On characteristic numbers of 24 dimensional string manifolds

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Abstract

In this paper, we study the Pontryagin numbers of 24 dimensional String manifolds. In particular, we find representatives of an integral basis of the String cobordism group at dimension 24, based on the work of Mahowald and Hopkins (The structure of 24 dimensional manifolds having normal bundles which lift to $BO[8]$, from "Recent progress in homotopy theory" (D. M. Davis, J. Morava, G. Nishida, W. S. Wilson, N. Yagita, editors), Contemp. Math. **293**, Amer. Math. Soc., Providence, RI, 89-110, 2002), Borel and Hirzebruch (Am J Math 80: 459–538, 1958) and Wall (Ann Math 75:163–198, 1962). This has immediate applications on the divisibility of various characteristic numbers of the manifolds. In particular, we establish the 2-primary divisibilities of the signature and of the modified signature coupling with the integral Wu class of Hopkins and Singer (J Differ Geom 70:329–452, 2005), and also the 3-primary divisibility of the twisted signature. Our results provide potential clues to understand a question of Teichner.

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1 Introduction

Let M be a $4m$ dimensional oriented closed manifold. M is called *Spin* if its second Stiefel–Whitney class vanishes: $\omega_2(M) = 0$. To investigate the geometry and topology of M, it is classical to study its characteristic numbers as cobordism invariants. Among others, there are two important types of characteristic numbers, namely *the twisted A-hat genus* $\widehat{A}(M, E)$ and *the twisted signature* $\mathrm{Sig}(M, E)$ for any given complex bundle E over M (see Appendix A.1 for explicit definitions). For instance, there are the twisted genera coupling with bundles naturally constructed from the tangent bundle TM of M

$$\widehat{A}(M, T^i \otimes \wedge^j \otimes S^k) := \widehat{A}(M, \otimes^i T_{\mathbb{C}}M \otimes \wedge^j (T_{\mathbb{C}}M) \otimes S^k(T_{\mathbb{C}}M)),$$
$$\mathrm{Sig}(M, T^i \otimes \wedge^j \otimes S^k) := \mathrm{Sig}(M, \otimes^i T_{\mathbb{C}}M \otimes \wedge^j (T_{\mathbb{C}}M) \otimes S^k(T_{\mathbb{C}}M)),$$

where $\wedge^j(T_{\mathbb{C}}M)$ and $S^k(T_{\mathbb{C}}M)$ are the j-th exterior and k-th symmetric powers of $T_{\mathbb{C}}M$ respectively.

As a twisted $\widehat{A}$-genus, the famous Witten genus $W(M)$ ([30]; see Appendix A.1 for explicit definition) possesses nice properties especially when M is *String*, that is, half of the first Pontryagin class vanishes: $\frac{p_1(M)}{2} = 0$. For instance, in the String case, the Witten genus $W(M)$ is a modular form of weight $2m$ over $SL(2, \mathbb{Z})$ with integral Fourier expansion ([31]). The homotopy theoretical refinement of the Witten genus on String manifolds leads to the theory of tmf (*topological modular form*) developed by Hopkins and Miller [12]. The String condition is the orientablity condition for this generalized cohomology theory.

String manifolds of dimension 24 are of special interest. For instance, in this dimension, one has (cf. page 85–87 in [11])

$$W(M) = \widehat{A}(M)\bar{\Delta} + \widehat{A}(M, T)\Delta,$$

where $\bar{\Delta} = E_4^3 - 744 \cdot \Delta$ with E_4 being the Eisenstein series of weight 4 and Δ being the modular discriminant of weight 12 (see Sect. 1.5 for definitions). Hirzebruch raised his prize question in [11] that whether there exists a 24 dimensional compact String manifold M such that $W(M) = \bar{\Delta}$ (or equivalently $\widehat{A}(M) = 1$, $\widehat{A}(M, T) = 0$) and the Monster group acts on M as self-diffeomorphisms. The existence of such manifold was confirmed by Mahowald–Hopkins [20]. Indeed, they determined the image of Witten genus at this dimension via *tmf*. However, the part of the question concerning the Monster group is still open.

In this paper, we study the Pontryagin numbers of 24 dimensional String manifolds from the perspective of algebraic topology. Combining the works of Mahowald–Hopkins [20], Borel–Hirzebruch [3] and Wall [29] we find representatives of an integral basis of the String cobordism group at dimension 24. This has immediate applications to the divisibility of various characteristic numbers of the 24 dimensional String manifolds. It also provides potential clue for understanding a question of Teichner (see Sect. 1.5).

1.1 Basis of string cobordism at dimension 24

Let Ω_{24}^{String} be the String cobordism group of dimension 24. By the calculation of Gorbounov–Mahowald [10], it is known that as a group

$$\Omega_{24}^{String} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

In [20] Mohowald–Hopkins determined two out of the four generators as the coimage of the Witten genus. In particular, by homotopy arguments they constructed two 24 dimensional String manifolds with explicit Pontryagin numbers, which we denote by M_1 and M_2 respectively in Sect. 3. It should be emphasized that the geometry of M_1 and M_2 is still mystery, which is crucial to the prize question of Hirzebruch. In Sects. 4 and 5, we construct the remaining two generators M_3 and M_4, and compute their Pontryagin numbers, respectively.

Our main result is that these 4 manifolds, M_1, M_2, M_3 and M_4, represent an integral basis of Ω_{24}^{String}. Indeed, this integral basis realizes a particular basis of all possible integral Pontryagin numbers of String 24-manifolds, consisting of $\widehat{A}(-)$, $\frac{1}{24}\widehat{A}(-, T)$, $\widehat{A}(-, \wedge^2)$, and $\frac{1}{8}\text{Sig}(-)$. Here, for any $M \in \Omega_{24}^{String}$, $\frac{1}{24}\widehat{A}(M, T) \in \mathbb{Z}$ was proved by Mahowald-Hopkins ([20]; also see the discussion in Sect. 1.5), and $\frac{1}{8}\text{Sig}(-) \in \mathbb{Z}$ is showed in Lemma 2.2 (cf. Sect. 7 of [20]). In particular, we completely understand the Pontryagin numbers of String manifolds at dimension 24.

Theorem 1 *The correspondence $\kappa : \Omega_{24}^{String} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ by*

$$\kappa(M) = (\widehat{A}(M), \frac{1}{24}\widehat{A}(M, T), \widehat{A}(M, \Lambda^2), \frac{1}{8}\text{Sig}(M))$$

is an isomorphism of abelian groups. Moreover, there exist two explicitly constructed manifolds $M_3, M_4 \in \ker W$ such that

$$K := \begin{pmatrix} \kappa(M_1) \\ \kappa(M_2) \\ \kappa(M_3) \\ \kappa(M_4) \end{pmatrix}^{\tau} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2^3 \cdot 3^3 \cdot 5 & 2^2 \cdot 3 \cdot 17 \cdot 1069 & -1 & 0 \\ 2^8 \cdot 3 \cdot 61 & 2^8 \cdot 5 \cdot 37 & 2^2 \cdot 7 & 1 \end{pmatrix}.$$

Two notable consequences of Theorem 1 are as follows.

Corollary 2 *For any rational homogeneous polynomial P in the Pontryagin classes with degree 24 and with $P(M) \in \mathbb{Z}$ for all $M \in \Omega_{24}^{String}$, there exist unique integers $a_1, \cdots, a_4$ such that*

$$P(M) = a_1\widehat{A}(M) + \frac{1}{24}a_2\widehat{A}(M, T) + a_3\widehat{A}(M, \wedge^2) + \frac{1}{8}a_4\text{Sig}(M).$$

Corollary 3 *The four manifolds M_i in Theorem 1 form a basis of the group Ω_{24}^{String}.*

1.2 2-primary divisibility of signature

Theorem 1 has strong implications on the characteristic numbers of 24-dimensional String manifolds. The first example concerns the signatures of the manifolds. Indeed, for 24 dimensional String manifold M, it is always true that (Lemma 2.2, or Section 7 of [20] by Mahowald–Hopkins)

$$8 \mid \text{Sig}(M),$$

which is optimal because $\text{Sig}(M_4) = 8$ as indicated in Theorem 1. Nevertheless, if we have more information about the topology of the manifold, say the divisibility of its second Pontryagin class, it can be expected that there is higher divisibility of the signature. To be precise, let m and n be two positive integer. By abuse of notation, we can define a new integer $(\frac{n}{m})$ by

$$\nu_p(\frac{n}{m}) = \begin{cases} \nu_p(n) - \nu_p(m) & \text{if } \nu_p(n) \geq \nu_p(m), \\ 0 & \text{if } \nu_p(n) < \nu_p(m), \end{cases}$$

where $\nu_p(k)$ denotes the exponent of the largest power of the prime p that divides k.

