SCHRÖDINGER EQUATIONS DEFINED BY A CLASS OF SELF-SIMILAR MEASURES

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Abstract. We study linear and nonlinear Schrödinger equations defined by fractal measures. Under the assumption that the Laplacian has compact resolvent, we prove that there exists a uniqueness weak solution for a linear Schrödinger equation, and then use it to obtain the existence and uniqueness of weak solution of a nonlinear Schrödinger equation. We prove that for a class of self-similar measures on $\mathbb{R}$ with overlaps, the Schrödinger equations can be discretized so that the finite element method can be applied to obtain approximate solutions. We also prove that the numerical solutions converge to the actual solution and obtain the rate of convergence.

Contents

1. Introduction

The Schrödinger operator in the fractal setting has been studied by a number of authors. Strichartz [27] studied the essential spectrum on the product of two copies of an infinite blowup of the Sierpinski gasket. Chen *et al.* studied the spectral asymptotics of the eigenvalues and Bohr's formula on several unbouned fractals. For Schrödinger operators defined by measures, typically self-similar measures, the authors [24] studied the bound states and Bohr's formula. It has become more and more apparent to physicists that spacetime exhibits fractal behavior. A multifractal spacetime

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model has recently been proposed by physicist Calcagni [4–6]. This is obtained by replacing the standard Lebesgue measure on a spacetime manifold by a Borel measure which is in general not absolutely continuous with respect to Lebesgue measure. Also, solution of the Schrödinger equation could play an important role in studying related mathematical problems. In fact, Hu and Zähle obtained heat kernel upper bound in metric measure spaces by using the solution of the Schrodinger equation [16]. Motivated by these, we study the Schrödinger equation defined by a fractal measure.

Let $U \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded open subset, and let μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. Let $L^2(U,\mu) := L^2(U,\mu,\mathbb{C})$ denote the space of measurable functions $u : U \to \mathbb{C}$ such that $\|u\|_\mu < \infty$ with

$$\|u\|_\mu := \Big(\int_U |u|^2 \, d\mu \Big)^{1/2}.$$

Let $H^1(U) := H^1(U,\mathbb{C})$ be the Sobolev space of complex-valued functions equipped with the norm

$$\|u\|_{H^1(U)} := \Big(\int_U |u|^2 \, dx + \int_U |\nabla u|^2 \, dx \Big)^{1/2}.$$

Let $H_0^1(U) := H_0^1(U,\mathbb{C})$ denote the completion of $C_c^\infty(U)$ in the H^1-norm, where $C_c^\infty(U)$ is the space of all complex-valued $C^\infty(U)$ functions with compact support in U. Throughout this paper, we regard $L^2(U,\mu)$ and $H_0^1(U)$ as real Hilbert spaces with the scalar product

$$(u,v)_\mu := \operatorname{Re} \int_U u\overline{v} \, d\mu \qquad \text{and} \qquad (u,v)_{H_0^1(U)} := \operatorname{Re} \int_U \nabla u \cdot \nabla \overline{v} \, dx,$$

respectively (see, e.g., [1,2]), where $\operatorname{Re}(z)$ denotes the real part of a complex number z and $\overline{v}$ denotes the conjugate function of v. It is known (see, e.g., [15]) that μ defines a Dirichlet Laplace operator Δ_μ^D(or simply Δ_μ), if the following *Poincaré inequality for a measure (PI)* holds: There exists some constant $C > 0$ such that

$$\int_U |u|^2 \, d\mu \leq C \int_U |\nabla u|^2 \, dx \qquad \text{for all } u \in C_c^\infty(U) \tag{1.1}$$

(see, e.g., [15, 20, 21]). The main purpose of this paper is to study the following linear Schrödinger equation defined by the Dirichlet Laplacian Δ_μ:

$$\begin{cases} i\partial_t u + \Delta_\mu u = f(t) & \text{on } U \times [0,T], \\ u = 0 & \text{on } \partial U \times [0,T], \\ u = g & \text{on } U \times \{t = 0\}, \end{cases} \tag{1.2}$$

where $u := u(t)$ is a Hilbert space valued function of t. We study the solution of equation (1.2) both theoretically and numerically.

We will describe the construction of the Laplacian $-\Delta_\mu$ in (1.2), as well as the associated nonnegative bilinear form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ (see (2.1)) in Subsection 2.1. Under the assumption $-\Delta_\mu$ has compact resolvent, we give an explicit formula for the weak solution of the Schrödinger equation (1.2) in Theorem 3.1. Let X be a Hilbert space with norm $\|\cdot\|$, we say a map $F : X \to X$ is *Lipschitz continuous* on X if there exists some constant $C > 0$ such that $\|F(u) - F(v)\| \leq C\|u - v\|$ for all $u, v \in X$. Using Theorem 3.1, we obtain our first main theorem.

Theorem 1.1. *Let $U \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded open subset, and let μ be a positive finite Borel measure with* $\operatorname{supp}(\mu) \subseteq \overline{U}$ *and $\mu(U) > 0$. Assume that μ satisfies (PI), $-\Delta_\mu$ has compact resolvent, and $F(\cdot)$ is Lipschitz continuous on* $\operatorname{dom}\mathcal{E}$*. If $g = \sum_{n=1}^{\infty} \alpha_n \varphi_n \in \operatorname{dom}\mathcal{E}$, then*

$$u(t) := \sum_{n=1}^{\infty} \alpha_n e^{-i\lambda_n t} \varphi_n - i \sum_{n=1}^{\infty} \Big(\int_0^t e^{-i\lambda_n (t-\tau)} \big(F(u(\tau)), \varphi_n\big)_\mu \, d\tau \Big) \varphi_n.$$

is the unique weak solution of the following non-linear schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_\mu u = F(u) & \text{on } U \times [0,T], \\ u = 0 & \text{on } \partial U \times [0,T], \\ u = g & \text{on } U \times \{t = 0\}. \end{cases} \tag{1.3}$$

As an example, let $F(u) = \sin u - mu$, $m \geq 0$ (see, e.g., [14]). Then $F(\cdot)$ is Lipschitz continuous on $\operatorname{dom}\mathcal{E}$.

We call a μ-measurable subset I of $\overline{U}$ a *cell (in $\overline{U}$)* if $\mu(I) > 0$. Clearly, $\overline{U}$ itself is a cell. Two cells I, J in U are *measure disjoint* with respect to μ if $\mu(I \cap J) = 0$. Let $I \subseteq \overline{U}$ be a cell. We call a finite family $\mathbf{P}$ of measure disjoint cells a *μ-partition* of I if $J \subseteq I$ for all $J \in \mathbf{P}$, and $\mu(I) = \sum_{J \in \mathbf{P}} \mu(J)$. A sequence of μ-partitions $(\mathbf{P}_k)_{k\geq 1}$ is *refining* if for any $J_1 \in \mathbf{P}_k$ and any $J_2 \in \mathbf{P}_{k+1}$, either $J_2 \subseteq J_1$ or they are measure disjoint, i.e., each member of $\mathbf{P}_{k+1}$ is a subset of some member of $\mathbf{P}_k$. Throughout this paper, $|E|$ denotes the diameter of a subset $E \subseteq \mathbb{R}^n$.

In order to discretize (1.2) and obtain numerical approximations of the weak solution, we will often impose the following additional conditions on μ: there exists a sequence of refining μ-partitions $(\mathbf{P}_k)_{k\geq 1} = (\{I_{k,\ell}\}_{\ell=0}^{N(k)})_{k\geq 1}$ of $\overline{U}$ such that

(P1) there exist some constant $\rho \in (0,1)$ and some integer m_0 satisfying $\max\{|J| : J \in \mathbf{P}_k\} \leq \rho^{k-m_0}$ for all $k \geq 1$;
(P2) for any $k \geq 1$, each cell $I \in \mathbf{P}_k$ is closed and connected;
(P3) for any $k \geq 2$ and any $0 \leq \ell \leq N(k)$, there exist similitudes $(\tau_{k,\ell,j})_{j=0}^{N(1)}$ of the form $\tau_{k,\ell,j}(x) = r_{k,\ell,j} x + b_{k,\ell,j}$ and constants $(c_{k,\ell,j})_{j=0}^{N(1)}$ such that $\tau_{k,\ell,j}(I_{1,j}) \subseteq I_{k,\ell}$, and

$$\mu|_{I_{k,\ell}} = \sum_{j=0}^{N(1)} c_{k,\ell,j} \cdot \mu|_{I_{1,j}} \circ \tau_{k,\ell,j}^{-1}. \tag{1.4}$$

Under assumption (P3), the μ measure of each cell in the partition can be computed by using (1.4), making it possible to discretize the Schrödinger equation (1.2). In order to discretize (1.2) and guarantee that the mass matrix that arises in the finite element method is positive definite (see Proposition 4.1), we assume that μ is a measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a,b]$.

Let $f(x,t) \equiv 0$ in (1.2). Multiplying the first equation in (1.2) by $\overline{v} \in \operatorname{dom}\mathcal{E}$, integrating both sides over $[a,b]$ with respect to $d\mu$, and then taking the real part, we obtain

$$\operatorname{Re} \int_a^b i\partial_t u(x,t) \overline{v}(x) \, d\mu = \operatorname{Re} \int_a^b \partial_x u(x,t) \overline{v}'(x) \, dx, \tag{1.5}$$

where $\partial_x u(x,t)$ and $\partial_t u(x,t)$ are the weak partial derivative of u with respect to x and t, respectively. Let $u_1(x,t)$ and $u_2(x,t)$ be the real and imaginary parts of $u(x,t)$, respectively. Then (1.5) can be rewritten as

$$\int_a^b \partial_t u_2(x,t)v(x)\,d\mu = -\int_a^b \partial_x u_1(x,t)v'(x)\,dx \tag{1.6}$$

and

$$\int_a^b \partial_t u_1(x,t)v(x)\,d\mu = \int_a^b \partial_x u_2(x,t)v'(x)\,dx \tag{1.7}$$

for all real-valued function $v \in \operatorname{dom}\mathcal{E}$.

Theorem 1.2. *Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a,b]$. Assume that there exists a sequence of refining partitions $(\mathbf{P}_k)_{k\ge1}$ satisfying conditions (P1)–(P3), and $\int_I x^j\,d\mu$, $I \in \mathbf{P}_1$, $j = 0,1,2$, can be evaluated explicitly. Then equations (1.6) and (1.7) can be discretized and the finite element method can be applied to yield a system of first-order ordinary differential equations, which has a unique solution that can be solved numerically.*

We are mainly interested in fractal measures μ. Let X be a non-empty compact subset of $\mathbb{R}^d$. Throughout this paper, an *iterated function system (IFS)* refers to a finite family of contractive similitudes $\{S_j\}_{j=1}^q$ defined on X, i.e.,

$$S_j(x) = \rho_j x + b_j, \qquad j = 1,\dots,q, \tag{1.8}$$

where $0 < \rho_j < 1$, and $b_j \in \mathbb{R}^d$. It is well-known that for each IFS $\{S_j\}_{j=1}^q$, there exists a unique non-empty compact subset $F \subseteq X$, called the *self-similar set*, such that

$$F = \bigcup_{j=1}^q S_j(F);$$

moreover, associated to each set of probability weights $\{w_j\}_{j=1}^q$ (i.e., $w_j > 0$ and $\sum_{j=1}^q w_j = 1$), there is a unique probability measure, called the *self-similar measure*, satisfying the following identity

$$\mu = \sum_{j=1}^q w_j \mu \circ S_j^{-1}$$

(see [11, 17]). An IFS $\{S_j\}_{j=1}^q$ is said to satisfy the *open set condition (OSC)* if there exists a non-empty bounded open set O such that $\bigcup_k S_k(O) \subseteq O$ and $S_k(O) \cap S_j(O) = \emptyset$ for all $k \ne j$. IFSs that do not satisfy (OSC), as well as all associated self-similar measures, are said to have overlaps.

