Chiral Equivariant Cohomology I

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ABSTRACT. We construct a new equivariant cohomology theory for a certain class of differential vertex algebras, which we call the chiral equivariant cohomology. A principal example of a differential vertex algebra in this class is the chiral de Rham complex of Malikov-Schechtman-Vaintrob of a manifold with a group action. The main idea in this paper is to synthesize the algebraic approach to classical equivariant cohomology due to H. Cartan[1], with the theory of differential vertex algebras, by using an appropriate notion of invariant theory. We also construct the vertex algebra analogues of the Mathai-Quillen isomorphism, the Weil and the Cartan models for equivariant cohomology, and the Chern-Weil map. We give interesting cohomology classes in the new theory that have no classical analogues.

Keywords: differential vertex algebras, equivariant de Rham theory, invariant theory, semi-infinite Weil algebra, Virasoro algebra.

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[1] Cartan's theory was further developed by Duflo-Kumar-Vergne [8] and Guillemin-Sternberg [18]. This paper follows closely the latter approach.

Contents

1. Introduction

1.1. Equivariant de Rham theory

For a topological space M equipped with an action of a compact Lie group G, the G-equivariant cohomology of M, denoted by $H^*_G(M)$, is defined to be $H^*((M \times \mathcal{E})/G))$, where $\mathcal{E}$ is any contractible topological space on which G acts freely. When M is a smooth manifold on which G acts by diffeomorphisms, there is a de Rham model of $H^*_G(M)$ due to H. Cartan [5][6], and developed further by Duflo-Kumar-Vergne [8] and Guillemin-Sternberg [18]. The treatment in [18] is simplified considerably by the use of supersymmetry [2][24][31][35][23], and will be the approach adopted in the present paper. Guillemin-Sternberg define the equivariant cohomology $H^*_G(A)$ of any G^*-algebra A, of which the algebra $\Omega(M)$ of smooth differential forms on M is an example. A G^*-algebra is a commutative superalgebra A equipped with an action of G, together with a compatible action of a certain differential Lie superalgebra $(\mathfrak{sg}, d)$ associated to the Lie algebra $\mathfrak{g}$ of G. Taking $A = \Omega(M)$ gives us the de Rham model of $H^*_G(M)$, and $H^*_G(\Omega(M)) = H^*_G(M)$ by an equivariant version of the de Rham theorem. A G^*-algebra (A, d) is a cochain complex, and the subalgebra of A which is both G-invariant and killed by $\mathfrak{sg}$ forms a subcomplex known as the basic subcomplex. $H^*_G(A)$ may be defined to be $H^*_{bas}(A \otimes W(\mathfrak{g}))$, where $W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$ is the Koszul complex of $\mathfrak{g}$. A change of variables [18] shows that $W(\mathfrak{g})$ is isomorphic to the subcomplex of $\Omega(EG)$ which is freely generated by the connection one-forms and curvature two-forms. Here EG is the total space of the classifying bundle of G and $\Omega(EG)$ is the de Rham complex of EG. This subcomplex is known as the Weil complex, and $H^*_{bas}(A \otimes W(\mathfrak{g}))$ is known as the *Weil model* for $H^*_G(A)$. Using an automorphism of the space $A \otimes W(\mathfrak{g})$ called the *Mathai-Quillen isomorphism*, one can construct the *Cartan model* which is often more convenient for computational purposes.

1.2. A vertex algebra analogue of G^*-algebras

Associated to G is a certain universal differential vertex algebra we call $O(\mathfrak{sg})$, which is analogous to $(\mathfrak{sg}, d)$. An $O(\mathfrak{sg})$-algebra is then a differential vertex algebra $\mathcal{A}$ equipped with an action of G together with a compatible action of $O(\mathfrak{sg})$. When G is connected, the G-action can be absorbed into the $O(\mathfrak{sg})$-action.

Associated to any Lie algebra $\mathfrak{g}$ is a $\mathbf{Z}_{\geq 0}$-graded vertex algebra $\mathcal{W}(\mathfrak{g})$ known as the *semi-infinite Weil complex* of $\mathfrak{g}$ (a.k.a. the $bc\beta\gamma$-system in physics [15]). When $\mathfrak{g}$ is finite dimensional, $\mathcal{W}(\mathfrak{g})$ is a vertex algebra which contains the classical Weil complex $W(\mathfrak{g})$ as the subspace of conformal weight zero, and is an example of an $O(\mathfrak{sg})$-algebra. It was studied in [10][1] in the context of semi-infinite cohomology of the loop algebra of $\mathfrak{g}$, which is a vertex algebra analogue of Lie algebra cohomology and an example of the general theory of semi-infinite cohomology developed in [9][11].

In [29], Malikov-Schechtman-Vaintrob constructed a sheaf $\mathcal{Q}_M$ of vertex algebras on any nonsingular algebraic scheme M, which they call the *chiral de Rham sheaf.* They also pointed out that the same construction can be done in the analytic and smooth categories (see remark 3.9 [29].) In this paper, we will carry out a construction that is equivalent in the smooth category. The space $\mathcal{Q}(M)$ of global sections of the MSV sheaf $\mathcal{Q}_M$ is a $\mathbf{Z}_{\geq 0}$-graded vertex algebra, graded by conformal weight, which contains the ordinary de Rham algebra $\Omega(M)$ as the subspace of conformal weight zero. There is a square-zero derivation $d_{\mathcal{Q}}$ on $\mathcal{Q}(M)$ whose restriction to $\Omega(M)$ is the ordinary de Rham differential d, and the inclusion of complexes $(\Omega(M), d) \hookrightarrow (\mathcal{Q}(M), d_{\mathcal{Q}})$ induces an isomorphism in cohomology. When M is a G-manifold, the algebra $\mathcal{Q}(M)$ is another example of an $O(\mathfrak{sg})$-algebra.

1.3. Vertex algebra invariant theory

For any vertex algebra V and any subalgebra $B \subset V$, there is a new subalgebra $Com(B, V) \subset V$ known as the *commutant* of B in V. This construction was introduced in [14] as a vertex algebra abstraction of a construction in representation theory [21] and in conformal field theory [17], known as the coset construction. It may be interpreted either as the vertex algebra analogue of the ordinary commutant construction in the theory of associative algebras, or as a vertex algebra notion of invariant theory. The latter interpretation was developed in [28], and is the point of view we adopt in this paper.

1.4. Chiral equivariant cohomology of an $O(\mathfrak{sg})$-algebra

Our construction of chiral equivariant cohomology synthesizes the three theories outlined above. We define the chiral equivariant cohomology $\mathbf{H}^*_G(\mathcal{A})$ of any $O(\mathfrak{sg})$-algebra $\mathcal{A}$ by replacing the main ingredients in the classical Weil model for equivariant cohomology with their vertex algebra counterparts. The commutant construction plays the same role that ordinary invariant theory plays in classical equivariant cohomology. We also construct the chiral analogues of the Mathai-Quillen isomorphism, the Cartan model for $\mathbf{H}^*_G(\mathcal{A})$, and a vertex algebra homomorphism $\kappa_G : \mathbf{H}^*_G(\mathbf{C}) \to \mathbf{H}^*_G(\mathcal{A})$ which is the chiral version of the Chern-Weil map. Here $\mathbf{C}$ is the one-dimensional trivial $O(\mathfrak{sg})$-algebra.

Specializing to $\mathcal{A} = \mathcal{Q}(M)$, for a G-manifold M, gives us a chiral equivariant cohomology theory of M which contains the classical equivariant cohomology. It turns out that there are other interesting differential vertex algebras $\mathcal{A}$, some of which are subalgebras of $\mathcal{Q}(M)$, for which $\mathbf{H}^*_G(\mathcal{A})$ can be defined and also contains the classical equivariant cohomology M. This will be the focus of a separate paper; see further remarks in the last section.

In the case where G is an n-dimensional torus T, we give a complete description of $\mathbf{H}^*_T(\mathbf{C})$. Working in the Cartan model, we show that for any $O(\mathfrak{st})$-algebra $\mathcal{A}$, $\mathbf{H}^*_T(\mathcal{A})$ is actually the cohomology of a much smaller subcomplex of the chiral Cartan complex, which we call the small chiral Cartan complex. Like the classical Cartan complex, the small chiral Cartan complex has the structure of a double complex, and there is an associated filtration and spectral sequence that computes $\mathbf{H}^*_T(\mathcal{A})$. For non-abelian G, we also construct a double complex structure in the Weil and Cartan models, and derive two corresponding spectral sequences.

