamma_j) \mathcal{H}^1(\partial E' \cap F_{j,\infty} \cap D_1). \quad (2.25)$$

Notice that since E_{∞} minimizes (2.23),

$$\mathcal{F}_{\infty}^{(1)}(E_{\infty}^{(1)}) \le \mathcal{F}_{\infty}^{(1)}(E'),$$

for any competitor E'.

Observe that $P_{\infty} \setminus \overline{E_{\infty}^{(1)}}$ satisfies a variational principle similar to that satisfied by $E_{\infty}^{(1)}$ but with $\pi - \gamma_j$ in place of γ_j . In case N > 1, we may simply "smooth out" the vertex of $(\pi_1 \cap \pi_2) \cap \overline{D_1}$ to decrease the functional $E_{\infty}^{(1)}$. Thus N = 1. Without loss of generality assume that $\gamma_1 \leq \gamma_2$.

To show that N = 1 in case 2 cannot happen, we first observe that if β_0 is the angle formed by $E_{\infty}^{(1)}$ and $F_{1,\infty}$ at 0, then $\beta_0 \geq \gamma_1$. Otherwise we may construct a competitor E' that has strictly smaller functional (2.25) as follows. Let $q_1 \in \partial D_{1/2} \cap (\partial E_{\infty}^{(1)} \setminus \partial C)$ and let q_2 be the point on $\partial E_{\infty}^{(1)} \cap F_{1,\infty}$ at distance ε from 0. Then let $E' = E_{\infty}^{(1)} \setminus H$, where H is the closed half plane with $0 \in H \setminus \partial H$ and $\{q_1, q_2\} \in \partial H$. If $\beta_0 < \gamma_1$, then it is easily checked (as illustrated in Figure 4) that

$$\mathcal{F}_{\infty}^{(1)}(E') < \mathcal{F}_{\infty}^{(1)}(E_{\infty}^{(1)}).$$



FIGURE 4. The construction of a competitor when $\beta_0 < \gamma_1$.

On the other hand, since $P_{\infty} \setminus E_{\infty}^{(1)}$ satisfies a similar variational principle with angle $\pi - \gamma_j$ in place of γ_j , we deduce that

$$\theta - \beta_0 \ge \pi - \gamma_2.$$

We therefore conclude that $\theta \ge \pi + \gamma_1 - \gamma_2$, contradiction. Thus case 2 is impossible.

In case 1, T_{∞} contains a single sheet of plane. Namely, there exists some plane $\pi_1 \subset \mathbf{R}^3$ such that $T_{\infty} = \pi_1 \cap C$.

Notice also that the plane $\pi_1 \subset \mathbf{R}^3$ should have constant contact angle along $F_{j,\infty}$, j = 1, 2:

$$\measuredangle(\pi_1, F_{1,\infty}) = \gamma_1, \quad \measuredangle(\pi_1, F_{2,\infty}) = \gamma_2, \tag{2.26}$$

since everywhere on $\partial \Sigma \cap (F_j \setminus L)$, Σ and F_j meet at constant contact angle γ_j . We point out that the angle assumption (2.9) is also a *necessary* and sufficient condition for the existence of $\pi_1 \subset \mathbf{R}^3$. As a consequence, $T_{\infty} = \pi_1 \cap \mathring{C}$ with π_1 uniquely determined by (2.26), independent of choice of the particular sequence of r_k chosen to construct T_{∞} . In other words, we have the strong property that the tangent cone is unique for T at q. \Box

Remark 2.10. This part of the argument relies heavily on the fact that T is two dimensional in two ways:

- The planes are the only minimal cones in \mathbf{R}^3 .
- A plane is uniquely determined by its intersection angles with two fixed planes.

Neither of these two statements is valid in higher dimensions.

Remark 2.11. The proof suggests that without the upper bound $\theta < \pi - |\gamma_1 - \gamma_2|$, the tangent cone of T at the corners could be a half plane through L_{∞} . Moreover, T_{∞} may depend on the choice of the sequences of r_k in its construction.

2.3.3. Curvature estimates and consequences. We prove a curvature estimate for Σ near the corner q. Combined with the uniqueness of tangent cone, we deduce that Σ must be graphical over its tangent plane at q. Then we may apply the PDE theory from [Lie88] to conclude that $\overline{\Sigma}$ is a $C^{1,\alpha}$ surface.

We begin with the following lemma, which is a consequence of the monotonicity formula.

Lemma 2.12. Let $C \in \mathbb{R}^3$ be an open subset enclosed by two planes F_1, F_2 with $\measuredangle(F_1, F_2) = \theta$. Let Σ be an area minimizing surface in C such that Σ intersects F_j at constant angles γ_1, γ_2 , and that $\mathcal{H}^2(\Sigma \cap B_0(R)) < C_0R^2$ holds for large R and some $C_0 > 0$. Assume also that

 $|\pi - (\gamma_1 + \gamma_2)| < \theta < \pi - |\gamma_1 - \gamma_2|.$

Then there is a plane $\pi_1 \subset \mathbf{R}^3$ such that $\Sigma = \pi_1 \cap C$.

Proof. Without loss of generality assume $0 \in \overline{\Sigma}$. Consider the tangent cone of Σ at ∞ and at 0. Since Σ satisfies the angle assumption (2.9), its tangent cone at 0, denoted by Σ_0 , exists and is planar. Now by the monotonicity formula (2.22) and the growth assumption $\mathcal{H}^2(\Sigma \cap B_0(R)) < C_0 R^2$, its tangent cone at infinity, denoted by Σ_{∞} , exists and is an area minimizing cone. Since Σ_0 and Σ_{∞} are both minimal cones in $C \subset \mathbf{R}^3$, they are parts of planes. However, the same argument as in the proof of Proposition 2.9 implies that $\Sigma_0 = \Sigma_{\infty} = \pi \cap C$, where π is the unique plane intersecting F_j at angle γ_j . Therefore the equality in the monotonicity formula holds, and $\Sigma = \Sigma_0 = \Sigma_{\infty}$ is planar.

We are ready to prove the curvature estimates:

Proposition 2.13. Let $\Sigma = \operatorname{spt}(T) \cap M$ be a minimizer of the variational problem (1.4). Let $L = F_1 \cap F_2$, $q \in \partial \Sigma \cap L$. Then the following curvature estimate holds:

$$|A_{\Sigma}|(x) \cdot \operatorname{dist}(x,q) \to 0, \qquad (2.27)$$

as $x \in \Sigma$ converges to q.

Proof. Assume, for the sake of contradiction, that there is $\delta > 0$ and a sequence of points $q_k \in \Sigma$ such that

$$\operatorname{dist}(q_k, q) = \varepsilon_k > 0, \quad \varepsilon_k |A_{\Sigma}|(q_k) = \delta_k > \delta$$

By a standard point-picking argument, we could also assume that

$$|A_{\Sigma}|(x) < \frac{2\delta_k}{\varepsilon_k}, \quad x \in C_{2\varepsilon_k}.$$
(2.28)

Consider the rescaled surfaces

$$\Sigma_k = \frac{\delta_k}{\varepsilon_k} (\Sigma - q_k) \subset \frac{\delta_k}{\varepsilon_k} (M - q_k).$$

Since $\delta_k > \delta$, $\varepsilon_k \to 0$, the ambient manifold M converges, in the sense of Gromov-Hausdorff, to (C, g_{Euclid}) . Since Σ_k is area minimizing, a subsequence (which we still denote by Σ_k) converges to an area minimizing surface Σ_{∞} . By (2.19), Σ_{∞} is nontrivial. We consider two different cases

- If $\limsup_k \delta_k = \infty$, then by taking a further subsequence (which we still denote by Σ_k), Σ_k converges to an area minimizing surface in \mathbf{R}^3 . Moreover, (2.28) guarantees that the $|A_{\Sigma_{\infty}}|(x) < 2$ for all $x \in \mathbf{R}^3$. Therefore the convergence $\Sigma_k \to \Sigma_{\infty}$ is, in fact, in C^{∞} . This produces a contradiction, since $|A_{\Sigma_k}|(0) = 1$ for all k, and Σ_{∞} is a plane through the origin.
- If $\limsup_k \delta_k < C < \infty$, then the sequence Σ_k converges to an area minimizing surface in the open set $C \subset \mathbf{R}^3$ enclosed by the two limit planes. This produces a similar contradiction, because $|A_{\Sigma_k}|(0) = 1$, $\Sigma_k \to \Sigma_\infty$ smoothly, and by Lemma 2.12, Σ_∞ is flat in its interior.