It is worth pointing out that for general self-similar measures with overlaps, it does not seem possible to discretize the Schrödinger equations (1.2) in the way described in the paper, and thus it is not clear how numerical approximations of the weak solution can be obtain. Theorem 1.2 provides a framework under which discretization can be performed.

Based on Theorem 1.2, we solve the Schrödinger equation (1.2) numerically for three different one-dimensional self-similar measures with overlaps, namely, the infinite Bernoulli convolution associated with the golden ratio, the three-fold convolution of the Cantor measure, and a class of self-similar measures that we call *essentially of finite type (EFT)* (see [23]). These measures share the common property that the support can be partitioned into a sequence of arbitrarily small intervals whose measures can be computed explicitly.

The following theorem shows that the approximate solutions converge to the actual weak solution and we also obtain a rate of convergence. See Subsection 2.1 and Definition 2.2 for the definitions of $\|\cdot\|_{\operatorname{dom}\mathcal{E}}$ and $\|\cdot\|_{2,\operatorname{dom}\mathcal{E}}$, respectively.

Theorem 1.3. *Assume the hypotheses of Theorem 1.2, let* $f \equiv 0$ *and* $g = \sum_{n=1}^{\infty} \alpha_n \varphi_n$ *in equation (1.2), and fix* $t \in [0,T]$. *If* $\sum_{n=1}^{\infty} |\alpha_n|^2 \lambda_n^3 < \infty$, *then the approximate solutions* u^m *obtained by the finite element method converge in* $L^2((a,b),\mu)$ *to the actual weak solution* u. *Moreover,*

$$\|u^m - u\|_\mu \le 2\big(\sqrt{T}\|u_t\|_{2,\operatorname{dom}\mathcal{E}} + \|u\|_{\operatorname{dom}\mathcal{E}}\big)\rho^{m/2},$$

where ρ *is the constant in condition (P1).*

The rest of this paper is organized as follows. Section 2 summarizes some notation, definitions and results that will be needed throughout the paper. We give the existence and uniqueness of solution of Schrödinger equation (1.2) in Section 3. Section 4 is devoted to the proof of Theorem 1.2. In Section 5, we apply Theorem 1.2 to three different self-similar measures with overlaps. The proof of Theorem 1.3 is given in Section 6.

2. Preliminaries

In this section, we summarize some notation, definitions and facts that will be used throughout the rest of the paper. For a Banach space X, we denote its topological dual by X'. For $v \in X'$ and $u \in X$, we let $\langle v, u\rangle = \langle v, u\rangle_{X',X} := v(u)$ denote the *dual pairing* of X' and X.

Definition 2.1. *Let* X *be a Banach space,* $u : (a,b) \subseteq \mathbb{R} \to X$, *and* $t_0 \in (a,b)$. *Then* u *is said to be* differentiable *at* t_0 *in the norm* $\|\cdot\|_X$ *if there exists* $v_0 \in X$ *such that*

$$\lim_{h\to 0}\left\|\frac{u(t_0+h)-u(t_0)}{h} - v_0\right\|_X = 0.$$

v_0 *is called the* strong derivative *of* u *at* t_0, *and we write*

$$v_0 = u_t(t_0) = \lim_{h\to 0}\frac{u(t_0+h)-u(t_0)}{h}.$$

Higher-order strong derivatives are defined similarly.

Note that if u is differentiable at t_0 in the norm $\|\cdot\|_X$, then it is continuous at t_0 in the norm $\|\cdot\|_X$.

Definition 2.2. *Let* X *be a separable Banach space with norm* $\|\cdot\|_X$. *Denote by* $L^p([0,T],X)$ *the space of all measurable functions* $u : [0,T] \to X$ *satisfying*

(1) $\|u\|_{L^p([0,T],X)} := \Big(\int_0^T \|u(t)\|_X^p\,dt\Big)^{1/p} < \infty$, *if* $1 \le p < \infty$, *and*
(2) $\|u\|_{L^\infty([0,T],X)} := \operatorname{esssup}_{0\le t\le T}\|u(t)\|_X < \infty$, *if* $p = \infty$.

If the interval $[0,T]$ is understood, we will abbreviate these norms as $\|u\|_{p,X}$ and $\|u\|_{\infty,X}$, respectively.

Let $U \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded open subset. For a function $\varphi : U \to \mathbb{R}$, we let φ' denote both its classical and weak derivatives. If $u \in L^2([0,T],X)$, where X is $H_0^1(U)$ or $L^2(U,\mu)$ etc., then for each fixed t, we denote by $u_x(x,t)$ (or ∇u) the classical or weak derivatives of u with respect to x.

Remark 2.1. *For each $1 \leq p \leq \infty$, $L^p([0,T],X)$ is a Banach space; moreover, $L^{p_2}([0,T],X) \subseteq L^{p_1}([0,T],X)$ if $1 \leq p_1 \leq p_2 \leq \infty$. Let X be a separable Banach space with inner product $(\cdot,\cdot)_X$. If $(X,(\cdot,\cdot)_X)$ is a separable Hilbert space, then $L^2([0,T],X)$ is a Hilbert space with the inner product*

$$(u,v)_{L^2([0,T],X)} := \int_0^T (u(t),v(t))_X \, dt.$$

Definition 2.3. *Let X be a Banach space. We define $C([0,T],X)$ to be the vector space of all continuous functions $u : [0,T] \to X$ such that*

$$\|u\|_{C([0,T],X)} := \max_{0 \leq t \leq T} \|u\|_X < \infty.$$

2.1. **Dirichlet Laplacian defined by a measure.** Let $U \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded open subset and μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. We assume that (PI) holds. In [15], (PI) implies that μ defines a Dirichlet operator in

$$L^2(U,\mu,\mathbb{R}) := \Big\{ u : U \to \mathbb{R} : \int_U |u|^2 \, dx < \infty \Big\}.$$

Similarly, we can define a Dirichlet operator in $L^2(U,\mu) := L^2(U,\mu,\mathbb{C})$, as follows. (PI) implies each equivalence class $u \in H_0^1(U)$ contains a unique (in the $L^2(U,\mu)$ sense) member $\hat{u}$ that belongs to $L^2(U,\mu)$ and satisfies both conditions below:

(1) there exists a sequence $\{u_n\}$ in $C_c^\infty(U)$ such that $u_n \to \hat{u}$ in $H_0^1(U)$ and $u_n \to \hat{u}$ in $L^2(U,\mu)$;
(2) $\hat{u}$ satisfies inequality (1.1).

We call $\hat{u}$ the $L^2(U,\mu)$-*representative* of u. Define a mapping $\iota : H_0^1(U) \to L^2(U,\mu)$ by $\iota(u) = \hat{u}$. ι is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(U)$ defined as $\mathcal{N} := \{ u \in H_0^1(U) : \|\iota(u)\|_\mu = 0 \}$. Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(U)$. Then $\iota : \mathcal{N}^\perp \to L^2(U,\mu)$ is injective. Unless explicitly stated otherwise, we will denote the $L^2(U,\mu)$-representative $\hat{u}$ simply by u.

Consider the non-negative bilinear form $\mathcal{E}(\cdot,\cdot)$ in $L^2(U,\mu)$ defined by

$$\mathcal{E}(u,v) := \operatorname{Re} \int_U \nabla u \cdot \nabla \overline{v} \, dx \tag{2.1}$$

with domain $\operatorname{dom} \mathcal{E} = \mathcal{N}^\perp$, or more precisely, $\iota(\mathcal{N}^\perp)$. (PI) implies that $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ is a closed quadratic form in $L^2(U,\mu)$. Hence there exists a non-negative self-adjoint operator A in $L^2(U,\mu)$ such that

$$\mathcal{E}(u,v) = \big(A^{1/2}u, A^{1/2}v\big)_\mu \quad \text{and} \quad \operatorname{dom} \mathcal{E} = \operatorname{dom}(A^{1/2})$$

(see, e.g., [12, Theorem 1.3.1]). We write $\Delta_\mu^D = -A$, and call it the *(Dirichlet) Laplacian* with respect to μ. If no confusion is possible, we denote Δ_μ^D simply by Δ_μ.

Let $u \in \operatorname{dom}\mathcal{E}$. Then $u \in \operatorname{dom}\Delta_\mu$ if and only if there exists a unique $f \in L^2(U,\mu)$ such that $\mathcal{E}(u,v) = (f,v)_\mu$ for all $v \in \operatorname{dom}\mathcal{E}$. In this case, $-\Delta_\mu u = f$. Throughout this paper, we let

$$\operatorname{dom}\mathcal{E} := \mathcal{N}^\perp \quad \text{and} \quad \|\cdot\|_{\operatorname{dom}\mathcal{E}} := \|\cdot\|_{H_0^1(U)}.$$

3. Extrapolation and weak solutions

In this section, we consider the existence and uniqueness of weak solution of equation (1.2). Let $U \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded open subset and μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. We assume that (PI) holds. Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be defined as in Section 2.1, and $-\Delta_\mu$ is the Dirichlet operator with respect to μ. By identifying $L^2(U,\mu)$ with $(L^2(U,\mu))'$, we have the following Gelfand triple (see, e.g., [13,30]):

$$\operatorname{dom}\mathcal{E} \hookrightarrow L^2(U,\mu) \cong (L^2(U,\mu))' \hookrightarrow (\operatorname{dom}\mathcal{E})',$$

where all the embeddings are continuous, injective, and dense. The embedding $L^2(U,\mu) \hookrightarrow (\operatorname{dom}\mathcal{E})'$ is given by

$$w \in L^2(U,\mu) \mapsto (w,\cdot)_\mu \in (L^2(U,\mu))' \subset (\operatorname{dom}\mathcal{E})'.$$

It follows that for any $u \in \operatorname{dom}\mathcal{E}$, there exists a unique $w \in (\operatorname{dom}\mathcal{E})'$ such that

$$\mathcal{E}(u,v) = \langle w, v\rangle \qquad \text{for all } v \in \operatorname{dom}\mathcal{E},$$

where throughout this paper, $\langle\cdot,\cdot\rangle$ denotes the pairing between $(\operatorname{dom}\mathcal{E})'$ and $\operatorname{dom}\mathcal{E}$. On the other hand, we note that the form $\mathcal{E}$ is coercive by (PI). Hence by the Lax-Milgram theorem, for every $w \in (\operatorname{dom}\mathcal{E})'$, there exists a unique $u \in \operatorname{dom}\mathcal{E}$ such that

$$\mathcal{E}(u,v) = \langle w, v\rangle \qquad \text{for all } v \in \operatorname{dom}\mathcal{E}.$$

Thus we can define a bijective operator L from $\operatorname{dom}\mathcal{E}$ to $(\operatorname{dom}\mathcal{E})'$ by

$$Lu = w, \tag{3.1}$$

and equip $(\operatorname{dom}\mathcal{E})'$ with the scalar product

$$(u,v)_{(\operatorname{dom}\mathcal{E})'} := \mathcal{E}(L^{-1}u, L^{-1}v).$$

Note that $\operatorname{dom} L = \operatorname{dom}\mathcal{E}$ and $\langle w, v\rangle = (w,v)_\mu$ for all $w \in (\operatorname{dom}\mathcal{E})'$ and $v \in \operatorname{dom}\mathcal{E}$. It follows that L is an extension of $-\Delta_\mu$. Throughout this paper, we equip $(\operatorname{dom}\mathcal{E})'$ with the norm

$$\|w\|_{(\operatorname{dom}\mathcal{E})'} := \|L^{-1}w\|_{\operatorname{dom}\mathcal{E}} \qquad \text{for } w \in (\operatorname{dom}\mathcal{E})'.$$

We remark that this norm is the standard norm in $(\operatorname{dom}\mathcal{E})'$, which is equivalent to the general norm in $(\operatorname{dom}\mathcal{E})'$ (see, e.g., [2]).

