STRONG DAMPING WAVE EQUATIONS DEFINED BY A CLASS OF SELF-SIMILAR MEASURES WITH OVERLAPS

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Abstract. The weak well-posedness of strong damping wave equations defined by fractal Laplacians is proved by using Galerkin method. These fractal Laplacians are defined by self-similar measures with overlaps, such as the well-known infinite Bernoulli convolution associated with the golden ratio, the three-fold convolution of the Cantor measure, and a class of self-similar measures that we call essentially of finite type. In general, the structure of self-similar measures with overlap are complicated and intractable. However, some important information about the structure of the three measures above can be obtained. We make use of these information to set up a framework for one-dimensional measures to discretize the equations, and use the finite element and central difference methods to obtain numerical approximations of the weak solutions. We also show that the numerical solutions converge to the actual solution and obtain the rate of convergence.

Contents

1. Introduction

Many phenomena in the real world are best modeled by some exotic geometric structures with a non-smooth appearance. The theory of fractals seeks to provide the mathematical framework for such powerful development. In the last decades years, analysis on fractals has shown an explosive development, due to numerous applications to problems arising in various fields, including physics, chemistry and biology. In this paper, we study the strong damping linear wave equations defined by fractal Laplacians associated with a class of self-similar measures with overlaps. Such measures have attracted considerable attentions because of their

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relation to classical analysis and their numerous unusual properties, such as non-integer spectral dimension [27, 29], Sub-Gaussian heat kernel [19], infinite wave propagation speed [30], and so on. Our long term goal is to combine ideas of Strichartz, including the celebrated Strichartz estimates, in a comprehensive study of nonlinear undamped and damping wave equations on fractals and fractalfolds. In our paper, we investigate the solution of the strong damping linear wave equation theoretically, and also provide numerical examples. We do not discuss the nonlinear wave equation directly, but rather develop approximating tools that may help in this study.

Let $U \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded open subset, and let μ be a positive finite Borel measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. It is known (see, e.g., [20]) that μ defines a Dirichlet Laplace operator Δ_μ, if the following *Poincaré inequality for a measure (PI)* holds: There exists some constant $C > 0$ such that

$$\int_U |u|^2 d\mu \leq C \int_U |\nabla u|^2 \, dx \qquad \text{for all } u \in C_c^\infty(U) \tag{1.1}$$

(see, e.g., [20,25,26]). We remark that if $n = 1$, then (PI) holds for any such μ, and thus Δ_μ is well-defined. More recently, the operator Δ_μ has been studied extensively in connection with fractal measures by authors including Fujita, M. Solomyak, Verbitsky, Naimark, Freiberg, Lobus, Zähle, Bird *et al.*, Hu *et al.*, Andrews *et al.*, Gu *et al.*, Ngai, Xie, and the first author (see [1–4, 8, 12–17, 19, 20, 26–30, 33, 37] and the references therein). Many of these papers study the spectral asymptotics of Δ_μ, while others study the associated wave, heat, and Schrödinger equations. In this paper, we study the following strong damping linear wave equation

$$\begin{cases} \partial_t^2 u - \alpha \Delta_\mu u - \sigma \Delta_\mu \partial_t u = f & \text{on } U \times [0, T], \\ u = 0 & \text{on } \partial U \times [0, T], \\ u(0) = g, \ \partial_t u(0) = h & \text{on } U \times \{t = 0\}, \end{cases} \tag{1.2}$$

where $\alpha \geq 1$ and $\sigma \geq 0$ are real parameters, and $u(t)$ is a Hilbert space valued function of t. In particular, if $\alpha = 1$ and $\sigma = 0$, then equation (1.2) becomes general undamped wave equation. In classical case, various forms of damping wave equations on a domain with smooth enough boundary have been studied extensively (see, e.g., [23, 32, 36]). For a class of one-dimensional fractal measures with overlaps, approximations of the solution of undamped linear wave equations have been studied in [3]. Recently, Dekkers *et al.* studied boundary valued problems for linear and nonlinear strong damping wave equations on arbitrary and fractal domain (see, [6, 7]).

The first objective of this paper is to obtain the weak well-posedness result of the strong damping wave equation (1.2) (see Definition 3.1 for definition of a weak solution). The main ingredient we used is the usual Galerkin method (see, [10] for detials). To perform the Galerkin method, we assume that $-\Delta_\mu$ has compact resolvent; that is, there exists a complete orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(U, \mu)$ such that $-\Delta_\mu \varphi_n = \lambda_n \varphi_n$ for all $n \geq 1$, where the eigenvalues satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$. Some sufficient conditions for the existence of an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of $L^2(U,\mu)$ consisting of the eigenfunctions of $-\Delta_\mu$ can be found in [20]. We remark that if $n=1$, then $-\Delta_\mu$ has compact resolvent. See Section 2 for the definition of $\operatorname{dom}\mathcal{E}$ and $L^2([0,T],L^2(U,\mu))$.

Theorem 1.1. *Let $U\subseteq\mathbb{R}^d$, $d\geq 1$, be a bounded open subset, and let μ be a positive finite Borel measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu)\subseteq\overline{U}$ and $\mu(U)>0$. Assume that μ satisfies (PI) and $-\Delta_\mu$ has compact resolvent. If $g\in\operatorname{dom}\mathcal{E}$, $h\in L^2(U,\mu)$ and $f\in L^2([0,T],L^2(U,\mu))$. Then the strong damping wave equation (1.2) has a unique weak solution.*

We also obtain some regularity results of the weak solution of equation (1.2)(see Theorem 3.3).

The second objective of this paper is to study equation (1.2) from a numerical point of view. We first introduce some definitions and notation that will be used. We call a μ-measurable closed and connected subset I of $\overline{U}$ a *cell (in $\overline{U}$)* if $\mu(I)>0$. Clearly, if $\overline{U}$ is connected, then $\overline{U}$ itself is a cell. Two cells I,J in $\overline{U}$ are *measure disjoint* with respect to μ if $\mu(I\cap J)=0$. Let $I\subseteq\overline{U}$ be a cell. We call a finite family $\mathbf{P}$ of measure disjoint cells a *μ-partition* of I if $J\subseteq I$ for all $J\in\mathbf{P}$, and $\mu(I)=\sum_{J\in\mathbf{P}}\mu(J)$. A sequence of μ-partitions $(\mathbf{P}_m)_{m\geq1}$ is *refining* if for any $m\geq 1$, each member of $\mathbf{P}_{m+1}$ is a proper subset of some member of $\mathbf{P}_m$. A sequence of μ-partitions $(\mathbf{P}_m)_{m\geq1}$ of $\overline{U}$ is *compatible* if (1) $(\mathbf{P}_m)_{m\geq1}$ is refining; (2) for any $m\geq 2$ and any $J\in\mathbf{P}_m$, there exist similitudes $(\tau_I)_{I\in\mathbf{P}_1}$ of the form $\tau_I(x)=r_Ix+b_I$ ($r_I\in(0,1), b_I\in\mathbb{R}^d$) and positive constants $(c_I)_{I\in\mathbf{P}_1}$ such that $\tau(I)\subseteq J$ and

$$\mu|_J=\sum_{I\in\mathbf{P}_1}c_I\cdot\mu|_I\circ\tau_I^{-1}. \tag{1.3}$$

Intuitively, (1.3) means that the μ measure of each cell in $\mathbf{P}_m$ for $m\geq 2$ can be expressed as a linear combination of $\{\mu(I):I\in\mathbf{P}_1\}$. We remark that (1.3) is more general than second-order (self-similar) identities, which were first introduced by Strichartz *et al.* in [35].

In order to discretize (1.2) and obtain numerical approximations of the weak solution, we will assume that there exists a sequence of compatible μ-partitions $(\mathbf{P}_m)_{m\geq1}$. Thus the μ measure of each island in the partition can be computed by using (1.3), making it possible to discretizing the strong damping wave equation (1.2). In order to guarantee that the mass matrix that arises in the finite element method is positive definite (see [3, Proposition 3.1]), we assume that μ is a measure on $\mathbb{R}$ with $\operatorname{supp}(\mu)=[a,b]$.

Let $f\equiv 0$ in equation (1.2). Multiplying the first equation in (1.2) by $v\in\operatorname{dom}\mathcal{E}$, integrating both sides with respect to $d\mu$, and then integrating by parts, we obtain

$$\begin{cases} -\alpha\displaystyle\int_a^b(\partial_x u)v'(x)\,dx-\sigma\int_a^b(\partial_x\partial_t u)v'(x)\,dx=\int_a^b(\partial_t^2u)v(x)d\mu,\\ u(0)=g,\ \partial_t u(0)=h,\end{cases} \tag{1.4}$$

where $\partial_x u$ and $\partial_t u$ are the weak partial derivative of u with respect to x and t, respectively.

Theorem 1.2. *Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) \subseteq [a,b]$. Assume that there exists a sequence of compatible μ-partitions $(\mathbf{P}_m)_{m\geq 1}$ of $[a,b]$ and $g, h \in \operatorname{dom}\mathcal{E}$. Then the equation (1.4) can be discretized to a system of second-order ordinary differential equations (4.6) by finite element method. Moreover, if $\operatorname{supp}(\mu) = [a,b]$ and the integrals $\int_I x^k\,d\mu$, $I \in \mathbf{P}_1$, $k = 0,1,2$, can be evaluated explicitly, then the equation (4.6) has a unique solution that can be solved numerically.*

The assumption $g, h \in \operatorname{dom}\mathcal{E}$ in Theorem 1.2 is used to guarantee that the initial conditions $u(0) = g$ and $\partial_t u(0) = h$ can be approximated by their linear interpolant. We will use the central difference method to solve the equation (4.6) in section 4.

We are mainly interested in fractal measures. Throughout this paper, an *iterated function system (IFS)* refers to a finite family of contractive similitudes $\{S_i\}_{i=1}^q$ defined on $\mathbb{R}^d$. It is well-known that for each IFS $\{S_i\}_{i=1}^q$ and probability weights $\{w_i\}_{i=1}^q$, there is a unique probability measure, called the *self-similar measure*, satisfying the following identity

$$\mu = \sum_{i=1}^{q} w_i \mu \circ S_i^{-1}$$

(see [11,21]). An IFS $\{S_i\}_{i=1}^q$ is said to satisfy the *open set condition (OSC)* if there exists a non-empty bounded open set O such that $\bigcup_i S_i(O) \subseteq O$ and $S_i(O) \cap S_j(O) = \emptyset$ for all $i \neq j$. IFSs that do not satisfy (OSC), as well as all associated self-similar measures, are said to have overlaps. For a class of self-similar measures with overlaps, the finite element method have been used to compute numerical approximations of the eigenvalues and eigenfunctions of the operator Δ_μ in [4].

Our study of the operator Δ_μ is mainly motivated by the effort to extend the current theory of analysis on fractals to include IFSs with overlaps. It is worth pointing out that for general self-similar measures with overlaps, it does not seem possible to discretize the strong damping wave equations (1.2) in the way described in the paper. Thus it is not clear how numerical approximations of the weak solution can be obtain. Theorem 1.2 provides a framework under which discretization can be performed.

