HOLONOMIC SYSTEMS FOR PERIOD MAPPINGS

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Dedicated to Professor Shing-Tung Yau on the occasion of his 65th birthday.

Abstract. Period mappings were introduced in the sixties [G] to study variation of complex structures of families of algebraic varieties. The theory of tautological systems was introduced recently [LSY][LY] to understand period integrals of algebraic manifolds. In this paper, we give an explicit construction of a tautological system for each component of a period mapping.

Contents

1. Setup

1.1. **Tautological systems.** We will follow notations in [HLZ]. Let G be a connected algebraic group over $\mathbb{C}$. Let X be a complex projective G-variety and let $\mathcal{L}$ be a very ample G-bundle over X which gives rise to a G-equivariant embedding

$$X \to \mathbb{P}(V),$$

where $V = \Gamma(X, \mathcal{L})^\vee$. Let $n = \dim V$. We assume that the action of G on X is locally effective, i.e. ker $(G \to \text{Aut }(X))$ is finite. Let $\hat{G} := G \times \mathbb{C}^\times$, whose Lie algebra is $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}e$, where e acts on V by identity. We denote by $Z : \hat{G} \to \text{GL}(V)$ the corresponding group representation, and by $Z : \hat{\mathfrak{g}} \to \text{End }(V)$ the corresponding Lie algebra representation. Note that under our assumption, $Z : \hat{\mathfrak{g}} \to \text{End }(V)$ is injective.

Let $\hat{\iota} : \hat{X} \subset V$ be the cone of X, defined by the ideal $I(\hat{X})$. Let $\beta : \hat{\mathfrak{g}} \to \mathbb{C}$ be a Lie algebra homomorphism. Then a *tautological system* as defined in [LSY][LY] is the cyclic D-module on $V^\vee$

$$\tau(G, X, \mathcal{L}, \beta) = D_{V^\vee}/D_{V^\vee}J(\hat{X}) + D_{V^\vee}(Z(x) + \beta(x), x \in \hat{\mathfrak{g}}),$$

where

$$J(\hat{X}) = \{\widehat{P} \mid P \in I(\hat{X})\}$$

is the ideal of the commutative subalgebra $\mathbb{C}[\partial] \subset D_{V^\vee}$ obtained by the Fourier transform of $I(\hat{X})$. Here $\widehat{P}$ denotes the Fourier transform of P.

Given a basis $\{a_1, \ldots, a_n\}$ of V, we have $Z(x) = \sum_{ij} x_{ij} a_i \frac{\partial}{\partial a_j}$, where (x_{ij}) is the matrix representing x in the basis. Since the a_i are also linear coordinates on $V^\vee$, we can view $Z(x) \in \mathrm{Der}\, \mathbb{C}[V^\vee] \subset D_{V^\vee}$. In particular, the identity operator $Z(e) \in \mathrm{End}\ V$ becomes the Euler vector field on $V^\vee$.

Let X be a d-dimensional compact complex manifold such that its anti-canonical line bundle $\mathcal{L} := \omega_X^{-1}$ is very ample. We shall regard the basis elements a_i of $V = \Gamma(X, \omega_X^{-1})^\vee$ as linear coordinates on $V^\vee$. Let $B := \Gamma(X, \omega_X^{-1})_{sm} \subset V^\vee$, which is Zariski open.

Let $\pi : \mathcal{Y} \to B$ be the family of smooth CY hyperplane sections $Y_a \subset X$, and let $\mathbb{H}^{\mathrm{top}}$ be the Hodge bundle over B whose fiber at $a \in B$ is the line $\Gamma(Y_a, \omega_{Y_a}) \subset H^{d-1}(Y_a)$. In [LY] the period integrals of this family are constructed by giving a canonical trivialization of $\mathbb{H}^{\mathrm{top}}$. Let Π be the period sheaf of this family, i.e. the locally constant sheaf generated by the period integrals.

Let G be a connected algebraic group acting on X.

Theorem 1.1 (See [LY]). *The period integrals of the family $\pi : \mathcal{Y} \to B$ are solutions to*

$$\tau \equiv \tau(G, X, \omega_X^{-1}, \beta_0)$$

where β_0 is the Lie algebra homomorphism with $\beta_0(\mathfrak{g}) = 0$ and $\beta_0(e) = 1$.

In [LSY] and [LY], it is shown that if G acts on X by finitely many orbits, then τ is regular holonomic.

1.2. **Period mapping.** Now we consider the period mapping of the family $\mathcal{Y}$:

$$\mathcal{P}^{p,k} : B \to \mathrm{Gr}(b^{p,k}, H^k(Y_{a_0}, \mathbb{C}))/\Gamma$$
$$a \mapsto [F^p H^k(Y_a, \mathbb{C})] \subset [H^k(Y_a, \mathbb{C})] \cong [H^k(Y_{a_0}, \mathbb{C})]$$

where $a_0 \in B$ is a fixed base point,

$$b^{p,k} := \dim F^p H^k(Y_a, \mathbb{C}) = \dim F^p H^k(Y_{a_0}, \mathbb{C}),$$

and Γ is the monodromy group acting on $\mathrm{Gr}(b^{p,k}, H^k(Y_{a_0}, \mathbb{C}))$.