When G is a simple, connected Lie group, we show that $\mathbf{H}^*_G(\mathbf{C})$ contains a vertex operator $\mathbf{L}(z)$ which has no classical analogue, and satisfies the Virasoro OPE relation. In particular, $\mathbf{H}^*_G(\mathbf{C})$ is an interesting non-abelian vertex algebra. This algebra plays the role of $H^*_G(\mathbf{C}) = S(\mathfrak{g}^*)^G$, the equivariant cohomology of a point, in the classical theory.

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2. Background

In this section we discuss the necessary background material in preparation for the main results to be developed in sections 2-7. Vertex algebras and modules have been discussed from various different points of view in [15][32][3][13][12][26][22][20]. We will follow the formalism developed in [26] and partly in [22]. We also carry out the construction of the chiral de Rham sheaf in the smooth category.

2.1. An interlude on vertex algebras

Let V be a vector space (always assumed defined over the complex numbers). Let z, w be formal variables. By $QO(V)$, we mean the space of all linear maps $V \to V((z)) := \{\sum_{n\in\mathbf{Z}} v(n)z^{-n-1} | v(n) \in V,\ v(n) = 0\ for\ n >> 0\}$. Each element $a \in QO(V)$ can be uniquely represented as a power series $a = a(z) := \sum_{n\in\mathbf{Z}} a(n)z^{-n-1} \in (End\ V)[[z, z^{-1}]]$, though the latter space is clearly much larger than $QO(V)$. We refer to $a(n)$ as the n-th Fourier mode of $a(z)$. If one regards $V((z))$ as a kind of "z-adic" completion of $V[z, z^{-1}]$, then $a \in QO(V)$ can be thought of as a map on $V((z))$ which is only defined on the dense subset $V[z,z^{-1}]$. *When V is equipped with a super vector space structure $V = V^0 \oplus V^1$ then an element $a \in QO(V)$ is assumed to be of the shape $a = a_0 + a_1$ where $a_i : V^j \to V^{i+j}((z))$ for $i, j \in \mathbf{Z}/2$.*

On $QO(V)$ there is a set of non-associative bilinear operations, $\circ_n$, indexed by $n \in \mathbf{Z}$, which we call the n-th circle products. They are defined by

$$a(w) \circ_n b(w) = Res_z a(z)b(w)\ i_{|z|>|w|}(z-w)^n - Res_z b(w)a(z)\ i_{|w|>|z|}(z-w)^n \in QO(V).$$

Here $i_{|z|>|w|} f(z,w) \in \mathbf{C}[[z,z^{-1},w,w^{-1}]]$ denotes the power series expansion of a rational function f in the region $|z| > |w|$. Be warned that $i_{|z|>|w|}(z-w)^{-1} \neq i_{|w|>|z|}(z-w)^{-1}$. As it is customary, we shall drop the symbol $i_{|z|>|w|}$ and just write $(z-w)^{-1}$ to mean the expansion in the region $|z| > |w|$, and write $-(w-z)^{-1}$ to mean the expansion in $|w| > |z|$. $Res_z(\cdots)$ here means taking the coefficient of z^{-1} of $(\cdots)$. It is easy to check that $a(w) \circ_n b(w)$ above is a well-defined element of $QO(V)$. *When V is equipped with a super vector space structure then the definition of $a \circ_n b$ above is replaced by one with the extra sign $(-1)^{|a||b|}$ in the second term.* Here $|a|$ is the $\mathbf{Z}/2$ grading of a homogeneous element $a \in QO(V)$.

The circle products are connected through the *operator product expansion* (OPE) formula ([26], Prop. 2.3): for $a, b \in QO(V)$, we have

$$a(z)b(w) = \sum_{n\geq 0} a(w) \circ_n b(w)\ (z-w)^{-n-1} + :a(z)b(w): \tag{2.1}$$

where

$$:a(z)b(w): = a(z)_- b(w) + (-1)^{|a||b|} b(w)a(z)_+$$
$$a(z)_- = \sum_{n<0} a(n)z^{-n-1}, \quad a(z)_+ = \sum_{n\geq 0} a(n)z^{-n-1}.$$

Note that $:a(w)b(w):$ is a well-defined element of $QO(V)$. It is called the Wick product of a and b, and it coincides with $a \circ_{-1} b$. The other negative circle products are related to this by

$$n!\ a(w) \circ_{-n-1} b(w) =: (\frac{d^n}{dw^n} a(w))\ b(w):. \tag{2.2}$$

For $a_1(z), ..., a_k(z) \in QO(V)$, it is convenient to define the k-fold iterated Wick product

$$:a_1(z)a_2(z)\cdots a_k(z): \stackrel{def}{=} :a_1(z)b(z):$$

where $b(z) =: a_2(z) \cdots a_k(z) :$. It is customary to rewrite (2.1) as

$$a(z)b(w) \sim \sum_{n \geq 0} a(w) \circ_n b(w)\ (z-w)^{-n-1}.$$

Thus $\sim$ means equal modulo the term $: a(z)b(w) :$. Note that when $a \circ_n b = 0$ for $n >> 0$ (which will be the case throughout this paper later), then formally $a(z)b(w)$ can be thought of as a kind of meromorphic function with poles along $z = w$. The product $a \circ_n b$ is formally $\oint_C a(z)b(w)(z-w)^n dz$ where C is a small circle around w (hence the name circle product).

From the definition, we see that

$$a(w) \circ_0 b(w) = [a(0), b(w)].$$

From this, it follows easily that $a\circ_0$ is a (graded) derivation of every circle product [25]. This property of the zeroth circle product will be used often later.

The set $QO(V)$ is a nonassociative algebra with the operations $\circ_n$ and a unit 1. We have $1 \circ_n a = \delta_{n,-1} a$ for all n, and $a \circ_n 1 = \delta_{n,-1} a$ for $n \geq -1$. We are interested in subalgebras $A \subset QO(V)$, i.e. linear subspaces of $QO(V)$ containing 1, which are closed under the circle products. In particular A is closed under formal differentiation

$$\partial a(w) = \frac{d}{dw} a(w) = a \circ_{-2} 1.$$

We shall call such a subalgebra a *circle algebra* (also called a quantum operator algebra in [26]).

Remark 2.1. *Fix a nonzero vector $\mathbb{1} \in V$ and let $a, b \in QO(V)$ such that $a(z)_+\mathbb{1} = b(z)_+\mathbb{1} = 0$ for $n \geq 0$. Then it follows immediately from the definition of the circle products that $(a \circ_p b)_+(z)\mathbb{1} = 0$ for all p. Thus if a circle algebra A is generated by elements a with the property that $a(z)_+\mathbb{1} = 0$, then every element in A has this property. In this case the vector $\mathbb{1}$ determines a linear map*

$$\chi : A \to V, \quad a \mapsto a(-1)\mathbb{1} = \lim_{z \to 0} a(z)\mathbb{1}$$

(called the creation map in [26]) having the following basic properties:

$$\chi(1) = \mathbb{1}, \quad \chi(a \circ_n b) = a(n)b(-1)\mathbb{1}, \quad \chi(\partial^p a) = p!\ a(-p-1)\mathbb{1}. \tag{2.3}$$

Definition 2.2. *We say that $a, b \in QO(V)$ circle commute if $(z-w)^N[a(z), b(w)] = 0$ for some $N \geq 0$. If N can be chosen to be 0, then we say that a, b commute. We say that $a \in QO(V)$ is a vertex operator if it circle commutes with itself.*

Definition 2.3. *A circle algebra is said to be commutative if its elements pairwise circle commute.*

Again when there is a $\mathbf{Z}/2$ graded structure, the bracket in the definition above means the super commutator. We will see shortly that the notion of a commutative circle algebra is essentially equivalent to the notion of a vertex algebra (see for e.g. [13]). An easy calculation gives the following very useful characterization of circle commutativity.

Lemma 2.4. *Given $N \geq 0$ and $a, b \in QO(V)$, we have*

$$\begin{aligned} & (z-w)^N[a(z), b(w)] = 0 \\ \Longleftrightarrow\ & [a(z)_+, b(w)] = \sum_{p=0}^{N-1} (a \circ_p b)(w)\ (z-w)^{-p-1} \\ \&\ \ & [a(z)_-, b(w)] = \sum_{p=0}^{N-1} (-1)^p (a \circ_p b)(w)\ (w-z)^{-p-1} \\ \Longleftrightarrow\ & [a(m), b(n)] = \sum_{p=0}^{N-1} \binom{m}{p} (a \circ_p b)(m+n-p) \quad \forall m, n \in \mathbf{Z}. \end{aligned}$$

Using this lemma, it is not difficult to show that for any circle commuting $a(z), b(z) \in QO(V)$ and $n \in \mathbf{Z}$, we have

$$a(z) \circ_n b(z) = \sum_{p \in \mathbf{Z}} (-1)^{p+1} (b(z) \circ_p a(z)) \circ_{n-p-1} 1. \tag{2.4}$$

Note that this is a finite sum by circle commutativity and the fact that $c(z) \circ_k 1 = 0$ for all $c(z) \in QO(V)$ and $k \geq 0$.