The following theorem shows that the approximate solutions obtained in Theorem 1.2 converge to the actual weak solution, and we also obtain a rate of convergence. Throughout this paper, $|E|$ denotes the diameter of a subset $E \subseteq \mathbb{R}^d$, and let $\operatorname{dom}\Delta_\mu$ denote the domain of the operator Δ_μ. See Section 2 for the definitions of $\|\cdot\|_{\operatorname{dom}\mathcal{E}}$ and $\|\cdot\|_{2,X}$, where X is a Hilbert space.

Theorem 1.3. *Assume the hypotheses of Theorem 1.2. Let $g \in \operatorname{dom}\Delta_\mu$, $h \in \operatorname{dom}\Delta_\mu$, $f \equiv 0$ in equation (1.2). If there exist constants $r \in (0,1)$ and $c > 0$ such that $\max\{|J| : J \in \mathbf{P}_m\} \leq c\,r^m$ for all $m \geq 1$, then the approximate solutions u^m obatained in Theorem 1.2, converge in $L^2([0,T], L^2([a,b],\mu))$ to the actual weak solution u of equation (1.2). Moreover,* *there exists constant $C := C(u, T, c,) > 0$ such that for all $m \geq 1$*

$$\|u^m - u\|_{2,L^2([a,b],\mu)} + \|\partial_t u^m - \partial_t u\|_{2,L^2([a,b],\mu)} \leq Cr^{m/2}. \tag{1.5}$$

We remark that the condition $g \in \operatorname{dom}\Delta_\mu, h \in \operatorname{dom}\Delta_\mu, f \equiv 0$ implies that $\partial_t u \in L^2([0,T], \operatorname{dom}\mathcal{E})$ (see, Theorem 3.3). It follows that $\partial_t u$ can be approximated by its linear interpolant for Lebesgue a.e. $t \in [0,T]$.

Based on Theorems 1.2 and 1.3, we solve the homogeneous strong damping wave equation (1.2) numerically for three different one-dimensional self-similar measures with overlaps. The first measure we study is the infinite Bernoulli convolution associated with the golden ratio:

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1}, \tag{1.6}$$

where $S_1(x) = \rho x$, $S_2(x) = \rho x + (1-\rho)$, and $\rho = (\sqrt{5}-1)/2$. The second measure is the three-fold convolution of the Cantor measure, which is defined by the following IFS with overlaps (see [27]):

$$S_i(x) = \frac{1}{3}x + \frac{2}{3}(i-1), \qquad i = 1,2,3,4,$$

together with probability weights $\{1/8, 3/8, 3/8, 1/8\}$. That is,

$$\mu = \frac{1}{8}\mu \circ S_1^{-1} + \frac{3}{8}\mu \circ S_2^{-1} + \frac{3}{8}\mu \circ S_3^{-1} + \frac{1}{8}\mu \circ S_4^{-1}. \tag{1.7}$$

The third family of self-similar measures that we call *essentially of finite type (EFT)* (see [29]) is defined by the following family of IFSs:

$$S_1(x) = r_1 x, \quad S_2(x) = r_2 x + r_1(1-r_2), \quad S_3(x) = r_2 x + 1 - r_2, \tag{1.8}$$

where the contraction ratios $r_1, r_2 \in (0,1)$ satisfy $r_1 + 2r_2 - r_1 r_2 \leq 1$, i.e., $S_2(1) \leq S_3(0)$. The first and second measures have been studied very extensively (see, e.g., [22, 27, 29]). They define Laplacians that exhibit many behaviors analogous to Laplacians on post-critically finite fractals, such as sub-Gaussian heat kernel estimates [19] and infinite wave propagation speed [30]. The third class is used in [29] to illustrate self-similar measures satisfying EFT.

Corollary 1.4. *Let μ be a positive finite Borel measure on $\mathbb{R}$. Assume that $g, h \in \operatorname{dom}\Delta_\mu$ and $f \equiv 0$.*

(a) *If μ is the infinite Bernoulli convolution associated with the golden ratio in (1.6), then equation (1.4) can be discretized into a system of ordinary differential equations, which has a unique solution that can be solved numerically. Moreover, the approximate solutions u^m converge in $L^2([0,T], L^2([0,1],\mu))$ to the actual weak solution u, and the inequality (1.5) holds.*

(b) *If μ is the three-fold convolution of the Cantor measure in (1.7), then the conclusions of part (a) also hold.*

(c) *If μ is a self-similar measure defined by the IFS (1.8) with $r_1 + 2r_2 - r_1 r_2 = 1$, then the conclusions of part (a) also hold.*

The assumption $r_1+2r_2-r_1r_2=1$ in Corollary 1.4 (c) is used to guarantee that $\text{supp}(\mu)=[0,1]$. The numerical results of above three different measures are shown in Figure 2.

The rest of this paper is organized as follows. Section 2 summarizes some notation, definitions and results that will be needed throughout the paper. In Section 3, we prove Theorem 1.1 and give some regularity results. Section 4 is devoted to the proof of Theorem 1.2. The proof of Theorem 1.3 and Corollary 1.4 are given in Section 5.

2. Preliminaries

In this section, we summarize some notation, definitions and facts that will be used throughout the rest of the paper. For a Banach space X, we denote its topological dual by X'. For $v \in X'$ and $u \in X$, we let $\langle v, u\rangle_{X',X} := v(u)$ denote the *dual pairing* of X' and X.

Definition 2.1. *Let X be a separable Banach space with norm $\|\cdot\|_X$. Denote by $L^p([0,T],X)$ the space of all measurable functions $u:[0,T]\to X$ satisfying*

(1) $\|u\|_{L^p([0,T],X)} := \Big(\int_0^T \|u(t)\|_X^p\,dt\Big)^{1/p} < \infty$, *if* $1\le p<\infty$, *and*
(2) $\|u\|_{L^\infty([0,T],X)} := \text{esssup}_{0\le t\le T}\|u(t)\|_X < \infty$, *if* $p=\infty$.

If the interval $[0,T]$ is understood, we will abbreviate these norms as $\|u\|_{p,X}$ and $\|u\|_{\infty,X}$, respectively.

Remark 2.1. *For each $1\le p\le\infty$, $L^p([0,T],X)$ is a Banach space; moreover, $L^{p_2}([0,T],X)\subseteq L^{p_1}([0,T],X)$ if $1\le p_1\le p_2\le\infty$. Let X be a separable Banach space with inner product $(\cdot,\cdot)_X$. If $(X,(\cdot,\cdot)_X)$ is a separable Hilbert space, then $L^2([0,T],X)$ is a Hilbert space with the inner product*

$$(u,v)_{L^2([0,T],X)} := \int_0^T (u(t),v(t))_X\,dt.$$

Definition 2.2. *let X be a Banach space. We define $C([0,T],X)$ (resp. $C^1([0,T],X)$) to be the vector space of all continuous (resp. C^1) functions $u:[0,T]\to X$ such that*

$$\|u\|_{C([0,T],X)} := \max_{0\le t\le T}\|u\|_X < \infty \quad (resp. \|u\|_{C^1([0,T],X)} := \|\partial_t u\|_{C([0,T],X)} < \infty).$$

Similarly we define $C^k([0,T],X)$ for all $k\ge 1$. $\|\cdot\|_{C^k([0,T],X)}$ is a norm.

Definition 2.3. *Let X be a Banach space and $u\in L^1([0,T],X)$.*

(1) *We say $v\in L^1([0,T],X)$ is the weak derivative of u, written $\partial_t u = v$, if*

$$\int_0^T \phi'(t)u(t)\,dt = -\int_0^T \phi(t)v(t)\,dt$$

for all scalar test functions $\phi\in C_c^\infty(0,T)$.

(2) *We say* $v \in L^1([0,T],X)$ *is the second weak derivative of* u, *written* $\partial_t^2 u = v$, *if*

$$\int_0^T \phi''(t)u(t)\,dt = \int_0^T \phi(t)v(t)\,dt$$

for all scalar test functions $\phi \in C_c^\infty(0,T)$.

Definition 2.4. *let* X *be a Banach space and* X' *its dual. We say a sequence* $\{u_m\}_{m=1}^\infty \subseteq X$ *converges weakly to* $u \in X$, *written* $u_m \rightharpoonup u$, *if*

$$\langle v, u_m\rangle \to \langle v, u\rangle$$

for each bounded linear functional $v \in X'$.

For convenience, we summarize the definition of the Dirichlet Laplacian on a bounded domain defined by a measure; details can be found in [20]. Let $U \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded open subset and μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. We assume that μ satisfies (PI) (see (1.1)). Then each equivalence class $u \in H_0^1(U)$ contains a unique (in the $L^2(U,\mu)$ sense) member $\hat{u}$ that belongs to $L^2(U,\mu)$ and satisfies both conditions below:

(1) there exists a sequence $\{u_n\}$ in $C_c^\infty(U)$ such that $u_n \to \hat{u}$ in $H_0^1(U)$ and $u_n \to \hat{u}$ in $L^2(U,\mu)$;
(2) $\hat{u}$ satisfies inequality (1.1).

We call $\hat{u}$ the $L^2(U,\mu)$-*representative* of u. Define a mapping $\iota : H_0^1(U) \to L^2(U,\mu)$ by $\iota(u) = \hat{u}$. ι is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(U)$ defined as $\mathcal{N} := \big\{u \in H_0^1(U) : \|\iota(u)\|_\mu = 0\big\}$. Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(U)$. Then $\iota : \mathcal{N}^\perp \to L^2(U,\mu)$ is injective. Unless explicitly stated otherwise, we will denote the $L^2(U,\mu)$-representative $\hat{u}$ simply by u.

Consider the non-negative bilinear form $\mathcal{E}(\cdot,\cdot)$ in $L^2(U,\mu)$ defined by

$$\mathcal{E}(u,v) := \int_U \nabla u \cdot \nabla v \, dx \tag{2.1}$$

with domain $\operatorname{dom}\mathcal{E} = \mathcal{N}^\perp$, or more precisely, $\iota(\mathcal{N}^\perp)$. (PI) implies that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a closed quadratic form in $L^2(U,\mu)$. Hence there exists a non-negative self-adjoint operator A in $L^2(U,\mu)$ such that

$$\mathcal{E}(u,v) = \big(A^{1/2}u, A^{1/2}v\big) \quad \text{and} \quad \operatorname{dom}\mathcal{E} = \operatorname{dom}(A^{1/2})$$

(see, e.g., [18, Theorem 1.3.1]). We write $\Delta_\mu^D = -A$, and call it the *(Dirichlet) Laplacian* with respect to μ. If no confusion is possible, we denote Δ_μ^D simply by Δ_μ. Let $u \in \operatorname{dom}\mathcal{E}$. Then $u \in \operatorname{dom}\Delta_\mu$ if and only if there exists a unique $f \in L^2(U,\mu)$ such that $\mathcal{E}(u,v) = (f,v)_\mu$ for all $v \in \operatorname{dom}\mathcal{E}$. In this case, $-\Delta_\mu u = f$. Throughout this paper, we let $\operatorname{dom}\mathcal{E} := \mathcal{N}^\perp$,

$$\|\cdot\|_{\operatorname{dom}\mathcal{E}} := \|\cdot\|_{H_0^1(U)} \quad \text{and} \quad \|\cdot\|_{\operatorname{dom}\Delta_\mu} := \|\Delta_\mu(\cdot)\|_{L^2(U,\mu)}.$$