Consider the local system $R^k\pi_*\mathbb{C}$, its stalk at $a \in B$ is $H^k(Y_a, \mathbb{C})$. The Gauss-Manin connection on the vector bundle $\mathcal{H}^k = R^k\pi_*\mathbb{C} \otimes \mathcal{O}_B$ has the property that

$$\nabla F^p \mathcal{H}^k \subset F^{p-1}\mathcal{H}^k \otimes \Omega_B.$$

If we choose a vector field $v \in \Gamma(B, TB)$, then $\nabla_v F^p \mathcal{H}^k \subset F^{p-1}\mathcal{H}^k$, i.e. $\nabla_v F^p H^k(Y_a, \mathbb{C}) \subset F^{p-1} H^k(Y_a, \mathbb{C})$.

Throughout this paper we shall consider the case $k = d-1$.

$\mathbb{H}^{\mathrm{top}}$ is the bundle on B whose fiber is

$$H^0(Y_a, \Omega^{d-1}) = H^{d-1,0}(Y_a) = F^{d-1}H^{d-1}(Y_a, \mathbb{C}).$$

Thus $\mathbb{H}^{\mathrm{top}} = F^{d-1}\mathcal{H}^{d-1}$. Theorem 1.1 tells us that its integral over a $(d-1)$-cycle on each fiber Y_{a_0} is governed by the tautological system τ.

We shall describe the Gauss-Manin connection explicitly below.

Let $(K_X^{-1})^\times$ denote the complement of the zero section in the total space K_X^{-1} of $\mathcal{L}$. Consider the principal $\mathbb{C}^\times$-bundle $(K_X^{-1})^\times \to X$ (with right action). Then there is a natural one-to-one correspondence between sections of ω_X^{-1} and $\mathbb{C}^\times$-equivariant morphisms $f : (K_X^{-1})^\times \to \mathbb{C}$, i.e. $f(m \cdot h^{-1}) = hf(m)$. We shall write f_a the function that represents the section a. Since $(K_X^{-1})^\times$ is a CY bundle over X by [LY], it admits a global non-vanishing top form $\hat{\Omega}$. Let x_0 be the vector field generated by $1 \in \mathbb{C} = \mathrm{Lie}(\mathbb{C}^\times)$. Then $\Omega := \iota_{x_0}\hat{\Omega}$ is a G-invariant $\mathbb{C}^\times$-horizontal form of degree d on $(K_X^{-1})^\times$. Moreover, since $\frac{\Omega}{f_a}$ is $G \times \mathbb{C}^\times$-invariant, it defines a family of meromorphic top form on X with pole along $V(f_a)$ [LY, Thm. 6.3].

Let $a_i^* \in \Gamma(X, \omega_X^{-1})$ be the dual basis of a_i. Let $f = \sum a_i a_i^*$ be the universal section of $V^\vee \times X \to K_X^{-1}$. Let $V(f)$ be the universal family of hyperplane sections, where $V(f_a) = Y_a$ is the zero locus of the section $f_a \equiv a \in V^\vee$. Let $U := V^\vee \times X - V(f)$ and $U_a = X - V(f_a)$. Let $\pi^\vee : U \to V^\vee$ denote the projection.

Let $\hat{\pi}^\vee : U \to B$ be the restriction of $\pi^\vee$ to B. Then there is a vector bundle $\hat{\mathcal{H}}^d := R^d(\hat{\pi}^\vee)_*\mathbb{C} \otimes \mathcal{O}_B$ whose fiber is $H^d(X - V(f_a))$ at $a \in B$. Then $\frac{\Omega}{f}$ is a global section of $F^d\hat{\mathcal{H}}^d$. And the Gauss-Manin connection on $\frac{\Omega}{f}$ is

$$\nabla_{\partial a_i} \frac{\Omega}{f} = \frac{\partial}{\partial a_i}\frac{\Omega}{f},$$

where $\partial_{a_i} := \frac{\partial}{\partial a_i}, i = 1, \ldots, n$.

Consider the residue map $\mathrm{Res} : H^d(X - V(f_a)) \to H^{d-1}(Y_a, \mathbb{C})$, it is shown in [LY] that $\mathrm{Res}\,\frac{\Omega}{f} \in \Gamma(B, \mathcal{H}^{d-1})$ is a canonical global trivialization of $\mathbb{H}^{\mathrm{top}} = F^{d-1}\mathcal{H}^{d-1}$. And similarly we have

$$\nabla_{\partial a_i} \mathrm{Res}\,\frac{\Omega}{f} = \frac{\partial}{\partial a_i}\mathrm{Res}\,\frac{\Omega}{f} = \mathrm{Res}\,\frac{\partial}{\partial a_i}\frac{\Omega}{f}.$$

In [HLZ, Coro. 2.2 and Lemma 2.6], it is shown that

Theorem 1.2. *If $\beta(\mathfrak{g}) = 0$ and $\beta(e) = 1$, there is a canonical surjective map*

$$\tau \twoheadrightarrow H^0\pi_+^\vee \mathcal{O}_U, \quad 1 \mapsto \frac{\Omega}{f}.$$

We now want to give an explicit description of each step of the Hodge filtration.

