Picard-Fuchs Equations for Relative Periods and Abel-Jacobi Map for Calabi-Yau Hypersurfaces

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Abstract

We study the variation of relative cohomology for a pair consisting of a smooth projective hypersurface and an algebraic subvariety in it. We construct an inhomogeneous Picard-Fuchs equation by applying a Picard-Fuchs operator to the holomorphic top form on a toric Calabi-Yau hypersurface, and deriving a general formula for the d-exact form on one side of the equation. We also derive a double residue formula, giving a purely algebraic way to compute the inhomogeneous Picard-Fuchs equations for Abel-Jacobi map, which has played an important role in recent study of D-branes [25]. Using the variation formalism, we prove that the relative periods of toric B-branes on a toric Calabi-Yau hypersurface satisfy the enhanced GKZ-hypergeometric system proposed in physics literature [6], and discuss the relations between the works [25] [21] [6] in recent study of open string mirror symmetry. We also give the general solutions to the enhanced hypergeometric system.

1 Introduction

Mirror symmetry connects symplectic geometry of Calabi-Yau manifold to complex geometry of its mirror manifold. In closed string theory, this has led to predictions on counting curves on projective Calabi-Yau threefolds [9][8]. In open string theory, mirror symmetry has led to predictions on counting holomorphic discs, first in the non-compact case studied in [4][3], and more recently in the compact quintic example, where the instanton sum of disc amplitude with non-trivial boundary on the real locus of the real quintic is shown to be identical to the normalized Abel-Jacobi map on the mirror quintic via mirror map [31][25][27].

In physics, the Abel-Jacobi map serves as the domain-wall tension of D-branes on the B-model, and is obtained via reduction of the holomorphic Chern-Simons action on curves [4]. It is conjectured to have remarkable integrality structure [26]. A key for calculating the Abel-Jacobi map is through inhomogeneous Picard-Fuchs equations [25]. Let X_z be a family of Calabi-Yau threefolds parameterized by variable z, and Ω_z be a family of nonzero holomorphic 3-forms on X_z. Assume that there is a family of pairs of holomorphic curves C_z^+, C_z^- in X_z. Let $\mathcal{D}(\partial_z)$ be a Picard-Fuchs operator. Then there exists a 2-form β_z such that

$$\mathcal{D}(\partial_z)\Omega_z = -d\beta_z \tag{1.1}$$

The exact term $d\beta_z$ does not contribute when it is integrated over a closed 3-cycle Γ in X_z. The so-called closed-string period $\int_\Gamma \Omega_z$ then satisfies a homogeneous Picard-Fuchs equation. In open string theory, it is necessary to consider the integral of Ω_z over a 3-chain Γ in X_z which is not closed, but whose boundary is $C^+ - C^-$. Because of contributions from the boundary, this so-called open-string period $\int_\Gamma \Omega_z$ satisfies an inhomogeneous Picard-Fuchs equation. Solving the equation gives a precise description of the Abel-Jacobi map up to closed-string periods. To study this map, β_z plays an essential role since it is this form that gives rise to one side of the inhomogeneous Picard-Fuchs equation:

$$\int_\Gamma \mathcal{D}(\partial_z)\Omega_z = -\int_{\partial\Gamma} \beta_z. \tag{1.2}$$

The inhomogeneous term on the right side turns out also to encodes important information for predicting the number of holomorphic disks on a mirror Calabi-Yau manifold.

There have been several proposals for constructing the inhomogeneous Picard-Fuchs equation and its solutions. In the case of 1-moduli family [25], it was done by first computing β_z using the Griffith-Dwork reduction procedure, and then by doing an explicit (but delicate) local analytic calculation of appropriate boundary integrals. Based on the notion of off-shell mirror symmetry, two other proposals [21][6] have been put forth. Roughly speaking, their setup begins with a family of divisors $Y_{z,u}$ which deforms in X_z under an additional parameter u. For each relative homology class $\Gamma \in H_3(X_z, Y_{z,u})$, one considers the integral

$$\int_\Gamma \Omega_z \tag{1.3}$$

which is called a relative period for B-brane. It is proposed that the open-string periods above be recovered as a certain critical value of the relative period, regarded as a function of u. To calculate the relative periods, [21] proposed a procedure similar to the Griffith-Dwork reduction. In [6], an enlarged polytope is proposed to encode both the geometry of the Calabi-Yau X_z and the B-brane geometry. This gives rise to a GKZ hypergeometric system for the relative periods, and a special solution at a critical point in u then leads to a solution to the original inhomogeneous Picard-Fuchs equation.

Our goal in this paper is to further develop the mathematical structures underlying inhomogeneous Picard-Fuchs equations and the Abel-Jacobi map, and to clarify the relationships between the three approaches mentioned above. Here is an outline. We begin, in section 2, with a description of a residue formalism for relative cohomology of a family of pairs (X_z, Y_z), including a number of variational formulas on the local system $H^n(X_z, Y_z)$. In section 3, we derive a general formula for the exact form (the β-term) appearing in the inhomogeneous Picard-Fuchs equation for toric Calabi-Yau hypersurfaces, generalizing GKZ-type differential equation to the level of differential forms instead of cohomology classes. This gives a much more uniform approach to computing the β-term than the Griffith-Dwork reduction. In section 4, we prove a purely algebraic a double residue formula for the inhomogeneous term of the Picard-Fuchs equation that governs the Abel-Jacobi map. This uniform approach also allows us to bypass the delicate local analytical calculation of boundary integrals in a previous approach [25][23]. In section 5, using the residue formalism in section 2, we give a simple interpretation of the relative version of the Griffith-Dwork reduction used in [21]. In particular, this gives a mathematical justification for the appearance of log divisor, and elucidates the relationship between relative periods and the Abel-Jacobi map. We also give a uniform description for the enhanced polytope method for describing toric B-brane geometry in a general toric Calabi-Yau hypersurface, and show that relative periods satisfy the corresponding enhanced GKZ system. Finally, we give a general formula, modeled on the closed string case [16][17], for solution to the enhanced GKZ system.

Acknowledgement. S.L. would like to thank J. Walcher for many stimulating discussions, and thank M.Soroush for answering many questions on his paper. After the completion of a preliminary draft of our paper, three other papers [2][5][15] with some overlap with ours have since been posted on the arXiv.

2 Variation of Relative Cohomology

Local System of Relative Cohomology and Gauss-Manin Connection

Let $\pi : \mathcal{X} \to S$ be a smooth family of n-dimensional projective varieties, and $\mathcal{Y} \to S$ be a family of smooth subvariety $\mathcal{Y} \subset \mathcal{X}$. Let $s \in S$ be a closed point, and denote by X_s, Y_s the corresponding fiber over s. Consider the family of relative cohomology class

$$H^n(X_s, Y_s)$$

given by the cohomology of the complex of pairs:

$$\Gamma(\Omega^n(X_s)) \oplus \Gamma(\Omega^{n-1}(Y_s))$$

with the differential

$$d(\alpha, \beta) = (d\alpha, \alpha|_{Y_s} - d\beta) \tag{2.1}$$

Here $\Omega^n(X_s)$ and $\Omega^{n-1}(Y_s)$ are sheaves of De Rham differential n-forms on X_s and $(n-1)$-form on Y_s, and Γ is the smooth global section. Therefore an element of $H^n(X_s, Y_s)$ is represented by a differential n-form on X_s whose restriction to Y_s is specified by an exact form.

Lemma 2.1. *$H^n(X_s, Y_s)$ forms a local system on S.*

Proof. The proof is similar to the case without $\mathcal{Y}$ by choosing a local trivialization of $\mathcal{X} \to S$ which also trivializes $\mathcal{Y} \to S$. See e.g.[30]. □

We denote this local system by $\mathcal{H}^n_{(\mathcal{X},\mathcal{Y})}$, and let ∇^{GM} be the Gauss-Manin connection. There's a well-defined natural pairing

$$\begin{array}{ccccc} H_n(X_s, Y_s) & \otimes & H^n(X_s, Y_s) & \to & \mathbb{C} \\ \Gamma & \otimes & (\alpha, \beta) & \mapsto & <\Gamma, (\alpha, \beta)> \equiv \int_\Gamma \alpha - \int_{\partial\Gamma} \beta. \end{array} \tag{2.2}$$

Given a family $(\alpha_s, \beta_s) \in H^n(X_s, Y_s)$ varying smoothly, which gives a smooth section of $\mathcal{H}^n_{(\mathcal{X},\mathcal{Y})}$ denoted by $[(\alpha_s, \beta_s)]$, and $\Gamma_s \in H_n(X_s, Y_s)$ a smooth family of relative cycles, we get a function on S given by the pairing

$$<\Gamma_s, (\alpha_s, \beta_s)> = \int_{\Gamma_s} \alpha_s - \int_{\partial\Gamma_s} \beta_s$$

Let v be a vector field on S. We consider the variation

$$\mathcal{L}_v <\Gamma_s, (\alpha_s, \beta_s)>$$

where $\mathcal{L}_v$ is the Lie derivative with respect to v. Suppose we have a lifting $\tilde{\alpha}, \tilde{\beta}$, which are differential forms on $\mathcal{X}, \mathcal{Y}$ respectively, such that

$$\tilde{\alpha}|_{X_s} = \alpha_s, \ \tilde{\beta}|_{Y_s} = \beta_s$$

and that Γ_s moves smoothly to form a cycle $\tilde{\Gamma}$ on $\mathcal{X}$:

$$\Gamma_s = \tilde{\Gamma} \cap X_s, \quad \partial\Gamma \subset \mathcal{Y}.$$

Let $\tilde{v}_{\mathcal{X}}$ be a lifting of v on $\mathcal{X}$, $\tilde{v}_{\mathcal{Y}}$ be a lifting of v on $\mathcal{Y}$.

Proposition 2.2 (Variation Formula)**.**

$$\mathcal{L}_v < \Gamma_s, (\alpha_s, \beta_s) >=< \Gamma_s, (\iota_{\tilde{v}_\mathcal{X}} \lrcorner d\tilde{\alpha}, \iota_{\tilde{v}_\mathcal{Y}} \lrcorner (d\tilde{\beta} - \tilde{\alpha})) > \tag{2.3}$$

where $\iota_{\tilde{v}_\mathcal{X}}$ *is the contraction with* $\tilde{v}_\mathcal{X}$*, and similarly for* $\iota_{\tilde{v}_\mathcal{Y}}$*.*

Proof. We fix a point $s_0 \in S$, and let $\sigma(t)$ be a local integral curve of v such that $\sigma(0) = s_0$. Let $\tilde{\Gamma}_{\sigma(t)}$ denote the one-dimensional family of cycles over $\sigma(s),\ \ 0 \leq s \leq t$, and we denote by Γ_t the cycle over the point $\sigma(t)$. Also let $\widetilde{\partial\Gamma}_{\sigma(t)}$ be the family of boundary cycle $\partial\Gamma_s$ over $\sigma(s), 0 \leq s \leq t$. Then we have

$$\partial\left(\tilde{\Gamma}_{\sigma(t)}\right) = \Gamma_t - \Gamma_0 - \widetilde{\partial\Gamma}_{\sigma(t)} \tag{2.4}$$

therefore

$$\int_{\tilde{\Gamma}_{\sigma(t)}} d\tilde{\alpha} = \int_{\Gamma_t} \tilde{\alpha} - \int_{\Gamma_0} \tilde{\alpha} - \int_{\widetilde{\partial\Gamma}_{\sigma(t)}} \tilde{\alpha}$$

Similarly

$$\int_{\widetilde{\partial\Gamma}} d\tilde{\beta} = \int_{\partial\Gamma_t} \tilde{\beta} - \int_{\partial\Gamma_0} \tilde{\beta}$$

Taking the derivative with respect to t, we get

$$\frac{\partial}{\partial t}\left(\int_{\Gamma_t} \tilde{\alpha} - \int_{\partial\Gamma_t} \tilde{\beta}\right) = \int_{\Gamma_t} \iota_{\tilde{v}_\mathcal{X}} \lrcorner d\tilde{\alpha} + \int_{\partial\Gamma_t} \iota_{\tilde{v}_\mathcal{Y}} \lrcorner \left(\tilde{\alpha} - d\tilde{\beta}\right)$$

The proposition follows. □

Note that $(\iota_{\tilde{v}_\mathcal{X}} \lrcorner d\tilde{\alpha}, \iota_{\tilde{v}_\mathcal{Y}} \lrcorner (d\tilde{\beta} - \tilde{\alpha}))$ is nothing but the Gauss-Manin connection

$$\nabla_v^{GM}[(\alpha_s, \beta_s)] = \left[\left((\iota_{\tilde{v}_\mathcal{X}} \lrcorner d\tilde{\alpha})|_{X_s}, (\iota_{\tilde{v}_\mathcal{Y}} \lrcorner (d\tilde{\beta} - \tilde{\alpha}))|_{Y_s}\right)\right] \tag{2.5}$$

and it is straightforward to check using the variation formula that the right side of (2.5) is independent of the choice of $\tilde{\alpha}, \tilde{\beta}, \tilde{v}_\mathcal{X}, \tilde{v}_\mathcal{Y}$, and that the connection is flat. The following corollary also follows from (2.5).

Corollary 2.3.

$$\mathcal{L}_v < \Gamma_s, (\alpha_s, \beta_s) >=< \Gamma_s, \nabla_v^{GM}[(\alpha_s, \beta_s)] > \tag{2.6}$$

Residue Formalism for Relative Cohomology

In this section, we assume that X_z moves as a family of hypersurfaces in a fixed $n+1$-dim ambient projective space M with defining equation $P_z = 0$. Here $P_z \in H^0(M, [D])$ for a fixed divisor class $[D]$, and z is holomorphic coordinate on S parametrizing the family. Let

$$\omega_z \in H^0(M, K_M(X_z))$$

be a rational (n+1,0)-form on M with pole of order one along X_z, then we get a famliy of holomorphic (n,0)-form on X_z given by

$$Res_{X_z}\omega_z \in H^0(X_z, K_{X_z})$$

and also a family of relative cohomology classes

$$(Res_{X_z}\omega_z, 0) \in H^n(X_z, Y_z)$$

Here our convention for Res is that if X_z is locally given by $w = 0$, and $\omega_z = \frac{dw}{w} \wedge \phi$, where ϕ is locally a smooth form, then $Res_{X_z}\omega_z = \phi|_{X_z}$. Note that the map Res_{X_z} : $H^0(M, K_M(X_z)) \to H^{n,0}(X_z)$ is at the level of forms, not only as cohomology classes since the order of pole is one. While in the residue formalism of ordinary cohomology, we can ignore exact forms to reduce the order of the pole [14], it is important to keep track of the order of the pole in considering periods of relative cohomology because of the boundary term.