Definition 3.1. *Use the notation above. Let $0 < T < \infty$. Assume $f \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$ and $g \in \operatorname{dom}\mathcal{E}$. A function $u \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$ with $\partial_t u \in L^\infty([0,T], (\operatorname{dom}\mathcal{E})')$ is a* weak solution *of the Schrödinger equation (1.2) if the following conditions are satisfied:*

(1) $\langle i\partial_t u, v\rangle - \mathcal{E}(u,v) = (f(t),v)_\mu$ *for each* $v \in \operatorname{dom}\mathcal{E}$ *and Lebesgue a.e.* $t \in [0,T]$*;*
(2) $u(0) = g$.

Remark 3.1. *Here are some comments on Definition 3.1.*

(a) *The boundary condition* $u|_{\partial U} = 0$ *in (1.2) is included in the assumption* $u(t) \in \operatorname{dom}\mathcal{E}$*. If* $u \in L^\infty([0,T],\operatorname{dom}\mathcal{E})$ *and* $\partial_t u \in L^\infty([0,T],(\operatorname{dom}\mathcal{E})')$*, then* $u \in C([0,T],L^2(U,\mu))$*, and thus the initial condition* $u(0) = g$ *makes sense.*
(b) *Condition (1) above is equivalent to*

$$i\partial_t u - Lu = f(t) \qquad \text{in } (\operatorname{dom}\mathcal{E})' \text{ for Lebesgue a.e. } t \in [0,T],$$

where L *is defined as in (3.1).*

We now assume that $-\Delta_\mu$ has compact resolvent and let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2(U,\mu)$ so that $-\Delta_\mu$ such that $-\Delta_\mu\varphi_n = \lambda_n\varphi_n$ for all $n \ge 1$, where $0 < \lambda_1 \le \cdots \le \lambda_n \le \lambda_{n+1} \le \cdots$ and $\lim_{n\to\infty}\lambda_n = \infty$. Some sufficient conditions for which $-\Delta_\mu$ has compact resolvent can be found in [9, 15, 20]. In particular, if $n = 1$, then $-\Delta_\mu$ has compact resolvent for any such μ. We remark that

$$\operatorname{dom}\mathcal{E} = \Big\{\sum_{n=1}^\infty a_n\varphi_n : \sum_{n=1}^\infty |a_n|^2\lambda_n < \infty\Big\} \text{ and } \operatorname{dom}\Delta_\mu = \Big\{\sum_{n=1}^\infty a_n\varphi_n : \sum_{n=1}^\infty |a_n|^2\lambda_n^2 < \infty\Big\}.$$

Since $\operatorname{dom}\mathcal{E} \hookrightarrow L^2(U,\mu) \hookrightarrow (\operatorname{dom}\mathcal{E})'$, $\{\varphi_n\}_{n=1}^\infty$ is also a complete orthonormal basis of $(\operatorname{dom}\mathcal{E})'$. Note that $w = \sum_{n=1}^\infty a_n\varphi_n \in (\operatorname{dom}\mathcal{E})'$ if and only if there exists a unique $L^{-1}w = \sum_{n=1}^\infty b_n\varphi_n \in \operatorname{dom}\mathcal{E}$ such that $\mathcal{E}(L^{-1}w, v) = \langle w, v\rangle$ for all $v \in \operatorname{dom}\mathcal{E}$. Substituting $v = \varphi_n$ for $n \ge 1$, we get $a_n = \langle w,\varphi_n\rangle = \mathcal{E}(L^{-1}w,\varphi_n) = b_n\lambda_n$, and so $w = \sum_{n=1}^\infty a_n\varphi_n \in (\operatorname{dom}\mathcal{E})'$ if and only if $\|w\|^2_{(\operatorname{dom}\mathcal{E})'} = \|L^{-1}w\|^2_{\operatorname{dom}\mathcal{E}} = \sum_{n=1}^\infty a_n^2/\lambda_n < \infty$. Therefore, for every $u = \sum_{n=1}^\infty a_n\varphi_n \in \operatorname{dom}\mathcal{E}$, we have $Lu = \sum_{n=1}^\infty a_n\lambda_n\varphi_n \in (\operatorname{dom}\mathcal{E})'$, and

$$(\operatorname{dom}\mathcal{E})' = \Big\{\sum_{n=1}^\infty a_n\varphi_n : \sum_{n=1}^\infty a_n^2/\lambda_n < \infty\Big\}.$$

We will describe the construction of the Laplacian $-\Delta_\mu$ in (1.2), as well as the associated nonnegative bilinear form $(\mathcal{E},\operatorname{dom}\mathcal{E})$ (see (2.1)) in Subsection 2.1. To give an explicit formula for the weak solution of the Schrödinger equation (1.2), we assume that $-\Delta_\mu$ has compact resolvent. Then there exists a complete orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2(U,\mu)$ such that $-\Delta_\mu\varphi_n = \lambda_n\varphi_n$ for all $n \ge 1$, where the eigenvalues satisfy $0 < \lambda_1 \le \cdots \le \lambda_n \le \lambda_{n+1} \le \cdots$ with $\lim_{n\to\infty}\lambda_n = \infty$. If $g \in L^2(U,\mu)$ and $f(t) \in L^2([0,T],L^2(U,\mu))$, we can write

$$g = \sum_{n=1}^\infty \alpha_n\varphi_n \quad \text{and} \quad f(t) = \sum_{n=1}^\infty \beta_n(t)\varphi_n, \tag{3.2}$$

where $\beta_n(t) = (f(t),\varphi_n)_\mu$ for $n \ge 1$. Let

$$u(t) := \sum_{n=1}^\infty \alpha_n e^{-i\lambda_n t}\varphi_n - i\sum_{n=1}^\infty \Big(\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\Big)\varphi_n \tag{3.3}$$

and

$$K(t) := -i\sum_{n=1}^{\infty}\alpha_n\lambda_n e^{-i\lambda_n t}\varphi_n - if(t) - \sum_{n=1}^{\infty}\lambda_n\Big(\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\Big)\varphi_n. \tag{3.4}$$

Theorem 3.1. *Let $U \subseteq \mathbb{R}^d$, $d \geq 1$, be a bounded open subset, and let μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{U}$ and $\mu(U) > 0$. Assume that μ satisfies (PI) and $-\Delta_\mu$ has compact resolvent. Let g, $f(t)$ $u(t)$ and $K(t)$ be defined as above. If $g \in \operatorname{dom}\mathcal{E}$ and $f(t) \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$, then*

(a) *$\partial_t u = K(t)$ in $(\operatorname{dom}\mathcal{E})'$ for Lebesgue a.e. $t \in [0,T]$.*
(b) *$u(t)$ is the unique weak solution of the Schrödinger equation (1.2).*
(c) *if $f \equiv 0$, then*

$$\|u(t)\|_\mu = \|g\|_\mu \quad \text{and} \quad \mathcal{E}(u,u) = \mathcal{E}(g,g) \qquad \text{for all } t \in [0,T].$$

If, in addition, $\sum_{n=1}^{\infty}|\alpha_n|^2\lambda_n^3 < \infty$, then $\partial_t u(t) \in \operatorname{dom}\mathcal{E}$ for Lebesgue a.e. $t \in [0,T]$.

Proof. Let g, $f(t)$, $u(t)$ and $K(t)$ be defined as in (3.2), (3.3) and (3.4), respectively. Since $g \in \operatorname{dom}\mathcal{E}$ and $f(t) \in L^\infty([0,T], \operatorname{dom}\mathcal{E})$, $u(t) \in C([0,T], \operatorname{dom}\mathcal{E})$ and $K(t) \in L^\infty([0,T], (\operatorname{dom}\mathcal{E})')$. In fact, using Hölder's inequality, we obtain

$$\begin{aligned}\|u(t)\|^2_{C([0,T],\operatorname{dom}\mathcal{E})} &= \max_{t\in[0,T]}\|u(t)\|^2_{\operatorname{dom}\mathcal{E}} \leq 2\Big(\sum_{n=1}^{\infty}|\alpha_n|^2\lambda_n + T\sum_{n=1}^{\infty}\lambda_n\int_0^T\big|\beta_n(\tau)\big|^2\,d\tau\Big)\\ &= 2\Big(\|g\|^2_{\operatorname{dom}\mathcal{E}} + T\|f(t)\|^2_{2,\operatorname{dom}\mathcal{E}}\Big) < \infty,\end{aligned}$$

and

$$\begin{aligned}\|K(t)\|^2_{\infty,(\operatorname{dom}\mathcal{E})'} &= \operatorname*{esssup}_{t\in[0,T]}\|K(t)\|^2_{(\operatorname{dom}\mathcal{E})'}\\ &\leq 3\Big(\sum_{n=1}^{\infty}|\alpha_n|^2\lambda_n + \|f(t)\|^2_{\infty,(\operatorname{dom}\mathcal{E})'} + \operatorname*{esssup}_{t\in[0,T]}\sum_{n=1}^{\infty}\lambda_n\Big|\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\Big|^2\Big)\\ &\leq 3\Big(\|g\|^2_{\operatorname{dom}\mathcal{E}} + \|f(t)\|^2_{\infty,(\operatorname{dom}\mathcal{E})'} + T\|f(t)\|^2_{2,\operatorname{dom}\mathcal{E}}\Big) < \infty.\end{aligned}$$

(a) Let δ satisfy $0 < 2\delta < T$ and $u(t) =: \sum_{n=1}^{\infty}c_n(t)\varphi_n$. For all $t \in [\delta, T-\delta]$ and each $h \in (-\delta,\delta)$, we have

$$\begin{aligned}u(t+h) - u(t) = &\sum_{n=1}^{\infty}\alpha_n\big(e^{-i\lambda_n(t+h)} - e^{-i\lambda_n t}\big)\varphi_n - i\sum_{n=1}^{\infty}\Big(\int_t^{t+h}e^{-i\lambda_n(t+h-\tau)}\beta_n(\tau)\,d\tau\Big)\varphi_n\\ &- i\sum_{n=1}^{\infty}\Big(\int_0^t\big(e^{-i\lambda_n(t+h-\tau)} - e^{-i\lambda_n(t-\tau)}\big)\beta_n(\tau)\,d\tau\Big)\varphi_n.\end{aligned}$$