Note that the spaces $\operatorname{dom}\mathcal{E}$, $L^2(U,\mu)$, $(\operatorname{dom}\mathcal{E})'$ form a Gelfand triple:

$$\operatorname{dom}\mathcal{E} \hookrightarrow L^2(U,\mu) \cong (L^2(U,\mu))' \hookrightarrow (\operatorname{dom}\mathcal{E})',$$

where we identify $L^2(U,\mu)$ with $(L^2(U,\mu))'$, and the embedding $L^2(U,\mu) \hookrightarrow (\operatorname{dom}\mathcal{E})'$ is given by

$$w \in L^2(U,\mu) \mapsto (w,\cdot)_\mu \in (L^2(U,\mu))' \subset (\operatorname{dom}\mathcal{E})'.$$

We now remark on the case when $-\Delta_\mu$ has compact resolvent. Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2(U,\mu)$ so that $-\Delta_\mu$ such that $-\Delta_\mu\varphi_n = \lambda_n\varphi_n$ for all $n \geq 1$, where $0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots$ and $\lim_{n\to\infty}\lambda_n = \infty$. Then the domains $\operatorname{dom}\mathcal{E}$ and $\operatorname{dom}\Delta_\mu$ can be expressed by using eigenfunctions and eigenvalues as

$$\operatorname{dom}\mathcal{E} = \left\{\sum_{n=1}^\infty a_n\varphi_n : \sum_{n=1}^\infty a_n^2\lambda_n < \infty\right\} \text{ and } \operatorname{dom}\Delta_\mu = \left\{\sum_{n=1}^\infty a_n\varphi_n : \sum_{n=1}^\infty a_n^2\lambda_n^2 < \infty\right\}.$$

Note that $f = \sum_{n=1}^\infty a_n\varphi_n \in (\operatorname{dom}\mathcal{E})'$ if and only if there exists a unique $u = \sum_{n=1}^\infty b_n\varphi_n \in \operatorname{dom}\mathcal{E}$ such that $\mathcal{E}(u,v) = \langle f, v\rangle$ for all $v \in \operatorname{dom}\mathcal{E}$, where, and throughout this paper, $\langle\cdot,\cdot\rangle$ denotes the pairing between $\operatorname{dom}\mathcal{E}$ and $(\operatorname{dom}\mathcal{E})'$. Letting $v = \varphi_k$, $k \geq 1$, it follows that

$$a_k = \langle f, \varphi_k\rangle = \mathcal{E}(u,\varphi_k) = b_k\lambda_k,$$

and so $f = \sum_{n=1}^\infty a_n\varphi_n \in (\operatorname{dom}\mathcal{E})'$ if and only if $\sum_{n=1}^\infty (a_n/\lambda_n)\varphi_n \in \operatorname{dom}\mathcal{E}$, i.e., $\sum_{n=1}^\infty a_n^2/\lambda_n < \infty$. Therefore,

$$(\operatorname{dom}\mathcal{E})' = \left\{u = \sum_{n=1}^\infty a_n\varphi_n : \sum_{n=1}^\infty a_n^2/\lambda_n < \infty\right\}.$$

Moreover, $\langle u, v\rangle = (u,v)_\mu$ for all $u \in \operatorname{dom}\mathcal{E}$ and $v \in (\operatorname{dom}\mathcal{E})'$.

3. Existence and uniqueness of weak solution

In this section, we consider the existence and uniqueness of weak solution of equation (1.2). Let $U \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded open subset, and let μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. Assume that (PI) (see (1.1)) holds, and let $-\Delta_\mu$ be the Dirichlet Laplace with respect to μ. Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be given as in (2.1).

Definition 3.1. *Let $f \in L^2([0,T], L^2(U,\mu))$, $g \in \operatorname{dom}\mathcal{E}$, and $h \in L^2(U,\mu)$. A function $u \in L^2([0,T], \operatorname{dom}\mathcal{E})$ with $\partial_t u \in L^2([0,T], \operatorname{dom}\mathcal{E})$ and $\partial_t^2 u \in L^2([0,T], (\operatorname{dom}\mathcal{E})')$ is a* weak solution *of strong damping wave equation (1.2) if it satisfies the following conditions:*

(i) $\langle\partial_t^2 u, v\rangle + \alpha\mathcal{E}(u,v) + \sigma\mathcal{E}(\partial_t u, v) = (f,v)_\mu$ *for all $v \in \operatorname{dom}\mathcal{E}$ and Lebesgue a.e. $t \in [0,T]$;*

(ii) $u(0) = g$ *and* $\partial_t u(0) = h$.

We use the Galerkin method to prove the existence and uniqueness of weak solution of (1.2). To perform the Galerkin method, we start by solving a finite dimensional approximation. We thus assume that $-\Delta_\mu$ has compact resolvent; that is, there exists a complete orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(U,\mu)$ such that $-\Delta_\mu\varphi_n = \lambda_n\varphi_n$ for all $n \geq 1$, $0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots$ and $\lim_{n\to\infty}\lambda_n = \infty$. For each positive integer m, define

$$u_m(t) := \sum_{k=1}^{m} \beta_{m,k}(t)\varphi_k, \tag{3.1}$$

where we will show that the coefficients $\{\beta_{m,k}(t)\}_{k=1}^m$ can be chosen to satisfy

$$\beta_{m,k}(0) = (g,\varphi_k)_\mu, \quad \beta'_{m,k}(0) = (h,\varphi_k)_\mu, \tag{3.2}$$

and for $t \in [0,T]$,

$$\left(\partial_t^2 u_m, \varphi_k\right)_\mu + \alpha\mathcal{E}(u_m,\varphi_k) + \sigma\mathcal{E}(\partial_t u_m,\varphi_k) = (f,\varphi_k)_\mu. \tag{3.3}$$

Note that $\beta_{m,k}(0)$ and $\beta'_{m,k}(0)$ are independent of m.

Proposition 3.1. *Assume the hypotheses of Theorem 1.1. Then for each $m \geq 1$, there exists a unique function $u_m(t)$ of the form (3.1) with the coefficients $\beta_{m,k}(t) \in H^2([0,T])$ $(k = 1,\dots,m)$ satisfying (3.2) and (3.3).*

Proof. Let $u_m(t)$ be defined as in (3.1). By the orthogonality of $\{\varphi_k\}_{k=1}^\infty$,

$$\left(\partial_t^2 u_m, \varphi_k\right)_\mu = \beta''_{m,k}(t), \quad k = 1,\dots,m. \tag{3.4}$$

Moreover, for $k = 1,\dots,m$,

$$\mathcal{E}(u_m,\varphi_k) = \lambda_k\beta_{m,k}(t), \quad \text{and} \quad \mathcal{E}(\partial_t u_m,\varphi_k) = \lambda_k\beta'_{m,k}(t). \tag{3.5}$$

Letting $f_k := (f,\varphi_k)_\mu$ for $k \geq 1$ and using (3.4), (3.5), we can convert equation (3.3) into the following linear system of ODEs

$$\beta''_{m,k}(t) + \alpha\lambda_k\beta_{m,k}(t) + \sigma\lambda_k\beta'_{m,k}(t) = f_k, \quad t \in [0,T],\ k = 1,\dots,m. \tag{3.6}$$

with the initial condition (3.2). Thus there exists a unique vector-valued function $\boldsymbol{\beta}_m(t) = (\beta_{m,1}(t),\dots,\beta_{m,m}(t))$ such that each $\beta_{m,k}(t) \in H^2([0,T])$ satisfies (3.2) and (3.6). □

We remark that if f_k is continuous on $[0,T]$ for some $k \geq 1$, then $\beta_{m,k}(t) \in C^2([0,T])$.

In order to obtain a weak solution u, we need to take the limit as $m \to \infty$. To this end, we will first obtain a key *energy estimate*. For convenience, throughout this section, we denote all generic constants, depending only on U and μ, by C.

Proposition 3.2. *Assume the hypotheses of Theorem 1.1, and let $u_m(t)$ be of form (3.1) satisfying (3.2) and (3.3). Then there exists a constant $C > 0$, depending only on U and μ,* *such that for all positive integer m,*

$$\max_{t\in[0,T]}\Big(\|u_m(t)\|^2_{\mathrm{dom}\,\mathcal{E}}+\|\partial_t u_m(t)\|^2_{L^2(U,\mu)}\Big)+\|\partial_t u_m\|^2_{2,\mathrm{dom}\,\mathcal{E}}+\|\partial_t^2 u_m\|^2_{2,(\mathrm{dom}\,\mathcal{E})'} \le C\big(\|f\|^2_{2,L^2(U,\mu)}+\|g\|^2_{\mathrm{dom}\,\mathcal{E}}+\|h\|^2_{\mu}\big). \tag{3.7}$$

Proof. Fix any $m\ge 0$. We multiply equation (3.3) by $\beta'_{m,k}(t)$, sum over $k=1,\ldots,m$ and use (3.1) to obtain

$$\big(\partial_t^2 u_m,\partial_t u_m\big)_\mu+\alpha\mathcal{E}(u_m,\partial_t u_m)+\sigma\mathcal{E}(\partial_t u_m,\partial_t u_m)=(f,\partial_t u_m)_\mu \qquad \text{for all } t\ge 0. \tag{3.8}$$

Note that

$$\frac{1}{2}\frac{d}{dt}\Big(\|\partial_t u_m(t)\|^2_\mu\Big)=\big(\partial_t^2 u_m,\partial_t u_m\big)_\mu \quad\text{and}\quad \frac{1}{2}\frac{d}{dt}\Big(\|u_m(t)\|^2_{\mathrm{dom}\,\mathcal{E}}\Big)=\mathcal{E}(u_m,\partial_t u_m). \tag{3.9}$$

Substituting (3.9) into (3.8), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\partial_t u_m(t)\|^2_\mu+\alpha\|u_m(t)\|^2_{\mathrm{dom}\,\mathcal{E}}\Big)+\sigma\|\partial_t u_m\|^2_{\mathrm{dom}\,\mathcal{E}}\\ &=(f,\partial_t u_m)_\mu\le\|f\|_\mu\|\partial_t u_m\|_\mu\le C\|f\|_\mu\|\partial_t u_m\|_{\mathrm{dom}\,\mathcal{E}}\le\frac{C}{2\sigma}\|f\|^2_\mu+\frac{\sigma}{2}\|\partial_t u_m\|^2_{\mathrm{dom}\,\mathcal{E}},\end{aligned} \tag{3.10}$$

where Cauchy-Schwarz, (PI) (see (1.1)) and Young inequality are used successively. Multiplying inequality (3.10) by the constant 2, and then integrating both sides with respect to time, we obtain

$$\begin{aligned}&\|\partial_t u_m(t)\|^2_\mu+\alpha\|u_m(t)\|^2_{\mathrm{dom}\,\mathcal{E}}+\sigma\int_0^t\|\partial_t u_m(\tau)\|^2_{\mathrm{dom}\,\mathcal{E}}\,d\tau\\ &\le\frac{C}{2\sigma}\int_0^t\|f(\tau)\|^2_\mu\,d\tau+\|u_m(0)\|^2_{\mathrm{dom}\,\mathcal{E}}+\|\partial_t u_m(0)\|^2_\mu\\ &\le C\Big(\|f(t)\|^2_{2,L^2(U,\mu)}+\|g\|^2_{\mathrm{dom}\,\mathcal{E}}+\|h\|^2_\mu\Big)\qquad\text{for } t\ge 0,\end{aligned} \tag{3.11}$$

where the fact $\|u_m(0)\|^2_{\mathrm{dom}\,\mathcal{E}}\le\|g\|^2_{\mathrm{dom}\,\mathcal{E}}$ and $\|\partial_t u_m(0)\|^2_\mu\le\|h\|^2_\mu$ are used in the last inequality. Since $t\in[0,T]$ was arbitrary, the inequality (3.11) implies that

$$\begin{aligned}&\max_{t\in[0,T]}\Big(\|u_m(t)\|^2_{\mathrm{dom}\,\mathcal{E}}+\|\partial_t u_m(t)\|^2_\mu\Big)+\|\partial_t u_m\|^2_{2,\mathrm{dom}\,\mathcal{E}}\\ &\le C\Big(\|f(t)\|^2_{2,L^2(U,\mu)}+\|g\|^2_{\mathrm{dom}\,\mathcal{E}}+\|h\|^2_\mu\Big).\end{aligned} \tag{3.12}$$