Many known commutative circle algebras can be constructed as follows. Start with a set $S \subset QO(V)$ and use this lemma to verify circle commutativity of the set. Then S generates a commutative circle algebra A by the next lemma [26][22].

Lemma 2.5. *Let $a, b, c \in QO(V)$ be such that any two of them circle commute. Then a circle commutes with all $b \circ_p c$.*

Proof: We have

$$[a(z_1), [b(z_2), c(z_3)]] = [[a(z_1), b(z_2)], c(z_3)]] \pm [b(z_2), [a(z_1), c(z_3)]]. \tag{2.5}$$

For $M, N \geq 0$, write $(z_1 - z_3)^{M+N} = ((z_1 - z_2) + (z_2 - z_3))^N (z_1 - z_3)^M$ and expand the first factor binomially. For $M, N >> 0$, each term $(z_1 - z_2)^i (z_2 - z_3)^{N-i} (z_1 - z_3)^M$ annihilates either the left side of (2.5) or the right side. Thus $(z_1 - z_3)^{M+N}$ annihilates (2.5). Multiplying (2.5) by $(z_2 - z_3)^p$, $p \geq 0$, and taking Res_{z_2}, we see that $(z_1 - z_3)^{M+N}[a(z_1), (b \circ_p c)(z_3)] = 0$. From this, we can also conclude that c circle commutes with all $b \circ_p a$, $p \geq 0$.

Now consider the case $p < 0$. For simplicity, we write $a = a(z), b = b(w), c = c(w)$. Suppose $(z - w)^N [a, b] = 0$. Differentiating $(z - w)^{N+1}[a, b] = 0$ with respect to w shows that a circle commutes with ∂b. By (2.2), it remains to show that a circle commutes with the Wick product $: bc :$. We have

$$[a, : bc :] = [a, b_-]c \pm b_-[a, c] + [a, c]b_+ \pm c[a, b_+].$$

For $M >> 0$, $(w - z)^M$ annihilates $[a, b], [a, c]$. In particular, $(w - z)^M [a, b_-] = (w - z)^M [b_+, a]$. It follows that

$$(w - z)^M [a, : bc :] = (w - z)^M [b_+, a]c \mp c(w - z)^M [b_+, a].$$

For $M >> 0$, the right side is zero by Lemma 2.4 because c circle commutes with all $b \circ_p a$, $p \geq 0$, and that $b \circ_p a = 0$ for $p >> 0$. $\square$

In the formulation Definition 2.3, many formal algebraic notions become immediately clear: a homomorphism is just a linear map that preserves all circle products and 1; a module over a circle algebra A is a vector space M equipped with a circle algebra homomorphism $A \to QO(M)$, etc. For example, every commutative circle algebra A is itself a faithful A-module, called *the left regular module*, as we now show. Define

$$\rho : A \to QO(A), \quad a \mapsto \hat{a}, \qquad \hat{a}(\zeta)b = \sum (a \circ_n b)\ \zeta^{-n-1}.$$

Lemma 2.6. *For $a, b \in A$, $m, n \in \mathbf{Z}$, we have*

$$[\hat{a}(m), \hat{b}(n)] = \sum_{p \geq 0} \binom{m}{p} \widehat{a \circ_p b}(m + n - p).$$

Proof: Applying the left side to a test vector $u \in A$, and using Lemma 2.4, we have

$$\begin{aligned}
&\hat{a}(m) \cdot \hat{b}(n) \cdot u(z) - \hat{b}(n) \cdot \hat{a}(m) \cdot u(z) \\
&= Res_{z_1} Res_{z_2} [a(z_2), b(z_1)] u(z)(z_2 - z)^m (z_1 - z)^n \\
&- Res_{z_2} Res_{z_1} u(z)[a(z_2), b(z_1)](-z + z_2)^m (-z + z_1)^n \\
&= Res_{z_1} Res_{z_2} \sum_{p \geq 0} (a \circ_p b)(z_1) u(z) \frac{(-1)^p}{p!} \left(\partial^p_{z_2} \delta(z_2, z_1)\right) (z_2 - z)^m (z_1 - z)^n \\
&- Res_{z_1} Res_{z_2} \sum_{p \geq 0} u(z)(a \circ_p b)(z_1) \frac{(-1)^p}{p!} \left(\partial^p_{z_2} \delta(z_2, z_1)\right) (-z + z_2)^m (-z + z_1)^n
\end{aligned}$$

where $\delta(z_1, z_2) = (z_1 - z_2)^{-1} + (z_2 - z_1)^{-1}$. By doing formal integration by parts and using the fact that $Res_{z_2} z_2^n \delta(z_2, z_1) = z_1^n$, the last expression becomes

$$\sum_p Res_{z_1} \binom{m}{p} (a \circ_p b)(z_1) u(z)(z_1 - z)^{m-p+n} - \sum_p Res_{z_1} \binom{m}{p} u(z)(a \circ_p b)(z_1)(-z + z_1)^{m-p+n}.$$

This is equal to the right side of our assertion applied on u. $\square$

Theorem 2.7. *ρ is an injective circle algebra homomorphism.*

Proof: We will consider the case without the $\mathbf{Z}/2$ grading. The argument carries over to superalgebra case with some sign changes, as usual. The map ρ is injective because $\hat{a}(-1)1 = a \circ_{-1} 1 = a$. Multiplying the formula in the preceding lemma by ζ^{-n-1} and summing over n, we find

$$[\hat{a}(m), \hat{b}(\zeta)] = \sum_{p \geq 0} \binom{m}{p} \widehat{a \circ_p b}(\zeta)\ \zeta^{m-p}. \tag{2.6}$$

On the other hand, it follows from the OPE formula that for $m \geq 0$,

$$[\hat{a}(m), \hat{b}(\zeta)] = \sum_{p \geq 0} \binom{m}{p} (\hat{a} \circ_p \hat{b})(\zeta)\ \zeta^{m-p}.$$

Specializing the two preceding formulas to $m = 0, 1, 2, ...$, we find that

$$\widehat{a \circ_p b} = \hat{a} \circ_p \hat{b}$$

for $p \geq 0$. This shows that ρ preserves the circle products $\circ_p$, $p \geq 0$. In particular (2.6) becomes

$$[\hat{a}(m), \hat{b}(\zeta)] = \sum_{p \geq 0} \binom{m}{p} (\hat{a} \circ_p \hat{b})(\zeta)\ \zeta^{m-p}$$

for all $m \in \mathbf{Z}$. This implies that $\hat{a}, \hat{b}$, circle commute, by Lemma 2.4.

Let A' be the (commutative) circle algebra generated by $\rho(A)$ in $QO(A)$. Since $\hat{a}(n)1 = a \circ_n 1 = 0$ for $a \in A$, $n \geq 0$, i.e. $\hat{a}_+ 1 = 0$, it follows that every element $\alpha \in A'$ has $\alpha_+ 1 = 0$ by Remark 2.1. Consider the creation map $\chi : A' \to A$, $\alpha \mapsto \alpha(-1)1$, which is clearly surjective because $\chi \circ \rho = id$. We also have $[\partial, \hat{a}(\zeta)]b = \frac{\partial}{\partial \zeta}\hat{a}(\zeta)b$, where $\partial b(z) = \frac{d}{dz} b(z)$. Applying the next lemma to the algebra $A' \subset QO(A)$, the vector $1 \in A$, and the linear map $\partial : A \to A$, we find that χ is an isomorphism with inverse ρ. (In particular this shows that $A' = \rho(A)$, hence $\rho(A)$ is closed under the circle products.) By Remark 2.1, we have

$$\chi(\hat{a} \circ_n \hat{b}) = a \circ_n b$$

for all n. Applying ρ to both sides yields that $\hat{a} \circ_n \hat{b} = \widehat{a \circ_n b}$ for all n. This shows that ρ preserves all circle products. $\square$