Let X be a projective variety of dimension d and Y a smooth hypersurface. We make the following hypothesis:

(*) For every $i > 0,\ k > 0, j \geq 0,$ we have

$$H^i(X,\Omega_X^j(kY)) = 0,$$

where $\Omega_X^j(kY) = \Omega_X^j \otimes \mathcal{O}_X(Y)^{\otimes k}$.

Theorem 1.3 (Griffiths)**.** *Let X be a projective variety and Y a smooth hypersurface. Assume (*) holds. Then for every integer p between 1 and d, the image of the natural map*

$$H^0(X,\Omega_X^d(pY)) \to H^d(X-Y,\mathbb{C})$$

which to a section α (viewed as a meromorphic form on X of degree d, holomorphic on $X-Y$ and having a pole of order less than or equal to p along Y) associates its de Rham cohomology class, is equal to $F^{d-p+1}H^d(X-Y)$ (See [Vo, II, p 160].)

Corollary 1.4. *Assume (*) holds for smooth CY hypersurfaces $Y_a \subset X$. Then the de Rham classes of*

$$\{\nabla_{\partial a_{\iota_1}} \cdots \nabla_{\partial a_{\iota_{p-1}}} \frac{\Omega}{f_a}\}_{1\leq \iota_1,\ldots,\iota_{p-1}\leq n}$$

generate the filtration $F^{d-p+1}H^d(U_a)$ for $1 \leq p \leq d$.

Proof. By our assumption X is projective and $\mathcal{O}_X(Y_a) = \omega_X^{-1}$ is very ample. By Theorem 1.3 it is sufficient to show that

$$\mathbb{C}\{\nabla_{\partial a_{\iota_1}} \cdots \nabla_{\partial a_{\iota_{p-1}}} \frac{\Omega}{f_a}\}_{1\leq \iota_1,\ldots,\iota_{p-1}\leq n} = H^0(X,\Omega_X^d(pY_a)) \tag{1.1}$$

Since Y_a are CY hypersurfaces, there is an isomorphism

$$\mathcal{O}_X \simeq \Omega_X^d(Y_a), \quad 1 \leftrightarrow \frac{\Omega}{f_a}.$$

Let $\mathcal{M}(pY_a)$ be the sheaf of meromorphic functions with a pole along Y_a of order less than or equal to p. Then there is an isomorphism

$$\mathcal{O}_X(Y_a)^{p-1} \simeq \mathcal{M}((p-1)Y_a), \quad g \leftrightarrow \frac{g}{f_a^{p-1}}.$$

Thus we have an isomorphism

$$\mathcal{O}_X \otimes \mathcal{O}_X(Y_a)^{p-1} \simeq \Omega_X^d(Y_a) \times \mathcal{M}((p-1)Y_a),\ 1\otimes g \leftrightarrow \frac{\Omega}{f_a}\frac{g}{f_a^{p-1}}.$$

Since

$$\Omega_X^d(pY_a) := \Omega_X^d \otimes \mathcal{O}_X(Y_a)^p \equiv \mathcal{O}_X(Y_a)^{p-1},$$

we have

$$H^0(X, \Omega^d_X(pY_a)) = \{g\frac{\Omega}{f_a^p} \mid g \in H^0(X, \mathcal{O}_X(Y_a)^{p-1}).\}$$

For $p = 1$, the statement is clearly true.

For $p = 2$, $f = \sum_i a_i a_i^*$, $\nabla_{\partial a_i} \frac{\Omega}{f_a} = \frac{\partial}{\partial a_i}\frac{\Omega}{f_a} = -\frac{a_i^*\Omega}{f_a^2}$. Since $\{\frac{a_i^*\Omega}{f_a^2} \mid a_i^* \in H^0(X, \omega_X^{-1})\} = H^0(X, \Omega^d_X(2Y_a))$, (1.1) is true.

Claim 1.5. *For a very ample line bundle L over X,*

$$H^0(X, L^k) \otimes H^0(X, L^l) \to H^0(X, L^{k+l})$$

is surjective.

Proof of claim. Since L is very ample, let $V := H^0(X, L)^\vee$, it follows that $L = \mathcal{O}_{\mathbb{P}V}(1)|_X$. Since restriction commutes with tensor product, $L^k = \mathcal{O}_{\mathbb{P}V}(k)|_X$. And since $H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(k)) \otimes H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(l)) \to H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(k+l))$ is surjective, it follows that $H^0(X, L^k) \otimes H^0(X, L^l) \to H^0(X, L^{k+l})$ is surjective. □

For $p = 3$, $\nabla_{\partial a_i}\nabla_{\partial a_j}\frac{\Omega}{f_a} = \frac{2a_i^*a_j^*\Omega}{f_a^3}$. Since $\mathcal{L} := \omega_X^{-1}$ is very ample, $H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}^2)$ is onto. So $\{a_i^*a_j^*\}_{1\le i,j\le n}$ generate $H^0(X, \mathcal{L}^2)$. Thus

$$\mathbb{C}\{\frac{a_i^*a_j^*\Omega}{f_a^3}\}_{1\le i,j\le n} = H^0(X, \Omega^d_X(3Y_a)),$$

and therefore (1.1) holds.