We choose a fixed open cover $\{U_\alpha\}$ of M and a partition of unity $\{\rho_\alpha\}$ subordinate to it. Let $P_{z,\alpha} = 0$ be the defining equation of X_s on U_α. Then we can write

$$\omega_z = \sum_\alpha \frac{d_M P_{z,\alpha}}{P_{z,\alpha}} \wedge \phi_{z,\alpha} \tag{2.7}$$

where $\phi_{z,\alpha}$ is a smooth $(n, 0)$-form with $\overline{supp(\phi_{z,\alpha})} \subset U_\alpha$. On the trivial family $M \times S$, we will use d_M to denote the differential along M only and use d to denote the differential on the total space. Let

$$\phi_z = \sum_\alpha \phi_{z,\alpha} \tag{2.8}$$

Then ϕ_z is a smooth form on $M \times S$ such that

$$\phi_z|_{X_z} = Res_{X_z}\omega_z$$

Consider the variation

$$\frac{\partial}{\partial z}\omega_z = d_M\left(\sum_\alpha \partial_z \log(P_{z,\alpha})\phi_{z,\alpha}\right) + \sum_\alpha \left(\frac{d_M P_{z,\alpha}}{P_{z,\alpha}} \wedge \partial_z\phi_{z,\alpha} - \frac{\partial_z P_{z,\alpha}}{P_{z,\alpha}} d_M\phi_{z,\alpha}\right)$$

Let

$$\tilde{\partial}_{z\mathcal{X}} = \frac{\partial}{\partial z} + n_{z,\mathcal{X}}$$

be a lifting of $\frac{\partial}{\partial z}$ to $\mathcal{X}$, where $n_{z,\mathcal{X}} \in \Gamma(T_M|_{X_z})$ is along the fiber, which is a normal vector field corresponding to the deformation of X_z in M with respect to z. Then in each U_α, we have

$$\left(\iota_{n_{X_z}} \lrcorner d_M P_{z,\alpha}\right)|_{X_z} = -\partial_z P_{z,\alpha}|_{X_z} \tag{2.9}$$

It follows easily that

$$\iota_{\tilde{\partial}_{z\mathcal{X}}} \lrcorner d\phi_z|_{X_z} = Res_{X_z}\left(\partial_z\omega_z - d_M\left(\partial_z \log(P_z)\phi_z\right)\right) \tag{2.10}$$

Note that since the transition function of $[D]$ is independent of z, $\partial_z \log(P_z)$ is globally well-defined. In general, $\partial_z\omega_z$ will have a pole of order two along X_z, but the substraction of $d_M\left(\partial_z \log(P_z)\phi_z\right)$ makes it logarithmic along X_z, hence the residue above is well-defined.

Next we choose arbitrary lifting of $\frac{\partial}{\partial z}$ to $\mathcal{Y}$, and write it as

$$\tilde{\partial}_{z,\mathcal{Y}} = \frac{\partial}{\partial z} + n_{z,\mathcal{Y}}$$

where $n_{z,\mathcal{X}} \in \Gamma(T_M|_{Y_z})$ is along the fiber, which is a normal vector field corresponding to the deformation of Y_z in M with respect to z. Then the variation formula implies that

Proposition 2.4 (Residue Variation Formula)**.**

$$\nabla^{GM}_{\partial_z}\left(Res_{X_z}\omega_z, 0\right) = \left(Res_{X_z}\left(\partial_z\omega_z - d_M\left(\partial_z\log(P_z)\phi_z\right)\right), -\iota_{n_z,\mathcal{Y}}\lrcorner\phi_z\right) \tag{2.11}$$

To see the effect of the second component on the right side, let us assume that $Y_z = X_z \cap H$, where H is a fixed hypersurface in M with defining equation $Q = 0$. Then we can choose a ϵ-tube $T_\epsilon(\Gamma)$ [14] of $\Gamma \in H_n(X_z, Y_z)$ with $\partial T_\epsilon(\Gamma) \subset H$. Hence

$$\begin{aligned}
\frac{1}{2\pi i}\int_{T_\epsilon(\Gamma)} -d_M\left(\partial_z\log(P_z)\phi_z\right) &= \frac{1}{2\pi i}\int_{T_\epsilon(\partial\Gamma)}\left(\partial_z\log(P_z)\phi_z\right)|_H \\
&= \int_{\partial\Gamma} Res_{Y_z}\left(\partial_z\log(P_z)\phi_z\right)|_H \\
&= -\int_{\partial\Gamma}\iota_{n_z,\mathcal{Y}}\lrcorner\phi_z
\end{aligned}$$

which cancels exactly the second component on the right side of (2.11). Therefore,

$$\partial_z < \Gamma, (Res_{X_z}\omega_z, 0) >= \frac{1}{2\pi i}\int_{T_\epsilon(\Gamma)}\partial_z\omega_z$$

We can localize the above observation and consider the following situation: on each U_α, suppose we can choose Q_α independent of z such that $P_{z,\alpha}, Q_\alpha$ are transversal and $Y_z \cap U_\alpha \subset \{Q_\alpha = 0, P_{z,\alpha} = 0\}$. Suppose we have a relative cycle $\Gamma \in H_n(X_z, Y_z)$ where we can choose a ϵ-tube $T_\epsilon(\Gamma)$ such that $\partial T_\epsilon(\Gamma) \cap U_\alpha$ lies in $\{Q_\alpha = 0\}$. We have the pairing

$$< \Gamma, (Res_{X_z}\omega_z, 0) >= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{T_\epsilon(\Gamma)}\omega_z$$

However, the right hand side doesn't depend on ϵ. In fact, let $T^\epsilon_\delta(\Gamma)$ be a solid annulus over Γ. By Stokes's theorem, we have

$$\int_{T_\epsilon(\Gamma)}\omega_z - \int_{T_\sigma(\Gamma)}\omega_z = \int_{\partial T^\epsilon_\delta(\Gamma)}\omega_z$$

since $\partial T^\epsilon_\delta(\Gamma) \cap U_\alpha \subset \{Q_\alpha = 0\}$ for each α, the above integral vanishes. Therefore the integral $\int_{T_\epsilon(\Gamma)}\omega_z$ doesn't depend on the position of the ϵ-tube if we impose the boundary condition as above. It follows immediately that

$$(\partial_z)^k < \Gamma, (Res_{X_z}\omega_z, 0) >= \frac{1}{2\pi i}\int_{T_\epsilon(\Gamma)}(\partial_z)^k\omega_z$$

where $\partial T_\epsilon(\Gamma) \cap U_\alpha$ lies in $\{Q_\alpha = 0\}$. Applying this to a Picard-Fuchs operator, we get

Proposition 2.5. *[Inhomogeneous Picard-Fuchs Equation] Let $\mathcal{D} = \mathcal{D}(\nabla^{GM}_{\partial_z})$ be a Picard-Fuchs operator, i.e.*

$$\mathcal{D}(\partial_z)\omega_z = -d\beta_z \tag{2.12}$$

for some rational $(n-1,0)$-form β_z with poles along X_z. Then under the above local choices, we have

$$\mathcal{D}(\partial_z) < \Gamma, (Res_{X_z}\omega_z, 0) >= \frac{1}{2\pi i}\int_{T_\epsilon(\partial\Gamma)}\beta_z. \tag{2.13}$$

In the next section, we will derive a general formula for β_z using toric method.

More generally, suppose that $\{Q_\alpha\}$ depends on z which is denoted by $Q_{z,\alpha}$, and we put our ϵ-tube $T_\epsilon(\Gamma_z)$ inside $\{Q_{z,\alpha}\}=0$ on U_α. Then

$$\int_{\Gamma_z} Res_{X_z}\omega_z = \frac{1}{2\pi i}\int_{T_\epsilon(\Gamma_z)}\omega_z = \sum_\alpha \frac{1}{2\pi i}\int_{T_\epsilon(\Gamma_z)}\rho_\alpha\omega_z$$

and the integration doesn't depend on ϵ assuming the boundary condition as above. Applying the variation formula (2.3) we get

$$\frac{\partial}{\partial z}\sum_\alpha \frac{1}{2\pi i}\int_{T_\epsilon(\Gamma_z)}\rho_\alpha\omega_z = \sum_\alpha \frac{1}{2\pi i}\int_{T_\epsilon(\Gamma_z)}\rho_\alpha\partial_z\omega_z - \sum_\alpha \frac{1}{2\pi i}\int_{T_\epsilon(\partial\Gamma_z)}\rho_\alpha\left(\iota_{n_{z,Q_\alpha}}\lrcorner\omega_z\right)|_{Q_{z,\alpha}=0}$$

where n_{z,Q_α} is the normal vector field corresponding to the deformation of $\{Q_{z,\alpha}=0\}$ inside M. In particular, we have $\iota_{n_{z,Q_\alpha}}\lrcorner dQ_{z,\alpha}|_{\{Q_{z,\alpha=0}\}} = -\partial_z Q_{z,\alpha}|_{\{Q_{z,\alpha=0}\}}$. Hence

$$\left(\iota_{n_{z,Q_\alpha}}\lrcorner\omega_z\right)|_{Q_{z,\alpha}=0} = -Res_{Q_{z,\alpha}=0}\left(\partial_z\log(Q_{z,\alpha})\omega_z\right)$$

Putting together the last three equations, we arrive at

Proposition 2.6 (cf. [32])**.**

$$\frac{\partial^k}{\partial z^k}\left(\int_{\Gamma_z}Res_{X_z}\omega_z\right) = \frac{1}{2\pi i}\int_{T_\epsilon(\Gamma_z)}\partial_z^k\omega_z + \sum_{l=1}^{k}\frac{\partial^{k-l}}{\partial z^{k-l}}\sum_\alpha\frac{1}{2\pi i}\int_{T_\epsilon(\partial\Gamma_z)}\rho_\alpha Res_{Q_{z,\alpha}=0}\left(\partial_z\log(Q_{z,\alpha})\frac{\partial^{l-1}}{\partial z^{l-1}}\omega_z\right)$$

3 Exact GKZ Differential Equation and Toric geometry

In this section, we study the Picard-Fuchs differential operators arising from a generalized GKZ hypergeometric systems [16][17] for toric Calabi-Yau hypersurfaces and derive a general formula for the β-term of an inhomogeneous Picard-Fuchs equation, from toric data.

We first consider the special case of a weighted projective space, where β-term will be much simpler than in the general case, which will be considered at the end of this section. Let $\mathbb{P}^4(\mathbf{w}) = \mathbb{P}^4(w_1,w_2,w_3,w_4,w_5)$. We assume that $w_5=1$ and it's of Fermat-type, i.e., $w_i|d$ for each i, where $d=w_1+w_2+w_3+w_4+w_5$. There's associated 4-dimensional integral convex polyhedron given by the convex hull of the integral vectors

$$\Delta = \left\{(x_1,\cdots,x_5)\in\mathbb{R}^5 | \sum_{i=1}^{5}w_ix_i = 0, x_i\geq -1\right\}$$

If we choose the basis $\{e_i=(1,0,0,0,-w_i), i=1..4\}$, then the vertices is given by

$$\begin{aligned}\Delta: \qquad & v_1 = (\frac{d}{w_1}-1,-1,-1,-1)\\ & v_2 = (-1,\frac{d}{w_2}-1,-1,-1)\\ & v_3 = (-1,-1,\frac{d}{w_3}-1,-1)\\ & v_4 = (-1,-1,-1,\frac{d}{w_4}-1)\\ & v_5 = (-1,-1,-1,-1)\end{aligned}$$

and the vertices of its dual polytope is given by

$$\Delta^* : \quad \begin{array}{l} v_1^* = (1,0,0,0) \\ v_2^* = (0,1,0,0) \\ v_3^* = (0,0,1,0) \\ v_4^* = (0,0,0,1) \\ v_5^* = (-w_1,-w_2,-w_3,-w_4) \end{array}$$

Let $v_k^*, k = 0, 1, 2, ..,$ be integral points of Δ^*, where $v_0^* = (0, 0, 0, 0)$. We write

$$f_{\Delta^*}(x) = \sum_{v_k^* \in \Delta^*} a_k X^{v_k^*} \tag{3.1}$$

which is the defining equation for our Calabi-Yau hypersurfaces in the anti-canonical divisor class. Here $X = \{X_1, X_2, X_3, X_4\}$ is the toric coordinate, $X^{v_k^*} = \prod_{j=1}^{4} X_j^{v_{k,j}^*}$. If we use homogeneous coordinate [10] $\{z_\rho, 1 \le \rho \le 5\}$ corresponding to the one-dim cone $\{v_\rho, 1 \le \rho \le 5\}$, then the toric coordinate can be written by homogeneous coordinate

$$X_j = \frac{z_j^{d/w_j}}{\prod_{\rho=1}^{5} z_\rho}, \quad j = 1, 2, 3, 4 \tag{3.2}$$

The relevant rational form with pole of order one along the hypersurface is given by

$$\begin{aligned} \Pi(a) &= \frac{\prod_{i=1}^{5} w_i}{d^3} \frac{1}{\sum_{v_k^* \in \Delta^*} a_k X^{v_k^*}} \prod_{j=1}^{4} \frac{dX_j}{X_j} \\ &= \frac{\Omega_0}{\sum_{v_k^* \in \Delta^*} a_k \prod_{\rho=1}^{5} z_\rho^{<v_k^*, v_\rho>+1}} \\ &= \frac{\Omega_0}{a_0 \prod_{\rho=1}^{5} z_\rho + \sum_{\rho=1}^{5} a_\rho z_\rho^{d/w_\rho} + \sum_{v_k^* \in \Delta^*, k>5} a_k \prod_{\rho=1}^{5} z_\rho^{<v_k^*, v_\rho>+1}} \end{aligned} \tag{3.3}$$

where $\Omega_0 = \sum_{\rho=1}^{5} (-1)^{\rho-1} w_\rho z_\rho dz_1 \wedge \cdots \hat{dz_\rho} \wedge \cdots dz_5$. Define the relation lattice by

$$L = \{l = (l_0, l_1, ...) \in \mathbb{Z}^{|\Delta^*|+1} | \sum_i l_i \bar{v}_i^* = 0\}, \quad \text{where} \;\; \bar{v}_i^* = (1, v_i^*), \;\; v_i^* \in \Delta^*$$

The moduli variable associated with the choice of a basis $\{l^{(k)}\}$ for L is given by [16]

$$x_k = (-1)^{l_0^{(k)}} a^{l^{(k)}}$$

The key idea here is to consider the following 1-parameter family of automorphisms

$$\phi_t : z_\rho \to (\frac{a_0}{a_\rho})^{\frac{w_\rho}{d} t} z_\rho, \quad 1 \le \rho \le 5.$$

Put

$$\Pi_t(a) = \phi_t^* \Pi(a).$$

It satisfies the differential equation

$$\partial_t\left(\phi_t^*\Pi(a)\right) = \mathcal{L}_V \phi_t^*\Pi(a)$$

where $V = \sum_{\rho=1}^{5} \frac{w_\rho}{d}(\log\frac{a_0}{a_\rho}) z_\rho \frac{\partial}{\partial z_\rho}$ is the generating vector field for ϕ_t, $\mathcal{L}_V$ is the Lie derivative. This is solved by

$$\phi_t^*\Pi(a) = e^{t\mathcal{L}_V}\Pi(a) \tag{3.4}$$

Define

$$\tilde{\Pi}(x) = a_0\Pi_1(a) \tag{3.5}$$

$\tilde{\Pi}(x)$ is a function of $\{x_k\}$ only. Indeed,

$$\tilde{\Pi}(x) = \frac{\Omega_0}{\prod_{\rho=1}^{5} z_\rho + \left(\prod_{\rho=1}^{5}(\frac{a_\rho}{a_0})^{w_\rho/d}\right)\sum_{\rho=1}^{5} z_\rho^{d/w_\rho} + \sum_{v_k^*\in\Delta^*, k>5}\frac{a_k}{a_0}\prod_{\rho=1}^{5}\left(\frac{a_0}{a_\rho}\right)^{\frac{w_\rho}{d}<v_k^*,v_\rho>}\prod_{\rho=1}^{5} z_\rho^{<v_k^*,v_\rho>+1}}$$

Since we have

$$\sum_{\rho=1}^{5} w_\rho v_\rho = 0,\quad v_k^* = \sum_{\rho=1}^{5}\frac{w_\rho}{d} < v_k^*, v_\rho > v_\rho^*$$

we see that both $\left(\prod_{\rho=1}^{5}(\frac{a_\rho}{a_0})^{w_\rho/d}\right)$ and $\frac{a_k}{a_0}\prod_{\rho=1}^{5}\left(\frac{a_0}{a_\rho}\right)^{\frac{w_\rho}{d}<v_k^*,v_\rho>}$ can be written in terms of x_k's as an algebraic function.