Fix any $v\in\mathrm{dom}\,\mathcal{E}$ with $\|v\|_{\mathrm{dom}\,\mathcal{E}}\le 1$, and write $v=v_1+v_2$, where $v_1\in\mathrm{span}\{\varphi_k\}_{k=1}^m$ and $(v_2,\varphi_k)_\mu=0$ for all $k=1,\ldots,m$. We first note that $\|v_1\|_{\mathrm{dom}\,\mathcal{E}}\le\|v\|_{\mathrm{dom}\,\mathcal{E}}\le 1$. Then equations (3.1) and (3.3) imply

$$\begin{aligned}\langle\partial_t^2 u_m,v\rangle&=(\partial_t^2 u_m,v)_\mu=(\partial_t^2 u_m,v_1)_\mu\\ &=(f,v_1)_\mu-\alpha\mathcal{E}(u_m,v_1)-\sigma\mathcal{E}(\partial_t u_m,v_1).\end{aligned}$$

Thus, by the Cauchy-Schwartz inequality and (PI), we have

$$\begin{aligned}|\langle\partial_t^2 u_m,v\rangle|&\le\|f\|_\mu\|v_1\|_\mu+\alpha\|u_m\|_{\mathrm{dom}\,\mathcal{E}}\|v_1\|_{\mathrm{dom}\,\mathcal{E}}+\sigma\|\partial_t u_m\|_{\mathrm{dom}\,\mathcal{E}}\|v_1\|_{\mathrm{dom}\,\mathcal{E}}\Big)\\ &\le C\Big(\|f\|_\mu+\|u_m\|_{\mathrm{dom}\,\mathcal{E}}+\|\partial_t u_m\|_{\mathrm{dom}\,\mathcal{E}}\Big).\end{aligned}$$

It follows that $\|\partial_t^2 u_m\|_{(\operatorname{dom}\mathcal{E})'} \leq C(\|f\|_\mu + \|u_m\|_{\operatorname{dom}\mathcal{E}} + \|\partial_t u_m\|_{\operatorname{dom}\mathcal{E}})$. Consequently, we deduce

$$\begin{aligned}\int_0^T \|\partial_t^2 u_m(t)\|^2_{(\operatorname{dom}\mathcal{E})'}\,dt &\leq C\int_0^T \Big(\|f\|_\mu^2 + \max_{t\in[0,T]}\|u_m(t)\|^2_{\operatorname{dom}\mathcal{E}} + \|\partial_t u_m\|^2_{\operatorname{dom}\mathcal{E}}\Big)\,d\tau \\ &\leq C\Big(\|f\|^2_{2,L^2(U,\mu)} + \|g\|^2_{\operatorname{dom}\mathcal{E}} + \|h\|^2_\mu\Big),\end{aligned} \tag{3.13}$$

where estimate (3.12) has been used in the last inequality. Combining (3.13) and (3.12), we obtain (3.7). □

Proof of Theorem 1.1. Using the energy estimate (3.7), we obtain a subsequence $\{u_{m_l}\}_{l=1}^\infty$, together with a function $u \in L^2([0,T], \operatorname{dom}\mathcal{E})$ satisfying $\partial_t u \in L^2([0,T], \operatorname{dom}\mathcal{E})$ and $\partial_t^2 u \in L^2([0,T], (\operatorname{dom}\mathcal{E})')$, such that

$$\begin{cases} u_{m_l} \rightharpoonup u & \text{in} \quad L^2([0,T], \operatorname{dom}\mathcal{E}), \\ \partial_t u_{m_l} \rightharpoonup \partial_t u & \text{in} \quad L^2([0,T], \operatorname{dom}\mathcal{E}), \\ \partial_t^2 u_{m_l} \rightharpoonup \partial_t^2 u & \text{in} \quad L^2([0,T], (\operatorname{dom}\mathcal{E})'). \end{cases} \tag{3.14}$$

Now, we fix an integer N and choose a function $v \in C^1([0,T], \operatorname{dom}\mathcal{E})$ of the form

$$v(t) = \sum_{k=1}^N d_k(t)\varphi_k, \qquad \text{where } \{d_k(t)\}_{k=1}^N \subseteq C^1([0,T]). \tag{3.15}$$

Letting $m \geq N$, multiplying (3.3) by $d_k(t)$, $k = 1, \cdots, N$, adding the equations up, and then integrating with respect to t, we get

$$\int_0^T \Big((\partial_t^2 u_m, v(t))_\mu + \alpha\mathcal{E}(u_m, v(t)) + \sigma\mathcal{E}(\partial_t u_m, v(t))\Big)\,dt = \int_0^T (f, v(t))_\mu\,dt. \tag{3.16}$$

Setting $m = m_l$, letting l tend to ∞, and using (3.14), we have

$$\int_0^T \Big((\partial_t^2 u, v(t))_\mu + \alpha\mathcal{E}(u, v(t)) + \sigma\mathcal{E}(\partial_t u, v(t))\Big)\,dt = \int_0^T (f, v(t))_\mu\,dt. \tag{3.17}$$

Since the set of functions of the form (3.15) is dense in $L^2([0,T], \operatorname{dom}\mathcal{E})$, (3.17) holds for all $v \in L^2([0,T], \operatorname{dom}\mathcal{E})$, and thus for all such v and Lebesgue a.e. $t \in [0,T]$,

$$(\partial_t^2 u, v)_\mu + \alpha\mathcal{E}\big(u, v\big) + \sigma\mathcal{E}\big(\partial_t u, v\big) = (f, v)_\mu.$$

Next we need to verify $u(0) = g$ and $\partial_t u(0) = h$. For this, in (3.17), we choose any function $v \in C^2([0,T], \operatorname{dom}\mathcal{E})$, with $v(T) = \partial_t v(T) = 0$. Applying integration by parts twice respect to t to the first term of (3.17), we can find

$$\begin{aligned}&\int_0^T \Big((u, \partial_t^2 v(t))_\mu + \alpha\mathcal{E}(u, v(t)) + \sigma\mathcal{E}(\partial_t u, v(t))\Big)\,dt \\ &= \int_0^T \big(f, v\big)_\mu\,dt - \big(u(0), \partial_t v(0)\big)_\mu + \big(\partial_t u(0), v(0)\big)_\mu.\end{aligned} \tag{3.18}$$

Similarly, using (3.16), we have

$$\int_0^T \Big((u_m, \partial_t^2 v(t))_\mu + \alpha \mathcal{E}\big(u_m, v(t)\big) + \sigma \mathcal{E}(\partial_t u_m, v(t)) \Big)\, dt$$
$$= \int_0^T \big(f, v\big)_\mu \, dt - \big(u_m(0), \partial_t v(0)\big)_\mu + \big(\partial_t u_m(0), v(0)\big)_\mu .$$

Setting $m = m_l$ and recall (3.2) and (3.14), we deduce

$$\int_0^T (u, \partial_t^2 v)_\mu + \alpha \mathcal{E}\big(u, v\big) + \sigma \mathcal{E}(\partial_t u, v)\, dt = \int_0^T \big(f, v\big)_\mu \, dt - \big(g, \partial_t v(0)\big)_\mu + \big(h, v(0)\big)_\mu . \tag{3.19}$$

Comparing (3.18) and (3.19), and noting that $v(0)$ and $v'(0)$ were arbitrary, we conclude that $u(0) = g$ and $\partial_t u(0) = h$. Therefore, u is a weak solution of strong damping wave equation (1.2).

It suffices to show that the only weak solution of (1.2) with $g = h = f \equiv 0$ is $u \equiv 0$ in $L^2([0,T], \operatorname{dom} \mathcal{E})$. To prove it , we take $\partial_t u \in L^2([0,T], \operatorname{dom} \mathcal{E})$. Then for Lebesgue a.e. $t \in [0,T]$,

$$\big(\partial_t^2 u, \partial_t u\big)_\mu + \alpha \mathcal{E}\big(u, \partial_t u\big) + \sigma \mathcal{E}(\partial_t u, \partial_t u) = \big(f, \partial_t u\big)_\mu = 0.$$

By assumption, we have $u(0) \equiv 0$ and $\partial_t u(0) \equiv 0$. Moreover, since

$$\frac{d}{dt}\Big(\frac{1}{2}\|\partial_t u\|_\mu^2 + \frac{\alpha}{2}\|u\|_{\operatorname{dom}\mathcal{E}}^2\Big) = \big(\partial_t^2 u, \partial_t u\big)_\mu + \alpha \mathcal{E}\big(u, \partial_t u\big),$$

we have for Lebesgue a.e. $s \in [0,T]$,

$$\frac{1}{2}\|\partial_t u(s)\|_\mu^2 + \frac{\alpha}{2}\|u(s)\|_{\operatorname{dom}\mathcal{E}}^2 + \sigma \int_0^s \|\partial_t u\|_{\operatorname{dom}\mathcal{E}}^2 \, dt = 0,$$

which implies $\partial_t u = 0$ in $L^2([0,T], \operatorname{dom} \mathcal{E})$ and $u = 0$ in $L^2([0,T], \operatorname{dom} \mathcal{E})$. □

Now, we give some regularity results of the weak solution of equation (1.2).