Theorem 4 *Let M be a* 24 *dimensional String manifold. If its* 8*-th Stiefel-Whitney class vanishes:* $\omega_8(M) = 0$, *then*

$$32 \mid \text{Sig}(M).$$

Furthermore, if the second Pontryagin class $p_2(M)$ is divisible by a positive integer n, then

$$\left(\frac{n^3}{2^2 \cdot 3^5 \cdot 5^3 \cdot 41}\right) \mid \text{Sig}(M).$$

Let us remark that it is known that for a String manifold M of any dimension, $6 \mid p_2(M)$ by Borel–Hirzebruch [3] (also see Li–Duan [15]).

1.3 2-primary divisibility of modified signature

Let $BSpin$ be the classifying space of the spinor groups in the stable range. In [13] Hopkins-Singer constructed a universal integral lift $\nu_{4k}^{Spin} \in H^{4k}(BSpin; \mathbb{Z})$ of the mod-2 Wu class with degree $4k$ for each positive integer k, and the total integral Spin Wu class

$$\nu_t^{Spin} = 1 + \nu_4^{Spin} + \nu_8^{Spin} + \nu_{12}^{Spin} + \cdots$$

has the characteristic series of the form

$$g(x) = 1 + \frac{1}{2}x^2 + \frac{11}{8}x^4 + \frac{37}{16}x^6 + \frac{691}{128}x^8 + \frac{2847}{256}x^{10} + \cdots.$$

In term of these classes we define for a Spin $8k$-manifold M the *modified signature* by the formulae

$$\text{Sig}(M, \nu) := \text{Sig}(M) - \langle \nu_{4k}^{Spin}(M) \cup \nu_{4k}^{Spin}(M), [M] \rangle, \quad \nu_{4k}^{Spin}(M) := f^*(\nu_{4k}^{Spin}),$$

where $f : M \to BSpin$ is the classifying map of the normal bundle of M in the stable range. It is a classical result that $\text{Sig}(M, \nu)$ is divisible by 8 for Spin manifolds. For our String manifold M however, we actually can get higher divisibility.

Theorem 5 *Let M be a* 24 *dimensional String manifold. Then*

$$32 \mid \text{Sig}(M, \nu).$$

The divisibility in Theorem 5 is optimal, as we shall see in its proof in Sect. 2 that the manifold $67M_3 + 3M_4 \in \Omega_{24}^{String}$ has the modified signature exactly equal to 32.

1.4 3-primary divisibility of twisted signature

In [4], Chen-Han studied the mod-3 congruence properties of certain twisted signature of 24 dimensional String manifolds. By the techniques of modular forms in index theory, they showed that

$$3 \mid \mathrm{Sig}(M, \wedge^2)$$

for any 24 dimensional String manifold M, and this is the best possible. By Theorem 1, it is easy to give a topological proof of this result by straightforward computation. Indeed, it can be showed that (see Remark 2.3)

$$96 \mid \mathrm{Sig}(M, \wedge^2)$$

in general, while for the generator M_1

$$3^2 \nmid \mathrm{Sig}(M_1, \wedge^2).$$

However, as in Theorem 4, with 3-primary divisibility of the second Pontryagin class $p_2(M)$, we can obtain higher divisibility for the twisted signature $\mathrm{Sig}(M, \wedge^2)$.

Theorem 6 *Let M be a* 24 *dimensional String manifold. If* $3^{k+1} \mid p_2(M)$ *($k \geq 1$), then*

$$3^{3k-1} \mid \mathrm{Sig}(M, \wedge^2).$$

1.5 Discussions on a question of Teichner

The original proof of Mahowald–Hopkins [20] on the fact (observed by Teichner [26]) that

$$24 \mid \widehat{A}(M, T) \tag{1.1}$$

for any 24 dimensional String manifold is of homotopy theoretical argument. It is based on the homotopy theory of Witten genus via tmf. Actually let Ω_{4k}^{String} be the string cobordism group in dimension $4k$. Let $MF_{2k}^{\mathbf{Z}}(SL(2, \mathbf{Z}))$ be the space of modular forms of weight $2k$ over $SL(2, \mathbf{Z})$ with integral Fourier expansion. The Witten genus W is the composition of the maps ([19]):

$$\Omega_{4k}^{String} \xrightarrow{\sigma} tmf^{-4k}(pt) \xrightarrow{e} MF_{2k}^{\mathbf{Z}}(SL(2, \mathbf{Z})) \ ,$$

where σ is the *refined Witten genus* and e is the edge homomorphism in a spectral sequence. Hopkins and Mahowald ([19]) show that σ is surjective. For $i, l \geq 0$, $j = 0, 1$, define

$$a_{i,j,l} = \begin{cases} 1 & i > 0, j = 0 \\ 2 & j = 1 \\ 24/\gcd(24, l) & i, j = 0 \end{cases}.$$

Hopkins and Mahowald also show that the image of e (and therefore the image of the Witten genus) has a basis given by monomials

$$a_{i,j,l} E_4(\tau)^i E_6(\tau)^j \Delta(\tau)^l, \quad i, l \geq 0, j = 0, 1, \tag{1.2}$$

where

$$E_4(\tau) = 1 + 240(q + 9q^2 + 28q^3 + \cdots),$$
$$E_6(\tau) = 1 - 504(q + 33q^2 + 244q^3 + \cdots)$$

are the Eisenstein series and $\Delta(\tau) = q \prod_{n \geq 0}(1 - q^n)^{24}$ is the modular discriminant. Their weights are 4,6, 12 respectively. In dimension 24, the image of the Witten genus is spanned by the monomials $E_4(\tau)^3$, $24\Delta(\tau)$ and since $\widehat{A}(M, T) - 24\widehat{A}(M)$ is the coefficient of q in the expansion of the Witten genus, $\widehat{A}(M, T)$ is divisible by 24. This observation was due to Teichner [26], who consequently raised the following question,

Question 7 Can we give a geometric proof of (1.1)?

Zhang [33] suggested that we may look at the geometry of this divisibility from the index theoretical point of view, that is, to study if we can express $\frac{1}{24}\widehat{A}(M, T)$ as an integral linear combination of indices of twisted Dirac operators or twisted signature operators.

Indeed, we are able to show with the help of computer program that for the generator M_1 of Hopkins–Mahowald, when $i + j + k \leq 5$, one has

$$24 \mid \widehat{A}(M_1, T^i \otimes \wedge^j \otimes S^k),$$
$$24 \mid \text{Sig}(M_1, T^i \otimes \wedge^j \otimes S^k). \tag{1.3}$$

This motivates us to conjecture that

Conjecture 8 *For any non-negative integer i, j and k,*

$$24 \mid \widehat{A}(M_1, T^i \otimes \wedge^j \otimes S^k),$$
$$24 \mid \text{Sig}(M_1, T^i \otimes \wedge^j \otimes S^k). \tag{1.4}$$

If the conjecture is true, then $\frac{1}{24}\widehat{A}(M_1, T)$ can not be written as a linear combination of $\widehat{A}(M_1, T^i \otimes \wedge^j \otimes S^k)$ or $\text{Sig}(M_1, T^i \otimes \wedge^j \otimes S^k)$ with integral coefficients. Otherwise, suppose we have an index formula for $\frac{1}{24}\widehat{A}(M, T)$ of this form, then it follows from (1.4) that $\frac{1}{24}\widehat{A}(M_1, T)$ must be divisible by 24. However, it is equal to -1 by Theorem 1, hence a contradiction. Indeed, from the discussion, if the conjecture is true, then for any $k \geq 2$, $\frac{1}{k}\widehat{A}(M_1, T)$ can not written as a linear combination of $\widehat{A}(M_1, T^i \otimes \wedge^j \otimes S^k)$ or $\text{Sig}(M_1, T^i \otimes \wedge^j \otimes S^k)$ with integral coefficients.