Given an integral point $l \in L$, consider the GKZ operator (it differs from the standard GKZ operator by a factor of $a_0\prod_{l_i>0} a_i^{l_i}$)

$$\begin{aligned}
D_l &= a_0\left\{\prod_{l_i>0} a_i^{l_i}\left(\frac{\partial}{\partial a_i}\right)^{l_i} - a^l\prod_{l_i<0} a_i^{-l_i}\left(\frac{\partial}{\partial a_i}\right)^{-l_i}\right\} \\
&= \left\{\prod_{j=1}^{l_0}(a_0\frac{\partial}{\partial a_0} - j)\prod_{i\neq 0, l_i>0}\prod_{j=0}^{l_i-1}(a_i\frac{\partial}{\partial a_i} - j) - a^l\prod_{j=1}^{-l_0}(a_0\frac{\partial}{\partial a_0} - j)\prod_{i\neq 0, l_i<0}\prod_{j=0}^{-l_i-1}(a_i\frac{\partial}{\partial a_i} - j)\right\} a_0 \\
&= \tilde{D}_l a_0
\end{aligned}$$

where we use the convention that $\prod_{i=1}^{m}(\cdots) = 1$ if $m \leq 0$. From

$$D_l\Pi(a) = 0$$

We get

$$e^{\mathcal{L}_V}\tilde{D}_l e^{-\mathcal{L}_V}\tilde{\Pi}(x) = 0 \tag{3.6}$$

Lemma 3.1.

$$e^{\mathcal{L}_V}(a_i\frac{\partial}{\partial a_i})e^{-\mathcal{L}_V} = a_i\frac{\partial}{\partial a_i} + \delta_{1\leq i\leq 5}\mathcal{L}_{\frac{w_i}{d}z_i\frac{\partial}{\partial z_i}} - \delta_{i,0}\mathcal{L}_{\sum_{\rho=1}^{5}\frac{w_\rho}{d}z_\rho\frac{\partial}{\partial z_\rho}}$$

here $\delta_{1\leq i\leq 5} = 1$ *if* $1 \leq i \leq 5$ *and otherwise* 0.

Proof. Since

$$\begin{aligned}\left[V, a_i\frac{\partial}{a_i}\right] &= \delta_{1\le i\le 5}\frac{w_i}{d}z_i\frac{\partial}{\partial z_i} - \delta_{i,0}\sum_{\rho=1}^{5}\frac{w_\rho}{d}z_\rho\frac{\partial}{\partial z_\rho}\\ \left[V,\left[V, a_i\frac{\partial}{a_i}\right]\right] &= 0\end{aligned}$$

The lemma follows from the formula

$$e^{\mathcal{L}_V}(a_i\frac{\partial}{\partial a_i})e^{-\mathcal{L}_V} = a_i\frac{\partial}{\partial a_i} + \sum_{k=1}^{\infty}\frac{1}{k!}\mathcal{L}_{(ad_V)^k\left(a_i\frac{\partial}{\partial a_i}\right)}$$

□

It follows from the lemma that

$$\left\{\prod_{j=1}^{l_0}(a_0\frac{\partial}{\partial a_0} - j)\prod_{i\neq 0, l_i>0}\prod_{j=0}^{l_i-1}(a_i\frac{\partial}{\partial a_i} - j + \delta_{1\le i\le 5}\mathcal{L}_{\frac{w_i}{d}z_i\frac{\partial}{\partial z_i}}) - a^l\prod_{j=1}^{-l_0}(a_0\frac{\partial}{\partial a_0} - j)\prod_{i\neq 0, l_i<0}\prod_{j=0}^{-l_i-1}(a_i\frac{\partial}{\partial a_i} - j + \delta_{1\le i\le 5}\mathcal{L}_{\frac{w_i}{d}z_i\frac{\partial}{\partial z_i}})\right\}\tilde{\Pi}(x) = 0 \quad (3.7)$$

where we have used $\mathcal{L}_{\sum_{\rho=1}^{5}\frac{w_\rho}{d}z_\rho\frac{\partial}{\partial z_\rho}}\tilde{\Pi}(x) = 0$ to eliminate the terms with $\mathcal{L}_{\sum_{\rho=1}^{5}\frac{w_\rho}{d}z_\rho\frac{\partial}{\partial z_\rho}}$.

Observe that each Lie derivative $\mathcal{L}_{\frac{w_i}{d}z_i\frac{\partial}{\partial z_i}}$ commutes with all other operators appearing on the left side of (3.7). So we can move every term involving $\mathcal{L}_{\frac{w_i}{d}z_i\frac{\partial}{\partial z_i}}$ to the right side, so that (3.7) can now be explicitly written as

$$\tilde{D}_l\tilde{\Pi}(x) = -\sum_{i=1}^{5}\mathcal{L}_{\frac{w_i}{d}z_i\frac{\partial}{\partial z_i}}\alpha_i$$

where the α_i are (easily computable) d-closed 4-forms depending on l. By the Cartan-Lie formula $\mathcal{L}_X = d\iota_X + \iota_X d$, we obtain the formula

Proposition 3.2. *[β-term Formula]*

$$\tilde{D}_l\tilde{\Pi}(x) = -d\beta_l$$

where $\beta_l = \sum_i \iota_{\frac{w_i}{d}z_i\frac{\partial}{\partial z_i}}\alpha_i$.

Next we consider the differential operators of the extended GKZ system induced by the automorphism of the ambient toric variety. The corresponds to the root of Δ^* [17]. Let

$$v_i^* \in R(\Delta^*), \quad < v_i^*, v_{\rho_i} >= -1, \quad < v_i^*, v_\rho >\ge 0 \text{ for } \rho\neq\rho_i$$

then we obtain an equation

$$\left\{\sum_{v_k^*\in\Delta^*}(< v_k^*, v_{\rho_i} > +1)a_{v_k^*}\frac{\partial}{\partial a_{v_k^*+v_i^*}}\right\}\frac{1}{a_0}e^{-\mathcal{L}_V}\tilde{\Pi}(x) = \mathcal{L}_{\left\{\prod_{\rho=1}^{5}z_\rho^{< v_i^*, v_\rho >}\right\}z_{\rho_i}\frac{\partial}{\partial z_{\rho_i}}}\frac{1}{a_0}e^{-\mathcal{L}_V}\tilde{\Pi}(x)$$

where in the formula we have identified $a_k \equiv a_{v_k^*}$ for convenience, and $a_{v_k^*+v_i^*}$ just corresponds to the point $v_k^*+v_i^*$. Note that

$$\begin{aligned} & a_{v_i^*}\left\{\sum_{v_k^*\in\Delta^*}(<v_k^*,v_{\rho_i}>+1)a_{v_k^*}\frac{\partial}{\partial a_{v_k^*+v_i^*}}\right\}\frac{1}{a_0} \\ = \quad & \left\{\sum_{v_k^*\in\Delta^*}(<v_k^*,v_{\rho_i}>+1)\frac{a_{v_k^*}a_{v_i^*}}{a_0 a_{v_i^*+v_k^*}}\left(a_{v_k^*+v_i^*}\frac{\partial}{\partial a_{v_k^*+v_i^*}}\right)\right\}-2\frac{a_{v_i^*}a_{-v_i^*}}{a_0^2} \end{aligned}$$

where the last term is zero if $-v_i^* \neq \Delta^*$. On the other hand, it's easy to compute

$$e^{\mathcal{L}_V}\mathcal{L}_{\left\{\prod\limits_{\rho=1}^{5} z_\rho^{<v_i^*,v_\rho>}\right\}z^{\rho_i}\frac{\partial}{\partial z^{\rho_i}}}e^{-\mathcal{L}_V}=\prod_{\rho=1}^{5}\left(\frac{a_0}{a_\rho}\right)^{\frac{w_\rho}{d}<v_i^*,v_\rho>}\mathcal{L}_{\left\{\prod\limits_{\rho=1}^{5} z_\rho^{<v_i^*,v_\rho>}\right\}z^{\rho_i}\frac{\partial}{\partial z^{\rho_i}}}$$

therefore we get the following

$$\begin{aligned} &\left\{\left\{\sum_{v_k^*\in\Delta^*}(<v_k^*,v_{\rho_i}>+1)\frac{a_{v_k^*}a_{v_i^*}}{a_0 a_{v_i^*+v_k^*}}\left(a_{v_k^*+v_i^*}\frac{\partial}{\partial a_{v_k^*+v_i^*}}+\delta_{1\le v_i^*+v_k^*\le 5}\mathcal{L}_{\frac{w_i}{d}z_{v_i^*+v_k^*}\frac{\partial}{\partial z_{v_i^*+v_k^*}}}\right)\right\}-2\frac{a_{v_i^*}a_{-v_i^*}}{a_0^2}\right\}\tilde{\Pi}(x) \\ &\qquad\qquad\qquad = \left(\frac{a_{v_i^*}}{a_0}\right)\prod_\rho\left(\frac{a_0}{a_\rho}\right)^{\frac{w_\rho}{d}<v_i^*,v_\rho>}\mathcal{L}_{\left\{\prod_\rho z_\rho^{<v_i^*,v_\rho>}\right\}z^{\rho_i}\frac{\partial}{\partial z^{\rho_i}}}\tilde{\Pi}(x) \end{aligned}$$

Again, by the Cartan-Lie formula we can easily write the right side as a d-exact form.

Example: $\mathbf{P}(1,1,1,1,1)$

We will compute the β-term for mirror quintic [31], where $w=(1,1,1,1,1)$, and

$$f_{\Delta^*}=a_0\prod_{i=1}^{5}z_i+\sum_{i=1}^{5}a_i z_i^5.$$

The relation lattice is generated by

$$l=(-5,1,1,1,1,1)$$

and the moduli variable is

$$.x=(-1)^{l_0}a^l=-\frac{\prod\limits_{i=1}^{5}a_i}{a_0^5} \tag{3.8}$$

Put

$$\tilde{\Pi}(x)=\frac{\omega}{\prod\limits_{i=1}^{5}z_i-x^{\frac{1}{5}}\sum\limits_{i=1}^{5}z_i^5} \tag{3.9}$$

Then our β-term formula Proposition 3.2 for the Picard-Fuchs equation yields

$$\left\{\prod_{i=1}^{5}\left(\Theta_x+\frac{1}{5}\mathcal{L}_{z_i\partial_{z_i}}\right)-x\prod_{i=1}^{5}(5\Theta_x+i)\right\}\tilde{\Pi}(x)=0 \tag{3.10}$$

or

$$\left(\Theta_x^5-x\prod_{i=1}^{5}(5\Theta_x+i)\right)\tilde{\Pi}(x)=-d\beta_x,\quad d\beta_x=\sum_{I\subset\{1,\ldots,5\},|I|\ge 1}\frac{\Theta_x^{5-|I|}}{5^{|I|}}\prod_{k\in I}(\mathcal{L}_{z_k\partial_k})\tilde{\Pi}(x).$$

Note that the sum of terms for $|I|=1$ in the expression of $d\beta_x$ is zero. A choice of β_x relevant in the next section will be

$$
\begin{aligned}
\beta_x &= \frac{\Theta_x^3}{5^2}\left\{z_1\iota_{\partial_1}\mathcal{L}_{z_2\partial_2}+z_3\iota_{\partial_3}\mathcal{L}_{z_4\partial_4}+(z_3\iota_{\partial_3}+z_4\iota_{\partial_4})(\mathcal{L}_{z_1\partial_1}+\mathcal{L}_{z_2\partial_2})\right\}\tilde{\Pi}(x) \\
&+\frac{\Theta_x^2}{5^3}\left\{(z_3\iota_{\partial_3}+z_4\iota_{\partial_4})\mathcal{L}_{z_1\partial_1}\mathcal{L}_{z_2\partial_2}+z_3\iota_{\partial_3}(\mathcal{L}_{z_1\partial_1}+\mathcal{L}_{z_2\partial z_2})\mathcal{L}_{z_4\partial_4}\right\}\tilde{\Pi}(x) \\
&+\frac{\Theta_x}{5^4}z_3\iota_{\partial_3}\mathcal{L}_{z_4\partial_4}\mathcal{L}_{z_1\partial_1}\mathcal{L}_{z_2\partial_2}\tilde{\Pi}(x)+\sum_{I\subset\{1,\ldots,4\},|I|\geq 1}\frac{\Theta_x^{4-|I|}}{5^{|I|+1}}z_5\iota_{\partial_5}\prod_{k\in I}(\mathcal{L}_{z_k\partial_k})\tilde{\Pi}(x)
\end{aligned}
$$

where ι_{∂_i} is the contraction with the vector $\frac{\partial}{\partial z_i}$.