With the curvature estimate, we may conclude the regularity discussion by concluding that Σ is graphical near the corner q:

Proposition 2.14. Let Σ be an energy minimizer of (1.4), $q \in \overline{\Sigma} \cap L$. Then Σ is a graph over the tangent plane at q, and its normal vector extends Hölder continuously to q; thus Σ is a $C^{1,\alpha}$ surface with corners.

Proof. We first prove that Σ is graphical near q. Embed a neighborhood of q isometrically into some Euclidean space \mathbf{R}^N . Take the unique plane $\pi_1 \subset T_q M$ obtained above such that the tangent cone of Σ at q is $\pi_1 \cap C$. Assume, for the sake of contradiction, that there is a sequence of points $q_k \in \Sigma$, dist $_M(q_k, q) \to 0$, and that the normal vectors N_k of $\Sigma \subset M$ at q_k is parallel to π_1 . Denote $\varepsilon_k = \text{dist}_M(q_k, q)$. Consider the rescaled surfaces $\Sigma_k = \varepsilon_k^{-1}(\Sigma - q)$. By the monotonicity formula (2.22) and the lower density bound (2.19), a subsequence of $\{\Sigma_k\}$ converges to the unique tangent cone $\pi_1 \cap C$ in the sense of varifolds. Notice that on Σ_k , the image of q_k under the homothety has unit distance to the origin. By taking a further subsequence (which we still denote by $\{(\Sigma_k, q_k, N_k)\}$), we may assume that $q_k \to q_\infty$, $N_k \to N_\infty$, and dist $\mathbf{R}^3(q_\infty, 0) = 1$. Now the curvature estimate (2.27) implies that,

 $|A_{\Sigma_{\infty}}|(x) < 2$, for all points $x \in \Sigma \cap B_{1/2}(q_{\infty})$.

For any point $x \in \Sigma_{\infty}$, and any curve *l* connecting q_{∞} and *x*, we have

$$|N_{\infty}(x) - N_{\infty}(q_{\infty})| < \int_{l} |A_{\Sigma_{\infty}}|(y)dy.$$

Therefore we conclude that, for points x on a neighborhood V of q_{∞} on Σ_{∞} ,

$$|N_{\infty}(x) - N_{\pi_1}| > \frac{1}{2}$$

where N_{π_1} is the unit normal vector of π_1 . This contradicts the fact that Σ_k converges to Σ_{∞} as varifolds.

Once we know that Σ is a graph over $T_q\Sigma$ near q, the result of [Lie88] directly applies, and we conclude that Σ has a Hölder continuous unit normal vector field up to q.

3. Non-rigid case

We prove Theorem 1.4 in this section. Let P be a polyhedron in \mathbb{R}^3 of cone or prism types. Assume, for the sake of contradiction, that there exists a P-type polyhedron (M^3, g) with $R(g) \ge 0$, $\overline{H} \ge 0$ and $\measuredangle_{ij}(M) < \measuredangle_{ij}(P)$. The strategy is to take the minimizer $\Sigma = \partial E$ of the (1.4). When M is of prism type, the existence and regularity of Σ follows from the maximum principle in Proposition 2.2. When M is of cone type, we need the extra assumption that $\mathcal{I} < 0$ to guarantee that $E \neq \emptyset$. Hence we prove the following:

Lemma 3.1. Let $P \subset \mathbf{R}^3$ be polyhedron of cone type, (M, g) be of P-type. Assume $\measuredangle_{ij}(M) < \measuredangle_{ij}(P)$, then the infimum \mathcal{I} appeared in (1.4) is negative.

Proof. As before let F_j , F'_j denote the side faces of M, P, respectively; B, B' denote their base faces. Assume, for the sake of contradiction, that

$$\mathcal{H}^2(\partial E \cap \mathring{M}) - \sum_{j=1}^k (\cos \gamma_j) \mathcal{H}^2(\partial E \cap F_j) \ge 0.$$
(3.1)

Notice that the inequality (3.1) is scaling invariant. Precisely, if $E \subset M$ satisfies (3.1), then under the homothety μ_r defined by $x \mapsto r(x-p)$, the set $(\mu_r)_{\#}(E) \subset (\mu_r)_{\#}(M)$ satisfies (3.1). Letting $r \to \infty$, the tangent cone T_pM of M at p should share the same property. Let $F_{j,\infty}$ denote the corresponding faces in T_pM . By assumption, $\mathcal{L}(F_{j,\infty}, F_{j+1,\infty}) < \mathcal{L}(F'_j, F'_{j+1})$. Therefore T_pM can be placed strictly inside the tangent cone of P at its vertex. By elementary Euclidean geometry, there exists a plane $\pi \subset \mathbb{R}^3$ such that π meets $F_{j,\infty}$ with angle $\gamma'_j > \gamma_j$. See Figure 5 for an illustration, where the dashed polyhedral cone is T_pM .



FIGURE 5. The tangent cone T_pM contained in P.

Let $\operatorname{proj}_{\pi}$ denote the projection $\mathbb{R}^3 \to \pi$. Then the Jacobian of $\operatorname{proj}_{\pi}$, restricted to each $F_{j,\infty}$, is $\cos \gamma'_j$. Denote E_{∞} the open domain enclosed by π and $F_{j,\infty}$, $j = 1, \dots, k$. By the area formula,

$$\mathcal{H}^2(\pi \cap \partial E_{\infty}) - \sum_{j=1}^k (\cos \gamma'_j) \mathcal{H}^2(F_{j,\infty} \cap \partial E_{\infty}) = 0.$$

Since $\gamma'_j > \gamma_j$, we conclude

$$\mathcal{H}^2(\pi \cap \partial E_\infty) - \sum_{j=1}^k (\cos \gamma_j) \mathcal{H}^2(F_{j,\infty} \cap \partial E_\infty) < 0,$$

contradiction.

In the proof we are going to need another simple fact from Euclidean geometry.

Lemma 3.2. Let P_i, Q_i, R_i , i = 1, 2, be six planes in \mathbb{R}^3 with the property that $\measuredangle(P_1, R_1) = \measuredangle(P_2, R_2), \ \measuredangle(Q_1, R_1) = \measuredangle(Q_2, R_2) \text{ and } \measuredangle(P_1, Q_1) \le \measuredangle(P_2, Q_2).$ Let $L_i = P_i \cap R_i, \ L'_i = Q_i \cap R_i, \ i = 1, 2.$ Then $\measuredangle(L_1, L'_1) \le \measuredangle(L_2, L'_2).$

Proof. The proof is very similar to that of Lemma 2.5. Let $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i$ be the unit normal vectors of $P_i, Q_i, R_i, i = 1, 2$, with the same choice of orientation, and such that $\mathbf{u}_i, \mathbf{v}_i$ are pointing out of the wedge region formed by P_i and Q_i . Denote $\gamma_1 = \measuredangle(P_1, R_1) = \measuredangle(P_2, R_2), \gamma_2 = \measuredangle(Q_1, R_1) = \measuredangle(Q_2, R_2), \theta = \measuredangle(P_1, Q_1), \text{ and } \theta' = \measuredangle(P_2, Q_2).$ Then $\measuredangle(L_i, L'_i)$ is given by the angle between $\mathbf{u}_i \times \mathbf{w}_i$ and $\mathbf{w}_i \times \mathbf{v}_i, i = 1, 2$. Notice that for $i = 1, 2, |\mathbf{u}_i \times \mathbf{w}_i| = \sin \gamma_1$, and $|\mathbf{w}_i \times \mathbf{v}_i| = \sin \gamma_2$. By assumptions, we have that

$$\mathbf{u}_1 \cdot \mathbf{v}_1 = -\cos\theta, \quad \mathbf{u}_2 \cdot \mathbf{v}_2 = -\cos\theta', \quad \mathbf{u}_i \cdot \mathbf{w}_i = \cos\gamma_1, \quad \mathbf{v}_i \cdot \mathbf{w}_i = \cos\gamma_2.$$

Therefore

$$(\mathbf{u}_1 \times \mathbf{w}_1) \cdot (\mathbf{w}_1 \times \mathbf{v}_1) = \cos \gamma_1 \cos \gamma_2 + \cos \theta'.$$

Hence if $\theta' \leq \theta$, $(\mathbf{u}_1 \times \mathbf{w}_1) \cdot (\mathbf{w}_1 \times \mathbf{v}_1) \geq (\mathbf{u}_2 \times \mathbf{w}_2) \cdot (\mathbf{w}_2 \times \mathbf{v}_2)$. This implies that $\measuredangle(L_1, L'_1) \leq \measuredangle(L_2, L'_2)$.