Lemma 2.8. *Let $A \subset QO(V)$ be a commutative circle algebra, $\mathbb{1} \in V$ a nonzero vector, and $D : V \to V$ a linear map such that $D\mathbb{1} = 0 = a_+\mathbb{1}$ and $[D, a(z)] = \partial a(z)$ for $a \in A$. If the creation map $\chi : A \to V$, $a \mapsto a(-1)\mathbb{1}$, is surjective then it is injective.*

Proof: By assumption, for $a \in A$, we have $Da(n)\mathbb{1} = -na(n-1)\mathbb{1}$. Thus if $a(-1)\mathbb{1} = 0$, then $a(-2)\mathbb{1} = 0$. Likewise $a(n)\mathbb{1} = 0$ for all $n < 0$. Since $a_+\mathbb{1} = 0$, it follows that $a\mathbb{1} = 0$. Since χ is surjective it suffices to show that $a(z)b(-1)\mathbb{1} = 0$ for arbitrary $b \in A$. Fix $N \geq 0$ with $(z-w)^N [a(z), b(w)] = 0$. Then

$$(z-w)^N a(z)b(w)\mathbb{1} = (z-w)^N b(w)a(z)\mathbb{1} = 0.$$

Since $b_+\mathbb{1} = 0$, we have $b(w)\mathbb{1} \to b(-1)\mathbb{1}$ as $w \to 0$. This shows that $z^N a(z)b(-1)\mathbb{1} = 0$, implying that $a(z)b(-1)\mathbb{1} = 0$. $\square$

The following are useful identities for circle commuting operators which measure the non-associativity and non-commutativity of the Wick product, and the failure of the positive circle products to be left and right derivations of the Wick product.

Lemma 2.9. *Let a, b, c be pairwise circle commuting, and $n \geq 0$. Then we have the identities*

$$: (: ab :)c : - : abc := \sum_{k\geq 0} \frac{1}{(k+1)!} \Big(: (\partial^{k+1}a)(b \circ_k c) : + (-1)^{|a||b|} : (\partial^{k+1}b)(a \circ_k c) : \Big)$$

$$a \circ_n (: bc :) - : (a \circ_n b)c : - (-1)^{|a||b|} : b(a \circ_n c) := \sum_{k=1}^{n} \binom{n}{k} (a \circ_{n-k} b) \circ_{k-1} c$$

$$(: ab :) \circ_n c = \sum_{k\geq 0} \frac{1}{k!} : (\partial^k a)(b \circ_{n+k} c) : + (-1)^{|a||b|} \sum_{k\geq 0} b \circ_{n-k-1} (a \circ_k c)$$

$$: ab : - (-1)^{|a||b|} : ba := \sum_{k\geq 0} \frac{(-1)^k}{(k+1)!} \partial^{k+1}(a \circ_k b).$$

Proof: By the preceding theorem, it suffices to show that $\hat{a}, \hat{b}, \hat{c}$ satisfy these identities. They can be checked as follows. First, apply the creation map χ to both sides and use (2.3) and Lemma 2.4. The calculations are straightforward, and details are left to the reader. □

Let A be a commutative circle algebra. A two-sided ideal of circle algebra A is a subspace I invariant under left and right operations by the circle products. In this case, there is a canonical homomorphism

$$A \to QO(A/I), \quad a \mapsto \bar{a}(\zeta), \ \bar{a}(n)(b + I) = \hat{a}(n)b + I = a \circ_n b + I.$$

This preserves the circle products, since the preceding theorem says that $A \to QO(A)$, $a \mapsto \hat{a}$, is a circle algebra homomorphism. Likewise for $a, b \in A$, we have that $\bar{a}, \bar{b}$ circle commute. Thus the image $\bar{A}$ of A in $QO(A/I)$ is a commutative circle algebra, and we have an exact sequence $0 \to I \to A \to \bar{A} \to 0$. We call $\bar{A}$ the *quotient algebra* of A by I.

Theorem 2.10. *If A is a commutative circle algebra, then $(A, 1, \partial, \rho)$ is a vertex algebra in the sense of [13] (without grading or Virasoro element).*

Proof: We know that the map $\rho : A \to QO(A)$, $a \mapsto \hat{a}$, has the property that $[\partial, \hat{a}(\zeta)]b = \frac{\partial}{\partial \zeta}\hat{a}(\zeta)b$. Moreover $\partial 1 = 0$ and that $\chi : \rho(A) \to A$, $\hat{a} \mapsto \hat{a}(-1)1 = a$, is the inverse of ρ. So it remains to verify the vertex algebra Jacobi identity:

$$Res_\zeta(\widehat{\hat{a}(\zeta)b})(w)\zeta^n(w+\zeta)^q = Res_z \hat{a}(z)\hat{b}(w)(z-w)^n z^q - Res_z \hat{b}(w)\hat{a}(z)(-w+z)^n z^q \quad (2.7)$$

for $n, q \in \mathbf{Z}$, $a, b \in A$. We will do this in several steps.

Case 1. $n \in \mathbf{Z}$, $q = 0$. The identity

$$\widehat{a \circ_n b} = \hat{a} \circ_n \hat{b}$$

is nothing but (2.7) in this case. For convenience, we will drop the ˆ from the notations temporarily.

Case 2. $n = 0$, $q = -1$. The right side of (2.7) becomes, using Lemma 2.4,

$$[a(-1), b] = \sum_{p \geq 0} (-1)^p (a \circ_p b)(w)\ w^{-p-1},$$

which agrees with the left side of (2.7).

Case 3. $n = -1$, $q = -1$. By direct computation, the right side of (2.7) is

$$\begin{aligned}\sum_{p \geq 0} a(-p-2) b(w) w^p - \sum_{p \geq 0} b(w) a(p-1) w^{-p-1} &= (a_- b - a(-1)b) w^{-1} + (b a_+ + b a(-1)) w^{-1} \\ &=: ab : w^{-1} - [a(-1), b] w^{-1}.\end{aligned}$$

This agrees with the left side of (2.7).

Case 4. $n = 0$, $q < 0$. Using integration by parts, the first term of the right side of (2.7) becomes

$$Res_z a(z) b(w) z^q = \frac{-1}{q+1} Res_z \partial a(z) b(w) z^{q+1}$$

for $q < -1$. Likewise for two other terms in (2.7). Thus this case can be reduced to Case 2.

Case 5. $n \in \mathbf{Z}$, $q \geq 0$. We have

$$\begin{aligned}(z-w)^n z^q &= (z-w)^{n+1} z^{q-1} + (z-w)^n z^{q-1} w \\ (-w+z)^n z^q &= (-w+z)^{n+1} z^{q-1} + (-w+z)^n z^{q-1} w \\ \zeta^n (w+\zeta)^q &= \zeta^{n+1} (w+\zeta)^{q-1} + \zeta^n (w+\zeta)^{q-1} w\end{aligned}$$

Using these identities, we can easily reduce this case to Case 1.

Case 6. $n \geq 0$, $q < 0$. We have

$$\begin{aligned}(z-w)^n z^q &= (z-w)^{n-1} z^{q+1} - (z-w)^{n-1} z^q w \\ (-w+z)^n z^q &= (-w+z)^{n-1} z^{q+1} - (-w+z)^{n-1} z^q w \\ \zeta^n (w+\zeta)^q &= \zeta^{n-1} (w+\zeta)^{q+1} - \zeta^{n-1} (w+\zeta)^q w\end{aligned}$$

Using this identities, we reduce this case to Case 4.

Case 7. $n < 0$, $q = -1$. Take (2.7) in Case 3, and operate on both sides by $\frac{d}{dw}$ repeatedly. We then get (2.7) for $n < 0$, $q = -1$.

Case 8. $n < 0$, $q < 0$. Using integration by parts, the first term of the right side of (2.7) becomes

$$Res_z a(z)b(w)(z-w)^n z^q = \frac{-1}{q+1} Res_z \partial a(z)b(w)(z-w)^n z^{q+1} + \frac{-n}{q+1} Res_z a(z)b(w)(z-w)^{n-1} z^{q+1}.$$

for $q < -1$. Likewise for two other terms in (2.7). Now we verify that this case reduces to Case 7.