Similarly, since $H^0(X, \mathcal{L}^{p-2}) \otimes H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}^{p-1})$ is surjective, by induction $a^*_{\iota_1}\cdots a^*_{\iota_{p-1}}$ generate $H^0(X, \mathcal{L}^{p-1})$. Therefore the de Rham classes of

$$\left\{\nabla_{\partial_{a_{\iota_1}}}\cdots\nabla_{\partial_{a_{\iota_{p-1}}}}\frac{\Omega}{f_a} = (-1)^{p-1}\frac{(p-1)!a^*_{\iota_1}\cdots a^*_{\iota_{p-1}}\Omega}{f_a^p}\right\}_{1\le\iota_1,\ldots,\iota_{p-1}\le n}$$

generate $F^{d-p+1}H^d(U_a)$. □

Corollary 1.6. *Assume (*) holds for smooth CY hypersurfaces $Y_a \subset X$. Then the de Rham classes of*

$$\{\nabla_{\partial_{a_{\iota_1}}}\cdots\nabla_{\partial_{a_{\iota_{p-1}}}} Res\,\frac{\Omega}{f_a}\}_{1\le\iota_1,\ldots,\iota_{p-1}\le n}$$

generate the filtration $F^{d-p}H^{d-1}_{van}(Y_a)$ for $1 \le p \le d$.

Proof. Consider the exact sequence

$$0 \to H^d_{\text{prim}}(X) \to H^d(X - V(f_a)) \xrightarrow{\text{Res}} H^{d-1}_{\text{van}}(Y_a, \mathbb{C}) \to 0,$$

it shows that $H^d(X - V(f_a))$ is mapped surjectively onto the vanishing cohomology of $H^{d-1}(Y_a, \mathbb{C})$ under the residue map. Since the residue map preserves the Hodge filtration, by Corollary 1.4 the result follows. □

The goal of this paper is to construct a regular holonomic differential system that governs the p-th derivative of period integrals for each $p \in \mathbb{Z}$. By the preceding corollary, this provides a differential system for "each step" of the period mapping of the family $\mathcal{Y}$.

2. Scalar system for first derivative

We shall use τ to denote interchangeably both the D-module and its left defining ideal. Let $P(\zeta) \in I(\hat{X})$ where $\zeta \in V$, then its Fourier transform $\widehat{P} = P(\partial_a)$, $a \in V^\vee$. Then the tautological system $\tau = \tau(G, X, \omega_X^{-1}, \beta_0)$ for $\Pi_\gamma(a)$ becomes the following system of differential equations:

$$\begin{cases} P(\partial_a)\phi(a) = 0 & \text{(2.1a)} \\ (\sum_{i,j} x_{ij} a_i \frac{\partial}{\partial a_j})\phi(a) = 0 & \text{(2.1b)} \\ (\sum_i a_i \frac{\partial}{\partial a_i} + 1)\phi(a) = 0. & \text{(2.1c)} \end{cases}$$

Let $\omega_a :=$ Res $\frac{\Omega}{f_a}$ for $a \in B$. Since the topology of Y_a doesn't change, we choose a $(d-1)$-cycle in $H^{d-1}(Y_{a_0}, \mathbb{C})$ for some $a_0 \in B$. Then the period integral becomes $\Pi_\gamma(a) = \int_\gamma \omega_a$. Then by Theorem 1.1, $\Pi_\gamma(a)$ are solutions of τ. By [BHLSY] and [HLZ], if X is a projective homogeneous space, then τ is complete, meaning that the solution sheaf agrees with the period sheaf.

The goal of this section is to write a system of scalar valued partial differential equations whose solution contains all the information of first order partial derivatives of period integrals.

2.1. **Vector valued system.**

Taking derivatives of equations in τ gives us a vector valued system of differential equations that involve all first order derivatives of period integrals.

Let $\phi_k(a) := \frac{\partial}{\partial a_k}\phi(a)$, $1 \le k \le n$.
From equation (2.1a) we have

$$\frac{\partial}{\partial a_k} p(\partial_a)\phi(a) = P(\partial_a)\frac{\partial}{\partial a_k}\phi(a) = P(\partial_a)\phi_k(a) = 0.$$

From equation (2.1b) we have

$$\begin{aligned}\frac{\partial}{\partial a_k}(\sum_{i,j} x_{ij} a_i \frac{\partial}{\partial a_j})\phi(a) &= (\sum_{i,j} x_{ij} a_i \frac{\partial}{\partial a_j}\frac{\partial}{\partial a_k})\phi(a) + (\sum_j x_{kj}\frac{\partial}{\partial a_j})\phi(a) \\ &= (\sum_{i,j} x_{ij} a_i \frac{\partial}{\partial a_j})\phi_k(a) + \sum_j x_{kj}\phi_j(a) = 0.\end{aligned}$$

From equation (2.1c) we have

$$\frac{\partial}{\partial a_k}(\sum_i a_i \frac{\partial}{\partial a_i} + 1)\phi(a) = (\sum_i a_i \frac{\partial}{\partial a_i}\frac{\partial}{\partial a_k} + \frac{\partial}{\partial a_k} + \frac{\partial}{\partial a_k})\phi(a)$$
$$= (\sum_i a_i \frac{\partial}{\partial a_i} + 2)\phi_k(a) = 0$$