Example: $\mathbf{P}(2,2,2,1,1)$

The mirror of degree 8 hypersurface in $\mathbf{P}(2,2,2,1,1)$ has two complex moduli. The integral points of its dual polytope Δ^* is given by

$$
\Delta^*: \quad
\begin{aligned}
v_0^* &= (0,0,0,0) \\
v_1^* &= (1,0,0,0) \\
v_2^* &= (0,1,0,0) \\
v_3^* &= (0,0,1,0) \\
v_4^* &= (0,0,0,1) \\
v_5^* &= (-2,-2,-2,-1) \\
v_6^* &= (-1,-1,-1,0)
\end{aligned}
$$

and in homogeneous coordinate

$$f_{\Delta^*} = a_0 z_1 z_2 z_3 z_4 z_5 + a_1 z_1^4 + a_2 z_2^4 + a_3 z_3^4 + a_4 z_4^8 + a_5 z_5^8 + a_6 z_4^4 z_5^4$$

A basis of relation lattice is given by

$$l^{(1)} = (-4,1,1,1,0,0,1), \quad l^{(2)} = (0,0,0,0,1,1,-2)$$

We get the moduli coordinates

$$x_1 = \frac{a_1 a_2 a_3 a_6}{a_0^4}, \; x_2 = \frac{a_4 a_5}{a_6^2}$$

The rational 4-form is given by

$$\tilde{\Pi}(x) = \frac{\Omega_0}{z_1 z_2 z_3 z_4 z_5 + x_1^{\frac{1}{4}} x_2^{\frac{1}{8}} (z_1^4 + z_2^4 + z_3^4 + z_4^8 + z_5^8) + x_1^{\frac{1}{4}} x_2^{-\frac{3}{8}} z_4^4 z_5^4}$$

The exact GKZ equation from $l^{(1)}$ and $l^{(2)}$ can be read

$$
\begin{aligned}
\left\{(\Theta_{x_1}-2\Theta_{x_2})\prod_{j=1}^{3}(\Theta_{x_1}+\frac{1}{4}\mathcal{L}_{z_i\partial_{z_i}})-x_1\prod_{j=1}^{4}(4\Theta_{x_1}+j)\right\}\tilde{\Pi}(x) &= 0 \\
\left\{\left(\Theta_{x_2}+\frac{1}{8}\mathcal{L}_{z_4\partial_{z_4}}\right)\left(\Theta_{x_2}+\frac{1}{8}\mathcal{L}_{z_5\partial_{z_5}}\right)-x_2\left(\Theta_{x_1}-2\Theta_{x_2}\right)\left(\Theta_{x_1}-2\Theta_{x_2}-1\right)\right\}\tilde{\Pi}(x) &= 0
\end{aligned}
$$

Example: $\mathbf{P}(7,2,2,2,1)$

The mirror of degree 14 hypersurface in $\mathbf{P}(7,2,2,2,1)$ has also two complex moduli. The integral points of its dual polytope Δ^* is given by

$$
\begin{aligned}
\Delta^* : \qquad & v_0^* = (0,0,0,0) \\
& v_1^* = (1,0,0,0) \\
& v_2^* = (0,1,0,0) \\
& v_3^* = (0,0,1,0) \\
& v_4^* = (0,0,0,1) \\
& v_5^* = (-7,-2,-2,-2) \\
& v_6^* = (-3,-1,-1,-1) \\
& v_7^* = (-4,-1,-1,-1) \\
& v_8^* = (-1,0,0,0)
\end{aligned}
$$

and in homogeneous coordinate

$$
f_{\Delta^*} = a_0 z_1 z_2 z_3 z_4 z_5 + a_1 z_1^2 + a_2 z_2^7 + a_3 z_3^7 + a_4 z_4^7 + a_5 z_5^{14} + a_6 z_1 z_5^7 + a_7 z_2 z_3 z_4 z_5^8 + a_8 z_2^2 z_3^2 z_4^2 z_5^2
$$

A basis of relation lattice is given by [17]

$$
\begin{aligned}
l^{(1)} &= (-1,0,0,0,0,-1,1,1,0), \quad l^{(2)} = (0,1,0,0,0,1,-2,0,0) \\
l^{(3)} &= (0,0,1,1,1,0,0,1,-4), \quad l^{(4)} = (0,0,0,0,0,1,0,-2,1)
\end{aligned}
$$

We get the "moduli" coordinates before eliminating the automorphism generated by the roots

$$
x_1 = -\frac{a_6 a_7}{a_0 a_5}, \; x_2 = \frac{a_1 a_5}{a_6^2}, \; x_3 = \frac{a_2 a_3 a_4 a_7}{a_8^4}, \; x_4 = \frac{a_5 a_8}{a_7^2}
$$

The rational 4-form is given by

$$
\tilde{\Pi}(x) = \frac{\Omega_0}{z_1 z_2 z_3 z_4 z_5 - x_1 x_2^{\frac{1}{2}} x_3^{\frac{1}{7}} x_4^{\frac{4}{7}} \left(z_1^2 + z_2^7 + z_3^7 + z_4^7 + z_5^{14}\right) - x_1 x_3^{\frac{1}{7}} x_4^{\frac{4}{7}} z_1 z_5^7 - x_1 x_2^{\frac{1}{2}} z_2 z_3 z_4 z_5^8 - x_1 x_2^{\frac{1}{2}} x_3^{-\frac{1}{7}} x_4^{\frac{3}{7}} z_2^2 z_3^2 z_4^2 z_5^2}
$$

We consider the GKZ operator corresponding to the following two relation vectors ([17])

$$
l_{\{1,5\}} = (0,1,0,0,0,1,-2,0,0), \quad l_{\{2,3,4,6\}} = (-1,0,1,1,1,0,1,0,-3)
$$

which gives two exact GKZ equations

$$
\begin{aligned}
\left\{\left(\Theta_{x_2} + \frac{1}{2}\mathcal{L}_{z_1 \partial_{z_1}}\right)\left(-\Theta_{x_1} + \Theta_{x_2} + \Theta_{x_4} + \frac{1}{14}\mathcal{L}_{z_5 \partial_{z_5}}\right) - x_2 \left(\Theta_{x_1} - 2\Theta_{x_2}\right)\left(\Theta_{x_1} - 2\Theta_{x_2} - 1\right)\right\} \tilde{\Pi}(x) &= 0 \\
\left\{\left(\Theta_{x_1} - 2\Theta_{x_2}\right) \prod_{i=2}^{4} \left(\Theta_{x_3} + \frac{1}{7}\mathcal{L}_{z_i \partial_{z_i}}\right) - x_1 x_3 x_4 \left(\Theta_{x_1} + 1\right) \prod_{i=0}^{2} \left(\Theta_{x_4} - 4\Theta_{x_3} - i\right)\right\} \tilde{\Pi}(x) &= 0
\end{aligned}
$$

and the roots v_7^*, v_8^* give two exact equations

$$
\begin{aligned}
\left\{\left(\Theta_{x_1} + \Theta_{x_3} - 2\Theta_{x_4}\right) - 2x_1 x_2 \left(\Theta_{x_1} - 2\Theta_{x_2}\right) - x_1 \left(-\Theta_{x_1} + \Theta_{x_2} + \Theta_{x_4} + \frac{1}{14}\mathcal{L}_{z_5 \partial_{z_5}}\right) + x_1 x_2^{\frac{1}{2}} \mathcal{L}_{z_5^7 \partial_{z_1}}\right\} \tilde{\Pi}(x) &= 0 \\
\left\{\left(\Theta_{x_4} - 4\Theta_{x_3}\right) - 2x_1^2 x_2 x_4 (\Theta_{x_1} + 1) - x_1 x_4 \left(\Theta_{x_1} + \Theta_{x_3} - 2\Theta_{x_4}\right) + x_1 x_2^{\frac{1}{2}} x_3^{-\frac{1}{7}} x_4^{\frac{3}{7}} \mathcal{L}_{z_2 z_3 z_4 z_5 \partial_{z_1}}\right\} \tilde{\Pi}(x) &= 0
\end{aligned}
$$

Calabi-Yau hypersurfaces in a general toric variety

For Calabi-Yau hypersurfaces of non-Fermat type, we can repeat the derivation of the β-term above by replacing the homogeneous coordinates by the toric coordinates X_i. Write the defining equation of a general Calabi-Yau hypersurface is in toric coordinates:

$$f_{\Delta^*}(X) = \sum_{v_k^* \in \Delta^*} a_k X^{v_k^*}.$$

Here we will assume that $v_0^* = (0,0,0,0), v_1^* = (1,0,0,0), v_2^*(0,1,0,0), v_3^* = (0,0,1,0), v_4^* = (0,0,0,1)$. Put

$$\Pi(a) = \frac{1}{f_{\Delta^*}(X)} \prod_{j=1}^{4} \frac{dX_j}{X_j}$$

As in the Fermat case above, we use the automorphism

$$\phi : X_i \to \frac{a_0}{a_i} X_i, \quad 1 \leq i \leq 4$$

to transform $\Pi(a)$ into a form parameterized only by moduli variables $x_k = (-1)^{l_0^{(k)}} a^{l^{(k)}}$. We can now repeat the derivation of the β-term for each GKZ operator D_l, using the form $\tilde{\Pi}(x) = a_0 \phi^* \Pi(a)$. To summarize, we have the following result.

Proposition 3.3. *Let $\tilde{\Pi}(x) = a_0 \phi^* \Pi(a)$, and for each $l \in L$, let*

$$\tilde{D}_l = \prod_{j=1}^{l_0} (a_0 \frac{\partial}{\partial a_0} - j) \prod_{i \neq 0, l_i > 0} \prod_{j=0}^{l_i - 1} (a_i \frac{\partial}{\partial a_i} - j) - a^l \prod_{j=1}^{-l_0} (a_0 \frac{\partial}{\partial a_0} - j) \prod_{i \neq 0, l_i < 0} \prod_{j=0}^{-l_i - 1} (a_i \frac{\partial}{\partial a_i} - j).$$

Then $\tilde{\Pi}(x)$ is killed by the differential operator obtained from $\tilde{D}_l$ by the substitutions:

$$\begin{aligned} a_0 \frac{\partial}{\partial a_0} &\mapsto a_0 \frac{\partial}{\partial a_0} - \mathcal{L}_{\sum_{j=1}^{4} X_j \frac{\partial}{\partial X_j}} \\ a_j \frac{\partial}{\partial a_j} &\mapsto a_j \frac{\partial}{\partial a_j} + \mathcal{L}_{X_j \frac{\partial}{\partial X_j}}, \quad 1 \leq j \leq 4 \\ a_j \frac{\partial}{\partial a_j} &\mapsto a_j \frac{\partial}{\partial a_j}, \quad j > 4. \end{aligned}$$

From this, the Cartan-Lie formula then explicitly yields a β-term for each GKZ operator:

$$\tilde{D}_l \tilde{\Pi}(x) = -d\beta_l.$$

4 Application to Abel-Jacobi Map And Inhomogeneous Picard-Fuchs Equation

In this section, we apply the inhomogeneous Picard-Fuchs equation Proposition 2.5 to study the Abel-Jacobi map for toric Calabi-Yau hypersurfaces that arise in open string mirror symmetry [25]. We will derive a purely algebraic double residue formula for computing the boundary integral of the β-term.

We will keep the same notation as in section 1: X_z will be a family of hypersurfaces moving in fixed 4-dimensional ambient space M parametrized by z, and we consider two fixed hypersurfaces $Y_1 : Q_1 = 0, Y_2 : Q_2 = 0$, where Q_1, Q_2 are sections of some fixed divisors. We consider a pair of family of curves C_z^+, C_z^- such that they are homologically equivalent as cycles

$$C_z^{\pm} \hookrightarrow X_z \cap Y_1 \cap Y_2, \quad [C_z^+] = [C_z^-] \in H_2(X_z, \mathbb{Z})$$

We denote by $\mathcal{C}^{\pm}$ he corresponding families

$$\begin{array}{ccc} \mathcal{C}^{\pm} & \hookrightarrow & \mathcal{X} \\ & \searrow & \downarrow \\ & & S \end{array}$$

Let $(\mathcal{H}^3_{\mathbb{Z}}, F^*\mathcal{H}^3_{\mathbb{C}})$ be the integral Hodge structure for the family $\mathcal{X} \to S$, and $\mathcal{J}^3$ be the intermediate Jacobian fibration

$$\mathcal{J}^3 = \frac{\mathcal{H}^3_{\mathbb{C}}}{F^2\mathcal{H}^3_{\mathbb{C}} \oplus \mathcal{H}^3_{\mathbb{Z}}}$$

The fiber of $\mathcal{J}^3$ over z can be identified with

$$J^3_z = \frac{\left(F^2 H^3(X_z, \mathbb{C})\right)^*}{H_3(X_z, \mathbb{Z})}.$$

The Abel-Jacobi map associated to $\mathcal{C}^{\pm}$ is a normal function of $\mathcal{J}^3$ given by

$$\int_{C_z^-}^{C_z^+} : S \to \mathcal{J}^3.$$

Suppose we are given a family of rational 4-form with pole of order one along X_z

$$\omega_z \in H^0(M, K_M(X_z)).$$

Then we obtain a section $Res_{X_z}\omega_z$ of $F^3\mathcal{H}^3_{\mathbb{C}}$. The integral that we will study is

$$\int_{C_z^-}^{C_z^+} Res_{X_z}\omega_z,$$

which is well-defined up to periods of closed cycles on X_z.

We can assume that $C_z^+ \cup C_z^-$ moves equisingularly above S, for otherwise we shrink S to some smaller open subset to achieve this. We can always choose local trivialization of $\mathcal{X}$ over a small disk around any point in S which also trivializes $\mathcal{C}^+ \cup \mathcal{C}^-$. Therefore it's easy to see that $H^3(X_z, Y_z)$ still forms a local system in this case by $Y_z = C_z^+ \cup C_z^-$ and the discussion in section one is still valid. Let $\{p_{z,A}\}$ be the singular points of $Y_1 \cap Y_2 \cap X_z$ corresponding to non-transversal intersections. It contains $C_z^+ \cap C_z^-$ and possibly some other points on C_z^+ or C_z^- when there are components of $Y_1 \cap Y_2$ other than $C_z^{\pm}$. *We assume that at each $p_{z,A}$, one of Y_1, Y_2 is intersecting transversely with X_z.* Therefore we can always put an ϵ-tube around each of $C_z^{\pm}$ such that the ϵ-tube lies on $Y_1 \cap Y_2$ outside a small disk centered at each $p_{z,A} \in C_z^+ \cup C_z^-$, where the ϵ-tube lies in one of Y_1 or Y_1 around that small disk.