This suggests that if we want to express $\frac{1}{24}\widehat{A}(M_1, T)$ as linear combination of indices of twisted Dirac operators or twisted signatures, one need to look at more types of twistings in addition to the bundles of the form $T^i \otimes \wedge^j \otimes S^k$.

1.6 Organization of the paper

The paper is organized as follows. In Sect. 2, we prove the 4 theorems in the introduction section by the knowledge presented in Sects. 3, 4 and 5. In Sect. 3 we summarize part of the work of Mahowald–Hopkins [20] on the coimage of Witten genus at dimension 24. In particular, we review the Pontryagin numbers of M_1 and M_2. In Sects. 4 and 5 we construct M_3 and M_4, and compute their Pontryagin numbers respectively. For M_4 more explicitly, we apply Wall's classification on $(n - 1)$-connected $2n$-manifolds [29] to construct M_4 as F_4-$\mathbb{O}P^2$-bundle (5.1), and apply the classical Borel–Hirzebruch algorithm [3] to calculate its Pontryagin classes. This is divided into 4 steps in Sect. 5. We end the paper with Appendix A.1 explaining the geometric and analytic aspects of twisted A-hats and twisted signatures together with their definitions.

2 Proof of Theorem 1, 4, 5 and 6

Before proving the main theorems stated in the Introduction, we summarize necessary results whose proofs are postponed to Sects. 3–5.

Theorem 2.1 *There exist four elements $M_i \in \Omega_{24}^{String}$, $1 \leq i \leq 4$, whose Pontryagin numbers and Witten genus are given by the table below:*

M	M_1	M_2	M_3	M_4
p_2^3	$2^{13} \cdot 3^5 \cdot 5^3$	$-2^{13} \cdot 3^5 \cdot 5^3 \cdot 41$	$2^7 \cdot 3^5 \cdot 5$	3888
p_3^2	$2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2$	$2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 31$	0	200
$p_2 p_4$	$2^{12} \cdot 3^5 \cdot 5^3$	$-2^{12} \cdot 3^5 \cdot 5^3 \cdot 41$	$2^5 \cdot 3^3 \cdot 5^3$	2868
p_6	$2^9 \cdot 3^4 \cdot 5^2 \cdot 89$	$-2^9 \cdot 3^4 \cdot 5^2 \cdot 11^2$	$2^5 \cdot 3^3 \cdot 5 \cdot 13$	1958
W	-24Δ	$\overline{\Delta}$	0	0

where $\bar{\Delta} = E_4^3 - 744 \cdot \Delta$ with E_4 the Eisenstein series of weight 4 and Δ the cusp form of weight 12.

Proof The manifolds M_1 and M_2 are constructed and studied by Mahowald–Hopkins [20] from the homotopy theoretical point of view, and their Pontryagin numbers and Witten genera are summarized in Theorem 3.2 and Lemma 3.3. In particular, M_1 and M_2 form a basis of the coimage of the Witten genus at dimension 24. On the other hand, M_3 and M_4 form a basis of the kernel of the Witten genus which are explicitly constructed in Sects. 4 and 5 respectively. Their Pontryagin numbers are summarized in Lemmas 4.2 and 5.12. □

Lemma 2.2 *Let M be a 24 dimensional String manifold. Then*

$$8 \mid Sig(M).$$

The lemma was implicitly proved by Mahowald–Hopkins [20] without statement. Here we give an alternative proof.

Proof Recall that any integral lift of the middle Wu class $v_{12}(M) \in H^{12}(M; \mathbb{Z}/2)$ is a characteristic element for the intersection form $I(M)$ of M over $\mathbb{Z}$. However, since M is String (by Bensen–Wood [1] or Duan [7])

$$0 = q_1(M) \equiv \omega_4(M) \bmod 2,$$

which implies that

$$\omega_6(M) = Sq^2 \omega_4(M) = 0$$

as well. It follows that

$$v_{12}(M) = \omega_6^2(M) = 0.$$

In particular, the trivial cohomology class 0 acts as an integral lift of $v_{12}(M)$ and hence is a characteristic. It follows that the intersection form $I(M)$ is of even type, and

$$Sig(M) \equiv I(M)(0, 0) = 0 \bmod 8.$$

□

Proof of Theorem 1, Corollarys 2 and 3 Recall we have 4 particular String manifolds M_1, M_2, M_3 and M_4 of dimension 24, the Pontryagin numbers of which are given in Theorem 2.1. With this information, it is straightforward to calculate the following 4 particular characteristic numbers of these 4 manifolds.

$$\begin{aligned}
&\widehat{A}(M_2) = 1, \widehat{A}(M_i) = 0, \quad \text{for } i \neq 2, \\
&\widehat{A}(M_1, T) = -24, \widehat{A}(M_i, T) = 0, \quad \text{for } i \neq 1, \\
&\widehat{A}(M_1, \wedge^2) = 1080, \widehat{A}(M_2, \wedge^2) = 218076, \widehat{A}(M_3, \wedge^2) = -1, \widehat{A}(M_4, \wedge^2) = 0, \\
&\text{Sig}(M_1) = 374784, \text{Sig}(M_1) = 378880, \text{Sig}(M_1) = 224, \text{Sig}(M_1) = 8.
\end{aligned} \tag{2.1}$$

In [20], Mahowald–Hopkins showed that for any 24 dimensional String manifold M

$$24 \mid \widehat{A}(M, T) \tag{2.2}$$

(this is also observed by Teichner [26]; cf. the discussions at Page 2961 in [4]). Together with Lemma 2.2, there exists a well defined homomorphism of abelain groups

$$\kappa := (\widehat{A}(-), \frac{1}{24}\widehat{A}(-, T), \widehat{A}(-, \wedge^2), \frac{1}{8}\text{Sig}(-)) : \Omega_{24}^{String} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}. \tag{2.3}$$

The values of κ (2.1) on the 4 manifolds M_i ($1 \leq i \leq 4$) are given by the matrix

$$K = (\kappa(M_1)^\tau, \kappa(M_2)^\tau, \kappa(M_3)^\tau, \kappa(M_4)^\tau) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1080 & 218076 & -1 & 0 \\ 46848 & 47360 & 28 & 1 \end{pmatrix}. \tag{2.4}$$

It is clear that $\det(K) = -1$. In particular, κ is an epimorphsim. On the other hand, by the calculation of Gorbounov–Mahowald [10], it is known that

$$\Omega_{24}^{String} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Hence, κ is indeed an isomorphism, and $\{M_1, M_2, M_3, M_4\}$ is an integral basis of Ω_{24}^{String}. We have showed Theorem 1 and Corollary 3. For Corollary 2, for any rational Pontryagin polynomial $P(-)$ we have

$$P(M) = a_1\widehat{A}(M) + a_2\frac{1}{24}\widehat{A}(M, T) + a_3\widehat{A}(M, \wedge^2) + a_4\frac{1}{8}\text{Sig}(M),$$

with some $a_i \in \mathbb{Q}$. If $P(M) \in \mathbb{Z}$ for any $M \in \Omega_{24}^{String}$. First choose $M = M_4$ and we have $P(M_4) = a_4$ by (2.4). It follows that $a_4 \in \mathbb{Z}$. Then choose $M = M_3$, and we have $P(M_3) = -a_3 + 28a_4$ by (2.4) which implies that $a_3 \in \mathbb{Z}$. Finally, $P(M_2) = a_1 + 218076a_3 + 47360a_4$ implies that $a_1 \in \mathbb{Z}$, while $P(M_1) = -a_2 + 1080a_3 + 46848a_4$ implies that $a_2 \in \mathbb{Z}$. This completes the proof of Corollary 2. □