It follows that

$$
\begin{aligned}
|c_n(t+h)-c_n(t)|^2 &= \Big|\alpha_n\big(e^{-i\lambda_n(t+h)}-e^{-i\lambda_n t}\big)-i\int_t^{t+h} e^{-i\lambda_n(t+h-\tau)}\beta_n(\tau)\,d\tau \\
&\qquad -i\int_0^t \big(e^{-i\lambda_n(t+h-\tau)}-e^{-i\lambda_n(t-\tau)}\big)\beta_n(\tau)\,d\tau\Big|^2 \\
&\le 3\bigg(|\alpha_n|^2\cdot\Big|e^{-i\lambda_n(t+h)}-e^{-i\lambda_n t}\Big|^2+\Big|\int_t^{t+h} e^{-i\lambda_n(t+h-\tau)}\beta_n(\tau)\,d\tau\Big|^2 \\
&\qquad +\Big|\int_0^t \big(e^{-i\lambda_n(t+h-\tau)}-e^{-i\lambda_n(t-\tau)}\big)\beta_n(\tau)\,d\tau\Big|^2\bigg) \\
&\le 3\bigg(h^2|\alpha_n|^2\lambda_n^2+h\int_t^{t+h}|\beta_n(\tau)|^2\,d\tau \\
&\qquad +T\int_0^T \big|e^{-i\lambda_n(t+h-\tau)}-e^{-i\lambda_n(t-\tau)}\big|^2\cdot|\beta_n(\tau)|^2\,d\tau\bigg) \\
&\le 3h^2\bigg(|\alpha_n|^2\lambda_n^2+\operatorname*{esssup}_{t\in[0,T]}|\beta_n(t)|^2+T\lambda_n^2\int_0^T|\beta_n(\tau)|^2\,d\tau\bigg) \\
&=: 3h^2\lambda_n M_n,
\end{aligned}
\tag{3.5}
$$

where the fact $|e^{-i\theta}-1|\le\theta$ for $\theta>0$ is used in the second and third inequalities. We remark that $\sum_{n=1}^{\infty} M_n=\|g\|^2_{\operatorname{dom}\mathcal{E}}+\|f(t)\|^2_{\infty,(\operatorname{dom}\mathcal{E})'}+T\|f(t)\|^2_{2,\operatorname{dom}\mathcal{E}}<\infty$. Let $K(t)=:\sum_{n=1}^{\infty}d_n(t)\varphi_n$. Using (3.4) and Hölder inequality, we have

$$
\begin{aligned}
|d_n(t)|^2 &= \Big|-i\alpha_n\lambda_n e^{-i\lambda_n t}-i\beta(t)-\lambda_n\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\Big|^2 \\
&\le 3\bigg(|\alpha_n|^2\lambda_n^2+\operatorname*{esssup}_{t\in[0,T]}|\beta(t)|^2+T\lambda_n^2\int_0^T|\beta_n(\tau)|^2\,d\tau\bigg)=3\lambda_n M_n.
\end{aligned}
\tag{3.6}
$$

Note that the classical derivative $c_n'(t)=d_n(t)$ for Lebesgue a.e. $t\in[0,T]$. It follows that

$$
s_n(t,h):=\frac{c_n(t+h)-c_n(t)}{h}-d_n(t)\to 0 \qquad \text{as} \qquad h\to 0 \tag{3.7}
$$

for Lebesgue a.e. $t\in[\delta,T-\delta]$ and $h\in(-\delta,\delta)$. Combining (3.5) and (3.6), we have for Lebesgue a.e. $t\in[\delta,T-\delta]$ and each $h\in(-\delta,\delta)$,

$$
\frac{|s_n(t,h)|^2}{\lambda_n}\le\frac{2}{\lambda_n}\bigg(\frac{|c_n(t+h)-c_n(t)|^2}{h^2}+|d_n(t)|^2\bigg)\le 12M_n. \tag{3.8}
$$

Using (3.8) and Weierstrass' M-test, we see the series $\sum_{n=1}^{\infty}|s_n(t,h)|^2/\lambda_n$ converges uniformly for all $h\in(-\delta,\delta)$ and Lebesgue a.e. $t\in[\delta,T-\delta]$. Thus for Lebesgue a.e. $t\in[\delta,T-\delta]$,

$$
\lim_{h\to 0}\Big\|\frac{u(t+h)-u(t)}{h}-K(t)\Big\|^2_{(\operatorname{dom}\mathcal{E})'}=\lim_{h\to 0}\sum_{n=1}^{\infty}|s_n(t,h)|^2/\lambda_n=\sum_{n=1}^{\infty}\lim_{h\to 0}|s_n(t,h)|^2/\lambda_n=0,
$$

where (3.7) is used in the last equality. It follows that $\partial_t u(t)=K(t)$ in $(\operatorname{dom}\mathcal{E})'$ for Lebesgue a.e. $t\in[\delta,T-\delta]$. The desired results follows by letting $\delta\to 0^+$.

(b) We first note that $u(0)=g$, and

$$Lu=\sum_{n=1}^{\infty}\alpha_n\lambda_n e^{-i\lambda_n t}\varphi_n - i\sum_{n=1}^{\infty}\lambda_n\Big(\int_0^t e^{-i\lambda_n(t-\tau)}\beta_n(\tau)\,d\tau\Big)\varphi_n, \tag{3.9}$$

where L is defined as in (3.1). Combining (3.9) and (a), we have $i\partial_t u(t)-Lu(t)=f(t)$ on $(\operatorname{dom}\mathcal{E})'$ for Lebesgue a.e. $t\in[0,T]$. It follows from Remark 3.1, that $u(t)$ is a weak solution of (1.2). It suffices to show the only solution of (1.2) with $f(t)\equiv g\equiv 0$ is that $u(t)\equiv 0$. Let u be a weak solution of (1.2) with $f(t)\equiv g\equiv 0$. To verify this, note that

$$\langle i\partial_t u, -i\overline{u}\rangle+\mathcal{E}(u,-i\overline{u})=0 \qquad \text{for Lebesgue a.e. } t\in[0,T]. \tag{3.10}$$

Since $\mathcal{E}(u,-i\overline{u})=0$, $u(0)=g\equiv 0$, and

$$\langle i\partial_t u,-i\overline{u}\rangle=\frac{1}{2}\frac{d}{dt}\|u(t)\|_\mu^2,$$

we obtain $\|u(t)\|_\mu^2=0$ for Lebesgue a.e. $t\in[0,T]$. It follows that $u=0$, which proves (b).

(c) Since $f\equiv 0$, by parts (a) and (b), we have $u(t)=\sum_{n=1}^{\infty}\alpha_n e^{-i\lambda_n t}\varphi_n$ and $\partial_t u(t)=-i\sum_{n=1}^{\infty}\alpha_n\lambda_n e^{-i\lambda_n t}\varphi_n$. It follows that

$$\|u(t)\|_\mu^2=\sum_{n=1}^{\infty}|\alpha_n|^2=\|g\|_\mu^2 \quad \text{and} \quad \mathcal{E}(u,u)=\sum_{n=1}^{\infty}|\alpha_n|^2\lambda_n=\mathcal{E}(g,g) \quad \text{for all } t\in[0,T].$$

Note that $\mathcal{E}(\partial_t u,\partial_t u)=\sum_{n=1}^{\infty}|\alpha_n|^2\lambda_n^3$ for Lebesgue a.e. $t\in[0,T]$. Thus the assumption $\sum_{n=1}^{\infty}|\alpha_n|^2\lambda_n^3<\infty$ implies $\partial_t u(t)\in\operatorname{dom}\mathcal{E}$ for Lebesgue a.e. $t\in[0,T]$. □

Now we prove Theorem 1.1. The main ingredients are Banach's fixed point theorem and Theorem 3.1.

Proof of Theorem 1.1. Given a function $u\in L^\infty([0,T],\operatorname{dom}\mathcal{E})$, set $h(t):=F(u(t))$. By the assumption on $F(\cdot)$, we see that $h\in L^\infty([0,T],\operatorname{dom}\mathcal{E})$. Consequently, Theorem 3.1 ensures that the linear Schrödinger equation

$$\begin{cases} i\partial_t w+\Delta_\mu w=h & \text{on } U\times[0,T],\\ w=0 & \text{on } \partial U\times[0,T],\\ w=g & \text{on } U\times\{t=0\}, \end{cases} \tag{3.11}$$

has a unique weak solution $w(t)\in L^\infty([0,T],\operatorname{dom}\mathcal{E})$ given by

$$w(t):=\sum_{n=1}^{\infty}\alpha_n e^{-i\lambda_n t}\varphi_n - i\sum_{n=1}^{\infty}\Big(\int_0^t e^{-i\lambda_n(t-\tau)}\big(h(\tau),\varphi_n\big)_\mu\,d\tau\Big)\varphi_n. \tag{3.12}$$

Define $A: L^\infty([0,T],\operatorname{dom}\mathcal{E})\to L^\infty([0,T],\operatorname{dom}\mathcal{E})$ by $A[u]=w$.

We now claim that if $T>0$ is small enough, then A is a contraction mapping from $L^\infty([0,T],\operatorname{dom}\mathcal{E})$ to $L^\infty([0,T],\operatorname{dom}\mathcal{E})$. Let $u(t),v(t)\in L^\infty([0,T],\operatorname{dom}\mathcal{E})$. We first get from (3.12) that for Lebesgue a.e. $0 \le t \le T$,

$$\begin{aligned}
\Big\| A[u(t)] - A[v(t)] \Big\|^2_{\operatorname{dom}\mathcal{E}} &= \sum_{n=1}^{\infty} \lambda_n \Big| \int_0^t e^{-i\lambda_n(t-\tau)} \big(F(u(\tau)) - F(v(\tau)), \varphi_n\big)_\mu \, d\tau \Big|^2 \\
&\le t \int_0^t \sum_{n=1}^{\infty} \lambda_n \Big| \big(F(u(\tau)) - F(v(\tau)), \varphi_n\big)_\mu \Big|^2 \, d\tau \\
&= t \int_0^t \Big\| F(u(\tau)) - F(v(\tau)) \Big\|^2_{\operatorname{dom}\mathcal{E}} \, d\tau \\
&\le CT \int_0^t \Big\| u(\tau) - v(\tau) \Big\|^2_{\operatorname{dom}\mathcal{E}} \, d\tau \le CT^2 \|u - v\|^2_{\infty, \operatorname{dom}\mathcal{E}},
\end{aligned} \tag{3.13}$$

where the assumption that $F(\cdot)$ is Lipschitz is used in the second inequality. It follows that

$$\|A[u] - A[v]\|_{\infty, \operatorname{dom}\mathcal{E}} \le \sqrt{C} T \|u - v\|_{\infty, \operatorname{dom}\mathcal{E}}.$$

Thus $A[\cdot]$ is a strict contraction, provided $T > 0$ is so small that $\sqrt{C}T = \gamma < 1$.

Given any $T > 0$, we select $T_1 > 0$ so small that $\sqrt{C}T_1 < 1$. We can then apply Banach's fixed point theorem to obtain a weak solution u of the nonlinear Schrödinger equation (1.3) that exists on the time interval $[0, T_1]$. Since $u(t) \in \operatorname{dom}\mathcal{E}$ for *a.e.* $0 \le t \le T_1$, we can find some $T_2 \in (T_1/2, T_1)$ such that $u(T_2) \in \operatorname{dom}\mathcal{E}$. We can then repeat the argument above to extend our solution u to the time interval $[T_2, T_3]$ such that $u(T_3) \in \operatorname{dom}\mathcal{E}$ and $T_3 \in [2T_2, T_1 + T_2]$. Repeating this process for a finite number of steps, we obtain a weak solution that exists on the entire interval $[0, T]$.