We also have a relation between derivatives $\frac{\partial}{\partial a_i}\frac{\partial}{\partial a_j}\phi(a) = \frac{\partial}{\partial a_j}\frac{\partial}{\partial a_i}\phi(a)$, which implies

$$\frac{\partial}{\partial a_i}\phi_j(a) = \frac{\partial}{\partial a_j}\phi_i(a).$$

Then we get a system of differential equations whose solutions are vector valued of the form $(\phi_1(a), \dots, \phi_n(a))$ as follows:

$$\begin{cases} P(\partial_a)\phi_k(a) = 0, \ \forall 1 \le k \le n & (2.2a) \\ (\sum_{i,j} x_{ij} a_i \frac{\partial}{\partial a_j})\phi_k(a) + \sum_j x_{kj}\phi_j(a) = 0, \ \forall 1 \le k \le n & (2.2b) \\ (\sum_i a_i \frac{\partial}{\partial a_i} + 2)\phi_k(a) = 0, \ \forall 1 \le k \le n & (2.2c) \\ \frac{\partial}{\partial a_i}\phi_j(a) = \frac{\partial}{\partial a_j}\phi_i(a), \ \forall 1 \le i, j \le n. & (2.2d) \end{cases}$$

Then by Theorem 1.1, $\left(\frac{\partial}{\partial a_1}\Pi_\gamma(a), \dots, \frac{\partial}{\partial a_n}\Pi_\gamma(a)\right)$ are solutions to system (2.2).

2.2. **Scalar valued system.** Now let $b_1, \dots, b_n$ be another copy of the basis of $V^\vee$. We now construct, by an elementary way, a system of differential equations over $V^\vee \times V^\vee$ that is equivalent to (2.2), but whose solutions are function germs on $V^\vee \times V^\vee$. Consider the system

$$\begin{cases} P(\partial_a)\phi(a,b) = 0 & (2.3a) \\ (\sum_{i,j} x_{ij} a_i \frac{\partial}{\partial a_j} + \sum_{i,j} x_{ij} b_i \frac{\partial}{\partial b_j})\phi(a,b) = 0 & (2.3b) \\ (\sum_i a_i \frac{\partial}{\partial a_i} + 2)\phi(a,b) = 0 & (2.3c) \\ \frac{\partial}{\partial a_i}\frac{\partial}{\partial b_j}\phi(a,b) = \frac{\partial}{\partial a_j}\frac{\partial}{\partial b_i}\phi(a,b), \ \forall 1 \le i, j \le n & (2.3d) \\ \frac{\partial}{\partial b_i}\frac{\partial}{\partial b_j}\phi(a,b) = 0, \ \forall 1 \le i, j \le n & (2.3e) \\ (\sum_i b_i \frac{\partial}{\partial b_i} - 1)\phi(a,b) = 0. & (2.3f) \end{cases}$$

Theorem 2.1. *By setting $\phi(a,b) = \sum_k b_k\phi_k(a)$ and $\phi_k(a) = \frac{\partial}{\partial b_k}\phi(a,b)$, the systems (2.2) and (2.3) are equivalent.*

Proof. First we show that if $(\phi_1(a),\dots,\phi_n(a))$ is a solution to system (2.2), let $\phi(a,b) = \sum_k b_k\phi_k(a)$, then $\phi(a,b)$ is a solution to system (2.3).

Since $\phi(a,b) = \sum_k b_k\phi_k(a)$, equation (2.2a) implies that

$$P(\partial_a)\phi(a,b) = P(\partial_a)\sum_k b_k\phi_k(a) = \sum_k b_k P(\partial_a)\phi_k(a) = 0,$$

thus equation (2.3a) holds.

Equation (2.3b) can be shown as follows:

$$\begin{aligned}
&(\sum_{i,j} x_{ij}a_i\frac{\partial}{\partial a_j} + \sum_{i,j} x_{ij}b_i\frac{\partial}{\partial b_j})(\sum_k b_k\phi_k(a)) \\
&= \sum_k b_k((\sum_{i,j} x_{ij}a_i\frac{\partial}{\partial a_j})\phi_k(a)) + \sum_{i,j} x_{ij}b_i\phi_j(a) \\
&= \sum_k b_k(-\sum_j x_{kj}\phi_j(a)) + \sum_{i,j} x_{ij}b_i\phi_j(a) = 0.
\end{aligned}$$

Equation (2.2c) shows that $\phi_k(a)$ is homogeneous of degree -2 in a, it implies that $\phi(a,b)$ is also homogeneous of degree -2 in a, which implies equation (2.3c).

Since $\frac{\partial}{\partial b_k}\phi(a,b) = \phi_k(a)$, equation (2.2d) is the same as saying that

$$\frac{\partial}{\partial a_i}\frac{\partial}{\partial b_j}\phi(a,b) = \frac{\partial}{\partial a_j}\frac{\partial}{\partial b_i}\phi(a,b) \quad \forall i,j,$$

which is equation (2.3d).

Since $\phi(a,b)$ is linear in b, equation (2.3e) holds. $\phi(a,b)$ is homogeneous of degree 1 in b, it implies equation (2.3f).

Next we show that if $\phi(a,b)$ is a solution to system (2.3), set $\phi_k(a) = \frac{\partial}{\partial b_k}\phi(a,b)$, then $(\phi_1(a),\dots,\phi_n(a))$ is a solution to system (2.2).