Proof of Theorem 4 First let us make a comment on the condition $\omega_8(M) = 0$. By Bensen–Wood [1] or Duan [7], it is known that

$$\frac{1}{2}p_1(M) = q_2(M) \equiv \omega_8(M) \mod 2.$$

Hence the condition $\omega_8(M) = 0$ is equivalent to $4 \mid p_2(M)$. To show that $32 \mid \text{Sig}(M)$ under this condition, let us recall from Theorem 2.1 that the characteristic numbers p_2^3 of the basis manifolds $\{M_1, M_2, M_3, M_4\}$ are

$$(p_2^3(M_1), p_2^3(M_2), p_2^3(M_3), p_2^3(M_4)) = (2^{13} \cdot 3^5 \cdot 5^3, -2^{13} \cdot 3^5 \cdot 5^3 \cdot 41, 2^7 \cdot 3^5 \cdot 5, 2^4 \cdot 3^5). \tag{2.5}$$

By Theorem 1, up to String cobrodism $M = \sum_{i=1}^{4} x_i M_i$ for some integral vector $(x_1, x_2, x_3, x_4) \in \mathbb{Z}^{\oplus 4}$. Hence

$$2^6 \mid p_2^3(M) = \sum_{i=1}^{4} x_i p_2^3(M_i),$$

which with (2.5) implies that

$$2^2 \mid x_4. \tag{2.6}$$

On the other hand, from (2.1) we have the signature vector

$$(\mathrm{Sig}(M_1), \mathrm{Sig}(M_2), \mathrm{Sig}(M_3), \mathrm{Sig}(M_4)) = (2^{11} \cdot 3 \cdot 61, \ 2^{11} \cdot 5 \cdot 37, \ 2^5 \cdot 7, \ 2^3). \tag{2.7}$$

Combining it with (2.6), it follows that $2^5 \mid \mathrm{Sig}(M)$. The second statement in the theorem can be proved by the same strategy, and we have completed the proof Theorem 4. □

Proof of Theorem 5 By Hopkins–Singer [13], it can be computed that for 24 dimensional String manifold M, the middle integral Spin Wu class

$$\nu_{12}^{\mathrm{Spin}}(M) = 5p_3(M).$$

Then with (2.7) and Theorem 2.1, it is straightforward to compute the value of the modified signatures of the basis manifolds

$$(\mathrm{Sig}(M_1, v), \mathrm{Sig}(M_2, v), \mathrm{Sig}(M_3, v), \mathrm{Sig}(M_4, v)) =$$
$$(-2^{10} \cdot 3 \cdot 826753, \ -2^{10} \cdot 5 \cdot 23 \cdot 668687, \ 2^5 \cdot 7, \ -2^7 \cdot 3 \cdot 13),$$

and the greatest common divisor

$$\mathrm{g.c.d.}(\mathrm{Sig}(M_1, v), \mathrm{Sig}(M_2, v), \mathrm{Sig}(M_3, v), \mathrm{Sig}(M_4, v)) = 32.$$

The theorem then follows immediately from Theorem 1. Moreover, $\mathrm{Sig}(67M_3 + 3M_4, v) = 32$ by direct computation. This verifies the remark after Theorem 5. □

Proof of Theorem 6 The theorem can be proved by the same strategy used in the proof of Theorem 4 with the value of the twisted signature of the manifolds

$$(\mathrm{Sig}(M_1, \wedge^2), \mathrm{Sig}(M_2, \wedge^2), \mathrm{Sig}(M_3, \wedge^2), \mathrm{Sig}(M_4, \wedge^2)) =$$
$$(2^{13} \cdot 3 \cdot 4013, \ -2^{13} \cdot 3^4 \cdot 1063, \ 2^7 \cdot 3 \cdot 7 \cdot 23, \ 2^5 \cdot 3 \cdot 23), \tag{2.8}$$

which can be computed directly from Theorem 2.1. □

Remark 2.3 Notice by (2.8) we also have

$$96 \mid \mathrm{Sig}(M, \wedge^2)$$

for any 24-dimensional String manifold M. This reproves the result of Chen–Han [4] that $3 \mid \mathrm{Sig}(M, \wedge^2)$ by different methods.

3 M_1 and $M_2 \in \text{Coim}(W)$

In this section, we review the information of two String manifolds M_1 and M_2 of dimension 24 constructed by Mahowald–Hopkins [20]. Let us start with Kervaire–Milnor's almost parallelizable manifolds. In [23] Kervaire–Milnor showed that there is an almost parallelizable manifold M_0^{4n} of dimension $4n$ with the top Pontryagin class

$$p_n(M_0^{4n}) = \text{denom}\left(\frac{B_{2n}}{4n}\right) \cdot a_n \cdot (2n-1)! \cdot x_{4n}, \tag{3.1}$$

where $x_{4n} \in H^{4n}(M_0^{4n})$ is the generator,

$$a_n = \begin{cases} 2 & n = \text{odd} \\ 1 & n = \text{even}, \end{cases}$$

and B_{2n} is the Bernoulli number. Then it is easy to calculate that for M_0^4

$$p_1(M_0^4) = 48x_4, \quad \text{Sig}(M_0^4) = 16, \tag{3.2}$$

for M_0^8

$$p_2(M_0^8) = 1440x_8, \quad \text{Sig}(M_0^8) = 224, \tag{3.3}$$

and for M_0^{12}

$$p_3(M_0^{12}) = 120960x_{12}, \quad \text{Sig}(M_0^{12}) = 7936. \tag{3.4}$$

The following proposition is well known.

Proposition 3.1

$$\Omega_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[M_0^4, M_0^8, M_0^{12}, \cdots].$$

□

From this proposition, Mahowald–Hopkins chose a particular basis for $\Omega_{24}^{SO} \otimes \mathbb{Q}$

$$\begin{aligned}
B_1 &= M_0^8 \times M_0^8 \times M_0^8, \\
B_2 &= \frac{1}{2}M_0^{12} \times \frac{1}{2}M_0^{12}, \\
B_3 &= M_0^8 \times M_0^{16}, \\
B_4 &= \frac{1}{2}M_0^{24}.
\end{aligned} \tag{3.5}$$

They called $\frac{1}{2}M_0^{8k+4}$ a *fake manifold* since it is not a proper manifold. Nevertheless, they showed that there is a proper manifold B_2 with its Pontryagin numbers equal to those of the square of $\frac{1}{2}M_0^{12}$. Among others, in [20] Mahawold–Hopkins determined the image of Witten genus at dimension 24. Recall that at this particular case, there is the famous formula of Hirzebruch (Page 85–87 in [11])

$$W(M) = \widehat{A}(M)\bar{\Delta} + \widehat{A}(M, T)\Delta, \tag{3.6}$$

for any 24 dimensional String manifold M, where

$$\bar{\Delta} = E_4^3 - 744 \cdot \Delta,$$

with E_4 the Eisenstein series of weight 4 and Δ the famous cusp form of weight 12.

Theorem 3.2 (Sect. 9 in *[20]*) *There exist two proper String manifold M_1 and M_2 of dimension* 24, *such that in the rational oriented cobordism ring*

$$M_1 = \frac{B_1 + B_2}{72}, \quad M_2 = \frac{-41B_1 + 31B_2}{72}. \tag{3.7}$$

Furthermore, the image of Witten genus at dimension 24

$$\text{Im}\big\{ W : \Omega_{24}^{String} \to \mathbb{Z}[[q]] \big\} \cong \mathbb{Z}\{M_1, M_2\}, \tag{3.8}$$

with

$$W(M_1) = -24\Delta, \quad W(M_2) = \bar{\Delta}. \tag{3.9}$$

□

Let us summarize the Pontryagin numbers of M_1 and M_2.

Lemma 3.3 *For M_1,*

$$p_2^3 = 2^{13} \cdot 3^5 \cdot 5^3, \quad p_3^2 = 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2, \quad p_2 p_4 = 2^{12} \cdot 3^5 \cdot 5^3, \quad p_6 = 2^9 \cdot 3^4 \cdot 5^2 \cdot 89,$$

and for M_2,

$$p_2^3 = -2^{13} \cdot 3^5 \cdot 5^3 \cdot 41, \quad p_3^2 = 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 31,$$
$$p_2 p_4 = -2^{12} \cdot 3^5 \cdot 5^3 \cdot 41, \quad p_6 = -2^9 \cdot 3^4 \cdot 5^2 \cdot 11^2.$$

□

4 $M_3 \in \text{Ker}(W)$

Since the image of the Witten genus is known, we are left to consider its kernel. There is an outstanding principle to attack it.