To prove the uniqueness, suppose that u and v are two weak solutions of the nonlinear Schrödinger equation (1.3). Then we have $A[u] = u$ and $A[v] = v$. It follows from the uniqueness of the fixed point of A that $u(t) = v(t)$ in $L^\infty([0, T_1], \operatorname{dom}\mathcal{E})$. Combining this argument with the extension argument above shows that $u(t) = v(t)$ in $L^\infty([0, T], \operatorname{dom}\mathcal{E})$. □

4. The finite element method

In this section, we let $f \equiv 0$ in equation (1.2), and use the finite element method to solve the homogeneous Schrödinger equation (1.2). Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a, b]$. Assume that there exists a sequence of refining partitions $(\mathbf{P}_k)_{k \ge 1} = (\{I_{k,\ell}\}_{\ell=0}^{N(k)})_{k \ge 1}$ satisfying conditions (P1)–(P3) in Section 1. Without loss of generality, we can write $I_{m,\ell} = [x_{m,\ell}, x_{m,\ell+1}]$ for $m \ge 1$ and $0 \le \ell \le N(m) - 1$. It is easy to see that $x_{m,0} = a$ and $x_{m,N(m)} = b$ for all $m \ge 1$.

We first note that the equations (1.6) and (1.7) are the integral form of the homogeneous Schrödinger equation (1.2). Now, we apply the finite element method to approximate the solution $u(x, t)$ satisfying (1.6) and (1.7) by

$$u^m(x, t) := \sum_{j=0}^{N(m)} (w_{1j}(t) + i w_{2j}(t)) \phi_j(x), \tag{4.1}$$

where, for $j = 0, 1, \ldots, N(m)$, $w_{1j}(t) := w^m_{1j}(t)$ and $w_{2j}(t) := w^m_{2j}(t)$ are real-valued functions to be determined, and $\phi_j(x) := \phi_{m,j}(x)$ are the standard piecewise linear *finite element basis functions* (also called *tent functions*) defined by

$$\phi_j(x) := \begin{cases} \dfrac{x - x_{m,j-1}}{x_{m,j} - x_{m,j-1}} & \text{if } x \in I_{m,j-1},\ j = 1,2,\dots,N(m), \\ \dfrac{x - x_{m,j+1}}{x_{m,j} - x_{m,j+1}} & \text{if } x \in I_{m,j},\ j = 0,1,\dots,N(m)-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Let $u_1^m(x,t)$ and $u_2^m(x,t)$ be the real and complex part of $u^m(x,t)$, respectively. We require $u^m(x,t)$ to satisfy equations (1.6) and (1.7) as follows:

$$\int_a^b \partial_t u_2^m(x,t)\phi_j(x)\,d\mu = -\int_a^b \partial_x u_1^m(x,t)\phi_j'(x)\,dx \tag{4.3}$$

and

$$\int_a^b \partial_t u_1^m(x,t)\phi_j(x)\,d\mu = \int_a^b \partial_x u_2^m(x,t)\phi_j'(x)\,dx. \tag{4.4}$$

Moreover, we require $u^m(x,t)$ to satisfy the Dirichlet boundary condition $u^m(a,t) = u^m(b,t) = 0$. We note that $\phi_\ell(a) = \phi_\ell(x_{m,0}) = 0$ and $\phi_j(b) = \phi_j(x_{m,N(m)}) = 0$ for all $\ell = 1,\dots,N(m)$ and $j = 0,1,\dots,N(m)-1$. Thus $w_{k0}(t) = w_{kN(m)}(t) = 0$ for all $t \in [0,T]$ and $k = 1,2$. Using this and substituting (4.1) into (4.3) and (4.4) gives

$$\sum_{\ell=1}^{N(m)-1} w_{2\ell}' \int_a^b \phi_\ell(x)\phi_j(x)\,d\mu = -\sum_{\ell=1}^{N(m)-1} w_{1\ell} \int_a^b \phi_\ell'(x)\phi_j'(x)\,dx \tag{4.5}$$

and

$$\sum_{\ell=1}^{N(m)-1} w_{1\ell}' \int_a^b \phi_\ell(x)\phi_j(x)\,d\mu = \sum_{\ell=1}^{N(m)-1} w_{2\ell} \int_a^b \phi_\ell'(x)\phi_j'(x)\,dx \tag{4.6}$$

for $1 \le j \le N(m)-1$. We define the mass matrix $\mathbf{M} = \mathbf{M}^{(m)} = (M_{\ell j}^{(m)})$ and stiffness matrix $\mathbf{K} = \mathbf{K}^{(m)} = (K_{\ell j}^{(m)})$, respectively, by

$$M_{\ell j}^{(m)} = \int_a^b \phi_\ell(x)\phi_j(x)\,d\mu \quad \text{and} \quad K_{\ell j}^{(m)} = \int_a^b \phi_\ell'(x)\phi_j'(x)\,dx,$$

where $1 \le \ell, j \le N(m)-1$. It follows from the definition of $\phi_j(x)$ that both $\mathbf{M}$ and $\mathbf{K}$ are tridiagonal. Let

$$\mathbf{w}_1(t) = \mathbf{w}_{1m}(t) := \begin{bmatrix} w_{11}(t) \\ \vdots \\ w_{1,N(m)-1}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2(t) = \mathbf{w}_{2m}(t) := \begin{bmatrix} w_{21}(t) \\ \vdots \\ w_{2,N(m)-1}(t) \end{bmatrix}.$$

Define

$$\hat{\mathbf{M}} := \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \hat{\mathbf{K}} := \begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ \mathbf{K} & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{w}} := \begin{bmatrix} \mathbf{w_1} \\ \mathbf{w_2} \end{bmatrix}.$$

Then (4.5) and (4.6) can be expressed in matrix form as

$$\hat{\mathbf{M}}\frac{d\hat{\mathbf{w}}}{dt} = -\hat{\mathbf{K}}\hat{\mathbf{w}}. \tag{4.7}$$

This gives us a system of first-order linear ODEs with constant coefficients. To solve it, we need to impose initial conditions. The initial condition $u(x,0) = g(x)$ for $a \le x \le b$ can be approximated by its linear interpolant $\widetilde{g}(x) = \sum_{j=1}^{N(m)-1} g(x_{m,j})\phi_j(x)$. Let $g_1(x)$ and $g_2(x)$ be the real and complex part of $g(x)$. Therefore, we set $w_{1j}(0) = g_1(x_{m,j})$ and $w_{2j}(0) = g_2(x_{m,j})$ for $1 \le j \le N(m) - 1$. This leads to the initial condition

$$\hat{\mathbf{w}}(0) = \hat{\mathbf{w}}_m(0) := \left[g_1(x_{m,1}), \cdots, g_1(x_{m,N(m)-1}), g_2(x_{m,1}), \cdots, g_2(x_{m,N(m)-1})\right]. \tag{4.8}$$

Consequently, we get the linear system

$$\hat{\mathbf{M}}\frac{d\hat{\mathbf{w}}}{dt} = -\hat{\mathbf{K}}\hat{\mathbf{w}}, \quad t > 0, \quad \text{and} \quad \hat{\mathbf{w}}(0) = \hat{\mathbf{w}}_{m,0}. \tag{4.9}$$

Since $\operatorname{supp}(\mu) = [a, b]$, [29, Proposition 4.1] implies that $\mathbf{M}$ is invertible. It follows that $\hat{\mathbf{M}}$ also is invertible. Thus the system in (4.9) has a unique solution. More precisely, the following proposition holds.

Proposition 4.1. *Assume that* $\operatorname{supp}(\mu) = [a, b]$*. Then the mass matrix* $\mathbf{M}$ *and the stiffness matrix* $\mathbf{K}$ *are positive definite. Consequently, (4.9) has a unique solution* $\hat{\mathbf{w}}(t)$*. Moreover,* $w_{1j}(t) \in C(0, T)$ *and* $w_{2j}(t) \in C(0, T)$ *for* $j = 1, \ldots, N(m) - 1$*.*

Proof of Theorem 1.2. The assertions hold by combining the derivations above and Proposition 4.1. □

We note that the matrix $\mathbf{K}$ can be computed directly. The authors have given the explicitly formula for $\mathbf{M}$ in [29], as follows. Let $c_{m,\ell,j}$ and $\tau_{m,\ell,j}$ be the constants and similitudes from condition (P3) in Section 1, respectively. Then for $1 \le \ell \le N(m) - 1$, we have

$$\begin{aligned} M^{(m)}_{\ell,\ell} &= (x_{m,\ell} - x_{m,\ell-1})^{-2} \cdot \sum_{j=0}^{N(1)} c_{m,\ell-1,j} \int_{I_{1,j}} \left(\tau_{m,\ell-1,j}(x) - x_{m,\ell-1}\right)^2 d\mu \\ &\quad + (x_{m,\ell} - x_{m,\ell+1})^{-2} \cdot \sum_{j=0}^{N(1)} c_{m,\ell,j} \int_{I_{1,j}} \left(\tau_{m,\ell,j}(x) - x_{m,\ell+1}\right)^2 d\mu. \end{aligned} \tag{4.10}$$

For $2 \le \ell \le N(m) - 1$, we obtain

$$M^{(m)}_{\ell,\ell-1} = -(x_{m,\ell} - x_{m,\ell-1})^{-2} \cdot \sum_{j=0}^{N(1)} c_{m,\ell-1,j} \int_{I_{1,j}} \left(\tau_{m,\ell-1,j}(x) - x_{m,\ell-1}\right)\left(\tau_{m,\ell-1,j}(x) - x_{m,\ell}\right) d\mu, \tag{4.11}$$

and $M^{(m)}_{\ell-1,\ell} = M^{(m)}_{\ell,\ell-1}$.

Define

$$\mathcal{J}_{k,j} := \int_{I_{1,j}} x^k \, d\mu, \quad k = 0, 1, 2, \text{ and } j = 0, \ldots, N(1). \tag{4.12}$$

Since each $\tau_{k,\ell,j}$ is of the form $\tau_{k,\ell,j}(x) = r_{k,\ell,j}x + b_{k,\ell,j}$, we can see that the matrix $\mathbf{M}$ is completely determined by the integrals $\mathcal{J}_{k,j}$, where $k = 0, 1, 2$ and $j = 0, \ldots, N(1)$. Hereafter, we assume that the constant $\mathcal{J}_{k,j}$ can be evaluated explicitly for $k = 0, 1, 2$ and $j = 0, \ldots, N(1)$.