This completes the proof. $\square$

Lemma 2.11. *If $(V, \mathbb{1}, D, Y)$ is a vertex algebra, then $Y(V) \subset QO(V)$ is a commutative circle algebra.*

Proof: Write $a(z) = Y(a, z)$, $b(z) = Y(b, z)$, for $a, b \in V$. The Jacobi identity implies that $Y(a(p)b, z) = a(z) \circ_p b(z)$ for all p, which shows that $Y(V)$ is closed under the circle products. The Jacobi identity also implies that the commutator relations in Lemma 2.4 hold for $a(z), b(z)$, which shows that $a(z), b(z)$ circle commute. This shows that $Y(V) \subset QO(V)$ is a commutative circle algebra. $\square$

Remark 2.12. *Thus the notion of a vertex algebra is abstractly equivalent to our notion of a commutative circle algebra. While the former theory emphasizes the quadruple of structures $(V, \mathbb{1}, D, Y)$ satisfying an infinite family of (Jacobi) identities, the latter theory emphasizes the circle products and circle commutativity, and shows that all other structures can be obtained canonically in any given commutative circle algebra. The latter theory will be more convenient for the purposes of this paper. Note that the formal algebraic notions such as modules, ideals, and quotients for vertex algebras [12] are equivalent to the corresponding notions for commutative circle algebras under this dictionary. We will refer to a commutative circle algebra simply as a vertex algebra throughout the rest of the paper.*

The left regular module guarantees that for any given abstract vertex algebra A, one can always embed A in $QO(A)$ in a canonical way. It is often convenient to pass between A and its image $\rho(A)$ in $QO(A)$. For example, we shall often denote the Fourier modes $\hat{a}(n)$ simply as $a(n)$. Thus when we say that a vertex operator $b(z)$ is annihilated by the

Fourier mode $a(n)$ of a vertex operator $a(z)$, we mean that $a \circ_n b = 0$. Here we regard b as being an element in the state space A, while a operates on the state space, and the map $a \mapsto \hat{a}$ is the state-operator correspondence.

Note that every commutative (super) algebra is canonically a vertex algebra where any two elements strictly (graded) commute. More generally we shall say that a vertex algebra is abelian if any two elements pairwise commute. Otherwise we say that the vertex algebra is non-abelian. If a, b are two vertex operators which commute, then their Wick product is the ordinary product and we write ab or $a(z)b(z)$.

2.2. Examples

We now give several constructions of known examples of vertex (super) algebras, all of which will be used extensively later.

Example 2.13. *Current algebras.*

Let $\mathfrak{g}$ be a Lie algebra equipped with a symmetric $\mathfrak{g}$-invariant bilinear form B, possibly degenerate. The loop algebra of $\mathfrak{g}$ is defined to be

$$\mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}],$$

with bracket given by $[ut^n, vt^m] = [u, v]t^{n+m}$. The form B determines a 1-dimensional central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}[t, t^{-1}]$ as follows:

$$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbf{C}\tau,$$

with bracket

$$[ut^n, vt^m] = [u, v]t^{n+m} + nB(u, v)\delta_{n+m,0}\tau.$$

$\hat{\mathfrak{g}}$ is equipped with the $\mathbf{Z}$-grading $deg(ut^n) = n$, and $deg(\tau) = 0$. Let $\mathfrak{g}_{\geq}$ be the subalgebra of elements of non-negative degree, and let

$$N(\mathfrak{g}, B) = \mathfrak{U}\hat{\mathfrak{g}} \otimes_{\mathfrak{g}_{\geq}} \mathbf{C}$$

where $\mathbf{C}$ is the $\mathfrak{g}_{\geq}$-module in which $\mathfrak{g}[t]$ acts by zero and τ by 1. Clearly $N(\mathfrak{g}, B)$ is graded by the non-positive integers. For $u \in \mathfrak{g}$, denote by $u(n)$ the linear operator on $N(\mathfrak{g}, B)$ representing ut^n, and put

$$u(z) = \sum_n u(n) z^{-n-1}.$$

Then for $u, v \in \mathfrak{g}$, we get

$$[u(z)_+, v(w)] = B(u,v)(z-w)^{-2} + [u,v](w)(z-w)^{-1}$$
$$[u(z)_-, v(w)] = -B(u,v)(w-z)^{-2} + [u,v](w)(w-z)^{-1}.$$

It follows immediately that $(z-w)^2[u(z), v(w)] = 0$. Thus the operators $u(z) \in QO(N(\mathfrak{g}, B))$ generate a vertex algebra [14][27][26], which we denote by $O(\mathfrak{g}, B)$. Consider the vector $\mathbb{1} = 1 \otimes 1 \in N(\mathfrak{g}, B)$, called the vacuum vector.

Lemma 2.14. *[27] The creation map $\chi : O(\mathfrak{g}, B) \to N(\mathfrak{g}, B)$, $a(z) \mapsto a(-1)\mathbb{1}$, is an $O(\mathfrak{g}, B)$-module isomorphism.*

Proof: We sketch a proof. By Remark 2.1, we have $\chi(a \circ_n b) = a(n)\chi(b)$, hence $O(\mathfrak{g}, B)$ is an $O(\mathfrak{g}, B)$-module homomorphism. Next, $U\hat{\mathfrak{g}}$ has a derivation defined by $D\tau = 0$, $D(ut^n) = -nut^{n-1}$, and it descends to a linear map on $N(\mathfrak{g}, B)$ such that $[D, u(z)] = \partial u(z)$. This implies that $[D, a] = \partial a$ for all $a \in O(\mathfrak{g}, B)$. Thus to show that χ is a linear isomorphism, it suffices to show that it is surjective, by Lemma 2.8. But this follows from PBW (see below). $\square$

It is convenient to identify the spaces $N(\mathfrak{g}, B)$ and $O(\mathfrak{g}, B)$ under this isomorphism. Obviously $O(\mathfrak{g}, B)$ contains the iterated Wick products

$$: u^{I_0} \partial u^{I_1} \cdots \partial^p u^{I_p} :$$

where u^I means the *symbol* $u_1(z)^{i_1} \cdots u_d(z)^{i_d}$, ∂u^I means the symbol $\partial u_1(z)^{i_1} \cdots \partial u_d(z)^{i_d}$, for a given multi-index $I = (i_1, .., i_d)$, and likewise for other multi-index monomials. Here the $u_1, .., u_d$ form a basis of $\mathfrak{g}$. Under the creation map the image of the iterated Wick products above are the vectors, up to nonzero scalars,

$$u(-1)^{I_0} u(-2)^{I_1} \cdots u(-p-1)^{I_p} \mathbb{1}$$

which form a PBW basis, indexed by $(I_0, I_1, I_2, ...)$, of the induced module $N(\mathfrak{g}, B)$. Note also that there is a canonical inclusion of linear spaces $\mathfrak{g} \hookrightarrow O(\mathfrak{g}, B)$, $u \mapsto u(z)$.

An even vertex operator J is called a current if $J(z)J(w) \sim \alpha\ (z-w)^{-2}$ for some scalar α. The formula for $[u(z)_+, v(w)]$ above implies the more familiar OPE relation

$$u(z)v(w) \sim B(u,v)(z-w)^{-2} + [u,v](w)\ (z-w)^{-1}.$$

In particular each $u(z)$ is a current (hence the name current algebra). The vertex algebra $O(\mathfrak{g},B)$ has the following universal property [27]. Suppose that A is any vertex algebra and $\phi:\mathfrak{g}\to A$ is a linear map such that $\phi(u)(z)\ \phi(v)(w)\sim B(u,v)(z-w)^{-2}+\phi([u,v])(w)\ (z-w)^{-1}$ for $u,v\in\mathfrak{g}$. Then there exists a unique vertex algebra homomorphism $O(\mathfrak{g},B)\to A$ sending $u(z)$ to $\phi(u)(z)$ for $u\in\mathfrak{g}$. In particular, any Lie algebra homomorphism $(\mathfrak{g},B)\to(\mathfrak{g}',B')$ preserving the bilinear forms induces a unique vertex algebra homomorphism $O(\mathfrak{g},B)\to O(\mathfrak{g}',B')$ extending $\mathfrak{g}\to\mathfrak{g}'$. It is also known [27] that any Lie algebra derivation $d:(\mathfrak{g},B)\to(\mathfrak{g},B)$ induces a unique vertex algebra derivation (i.e. a graded derivation of all circle products) $\mathbf{d}:O(\mathfrak{g},B)\to O(\mathfrak{g},B)$ with $u(z)\mapsto(du)(z)$.

When $\mathfrak{g}$ is a finite-dimensional Lie algebra, $\mathfrak{g}$ possesses a canonical invariant, symmetric bilinear form, namely, the Killing form $\kappa(u,v)=Tr\big(ad(u)\cdot ad(v)\big)$. In this case, the current algebra $O(\mathfrak{g},\lambda\kappa)$ is said to have a *Schwinger charge* λ [33].