Theorem 4.1 (Jung and Dessai; *[6]*) *The ideal of $\Omega_*^{String} \otimes \mathbb{Q}$, consisting of bordism classes of Caley plane bundles with connected structure groups, is precisely the kernel of the rational Witten genus.* □

The local version of the theorem was proved by McTague [21] for the localization of Witten genus away from 6. The simplest Caley plane bundles are, of course, the trivial ones. Let us define

$$M_3 := M_0^8 \times \mathbb{O}P^2, \tag{4.1}$$

where M_0^8 is the almost parallelizable manifold (3.1) of dimension 8, and $\mathbb{O}P^2$ is *the Caley plane or the octonionic projective plane*. The cell structure of $\mathbb{O}P^2$ is clear from its cohomology ring

$$H^*(\mathbb{O}P^2) \cong \mathbb{Z}[u_8]/u_8^3, \tag{4.2}$$

where $\deg(u_8) = 8$. Further, $\mathbb{O}P^2$ is a 16 dimensional manifold with the total Pontryagin class (Theorem 19.4 in [3])

$$p(\mathbb{O}P^2) = 1 + 6u_8 + 39u_8^2. \tag{4.3}$$

Hence, we can compute all the Pontryagin classes of M_3. It is clear that

$$W(M_3) = 0,$$

but there is the particular twisted genus

$$\widehat{A}(M_3, \wedge^2) = -1. \tag{4.4}$$

Let us summarize the Pontryagin numbers of M_3.

Lemma 4.2 *For M_3,*

$$p_2^3 = 2^7 \cdot 3^5 \cdot 5, \quad p_3^2 = 0, \quad p_2 p_4 = 2^5 \cdot 3^3 \cdot 5^3, \quad p_6 = 2^5 \cdot 3^3 \cdot 5 \cdot 13.$$

□

5 $M_4 \in \mathbf{Ker}(W)$

We continue to construct particular String manifolds of dimension 24 in the kernel of Witten genus. In this non-trivial case, we need to construct an appropriate closed 8-manifold N^8 and apply the pullback diagram

$$\begin{array}{ccc}
\mathbb{O}P^2 & =\!=\!=\!=\!= & \mathbb{O}P^2 \\
\downarrow & & \downarrow \\
M^{24} & \xrightarrow{\tilde{f}} & BSpin(9) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Theta} \\
N^8 & \xrightarrow{f} & BF_4,
\end{array} \tag{5.1}$$

where F_4 is the exceptional Lie group, and

$$\mathbb{O}P^2 \to BSpin(9) \xrightarrow{\Theta} BF_4 \tag{5.2}$$

is called *the universal F_4-$\mathbb{O}P^2$-bundle* following Klaus [17]. It exists since $Spin(9)$ is the subgroup of F_4 with the quotient

$$F_4/Spin(9) \cong \mathbb{O}P^2.$$

The pullback bundle π is called *an F_4-$\mathbb{O}P^2$-bundle*, as a generalization of $PSp(3)$-$\mathbb{H}P^2$-*bundles* of Kreck–Stolz [18]. These bundles were studied by Borel–Hirzebruch [3] in general context. In particular, Borel–Hirzebruch [3] developed a theory with associated algorithm to compute the Pontrygin classes of such bundles. In the following, we will first recall the Borel–Hirzebruch algorithm, and then construct an appropriate M_4 step by step.

5.1 Borel–Hirzebruch algorithm

Given any fibre bundle

$$F \to E \xrightarrow{p} B \tag{5.3}$$

with structural group G, and F, E, B are all manifolds. Set $\dim F = n$. There is the induced bundle

$$\mathbb{R}^n \to E \times_G TF \to E, \tag{5.4}$$

where the action of G on the tangent bundle TF is induced from that on F. The bundle (5.4), denoted by p^Δ as in [17], is called *the bundle along the fibre associated to the bundle p* (5.3). In particular, it is easy to see that

$$TE \cong p^\Delta \oplus p^*(TB). \tag{5.5}$$

Now let G be a compact connected Lie group with subgroup H. The principal bundle

$$H \to G \to G/H$$

can be extended twice to the right, and we have the fibre bundle

$$G/H \to BH \xrightarrow{\Theta} BG \tag{5.6}$$

Let S be the maximal torus of H. The inclusion of the maximal torus induces a map of classifying maps

$$\rho : BS \to BH \tag{5.7}$$

Theorem 5.1 (Special version of Theorem *10.7* in [3]; the universal case) *Let $S \le H \le G$ and ρ as above. Denote by*

$$\{\pm b_j\}_{j=1}^k$$

the set of the roots of G with respect to S, which are complementary to those of H (view $b_j \in H^2(BS; \mathbb{Z})$). Then the Pontryagin class of the bundle along the fibre Θ^Δ, associated to the fibre bundle Θ, is determined by

$$\rho^*(p(\Theta^\Delta)) = \prod_{j=1}^k (1 + b_j^2). \tag{5.8}$$

$\square$

5.2 Step 1: compute the Pontryagin class of the bundle along the fibre Θ^Δ

Now let us restrict ourselves to consider the bundle (5.2)

$$\mathbb{O}P^2 \to BSpin(9) \xrightarrow{\Theta} BF_4.$$

Recall that F_4 and $Spin(9)$ are of both rank 4, and there is a maximal torus

$$S \cong T^4 \hookrightarrow Spin(9),$$

which is also the maximal torus of F_4 via the inclusion $Spin(9) \hookrightarrow F_4$. Denote

$$H^2(BS; \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, x_3, x_4].$$

It is known that the roots of $Spin(9)$, with respect to S, are

$$\pm x_i \pm x_j \ (1 \le j < j \le 4), \quad \pm x_1, \pm x_2, \pm x_3, \pm x_4,$$

while the complementary root of F_4 are

$$\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4).$$

Let $r_i = \frac{1}{2}(x_1 \pm x_2 \pm x_3 \pm x_4)$. By Theorem 5.1, we have that

$$\rho^*(p(\Theta^\Delta)) = \prod_{i=1}^{8}(1 + r_i^8), \tag{5.9}$$

where $\rho : BS \to BSpin(9)$. On the other hand, we know that

$$\prod_{i=1}^{4}(1 + x_i^2) = \rho^*(1 + p_1 + p_2 + p_3 + p_4),$$

where $p_i \in H^{4i}(BSpin(9))$ is the i-th Pontryagin class. Hence, by straightforward calculation we obtain the following

Proposition 5.2 *The Pontryagin class of the bundle along the fibre associated to the universal F_4-$\mathbb{O}P^2$-bundle Θ is*

$$\begin{aligned}
p(\Theta^\Delta) = {}& 1 + (2p_1) + \left(-p_2 + \frac{7}{4}p_1^2\right) + \left(2p_3 - \frac{3}{2}p_1 p_2 + \frac{7}{8}p_1^3\right) \\
& + \left(-\frac{17}{2}p_4 + 2p_1 p_3 + \frac{3}{8}p_2^2 - \frac{15}{16}p_1^2 p_2 + \frac{35}{128}p_1^4\right) \\
& + \left(-\frac{5}{2}p_1 p_4 - p_2 p_3 + \frac{3}{4}p_1^2 p_3 + \frac{3}{8}p_1 p_2^2 - \frac{5}{16}p_1^3 p_2 + \frac{7}{128}p_1^5\right) \\
& + \left(-\frac{7}{4}p_2 p_4 + \frac{5}{16}p_1^2 p_4 + p_3^2 - \frac{1}{2}p_1 p_2 p_3 + \frac{1}{8}p_1^3 p_3 - \frac{1}{16}p_2^3\right. \\
& \left. + \frac{9}{64}p_1^2 p_2^2 - \frac{15}{256}p_1^4 p_2 + \frac{7}{1024}p_1^6\right).
\end{aligned} \tag{5.10}$$

$\square$

5.3 Step 2: the appropriate base manifold and classifying map (N^8, f)

At this step, we construct an appropriate Spin manifold N^8 of dimension 8 as the base manifold of the F_4-$\mathbb{O}P^2$-bundle π in (5.1). We need Wall's $(n-1)$-connected $2n$-manifolds with $n = 4$ [29] (also see [7]).