Next, we discuss the solution of the linear system (4.7). Let $\hat{\mathbf{w}}_n := \hat{\mathbf{w}}(t_n)$, $n \ge 0$, and use the central difference method to solve the equation (4.9). We approximate the derivative as

$$\hat{\mathbf{w}}'(t_n) \approx \frac{\hat{\mathbf{w}}_{n+1} - \hat{\mathbf{w}}_n}{\Delta t} \quad \text{and} \quad \hat{\mathbf{w}}(t_n) \approx \frac{\hat{\mathbf{w}}_{n+1} + \hat{\mathbf{w}}_n}{2}. \tag{4.13}$$

Substituting (4.13) into (4.7) yields

$$\hat{\mathbf{M}}\frac{\hat{\mathbf{w}}_{n+1}-\hat{\mathbf{w}}_n}{\Delta t}=-\hat{\mathbf{K}}\frac{\hat{\mathbf{w}}_{n+1}+\hat{\mathbf{w}}_n}{2}.$$

Since $\mathbf{M}$ and $\mathbf{K}$ are positive definite, so is $2\hat{\mathbf{M}}+(\Delta t)\hat{\mathbf{K}}$ for all $\Delta t>0$. Thus, $\hat{\mathbf{w}}_{n+1}=(2\hat{\mathbf{M}}+(\Delta t)\hat{\mathbf{K}})^{-1}(2\hat{\mathbf{M}}-(\Delta t)\hat{\mathbf{K}})\hat{\mathbf{w}}_n$. Therefore, equation (4.7) becomes

$$\begin{cases}\hat{\mathbf{w}}_{n+1}=(2\hat{\mathbf{M}}+(\Delta t)\hat{\mathbf{K}})^{-1}(2\hat{\mathbf{M}}-(\Delta t)\hat{\mathbf{K}})\hat{\mathbf{w}}_n, & n=0,1,2,\ldots,\\ \hat{\mathbf{w}}_0=\hat{\mathbf{w}}(t_0)=\hat{\mathbf{w}}(0), & \\ t_n=n\Delta t. & .\end{cases}\tag{4.14}$$

To solve this system, fix Δt and substitute the initial condition $\hat{\mathbf{w}}_0$ from (4.8) into the first equation in (4.14) to get $\hat{\mathbf{w}}_1$. Then $\hat{\mathbf{w}}_{n+1}$ can be computed recursively.

5. Fractal measures defined by iterated function systems

In this section, we solve the homogeneous Schrödinger equation (1.2) numerically for three different measures. These measures are defined by IFSs with overlaps and satisfy the conditions (P1)-(P3) in Section 1 (see [29, Proposition 5.1 and Section 5.3]). In the first and second cases, the measures satisfy a family of second-order self-similar identities. These identities were first introduced by Strichartz *et al.* [28] to approximate the density of the infinite Bernoulli convolution associated with the golden ratio.

Let $\{S_j\}_{j=1}^q$ be an IFS of contractive similitudes on $\mathbb{R}$, and let μ be the associated self-similar measure. Assume that $\operatorname{supp}(\mu)=[a,b]$. Define an auxiliary IFS

$$T_j(x)=r_jx+d_j,\qquad j=1,2,\ldots,N,$$

where $n_j\in\mathbb{N}$, $d_j\in\mathbb{R}$. We assume that μ satisfies a family of *second-order (self-similar) identities* with respect to $\{T_j\}_{j=1}^N$ (see [18]), that is, the following conditions hold:

(1) $\operatorname{supp}(\mu)\subseteq\bigcup_{j=1}^N T_j(\operatorname{supp}(\mu))$, and
(2) for each Borel subset $A\subseteq\operatorname{supp}(\mu)$ and $0\le\ell,j\le N$, $\mu(T_\ell T_j A)$ can be expressed as a linear combination of $\{\mu(T_kA):k=1,\ldots,N\}$. In matrix form,

$$\mu(T_\ell T_jA)=\mathbf{e}_\ell M_j\begin{bmatrix}\mu(T_1A)\\ \vdots\\ \mu(T_NA)\end{bmatrix},\quad \ell,j=1,\ldots,N,\tag{5.15}$$

where $\mathbf{e}_\ell$ is the ℓth row of the $N\times N$ identity matrix and M_j is some $N\times N$ matrix independent of A.

Furthermore, assume that $\{T_j\}_{j=1}^N$ satisfies (OSC). Define

$$\mathbf{P}_k:=\left\{T_{\boldsymbol{j}}([a,b]):\boldsymbol{j}\in\{1,\ldots,N\}^k\right\}\qquad\text{for } k\ge 1.$$

[29, Proposition 5.1] tells us that $(\mathbf{P}_k)_{k\geq 1}$ is a sequence of refining μ-partitions of $[a,b]$ satisfying conditions (P1)–(P3) in Section 1. Moreover, the matrix $\mathbf{M}$ is completely determined by the integrals $\int_a^b x^k\,d\mu\circ T_j$, $k=0,1,2$, $j=1,\dots,N$.

5.1. **Infinite Bernoulli convolution associated with the golden ratio.** We consider the infinite Bernoulli convolution associated with the golden ratio:

$$\mu=\frac{1}{2}\mu\circ S_1^{-1}+\frac{1}{2}\mu\circ S_2^{-1}, \tag{5.16}$$

where

$$S_1(x)=\rho x,\qquad S_2(x)=\rho x+(1-\rho),\qquad \rho=(\sqrt{5}-1)/2.$$

We note that $\operatorname{supp}(\mu)=[0,1]$. Strichartz *et al.* [28] showed that μ satisfies a family of second-order identities with respect to the following auxiliary IFS:

$$T_1(x):=\rho^2 x,\qquad T_2(x):=\rho^3 x+\rho^2,\qquad T_3(x):=\rho^2 x+\rho.$$

Moreover, μ satisfies the following second-order identities [18]: for each Borel $A\subseteq[0,1]$,

$$\begin{bmatrix}\mu(T_1T_jA)\\ \mu(T_2T_jA)\\ \mu(T_3T_jA)\end{bmatrix}=M_j\begin{bmatrix}\mu(T_1A)\\ \mu(T_2A)\\ \mu(T_3A)\end{bmatrix},\quad j=1,2,3,$$

where

$$M_1=\frac{1}{8}\begin{bmatrix}2&0&0\\1&2&0\\0&4&0\end{bmatrix},\quad M_2=\frac{1}{4}\begin{bmatrix}0&1&0\\0&1&0\\0&1&0\end{bmatrix},\quad M_3=\frac{1}{8}\begin{bmatrix}0&4&0\\0&2&1\\0&0&2\end{bmatrix}.$$

This can be used to compute the measure of suitable subintervals of $[0,1]$. The integrals $\int_0^1 x^k\,d\mu\circ T_j$, $k=0,1,2$, $j=1,2,3$, have been calculated in [8, Section 5]. We can thus calculate the entries of the mass matrix $\mathbf{M}$ and solve the linear system (4.7). The result is shown in Figure 1. Here we choose g to have small support so that it models the Dirac delta function.

5.2. **Three-fold convolution of the Cantor measure.** We consider the following three-fold convolution of the Cantor measure studied in [18,22,23]. The three-fold convolution of the Cantor measure μ is the self-similar measure defined by the following IFS with overlaps (see [22]):

$$S_j(x)=\frac{1}{3}x+\frac{2}{3}(j-1),\qquad j=1,2,3,4,$$

together with probability weights $\{1/8,3/8,3/8,1/8\}$. That is,

$$\mu=\frac{1}{8}\mu\circ S_1^{-1}+\frac{3}{8}\mu\circ S_2^{-1}+\frac{3}{8}\mu\circ S_3^{-1}+\frac{1}{8}\mu\circ S_4^{-1}. \tag{5.17}$$

Note that $\operatorname{supp}(\mu)=[0,3]$. It is shown in [18] that μ satisfies a family of second-order identities with respect to the following auxiliary IFS

$$T_1(x):=\frac{1}{3}x,\qquad T_2(x)=\frac{1}{3}x+1,\quad T_3(x)=\frac{1}{3}x+2. \tag{5.18}$$

In fact, for each Borel $A\subseteq[0,3]$,

$$\begin{bmatrix}\mu(T_1T_jA)\\ \mu(T_2T_jA)\\ \mu(T_3T_jA)\end{bmatrix}=M_j\begin{bmatrix}\mu(T_1A)\\ \mu(T_2A)\\ \mu(T_3A)\end{bmatrix},\quad j=1,2,3,$$

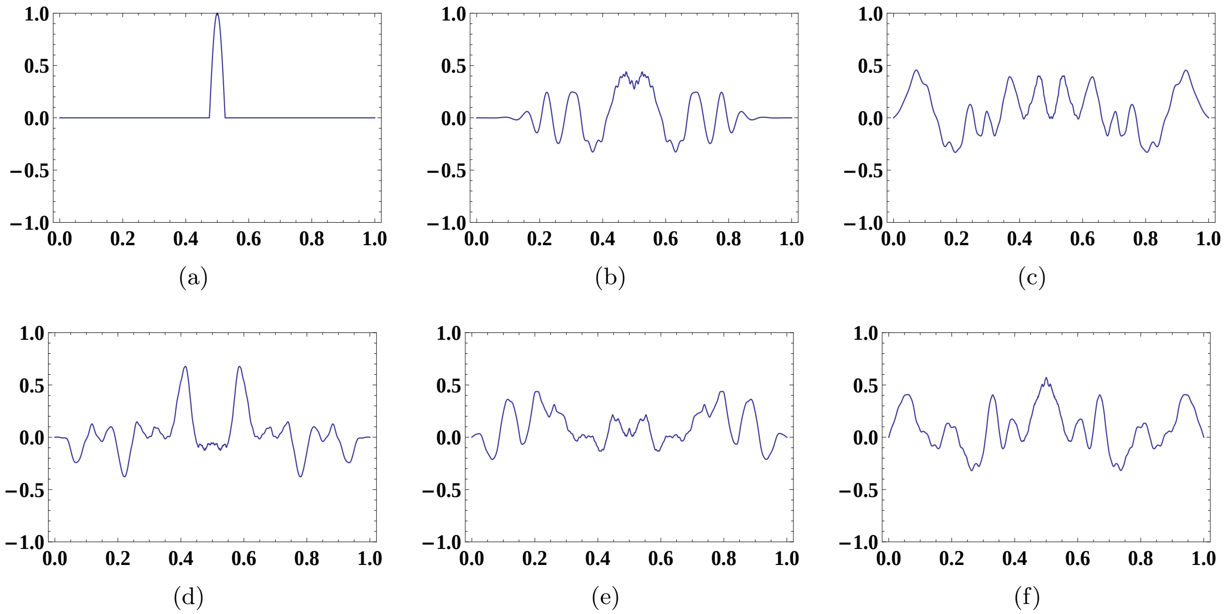

FIGURE 1. Figure for numerical solutions of the homogeneous Schrödinger equation defined by the infinite Bernoulli convolution associated with the golden ratio. The initial condition is given by the function $g(x) := \sin(20\pi(x-0.475)) + i\sin(20\pi(x-0.475))$ for $x \in (0.475, 0.525)$, and $g(x) := 0$ otherwise. Here $\Delta t = 0.0001$. From (a) to (f), the values of t are 0.0, 0.001, 0.002, 0.004, 0.008, 0.02.

where M_1, M_2, M_3 are given by

$$M_1 = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad M_2 = \frac{1}{8}\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \frac{1}{8}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The integrals $\int_0^1 x^k \, d\mu \circ T_j$, $k = 0, 1, 2$, $j = 1, 2, 3$, have been calculated in [3, Section 4.3]. We can thus calculate the entries of the mass matrix $\mathbf{M}$ and solve the linear system (4.7). The result is shown in Figure 2.