Suppose $\mathcal{D} = \mathcal{D}(\partial_z)$ is a Picard-Fuchs operator, and

$$\mathcal{D}(\partial_z)\omega_z = -d\beta_z$$

Applying (2.13) and using the fact that β_z is of type $(3,0)$, we get

$$\begin{aligned}\mathcal{D}(\partial_z)\int_{C_z^-}^{C_z^+} Res_{X_z}\omega_z &= \frac{1}{2\pi i}\int_{T_\epsilon(C_z^+ - C_z^-)}\beta_z \\ &= \sum_{p_{z,A}\in C_z^+}\frac{1}{2\pi i}\int_{T_\epsilon(D_{z,A}^+)}\beta_z - \sum_{p_{z,A}\in C_z^-}\frac{1}{2\pi i}\int_{T_\epsilon(D_{z,A}^-)}\beta_z\end{aligned}$$

where $D_{z,A}^{\pm}$ is a small disk around $p_{z,A}\in C_z^{\pm}$ such that $T_\epsilon(\partial D_{z,A}^{\pm})\in Y_1\cap Y_2$.

Fix a $p_{z,A}\in C_z^+$ and let $P_{z,A}=0$ be the local defining equation for X_z near $p_{z,A}$. We assume that $T_\epsilon(D_{z,A}^+)\subset Y_1$. Then β_z restricts to Y_1 becoming a top $(3,0)$-form, and we can write

$$\beta_z|_{Y_1} = d\eta_{z,A} \tag{4.1}$$

for some local holomorphic $(2,0)$-form $\eta_{z,A}$ on Y_1, defined near $p_{z,A}$ and with poles along $Y_1\cap X_z$. Hence

$$\begin{aligned}\frac{1}{2\pi i}\int_{T_\epsilon(D_{z,A}^+)}\beta_z &= -\frac{1}{2\pi i}\int_{T_\epsilon(\partial D_{z,A}^+)}\eta_{z,A} \\ &= -\int_{\partial D_{z,A}^+} Res_{C_{z,A}^+}(\eta_{z,A}|_{Y_1\cap Y_2}) \\ &= -2\pi i\ Res_{p_{z,A}}Res_{C_z^+}(\eta_{z,A}|_{Y_1\cap Y_2})\end{aligned}$$

Thus we arrive at the following "localization" formula

Theorem 4.1 (Double Residue Formula)**.** *Under the stated assumption above, we have*

$$\begin{aligned}\mathcal{D}(\partial_z)\frac{1}{2\pi i}\int_{C_z^-}^{C_z^+} Res_{X_z}\omega_z &= -\sum_{p_{z,A}\in C_z^+} Res_{p_{z,A}}Res_{C_z^+}(\eta_{z,A}|_{Y_1\cap Y_2}) \\ &\quad + \sum_{p_{z,A}\in C_z^-} Res_{p_{z,A}}Res_{C_z^+}(\eta_{z,A}|_{Y_1\cap Y_2})\end{aligned} \tag{4.2}$$

Note that in this double residue formula, we can throw away those terms in $\eta_{z,A}$ with pole of order one along X_z. In fact, since X_z and Y_1 intersects transversely locally at $p_{z,A}$, we can write locally

$$\beta_z|_{Y_1} = d(\xi_{z,A}) + \alpha_{z,A}$$

where $\alpha_{z,A}$ has pole of order one along $X_z\cap Y_1$. Since $d\alpha_{z,A}=0$, we can furthermore write

$$\alpha_{z,A} = d(\alpha'_{z,A})$$

where $\alpha'_{z,A}$ is logrithmic along $X_z\cap Y_1$. Then

$$Res_{C_z^+}\alpha'_{z,A}$$

will be holomorphic on C_z^+ around $p_{z,A}$, which is killed by taking another residue at $p_{z,A}$. This leads to the following algebraic algorithm to calculate $\eta_{z,A}$ in practice: if locally around $p_{z,A}$ inside Y_1 we have

$$\beta_z|_{Y_1} = \frac{dP_{z,A}}{P_{z,A}^l}\wedge\zeta, \quad l>1$$

where ζ has no pole along $X_z \cap Y_1$, then we can reduce the order of pole by

$$\beta_z|_{Y_1} = -d\left(\frac{\zeta}{(l-1)P_{z,A}^{l-1}}\right) + \frac{d\zeta}{(l-1)P_{z,A}^{l-1}}$$

and apply this procedure until we get

$$\beta_z|_{Y_1} = d(\psi_{z,A}) + \varphi_{z,A}$$

where $\psi_{z,A}$ is of the form $\frac{\zeta_{l-1}}{P_{z,A}^{l-1}} + \frac{\zeta_{l-2}}{P_{z,A}^{l-2}} + \cdots + \frac{\zeta_1}{P_{z,A}}$, and $\varphi_{z,A}$ has pole of order one. Then we can simply use $\psi_{z,A}$ instead of η_{z_A} in (4.2) to compute the double residue.

Example: Mirror Quintic

We will apply formula (4.2) to the example studied in [25] where the inhomogeneous Picard-Fuchs equation for the Abel-Jacobi map was computed by using a much more elaborate analytic method.

Consider the one-parameter family of quintic in $\mathbb{P}^4$ defined by

$$X_x = \{P_x = \prod_{i=1}^{5} z_i - x^{1/5} \sum_{i=1}^{5} z_i^5 = 0\} \tag{4.3}$$

Let $\omega = \iota_\theta \lrcorner \Omega,\ \Omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_5,\ \theta = \sum_{i=1}^{5} z_i \frac{\partial}{\partial z_i}$, and the family of holomorphic 3-form on X_x is given by the residue map

$$\Omega_x = Res_{P_x} \frac{\omega}{P_x}$$

and consider a pair of families of curves

$$C_x^{\pm} = \{z_1 + z_2 = 0, z_3 + z_4 = 0, z_1 z_3 \pm \sqrt{x^{1/5}} z_5^2 = 0\}$$

We denote

$$Y_1 = \{z_1 + z_2 = 0\}, \quad Y_2 = \{z_3 + z_4 = 0\}$$

then Y_1 is transversal with P_x except at p_1, and Y_2 is transversal with P_x except at p_2, where

$$p_1 = [1, -1, 0, 0, 0], \quad p_2 = [0, 0, 1, -1, 0]$$

We consider the Picard-Fuchs operator from the GKZ-system and by (3.10)

$$D(\partial_x) = \Theta_x^5 - x \prod_{i=1}^{5} (5\Theta_x + i), \quad d\beta_x = \sum_{I \subset \{1,\dots,5\}, |I| \geq 1} \frac{\Theta_x^{5-|I|}}{5^{|I|}} \prod_{k \in I} (\mathcal{L}_{z_k \partial_k}) \tilde{\Pi}(x)$$

We choose β to be

$$\begin{aligned}
\beta \quad = \quad & \frac{\Theta_x^3}{5^2} \{z_1 \iota_{\partial_1} \mathcal{L}_{z_2 \partial_2} + z_3 \iota_{\partial_3} \mathcal{L}_{z_4 \partial_4} + (z_3 \iota_{\partial_3} + z_4 \iota_{\partial_4})(\mathcal{L}_{z_1 \partial_1} + \mathcal{L}_{z_2 \partial_2})\} \tilde{\Pi}(x) \\
& + \frac{\Theta_x^2}{5^3} \{(z_3 \iota_{\partial_3} + z_4 \iota_{\partial_4}) \mathcal{L}_{z_1 \partial_1} \mathcal{L}_{z_2 \partial_2} + z_3 \iota_{\partial_3} (\mathcal{L}_{z_1 \partial_1} + \mathcal{L}_{z_2 \partial z_2}) \mathcal{L}_{z_4 \partial_4}\} \tilde{\Pi}(x) \\
& + \frac{\Theta_x}{5^4} z_3 \iota_{\partial_3} \mathcal{L}_{z_4 \partial_4} \mathcal{L}_{z_1 \partial_1} \mathcal{L}_{z_2 \partial_2} \tilde{\Pi}(x) + \sum_{I \subset \{1,\dots,4\}, |I| \geq 1} \frac{\Theta_x^{4-|I|}}{5^{|I|+1}} z_5 \iota_{\partial_5} \prod_{k \in I} (\mathcal{L}_{z_k \partial_k}) \tilde{\Pi}(x)
\end{aligned}$$

Then the calculation of Abel-Jacobi map is localized near p_1, p_2 by formula (4.2).

• Near p_2. We have

$$\beta|_{Y_1} = \frac{\Theta_x^3}{5^2}\left(z_1\iota_{\partial_1}\mathcal{L}_{z_2\partial_2}\tilde{\Pi}(x)\right)|_{Y_1}$$

Note that we can move Θ_x outside the integral in the local calculation of double residue. If we choose local coordinate such that $z_3 = 1, z_2 = -z_1$, then

$$\begin{aligned}\left(z_1\iota_{\partial_1}\mathcal{L}_{z_2\partial_2}\tilde{\Pi}(x)\right)|_{Y_1} &= -\frac{x^{1/5}z_1(1+z_4^5+z_5^5+5z_1^5)dz_4dz_1dz_5}{\left[-x^{1/5}(1+z_4^5+z_5^5)-z_1^2z_4z_5\right]^2} \\ &= d\left(\frac{x^{1/5}z_1(1+z_4^5+z_5^5+5z_1^5)dz_1\wedge dz_5}{(-5x^{1/5}z_4^4-z_1^2z_5)\left[-x^{1/5}(1+z_4^5+z_5^5)-z_1^2z_4z_5\right]}\right) + \text{order one term}\end{aligned}$$

Then the relevant residue is given by

$$Res_{p_2}Res_{C_z^{\pm}}\left(\frac{x^{1/5}z_1(1+z_4^5+z_5^5+5z_1^5)dz_1\wedge dz_5}{(-5x^{1/5}z_4^4-z_1^2z_5)\left[-x^{1/5}(1+z_4^5+z_5^5)-z_1^2z_4z_5\right]}\right)|_{z_4=-1}$$

which is easily computed to be zero.

• Near p_1. We have

$$\beta|_{Y_2} = \frac{\Theta_x^3}{5^2}\left(z_3\iota_{\partial_3}\mathcal{L}_{z_4\partial_4}\tilde{\Pi}(x)\right) + \frac{\Theta_x^2}{5^3}d\left((\iota_{z_1\partial_1}+\iota_{z_2\partial z_2})z_3\iota_{\partial_3}\mathcal{L}_{z_4\partial_4}\tilde{\Pi}(x)\right) + \frac{\Theta_x}{5^4}d\left(\iota_{z_1\partial_1}\iota_{z_3\partial_3}\mathcal{L}_{z_4\partial_4}\mathcal{L}_{z_2\partial_2}\tilde{\Pi}(x)\right)$$

The first term will be zero after double residue by the same calculation as above. The contribution of the second term is also zero since

$$\left((\iota_{z_1\partial_1}+\iota_{z_2\partial z_2})z_3\iota_{\partial_3}\mathcal{L}_{z_4\partial_4}\tilde{\Pi}(x)\right)|_{Y_1\cap Y_2} = 0$$

Therefore we get

$$\begin{aligned}D(\partial_x)\frac{1}{2\pi i}\int_{C_x^-}^{C_x^+}\Omega_x &= -\frac{\Theta_x}{5^4}Res_{p_1}Res_{C_z^+}\left(\iota_{z_1\partial_1}\iota_{z_3\partial_3}\mathcal{L}_{z_4\partial_4}\mathcal{L}_{z_2\partial_2}\tilde{\Pi}(x)\right)|_{Y_1\cap Y_2} \\ &\quad +\frac{\Theta_x}{5^4}Res_{p_1}Res_{C_z^-}\left(\iota_{z_1\partial_1}\iota_{z_3\partial_3}\mathcal{L}_{z_4\partial_4}\mathcal{L}_{z_2\partial_2}\tilde{\Pi}(x)\right)|_{Y_1\cap Y_2}\end{aligned}$$

The double residue now is easy to compute and we get

$$Res_{p_1}Res_{C_x^{\pm}}\left(\iota_{z_1\partial_1}\iota_{z_3\partial_3}\mathcal{L}_{z_4\partial_4}\mathcal{L}_{z_2\partial_2}\tilde{\Pi}(x)\right)|_{Y_1\cap Y_2} = \mp\frac{375}{8}\sqrt{x}$$

therefore

$$D(\partial_x)\frac{1}{2\pi i}\int_{C_x^-}^{C_x^+}\Omega_x = \Theta_x\frac{3}{20}\sqrt{x} \tag{4.4}$$

To compare our result with [25], we use the normalization

$$\begin{aligned}\hat{\Omega}_x &= -\left(\frac{5}{2\pi i}\right)^3\Omega_x, \\ \mathcal{L} &= \Theta_x^4 - 5x(5\Theta_x+1)(5\Theta_x+2)(5\Theta_x+3)(5\Theta_x+4) \\ D(\partial_x) &= \Theta_x\mathcal{L}\end{aligned}$$

then we have

$$\Theta_x \mathcal{L} \int_{C_x^-}^{C_x^+} \hat{\Omega}_x = \Theta_x \frac{75}{16\pi^2} \sqrt{x} \tag{4.5}$$

It follows that

$$\mathcal{L} \int_{C_x^-}^{C_x^+} \hat{\Omega}_x = \frac{75}{16\pi^2} \sqrt{x} + c$$

for some constant c. Since the monodromy $x \to e^{2\pi i} x$ will switch the curves C_x^+ and C_x^-, the constant c is zero. We therefore obtain the inhomogeneous term differs from the result of [25] by a factor of 5, which is exactly the order of stabilizer of $C_x^\pm$ under the finite quotient that yields the mirror quintic.

5 GKZ System and Picard-Fuchs Equation for Relative Cohomology

In this section, W will be fixed ambient space, $X_z = \{P_z = 0\}$ a family of hypersurfaces parametrized by z, and $D_u = \{Q_u = 0\}$ a family of hypersurfaces parametrized by u. Here P_z, Q_u moves in the linear systems of certain possibly different fixed line bundles on W. Let $Y_{z,u} \subset X_z$ be the intersection $X_z \cap D_u$, and we consider the local system of relative cohomology

$$H^3(X_z, Y_{z,u})$$

parametrized by both z and u. Given a rational form ω_z in W with pole of order one along X_z, and choose a smooth family of relative cycles $\Gamma_{z,u} \in H_3(X_z, Y_{z,u})$. We will study the Picard-Fuchs equation for the relative period

$$\Pi(z,u) = \int_{\Gamma_{z,u}} Res_{X_z} \omega_z. \tag{5.1}$$

We can assume that the relative cycle $\Gamma_{z,u}$ is away from any non-transversal intersection point of X_z, Q_u, since we can always move the relative cycle away from those points without changing its cohomology class.