Theorem 3.3. *Assume the hypotheses of Theorem 1.1. Let u be the weak solution of the strong damping wave equation (1.2). Then*

(a) $u \in L^\infty([0,T], \operatorname{dom} \mathcal{E})$, $\partial_t u \in L^\infty([0,T], L^2(U,\mu)) \cap L^2([0,T], \operatorname{dom} \mathcal{E})$, *and*

$$\operatorname{ess\,sup}_{t\in[0,T]}\big(\|u(t)\|_{\operatorname{dom}\mathcal{E}}^2 + \|\partial_t u(t)\|_\mu^2\big) + \|\partial_t u\|_{2,\operatorname{dom}\mathcal{E}}^2 + \|\partial_t^2 u\|_{2,(\operatorname{dom}\mathcal{E})'}^2$$
$$\le C\big(\|f\|_{2,L^2(U,\mu)}^2 + \|g\|_{\operatorname{dom}\mathcal{E}}^2 + \|h\|_\mu^2\big).$$

(b) *If, in addition,* $g \in \operatorname{dom} \Delta_\mu$ *and* $h \in \operatorname{dom} \mathcal{E}$, *then* $u \in L^\infty([0,T], \operatorname{dom} \Delta_\mu), \partial_t u \in L^\infty([0,T], \operatorname{dom} \mathcal{E}) \cap L^2([0,T], \operatorname{dom} \Delta_\mu), \partial_t^2 u \in L^2([0,T], L^2(U,\mu))$ *and*

$$\operatorname{ess\,sup}_{t\ge 0}\big(\|u(t)\|_{\operatorname{dom}\Delta_\mu}^2 + \|\partial_t u(t)\|_{\operatorname{dom}\mathcal{E}}^2\big) + \|\partial_t u\|_{2,\operatorname{dom}\Delta_\mu}^2$$
$$\le C\big(\|f\|_{2,L^2(U,\mu)}^2 + \|g\|_{\operatorname{dom}\Delta_\mu}^2 + \|h\|_{\operatorname{dom}\mathcal{E}}^2\big). \tag{3.20}$$

(c) *If* $g \in \operatorname{dom} \Delta_\mu$, $h \in \operatorname{dom} \Delta_\mu$, $f \equiv 0$, *then* $\partial_t^2 u \in L^2([0,T], \operatorname{dom} \mathcal{E})$.

Proof. (a) Passing to limits in (3.7) as $m = m_l \to \infty$, we deduce part (a).

(b) Let $u_m(t)$ be given as in Proposition 3.1. We multiply equation (3.3) by $\lambda_k \beta'_{m,k}(t)$ and sum over $k = 1, \cdots, m$. Then we have

$$(\partial_t^2 u_m, -\Delta_\mu \partial_t u_m)_\mu + \alpha \mathcal{E}(u_m, -\Delta_\mu \partial_t u_m) + \sigma \mathcal{E}(\partial_t u_m, -\Delta_\mu \partial_t u_m) = (f, -\Delta_\mu \partial_t u_m)_\mu. \quad (3.21)$$

We remark that the left-hand side of (3.21) equals

$$\frac{1}{2}\frac{d}{dt}\Big(\|\partial_t u_m\|^2_{\operatorname{dom}\mathcal{E}} + \alpha \|u_m\|^2_{\operatorname{dom}\Delta_\mu}\Big) + \sigma \|\partial_t u_m\|^2_{\operatorname{dom}\Delta_\mu}.$$

It follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\partial_t u_m\|^2_{\operatorname{dom}\mathcal{E}} + \alpha \|u_m\|^2_{\operatorname{dom}\Delta_\mu}\Big) + \sigma \|\partial_t u_m\|^2_{\operatorname{dom}\Delta_\mu} \\
&\le |(f, -\Delta_\mu \partial_t u_m)_\mu| \le \|f\|_\mu \cdot \|\Delta_\mu \partial_t u_m\|_\mu = \|f\|_\mu \cdot \|\partial_t u_m\|_{\operatorname{dom}\Delta_\mu} \\
&\le \frac{2}{\sigma}\|f\|^2_\mu + \frac{\sigma}{2}\|\partial_t u_m\|^2_{\operatorname{dom}\Delta_\mu}.
\end{aligned} \quad (3.22)$$

Integrating over $[0, t]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\partial_t u_m(t)\|^2_{\operatorname{dom}\mathcal{E}} + \frac{\alpha}{2}\|u_m(t)\|^2_{\operatorname{dom}\Delta_\mu} + \frac{\sigma}{2}\int_0^t \|\partial_t u_m(\tau)\|^2_{\operatorname{dom}\Delta_\mu}\, d\tau \\
&\le \frac{2}{\sigma}\int_0^t \|f(\tau)\|^2_\mu \, d\tau + \frac{\alpha}{2}\|u_m(0)\|^2_{\operatorname{dom}\Delta_\mu} + \frac{1}{2}\|\partial_t u_m(0)\|^2_{\operatorname{dom}\mathcal{E}} \\
&\le \frac{2}{\sigma}\|f\|^2_{2,L^2(U,\mu)} + \frac{\alpha}{2}\|g\|^2_{\operatorname{dom}\Delta_\mu} + \frac{1}{2}\|h\|^2_{\operatorname{dom}\mathcal{E}} \qquad \text{for all } 0 < t \le T.
\end{aligned} \quad (3.23)$$

Taking the weak limit of a subsequence of $\{u_m\}$, we find

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in[0,T]}\Big(\|\partial_t u(t)\|^2_{\operatorname{dom}\mathcal{E}} + \|u(t)\|^2_{\operatorname{dom}\Delta_\mu}\Big) + \|\partial_t u\|^2_{2,\operatorname{dom}\Delta_\mu} \\
&\le C\Big(\|f\|^2_{2,L^2(U,\mu)} + \|g\|^2_{\operatorname{dom}\Delta_\mu} + \|h\|^2_{\operatorname{dom}\mathcal{E}}\Big).
\end{aligned} \quad (3.24)$$

Hence, (3.20) holds, $u \in L^\infty([0,T], \operatorname{dom}\Delta_\mu)$ and $\partial_t u \in L^\infty([0,T], \operatorname{dom}\mathcal{E}) \cap L^2([0,T], \operatorname{dom}\Delta_\mu)$. It follows that $\partial_t^2 u \in L^2([0,T], L^2(U,\mu))$.

(c) Let $w(t) := \partial_t u$. It is easy to check that $w(t)$ is the unique weak solution of the following boundary value problem

$$\partial_t^2 w - \alpha\Delta_\mu w - \sigma \Delta_\mu \partial_t w = 0, \quad w(0) = h, \quad \partial_t w(0) = \alpha \Delta_\mu g + \sigma \Delta_\mu h, \quad w|_{\partial U} = 0.$$

Using part (a), we have $\partial_t w \in L^2([0,T], \operatorname{dom}\mathcal{E})$. That is, $\partial_t^2 u \in L^2([0,T], \operatorname{dom}\mathcal{E})$. □

4. The finite element method

In this section, we let $f \equiv 0$ in equation (1.2), and use the finite element method to solve the homogeneous strong damping wave equaion (1.2). Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a, b]$. Assume that there exists a sequence of compatible

μ-partitions $(\mathbf{P}_m)_{m\geq 1} = (\{I_{m,i}\}_{i=1}^{N(m)})_{m\geq 1}$ of $[a,b]$. We thus can write $I_{m,i} = [x_{m,i-1}, x_{m,i}]$ for $m \geq 1$ and $1 \leq i \leq N(m)$. It is easy to see that $x_{m,0} = a$ and $x_{m,N(m)} = b$ for all $m \geq 1$.

We apply the finite element method to approximate the weak solution $u(t)$ satisfying (1.4) by

$$u^m(t) = \sum_{i=0}^{N(m)} w_{m,i}(t)\phi_{m,i}, \tag{4.1}$$

where, for $i = 0, 1, \ldots, N(m)$, $w_{m,i}(t)$ are functions to be determined and $\phi_{m,i}$ are the standard piecewise linear *finite element basis functions* (also called *tent functions*) defined by

$$\phi_{m,i}(x) := \begin{cases} \dfrac{x - x_{m,i-1}}{x_{m,i} - x_{m,i-1}} & \text{if } x \in I_{m,i},\ i = 1, 2, \ldots, N(m), \\ \dfrac{x - x_{m,i+1}}{x_{m,i} - x_{m,i+1}} & \text{if } x \in I_{m,i+1},\ i = 0, 1, \ldots, N(m) - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Fix any $m \geq 1$. We require $u^m(t)$ to satisfy the integral form of the homogeneous strong damping wave equation

$$\int_a^b \partial_t^2 u^m(t)\phi_{m,j}\, d\mu = -\alpha \int_a^b \partial_x u^m(t)\phi'_{m,j}\, dx - \sigma \int_a^b \partial_x\partial_t\big(u^m(t)\big)\phi'_{m,j}\, dx \tag{4.3}$$

for $j = 1, \ldots, N(m)-1$, and the Dirichlet boundary condition $u^m(t) = 0$ on $\{a,b\}\times[0,T]$. We note that $\phi_{m,i}(a) = \phi_{m,i}(x_{m,0}) = 0$ and $\phi_{m,j}(b) = \phi_{m,j}(x_{m,N(m)}) = 0$ for all $i = 1, \ldots, N(m)$ and $j = 0, 1, \ldots, N(m) - 1$. Thus $w_{m,0}(t) = w_{m,N(m)}(t) = 0$ for all $t \in [0,T]$. Using this and substituting (4.1) into (4.3) gives

$$\begin{aligned} &\sum_{i=1}^{N(m)-1} w''_{m,i} \int_a^b \phi_{m,i}(x)\phi_{m,j}(x)\, d\mu \\ &= -\alpha \sum_{i=1}^{N(m)-1} w_{m,i} \int_a^b \phi'_{m,i}(x)\phi'_{m,j}(x)\, dx - \sigma \sum_{i=1}^{N(m)-1} w'_{m,i} \int_a^b \phi'_{m,i}(x)\phi'_{m,j}(x)\, dx \end{aligned} \tag{4.4}$$

for $1 \leq j \leq N(m) - 1$. We define the mass matrix $\mathbf{M} = \mathbf{M}^{(m)} = (M_{ij}^{(m)})$ and stiffness matrix $\mathbf{K} = \mathbf{K}^{(m)} = (K_{ij}^{(m)})$, respectively, by

$$M_{ij}^{(m)} = \int_a^b \phi_{m,i}(x)\phi_{m,j}(x)\, d\mu \quad \text{and} \quad K_{ij}^{(m)} = \int_a^b \phi'_{m,i}(x)\phi'_{m,j}(x)\, dx,$$

where $1 \leq i, j \leq N(m) - 1$. It follows from the definition of $\phi_{m,j}(x)$ that both $\mathbf{M}$ and $\mathbf{K}$ are tridiagonal. Let

$$\mathbf{w}(t) = \mathbf{w}_m(t) := \begin{bmatrix} w_{m,1}(t) \\ \vdots \\ w_{m,N(m)-1}(t) \end{bmatrix}.$$

Then (4.4) can be put into matrix form as

$$\mathbf{M}\mathbf{w}'' = -\alpha\mathbf{K}\mathbf{w} - \sigma\mathbf{K}\mathbf{w}'.$$

This gives us a system of second-order linear ODEs with constant coefficients. To solve it, we need to impose initial conditions. The initial condition $u(0) = g \in \operatorname{dom}\mathcal{E}$ and $\partial_t u(0) = h \in \operatorname{dom}\mathcal{E}$ for $a < x < b$ can be approximated by its linear interpolant $\widetilde{g}(x) = \sum_{i=1}^{N(m)-1} g(x_{m,i})\phi_{m,i}(x)$ and $\widetilde{h}(x) = \sum_{i=1}^{N(m)-1} h(x_{m,i})\phi_{m,i}(x)$. Therefore, we set $w_{m,i}(0) = g(x_{m,i})$ and $w'_{m,i}(0) = h(x_{m,i})$. This leads to the initial condition

$$\mathbf{w}(0) = \mathbf{w}_m(0) := \begin{bmatrix} g(x_{m,1}) \\ \vdots \\ g(x_{m,N(m)-1}) \end{bmatrix} \quad \text{and} \quad \mathbf{w}'(0) = \mathbf{w}'_m(0) := \begin{bmatrix} h(x_{m,1}) \\ \vdots \\ h(x_{m,N(m)-1}) \end{bmatrix}. \tag{4.5}$$

Consequently, we get the linear system

$$\begin{cases} \mathbf{M}\mathbf{w}'' = -\alpha\mathbf{K}\mathbf{w} - \sigma\mathbf{K}\mathbf{w}', & 0 < t \le T, \\ \mathbf{w}(0) = \mathbf{w}_m(0), \quad \mathbf{w}'(0) = \mathbf{w}'_m(0). \end{cases} \tag{4.6}$$

The matrix $\mathbf{K}$ can be computed directly. Since $(\mathbf{P}_m)_{m\ge1}$ is a sequence of compatible μ-partitions of $[a, b]$, we can see that the matrix $\mathbf{M}$ is completely determined by

$$\left\{ \int_I x^k \, d\mu : I \in \mathbf{P}_1, k = 0, 1, 2 \right\}. \tag{4.7}$$

Hereafter, we assume that the integrals in (4.7) can be evaluated explicitly for $k = 0, 1, 2$ and $I \in \mathbf{P}_1$. Thus the matrix $\mathbf{M}$ can be computed. Since $\operatorname{supp}(\mu) = [a, b]$, [3, Proposition 3.1] implies that $\mathbf{M}$ is invertible. It follows that the system in (4.6) has a unique solution $\mathbf{w}(t)$ that can be solved numerically. Moreover, $w_{m,j}(t) \in C^2(0, T)$ for $m \ge 1$ and $j = 1, \dots, N(m) - 1$.