It is easy to see that if B_1,B_2 are bilinear forms on $\mathfrak{g}$, and M_1,M_2 are $O(\mathfrak{g},B_1)$-, $O(\mathfrak{g},B_2)$-modules respectively, then $M_1\otimes M_2$ is canonically an $O(\mathfrak{g},B_1+B_2)$-module. In particular, tensor products of $O(\mathfrak{g},0)$-modules are again $O(\mathfrak{g},0)$-modules.

There is a verbatim construction for any Lie super algebra equipped with an invariant form.

Example 2.15. *Semi-infinite symmetric and exterior algebras.*

Let V be a finite dimensional vector space. Regard $V\oplus V^*$ as an abelian Lie algebra. Then its loop algebra has a one-dimensional central extension by $\mathbf{C}\tau$ with bracket

$$[(x,x')t^n,(y,y')t^m]=(\langle y',x\rangle-\langle x',y\rangle)\delta_{n+m,0}\tau,$$

which is a Heisenberg algebra, which we denote by $\mathfrak{h}=\mathfrak{h}(V)$. Let $\mathfrak{b}\subset\mathfrak{h}$ be the subalgebra generated by τ, $(x,0)t^n$, $(0,x')t^{n+1}$, for $n\geq 0$, and let $\mathbf{C}$ be the one-dimensional $\mathfrak{b}$-module on which each $(x,0)t^n$, $(0,x')t^{n+1}$ act trivially and the central element τ acts by the identity. Consider the $\mathfrak{U}\mathfrak{h}$-module $\mathfrak{U}\mathfrak{h}\otimes_{\mathfrak{b}}\mathbf{C}$. The operators representing $(x,0)t^n,(0,x')t^{n+1}$ on this module are denoted by $\beta^x(n),\gamma^{x'}(n)$, and the Fourier series

$$\beta^x(z)=\sum\beta^x(n)z^{-n-1},\quad \gamma^{x'}(z)=\sum\gamma^{x'}(n)z^{-n-1}\in QO(\mathfrak{U}\mathfrak{h}\otimes_{\mathfrak{b}}\mathbf{C})$$

have the properties

$$[\beta^x_+(z),\gamma^{x'}(w)]=\langle x',x\rangle(z-w)^{-1},\quad [\beta^x_-(z),\gamma^{x'}(w)]=\langle x',x\rangle(w-z)^{-1}.$$

It follows that $(z-w)[\beta^x(z),\gamma^{x'}(w)]=0$. Moreover the $\beta^x(z)$ commute; likewise for the $\gamma^{x'}(z)$. Thus the $\beta^x(z),\gamma^{x'}(z)$ generate a vertex algebra $\mathcal{S}(V)$. This algebra was introduced in [FMS], and is known as a fermionic ghost system, or a $\beta\gamma$-system, or a semi-infinite symmetric algebra. By using the Lie algebra derivation $D:\mathfrak{h}\to\mathfrak{h}$ defined by $(x,0)t^n\mapsto -n(x,0)t^{n-1}$, $(0,x')t^{n+1}\mapsto -n(0,x')t^n$, $\tau\mapsto 0$, one can easily show, as in the case of $O(\mathfrak{g},B)$, that the creation map $\mathcal{S}(V)\to\mathfrak{U}\mathfrak{h}\otimes_{\mathfrak{b}}\mathbf{C}$, $a(z)\mapsto a(-1)1\otimes 1$, is a linear isomorphism, and that the $\beta^x,\gamma^{x'}$ have the OPE relation

$$\beta^x(z)\gamma^{x'}(w)\sim\langle x',x\rangle(z-w)^{-1}.$$

By the PBW theorem, it is easy to see that the vector space $\mathfrak{U}\mathfrak{h}\otimes_{\mathfrak{b}}\mathbf{C}$ has the structure of a polynomial algebra with generators given by the negative Fourier modes $\beta^x(n),\gamma^{x'}(n)$, $n<0$, which are linear in $x\in V$ and $x'\in V^*$.

We can also regard $V\oplus V^*$ as an odd abelian Lie (super) algebra, and consider its loop algebra and a one-dimensional central extension by $\mathbf{C}\tau$ with bracket

$$[(x,x')t^n,(y,y')t^m]=(\langle y',x\rangle+\langle x',y\rangle)\delta_{n+m,0}\tau.$$

Call this $\mathbf{Z}$-graded algebra $\mathfrak{j}=\mathfrak{j}(V)$, and form the induced module $\mathfrak{U}\mathfrak{j}\otimes_{\mathfrak{a}}\mathbf{C}$. Here $\mathfrak{a}$ is the subalgebra of $\mathfrak{j}$ generated by τ, $(x,0)t^n$, $(0,x')t^{n+1}$, for $n\geq 0$, and $\mathbf{C}$ is the one-dimensional $\mathfrak{a}$-module on which $(x,0)t^n$, $(0,x')t^{n+1}$ act trivially and τ acts by 1. Then there is clearly a vertex algebra $\mathcal{E}(V)$, analogous to $\mathcal{S}(V)$, and generated by odd vertex operators $b^x(z),c^{x'}(z)\in QO(\mathfrak{U}\mathfrak{j}\otimes_{\mathfrak{a}}\mathbf{C})$ with OPE

$$b^x(z)c^{x'}(w)\sim\langle x',x\rangle(z-w)^{-1}.$$

This vertex algebra is known as a bosonic ghost system, or bc-system, or a semi-infinite exterior algebra. Again the creation map $\mathcal{E}(V)\to\mathfrak{U}\mathfrak{j}\otimes_{\mathfrak{a}}\mathbf{C}$, $a(z)\mapsto a(-1)1\otimes 1$, is a linear isomorphism. As in the symmetric case, the vector space $\mathfrak{U}\mathfrak{j}\otimes_{\mathfrak{a}}\mathbf{C}$ has the structure of an odd polynomial algebra with generators given by the negative Fourier modes $b^x(n),c^{x'}(n)$, $n<0$, which are linear in $x\in V$ and $x'\in V^*$.

A lot of subsequent computations involve taking OPE of iterated Wick products of vertex operators in

$$\mathcal{W}(V):=\mathcal{E}(V)\otimes\mathcal{S}(V).$$

There is a simple tool from physics, known as Wick's theorem, that allows us to compute $A(z)B(w)$ easily where each of A,B has the shape $:a_1\cdots a_p:$ where a_i is one of the generators of $\mathcal{W}(V)$, or their higher derivatives. For an introduction to Wick's theorem, see [20]. Here is a typical computation by Wick's theorem by "summing over all possible contractions":

$$\begin{aligned}(:a_1(z)a_2(z):)(:a_3(w)a_4(w):) &\sim \langle a_2a_3\rangle\langle a_1a_4\rangle + (-1)^{|a_2||a_3|}\langle a_1a_3\rangle\langle a_2a_4\rangle \\ &+ \langle a_2a_3\rangle :a_1(z)a_4(w): + :a_2(z)a_3(w): \langle a_1a_4\rangle \\ &+(-1)^{|a_2||a_3|}\langle a_1a_3\rangle :a_2(z)a_4(w): +(-1)^{|a_2||a_3|} :a_1(z)a_3(w): \langle a_2a_4\rangle.\end{aligned}$$

Here the a_i are homogeneous vertex operators with OPE $a_i(z)a_j(w) \sim \langle a_ia_j\rangle$, where the symbol $\langle a_ia_j\rangle$ denotes something of the form *const.* $(z-w)^{-p}$ depending on i,j. To get the final answer for the OPE, one formally expands each $:a_i(z)a_j(w):$ on the right side above in powers of $(z-w)$, i.e. replacing it formally by $:a_i(w)a_j(w): + :\partial a_i(w)a_j(w):(z-w)+\cdots$.

Now let $\mathfrak{g}$ be a Lie algebra and V be a finite-dimensional $\mathfrak{g}$-module via the homomorphism $\rho:\mathfrak{g}\to End\ V$. Associated to ρ is a $\mathfrak{g}$-invariant bilinear form B on $\mathfrak{g}$ given by $B(u,v) = Tr\big(\rho(u)\rho(v)\big)$.