We first consider the variation with respect to z. We choose ϵ-tube of $\Gamma_{z,u}$ such that $\partial T_\epsilon(\Gamma_{z,u})$ lies inside D_u. As shown in section 2, we have

$$(\partial_z)^k \int_{\Gamma_{z,u}} Res_{X_z} \omega_z = (\partial_z)^k \frac{1}{2\pi i} \int_{T_\epsilon(\Gamma_{z,u})} \omega_z = \frac{1}{2\pi i} \int_{T_\epsilon(\Gamma_{z,u})} (\partial_z)^k \omega_z \tag{5.2}$$

and the above integration is independent of ϵ. Let $\mathcal{D}_z$ be a Picard-Fuchs differential operator and let

$$\mathcal{D}_z \omega_z = -d\beta_z.$$

Then formula (5.2) yields

$$\begin{aligned} \mathcal{D}_z \Pi(z,u) &= \frac{1}{2\pi i} \int_{T_\epsilon(\Gamma_{z,u})} -d\beta_z = \frac{1}{2\pi i} \int_{T_\epsilon(\partial\Gamma_{z,u})} \beta_z|_{D_u} \\ &= \int_{(\partial\Gamma_{z,u})} Res_{Y_{z,u}} \beta_z|_{D_u} = \int_{(\partial\Gamma_{z,u})} Res_{Y_{z,u}} Res_{D_u} d\log Q_u \wedge \beta_z \end{aligned} \tag{5.3}$$

Note that $d\log Q_u \wedge \beta_z$ is not well-defined global form, but $Res_{D_u} d\log Q_u \wedge \beta_z$ is well-defined.

Next we fix z and consider the variation with respect to u. Choose a lifting of $\frac{\partial}{\partial u}$ to $\mathcal{Y}$

$$\frac{\partial}{\partial u} + n_{\mathcal{Y}_{z,u}}$$

where $n_{\mathcal{Y}_{z,u}}$ is the normal vector field corresponding to the deformation of $Y_{z,u}$ inside X_z with respect to u. Then using (2.11) and keeping the same notation as in section 2, we get

$$\partial_u \Pi(z,u) = \int_{\partial\Gamma_{z,u}} \iota_{n_{\mathcal{Y}_{z,u}}} \lrcorner \phi_z|_{X_z}$$

Since

$$\partial_u Q_u|_{Y_{z,u}} = -\left(\iota_{n_{\mathcal{Y}_{z,u}}} \lrcorner d_M Q_u\right)|_{Y_{z,u}}$$

and ϕ_z is holomorphic 3-form, we have

$$\begin{aligned} \left(\iota_{n_{\mathcal{Y}_{z,u}}} \lrcorner \phi_z|_{X_z}\right)|_{Y_{z,u}} &= -Res_{Y_{z,u}}((\partial_u \log Q_u)\phi_z)|_{X_z} \\ &= -Res_{Y_{z,u}} Res_{X_z}(\partial_u \log Q_u)\omega_z \\ &= Res_{Y_{z,u}} Res_{Q_u}(\partial_u \log Q_u)\omega_z. \end{aligned}$$

It follows that

$$\partial_u \Pi(z,u) = \int_{\partial\Gamma_{z,u}} Res_{Y_{z,u}} Res_{Q_u}(\partial_u \log Q_u)\omega_z \tag{5.4}$$

Note that $(\partial_u \log Q_u)\omega_z$ is globally well-defined rational form with pole along $X_z \cup D_u$. To summarize, we have

Theorem 5.1 (Variations of Relative Periods)**.** *Let X_z, $Y_{z,u}$, $\Gamma_{z,u}$, D_u, $\mathcal{D}_z$ and β_z be as given above. Then the relative period $\Pi(z,u) = \int_{\Gamma_{z,u}} Res_{X_z}\omega_z$ satisfies the equations*

$$\begin{aligned} \partial_z \Pi(z,u) &= \int_{T_\epsilon(\Gamma_{z,u})} \partial_z \omega_z, \quad \text{where } T_\epsilon(\partial\Gamma_{z,u}) \subset D_u = \{Q_u = 0\} \\ \partial_u \Pi(z,u) &= \int_{\partial\Gamma_{z,u}} Res_{Y_{z,u}} Res_{D_u}(\partial_u \log Q_u)\omega_z \\ \mathcal{D}_z \Pi(z,u) &= \int_{\partial\Gamma_{z,u}} Res_{Y_{z,u}} \beta|_{Q_u} = \int_{\partial\Gamma_{z,u}} Res_{Y_{z,u}} Res_{D_u} d\log Q_u \wedge \beta_z. \end{aligned}$$

Griffith-Dwork Reduction

Notice that if we pick a fixed Q_0, then we can formally write

$$\begin{aligned} Res_{D_u} d\log Q_u \wedge \beta_z &= Res_{D_u}\left(d\left(\log\left(\frac{Q_u}{Q_0}\right)\ \beta_z\right) - \log\frac{Q_u}{Q_0}\ d\beta_z\right) \\ &= Res_{D_u}\left(d\left(\log\left(\frac{Q_u}{Q_0}\right)\ \beta_z\right) + \log\left(\frac{Q_u}{Q_0}\right)\ \mathcal{D}_z\omega_z\right) \end{aligned}$$

then we get

$$\begin{aligned} \mathcal{D}_z \Pi(z,u) &= \int_{\partial\Gamma_{z,u}} Res_{Y_{z,u}} Res_{D_u} \mathcal{D}_z\left(\log\left(\frac{Q_u}{Q_0}\right)\ \omega_z\right) \\ \partial_u \Pi(z,u) &= \int_{\partial\Gamma_{z,u}} Res_{Y_{z,u}} Res_{D_u} \partial_u\left(\log\left(\frac{Q_u}{Q_0}\right)\ \omega_z\right) \end{aligned}$$

Therefore one can derive a Picard-Fuchs equation for $\Pi(z,u)$ using Griffith-Dwork reduction procedure [14] by starting with $\log\left(\frac{Q_u}{Q_0}\right)\ \omega_z$. This procedure of adding Log-Q is proposed in physics recently [21]. The theorem above shows that such a procedure is mathematical justified. We refer to [21] for applications of this procedure to specific examples.

Enhanced Polytope and GKZ System

We now apply the method developed in the preceding section to study toric B-branes on toric Calabi-Yau hypersurfaces. For physical motivations of B-brane geometry on Calabi-Yau manifolds and their mirror transformation, see [4]. The GKZ-system associated with an enhanced polytope as a way to understand toric B-brane geometry is proposed in [6]. We will give a uniform derivation of the general enhanced GKZ system by using the variation formula we have developed earlier.

We consider Calabi-Yau hypersurfaces in four dimensional toric variety M. The results extend easily to arbitrary dimensions. As in section 2, we assume that the anti-canonical sections $H^0(M, K_M^{-1})$ has a monomial basis in toric coordinates X_i, whose exponents are exactly the set of integral points of an integral polytope Δ^*. The defining equation for a Calabi-Yau hypersurface X_a^* in M in the toric coordinates has the form

$$f_{\Delta^*}(a) = \sum_{v_i^* \in \Delta^*} a_i X^{v_i^*} \tag{5.5}$$

where v_0^* is the origin by convention, and the relevant rational form with pole along the hypersurface is given by

$$\omega_a = \frac{1}{f_{\Delta^*}(a)} \prod_{i=1}^{4} \frac{dX_i}{X_i}$$

Let

$$L = \{l = (l_0, l_1, ...) \in \mathbb{Z}^{|\Delta^*|} | \sum_i l_i \bar{v}_i^* = 0\}, \quad \text{where} \quad \bar{v}_i^* = (1, v_i^*)$$

be the relation lattice. Next we briefly recall the so-called B-brane geometry introduced in physics. In order to describe the so-called superpotential in GLSM [33][20][4], one introduces an extra variable P, and let

$$\bar{X} = (P, X) = (P, X_1, X_2, X_3, X_4).$$

Obviously we have

$$\sum_i l_i = 0, \quad \prod_{v_i^* \in \Delta^*} \left(\bar{X}^{\bar{v}_i^*} \right)^{l_i} = 1, \quad l \in L. \tag{5.6}$$

The lattice L (denoted as "Q" in [4]) is thought of as the toric data that encodes the Calabi-Yau geometry. The "brane" data is specified by one additional lattice vector *not* in L:

$$q = \{q_i\}_{v_i^* \in \Delta^*} \in \mathbb{Z}^{|\Delta^*|}, \quad \sum_i q_i = 0. \tag{5.7}$$

(In principle, one can consider a brane configuration involving more than one lattice vectors. But for simplicity we will consider the case of only one such vector.) The B-brane corresponding to a choice of B-brane vector q is the divisor $Y_{a,b}^* = X_a^* \cap D_b$, where D_b is the closure in M of the locus

$$h(b) := b_0 + b_1 \prod_{v_i^* \in \Delta^*} \left(X^{v_i^*} \right)^{q_i} = 0. \tag{5.8}$$

Here $[b_0, b_1]$ is called the open string modulus. Under open string mirror symmetry, the B-model data (L, q) corresponds to certain A-model data that includes a special lagrangian cycle and a mirror manifold [4].

Next, we describe Picard-Fuchs equations for the relative periods of the family of pairs $(X_a^*, Y_{a,b}^*)$. Almost the entire *general theory* of periods in closed string mirror symmetry [16][17] turns out to carry over to the current open string context, with few modifications. For a given relative cycle $\Gamma_{a,b} \in H_3(X_a^*, Y_{a,b}^*)$, the corresponding relative period is

$$\Pi(a,b) = \int_{\Gamma_{a,b}} \frac{1}{f_{\Delta^*}(a)} \prod_{i=1}^{4} \frac{dX_i}{X_i}. \tag{5.9}$$

Consider the automorphism on M given by the torus action

$$\phi(\lambda) : X_k \to \lambda_k X_k, \ k = 1, .., 4.$$

This induces a group action on the family of pairs $(X_a, Y_{a,b})$, hence on the parameter space. The induced transformation on the parameters (a, b) is given by

$$a_i \to \left(\prod_{k=1}^{4} \lambda_k^{v_{i,k}^*}\right) a_i, \ b_0 \to b_0, \ b_1 \to \left(\prod_{v_i^* \in \Delta^*} \left(\prod_{k=1}^{4} \lambda_k^{v_{i,k}^*}\right)^{q_i}\right) b_1.$$

Since the relative periods are invariant under the transformation, it follows that

$$\Pi\left(\left(\prod_{k=1}^{4} \lambda_k^{v_{i,k}^*}\right) a_i, b_0, \left(\prod_{v_i^* \in \Delta^*} \left(\prod_{k=1}^{4} \lambda_k^{v_{i,k}^*}\right)^{q_i}\right) b_1\right) = \Pi(a,b) \tag{5.10}$$

or equivalently

$$\sum_{v_i^* \in \Delta^*} v_{i,k}^* \left(a_i \frac{\partial}{\partial a_i} + q_i b_1 \frac{\partial}{\partial b_1}\right) \Pi(a,b) = 0, \quad k = 1, .., 4. \tag{5.11}$$

Put

$$\mathcal{L}_k \quad = \quad \sum_{v_i^* \in \Delta^*} \bar{v}_{i,k}^* \left(a_i \frac{\partial}{\partial a_i} + q_i b_1 \frac{\partial}{\partial b_1}\right) - \beta_k, \ k = 0, 1, .., 4 \tag{5.12}$$

where $\beta = (-1, 0, 0, 0, 0)$. Then we get

$$\mathcal{L}_k \Pi(a,b) = 0 \tag{5.13}$$

Given $l \in L$, let

$$\mathcal{D}_l = \prod_{l_i > 0} \left(\frac{\partial}{\partial a_i}\right)^{l_i} - \prod_{l_i < 0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i}$$

since $\mathcal{D}_l \left(\frac{1}{f_{\Delta^*}(a)} \prod_{i=1}^4 \frac{dX_i}{X_i}\right) = 0$ is now exact equation, it follows from Theorem 5.1 that

$$\mathcal{D}_l \Pi(a,b) = 0 \tag{5.14}$$

There is another set of differential equation given by variation of the open string modulus. By Theorem 5.1 again,

$$\frac{\partial}{\partial b_0} \Pi(a,b) \quad = \quad \int_{\partial \Gamma_{a,b}} Res_{Y_{a,b}} Res_{D_b} \frac{1}{f_{\Delta^*}(a) h(b)} \prod_{i=1}^{4} \frac{dX_i}{X_i} \tag{5.15}$$

$$\frac{\partial}{\partial b_1} \Pi(a,b) \quad = \quad \int_{\partial \Gamma_{a,b}} Res_{Y_{a,b}} Res_{D_b} \frac{\prod_i \left(X^{v_i^*}\right)^{q_i}}{f_{\Delta^*}(a) h(b)} \prod_{i=1}^{4} \frac{dX_i}{X_i} \tag{5.16}$$

It follows that

$$\left(b_0\frac{\partial}{\partial b_0}+b_1\frac{\partial}{\partial b_1}\right)\Pi(a,b)=0. \tag{5.17}$$

This can also be seen as a consequence of the invariance of the divisor D_b under rescaling $b_0\to\lambda b_0, b_1\to\lambda b_1$.

Next, to get a full set of equations governing the relative periods, we introduce the following [6]. Given the vector q above, the enhanced polytope $\hat{\Delta}^*$ is the convex hull of the following points in $\mathbb{Z}^5$:

$$\hat{v}_i^*=(v_i^*;0), v_i^*\in\Delta^*, \quad w_0=(0;1),\ w_1=(\sum_{v_i^*\in\Delta^*}q_iv_i^*;1). \tag{5.18}$$

Consider their images under the map $\mathbb{Z}^5\to\mathbb{Z}^6$, $w\mapsto\bar{w}=(w,1)$:

$$\bar{\hat{v}}_i^*=(1;v_i^*;0), v_i^*\in\Delta^*, \quad \bar{w}_0=(1;0;1),\ \bar{w}_1=(1;\sum_{v_i^*\in\Delta^*}q_iv_i^*;1) \tag{5.19}$$

and define the enhanced relation lattice by

$$\hat{L}=\left\{\hat{l}=(\hat{l}^c;\hat{l}^o)\in\mathbb{Z}^{|\Delta^*|+2}|\sum_{v_i^*\in\Delta^*}\hat{l}_i^c\bar{\hat{v}}_i^*+\sum_{k=0,1}\hat{l}_k^o\bar{w}_k=0\right\}. \tag{5.20}$$

(The subscript "c", "o" stand for "closed" and "open" respectively.)