Now we prove Theorem 1.4.

Proof. Assume, for the sake of contradiction, that $\angle i_{ij}(M) < \angle i_{ij}(P)$. By Theorem 2.1 and Lemma 3.1, the infimum in (1.4) is achieved by an open set E, with $\Sigma = \partial E \cap \mathring{M}$ a smooth surface which is $C^{1,\alpha}$ up to its corners for some $\alpha \in (0, 1)$. By the first variation formula (2.1), Σ is capillary minimal. We apply the second variational formula (2.2) and conclude

$$\int_{\Sigma} [|\nabla f|^2 - (|A|^2 + \operatorname{Ric}(N, N))f^2] d\mathcal{H}^2 - \int_{\partial \Sigma} Qf^2 d\mathcal{H}^1 \ge 0, \quad (3.2)$$

for any C^2 function f compactly supported away from the corners, where on $\partial \Sigma \cap F_j$,

$$Q = \frac{1}{\sin \gamma_j} \operatorname{II}(\overline{\nu}, \overline{\nu}) + (\cot \gamma_j) A(\nu, \nu).$$

Since the surface Σ is $C^{1,\alpha}$ to its corners, its curvature |A| is square integrable. Hence by a standard approximation argument we conclude that the above inequality holds for the constant function f = 1. We have

$$-\int_{\Sigma} (|A|^{2} + \operatorname{Ric}(N, N)) - \sum_{j=1}^{n} \int_{\partial \Sigma \cap F_{j}} \left[\frac{1}{\sin \gamma_{j}} \operatorname{II}(\overline{\nu}, \overline{\nu}) + \cot \gamma_{j} A(\nu, \nu) \right] \ge 0. \quad (3.3)$$

Applying the Gauss equation on Σ , we have

$$|A|^{2} + \operatorname{Ric}(N, N) = \frac{1}{2}(R - 2K_{\Sigma} + |A|^{2}), \qquad (3.4)$$

where R is the scalar curvature of M, K_{Σ} is the Gauss curvature of Σ .

By the Gauss-Bonnet formula for $C^{1,\alpha}$ surfaces with piecewise smooth boundary components, we have that

$$\int_{\Sigma} K_{\Sigma} d\mathcal{H}^2 + \int_{\partial \Sigma} k_g d\mathcal{H}^1 + \sum_{j=1}^n (\pi - \alpha_j) = 2\pi \chi(\Sigma) \le 2\pi, \qquad (3.5)$$

here k_g is the geodesic curvature of $\partial \Sigma \subset \Sigma$, and α_j are the interior angles of Σ at the corners. By Lemma 3.2, $\alpha_j < \alpha'_j$, where α'_j is the corresponding interior angle of the base face of the Euclidean polyhedron P. Since $\sum_{j=1}^{k} (\pi - \alpha'_j) = 2\pi$, we conclude $\sum_{j=1}^{k} (\pi - \alpha_j) > 2\pi$. As a result, we have that

$$-\int_{\Sigma} K_{\Sigma} d\mathcal{H}^2 > \int_{\partial \Sigma} k_g d\mathcal{H}^1.$$
(3.6)

Combining (3.3), (3.4) and (3.6) we conclude that

$$\int_{\Sigma} \frac{1}{2} \left(R + |A|^2 \right) d\mathcal{H}^2 + \sum_{j=1}^n \int_{\partial \Sigma \cap F_j} \left[\frac{1}{\sin \gamma_j} \operatorname{II}(\overline{\nu}, \overline{\nu}) + \cot \gamma_j A(\nu, \nu) + k_g \right] d\mathcal{H}^1 < 0. \quad (3.7)$$

To finish the proof, let us analyze the last integrand in (3.7). Fix one j and consider $\partial \Sigma \cap F_j$. For convenience let $\gamma = \gamma_j$. We make the following

Claim.

$$II(\overline{\nu},\overline{\nu}) + \cos\gamma A(\nu,\nu) + \sin\gamma k_g = \overline{H}, \qquad (3.8)$$

where \overline{H} is the mean curvature of ∂M in M.

Let T be the unit tangential vector of $\partial \Sigma$. Since Σ is minimal, $A(\nu, \nu) = -A(T,T)$. Therefore

$$\cos \gamma A(\nu, \nu) + \sin \gamma k_g = -\cos \gamma A(T, T) + \sin \gamma k_g$$
$$= -\langle \nabla_T T, \cos \gamma N + \sin \gamma \nu \rangle$$
$$= -\langle \nabla_T T, X \rangle$$
$$= \operatorname{II}(T, T).$$

Since T and $\overline{\nu}$ form an orthonormal basis of ∂M , we have

$$\mathrm{II}(\overline{\nu},\overline{\nu}) + \cos\gamma A(\nu,\nu) + \sin\gamma k_g = \mathrm{II}(T,T) + \mathrm{II}(\overline{\nu},\overline{\nu}) = \overline{H}$$

The claim is proved.

To finish the proof, we note that (3.7) implies that

$$\int_{\Sigma} \frac{1}{2} \left(R + |A|^2 \right) d\mathcal{H}^2 + \sum_{j=1}^n \int_{\partial \Sigma \cap F_j} \frac{1}{\sin \gamma_j} \overline{H} d\mathcal{H}^1 < 0, \tag{3.9}$$

contradicting the fact that the scalar curvature R of M and the surface mean curvature \overline{H} of $\partial M \subset M$ are nonnegative.

4. Rigidity

In this section we prove Theorem 1.5. Rigidity properties of minimal and area-minimizing surfaces have attracted lots of interests in recent years. Following the Schoen-Yau proof of the positive mass theorem, Cai-Galloway [CG00] studied the rigidity of area-minimizing tori in three-manifolds in nonnegative scalar curvature. The case of area-minimizing spheres was carried out by Bray-Brendle-Neves [BBN10]. Their idea is to study constant mean curvature (CMC) foliation around an infinitesimally rigid area-minimizing surface, and obtain a local splitting result for the manifold. It is very robust and applies to a wide variety of rigidity analysis: in the case of negative [Nun13] scalar curvature, and for area-minimizing surfaces with boundary [Amb15] (see also [MM15]). We adapt their idea for our rigidity analysis, and perform a dynamical analysis for foliations with constant mean curvature capillary surfaces. The new challenge here is that, when M is of cube type, the energy minimizer of (1.4) may be empty. In this case the tangent cone $T_p M$ coincides with that of the Euclidean model P, and $\mathcal{I} = 0$. Our strategy, motivated by the earlier work of Ye [Ye91], is to construct a constant mean curvature foliation near the vertex p, such that the mean curvature on each leaf converges to zero when approaching p.

4.1. Infinitesimally rigid minimal capillary surfaces. Assume the energy minimizer $\Sigma = \partial E \cap \mathring{M}$ exists for (1.4). Tracing equality in the proof in Section 3, we conclude that

$$\chi(\Sigma) = 0, \quad R_M = 0, \quad |A| = 0 \quad \text{on } \Sigma$$

$$\overline{H} = 0 \quad \text{on } \partial\Sigma, \qquad \alpha_j = \alpha'_j \quad \text{at the corners of } \Sigma.$$
(4.1)

Moreover, by the second variation formula (2.2),

$$Q(f,f) = -\int_{\Sigma} (f\Delta f + (|A|^2 + \operatorname{Ric}(N,N))f^2) d\mathcal{H}^{n-1} + \sum_{j=1}^k \int_{\partial\Sigma\cap F_j} f\left(\frac{\partial f}{\partial\nu} - Qf\right) d\mathcal{H}^{n-2} \ge 0,$$

with Q(1,1) = 0. We then conclude that for any C^2 function f compactly supported away from the vertices of Σ , Q(1, f) = 0. By choosing appropriate g, we further conclude that

$$\operatorname{Ric}(N,N) = 0 \quad \text{on } \Sigma, \quad \frac{1}{\sin \gamma_j} \operatorname{II}(\overline{\nu},\overline{\nu}) + \cot \gamma_j A(\nu,\nu) = 0 \quad \text{on } \partial \Sigma \cap F_j.$$

Combining with (3.4) and (3.8), we conclude that

$$K_{\Sigma} = 0 \quad \text{on } \Sigma, \quad k_q = 0 \quad \text{on } \partial \Sigma.$$
 (4.2)

Call a surface Σ satisfying (4.1) and (4.2) *infinitesimally rigid*. Notice that such a surface is isometric to an flat k-polygon in \mathbb{R}^2 .