Proof of Theorem 1.2. The desired results follow from the derivations above. □

Now, we use the central difference method to solve the equation (4.6). Let $\mathbf{w}_n := \mathbf{w}(t_n)$ for $n \ge -1$. We approximate the derivative as

$$\mathbf{w}''(t_n) \approx \frac{\mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1}}{(\Delta t)^2} \quad \text{and} \quad \mathbf{w}'(t_n) \approx \frac{\mathbf{w}_{n+1} - \mathbf{w}_{n-1}}{2\Delta t}. \tag{4.8}$$

Substituting (4.8) into (4.6) yields

$$\mathbf{M}\frac{\mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1}}{(\Delta t)^2} = -\alpha\mathbf{K}\mathbf{w}_n - \sigma\mathbf{K}\frac{\mathbf{w}_{n+1} - \mathbf{w}_{n-1}}{2\Delta t}. \tag{4.9}$$

Since $\mathbf{M}$ and $\mathbf{K}$ are positive definite, so is $2\mathbf{M} + \sigma\Delta t\mathbf{K}$ for all $\Delta t > 0$. It follows from (4.9) that

$$\mathbf{w}_{n+1} = \big(2\mathbf{M} + \sigma\Delta t\mathbf{K}\big)^{-1}\Big(\big(4\mathbf{M} - 2\alpha(\Delta t)^2\mathbf{K}\big)\mathbf{w}_n + (\sigma\Delta t\mathbf{K} - 2\mathbf{M})\mathbf{w}_{n-1}\Big).$$

In particular, $\mathbf{w}_1 = \big(2\mathbf{M} + \sigma\Delta t\mathbf{K}\big)^{-1}\Big(\big(4\mathbf{M} - 2\alpha(\Delta t)^2\mathbf{K}\big)\mathbf{w}_0 + (\sigma\Delta t\mathbf{K} - 2\mathbf{M})\mathbf{w}_{-1}\Big)$. Using $\mathbf{w}'_0 = \big(\mathbf{w}_1 - \mathbf{w}_{-1}\big)/(2\Delta t)$, we get

$$\mathbf{w}_1 = (4\mathbf{M})^{-1}\Big(\big(4\mathbf{M} - 2\alpha(\Delta t)^2\mathbf{K}\big)\mathbf{w}_0 + 2\Delta t(2\mathbf{M} - \sigma\Delta t\mathbf{K})\mathbf{w}'_0\Big).$$

Therefore, the equation (4.6) becomes

$$\begin{cases} \mathbf{w}_{n+1} = \big(2\mathbf{M} + \sigma\Delta t\mathbf{K}\big)^{-1}\Big(\big(4\mathbf{M} - 2\alpha(\Delta t)^2\mathbf{K}\big)\mathbf{w}_n + (\sigma\Delta t\mathbf{K} - 2\mathbf{M})\mathbf{w}_{n-1}\Big), \\ \mathbf{w}_1 = (4\mathbf{M})^{-1}\Big(\big(4\mathbf{M} - 2\alpha(\Delta t)^2\mathbf{K}\big)\mathbf{w}_0 + 2\Delta t(2\mathbf{M} - \sigma\Delta t\mathbf{K})\mathbf{w}_0'\Big) \\ \mathbf{w}_0 = \mathbf{w}(0), \quad \mathbf{w}_0' = \mathbf{w}'(0), \\ t_n = n\Delta t. \end{cases} \tag{4.10}$$

To solve this system, fix $\Delta t, \alpha, \sigma$ and substitute the initial condition $\mathbf{w}_0$ and $\mathbf{w}_0'$ from (4.5) into the first and second equations in (4.10) to get $\mathbf{w}_1$. Then $\mathbf{w}_{n+1}$ can be computed recursively. In next section, applying (4.10), we will give the numerical results of three different measures in Corollary 1.4.

5. Convergence of numerical approximations

In this section, we prove the convergence of numerical approximations of the strong damping wave equation (1.2). Some of our results are obtained by modifying similar ones in [3,34]. Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a, b]$. Assume that there exists a sequence of compatible μ-partitions $(\mathbf{P}_m)_{m\geq 1}$ such that $\int_I x^k\, d\mu$, $I \in \mathbf{P}_1$, $k = 0, 1, 2$, can be evaluated explicitly. Let V_m be the set of end-points of all level-m subintervals in $\mathbf{P}_m$, and arrange its element so that $V_m = \{x_{m,i} : i = 0, \dots, N(m)\}$ with $x_{m,i} < x_{m,i+1}$ for $i = 0, 1, \dots, N(m) - 1$, $x_{m,0} = a$ and $x_{m,N(m)} = b$. Let S^m be the space of continuous piecewise linear functions on $[a, b]$ with nodes V_m, and let

$$S_D^m = \{u \in S^m : u(a) = u(b) = 0\}$$

be the subspaces of S^m consisting of functions satisfying the Dirichlet boundary condition.

We choose the basis of S^m consisting of the tent functions $\{\phi_{m,i}\}_{i=0}^{N(m)}$ defined in (4.2) and choose the basis $\{\phi_{m,i}\}_{i=1}^{N(m)-1}$ for S_D^m. The linear map $\mathcal{F}_m : \operatorname{dom}\mathcal{E} \to S_D^m$ defined by

$$\mathcal{F}_m u(x) = \sum_{i=1}^{N^m - 1} u(x_{m,i})\phi_{m,i}(x) \qquad u \in \operatorname{dom}\mathcal{E} \text{ and } m \geq 1,$$

is called the *Rayleigh-Ritz projection* with respect to V_m. Then for any $u \in \operatorname{dom}\mathcal{E}$, $\mathcal{F}_m u$ is the component of u in the subspace S_D^m, $u - \mathcal{F}_m u$ vanishes on the boundary $\{a, b\}$, and

$$\mathcal{E}(u - \mathcal{F}_m u, v) = 0 \tag{5.1}$$

for all $v \in S_D^m$ (see, [34, Theorem 1.1]). Let

$$\|V_m\| := \max\{|x_{m,i} - x_{m,i-1}| : 1 \leq i \leq N(m)\}$$

be the *norm* of V_m for $m \geq 1$. We remark that

$$\|\mathcal{F}_m u - u\|_\mu \leq 2\|V_m\|^{1/2}\|u\|_{\operatorname{dom}\mathcal{E}} \tag{5.2}$$

for all $u \in \operatorname{dom}\mathcal{E}$ and $m \geq 1$ (see [3, Lemma 5.3]).

Throughout the rest of this section, let $g \in \operatorname{dom}\Delta_\mu, h \in \operatorname{dom}\Delta_\mu, f \equiv 0$, and $u(t)$ be the solution of the corresponding homogeneous strong damping wave equation (1.2). Then

$$(\partial_t^2 u, v)_\mu + \alpha\mathcal{E}(u,v) + \sigma\mathcal{E}(\partial_t u, v) = 0 \quad \text{for all } v \in \operatorname{dom}\mathcal{E} \text{ and Lebesgue a.e. } t \in [0,T]. \tag{5.3}$$

By Theorem 3.3, we have $\partial_t^2 u \in L^2([0,T], \operatorname{dom}\mathcal{E})$. As in Section 4,

$$u^m(t) = \sum_{i=1}^{N(m)-1} w_{m,i}(t)\phi_{m,i}.$$

Thus $u^m(t)$ satisfies

$$(\partial_t^2 u^m, v^m)_\mu + \alpha\mathcal{E}(u^m, v^m) + \sigma\mathcal{E}(\partial_t u^m, v^m) = 0 \tag{5.4}$$

for all $v^m \in S_D^m$ and Lebesgue a.e. $t \in [0,T]$, $u^m(0) = \mathcal{F}_m g$, and $\partial_t u^m(0) = \mathcal{F}_m h$. Finally, define

$$e(t) := e^m(t) = \mathcal{F}_m u(t) - u^m(t).$$

We remark that $e(0) = \partial_t e(0) \equiv 0$.

Lemma 5.1. *Let $\mathcal{F}_m, u, u^m, e$ be as above. Then for Lebesgue a.e. $t \in [0,T]$,*

$$(\partial_t^2 e, \partial_t e)_\mu + \alpha\mathcal{E}(e, \partial_t e) + \sigma\mathcal{E}(\partial_t e, \partial_t e) = (\mathcal{F}_m\partial_t^2 u - \partial_t^2 u, \partial_t e)_\mu. \tag{5.5}$$

Proof. We first note that the functions $\partial_t e$, $\partial_t^2 e$, $\partial_t\big(\mathcal{F}_m u\big) = \mathcal{F}_m\partial_t u$, and $\partial_t^2\big(\mathcal{F}_m u\big) = \mathcal{F}_m\partial_t^2 u$ all belong to S_D^m. Thus substituting $\partial_t e$ for v in (5.3) and for v^m in (5.4), we get

$$(\partial_t^2 u, \partial_t e)_\mu + \alpha\mathcal{E}(u, \partial_t e) + \sigma\mathcal{E}(\partial_t u, \partial_t e) = 0$$

and

$$(\partial_t^2 u^m, \partial_t e)_\mu + \alpha\mathcal{E}(u^m, \partial_t e) + \sigma\mathcal{E}(\partial_t u^m, \partial_t e) = 0$$

for Lebesgue a.e. $t \in [0,T]$. Subtracting these equations gives $(\partial_t^2 u - \partial_t^2 u^m, \partial_t e)_\mu + \alpha\mathcal{E}(u - u^m, \partial_t e) + \sigma\mathcal{E}(\partial_t u - \partial_t u^m, \partial_t e) = 0$ for Lebesgue a.e. $t \in [0,T]$. Equivalently,

$$(\partial_t^2 u - \partial_t^2 u^m, \partial_t e)_\mu + \alpha\mathcal{E}(u - \mathcal{F}_m u + \mathcal{F}_m u - u^m, \partial_t e) + \sigma\mathcal{E}(\partial_t u - \mathcal{F}_m\partial_t u + \mathcal{F}_m\partial_t u - \partial_t u^m, \partial_t e) = 0,$$

which, together with the fact $\mathcal{E}(u - \mathcal{F}_m u, \partial_t e) = 0$ and $\mathcal{E}(\partial_t u - \mathcal{F}_m\partial_t u, \partial_t e) = 0$ (see (5.1)), yields

$$(\partial_t^2 u - \partial_t^2 u^m, \partial_t e)_\mu + \alpha\mathcal{E}(\mathcal{F}_m u - u^m, \partial_t e) + \sigma\mathcal{E}(\mathcal{F}_m\partial_t u - \partial_t u^m, \partial_t e) = 0.$$