Lemma 2.16. *$\rho:\mathfrak{g}\to End\ V$ induces a vertex algebra homomorphism $\rho_{\mathcal{S}}: O(\mathfrak{g},-B)\to \mathcal{S}(V)$.*

Proof: Let $\rho^*:\mathfrak{g}\to V^*$ be the dual module, let $\langle,\rangle$ denote the pairing between V and V^*. Choose a basis $x_1,\dots,x_n$ of V, and let $x_1',\dots,x_n'$ be the dual basis. Put

$$\Theta^u_{\mathcal{S}}(z) = -:\beta^{\rho(u)x_i}(z)\gamma^{x_i'}(z):$$

(summing over i, as usual). We need to show that the following OPE holds:

$$\Theta^u_{\mathcal{S}}(z)\Theta^v_{\mathcal{S}}(w) \sim -B(u,v)(z-w)^{-2} + \Theta^{[u,v]}_{\mathcal{S}}(w)(z-w)^{-1}. \tag{2.8}$$

By Wick's theorem,

$$\begin{aligned}\Theta^u_{\mathcal{S}}(z)\Theta^v_{\mathcal{S}}(w) &= \Big(-:\beta^{\rho(u)x_i}(z)\gamma^{x_i'}(z):\Big)\Big(-:\beta^{\rho(v)x_j}(w)\gamma^{x_j'}(w):\Big) \\ &= -\langle\rho(u)x_i,x_j'\rangle\langle\rho(v)x_j,x_i'\rangle(z-w)^{-2} - \langle\rho(v)x_j,x_i'\rangle :\beta^{\rho(u)x_i}(w)\gamma^{x_j'}(w):(z-w)^{-1} \\ &+ \langle\rho(u)x_i,x_j'\rangle :\beta^{\rho(v)x_j}(w)\gamma^{x_i'}(w):(z-w)^{-1}\end{aligned}$$

which yields the right side of (2.8). $\square$

Likewise we have the fermionic analogues $\mathcal{E}(V)$ of $\mathcal{S}(V)$, and $\Theta^u_{\mathcal{E}}$ of $\Theta^u_{\mathcal{S}}$ with

$$\Theta^u_{\mathcal{E}}(z) =: b^{\rho(u)x_i}c^{x_i'}: .$$

A verbatim computation with gives

Lemma 2.17. *$\rho : \mathfrak{g} \to End\ V$ induces a vertex algebra homomorphism $\rho_{\mathcal{E}} : O(\mathfrak{g}, B) \to \mathcal{E}(V)$.*

Now let's specialize to the case where V is the adjoint module of $\mathfrak{g}$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra. Then $\mathcal{W}(\mathfrak{g}) = \mathcal{E}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g})$ is called the semi-infinite Weil algebra of $\mathfrak{g}$. This algebra has been studied by numerous authors (see e.g. [10][1][16]). By the two preceding lemmas, we have a vertex algebra homomorphism $O(\mathfrak{g}, 0) \to \mathcal{W}(\mathfrak{g})$, with $u(z) \mapsto \Theta^u_{\mathcal{E}}(z) \otimes 1 + 1 \otimes \Theta^u_{\mathcal{S}}(z)$ for $u \in \mathfrak{g}$.

Example 2.18. *Virasoro elements.*

Let A be a vertex algebra. We call a vertex operator $L \in A$ a Virasoro element if

$$L(z)L(w) \sim \frac{c}{2}(z-w)^{-4} + 2L(w)\ (z-w)^{-2} + \partial L(w)\ (z-w)^{-1}$$

where c is a scalar called the central charge of L. One often further requires that $L(w)\circ_1$ acts diagonalizably on A and that $L(w)\circ_0$ acts by ∂. If these two conditions hold, then A, equipped with L, is called a conformal vertex algebra of central charge c. A vertex operator $a \in A$ is said to be primary (with respect to L) of conformal weight $\Delta \in \mathbf{C}$ if

$$L(z)a(w) \sim \ \Delta\ a(w)\ (z-w)^{-2} + \partial a(w)\ (z-w)^{-1}.$$

A vertex operator $a \in A$ is said to be quasi-primary of conformal weight $\Delta \in \mathbf{Z}_{>}$ if

$$L(z)a(w) \sim \ \alpha\ (z-w)^{-\Delta-2} + \Delta\ a(w)\ (z-w)^{-2} + \partial a(w)\ (z-w)^{-1}$$

for some scalar α.

For example if $\mathfrak{g}$ is a finite-dimensional simple Lie algebra then the vertex algebra $O(\mathfrak{g}, \lambda\kappa)$ has a Virasoro element given by *Sugawara-Sommerfield formula*

$$L(z) = \frac{1}{2\lambda+1}\sum_i : x^i(z)x^i(z) :$$

where the x^i is an orthornormal basis of $(\mathfrak{g}, \kappa)$. This Virasoro element has central charge $\frac{2\lambda\ dim\ \mathfrak{g}}{2\lambda+1}$ provided of course that the denominator is nonzero. (Note that we have chosen a normalization so that we need not explicitly mention the dual Coxeter number of $\mathfrak{g}$.) More generally if $(\mathfrak{g}, B)$ is any finite dimensional Lie algebra with a non-degenerate invariant form then $O(\mathfrak{g}, \lambda B)$ admits a Virasoro element L for all but finitely many values of λ [27]. The Virasoro element above is characterized by property that for every $x \in \mathfrak{g}$, the vertex operator $x(z)$ is primary of conformal weight 1.

Example 2.19. *Topological vertex algebras.*

This notion was introduced in [25] Definition 3.4, where we call these objects TVA. It is an abstraction based on examples from physics (see e.g. [4][7][25]). A topological vertex algebra is a vertex algebra A equipped with four distinguished vertex operators L, F, J, G, where L is a Virasoro element with central charge zero, F is an even current which is a conformal weight one quasi-primary (with respect to L), J an odd conformal weight one primary with $J(0)^2 = 0$, and G an odd conformal weight two primary, such that

$$J(0)G = L, \quad F(0)J = J, \quad F(0)G = -G.$$

In special cases, further conditions are often imposed, such as that $J(z)$ commutes with itself, or that $G(z)$ commutes with itself, which we do not require here. Note also that we do not require that the Fourier mode $L(1)$ acts diagonalizably on A. There are numerous examples arising from physics. One of the simplest is given by $\mathcal{W}(\mathbf{C})$, which has four generators b, c, β, γ. If we put (suppressing z) $L = - : b\partial c : + : \beta\partial\gamma :$, $F = - : bc :$, $J =: c\beta :$, $G =: b\partial\gamma :$, then it is straightforward to check that they give $\mathcal{W}(\mathbf{C})$ the structure of a TVA. The same vertex algebra $\mathcal{W}(\mathbf{C})$ supports many TVA structures. One can twist the one above by using the current F to get some other TVA structures on $\mathcal{W}(\mathbf{C})$. Another such example is given in [16].

2.3. Differential and graded structures

The vertex algebras we consider here typically come equipped with a number of graded structures. The $\mathbf{Z}/2$-graded structure on a vertex algebra often arises from a $\mathbf{Z}$-grading we call degree. The semi-infinite exterior algebra $\mathcal{E}(V)$ is one such example, where the odd generators $b^x, c^{x'}$ are assigned degrees -1 and +1 respectively. Then $\mathcal{E}(V)$ is a direct sum of subspaces consisting of degree homogeneous elements. Like the $\mathbf{Z}/2$-graded structure in this case, the degree structure on a vertex algebra is additive under the circle products. In general, we say that a vertex algebra $\mathcal{A}$ is *degree graded* if it is $\mathbf{Z}$-graded $\mathcal{A} = \oplus_{p \in \mathbf{Z}} \mathcal{A}^p$, and the degree is additive under the circle products. We denote the degree by $deg = deg_{\mathcal{A}}$.

In addition to the degree grading, the vertex algebras we consider often come equipped with another $\mathbf{Z}$-grading we call weight. In the example $\mathcal{E}(V)$, the vertex operator $b^x, c^{x'}$

can be assigned weights 1 and 0 respectively. Then $\mathcal{E}(V)$ is a direct sum of subspaces consisting of weight homogeneous elements. The weight structure is not additive under the circle products in this case. But rather, we have $wt(a \circ_n b) = wt(a) + wt(b) - n - 1$. In general, we say that a vertex algebra $\mathcal{A}$ *weight graded* if it is $\mathbf{Z}$-graded $\mathcal{A} = \oplus_{n\in\mathbf{Z}}\mathcal{A}[n]$, and the nth circle product has weight $-n-1$. We denote the weight by $wt = wt_{\mathcal{A}}$. Note that there can be several different weight structures on the same vertex algebra.