Definition 5.3 Let $A = \{a_{ij}\}_{n \times n}$ be a unimodular symmetric integral matrix of rank n, $b = (b_1, b_2, \cdots, b_n)$ be a sequence of integers of length n. The pair (A, b) is called a *Wall pair* if it satisfies the congruent condition

$$a_{ii} \equiv b_i \bmod 2, \quad 1 \le i \le n. \tag{5.11}$$

It is natural to ask that which Wall pairs (A, b) can be realized as the pair $(I(N^8), q_1(N^8))$ of a Wall manifold N^8; here $I(N^8)$ is the intersection form of N^8 and $q_1(N^8) = \frac{1}{2}p_1(N^8)$ is the first Spin class of N^8 (also see (5.21) and (5.22)).

Theorem 5.4 (Theorem 4 in [29]; also see Theorem 10.11 and 10.13 in [7]) *For any Wall pair (A, b) such that*

$$\mathrm{Sig}(A) \equiv bAb^\tau \bmod 224, \tag{5.12}$$

there exists a smooth manifold N^8 such that under a certain choice of basis of

$$H^4(N^8; \mathbb{Z}) \cong \mathbb{Z}\{x_1, \cdots, x_n\},$$

the intersection form $I(N^8)$ is represented by the matrix A, and the first Spin class $q_1(N^8)$ is represented by b; in other word,

$$I(N^8)(x_i, x_j) = a_{ij}, \quad \text{and } q_1(N_8) = b_1 x_1 + \cdots b_n x_n.$$

Proof The theorem has been proved in [7] based on [29]. Indeed Wall [29] showed that for each Wall pair (A, b) there exists a closed 8 dimensional topological manifold N^8 such that its intersection form is represented by A and its first Spin class is represented by b. Moreover, $M = W \cup_h D^8$ where W is a 3-connected smooth manifold with boundary ∂W a homotopy 7-sphere and $h : \partial W \to \partial D^8$ is a homeomorphism. To show that N^8 is smooth, we may compute the Eells–Kuiper μ-invariant [8] of the boundary ∂W

$$\mu(\partial W) \equiv \frac{b A b^\tau - \text{Sig}(A)}{224} \mod 1.$$

Since μ-invariant is a complete invariant for homotopy 7-spheres, we see that ∂W is diffeomorphic to the standard S^7, and hence N^8 is smooth. $\qquad\square$

We now apply Theorem 5.4 to construct an appropriate N^8 such that after particular pullback f the total space M^{24} in Diagram (5.1) will be a String manifold. For that, we may choose

$$A = \text{diag}(H, E_8), \quad b = (2, 2, 0, \cdots, 0) \tag{5.13}$$

where

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

are the hyperbolic matrix of rank 2 and the Cartan matrix of the exceptional Lie group E_8 respectively. It is then clear that the conditions (5.11) and (5.12) are satisfied, and even better

$$\text{Sig}(A) = b A b^\tau = 8. \tag{5.14}$$

Hence, by Theorem 5.4 there exists a smooth N^8 such that

$$H^4(N^8) \cong \mathbb{Z}\{a_1, a_2, b_1, \cdots b_8\}, \tag{5.15}$$
$$q_1(N^8) = 2(a_1 + a_2), \tag{5.16}$$

and under the basis $\{a_1, a_2, b_1, \cdots b_8\}$ the intersection form of N^8 is represented by A in (5.13). In particular, we can use the Hirzebruch signature formula to calculate the second Pontryagin class of N^8.

Lemma 5.5

$$p(N^8) = 1 + 4(a_1 + a_2) + 56 a_1 a_2.$$

$\qquad\square$

In Diagram (5.1), by Lemma 5.6 below let us choose

$$f : N^8 \to BF_4$$

such that

$$f^*(x_4) = -(a_1 + a_2), \tag{5.17}$$

where $x_4 \in H^4(BF_4)$ is the generator such that (cf. (5.24))

$$\Theta^*(x_4) = q_1 \in H^4(BSpin(9)). \tag{5.18}$$

We notice that by Proposition 5.2

$$p_1(\Theta^\Delta) = 2p_1 = 4q_1 \in H^4(BSpin(9))$$

Hence, by (5.5) we have

$$\begin{aligned}
p_1(M^{24}) &= p_1(\pi^\Delta) + \pi^*(p_1(N^8)) \\
&= \tilde{f}^*(p_1(\Theta^\Delta)) + 4(a_1 + a_2) \\
&= 4\tilde{f}^*(q_1) + 4(a_1 + a_2) \\
&= 4\tilde{f}^* \circ \Theta^*(x_4) + 4(a_1 + a_2) \\
&= 4\pi^* \circ f^*(q_1) + 4(a_1 + a_2) \\
&= 0.
\end{aligned}$$

Hence, M^{24} is a String manifold, and from now on we may denote this particular String manifold by M_4.

Lemma 5.6 *There is a natural isomorphism of sets*

$$[N^8, BF_4] \cong \left[\bigvee_{i=1}^2 S_{a_i}^4 \vee \bigvee_{j=1}^8 S_{b_j}^4, BF_4 \right] \cong \mathbb{Z}^{\oplus 10},$$

where $S_{a_i}^4$ ($i = 1, 2$) and $S_{b_j}^4$ ($1 \leq j \leq 8$) represents the cohomology class a_i and b_j in (5.15) respectively.

Proof By the computation of Mimura [24], it is known that

$$\pi_i(BF_4) = 0, \quad 0 \leq i \leq 8, \text{ and } i \neq 4.$$

Then by applying the functor $[-, BF_4]$ to the cofibre sequence determined the attaching map of N^8, we get an exact sequence. From that the lemma follows easily. □

5.4 Step 3: determine the pullback image of $H^*(BSpin(9))$

In the last step, we have constructed the String manifold M_4 as the total space of the F_4-$\mathbb{O}P^2$-bundle over the particular Wall manifold N^8, via the pullback diagram

$$
\begin{array}{ccc}
\mathbb{O}P^2 & \!=\!=\!=\! & \mathbb{O}P^2 \\
\downarrow & & \downarrow \\
M_4 & \xrightarrow{\ \tilde{f}\ } & BSpin(9) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Theta} \\
N^8 & \xrightarrow{\ f\ } & BF_4,
\end{array}
\tag{5.19}
$$

such that

$$
f^*(x_4) = -(a_1 + a_2).
$$

It is clear that

$$
H^*(M_4) \cong H^*(N)[u_8]/\langle u_8^3 - t a_1 a_2 u_8^2 \rangle, \tag{5.20}
$$

for some $t \in \mathbb{Z}$, and $a_1 a_2 u_8^2 \in H^{24}(M_4)$ is a generator. In order to compute the Pontryagin class of M_4, we need to determine the image of $H^*(BSpin(9))$ under $\tilde{f}^*$.