5.3. **A class of self-similar measures satisfying (EFT).** In this subsection, we consider the following family of IFSs:

$$S_1(x) = r_1 x, \quad S_2(x) = r_2 x + r_1(1 - r_2), \quad S_3(x) = r_2 x + 1 - r_2, \tag{5.19}$$

where the contraction ratios $r_1, r_2 \in (0, 1)$ satisfy $r_1 + 2r_2 - r_1 r_2 \le 1$, i.e., $S_2(1) \le S_3(0)$. The Hausdorff dimension of the self-similar sets is computed in [19]. The multifractal properties and spectral dimension of the corresponding self-similar measures are recently studied in [10, 23, 25].

Let μ be a self-similar measure defined by an IFS in (5.19) and a probability vector $(p_i)_{i=1}^3$. Let $I_{1,1} := S_1(X) \bigcup S_2(X)$ and $I_{1,0} := S_3(X)$, where $X = [0, 1]$. In order to define a sequence of refining μ-partitions of $[0, 1]$, we adopt the definition of an island from [23]. Let $\mathcal{M}_k := \{1, 2, 3\}^k$

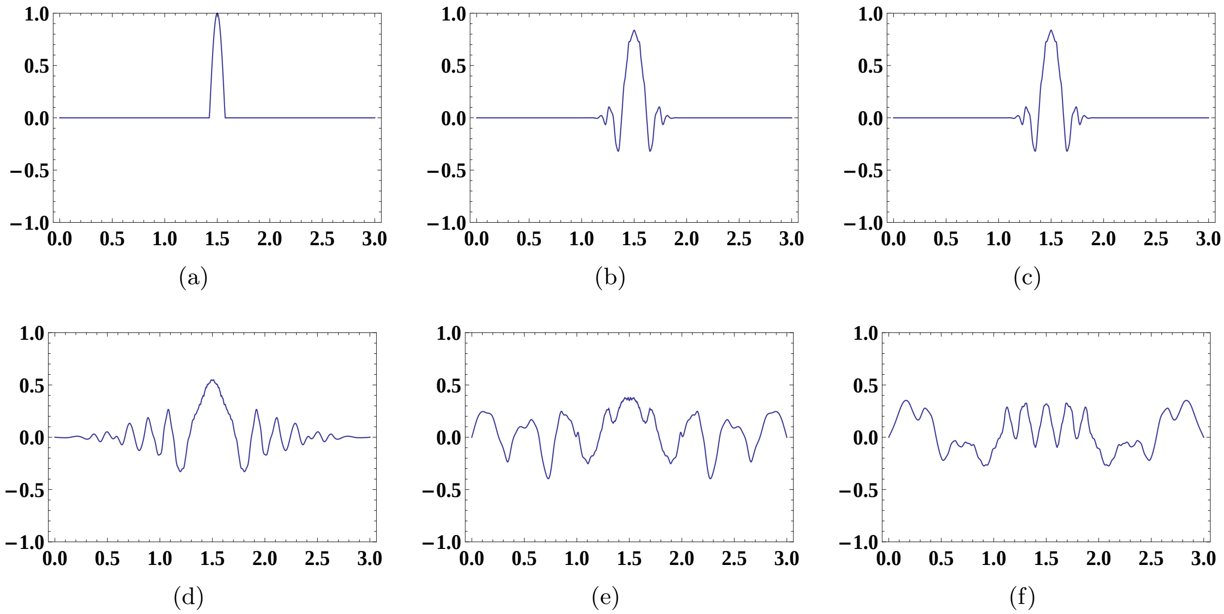

FIGURE 2. Figure for numerical solutions of the homogeneous Schrödinger equation defined by the three-fold convolution of the Cantor measure. The initial condition is given by the function $g(x) = \sin(20\pi(x/3 - 0.475)) + i\sin(20\pi(x/3 - 0.475)), x \in (1.425, 1.575)$, and $g(x) = 0$ otherwise. Here $\Delta t = 0.0001$. From (a) to (f), the values of t are 0.0, 0.001, 0.002, 0.004, 0.008, 0.02.

for $k \geq 1$ and $\mathcal{M}_0 := \emptyset$. A closed subset $I \subseteq [0,1]$ is called a *level-k island* with respect to $\{\mathcal{M}_k\}$ if the following conditions hold:

(1) there exists a finite sequence of indexes $\boldsymbol{i}_0, \boldsymbol{i}_1, \dots, \boldsymbol{i}_n$ in $\mathcal{M}_k$ such that $S_{\boldsymbol{i}_k}(0,1) \cap S_{\boldsymbol{i}_{k+1}}(0,1) \neq \emptyset$ for all $k = 0, \dots, n-1$, and $I = \bigcup_{k=0}^{n} S_{\boldsymbol{i}_k}([0,1])$;
(2) for any $\boldsymbol{j} \in \mathcal{M}_k \setminus \{\boldsymbol{i}_0, \dots, \boldsymbol{i}_n\}$ and any $k \in \{0, \dots, n\}$, $S_{\boldsymbol{j}}(0,1) \cap S_{\boldsymbol{i}_k}(0,1) = \emptyset$.

Intuitively, for each level-k island I, I° is a connected component of $S_{\mathcal{M}_k}(0,1) := \bigcup_{\boldsymbol{i} \in \mathcal{M}_k} S_{\boldsymbol{i}}(0,1)$ (see Figure 3). For $k \geq 1$, define

$$\mathbf{P}_k := \big\{ I : I \text{ is a level-}k \text{ island with respect to } \{\mathcal{M}_k\} \ \big\}. \tag{5.20}$$

It follows from [29, Section 5.3] that $(\mathbf{P}_k)_{k \geq 1}$ is a sequence of refining μ-partitions of $[0,1]$ satisfying conditions (P1)–(P3) in Section 1.

If $S_2(1) = S_3(0)$, then $\operatorname{supp}(\mu) = [0,1]$. In particular, if $r_1 = 1/2$ and $r_2 = 1/3$, then $S_2(1) = S_3(0)$. For two cases: $p_1 = p_2 = p_3 = 1/3$ and $p_1 = 2/3, p_2 = p_3 = 1/6$, the authors have calculated the integrals $\int_{I_{1,j}} x^k \, d\mu$ in [29], where $j = 0,1$ and $k = 0,1,2$. We can thus calculate the entries of the mass matrix $\mathbf{M}$ and solve the linear system (4.7). The results are shown in Figures 4 and 5.

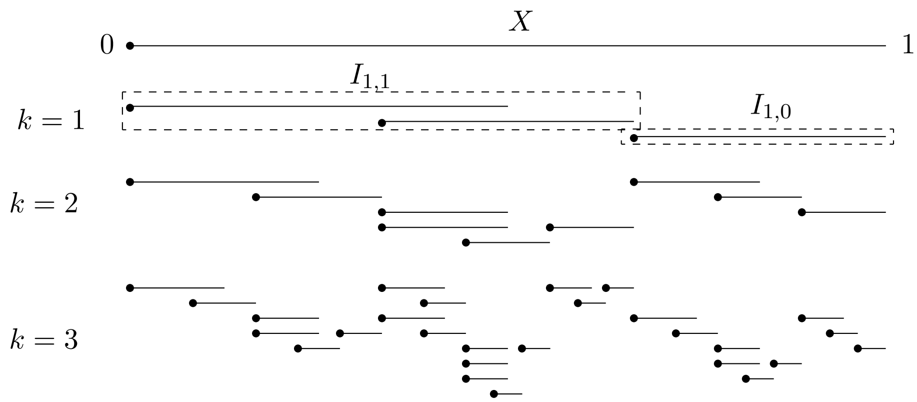

FIGURE 3. μ-partitions $\mathbf{P}_k$ for $k = 1, 2, 3$, where $\mathbf{P}_k$ is defined as in (5.20). Cells that are labeled consist of line segments enclosed by a box. The figure is drawn with $r_1 = 1/2$ and $r_2 = 1/3$.

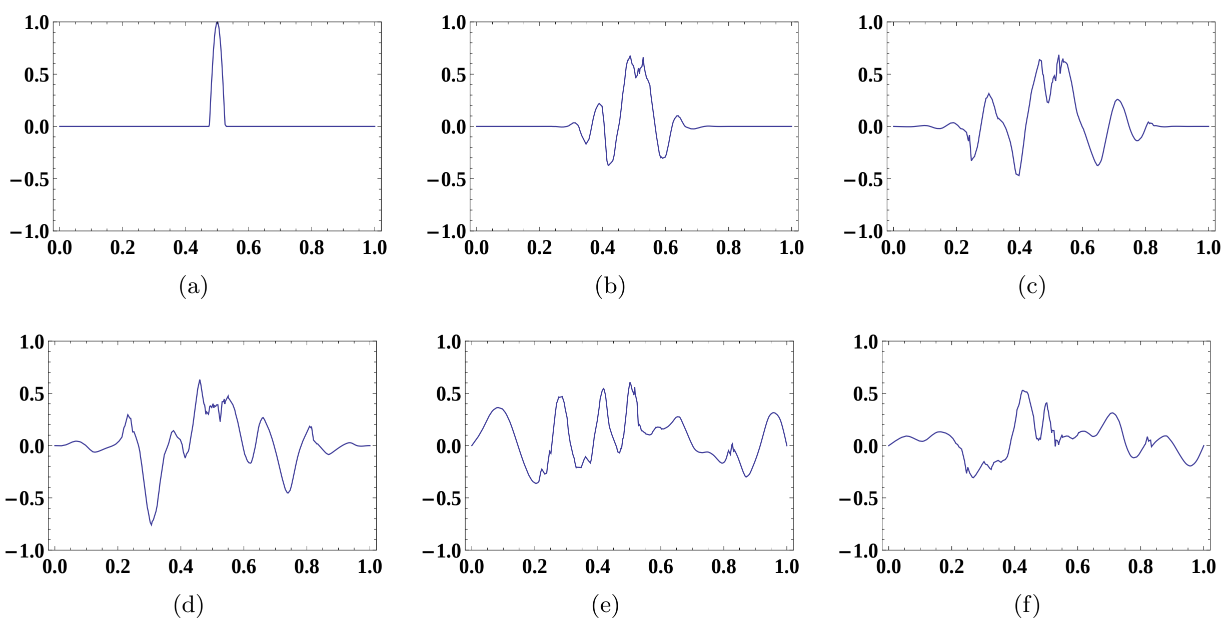

FIGURE 4. Numerical solutions of the homogeneous Schrödinger equation (1.2) corresponding to the self-similar measure defined by the IFS in (5.19) with probability weights $p_1 = p_2 = p_3 = 1/3$. The initial condition, the values of Δt and t are the same as those in Figure 1.

6. CONVERGENCE OF NUMERICAL APPROXIMATIONS

In this section, we prove the convergence of numerical approximations of the homogeneous Schrödinger equation (1.2). Some of our results are obtained by modifying similar ones in [26].