It follows that

$$(\mathcal{F}_m\partial_t^2 u - \partial_t^2 u^m, \partial_t e)_\mu + \alpha\mathcal{E}(\mathcal{F}_m u - u^m, \partial_t e) + \sigma\mathcal{E}(\mathcal{F}_m\partial u_t - \partial_t u^m, \partial_t e) = (\mathcal{F}_m\partial_t^2 u - \partial_t^2 u, \partial_t e)_\mu$$

for Lebesgue a.e. $t \in [0,T]$. Thus equation (5.5) holds. □

Theorem 5.2. *Assume the hypotheses of Lemma 5.1. If there exist constants $r \in (0,1)$ and $c > 0$ such that $\max\{|I| : I \in \mathbf{P}_m\} \le c\,r^m$ for all $m \ge 1$, then there exists constant $C > 0$ such that*

$$\|\mathcal{F}_m u - u^m\|_{2,L^2([a,b],\mu)} + \|\mathcal{F}_m\partial_t u - \partial_t u^m\|_{2,L^2([a,b],\mu)} \le C r^{m/2}\|\partial_t^2 u\|_{2,\operatorname{dom}\mathcal{E}} \quad \textit{for all } m \ge 1.$$

Proof. Fix any $m \geq 1$. The first and second terms in (5.5) can be rewritten as

$$(\partial_t^2 e, \partial_t e)_\mu = \frac{1}{2}\frac{d}{dt}\big(\|\partial_t e\|_\mu^2\big) = \|\partial_t e\|_\mu \cdot \frac{d}{dt}\big(\|\partial_t e\|_\mu\big), \qquad \alpha\mathcal{E}(e, \partial_t e) = \frac{\alpha}{2}\frac{d}{dt}\big(\|e\|_{\mathrm{dom}\,\mathcal{E}}^2\big),$$

respectivey, and the term $\sigma\mathcal{E}(\partial_t e, \partial_t e) \geq 0$. Thus (5.5) leads to

$$\|\partial_t e\|_\mu \cdot \frac{d}{dt}\big(\|\partial_t e\|_\mu\big) \leq \Big|(\mathcal{F}_m \partial_t^2 u - \partial_t^2 u, \partial_t e)_\mu\Big| \leq \|\mathcal{F}_m \partial_t^2 u - \partial_t^2 u\|_\mu \cdot \|\partial_t e\|_\mu$$

and

$$\frac{\alpha}{2}\frac{d}{dt}\|e\|_{\mathrm{dom}\,\mathcal{E}}^2 \leq \Big|(\mathcal{F}_m \partial_t^2 u - \partial_t^2 u, \partial_t e)_\mu\Big| \leq \|\mathcal{F}_m \partial_t^2 u - \partial_t^2 u\|_\mu \cdot \|\partial_t e\|_\mu$$

for Lebesgue a.e. $t \in [0, T]$. It follows that

$$\frac{d}{dt}\big(\|\partial_t e\|_\mu\big) \leq \|\mathcal{F}_m \partial_t^2 u - \partial_t^2 u\|_\mu \quad \text{and} \quad \frac{d}{dt}\big(\|e\|_{\mathrm{dom}\,\mathcal{E}}^2\big) \leq \frac{2}{\alpha}\|\mathcal{F}_m \partial_t^2 u - \partial_t^2 u\|_\mu \cdot \|\partial_t e\|_\mu \tag{5.6}$$

for Lebesgue a.e. $t \in [0, T]$. Integrating both sides of two inequalities in (5.6) with respect to τ from 0 to t, respectively, we get

$$\begin{aligned} \|\partial_t e(t)\|_\mu &= \|\partial_t e(t)\|_\mu - \|\partial_t e(0)\|_\mu = \int_0^t \frac{d}{d\tau}\Big(\|\partial_\tau e(\tau)\|_\mu\Big)\, d\tau \\ &\leq \int_0^T \|\mathcal{F}_m \partial_\tau^2 u(\tau) - \partial_\tau^2 u(\tau)\|_\mu \, d\tau \leq \sqrt{T}\|\mathcal{F}_m \partial_\tau^2 u - \partial_\tau^2 u\|_{2, L^2([a,b],\mu)} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \|e(t)\|_{\mathrm{dom}\,\mathcal{E}}^2 &= \|e(t)\|_{\mathrm{dom}\,\mathcal{E}}^2 - \|e(0)\|_{\mathrm{dom}\,\mathcal{E}}^2 = \int_0^t \frac{d}{d\tau}\Big(\|e(\tau)\|_{\mathrm{dom}\,\mathcal{E}}^2\Big)\, d\tau \\ &\leq \frac{2}{\alpha}\int_0^T \Big(\|\mathcal{F}_m \partial_\tau^2 u(\tau) - \partial_\tau^2 u(\tau)\|_\mu \cdot \|\partial_\tau e(\tau)\|_\mu\Big)\, d\tau, \end{aligned} \tag{5.8}$$

for Lebesgue a.e. $t \in [0, T]$, where the fact $e(0) = \partial_t e(0) \equiv 0$ is used. Since $\partial_t^2 u \in L^2([0, T], \mathrm{dom}\,\mathcal{E})$, (5.2) and the assumption $\max\{|I| : I \in \mathbf{P}_m\} \leq c\, r^m$ imply

$$\|\mathcal{F}_m \partial_t^2 u - \partial_t^2 u\|_\mu \leq 2\|V_m\|^{1/2}\|\partial_t^2 u\|_{\mathrm{dom}\,\mathcal{E}} \leq 2\sqrt{c}\, r^{m/2}\|\partial_t^2 u\|_{\mathrm{dom}\,\mathcal{E}} \tag{5.9}$$

for Lebesgue a.e. $t \in [0, T]$. Combining (5.7) and (5.9), we have

$$\|\partial_t e(t)\|_\mu \leq 2\sqrt{cT} r^{m/2}\|\partial_t^2 u\|_{2, \mathrm{dom}\,\mathcal{E}} \tag{5.10}$$

for Lebesgue a.e. $t \in [0, T]$. Furthermore, using (PI), (5.8), (5.9) and (5.10), we obtain

$$\begin{aligned} \|e(t)\|_\mu^2 \leq C_1\|e(t)\|_{\mathrm{dom}\,\mathcal{E}}^2 &\leq \frac{8C_1 c\sqrt{T} r^m}{\alpha}\|\partial_t^2 u\|_{2, \mathrm{dom}\,\mathcal{E}} \cdot \|\partial_t^2 u\|_{2, \mathrm{dom}\,\mathcal{E}} \\ &\leq \frac{8C_1 cT r^m}{\alpha}\|\partial_t^2 u\|_{2, \mathrm{dom}\,\mathcal{E}}^2 \end{aligned} \tag{5.11}$$

for Lebesgue a.e. $t \in [0, T]$ and some constant $C_1 > 0$. Hence, the desired inequality follows from (5.10) and (5.11). □

Proof of Theorem 1.3. Combining Theorem 5.2 and (5.2), we have

$$\begin{aligned}
&\|u^m - u\|_{2,L^2([a,b],\mu)} + \|\partial_t u^m - \partial_t u\|_{2,L^2([a,b],\mu)} \\
&\le \|u^m - \mathcal{F}_m u\|_{2,L^2([a,b],\mu)} + \|\mathcal{F}_m u - u\|_{2,L^2([a,b],\mu)} \\
&\quad + \|\partial_t u^m - \mathcal{F}_m \partial_t u\|_{2,L^2([a,b],\mu)} + \|\mathcal{F}_m \partial_t u - \partial_t u\|_{2,L^2([a,b],\mu)} \\
&\le C r^{m/2} \|\partial_t^2 u\|_{2,\operatorname{dom}\mathcal{E}} + 2\sqrt{cT} r^{m/2} \Big(\|u\|_{2,\operatorname{dom}\mathcal{E}} + \|\partial_t u\|_{2,\operatorname{dom}\mathcal{E}} \Big) \\
&\le C_* \big(\|\partial_t^2 u\|_{2,\operatorname{dom}\mathcal{E}} + \|\partial_t u\|_{2,\operatorname{dom}\mathcal{E}} + \|u\|_{2,\operatorname{dom}\mathcal{E}} \big) r^{m/2},
\end{aligned}$$

for some constant C_*, which completes the proof. □

Next, we shall prove Corollary 1.4. The following notion play an essential role in our argument. Let $\{S_i\}_{i=1}^q$ be an IFS of contractive similitudes on $\mathbb{R}$, and let μ be the associated self-similar measure. Let $\{T_j\}_{j=1}^{\ell}$ be an auxiliary IFS. We say μ satisfies a family of *second-order (self-similar) identities* with respect to $\{T_j\}_{j=1}^{\ell}$ (see [22]), that is,

(1) $\operatorname{supp}(\mu) \subseteq \bigcup_{j=1}^{\ell} T_j(\operatorname{supp}(\mu))$, and
(2) for each Borel subset $A \subseteq \operatorname{supp}(\mu)$ and $0 \le i, j \le \ell$, $\mu(T_i T_j A)$ can be expressed as a linear combination of $\{\mu(T_k A) : k = 1, \dots, \ell\}$ as

$$\mu(T_i T_j A) = \sum_{k=1}^{\ell} c_k \mu(T_k A),$$

where $c_k := c_k(i,j)$ are independent of A.

We remark that if $\{S_i\}_{i=1}^q$ satisfies (OSC), then μ satisfies a family of second-order (self-similar) identities with respect to $\{S_i\}_{i=1}^q$ itself. Let $\mathcal{M}_k := \{1, \cdots, q\}^k$ for $k \ge 1$ and $\mathcal{M}_0 := \emptyset$. Let $[a,b]$ be the minimum closed interval satisfying $\operatorname{supp}(\mu) \subseteq [a,b]$. A closed subset $I \subseteq [a,b]$ is called a *level-k island* with respect to $\{\mathcal{M}_k\}$ if the following conditions hold:

(1) there exists a finite sequence of indexes $\boldsymbol{i}_0, \boldsymbol{i}_1, \dots, \boldsymbol{i}_n$ in $\mathcal{M}_k$ such that $S_{\boldsymbol{i}_k}(a,b) \cap S_{\boldsymbol{i}_{k+1}}(a,b) \neq \emptyset$ for all $k = 0, \dots, n-1$, and $I = \bigcup_{k=0}^{n} S_{\boldsymbol{i}_k}[a,b]$;
(2) for any $\boldsymbol{j} \in \mathcal{M}_k \setminus \{\boldsymbol{i}_0, \dots, \boldsymbol{i}_n\}$ and any $k \in \{0, \dots, n\}$, $S_{\boldsymbol{j}}(a,b) \cap S_{\boldsymbol{i}_k}(a,b) = \emptyset$.