Equation (2.3e) tells us that $\phi(a,b)$ is linear in b, i.e. there exists functions $h_k(a)$ and $g(a)$ on $V^\vee$ such that

$$\phi(a,b) = \sum_k b_k h_k(a) + g(a).$$

Equation (2.3f) shows that $\phi(a,b)$ is homogeneous of degree 1 in b, which implies that $g = 0$ and

$$\phi(a,b) = \sum_k b_k h_k(a).$$

Now $\phi_k(a) = \frac{\partial}{\partial b_k}\phi(a,b) = \frac{\partial}{\partial b_k}(\sum_k b_k h_k(a)) = h_k(a)$ and we thus can write

$$\phi(a,b) = \sum_k b_k\phi_k(a).$$

Equation (2.3a) shows that

$$P(\partial_a)\sum_k b_k\phi_k(a) = \sum_k b_k(P(\partial_a)\phi_k(a)) = 0.$$

Since the b_i's are linearly independent, this further shows that

$$P(\partial_a)\phi_k(a) = 0 \quad \forall k,$$

which coincides with equation (2.2a).

From equation (2.3b) we can see that

$$\begin{aligned}
&(\sum_{i,j} x_{ij}a_i\frac{\partial}{\partial a_j} + \sum_{i,j} x_{ij}b_i\frac{\partial}{\partial b_j})(\sum_k b_k\phi_k(a))\\
&= \sum_k b_k(\sum_{i,j} x_{ij}a_i\frac{\partial}{\partial a_j})\phi_k(a) + \sum_{i,j} x_{ij}b_i\phi_j(a)\\
&= \sum_k b_k((\sum_{i,j} x_{ij}a_i\frac{\partial}{\partial a_j})\phi_k(a) + \sum_j x_{kj}\phi_j(a)) = 0.
\end{aligned}$$

Since the b_i's are linearly independent, we have

$$(\sum_{i,j} x_{ij}a_i\frac{\partial}{\partial a_j})\phi_k(a) + \sum_j x_{kj}\phi_j(a) = 0, \quad \forall k,$$

which is equation (2.2b).

Equation (2.3c) shows that $\sum_k b_k\phi_k(a)$ is homogeneous of degree -2 in a, which implies that $\phi_k(a)$ is homogeneous of degree -2 in a as well, which would imply equation (2.2c).

It's also clear that equation (2.3d) implies (2.2d).

Therefore the two systems are equivalent in the above sense. □

Therefore $\sum_k b_k\frac{\partial}{\partial a_k}\Pi_\gamma(a)$ are solutions to system (2.3).

2.3. **Regular holonomicity of the new system.** In this section we will show that system (2.3) is regular holonomic, by extending the proof for the original tautological system in paper [LSY].

Let $\mathcal{M} := D_{V^\vee\times V^\vee}/\mathcal{J}$ where $\mathcal{J}$ is the left ideal generated by the operators in system (2.3).

Theorem 2.2. *Assume that the G-variety X has only a finite number of G-orbits. Then the D-module $\mathcal{M}$ is regular holonomic.*

Proof. Consider the Fourier transform:

$$\widehat{a_i} = \frac{\partial}{\partial\zeta_i},\ \widehat{b_i} = \frac{\partial}{\partial\xi_i},\ \widehat{\frac{\partial}{\partial a_i}} = -\zeta_i,\ \widehat{\frac{\partial}{\partial b_i}} = -\xi_i.$$

where $\zeta, \xi \in V$. The Fourier transform of the D-module $\mathcal{M} = D_{V^\vee \times V^\vee}/\mathcal{J}$ is $\widehat{\mathcal{M}} = D_{V\times V}/\widehat{\mathcal{J}}$, where $\widehat{\mathcal{J}}$ is the $D_{V\times V}$-ideal generated by the following operators:

$$\begin{cases} P(\zeta) & \text{(2.4a)} \\ \zeta_i\xi_j - \zeta_j\xi_i, \ \forall 1 \le i,j \le n & \text{(2.4b)} \\ \xi_i\xi_j, \ \forall 1 \le i,j \le n & \text{(2.4c)} \\ \sum_{i,j} x_{ij}\frac{\partial}{\partial \zeta_i}\zeta_j + \sum_{i,j} x_{ij}\frac{\partial}{\partial \xi_i}\xi_j = \sum_{i,j} x_{ji}\zeta_i\frac{\partial}{\partial \zeta_j} + \sum_{i,j} x_{ji}\xi_i\frac{\partial}{\partial \xi_j} + \sum 2x_{ii} & \text{(2.4d)} \\ \sum_i \frac{\partial}{\partial \zeta_i}(-\zeta_i) + 2 = \sum_i -\zeta_i\frac{\partial}{\partial \zeta_i} - n + 2 & \text{(2.4e)} \\ \sum_i \frac{\partial}{\partial \xi_i}(-\xi_i) - 1 = \sum_i -\xi_i\frac{\partial}{\partial \xi_i} - n - 1. & \text{(2.4f)} \end{cases}$$