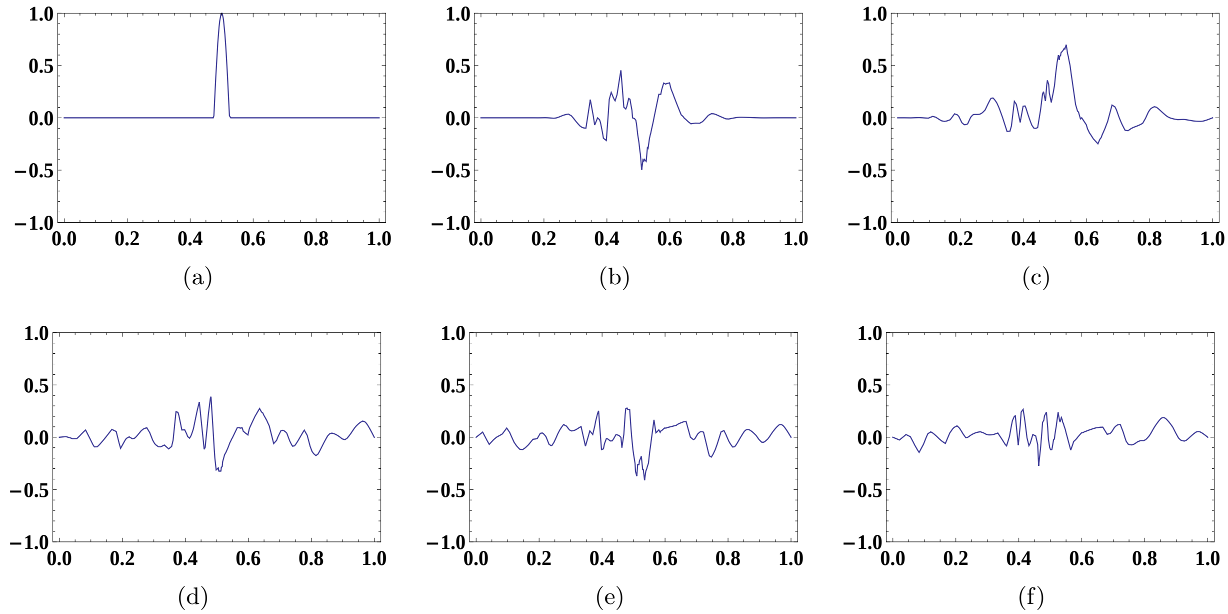

FIGURE 5. Numerical solutions of the homogeneous Schrödinger equation (1.2) corresponding to the self-similar measure defined by the IFS in (5.19) with probability weights $p_1 = 2/3, p_2 = p_3 = 1/6$. The initial condition, the values of Δt and t are the same as those in Figure 1.

Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a, b]$. In this case, there exists a complete orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of $L^2(U, \mu)$ such that $-\Delta_\mu \varphi_n = \lambda_n \varphi_n$ for all $n \geq 1$, where the eigenvalues satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$. Assume that there exists a sequence of refining partitions $(\mathbf{P}_k)_{k\geq 1}$ satisfying conditions (P1)–(P3) in Section 1. Let V_m be the set of end-points of all level-m sub-intervals, and arrange its element so that $V_m = \{x_{m,\ell} : \ell = 0, \ldots, N(m)\}$ with $x_{m,\ell} < x_{m,\ell+1}$ for $\ell = 0, 1, \ldots, N(m) - 1$, $x_{m,0} = a$ and $x_{m,N(m)} = b$. Let S^m be the space of continuous piecewise linear functions on $[a, b]$ with nodes V_m, and let

$$S_D^m = \{u \in S^m : u(a) = u(b) = 0\}$$

be the subspaces of S^m consisting of functions satisfying the Dirichlet boundary condition.

We choose the basis of S^m consisting of the tent functions $\{\phi_\ell\}_{\ell=0}^{N(m)}$ defined in (4.2) and choose the basis $\{\phi_\ell\}_{\ell=1}^{N(m)-1}$ for S_D^m. The linear map $\mathcal{P}_m : \operatorname{dom} \mathcal{E} \to S_D^m$ defined by

$$\mathcal{P}_m v = \sum_{\ell=1}^{N^m-1} v(x_{m,\ell})\phi_\ell(x), \quad v \in \operatorname{dom} \mathcal{E},$$

is called the *Rayleigh-Ritz projection* with respect to V_m. Let

$$\|V_m\| := \max\{|x_{m,\ell} - x_{m,\ell-1}| : 1 \leq \ell \leq N(m)\}$$

be the *norm* of V_m for $m \geq 1$.

Lemma 6.1. *For $m \geq 1$, let V_m and $\mathcal{P}_m$ be defined as above. Then for any $u \in \operatorname{dom}\mathcal{E}$, $\mathcal{P}_m u$ is the component of u in the subspace S_D^m, $u - \mathcal{P}_m u$ vanishes on the boundary $\{a, b\}$, and*

$$\mathcal{E}(u - \mathcal{P}_m u, v) = 0 \qquad \text{for all } v \in S_D^m. \tag{6.1}$$

Proof. The proof can be found in [26]. □

Throughout the rest of this section, let $g = \sum_{n=1}^{\infty} \alpha_n \varphi_n \in \operatorname{dom}\mathcal{E}$ satisfy $\sum_{n=1}^{\infty} |\alpha_n|^2 \lambda_n^3 < \infty$ and let $f = 0$, and u be the solution of the corresponding homogeneous Schrödinger equation (1.2). Then

$$(iu_t, v)_\mu - \mathcal{E}(u, v) = 0 \qquad \text{for all } v \in \operatorname{dom}\mathcal{E}. \tag{6.2}$$

By Theorem 3.1(c), $u_t \in \operatorname{dom}\mathcal{E}$. As in Section 4,

$$u^m(x,t) = \sum_{\ell=1}^{N(m)-1} \big(w_{1j}(t) + iw_{2j}(t)\big)\phi_\ell(x).$$

Thus it follows from the derivations in Section 4 that u^m satisfies

$$(iu_t^m, v^m)_\mu + \mathcal{E}(u^m, v^m) = 0 \qquad \text{for all } v^m \in S_D^m, \tag{6.3}$$

and $u^m(x,0) = \sum_{\ell=1}^{N(m)-1} g(x_{m,\ell})\phi_\ell(x)$. Finally, define

$$e(x,t) := e^m(x,t) = \mathcal{P}_m u(x,t) - u^m(x,t).$$

Lemma 6.2. *Let u, u^m, e be as above. Then*

$$(e_t, e)_\mu = (\mathcal{P}_m u_t - u_t, e)_\mu. \tag{6.4}$$

Proof. We first note that the functions e, e_t and $(\mathcal{P}_m u)_t = \mathcal{P}_m u_t$ all belong to S_D^m. Thus substituting ie for v in (6.2) and for v^m in (6.3), we get

$$(iu_t, ie)_\mu + \mathcal{E}(u, ie) = 0 \quad \text{and} \quad (iu_t^m, ie)_\mu + \mathcal{E}(u^m, ie) = 0.$$

Subtracting these equations gives $(i(u_t - u_t^m), ie)_\mu + \mathcal{E}(u - u^m, ie) = 0$. Using the fact that $(i(u_t - u_t^m), ie)_\mu = ((u_t - u_t^m), e)_\mu$, we get

$$(u_t - \mathcal{P}_m u_t + \mathcal{P}_m u_t - u_t^m, e)_\mu + \mathcal{E}(u - \mathcal{P}_m u + \mathcal{P}_m u - u^m, ie) = 0,$$

which, together with the fact $\mathcal{E}(u - \mathcal{P}_m u, ie) = 0$ (Lemma 6.1), yields

$$(\mathcal{P}_m u_t - u_t^m, e)_\mu + \mathcal{E}(\mathcal{P}_m u - u^m, ie) = (\mathcal{P}_m u_t - u_t, e)_\mu.$$

The desired result follows from the equalities $\mathcal{E}(\mathcal{P}_m u - u^m, ie) = \mathcal{E}(e, ie) = 0$ and $\mathcal{P}_m u_t - u_t^m = e_t$. □

Lemma 6.3. ([3, Lemma 5.3]) *Assume the hypotheses of Lemma 6.1, and let $v \in \operatorname{dom}\mathcal{E}$. Then*

$$\|\mathcal{P}_m v - v\|_\mu \leq 2\|V_m\|^{1/2}\|v\|_{\operatorname{dom}\mathcal{E}} \qquad \text{for all } m \geq 1. \tag{6.5}$$

Theorem 6.4. *Assume the hypotheses of Lemma 6.2, and let ρ be the constant in condition (P1). Then*

$$\|\mathcal{P}_m u - u^m\|_\mu \leq 2\sqrt{T}\rho^{m/2}\|u_t\|_{2,\operatorname{dom}\mathcal{E}}.$$

Proof. The proof is similar to that of [29, Theorem 6.4]; we include it here for completeness. The left side of (6.4) can be rewritten as

$$(e_t, e)_\mu = \frac{1}{2}\frac{d}{dt}\big(\|e\|_\mu^2\big) = \|e\|_\mu \cdot \frac{d}{dt}\big(\|e\|_\mu\big).$$

Thus (6.4) leads to

$$\|e\|_\mu \cdot \big(\|e\|_\mu\big)_t = (\mathcal{P}_m u_t - u_t, e)_\mu \le \|\mathcal{P}_m u_t - u_t\|_\mu \cdot \|e\|_\mu,$$

and hence

$$\big(\|e\|_\mu\big)_t \le \|\mathcal{P}_m u_t - u_t\|_\mu. \tag{6.6}$$

Integrating the left side of (6.6) with respect to τ from 0 to t, we get

$$\int_0^t \frac{d}{d\tau}\big(\|e(\tau)\|_\mu\big)\,d\tau = \|e(t)\|_\mu - \|e(0)\|_\mu = \|e(t)\|_\mu, \tag{6.7}$$

where the fact $e(0) = \mathcal{P}_m u(x,0) - u^m(x,0) = \mathcal{P}_m g(x) - \sum_{\ell=1}^{N(m)-1} g(x_{m,\ell})\phi_\ell(x) = 0$ is used in the last equality. Combining (6.6), (6.7), Lemma 6.3 and Hölder's inequality, we have

$$\begin{aligned}
\|e(t)\|_\mu &\le \int_0^t \|\mathcal{P}_m u_t(\tau) - u_t(\tau)\|_\mu\,d\tau && \text{(by (6.6) and (6.7))}\\
&\le \int_0^T 2\|V_m\|^{1/2}\|u_t\|_{\mathrm{dom}\,\mathcal{E}}\,d\tau && \text{(by Lemma 6.3)}\\
&\le 2\sqrt{T}\|V_m\|^{1/2}\|u_t\|_{2,\mathrm{dom}\,\mathcal{E}} && \text{(by Hölder's inequality)}\\
&\le 2\sqrt{T}\rho^{m/2}\|u_t\|_{2,\mathrm{dom}\,\mathcal{E}},
\end{aligned}$$

proving the desired result. □

Proof of Theorem 1.3. Combining Theorem 6.4 and Lemma 6.3, we have, for each fixed $t \in [0,T]$,

$$\begin{aligned}
\|u^m - u\|_\mu &\le \|u^m - \mathcal{P}_m u\|_\mu + \|\mathcal{P}_m u - u\|_\mu\\
&\le 2\sqrt{T}\rho^{m/2}\|u_t\|_{2,\mathrm{dom}\,\mathcal{E}} + 2\rho^{m/2}\|u\|_{\mathrm{dom}\,\mathcal{E}}\\
&\le 2\big(\sqrt{T}\|u_t\|_{2,\mathrm{dom}\,\mathcal{E}} + \|u\|_{\mathrm{dom}\,\mathcal{E}}\big)\rho^{m/2},
\end{aligned}$$

which completes the proof. □

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