First, by the computation of Duan [7] it is known that (cf. Thomas [25] and Benson–Wood [1])

$$
H^*(BSpin(9)) \cong \mathbb{Z}[q_1, q_2, q_3, q_4] \oplus \text{(the 2 torsion part)}, \tag{5.21}
$$

where q_i is called the *the i-th universal Spin class* with $\deg(q_i) = 4i$. The Spin classes determine the Pontryagin classes, in which way they illustrate the divisibility of Pontryagin classes of Spin manifolds. In the low dimensions, the conversion formulae are

$$
\begin{aligned}
p_1 &= 2q_1, \\
p_2 &= 2q_2 + q_1^2, \\
p_3 &= q_3, \\
p_4 &= 2q_4 + q_2^2 - 2q_1 q_3.
\end{aligned}
\tag{5.22}
$$

On the other hand, it is also known that

$$
H^*(BF_4) \cong \mathbb{Z}[x_4, x_{12}, x_{16}] \oplus \text{(the torsion part)}, \tag{5.23}
$$

where $\deg(x_i) = i$. Since $\mathbb{O}P^2$ is 7-connected and BF_4 is 3-connected, the fibre bundle

$$
\mathbb{O}P^2 \xrightarrow{i} BSpin(9) \xrightarrow{\Theta} BF_4
$$

is a cofibre sequence up to degree 11, by the dual Blakers-Massey theorem or a simple argument of the Serre spectral sequence. In particular, we have

$$
\Theta^*(x_4) = q_1, \tag{5.24}
$$

and there is an exact sequence

$$
\to 0 = H^7(\mathbb{O}P^2) \to H^8(BF_4) \xrightarrow{\Theta^*} H^8(BSpin(9))
$$

$$
\xrightarrow{i^*} H^8(\mathbb{O}P^2) \to H^9(BF_4) \to H^9(BSpin(9)) = 0.
$$

Since Θ^* maps $H^8(BF_4) \cong \mathbb{Z}\{x_4^2\}$ isomorphically onto $\mathbb{Z}\{q_1^2\} \leq H^8(BSpin(9))$ and $H^9(BF_4) \cong \mathbb{Z}/3$ by Toda [27], the above exact sequence implies the short exact sequence

$$0 \to \mathbb{Z}\{q_2\} \to H^8(\mathbb{O}P^2) \cong \mathbb{Z}\{u_8\} \to \mathbb{Z}/3 \to 0.$$

Hence

$$i^*(q_2) = 3u_8, \tag{5.25}$$

which implies that

$$\tilde{f}^*(q_2) = 3u_8 + ka_1a_2, \tag{5.26}$$

for some $k \in \mathbb{Z}$. In order to determine the image of q_3 and q_4 under $\tilde{f}$, we need to use the Weyl invariants of F_4.

Theorem 5.7 (Borel [2]) *Let G be a compact Lie group with a maximal torus T and Weyl group W_G. The inclusion $T \hookrightarrow G$ induces an isomorphism*

$$H^*(BG; \mathbb{Q}) \cong H^*(BT; \mathbb{Q})^{W_G}.$$

$\square$

We borrow the notations from Step 1. Let

$$\prod_{i=1}^{4}(1 + x_i^2) = 1 + p_1 + p_2 + p_3 + p_4,$$

and

$$r_i = \frac{1}{2}(x_1 \pm x_2 \pm x_3 \pm x_4), \quad 1 \leq i \leq 8.$$

Let

$$I_{2k} = \sum_{i=1}^{4} x_i^{2k} + \sum_{j=1}^{8} r_j^{2k}.$$

It is known that (for instance, see [22], or [28])

$$H^*(BS; \mathbb{Q})^{W_{F_4}} = \mathbb{Q}[I_2, I_6, I_8, I_{12}].$$

From this, it is not hard to show that (Section 19 in [3])

$$H^{\leq 16}(BS; \mathbb{Q})^{W_{F_4}} \cong \mathbb{Q}^{\leq 16}[p_1, -6p_3 + p_1p_2, 12p_4 + p_2^2 - \frac{1}{2}p_1^2p_2]. \tag{5.27}$$

Lemma 5.8

$$\tilde{f}^*(-6p_3 + p_1p_2) = 0, \quad \tilde{f}^*(12p_4 + p_2^2 - \frac{1}{2}p_1^2p_2) = 0.$$

Proof We have the commutative diagram

$$
\begin{array}{ccccc}
H^*(M_4) & \xleftarrow{\tilde{f}} & H^*(BSpin(9)) & \xrightarrow{\rho^*} & H^*(BS)^{W_{Spin(9)}} \\
{\scriptstyle \pi^*}\big\uparrow & & {\scriptstyle \Theta^*}\big\uparrow & & \big\uparrow \\
H^*(N^8) & \xleftarrow{f^*} & H^*(BF_4) & \xrightarrow{\rho^*} & H^*(BS)^{W_{F_4}},
\end{array}
\tag{5.28}
$$

which particularly implies that $\tilde{f}^* \circ \Theta^*(x) = 0$ for any x with $\deg(x) > 8$. Then by Theorem 5.7 and (5.27), the lemma follows easily. $\square$

We can now determine the pullback image of $H^*(BSpin(9))$ through $\tilde{f}$.

Lemma 5.9

$$\begin{aligned}
\tilde{f}^*(q_1) &= -(a_1 + a_2), \\
\tilde{f}^*(q_2) &= 3u_8 + ka_1a_2, \\
\tilde{f}^*(q_3) &= -2(a_1 + a_2)u_8, \\
\tilde{f}^*(q_4) &= -6u_8^2 + 4a_1a_2u_8 - 4ka_1a_2u_8.
\end{aligned}$$

Proof The image of q_1 and q_2 were determined already. For the other two, we only need to use the conversion formulas (5.22) to rewrite the two equalities in Lemma 5.8 in Spin classes, and then solve $\tilde{f}^*(q_3)$ and $\tilde{f}^*(q_4)$ from them directly. □

5.5 Step 4: compute the Pontryagin numbers of M_4

We are now in a position to compute the Pontryagin numbers of M_4. First, let us translate the image of $H^*(BSpin(9))$ under $\tilde{f}$, obtained in Lemma 5.9, in terms of Pontryagin classes by using the conversion formulas (5.22).

Lemma 5.10

$$\begin{aligned}
\tilde{f}^*(p_1) &= -2(a_1 + a_2), \\
\tilde{f}^*(p_2) &= 6u_8 + 2(k + 1)a_1a_2, \\
\tilde{f}^*(p_3) &= -2(a_1 + a_2)u_8, \\
\tilde{f}^*(p_4) &= -3u_8^2 - 2ka_1a_2u_8.
\end{aligned}$$

□

By (5.5), we know that

$$p(M_4) = \pi^*(p(N^8)) \cdot \tilde{f}^*(p(\Theta^\Delta)). \tag{5.29}$$

With Lemma 5.5 for $p(N^8)$, Proposition 5.10 for $p(\Theta^\Delta)$ and Lemma 5.10 for $\tilde{f}$, it is now straightforward to calculate the Pontryagin class of M_4.

Lemma 5.11

$$\begin{aligned}
p(M_4) = 1 &+ (36a_1a_2 - 2ka_1a_2 - 6u_8) - 10(a_1 + a_2)u_8 \\
&+ (-244a_1a_2 + 26ka_1a_2 + 39u_8)u_8 \\
&+ 126(a_1 + a_2)u_8^2 + (1958a_1a_2 + 18ka_1a_2 + 18u_8)u_8^2.
\end{aligned}$$

□

We can now determine the Pontryagin numbers of M_4.

Lemma 5.12 *For M_4,*

$$p_2^3 = 3888, \quad p_3^2 = 200, \quad p_2p_4 = 2868, \quad p_6 = 1958.$$

Proof Recall by (5.20) $u_8^3 = ta_1a_2u_8^2$ and $a_1a_2u_8^2$ is a generator of $H^{24}(M)$. By Lemma 5.11, it is straightforward to calculate that

$$p_2^3 = 3888 - 216(k+t), \quad p_3^2 = 200, \quad p_2p_4 = 2868 - 234(k+t), \quad p_6 = 1958 + 18(k+t). \tag{5.30}$$

Hence, by Hirzebruch's signature theorem, it is easy to calculate that

$$\text{Sig}(M_4) = 8 + \frac{9590138}{70945875}(k + t). \tag{5.31}$$

On the other hand, by a theorem of Chern-Hirzebruch-Serre [5] we know that

$$\text{Sig}(M_4) = \text{Sig}(N^8)\text{Sig}(\mathbb{O}P^2) = 8. \tag{5.32}$$

Hence from (5.31) $k + t = 0$, and the lemma follows from (5.30). □

Appendix A: Twisted *A*-hats and twisted signatures

Let M be a $4m$ dimensional oriented closed smooth manifold. There are two important characteristic numbers, namely *the (twisted) A-hat genus* and *the (twisted) signature*, which are the topological pillars of the Atiyah–Singer index theory.