Consider the $G \times \mathbb{C}^\times \times \mathbb{C}^\times$-action on $V \times V$ where G acts diagonally and each $\mathbb{C}^\times$ acts on V by scaling. Consider the ideal $\mathcal{I}$ generated by (2.4a), (2.4b) and (2.4c). Operator (2.4c) tells us that $\xi_i = 0$ for all i, thus $(V \times V)/\mathcal{I} \subset V \times \{0\}$. The ideal $\mathcal{I}$ also contains (2.4a), thus $(V \times V)/\mathcal{I} = \hat{X} \times \{0\}$, which is an algebraic variety. Operator (2.4d) comes from the G-action on $V \times V$, operators (2.4e) and (2.4f) come from each copy of $\mathbb{C}^\times$-action on V. Now from our assumption the G-action on X has only a finite number of orbits, thus when lifting to $\hat{X} \times \{0\} \subset V \times V$ there are also finitely many $G \times \mathbb{C}^\times \times \mathbb{C}^\times$-orbits. Therefore $\widehat{\mathcal{M}}$ is a twisted $G \times \mathbb{C}^\times \times \mathbb{C}^\times$-equivariant coherent $D_{V^\vee \times V^\vee}$-module in the sense of [Ho] whose support $\operatorname{Supp}\widehat{\mathcal{M}} = \hat{X} \times \{0\}$ consists of finitely many $G \times \mathbb{C}^\times \times \mathbb{C}^\times$-orbits. Thus the $\widehat{\mathcal{M}}$ is regular holonomic [Bo].

The D-module $\mathcal{M} = D_{V^\vee \times V^\vee}/\mathcal{J}$ is homogeneous since the ideal $\mathcal{J}$ is generated by homogeneous elements under the graduation $\deg \frac{\partial}{\partial a_i} = \deg \frac{\partial}{\partial b_i} = -1$ and $\deg a_i = \deg b_i = 1$. Thus $\mathcal{M}$ is regular holonomic since its Fourier transform $\widehat{\mathcal{M}}$ is regular holonomic [Br]. □

3. Scalar systems for higher derivatives

Now we take derivative of system (2.2) and get a new scalar valued system whose solution consists of n^2 functions $\phi_{lk} := \frac{\partial}{\partial a_l}\frac{\partial}{\partial a_k}\phi(a)$, $1 \le l,k \le n$.

$$
\begin{cases}
P(\partial_a)\phi_{lk}(a) = 0, \ \forall 1 \le k, l \le n & (3.1a) \\
(\sum_{i,j} x_{ij} a_i \frac{\partial}{\partial a_j})\phi_{lk}(a) + \sum_j x_{lj}\phi_{jk}(a) + \sum_j x_{kj}\phi_{lj}(a) = 0, \ \forall k, l & (3.1b) \\
(\sum_i a_i \frac{\partial}{\partial a_i} + 3)\phi_{lk}(a) = 0, \ \forall 1 \le k, l \le n & (3.1c) \\
\phi_{lk}(a) = \phi_{kl}(a), \ \forall 1 \le k, l \le n & (3.1d) \\
\frac{\partial}{\partial a_i}\phi_{jk}(a) = \frac{\partial}{\partial a_j}\phi_{ki}(a) = \frac{\partial}{\partial a_k}\phi_{ij}(a), \ \forall 1 \le i, j, k \le n. & (3.1e)
\end{cases}
$$

And considering $\phi(a, b) := \sum_{l,k} b_l b_k \phi_{l,k}(a)$, we get a new system:

$$
\begin{cases}
P(\partial_a)\phi(a, b) = 0 & (3.2a) \\
(\sum_{i,j} x_{ij} a_i \frac{\partial}{\partial a_j} + \sum_{i,j} x_{ij} b_i \frac{\partial}{\partial b_j})\phi(a, b) = 0 & (3.2b) \\
(\sum_i a_i \frac{\partial}{\partial a_i} + 3)\phi(a, b) = 0 & (3.2c) \\
(\sum_i b_i \frac{\partial}{\partial b_i} - 2)\phi(a, b) = 0 & (3.2d) \\
\frac{\partial}{\partial b_i}\frac{\partial}{\partial b_j}\frac{\partial}{\partial b_k}\phi(a, b) = 0, \ \forall 1 \le i, j, k \le n & (3.2e) \\
\frac{\partial}{\partial a_i}\frac{\partial}{\partial b_j}\frac{\partial}{\partial b_k}\phi(a, b) = \frac{\partial}{\partial a_j}\frac{\partial}{\partial b_k}\frac{\partial}{\partial b_i}\phi(a, b) = \frac{\partial}{\partial a_k}\frac{\partial}{\partial b_i}\frac{\partial}{\partial b_j}\phi(a, b), \forall i, j, k. & (3.2f)
\end{cases}
$$

Similar to the previous case, we have:

Proposition 3.1. *By setting $\phi(a, b) = \sum_{l,k} b_l b_k \phi_{l,k}(a)$ and $\phi_{lk}(a) = \frac{\partial}{\partial b_l}\frac{\partial}{\partial b_k}\phi(a, b)$, the systems* (3.1) *and* (3.2) *are equivalent.*