Next, we construct a local foliation by CMC capillary surfaces Σ_t . Take a vector field Y defined in a neighborhood of Σ , such that Y is tangential when restricted to ∂M . Let $\phi(x, t)$ be the flow of Y. Precisely, we have:

Proposition 4.1. Let Σ^2 be a properly embedded, two-sided, minimal capillary surface in M^3 . If Σ is infinitesimally rigid, then there exists $\varepsilon > 0$ and a function $w : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbf{R}$ such that, for every $t \in (-\varepsilon, \varepsilon)$, the set

$$\Sigma_t = \{\phi(x, w(x, t) : x \in \Sigma)\}$$

is a capillary surface with constant mean curvature H(t) that meets F_j at constant angle γ_j . Moreover, for every $x \in \Sigma$ and every $t \in (-\varepsilon, \varepsilon)$,

$$w(x,0) = 0, \quad \int_{\Sigma} (w(x,t) - t) d\mathcal{H}^2 = 0 \quad and \quad \frac{\partial}{\partial t} w(x,t) \Big|_{t=0} = 1.$$

Thus, by possibly choosing a smaller ε , $\{\Sigma_t\}_{t \in (-\varepsilon,\varepsilon)}$ is a foliation of a neighborhood of $\Sigma_0 = \Sigma$ in M.

Our proof goes by an argument involving the inverse function theorem, and is essentially taken from [BBN10] and [Amb15]. We do, however, need an elliptic theory on cornered domains. This is done by Lieberman [Lie89]. The following Schauder estimate is what we need:

Theorem 4.2 (Lieberman, [Lie89]). Let $\Sigma^2 \subset \mathbf{R}^3$ be an open polygon with interior angles less than π . Let L_1, \dots, L_k be the edges of Σ . Then there exists some $\alpha > 0$ depending only on the interior angles of Σ , such that if $f \in C^{0,\alpha}(\overline{\Sigma}), \ g|_{\overline{L_i}} \in C^{0,\alpha}(\overline{L_j})$, then the Neumann boundary problem

$$\begin{cases} \Delta u = f & \text{in } \Sigma\\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial \Sigma \end{cases}$$

$$(4.3)$$

has a solution u with $\int_{\Sigma} u = 0$, and $u \in C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\overline{\Sigma})$. Moreover, the Schauder estimate holds:

$$|u|_{2,\alpha,\Sigma} + |u|_{1,\alpha,\overline{\Sigma}} \le C(|f|_{0,\alpha,\overline{\Sigma}} + \sum_{j=1}^{k} |g|_{0,\alpha,L_j}).$$

We now prove Proposition 4.1.

Proof. For a function $u \in C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\overline{\Sigma})$, consider the surface $\Sigma_u = \{\phi(x, u(x)) : x \in \Sigma\}$, which is properly embedded if $|u|_0$ is small enough. We use the subscript u to denote the quantities associated to Σ_u . For instance, H_u denotes the mean curvature of Σ_u , N_u denotes the unit normal vector field of Σ_u , and X_u denotes the restriction of X onto Σ_u . Then $\Sigma_0 = \Sigma$, $H_0 = 0$, and $\langle N_u, X_u \rangle = \cos \gamma_j$ along $\partial \Sigma \cap F_j$.

Consider the Banach spaces

$$F = \left\{ u \in C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\overline{\Sigma}) : \int_{\Sigma} u = 0 \right\},$$
$$G = \left\{ u \in C^{0,\alpha}(\Sigma) : \int_{\Sigma} u = 0 \right\}, \quad H = \left\{ u \in L^{\infty}(\partial\Sigma) : u|_{\overline{L_j}} \in C^{0,\alpha}(\overline{L_j}) \right\}.$$

Given small $\delta > 0$ and $\varepsilon > 0$, define the map $\Psi : (-\varepsilon, \varepsilon) \times (B_0(\delta) \subset F) \to G \times H$ given by

$$\Psi(t,u) = \left(H_{t+u} - \frac{1}{|\Sigma|} \int_{\Sigma} H_{t+u}, \langle N_{t+u}, X_{t+u} \rangle - \cos \gamma\right),$$

where $\gamma = \gamma_j$ on $\partial \Sigma \cap \check{F}_j$.

In order to apply the inverse function theorem, we need to prove that $D_u \Psi|_{(0,0)}$ is an isomorphism when restricted to $\{0\} \times F$. In fact, for any $v \in F$,

$$D_u \Psi|_{(0,0)}(0,v) = \frac{d}{ds} \bigg|_{s=0} \psi(0,sv) = \left(\Delta v - \frac{1}{|\Sigma|} \int_{\partial \Sigma} \frac{\partial v}{\partial \nu}, -\frac{\partial v}{\partial \nu}\right)$$

The calculation is given in Lemma A.2 and Lemma A.3 in the appendix. Now the fact that $D_u \Psi|_{(0,0)}$ is an isomorphism follows from Theorem 4.2. The rest of the proof is the same as Proposition 10 in [Amb15], which we will omit here.

4.2. **CMC capillary foliation near the vertex.** When (M^3, g) is of cone type with vertex p, we have proved that \mathcal{I} is realized by a minimizer $\partial E \neq \emptyset$ when $\mathcal{I} < 0$. Now it is obvious from the definition that $\mathcal{I} \leq 0$. However, in the case that $\mathcal{I} = 0$, it is a priori possible that the minimizer $E = \emptyset$. Assume $\mathcal{I} = 0$. We investigate this case with a different approach.

Notice that, as a consequence of Lemma 3.1, $\mathcal{I} = 0$ implies that

$$\measuredangle(F_j, F_{j+1})|_p = \measuredangle(F'_j, F'_{j+1}),$$

where F'_j is the corresponding face of the Euclidean model P. Recall that in the Euclidean model P', its base face B' intersects F'_j at angle γ_j . Thus P is foliated by a family of planes parallel to B', where each leaf is minimal, and meets F'_j at constant angle γ_j . We generalize this observation to arbitrary Riemannian polyhedra, and obtain:

Theorem 4.3. Let (M^3, g) be a cone type Riemannian polyhedron with vertex p. Let $P \subset \mathbf{R}^3$ be a polyhedron with vertex p', such that the tangent cones (T_pM, g_p) and $(T_{p'}P, g_{Euclid})$ are isometric. Denote $\gamma_1, \dots, \gamma_k$ the angles between the base face and the side faces of P. Then there exists a small neighborhood U of p in M, such that U is foliated by surfaces $\{\Sigma_{\rho}\}_{\rho \in (0,\varepsilon)}$ with the properties that:

(1) for each $\rho \in (0, \varepsilon)$, Σ_{ρ} meet the side face F_i at constant angle γ_i ;

(2) each Σ_{ρ} has constant mean curvature λ_{ρ} , and $\lambda_{\rho} \to 0$ as $\rho \to 0$.

Remark 4.4. Before proceeding to the proof, let us remark that the local foliation structure of Riemannian manifolds has been a thematic program in geometric analysis, and has deep applications to mathematical general relativity. See: Ye [Ye91] for spherical foliations around a point; Huisken-Yau [HY96] for foliations in asymptotically flat spaces; Mahmoudi-Mazzeo-Pacard [MP05][MMP06] for foliations around general minimal submanifolds.

Remark 4.5. As a technical remark, let us recall that in all of the aforementioned foliation results, some extra conditions are necessary (e.g. Ye's result required the center point to be a non-degenerate critical point of scalar curvature; Mahmoudi-Mazzeo-Pacard needed the minimal submanifold to be non-degenerate critical point for the volume functional). However, in our result, no extra condition is needed. Geometrically, this is because in the tangent cone $T_p M \subset \mathbf{R}^3$, the desired foliation is unique.