Proposition 5.2. *(Enhanced GKZ System) Put*

$$\begin{aligned}\hat{\mathcal{L}}_k &= \sum_{v_i^*\in\Delta^*}\bar{\hat{v}}_{i,k}^*a_i\frac{\partial}{\partial a_i}+\bar{w}_{0,k}b_0\frac{\partial}{\partial b_0}+\bar{w}_{1,k}b_1\frac{\partial}{\partial b_1}-\hat{\beta}_k, \qquad k=0,..,5\\ \hat{\mathcal{D}}_{\hat{l}} &= \prod_{l_i^c>0}\left(\frac{\partial}{\partial a_i}\right)^{l_i^c}\prod_{l_k^o>0}\left(\frac{\partial}{\partial b_k}\right)^{l_k^o}-\prod_{l_i^c<0}\left(\frac{\partial}{\partial a_i}\right)^{-l_i^c}\prod_{l_k^o<0}\left(\frac{\partial}{\partial b_k}\right)^{-l_k^o}, \quad \hat{l}\in\hat{L}\end{aligned}$$

where $\hat{\beta}=(-1,0,0,0,0,0)$. Then the relative periods $\Pi(a,b)$ are solutions to the following enhanced GKZ system:

$$\hat{\mathcal{L}}_k\Pi(a,b) = 0, \qquad k=0,..,5 \tag{5.21}$$

$$\hat{\mathcal{D}}_{\hat{l}}\Pi(a,b) = 0, \qquad \hat{l}\in\hat{L} \tag{5.22}$$

Proof. The first order equations are simply restating (5.12) and (5.17). Note that in (5.15) and (5.16) the integrals are over exact, hence closed cycles. Thus there is no contributions from the variations of a,b when we differentiate them with respect to the a,b. It follows that when the operator $\hat{\mathcal{D}}_{\hat{l}}$ is applied to them, we can interchange the order of the operator and integration. Since the operator kills the integrand, it follows easily that

$$\hat{\mathcal{D}}_{\hat{l}}\frac{\partial}{\partial b_0}\Pi(a,b)=\hat{\mathcal{D}}_{\hat{l}}\frac{\partial}{\partial b_1}\Pi(a,b)=0. \tag{5.23}$$

Since $\hat{D}_{\hat{l}}\Pi(a,b)$ is homogeneous of degree $-|l_0^o|$ in b_0,b_1, this equation implies that

$$\hat{D}_{\hat{l}}\Pi(a,b)=0. \tag{5.24}$$

□

We now construct a general solution to the enhanced GKZ system above. The main idea grew out of early attempts to generalize the known examples from mirror symmetry (see for e.g. [7].) One of the main problems in closed string mirror symmetry for toric hypersurfaces (and complete intersections) in a complete toric variety was to construct the so-called large radius limit and all analytic solutions to the extended GKZ system [16][17] with $\beta = (-1, 0, .., 0)$. It was proved [18] that if the projective toric variety $\mathbb{P}_\Sigma$ is semi-positive, then a large radius limit corresponds exactly to the canonical triangulation associated to the fan of $\mathbb{P}_\Sigma$, and that the unique powers series solution is given by a Gamma series. To get other analytic solutions, our idea was then to *deform* this Gamma series, as in eqn. (3.5) of [17]. For by the analyticity of Gamma function, the deformed Gamma series is a priori a solution to the GKZ system modulo a set of relations in the deformation parameters $D_0, .., D_p$. Now if all the relations lie in the ring $\mathbb{C}[D_0, .., D_p]$, then it follows easily that the deformed Gamma series would be a solution modulo the Stanley-Reisner ideal of a maximal triangulation. It was then shown that the semi-positivity condition on $\mathbb{P}_\Sigma$ guarantees that the relations lie in the appropriate ring. The Stanley-Reisner ring is the cohomology ring of $\mathbb{P}_\Sigma$, and a complete set of solutions is parameterized by it [17]. Similar problems for general point configurations in $\mathbb{Z}^n$ has also been studied [1][29].

We now formulate our general solutions to the enhanced GKZ system. We will show that the results on closed string mirror symmetry discussed above essentially carries over to the open string context with some modifications. We begin with a brief review of the general setup. Fix a possibly incomplete simplicial fan Σ in $\mathbb{Z}^n$ such that every maximal cone is n dimensional and that the support $|\Sigma|$ of Σ is convex. Let $\Sigma(1) = \{\mu_1, .., \mu_p\}$ be the set of integral generators of the 1-cones, and put $\mu_0 = 0$ and $\nabla = conv(\Sigma(1) \cup 0)$. The Stanley-Reisner ideal of the fan Σ is denoted by $SR_\Sigma \subset \mathbb{C}[D_1, .., D_p]$. Put $\mathcal{A} = \{\bar{\mu}_0, .., \bar{\mu}_p\} \subset (1, \mathbb{Z}^n)$ with $\bar{w} = (1, w)$, and let $L \subset \mathbb{Z}^{p+1}$ be the lattice of relations of $\mathcal{A}$. We shall *assume that the first Chern class* $c_1(\mathbb{P}_\Sigma)$ *of the toric variety* $\mathbb{P}_\Sigma$ *is semi-positive.*

Proposition 5.3. *For each n-cone σ in Σ, $\sigma \cap \nabla$ is an n-simplex whose vertices are μ_0 and the n generators of σ. Such simplices $\sigma \cap \nabla$ form a triangulation $\mathcal{T}_\Sigma$ of ∇.*

Proof. The class $c_1(\mathbb{P}_\Sigma)$ is represented by the piecewise linear function α_Σ with value 1 at each $\mu_1, .., \mu_p$. That $c_1(\mathbb{P}_\Sigma)$ is semi-positive means that α_Σ is convex. The argument of Theorem 4.10 [18] applied to Σ shows that each μ_i must be on the boundary of ∇. This implies that $\sigma \cap \nabla$ is an n-simplex whose vertices are μ_0 and the n generators of σ. Since the support $|\Sigma|$ is convex, such n-simplices must fill up all of ∇. That they form a triangulation follows easily from that Σ is a fan. □

The triangulation of ∇ above gives $conv(\mathcal{A}) = (1, \nabla)$ a triangulation, which we also denote by $\mathcal{T}_\Sigma$. We now consider the GKZ $\mathcal{A}$-hypergeometric system with $\beta = (-1, 0, .., 0)$. Let $C(\mathcal{T}_\Sigma) \subset L^*$ be the cone of $\mathcal{T}_\Sigma$ convex piecewise linear function modulo linear functions. Consider the following deformed Gamma series (cf. eqn. (3.5) of [17]):

$$B_\Sigma(a) = \frac{1}{a_0} \sum_{l \in C(\mathcal{T}_\Sigma)^\vee} \frac{\Gamma(-l_0 - D_0 + 1)}{\prod_{j=1}^p \Gamma(l_j + D_j + 1)} a^{l+D}. \tag{5.25}$$

For $l \in L$, put $B_l = \frac{\Gamma(-l_0 - D_0 + 1)}{\prod_{j=1}^p \Gamma(l_j + D_j + 1)}$, which takes value in $\frac{1}{D_0}\mathbb{C}[[D_0, .., D_p]]$. Then $\sum_{l \in L} B_l a^{l+D}/a_0$ is a constant multiple of the GKZ Gamma series with $(D_0, .., D_p) = (1 + \gamma_0, \gamma_1, .., \gamma_p)$ (see section 3 [17]), and it is killed by the operators $\mathcal{D}_l$ of the GKZ system [13]. Note that $C(\mathcal{T}_\Sigma) = \cap_{I \in \mathcal{T}_\Sigma} K(I)^\vee$ [13]. By the usual product identity of Gamma function, it follows easily that B_l is zero modulo SR_Σ for $l \notin C(\mathcal{T}_\Sigma)^\vee$, and so $B_\Sigma(a)$ modulo SR_Σ is killed by the $\mathcal{D}_l$. Let $J_\Sigma \subset \mathbb{C}[D_0, .., D_p]$ be ideal generated by SR_Σ and the linear forms $\sum_i \bar{\mu}_i D_i$.

Theorem 5.4. *Suppose that $c_1(\mathbb{P}_\Sigma)$ is semi-positive and that $\mathcal{T}_\Sigma$ is regular. Then the deformed Gamma series $B_\Sigma(a)$ modulo J_Σ gives analytic solutions to the $\beta = (-1, 0, .., 0)$ GKZ $\mathcal{A}$-hypergeometric system on some domain (i.e. near a large radius limit.) If $\mathbb{P}_\Sigma$ is nonsingular, then this gives a complete set of solutions.*

Proof. By the argument of Theorem 4.10 [18] applied to Σ again, we find that every $\mathcal{T}_\Sigma$-primitive relation l has $l_0 \leq 0$. Now, Proposition 4.8 [18] shows that $C(\mathcal{T}_\Sigma)^\vee$ is generated by the $\mathcal{T}_\Sigma$-primitive relations. It follows that $B_l \in \mathbb{C}[[D_0, .., D_p]]$ for $l \in C(\mathcal{T}_\Sigma)^\vee$. Thus $B_\Sigma(a)$ modulo J_Σ is well-defined. Since $\mathcal{T}_\Sigma$ is regular, $C(\mathcal{T}_\Sigma)$ is a maximal cone in the secondary fan of $\mathcal{A}$. If this cone is not regular, we can subdivide it to obtain a regular maximal cone in it. The dual of this regular cone contains $C(\mathcal{T}_\Sigma)$ and is generated by an integral basis $l^{(1)}, .., l^{(p-n)}$ of L. The regular cone corresponds to an affine variety with coordinates $x_k = (-1)^{l_0^{(k)}} a^{l^{(k)}}$, and $B_\Sigma(a)$ becomes

$$\frac{1}{a_0} \sum_{m=(m_1,..,m_{p-n}) \in \mathbb{Z}_{\geq 0}^{p-n}} B_{\sum_k m_k l^{(k)}} x^m.$$

By the usual product identity of Gamma function, it follows easily that this converges for small x.

Since expanding $B_\Sigma(a)$ as a linear combination of monomials in the D_i (say, up to any degree N) obviously yields coefficients that are linearly independent, taking $B_\Sigma(a)$ modulo J_Σ yields a series of the form $\sum_i w_i(a)\alpha_i$ where the coefficients w_i are also independent; here the α_i are any homogeneous basis of $\mathbb{C}[D_0, .., D_p]/J_\Sigma$. In other words, $B_\Sigma(a)$ modulo J_Σ yields *dim* $\mathbb{C}[D_0, .., D_p]/J_\Sigma$ independent solutions. Now, for Σ is regular the space of solutions has dimension $vol(\nabla)$, by a theorem of [13]. This is the number of maximal cones in Σ, which coincides with *dim* $\mathbb{C}[D_0, .., D_p]/J_\Sigma$ by the next theorem. □

Remark 5.5. *Given a fan Σ, it is easy to check in practice the semi-positivity of $c_1(\mathbb{P}_\Sigma)$ and regularity of $\mathcal{T}_\Sigma$. Neither condition implies the other in general. The semi-positivity condition holds iff every primitive relation (which is easy to compute) l has $l_0 \leq 0$. Since the primitive relations generates $C(\mathcal{T}_\Sigma)^\vee$, it is also easy to decide if this cone is strongly convex. Regularity of $\mathcal{T}_\Sigma$ is equivalent to $C(\mathcal{T}_\Sigma)^\vee$ being strongly convex.*

Let Σ be a possibly incomplete simplicial fan in $N = \mathbb{Z}^n$, such that every maximal cone is n dimensional, that $|\Sigma|$ is convex, and that there exists a strongly convex continuous piecewise linear function $\psi : |\Sigma| \to \mathbb{R}$. The next theorem gives the chow ring of the toric variety $\mathbb{P}_\Sigma$ and generalizes a well-known result for complete simplicial toric varieties.

Theorem 5.6. *Under the assumptions above on Σ, the Chow ring $A_*(\mathbb{P}_\Sigma, \mathbb{Q})$ is isomorphic $\mathbb{Q}[D_1, .., D_p]/J_\Sigma$, whose dimension is the number of maximal cones in Σ.*

Our overall strategy follows [12], but one key step requires a modification that borrows an idea in the proof of [11] which dealt with the case of complete toric varieties. Let's introduce some vocabulary. Let $\{\sigma_i\}_{i\in I}$ be the set of n-cones in Σ. Let $m_i \in N^\vee$ be such that $\psi|\sigma_i = m_i$. By the strong convexity assumption, the m_i are pairwise distinct. So, we can find a point x_0 in the interior of $|\Sigma|$, so that the values $m_i(x_0)$ are pairwise distinct. We order the index set I and identify it with $\{1, 2, .., |I|\}$ in this order. We say that a codimension one face of the cone σ_i is a *shared wall* of σ_i, if it is of the form $\sigma_k \cap \sigma_i$, and a *free wall* if it is not. For each $i \in I$, let τ_i *be the intersection of all free walls of σ_i and all shared walls $\sigma_k \cap \sigma_i$ with $k > i$.* The following lemma generalizes the key property (*) in section 5.2 [12].

Lemma 5.7. *(Order Lemma) If $\tau_i \subset \sigma_j$ then $j \geq i$.*

Proof. We begin with some basic facts. Let $i, k \in I$.

(1) $m_k - m_i \in \sigma_i^\vee$. In particular, $x_0 \notin \sigma_i$ for $i > 1$. This follows from the convexity of ψ and that $m_1(x_0) < m_2(x_0) < \cdots$.

(2) Under the inclusion reversing correspondence between the faces of σ_i and those of $\sigma_i^\vee$, a wall of σ_i corresponds to an edge of $\sigma_i^\vee$ of the form $\mathbb{R}_+ m$ where $m \in N^\vee$ is an inward pointing normal to the wall. Note that we can pick $m = m_k - m_i$, if the wall is $\sigma_k \cap \sigma_i$. Note also that $x \in \sigma_i$ iff $m(x) \geq 0$ for every edge $\mathbb{R}_+ m$ of $\sigma_i^\vee$.