Note that the term island is adopted from [29]. Intuitively, for each level-k island I, I° is a connected component of $S_{\mathcal{M}_k}(a,b) := \bigcup_{\boldsymbol{i} \in \mathcal{M}_k} S_{\boldsymbol{i}}(a,b)$.

Proof of Corollary 1.4. (a) Let μ be the infinite Bernoulli convolution associated with the golden ratio. Noting that $\operatorname{supp}(\mu) = [0,1]$. Strichartz *et al.* [35] showed that μ satisfies a family of second-order identities with respect to the following auxiliary IFS:

$$T_1(x) := \rho^2 x, \qquad T_2(x) := \rho^3 x + \rho^2, \qquad T_3(x) := \rho^2 x + \rho,$$

where $\rho = (\sqrt{5}-1)/2$. Since $\{T_i\}_{i=1}^3$ satisfies (OSC), we have $(\mathbf{P}_m)_{m\geq 1}$ is a sequence of compatible μ-partitions of $[0,1]$, where $\mathbf{P}_m := \Big\{T_{\boldsymbol{j}}([a,b]) : \boldsymbol{j} \in \{1,2,3\}^m\Big\}$. It follows that $\max\{|I| : I \in \mathbf{P}_m\} \leq \rho^{2m}$ for all $m \geq 1$. Note that the integrals $\int_0^1 x^k d\mu \circ T_j$, $k = 0,1,2$, $j = 1,2,3$, have been calculated in [4, Section 5]. Since

$$\int_{T_j[a,b]} x^k d\mu = \int_a^b (T_j x)^k d\mu \circ T_j \qquad \text{for all } k = 0,1,2, j = 1,2,3,$$

we have the integrals $\int_{T_j[0,1]} x^k \, d\mu$ can be evaluated explicitly for $k = 0,1,2$ and $j = 1,2,3$. Hence, the desired results follow from Theorems 1.2 and 1.3.

(b) The proof is similar to that of part (a). Let μ be the three-fold convolution of the Cantor measure. Note that $\operatorname{supp}(\mu) = [0,3]$. Lau and Ngai [22] shown that μ satisfies a family of second-order identities with respect to the following auxiliary IFS:

$$T_1(x) := \frac{1}{3}x, \qquad T_2(x) = \frac{1}{3}x + 1, \quad T_3(x) = \frac{1}{3}x + 2. \tag{5.12}$$

The integrals $\int_0^1 x^k d\mu \circ T_j$, $k = 0,1,2$, $j = 1,2,3$, have been calculated in [3, Section 4.3].

(c) Let μ be a self-similar measure defined by an IFS in (1.8) and a probability vector $(p_i)_{i=1}^3$. Since $r_1 + 2r_2 - r_1 r_2 = 1$, $\operatorname{supp}(\mu) = [0,1]$. Let $\mathcal{M}_k := \{1,2,3\}^k$ for $k \geq 1$ and $\mathcal{M}_0 := \emptyset$. For $k \geq 1$, define

$$\mathbf{P}_k := \Big\{ I : I \text{ is a level-}k \text{ island with respect to } \{\mathcal{M}_k\} \Big\} \tag{5.13}$$

(see Figure 1). We note that $\mathbf{P}_1 = \{I_{1,1}, I_{1,0}\}$, where $I_{1,1} := S_1[0,1] \bigcup S_2[0,1]$ and $I_{1,0} := S_3[0,1]$. By [29, Lemma 3.5] and the proof of [29, Example 3.3], we can see that for any $k \geq 2$ and any $I \in \mathbf{P}_k$, conditions (i) or (ii) below holds:

(i) there exist $\boldsymbol{j} \in \mathcal{M}_{k-1}$, $i \in \{0,1\}$, and a constant $c > 0$ such that $S_{\boldsymbol{j}}(I_{1,i}) = I$ and $\mu|_I = c \cdot \mu|_{I_{1,i}} \circ S_{\boldsymbol{j}}^{-1}$;

(ii) there exist some $\boldsymbol{j}_0, \boldsymbol{j}_1 \in \mathcal{M}_{k-1}$ and positive constants c_0, c_1 such that $S_{\boldsymbol{j}_0}(I_{1,0}) \subseteq I$, $S_{\boldsymbol{j}_1}(I_{1,1}) = I$ and $\mu|_I = c_0 \cdot \mu|_{I_{1,0}} \circ S_{\boldsymbol{j}_0}^{-1} + c_1 \cdot \mu|_{I_{1,1}} \circ S_{\boldsymbol{j}_1}^{-1}$.

It follows that $(\mathbf{P}_m)_{m\geq 1}$ is a sequence of compatible μ-partitions of $[0,1]$. Moreover, $\max\{|I| : I \in \mathbf{P}_m\} \leq (r_1 + r_2)^m$ for all $m \geq 1$.

By Theorems 1.2 and 1.3, it suffices to prove that the integrals $\int_{I_{1,i}} x^k \, d\mu$, $i = 1,2$, $k = 0,1,2$, can be calculated. Using formula

$$\int_0^1 x^k \, d\mu = \sum_{i=1}^3 p_i \int_0^1 \big(S_i(x)\big)^k \, d\mu,$$

we can calculate the integrals $\int_0^1 x^k d\mu$ for all $k = 0, 1, 2$. Combining it and the following formulas

$$\int_{I_{1,0}} x^k \, d\mu = \sum_{i=1}^{3} p_i \int_{S_i^{-1}(I_{1,0})} \big(S_i(x)\big)^k \, d\mu = p_3 \int_0^1 \big(S_3(x)\big)^k \, d\mu,$$

$$\int_{I_{1,1}} x^k \, d\mu = \sum_{i=1}^{3} p_i \int_{S_i^{-1}(I_{1,0})} \big(S_i(x)\big)^k \, d\mu = p_1 \int_0^1 \big(S_1(x)\big)^k \, d\mu + p_2 \int_0^1 \big(S_2(x)\big)^k \, d\mu,$$

we can find the exact value of the integrals in $\int_{I_{1,i}} x^k \, d\mu$, $i = 1, 2$, $k = 0, 1, 2$, which completes the proof. $\square$

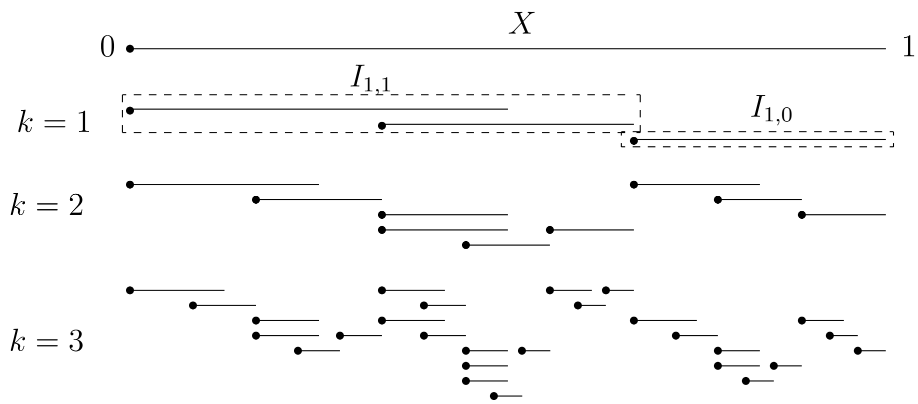

FIGURE 1. μ-partitions $\mathbf{P}_k$ for $k = 1, 2, 3$, where $\mathbf{P}_k$ is defined as in (5.13). Cells that are labeled consist of line segments enclosed by a box. The figure is drawn with $r_1 = 1/2$ and $r_2 = 1/3$.

The numerical results of three different measures in Corollary 1.4 are shown in Figure 2.

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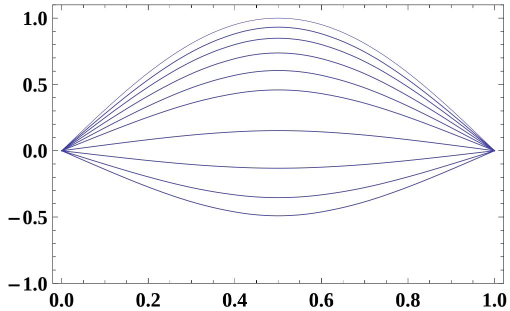
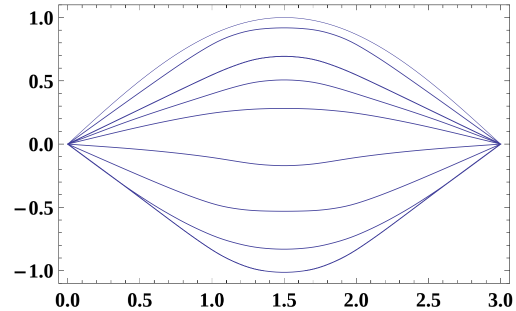
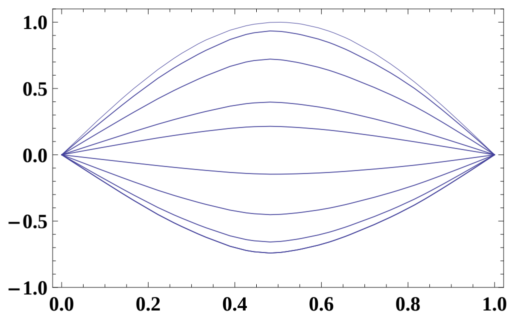


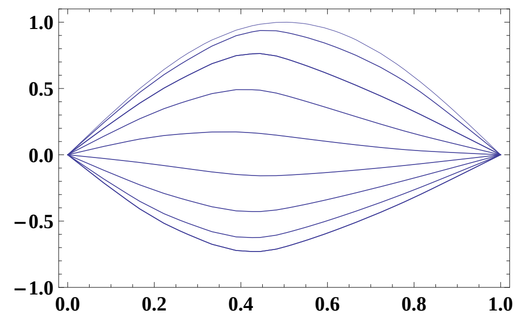
(a) Infinite Bernoulli convolution associated with the golden ratio. From top to bottom, the values of t are 0.0, 0.1, 0.15, 0.2, 0.25, 0.3, 0.4, 0.5, 0.6, 0.7.



(b) Three-fold convolution of the Cantor measure. From top to bottom, the values of t are 0.0, 0.2, 0.4, 0.5, 0.6, 0.8, 1, 1.2, 1.5.



(c) The self-similar measure defined by IFS (1.8) and probabilty weights $p_1 = p_2 = p_3 = 1/3$. From top to bottom, the values of t are 0.0, 0.1, 0.2, 0.3, 0.35, 0.45, 0.55, 0.65, 0.75.



(d) The self-similar measure defined by IFS (1.8) and probabilty weights $p_1 = 2/3$, $p_2 = p_3 = 1/6$. From top to bottom, the values of t are 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8

FIGURE 2. Numerical solutions of the strong damping wave equation (1.2) corresponding to the self-similar measure in Corollary 1.4. The time step Δt in equation (4.10) is taken to be 0.001. The initial data $g(x) = \sin(\pi x)$ in (a), (c), (d), $g = \sin((\pi/3)x)$ in (b); $h \equiv 0$; the constants $\alpha = 2$ and $\sigma = 0.1$.

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