Proof. Here we check for (3.2b) and the rest is clear. Let $\phi(a, b) = \sum_{l,k} b_l b_k \phi_{l,k}(a)$, then the left-hand side of (3.2b) becomes

$$
\begin{aligned}
&\sum_{i,j,k,l} b_k b_l x_{ij} a_i \frac{\partial}{\partial a_j}\phi_{l,k}(a) + \sum_{i,j,k,l} x_{ij}\delta_{kj} b_i b_l \phi_{lk}(a) + \sum_{i,j,k,l} x_{ij}\delta_{kl} b_i b_k \phi_{lk}(a) \\
= &\sum_{i,j,k,l} b_k b_l x_{ij} a_i \frac{\partial}{\partial a_j}\phi_{l,k}(a) + 2\sum_{j,k,l} x_{lj} b_k b_l \phi_{jk}(a)
\end{aligned}
$$

And (3.1b) implies that

$$\begin{aligned}
&-\sum_{k,l} b_k b_l \sum_{i,j} x_{ij} a_i \frac{\partial}{\partial a_j}\phi_{lk}(a)\\
=&\sum_{k,l} b_k b_l \sum_{j} x_{lj}\phi_{jk}(a) + \sum_{k,l} b_k b_l \sum_{j} x_{kj}\phi_{lj}(a)\\
=&2\sum_{j,k,l} b_k b_l x_{lj}\phi_{jk}(a)
\end{aligned}$$

Therefore the left-hand side of (3.2b) equals 0. The reverse direction is also clear. ☐

Proposition 3.2. *Assume that the G-variety X has only a finite number of G-orbits, then system* (3.2) *is regular holonomic.*

The proof of Theorem 2.2 follows here.

In general, for p-th derivatives of $\phi(a)$ satisfying τ, we can construct a new system as follows:

$$\begin{cases}
P(\partial_a)\phi(a,b) = 0 & (3.3a)\\
\left(\sum_{i,j} x_{ij} a_i \frac{\partial}{\partial a_j} + \sum_{i,j} x_{ij} b_i \frac{\partial}{\partial b_j}\right)\phi(a,b) = 0 & (3.3b)\\
\left(\sum_{i} a_i \frac{\partial}{\partial a_i} + 1 + p\right)\phi(a,b) = 0 & (3.3c)\\
\left(\sum_{i} b_i \frac{\partial}{\partial b_i} - p\right)\phi(a,b) = 0 & (3.3d)\\
\frac{\partial}{\partial b_{k_1}}\cdots\frac{\partial}{\partial b_{k_p}}\phi(a,b) = 0 \quad \forall 1 \le k_i \le n & (3.3e)\\
\frac{\partial}{\partial a_{k_1}}\frac{\partial}{\partial b_{k_2}}\cdots\frac{\partial}{\partial b_{k_p}}\phi(a,b) = \frac{\partial}{\partial a_{k_{l_1}}}\frac{\partial}{\partial b_{k_{l_2}}}\cdots\frac{\partial}{\partial b_{k_{l_p}}}\phi(a,b) & (3.3f)
\end{cases}$$

where $1 \le k_i \le n$ and $\{l_1, \ldots, l_p\}$ is any permutation of $\{1, \ldots, p\}$.

The relationship between $\phi(a,b)$ and the derivatives of $\phi(a)$ is:

$$\phi(a,b) = \sum_{k_1,\ldots,k_p} b_{k_1}\cdots b_{k_p}\frac{\partial}{\partial a_{k_1}}\cdots\frac{\partial}{\partial a_{k_p}}\phi(a)$$

.

From our previous argument it is clear that if G acts on X with finitely many orbits, this system is regular holonomic.

By Theorem 1.1, $\sum_{k_1,\ldots,k_p} b_{k_1}\cdots b_{k_p}\frac{\partial}{\partial a_{k_1}}\cdots\frac{\partial}{\partial a_{k_p}}\Pi_\gamma(a)$ are solutions to system (3.3).

4. Concluding remarks

We conclude this paper with some remarks about our new differential systems for period mappings.

For the family of CYs $\mathcal{Y}$, since the period mapping is given by higher derivatives of the periods of $(d-1,0)$ forms, any information about the period mapping can in principle be derived from period integrals, albeit somewhat indirectly. However, the point here is that an explicit regular holonomic system for the full period mapping would give us a way to study the structure of this mapping by D-module techniques directly. Thanks to the Riemann-Hilbert correspondence, these techniques have proven to be a very fruitful approach to geometric questions about the family $\mathcal{Y}$ (e.g. degenerations, monodromy, etc.) [BHLSY, HLZ] when applied to τ. The new differential system that we have constructed for the period mapping is in fact nothing but a tautological system. Namely, it is a regular holonomic D-module defined by a polynomial ideal together with a set of first order symmetry operators – conceptually of the same type as τ. It is therefore *directly* amenable to the same tools (Fourier transforms, Riemann-Hilbert, Lie algebra homology, etc.) we applied to investigate τ itself. The hope is that understanding the structure of the new D-module will shed new light on Hodge-theoretic questions about the family $\mathcal{Y}$. We would like to return to these questions in a future paper.

5. Acknowledgement

We dedicate this paper to our mentor, teacher and colleague Professor Shing-Tung Yau on the occasion of his 65th birthday, 60 years after the celebrated Calabi conjecture. The authors AH and BHL also thank their collaborators S. Bloch, R. Song, D. Srinivas, S.-T. Yau and X. Zhu for their contributions to the foundation of the theory of tautological systems, upon which this paper is based. Finally, we thank the referee for corrections and helpful comments. BHL is partially supported by NSF FRG grant DMS 1159049.

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