Now given $\tau_i \subset \sigma_j$, for some j. We consider the fan $star(\tau_i)$ in $N/\mathbb{R}\tau_i$, and note that $\psi - m_i$ induces a convex piecewise linear function on $star(\tau_i)$. We denote by $\bar{\sigma}_i, \bar{\sigma}_j$ the images of σ_i, σ_j in $star(\tau_i)$, and $\bar{x}_0$ the image of x_0 in $N/\mathbb{R}\tau_i$. Then $\bar{\sigma}_i^\vee = \sigma_i^\vee \cap \tau_i^\perp$. Let m be an edge of $\bar{\sigma}_i^\vee$. Then m corresponds to free wall in σ_i or a shared wall $\sigma_k \cap \sigma_i$ with $k > i$. In the first case, we have $m(x_0) \geq 0$ since x_0 lies in $|\Sigma|$ which is convex by assumption, and in the second case, $m(x_0) = m_k(x_0) - m_i(x_0) > 0$ since $k > i$. It follows that $\bar{x}_0 \in \bar{\sigma}_i$. Therefore $(m_j - m_i)(\bar{x}_0) \geq 0$ by the convexity of $\psi - m_i$, i.e., $m_j(x_0) \geq m_i(x_0)$. It follows that $j \geq i$. $\square$

Proof of Theorem:
Claim: For each cone γ in Σ, there's a unique $i = i(\gamma)$ such that $\tau_i \subset \gamma \subset \sigma_i$. And if $\gamma \subset \gamma'$, then $i(\gamma) \leq i(\gamma')$.
Uniqueness follows from the Order Lemma above. For the existence, let $i(\gamma)$ be the minimal such that $\gamma \subset \sigma_i$. If $\gamma = \sigma_i$, then we're done, otherwise, we write γ as intersection of (n-1)-dim faces. Then it's easy to see that by the minimal property of $i(\gamma)$, we have $\tau_i \subset \gamma$. The claim is proved.

It follows from the argument in section 5.2 [12] that

$$A_*(P_\Sigma) = H_*^{BM}(P_\Sigma) = span_{\mathbb{Q}}\{[V(\tau_i)]_i\}$$

where P_Σ is the toric variety associated to Σ. Now we consider the surjection

$$A/J_\Sigma \to A_*(P_\Sigma).$$

where $A = \mathbb{Q}[D_1, .., D_p]$. The "algebraic moving lemma" in section 5.2 [12] continues to work in this the incomplete case, and the proof there shows that $\{p(\tau_i)\}$ generates A/J_Σ as $\mathbb{Q}$-module, where $p(\tau_i)$ is the monomial corresponds to the cone τ_i. By comparing the dimensions, we get the isomorphism

$$A/J_\Sigma \simeq A_*(P_\Sigma)$$

In particular, $\dim_{\mathbb{Q}} A/J_\Sigma$ equals the number of maximal cones in Σ. $\square$

We now apply our results to solve the enhanced GKZ system in open string mirror symmetry as a special case. Let Σ^* be a complete regular fan in $\mathbb{R}^4$, $\Sigma^*(1)$ be the set of generators of its 1-dimensional cones, and assume that $\Sigma^*(1) = \Delta^*$. Let $\hat{\Delta}^*$ be the enhanced polytope as before. For each cone $\sigma \in \Sigma^*$, we get a cone $\hat{\sigma}$ in $\mathbb{R}^5$ generated by $(\sigma, 0)$ and the vector w_1. The set of cones $\hat{\sigma}$ obtained this way, together with 0, form a fan whose support is the half space $\mathbb{R}^4 \times \mathbb{R}_{\geq}$. Since w_0 lies in this half space, there is a canonical way to subdivide the fan into a regular incomplete fan $\hat{\Sigma}$, so that $\hat{\Sigma}(1) = \hat{\Delta}^*$. We shall *assume that $c_1(\mathbb{P}_{\hat{\Sigma}})$ is semi-positive and that $\mathcal{T}_\Sigma$ is regular.* Note that this implies that $c_1(\mathbb{P}_{\Sigma^*})$ is also semi-positive. But the converse is not true. In fact, semi-positivity of $c_1(\mathbb{P}_{\hat{\Sigma}})$ put a strong constraint on the B-brane vector q.

Corollary 5.8. *Suppose that $c_1(\mathbb{P}_{\hat{\Sigma}})$ is semi-positive and $\mathcal{T}_{\hat{\Sigma}}$ is regular. Then $B_{\hat{\Sigma}}(a)$ modulo J_Σ gives a complete set of solutions to the enhanced GKZ system.*

Example: B-brane on Mirror Quintic

Consider the one-parameter family of mirror quintic given in homogeneous coordinate by

$$\prod_{i=1}^{5} z_i - x^{1/5} \sum_{i=1}^{5} z_i^5 = 0 \tag{5.26}$$

and one-parameter family of B-branes on it given by

$$z_5^4 - \phi z_1 z_2 z_3 z_4 = 0 \tag{5.27}$$

Note that this divisor is invariant under the $(\mathbb{Z}_5)^3$ action [9], hence descends to a hypersurface on the mirror quintic. The integral points of the polytope Δ^* are

$$\begin{aligned}
\Delta^* : \quad & v_0^* = (0,0,0,0) \\
& v_1^* = (1,0,0,0) \\
& v_2^* = (0,1,0,0) \\
& v_3^* = (0,0,1,0) \\
& v_4^* = (0,0,0,1) \\
& v_5^* = (-1,-1,-1,-1).
\end{aligned}$$

The toric coordinates X_i are related to homogeneous coordinates z_i by

$$X_i = \frac{z_i^5}{\prod_{i=1}^5 z_i}, \quad i = 1, ..., 4. \tag{5.28}$$

Hence the B-brane divisor is given in toric coordinates by

$$X^{v_5^* - v_0^*} - \phi = 0,$$

which corresponds to the B-brane vector

$$q = (-1, 0, 0, 0, 0, 1).$$

The integral points in the enhanced polytope are

$$\begin{aligned}
\hat{\Delta}^* : \quad & \hat{v}_0^* = (0,0,0,0;0) \\
& \hat{v}_1^* = (1,0,0,0;0) \\
& \hat{v}_2^* = (0,1,0,0;0) \\
& \hat{v}_3^* = (0,0,1,0;0) \\
& \hat{v}_4^* = (0,0,0,1;0) \\
& \hat{v}_5^* = (-1,-1,-1,-1;0) \\
& w_0 = (0,0,0,0;1) \\
& w_1 = (-1,-1,-1,-1;1)
\end{aligned}$$

The relation lattice is generated by

$$l^{(0)} = (-1,0,0,0,0,1,1,-1) \tag{5.29}$$
$$l^{(1)} = (-5,1,1,1,1,1,0,0) \tag{5.30}$$

x above is one of the moduli variable, and the relation between ϕ and u can be read

$$u = x^{1/5}\phi$$

The Picard-Fuchs equation from $l^{(0)}, l^{(1)}$ can be read

$$\{(\theta_x+\theta_u)\,\theta_u - u\,((5\theta_x+\theta_u+1))\,\theta_u\}\,\tilde{\Pi}(x,u)=0$$
$$\left\{\theta_x^4(\theta_x+\theta_u) - x\prod_{i=1}^{5}(5\theta_x+\theta_u+i)\right\}\tilde{\Pi}(x,u)=0$$

where $\theta_x = x\frac{\partial}{\partial x}, \theta_u = u\frac{\partial}{\partial u}$. This is equivalent to the ones in [6]. To get the complete equations from GKZ system, we need one more equation obtained from $l^{(1)} - l^{(0)}$ [6].

Large Radius Limit

The following maximal triangulation $\mathcal{T} = \mathcal{T}_{\hat{\Sigma}}$ of $\hat{\Delta}^*$ corresponds to the large radius limit. Its maximal simplices correspond to the following maximal cones in $\hat{\Sigma}$

$$< \hat{v}_0^*, \hat{v}_1^*, \hat{v}_2^*, \hat{v}_3^*, v_4^*, w_0 >$$
$$< \hat{v}_0^*, \hat{v}_1^*, \hat{v}_2^*, \hat{v}_3^*, v_5^*, w_1 >$$
$$< \hat{v}_0^*, \hat{v}_1^*, \hat{v}_2^*, \hat{v}_4^*, v_5^*, w_1 >$$
$$< \hat{v}_0^*, \hat{v}_1^*, \hat{v}_3^*, \hat{v}_4^*, v_5^*, w_1 >$$
$$< \hat{v}_0^*, \hat{v}_2^*, \hat{v}_3^*, \hat{v}_4^*, v_5^*, w_1 >$$
$$< \hat{v}_0^*, \hat{v}_1^*, \hat{v}_2^*, \hat{v}_3^*, w_0, w_1 >$$
$$< \hat{v}_0^*, \hat{v}_1^*, \hat{v}_2^*, \hat{v}_4^*, w_0, w_1 >$$
$$< \hat{v}_0^*, \hat{v}_1^*, \hat{v}_3^*, \hat{v}_4^*, w_0, w_1 >$$
$$< \hat{v}_0^*, \hat{v}_2^*, \hat{v}_3^*, \hat{v}_4^*, w_0, w_1 >$$

where generators of each cone is given in the bracket $< \cdots >$. It follows that the cone $C(\mathcal{T})$ in the secondary fan corresponding to $\mathcal{T}$ [13] [17] has the dual cone

$$C(\mathcal{T})^\vee = \mathbb{Z}_{\geq 0}(l^{(1)} - l^{(0)}) + \mathbb{Z}_{\geq 0} l^{(0)}$$

From this, it is easy to see that $c_1(\mathbb{P}_{\hat{\Sigma}})$ is semi-positive and that $\mathcal{T}$ is regular. Local coordinates are given by

$$z_1 = \frac{a_1 a_2 a_3 a_4 b_1}{a_0^4 b_0}, \quad z_2 = \frac{a_5 b_0}{a_0 b_1}$$

where $z_1 = 0, z_2 = 0$ is the large radius limit point. We use variables $D_0, D_1, \ldots, D_5$ to represent $\hat{v}_0^*, \hat{v}_1^*, \ldots, \hat{v}_5^*$ and D_6, D_7 to represent w_0, w_1. The primitive collections gives the generators of the Stanley-Reisner ideal

$$D_1 D_2 D_3 D_4 D_7, D_5 D_6, D_1 D_2 D_3 D_4 D_5$$

There're only two independent D_i's after imposing the linear relations as in the above discussion on deformed Gamma series. Put

$$E_1 = D_1, E_2 = D_5.$$

Then the deformed Gamma series can be written as

$$B_{\hat{\Delta}^*}(z_1, z_2) = \frac{1}{a_0}\sum_{m,n\geq 0} \frac{(-1)^m \Gamma(4m+n+4E_1+E_2+1)\sin\pi(E_1-E_2)}{\Gamma(m+E_1+1)^4 \Gamma(n+E_2+1)\pi(m-n+E_1-E_2)} z_1^{m+E_1} z_2^{n+E_2}$$

where we have used Gamma function identity: $\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z}$. The generators of Stanley-Reisner ideal can be written modulo the linear relations as $E_1^4(E_1-E_2), E_2(E_1-$

$E_2), E_1^4 E_2$. Then $B_{\hat{\Sigma}}(z_1, z_2)$ satisfies the GKZ equations if it's viewed as taking values in the ring

$$\mathbb{Q}[E_1, E_2]/(E_1^4(E_1 - E_2), E_2(E_1 - E_2), E_1^4 E_2)$$

which has length 9 equal to $vol(conv((1, \hat{\Delta}^*)))$. Therefore we have obtained the 9-dimensional solution space to the enhanced GKZ system. There is one regular solution which coincides with the regular closed string period for the mirror quintic at the large radius limit:

$$\omega_0(z_1, z_2) = \sum_{m \geq 0} \frac{(-1)^m \ (5m)!}{(m!)^5} (z_1 z_2)^m$$

and two solutions with single log behavior

$$\omega_1(z_1, z_2) = \omega_0(z_1, z_2) \ln z_1 z_2 + 5 \sum_{m \geq 0} \frac{(-1)^m \ (5m)! \sum_{j=m+1}^{5m} \frac{1}{j}}{(m!)^5} (z_1 z_2)^m$$

$$\omega_2(z_1, z_2) = \omega_0(z_1, z_2) \ln z_2 + \sum_{m \geq 0} \frac{(-1)^m \ (5m)! \sum_{j=m+1}^{5m} \frac{1}{j}}{(m!)^5} (z_1 z_2)^m - \sum_{m \geq 0, n \geq 0, m \neq n} \frac{(-1)^m (4m+n)!}{(m!)^4 n! (m-n)} z_1^m z_2^n$$

$\omega_1(z_1, z_2)$ corresponds to the closed period, and $\omega_2(z_1, z_2)$ now comes from the relative period. The so-called open-closed mirror map can be similarly obtained as in the closed string case by normalizing

$$t_1 = \omega_1(z_1, z_2)/\omega_0(z_1, z_2), t_2 = \omega_2(z_1, z_2)/\omega_0(z_1, z_2)$$

Not all of our solutions to the enhanced GKZ system come from relative periods. In closed string mirror symmetry, the periods of toric Calabi-Yau hypersurfaces in $\mathbb{P}_{\Sigma}^{(4)}$ near the large radius limit correspond to solutions with no more than $(log)^3$ behavior, while the full solution space to the corresponding GKZ system include functions with up to $(log)^4$. A key observation in [17] was that a natural way to get the periods from the deformed Gamma series is by multiplying it by the Calabi-Yau divisor $c_1(\mathbb{P}_{\Sigma})$. This has the effect of killing off the $(log)^4$ terms in B_{Σ}. We expect that the same phenomenon happens for relative periods. Namely, they should correspond exactly to the solution $c_1(\mathbb{P}_{\hat{\Sigma}}) B_{\hat{\Sigma}}$. Again, this does have the expected effect of killing off the $(log)^5$ terms in $B_{\hat{\Sigma}}$. In the example above this procedure yields 7 independent solutions as in [6] [21]. This will be studied in greater generality in a follow up paper.

Relative Periods And Abel-Jacobi Map

Consider the relative period $\Pi(z, u)$ given by the family of pairs $(X_z, Y_{z,u})$ as before. From the double residue formula

$$\partial_u \Pi(z, u) = \int_{\partial \Gamma_{z,u}} Res_{Y_{z,u}} Res_{D_u} (\partial_u \log Q_u) \omega_z$$

we see that if u_0 is some point where $\partial \Gamma_{z,u_0} = C^+ - C-$ is a pair of holomorphic curves, then

$$\partial_u \Pi(z, u)|_{z,u_0} = 0$$

since the integrand is a form of type $(2, 0)$. Therefore the loci corresponding to Abel-Jacobi map lies in the critical loci of relative period with respect to the deformation of the divisor. This corresponds in physics the statement that the critical point of the off-shell superpotential will give the D-brane domain wall tension. This was carried out in details for the mirror quintic example in [21] and [6]. Note that in their example, the pair of curves C^+, C^- lie inside the transversal-intersection loci of $X_z \cap D_u$ except at two fixed points. The double residue formula doesn't work in a straight-forward way. But it can be seen that by blowing up twice at those two points, the proper transformation of X_z, Q_u will intersect transversally at all points on the proper transformation of $C^{\pm}$. The double residue formula can then be applied to the blown up configuration. The Abel-Jacobi map will still be the critical value of some of the relative periods.

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