Proof. Let U be a small neighborhood of p in M. Take a local diffeomorphism $\varphi : P \to U$, such that the pull back metric φ^*g and g_{Euclid} are C^1 close. Place the vertex p' of P at the origin of \mathbf{R}^3 . In local coordinates on \mathbf{R}^3 , the above requirement is then equivalent to

$$g_{ij}(0) = g_{ij,k} = 0, \quad g_{ij}(x) = o(|x|), g_{ij,k}(x) = o(1) \text{ for } |x| < \rho_0.$$

 φ may be constructed, for instance, via geodesic normal coordinates.

Denote $\overline{M} \subset \mathbf{R}^3$ the tangent cone of M at p. By assumption, the dihedral angles $\angle (F_i, F_j)|_p = \angle (F'_i, F'_j)$. Let π be the plane in \mathbf{R}^3 such that in Euclidean metric, π and F_j meet at constant angle γ_j . For $\rho \in (0, 1]$, let π_ρ be the plane that is parallel to π and has distance ρ to 0. Let Σ_ρ be the intersection of π_ρ with the interior of the cone T_pM . Denote X the outward pointing unit normal vector field on $\partial \overline{M}$, N_ρ the unit vector field of $\Sigma_\rho \subset \overline{M}$ pointing towards 0. Denote Y the vector field such that for each $x \in \Sigma_\rho$, Y(x) is parallel to \vec{x} . Moreover, we require that the flow of Y parallel translates $\{\Sigma_\rho\}$, and Y(x) is tangent to $\partial \overline{M}$ when $x \in \partial \overline{M}$. Let $\phi(x,t)$ be the flow of Y. For a function $u \in C^{2,\alpha}(\Sigma_1) \cap C^{1,\alpha}(\overline{\Sigma_1})$ (Σ_1 is parallel to π , and of distance 1 to the origin), define the perturbed surface

$$\Sigma_{\rho,u} = \{\phi(\rho x, u(\rho x)) : x \in \Sigma_1\}.$$

Since $\Sigma_{\rho} = \rho \Sigma_1$, the surface $\Sigma_{\rho,u}$ is a small perturbation of Σ_{ρ} , and is properly embedded, if $|u|_0$ is sufficiently small.

We use the subscript ρ to denote geometric quantities related to Σ_{ρ} , and the subscript (ρ, u) to denote geometric quantities related to the perturbed surfaces $\Sigma_{\rho,u}$, both in the metric φ^*g . In particular, $H_{\rho,u}$ denotes the mean curvature of $\Sigma_{\rho,u}$, and $N_{\rho,u}$ denotes the unit normal vector field of $\Sigma_{\rho,u}$ pointing towards 0. It follows from Lemma A.1 and Lemma A.2 that we have the following Taylor expansion of geometric quantities.

$$H_{\rho,u} = H_{\rho} + \frac{1}{\rho^2} \Delta_{\rho} u + (\operatorname{Ric}(N_{\rho}, N_{\rho}) + |A_{\rho}|^2) u + L_1 u + Q_1(u)$$

$$\langle X_{\rho,u}, N_{\rho,u} \rangle = \langle X_{\rho}, u_{\rho} \rangle - \frac{\sin \gamma_j}{\rho} \frac{\partial u}{\partial \nu_{\rho}}$$

$$+ (\cos \gamma_j A(\nu_{\rho}, \nu_{\rho}) + \operatorname{II}(\overline{\nu_{\rho}}, \overline{\nu_{\rho}})) u + L_2 u + Q_2(u).$$

(4.4)

Let us explain (4.4) a bit more. Q_1, Q_2 are terms that are at least quadratic in u. The functions L_1, L_2 exhibit how the mean curvature H_{ρ} and the contact angle γ_j deviate from being constant. In particular, they are bounded in the following manner:

$$\begin{split} L_1 &\leq C |\nabla_{\rho} H_{\rho}| |Y| \leq C |g|_{C^2} < C, \quad L_2 \leq C |\nabla_{\rho} \langle X_{\rho}, N_{\rho} \rangle \, ||Y| < C |g|_{C^1} < C. \end{split}$$

The operator Δ_{ρ} is the Laplace operator on Σ_{ρ} . At $x \in \Sigma_{\rho}$,

$$\Delta_{\rho} = \frac{1}{\sqrt{\det(g)}} \partial_i \left(\sqrt{\det(g)} g^{ij} \partial_j \right).$$

In particular, Δ_{ρ} converges to the Laplace operator on \mathbf{R}^2 as $\rho \to 0$. In local coordinates, it is not hard to see that

$$|H_{\rho}| \le C|g|_{C^1} = o(1), \quad |\langle X_{\rho}, N_{\rho} \rangle - \cos \gamma_j| \le |g|_{C^0} = o(\rho).$$

Denote $D_{\rho} = \langle X_{\rho}, N_{\rho} \rangle - \cos \gamma_j$. Letting $H_{\rho,u} \equiv \lambda$, we deduce from (4.4) that we need to solve for u from

$$\begin{cases} \Delta_{\rho}u + \rho^{2}L_{1}u + \rho^{2}Q_{1}(u) = \rho^{2}(\lambda - H_{\rho}) & \text{in } \Sigma_{1}, \\ \frac{\partial u}{\partial \nu_{\rho}} = \rho D_{\rho} + \rho L_{2}u + \rho Q_{2}(u) & \text{on } \partial \Sigma_{1}. \end{cases}$$

$$(4.5)$$

We use inverse function theorem as in the proof of Proposition 4.1. Precisely, denote the operator

$$\begin{cases} \mathcal{L}_{\rho}(u) = \Delta_{\rho}u - \rho^{2}L_{1}u - \rho^{2}Q_{1}(u) + \rho^{2}H_{\rho}, \\ \mathcal{B}_{\rho}(u) = \frac{\partial u}{\partial\nu_{\rho}} - \rho D_{\rho} - \rho L_{2}u - \rho Q_{2}(u), \end{cases}$$

and consider the Banach spaces

$$F = \left\{ u \in C^{2,\alpha}(\Sigma_1) \cap C^{1,\alpha}(\overline{\Sigma_1}) : \int_{\Sigma_1} u = 0 \right\},$$
$$G = \left\{ u \in C^{0,\alpha}(\Sigma_1) : \int_{\Sigma_1} u = 0 \right\}, H = \left\{ u \in L^{\infty}(\partial \Sigma_1) : u |_{\overline{L_j}} \in C^{0,\alpha}(\overline{L_j}) \right\}.$$

Again we use L_1, \dots, L_j to denote the edges of Σ_1 .

For a small $\delta > 0$, let $\Psi : (-\varepsilon, \varepsilon) \times (B_{\delta}(0) \subset F) \to G \times H$ given by

$$\Psi(\rho, u) = \left(\mathcal{L}_{\rho}(u) - \frac{1}{|\Sigma_1|} \int_{\Sigma_1} \mathcal{L}_{\rho}(u) d\mathcal{H}^2, \mathcal{B}_{\rho}(u)\right)$$

By the asymptotic behavior as $\rho \to 0$ discussed above, the linearized operator $D_u \Psi|_{(0,0)}$, when restricted to $\{0\} \times F$, is given by

$$D_u \Psi|_{(0,0)}(0,v) = \frac{d}{ds} \bigg|_{s=0} \Psi(0,sv) = \left(\Delta v - \int_{\Sigma_1} \Delta v, \frac{\partial v}{\partial \nu}\right).$$

By Theorem 4.2, for some $\alpha \in (0,1)$, $D_u \Psi|_{(0,0)}$ is an isomorphism when restricted to $\{0\} \times F$. We therefore apply the inverse function theorem and conclude that, for small $\varepsilon > 0$, there exists a C^1 map between Banach spaces $\rho \in (-\varepsilon, \varepsilon) \mapsto u(\rho) \in B_{\delta}(0) \subset F$ for every $\rho \in (-\varepsilon, \varepsilon)$, such that $\Psi(\rho, u(\rho)) = (0, 0)$. Thus the surface $\Sigma_{\rho, u(\rho)}$ is minimal, and meets F_j at constant angle γ_j .