Equip M with a Riemannian metric g^{TM}. Let ∇^{TM} be the associated Levi-Civita connection on TM and $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM}. ∇^{TM} extends canonically to a Hermitian connection $\nabla^{T_{\mathbf{C}}M}$ on $T_{\mathbf{C}}M = TM \otimes \mathbf{C}$, the complexification of TM.

Let $\widehat{A}(TM, \nabla^{TM})$ be the Hirzebruch $\widehat{A}$-form defined by (cf. [32])

$$\widehat{A}(TM, \nabla^{TM}) = \det^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}\right). \tag{A.1}$$

Let E be a Hermitian vector bundles over M carrying a Hermitian connection ∇^E. Let $R^E = (\nabla^E)^2$ be the curvature of ∇^E. The Chern character form (cf. [32]) is defined as

$$\text{ch}(E, \nabla^E) = \text{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]. \tag{A.2}$$

The $\widehat{A}$-*genus* and the *twisted* $\widehat{A}$-*genus* are defined respectively as

$$\begin{aligned}
\widehat{A}(M) &= \int_M \widehat{A}(TM, \nabla^{TM}), \\
\widehat{A}(M, E) &= \int_M \widehat{A}(TM, \nabla^{TM})\text{ch}(E, \nabla^E).
\end{aligned} \tag{A.3}$$

When M is spin, let $S(TM) = S_+(TM) \oplus S_-(TM)$ denote the bundle of complex spinors associated to the Spin structure. Then $S(TM)$ carries induced Hermitian metric and connection preserving the above $\mathbf{Z}_2$-grading. Let

$$D_{\pm} : \Gamma(S_{\pm}(TM)) \to \Gamma(S_{\mp}(TM))$$

denote the induced Spin Dirac operators (cf. [14]). By the Atiyah-Singer index theorem,

$$\widehat{A}(M) = \text{Ind}(D),$$
$$\widehat{A}(M, E) = \text{Ind}(D \otimes E). \tag{A.4}$$

Let $\widehat{L}(TM, \nabla^{TM})$ be the Hirzebruch characteristic form defined by (cf. [16,32])

$$\widehat{L}(TM, \nabla^{TM}) = \det{}^{1/2} \left(\frac{\frac{\sqrt{-1}}{2\pi} R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{4\pi} R^{TM}\right)} \right). \tag{A.5}$$

Note that $\widehat{L}(TM, \nabla^{TM})$ defined here is different from the classical Hirzebruch L-form defined by

$$L(TM, \nabla^{TM}) = \det{}^{1/2} \left(\frac{\frac{\sqrt{-1}}{2\pi} R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{2\pi} R^{TM}\right)} \right).$$

However they give same top (degree $4m$) forms and therefore

$$\int_M \widehat{L}(TM, \nabla^{TM}) = \int_M L(TM, \nabla^{TM}). \tag{A.6}$$

We would also like to point out that our $\widehat{L}$ is different from the $\widehat{\mathbf{L}}$ in page 233 of [14].

Let $\text{ch}(E, \nabla^E) = \sum_{i=0}^{2m} \text{ch}^i(E, \nabla^E)$ such that $\text{ch}^i(E, \nabla^E)$ is the degree $2i$ component. Define

$$\text{ch}_2(E, \nabla^E) = \sum_{i=0}^{2m} 2^i \text{ch}^i(E, \nabla^E). \tag{A.7}$$

It's not hard to see that

$$\int_M \widehat{L}(TM, \nabla^{TM}) \text{ch}(E, \nabla^E) = \int_M L(TM, \nabla^{TM}) \text{ch}_2(E, \nabla^E). \tag{A.8}$$

Let $\Lambda_{\mathbb{C}}(T^*M)$ be the complexified exterior algebra bundle of TM. Let $\langle \, , \, \rangle_{\Lambda_{\mathbb{C}}(T^*M)}$ be the Hermitian metric on $\Lambda_{\mathbb{C}}(T^*M)$ induced by g^{TM}. Let dv be the Riemannian volume form associated to g^{TM}. Then $\Gamma(M, \Lambda_{\mathbb{C}}(T^*M))$ has a Hermitian metric such that for $\alpha, \alpha' \in \Gamma(M, \Lambda_{\mathbb{C}}(T^*M))$,

$$\langle \alpha, \alpha' \rangle = \int_M \langle \alpha, \alpha' \rangle_{\Lambda_{\mathbb{C}}(T^*M)} \, dv.$$

For $X \in TM$, let $c(X)$ be the Clifford action on $\Lambda_{\mathbb{C}}(T^*M)$ defined by $c(X) = X^* - i_X$, where $X^* \in T^*M$ corresponds to X via g^{TM}. Let $\{e_1, e_2, \cdots, e_{2n}\}$ be an oriented orthogonal basis of TM. Set

$$\Omega = (\sqrt{-1})^n c(e_1) \cdots c(e_{2n}).$$

Then one can show that Ω is independent of the choice of the orthonormal basis and $\Omega_E = \Omega \otimes 1$ is a self-adjoint operator on $\Lambda_{\mathbb{C}}(T^*M) \otimes E$ such that $\Omega_E^2 = \text{Id}|_{\Lambda_{\mathbb{C}}(T^*M) \otimes E}$.

Let d be the exterior differentiation operator and d^* be the formal adjoint of d with respect to the Hermitian metric. The operator

$$D_{Sig} := d + d^* = \sum_{i=1}^{2n} c(e_i) \nabla_{e_i}^{\Lambda_{\mathbb{C}}(T^*M)} : \Gamma(M, \Lambda_{\mathbb{C}}(T^*M)) \to \Gamma(M, \Lambda_{\mathbb{C}}(T^*M))$$

is the signature operator and the more general twisted signature operator is defined as (cf. [9])

$$D_{Sig} \otimes E := \sum_{i=1}^{2n} c(e_i) \nabla_{e_i}^{\Lambda_{\mathbb{C}}(T^*M) \otimes E} : \Gamma(M, \Lambda_{\mathbb{C}}(T^*M) \otimes E) \to \Gamma(M, \Lambda_{\mathbb{C}}(T^*M) \otimes E).$$

The operators $D_{Sig} \otimes E$ and Ω_E are anti-commutative. If we decompose $\Lambda_{\mathbb{C}}(T^*M) \otimes E = \Lambda_{\mathbb{C}}^+(T^*M) \otimes E \oplus \Lambda_{\mathbb{C}}^-(T^*M) \otimes E$ into ± 1 eigenspaces of Ω_E, then $D_{Sig} \otimes E$ decomposes to define

$$(D_{Sig} \otimes E)^{\pm} : \Gamma(M, \Lambda_{\mathbb{C}}^{\pm}(T^*M) \otimes E) \to \Gamma(M, \Lambda_{\mathbb{C}}^{\mp}(T^*M) \otimes E). \tag{A.9}$$

The *twisted signature* of M is defined as the index of the operator $(D_{Sig} \otimes E)^+$ denoted by $\mathrm{Sig}(M, E)$,

$$\mathrm{Sig}(M, E) = \mathrm{Ind}((D_{Sig} \otimes E)^+). \tag{A.10}$$

By the Atiyah-Singer index theorem,

$$\mathrm{Sig}(M, E) = \int_M \widehat{L}(TM, \nabla^{TM}) \mathrm{ch}(E, \nabla^E).$$

Note that in the book [14] (Theorem 13.9), the following formula is given

$$\mathrm{Sig}(M, E) = \int_M L(TM, \nabla^{TM}) \mathrm{ch}_2(E, \nabla^E).$$

There is an important twisted $\widehat{A}$-genus, namely the Witten genus [30] by coupling $\widehat{A}(M)$ with *the Witten bundle* [30]

$$\Theta(T_{\mathbb{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^{2n}}(\widetilde{T_{\mathbb{C}}M}), \quad \text{with} \quad \widetilde{T_{\mathbb{C}}M} = TM \otimes \mathbb{C} - \mathbb{C}^{4m}.$$

The Witten genus then can defined as

$$W(M) = \left\langle \widehat{A}(TM) \mathrm{ch}(\Theta(T_{\mathbb{C}}M)), [M] \right\rangle.$$

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