By definition, u(0) is the zero function. Denote $v = \frac{\partial u(\rho)}{\partial \rho}$. Differentiating (4.5) with respect to ρ and evaluating at $\rho = 0$, we deduce

$$\begin{cases} \Delta v = 0 & \text{in } \Sigma_1, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Sigma_1. \end{cases}$$
(4.6)

Therefore v is also the zero function. Thus we conclude that

$$|u|_{1,\alpha,\overline{\Sigma_1}} = o(\rho),$$

for $|\rho| < \rho_0$.

Therefore the surfaces $\Sigma_{\rho,u(\rho)}$ is a foliation of a small neighborhood of p. Moreover, integrating (4.5) over Σ_1 , we find that the constant mean curvature of $\Sigma_{\rho,u(\rho)}$ satisfies

$$\lambda_{\rho} = \frac{1}{\rho^2} \int_{\Sigma_1} \Delta u + \int_{\Sigma_1} (L_1 u + Q_1(u) + H_{\rho})$$

= $\frac{1}{\rho^2} \int_{\partial \Sigma_1} \frac{\partial u}{\partial \nu} + \int_{\Sigma_1} (L_1 u + Q_1(u) + H_{\rho}) + o(1)$ (4.7)
= $\frac{1}{\rho} \int_{\partial \Sigma_1} (D_{\rho} + L_2 u + Q_2(u)) + \int_{\Sigma_1} (L_1 u + Q_1(u) + H_{\rho}) + o(1).$

Since

$$D_{\rho} = o(\rho), \quad |u|_{1,\alpha,\overline{\Sigma_1}} = o(\rho), \quad H_{\rho} = o(1),$$

we conclude that $\lambda_{\rho} \to 0$, as $\rho \to 0$.

4.3. Local splitting. We analyze the CMC capillary foliations developed above to prove a local splitting theorem, thus prove Theorem 1.5. We need the extra assumption (1.2) that

$$\gamma_j \le \pi/2, j = 1, \cdots, k \text{ or } \gamma_j \ge \pi/2, j = 1, \cdots, k.$$

First notice that, if $P \subset \mathbf{R}^3$ is a cone, then (1.2) is possible only when $\gamma_j \leq \pi/2, j = 1, \dots, k$; if P is a prism and $\gamma_j > \pi/2$, then instead of (1.4),

we consider, for $E_1 = M \setminus \overline{E}$,

$$\mathcal{F}(E_1) = \mathcal{H}^2(\partial E_1 \cap \mathring{M}) - \sum_{j=1}^k (\cos \gamma_j) \mathcal{H}^2(\partial E_1 \cap F_j), \qquad (4.8)$$

and reduce the problem to the case where $\gamma_j \leq \pi/2$. Thus we always assume $\gamma_j \leq \pi/2, \ j = 1, \cdots, k$.

Under the same conventions as before, assume we have a local CMC capillary foliation $\{\Sigma_{\rho}\}_{\rho \in I}$, where as ρ increase, Σ_{ρ} moves in the direction of N_{ρ} . We will take I to be $(-\varepsilon, \varepsilon)$, $(-\varepsilon, 0)$ or $(0, \varepsilon)$, according to the location of the foliation. We prove the following differential inequality for the mean curvature $H(\rho)$.

Proposition 4.6. There exists a nonnegative continuous function $C(\rho) \ge 0$ such that

$$H'(\rho) \ge C(\rho)H(\rho).$$

Proof. Let $\psi : \Sigma \times I \to M$ parametrizes the foliation. Denote $Y = \frac{\partial \psi}{\partial t}$. Let $v_{\rho} = \langle Y, N_{\rho} \rangle$ be the lapse function. Then by Lemma A.1 and Lemma A.2, we have

$$\frac{d}{d\rho}H(\rho) = \Delta_{\rho}v_{\rho} + (\operatorname{Ric}(N_{\rho}, N_{\rho}) + |A_{\rho}|^2)v_{\rho} \quad \text{in } \Sigma_{\rho},$$
(4.9)

$$\frac{\partial v_{\rho}}{\partial \nu_{\rho}} = \left[(\cot \gamma_j) A_{\rho}(\nu_{\rho}, \nu_{\rho}) + \frac{1}{\sin \gamma_j} \operatorname{II}(\overline{\nu_{\rho}}, \overline{\nu_{\rho}}) \right] v_{\rho} \quad \text{on } \partial \Sigma_{\rho} \cap F_j.$$
(4.10)

By shrinking the interval I if possible, we may assume $v_{\rho} > 0$ for $\rho \in I$. Multiplying $\frac{1}{v_{\rho}}$ on both sides of (4.9) and integrating on Σ_{ρ} , we deduce that

$$H'(\rho) \int_{\Sigma_{\rho}} \frac{1}{v_{\rho}} = \int_{\Sigma_{\rho}} \frac{|\nabla v_{\rho}|^2}{v_{\rho}^2} d\mathcal{H}^2 + \frac{1}{2} \int_{\Sigma_{\rho}} (R + |A|^2 + H^2) d\mathcal{H}^2 - \int_{\Sigma_{\rho}} K_{\Sigma_{\rho}} d\mathcal{H}^2 + \sum_{j=1}^k \int_{\partial\Sigma_{\rho}\cap F_j} \left[\cot\gamma_j A_{\rho}(\nu_{\rho}, \nu_{\rho}) + \frac{1}{\sin\gamma_j} \operatorname{II}(\overline{\nu_{\rho}}, \overline{\nu_{\rho}}) \right] d\mathcal{H}^1 \geq - \int_{\Sigma_{\rho}} K_{\Sigma_{\rho}} d\mathcal{H}^2 + \sum_{j=1}^k \int_{\partial\Sigma_{\rho}\cap F_j} \left[\cot\gamma_j A_{\rho}(\nu_{\rho}, \nu_{\rho}) + \frac{1}{\sin\gamma_j} \operatorname{II}(\overline{\nu_{\rho}}, \overline{\nu_{\rho}}) \right] d\mathcal{H}^1.$$

$$(4.11)$$

Using the Gauss-Bonnet formula and Lemma 3.2,

$$-\int_{\Sigma_{\rho}} K_{\Sigma_{\rho}} d\mathcal{H}^2 \ge \int_{\partial \Sigma_{\rho}} k_g d\mathcal{H}^1.$$
(4.12)

As in (3.8), we also have

$$k_g + \cot \gamma_j A(\nu_\rho, \nu_\rho) + \frac{1}{\sin \gamma_j} \operatorname{II}(\overline{\nu_\rho}, \overline{\nu_\rho}) = (\cot \gamma_j) H(\rho) + \frac{1}{\sin \gamma_j} \overline{H}, \quad (4.13)$$

on $\partial \Sigma_{\rho} \cap F_j$. Combining these, we deduce

$$H'(\rho) \int_{\Sigma_{\rho}} \frac{1}{v_{\rho}} \geq \sum_{j=1}^{k} \int_{\partial\Sigma_{\rho}\cap F_{j}} \left[(\cot \gamma_{j})H(\rho) + \frac{1}{\sin \gamma_{j}}\overline{H} \right] d\mathcal{H}^{1}$$

$$\geq \left[\sum_{j=1}^{k} (\cot \gamma_{j})\mathcal{H}^{1}(\partial\Sigma_{\rho}\cap F_{j}) \right] H(\rho).$$

$$(4.14)$$

Take $C(\rho) = \sum_{j=1}^{k} (\cot \gamma_j) \mathcal{H}^1(\partial \Sigma_{\rho} \cap F_j)$. The proposition is proved.

We are now ready to prove Theorem 1.5.

Proof. If (M^3, g) is of prism type, or if (M^3, g) is of cone type with $\mathcal{I} < 0$, then the variational problem (1.4) has a nontrivial solution E with a $C^{1,\alpha}$ boundary Σ . Therefore Σ is infinitesimally rigid minimal capillary, and there is a CMC capillary foliation $\{\Sigma_{\rho}\}_I$ around Σ , where $I = (-\varepsilon, \varepsilon)$ if $\Sigma \subset \mathring{M}$, $I = [0, \varepsilon)$ if $\Sigma = B_1$, and $I = (-\varepsilon, 0]$ if $\Sigma = B_2$. By Proposition 4.6, the mean curvature $H(\rho)$ of Σ_{ρ} satisfies

$$\begin{cases} H(0) = 0\\ H'(\rho) \ge C(\rho)H(\rho) \end{cases}$$

where $C(\rho) \ge 0$. By standard ordinary differential equation theory,

$$H(\rho) \ge 0$$
 when $\rho \ge 0$, $H(\rho) \le 0$ when $\rho \le 0$.

Denote E_{ρ} the corresponding open domain in M. Since each Σ_{ρ} meets F_j at constant angle γ_j , the first variation formula (2.1) implies that

$$F(\rho_1) - F(\rho_2) = -\int_{\rho_2}^{\rho_1} d\rho \int_{\Sigma_{\rho}} H(\rho) v_{\rho} d\mathcal{H}^2$$

We then conclude that for $\delta > 0$,

$$F(\delta) \le F(0), \qquad F(-\delta) \le F(0).$$

However, $\Sigma_0 = \Sigma$ minimizes the functional (1.4). Therefore in a neighborhood of Σ , $F(\rho) = F(0)$, $H(\rho) \equiv 0$. Tracing back the equality conditions, we find that

 $v_{\rho} \equiv \text{constant}, \quad \text{each } \Sigma_{\rho} \text{ is infinitesimally rigid.}$

It is then straightforward to check that the normal vector fields of Σ_{ρ} is parallel (see [BBN10] or [MM15]). In particular, its flow is a flow by isometries and therefore provides the local splitting. Since M is connected, this splitting is also global, and we conclude that (M^3, g) is isometric to a flat polyhedron in \mathbb{R}^3 .

If (M^3, g) is of cone type with $\mathcal{I} = 0$, then by Theorem 4.3, there is a CMC capillary foliation $\{\Sigma_{\rho}\}_{\rho \in (-\varepsilon, 0)}$ near the vertex, with $H(\rho) \to 0$ as $\rho \to 0$. By Proposition 4.6, the mean curvature $H(\rho)$ satisfies

$$\begin{cases} H'(\rho) \ge C(\rho)H(\rho) & \rho \in (-\varepsilon, 0) \\ H(\rho) \to 0 & \rho \to 0. \end{cases}$$

Since $C(\rho) \ge 0$, we conclude that $H(\rho) \le 0$, $\rho \in (-\varepsilon, 0)$. Let E_{ρ} be the open subset bounded by Σ_{ρ} . Take $0 < \eta < \delta$, then

$$F(-\eta) - F(-\delta) = -\int_{-\delta}^{-\eta} d\rho \int_{\Sigma_{\rho}} Hv_{\rho} d\mathcal{H}^2 \ge 0 \quad \Rightarrow \quad F(-\delta) \le F(-\eta).$$

Letting $\eta \to 0$, we have

$$F(-\delta) \le 0.$$

As before, we conclude that $F(\rho) \equiv 0$ for $\rho \in (-\varepsilon, 0)$, and that each leaf Σ_{ρ} is infinitesimally rigid. Thus (M^3, g) admits a global splitting of flat k-polygon in \mathbb{R}^2 , and hence is isometric to a flat polyhedron in \mathbb{R}^3 . \Box

APPENDIX A.

We provide some general calculation for infinitesimal variations of geometric quantities of properly immersed hypersurfaces under variations of the ambient manifold (M^{n+1}, g) that leave the boundary of the hypersurface inside ∂M . We also refer the readers to the thorough treatment in [RS97] and [Amb15] (warning: the choice of orientation for the unit normal vector field N in [Amb15] is the opposite to ours).

We keep the notations used in Section 2.1 and for each $t \in (-\varepsilon, \varepsilon)$, we use the subscript t for the terms related to Σ_t . Recall that $Y = \frac{\partial \Psi(t, \cdot)}{\partial t}$ is the deformation vector field. Denote Y_0 the tangent part of Y on Σ , Y_0 the tangent part of Y on $\partial \Sigma$. Let $v = \langle Y, N \rangle$. For $q \in \Sigma$, let e_1, \dots, e_n be an orthonormal basis of $T_q \Sigma$, and let $e_i(t) = d\Psi_t(e_i)$. Let S_0, S_1 be the shape operators of $\Sigma \subset M$ and $\partial M \subset M$. Precisely, $S_0(Z_1) = -\nabla_{Z_1}N$, $S_1(Z_2) = \nabla_{Z_2}X$. We have:

Lemma A.1 (Lemma 4.1(1) of [RS97], Proposition 15 of [Amb15]).

$$\nabla_Y N = -\nabla^\Sigma v - S_0(Y_0). \tag{A.1}$$

We use Lemma A.1 to calculate the evolution of the contact angle along the boundary.

Lemma A.2. Let γ denote the contact angle between Σ and F_i . Then

$$\frac{d}{dt}\Big|_{t=0} \langle N_t, X_t \rangle = -\sin\gamma \frac{\partial v}{\partial \nu} + (\cos\gamma)A(\nu,\nu)v + \mathrm{II}(\overline{\nu},\overline{\nu})v + \left\langle L, \nabla^{\partial\Sigma}\gamma_j \right\rangle v,$$
(A.2)

where L is a bounded vector field on $\partial \Sigma$.

A POLYHEDRON COMPARISON THEOREM IN POSITIVE SCALAR CURVATURE 31

In particular, if each Σ_t meets F_j at constant angle γ_j , then on F_j ,

$$\frac{\partial v_t}{\partial \nu_t} = \left[(\cot \gamma_j) A_t(\nu_t, \nu_t) + \frac{1}{\sin \gamma_j} \operatorname{II}(\overline{\nu_t}, \overline{\nu_t}) \right] v_t.$$

Proof. Let us fix one boundary face F_j and denote γ_j by γ . By Lemma A.1,

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \langle N_t, X_t \rangle &= \langle \nabla_Y N, X \rangle + \langle N, \nabla_Y X \rangle \\ &= - \left\langle \nabla^\Sigma v, X \right\rangle - \left\langle S_0(Y_0), X \right\rangle + \left\langle N, \nabla_Y X \right\rangle. \end{aligned}$$

On ∂M , Y decomposes into $Y = Y_1 - \frac{v}{\sin \gamma} \overline{\nu}$. Notice that since $X = \cos \gamma N + \sin \gamma N$,

$$\langle S_0(Y_0), X \rangle = \langle S_0(Y_0), \cos \gamma N + \sin \gamma \nu \rangle = \sin \gamma A(Y_0, \nu).$$

We also have the vector decomposition on ∂M with respect to the orthonormal basis $\overline{\nu}, X$:

$$N = \cos \gamma X - \sin \gamma \overline{\nu}, \qquad \nu = \cos \gamma \overline{\nu} + \sin \gamma X. \tag{A.3}$$

Since $\langle X, X \rangle = 1$ along ∂M , we have $\langle X, \nabla_Z X \rangle = 0$ for any vector Z on ∂M . We have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \langle N_t, X_t \rangle &= -\sin\gamma \frac{\partial v}{\partial \nu} - \langle S_0(Y_0), X \rangle \\ &+ \left\langle \cos\gamma X - \sin\gamma \overline{\nu}, \nabla_{Y_1 - \frac{v}{\sin\gamma} \overline{\nu}} X \right\rangle \\ &= -\sin\gamma \frac{\partial v}{\partial \nu} - \sin\gamma A(Y_0, \nu) - \sin\gamma \left\langle \overline{\nu}, \nabla_{Y_1} X \right\rangle + \left\langle \overline{\nu}, \nabla_{\overline{\nu}} X \right\rangle v. \end{aligned}$$

Now we deal with the second and the third terms above. Notice that on $\partial \Sigma \cap F_j$,

$$Y_0 = Y_1 - (\cot \gamma) v\nu.$$

Thus $A(Y_0, \nu) = A(Y_1, \nu) - (\cot \gamma)vA(\nu, \nu) = -\langle \nabla_{Y_1} N, \nu \rangle - (\cot \gamma)A(\nu, \nu)v$. On the other hand, using the vector decomposition (A.3), we find

$$\begin{split} \langle \nabla_{Y_1} N, \nu \rangle &= \langle \nabla_{Y_1} (\cos \gamma X - \sin \gamma \overline{\nu}), \cos \gamma \overline{\nu} + \sin \gamma X \rangle \\ &= \cos^2 \gamma \left\langle \nabla_{Y_1} X, \overline{\nu} \right\rangle - \sin^2 \gamma \left\langle \nabla_{Y_1} \overline{\nu}, X \right\rangle + \left\langle L, \nabla^{\partial \Sigma} \gamma \right\rangle. \\ &= \left\langle \nabla_{Y_1} X, \overline{\nu} \right\rangle + \left\langle L, \nabla^{\partial \Sigma} \gamma \right\rangle. \end{split}$$

Here L is a vector field along $\partial \Sigma$, and $|L| \leq C = C(Y, X, \nu)$. Thus we conclude that

$$\frac{d}{dt}\Big|_{t=0} \langle N_t, X_t \rangle = -\sin\gamma \frac{\partial v}{\partial \nu} + (\cos\gamma) A(\nu, \nu) v + \mathrm{II}(\overline{\nu}, \overline{\nu}) v + \left\langle L, \nabla^{\partial \Sigma} \gamma \right\rangle,$$

as desired.

The evolution equation of the mean curvature has been studied in many circumstances. We refer the readers to the thorough calculation in Proposition 16, [Amb15]:

Lemma A.3 (Proposition 16 of [Amb15]). Let H_t be the mean curvature of Σ_t . Then

$$\frac{d}{dt}\Big|_{t=0} H_t = \Delta_{\Sigma} v + (\operatorname{Ric}(N, N) + |A|^2)v - \langle \nabla_{\Sigma} H, Y_0 \rangle.$$

In particular, if each Σ_t has constant mean curvature, then

$$\frac{d}{dt}H_t = \Delta_{\Sigma_t}v_t + (\operatorname{Ric}(N_t, N_t) + |A_t|^2)v_t.$$

References

- [Ale51] A. D. Aleksandrov, A theorem on triangles in a metric space and some of its applications, Trudy Mat. Inst. Steklov., v 38, Trudy Mat. Inst. Steklov., v 38, Izdat. Akad. Nauk SSSR, Moscow, 1951, pp. 5–23. MR 0049584
- [All72] William K. Allard, On the first variation of a varifold, Ann. of Math. (2) 95 (1972), 417–491. MR 0307015
- [All75] _____, On the first variation of a varifold: boundary behavior, Ann. of Math. (2) **101** (1975), 418–446. MR 0397520
- [Amb15] Lucas C. Ambrozio, Rigidity of area-minimizing free boundary surfaces in mean convex three-manifolds, J. Geom. Anal. 25 (2015), no. 2, 1001–1017. MR 3319958
- [BBN10] Hubert Bray, Simon Brendle, and Andre Neves, Rigidity of area-minimizing two-spheres in three-manifolds, Comm. Anal. Geom. 18 (2010), no. 4, 821–830. MR 2765731
- [CC97] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below. I, J. Differential Geom. 46 (1997), no. 3, 406–480. MR 1484888
- [CC00a] _____, On the structure of spaces with Ricci curvature bounded below. II, J. Differential Geom. 54 (2000), no. 1, 13–35. MR 1815410
- [CC00b] _____, On the structure of spaces with Ricci curvature bounded below. III, J. Differential Geom. 54 (2000), no. 1, 37–74. MR 1815411
- [CG00] Mingliang Cai and Gregory J. Galloway, Rigidity of area minimizing tori in 3manifolds of nonnegative scalar curvature, Comm. Anal. Geom. 8 (2000), no. 3, 565–573. MR 1775139
- [CN12] T. H. Colding and A. Naber, Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications, Ann. of Math. (2) 176 (2012), no. 2, 1173–1229. MR 2950772
- [CN13] J. Cheeger and A. Naber, Lower bounds on Ricci curvature and quantitative behavior of singular sets, Invent. Math. 191 (2013), no. 2, 321–339. MR 3010378
- [DPM15] G. De Philippis and F. Maggi, Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law, Arch. Ration. Mech. Anal. 216 (2015), no. 2, 473–568. MR 3317808
- [Fin86] R. Finn, Equilibrium capillary surfaces, Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Springer, 1986.
- [Gro14] Misha Gromov, Dirac and Plateau billiards in domains with corners, Cent. Eur. J. Math. 12 (2014), no. 8, 1109–1156. MR 3201312
- [HY96] G. Huisken and S.-T. Yau, Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature, Inventiones Mathematicae 124 (1996), 281–311.

A POLYHEDRON COMPARISON THEOREM IN POSITIVE SCALAR CURVATURE 33

- [Lie88] Gary M. Lieberman, Hölder continuity of the gradient at a corner for the capillary problem and related results, Pacific J. Math. 133 (1988), no. 1, 115–135. MR 936359
- [Lie89] _____, Optimal Hölder regularity for mixed boundary value problems, J. Math. Anal. Appl. 143 (1989), no. 2, 572–586. MR 1022556
- [LM17] C. Li and C. Mantoulidis, *Positive scalar curvature with skeleton singularities*, ArXiv e-prints (2017).
- [LV09] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Ann. of Math. (2) 169 (2009), no. 3, 903–991. MR 2480619
- [LZ17] M. Li and X. Zhou, A maximum principle for free boundary minimal varieties of arbitrary codimension, ArXiv e-prints (2017).
- [Mia02] P. Miao, Positive mass theorem on manifolds admitting corners along a hypersurface, Adv. Theor. Math. Phys. 6 (2002), no. 6, 1163–1182 (2003). MR 1982695
- [MM15] Mario Micallef and Vlad Moraru, Splitting of 3-manifolds and rigidity of areaminimising surfaces, Proc. Amer. Math. Soc. 143 (2015), no. 7, 2865–2872. MR 3336611
- [MMP06] F. Mahmoudi, R. Mazzeo, and F. Pacard, Constant mean curvature hypersurfaces condensing on a submanifold, Geom. Funct. Anal. 16 (2006), no. 4, 924–958. MR 2255386
- [MP05] Rafe Mazzeo and Frank Pacard, Foliations by constant mean curvature tubes, Comm. Anal. Geom. 13 (2005), no. 4, 633–670. MR 2191902
- [Nun13] Ivaldo Nunes, Rigidity of area-minimizing hyperbolic surfaces in threemanifolds, J. Geom. Anal. 23 (2013), no. 3, 1290–1302. MR 3078354
- [RS97] Antonio Ros and Rabah Souam, On stability of capillary surfaces in a ball, Pacific J. Math. 178 (1997), no. 2, 345–361. MR 1447419
- [Sim80] Leon Simon, Regularity of capillary surfaces over domains with corners, Pacific J. Math. 88 (1980), no. 2, 363–377. MR 607984
- [Sim87] _____, A strict maximum principle for area minimizing hypersurfaces, J. Differential Geom. 26 (1987), no. 2, 327–335. MR 906394
- [ST02] Yuguang Shi and Luen-Fai Tam, Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature, J. Differential Geom. 62 (2002), no. 1, 79–125. MR 1987378
- [ST16] Y. Shi and L.-F. Tam, *Scalar curvature and singular metrics*, ArXiv e-prints (2016).
- [Stu06a] K.-T. Sturm, A curvature-dimension condition for metric measure spaces, C.
 R. Math. Acad. Sci. Paris 342 (2006), no. 3, 197–200. MR 2198193
- [Stu06b] _____, On the geometry of metric measure spaces. I, Acta Math. 196 (2006), no. 1, 65–131. MR 2237206
- [Stu06c] _____, On the geometry of metric measure spaces. II, Acta Math. **196** (2006), no. 1, 133–17 7. MR 2237206
- [SW89] Bruce Solomon and Brian White, A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals, Indiana Univ. Math. J. 38 (1989), no. 3, 683–691. MR 1017330
- [SY79] R. Schoen and S. T. Yau, On the structure of manifolds with positive scalar curvature, Manuscripta Math. 28 (1979), no. 1-3, 159–183. MR 535700
- [Tay77] Jean E. Taylor, Boundary regularity for solutions to various capillarity and free boundary problems, Comm. Partial Differential Equations 2 (1977), no. 4, 323–357. MR 0487721
- [Whi10] Brian White, The maximum principle for minimal varieties of arbitrary codimension, Comm. Anal. Geom. 18 (2010), no. 3, 421–432. MR 2747434

[Wic14]	Neshan Wickramasekera, A sharp strong maximum principle and a sharp unique
	continuation theorem for singular minimal hypersurfaces, Calc. Var. Partial Dif-
	ferential Equations 51 (2014), no. 3-4, 799–812. MR 3268871

[Ye91] Rugang Ye, Foliation by constant mean curvature spheres., Pacific J. Math. 147 (1991), no. 2, 381–396.

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY *E-mail address:* rchlch@stanford.edu