We say that a vertex algebra $\mathcal{A}$ is *degree-weight graded* if it is both degree and weight graded and the gradings are compatible, i.e. $\mathcal{A}[n] = \oplus_{p\in\mathbf{Z}}\mathcal{A}^p[n]$, $\mathcal{A}^p = \oplus_{n\in\mathbf{Z}}\mathcal{A}^p[n]$, where $\mathcal{A}^p[n] = \mathcal{A}^p \cap \mathcal{A}[n]$.

As a consequence of Lemma 2.9, if a vertex algebra $\mathcal{A}$ is weight graded and has no negative weight elements, then $\mathcal{A}[0]$ is a commutative associative algebra with product $\circ_{-1}$ and unit 1. Almost all vertex algebras in this paper have this property.

If a vertex algebra $\mathcal{A}$ comes equipped with a Virasoro element L where $L\circ_1$ acts diagonalizably on $\mathcal{A}$ with integer eigenvalues, then the eigenspace decomposition defines a weight grading on $\mathcal{A}$.

We call a pair $(\mathcal{A}, \delta)$ a differential vertex algebra if $\mathcal{A}$ is a vertex algebra equipped with a linear map which is vertex algebra derivation, i.e. a super-derivation of each circle product, such that $\delta^2 = 0$. If, furthermore, $\mathcal{A}$ is degree graded, then we assume that δ is a degree $+1$ linear map. If, furthermore, $\mathcal{A}$ is weight graded, then we assume that δ is a weight 0 linear map. The categorical notion of homomorphisms and modules of differential vertex algebras are defined in an obvious way.

2.4. The commutant construction

This is a way to construct interesting vertex subalgebras of a given vertex algebra, and it is the vertex algebra analogue of the commutant construction in the theory of associative algebras.

Let A be a vertex algebra and $S \subset A$ any subset. The commutant of S in A is the space

$$Com(S, A) = \{a(z) \in A|\ b(z) \circ_n a(z) = 0,\ \forall b(z) \in S,\ n \geq 0\}.$$

It is a vertex subalgebra of A: this follows from the fact that for any elements a, b in a vertex algebra, we have $[b(z), a(w)] = 0$ iff $b(z) \circ_n a(z) = 0$ for $n \geq 0$, which is an immediate consequence of Lemma 2.4. From this, it also follows that if C is the vertex algebra generated by the set S, then

$$Com(C, A) = Com(S, A).$$

Clearly if $S \subset S' \subset A$, then we have $Com(S', A) \subset Com(S, A)$.

The commutant subalgebra $Com(C, A)$ has a second interpretation. It can be thought of as a vertex algebra analogue of the ring of invariants in a commutative ring with a Lie group or a Lie algebra action. First we can think of A as a C-module via the left regular action of A. Then $Com(C, A)$ is the subalgebra of A annihilated by all $\hat{c}(n)$, $c \in C$, $n \geq 0$. If C is a homomorphic image of a current algebra $O(\mathfrak{g}, B)$, then $Com(C, A) = A^{\mathfrak{g}_{\geq}}$ where the right side is the subspace of A annihilated by $u(n)$, $u \in \mathfrak{g}$, $n \geq 0$. The invariant theory point of view of the commutant construction is developed in [28]. In our construction of the chiral equivariant cohomology later, the commutant subalgebra will play the role of the classical algebra of invariants in the classical equivariant cohomology.

2.5. A vertex algebra for each open set

Notations. Here U, U', V, V' will denote open sets in $\mathbf{R}^n$, $\gamma : \mathbf{R}^n \to \mathbf{R}$ an arbitrary linear coordinate, and $\gamma^i : \mathbf{R}^n \to \mathbf{R}$ the i-th standard coordinate. The space of smooth complex-valued differential forms $\Omega(U)$ can be thought of as the space of functions on a super manifold. Without digressing into super geometry, it suffices to think of the linear coordinates γ^i (restricted to U) as even variables, and the coordinate one-forms $c^i := d\gamma^i$ as odd variables. We can regard the $\beta^i = \frac{\partial}{\partial \gamma^i}$ as even vector fields acting as derivations, and the $b^i = \frac{\partial}{\partial c^i}$ as odd vector fields acting as odd derivations, on the function space $\Omega(U)$.

Let $C = \mathbf{C}\{\frac{\partial}{\partial \gamma^i}, \frac{\partial}{\partial c^i} : 1 \leq i \leq n\} = C_0 \oplus C_1$ denote the $\mathbf{C}$-span of the constant vector fields. It is an abelian Lie (super) algebra acting by derivations on the commutative super algebra $\Omega(U)$. We now apply the current algebra construction to the semi-direct product Lie algebra

$$\Lambda(U) := C \triangleright \Omega(U)$$

equipped with the zero bilinear form 0. Thus we consider the loop algebra $\Lambda(U)[t, t^{-1}]$ and its module

$$\mathcal{V}(U) := N(\Lambda(U), 0)$$

For any $x \in \Lambda(U)$, denote by $x(k) \in End\ \mathcal{V}(U)$ the operator representing the $xt^k \in \Lambda(U)[t, t^{-1}]$, and form $x(w) = \sum x(k) w^{-k-1} \in QO(\mathcal{V}(U))$. Then the $x(w)$ generates the vertex (super) algebra $O(\Lambda(U), 0)$ defined in Example 2.13. As before, we identify this as a linear space with $\mathcal{V}(U)$. Recall that we have a linear inclusion $\Lambda(U) \hookrightarrow \mathcal{V}(U)$, $x \mapsto x(w)$.

In particular for any function $f\in C^\infty(U)\subset\Omega(U)\subset\Lambda(U)$ and any vector field $\beta\in C_0$, we have

$$\beta(z)f(w)\sim\beta(f)(w)\ (z-w)^{-1}.$$

The vertex algebra $\mathcal{V}(U)$ is too large. For example the constant function 1 gives a vertex operator $1(w)$ which is not equal to id. If f,g are two smooth functions then the vertex operators $(fg)(w)$ and $:f(w)g(w):$ are not the same. Let $\mathcal{I}(U)$ be the two-sided ideal in $\mathcal{V}(U)$ generated by the vertex operators

$$\frac{d}{dw}f(w)-\frac{d}{dw}\gamma^i(w)\ \frac{\partial f}{\partial\gamma^i}(w)-\frac{d}{dw}c^i(w)\ \frac{\partial f}{\partial c^i}(w),\quad (fg)(w)-f(w)g(w),\quad 1(w)-id$$

where $f,g\in\Omega(U)$. (As always, the repeated index i is summed over $i=1,..,n$, unless said otherwise.) We put

$$\mathcal{Q}(U):=\mathcal{V}(U)/\mathcal{I}(U).$$

Note that $\Lambda(U)$ becomes a $\mathbf{Z}_+$-graded Lie super algebra if we declare $C,\Omega(U)$ to have weight 1,0 respectively. This induces a $\mathbf{Z}$-grading on $\mathcal{Q}(U)$, and a canonical surjection [2] $\mathcal{Q}(U)[0]\rightarrow\Omega(U)$. On $\mathcal{Q}(U)$, we also have

$$b^i(z)c^j(w)\sim\delta_{ij}(z-w)^{-1}.$$

Let Γ be the vertex algebra generated by $\beta(w)$ with $\beta\in C_0$, and the $f(w)$ with $f\in C^\infty(U)$, subject to the relations

$$\beta(z)f(w)\sim\beta(f)(w)\ (z-w)^{-1}$$

and with

$$\frac{d}{dw}f(w)-\frac{d}{dw}\gamma^i(w)\ \frac{\partial f}{\partial\gamma^i}(w),\quad (fg)(w)-f(w)g(w),\quad 1(w)-id$$

being set to zero for all $f,g\in C^\infty(U)$. Let B be the vertex algebra generated by the $b^i(w),c^i(w)$, subject to the relations $b^i(z)c^j(w)\sim\delta_{ij}(z-w)^{-1}$. We claim that there is a canonical isomorphism

$$\mathcal{Q}(U)\cong\Gamma\otimes B$$

---

[2] We thank B. Song for pointing this out. If we declare that $\beta^i(z),b^i(z),c^i(z)$ have weights 1,1,0 respectively, $f(z)$ has weight 0 for $f\in C^\infty(U)$, then $\mathcal{I}(U)$ is homogeneous ideal in $N(\Lambda(U),0)$. Hence $\mathcal{Q}(U)[0]\rightarrow\Omega(U)$, $g(z)\mapsto g$, is well-defined and surjective.

Since each $f \in \Omega(U)$ can be uniquely written as $f_I c^I$ where $f_I \in C^\infty(U)$ and $c^I = c^{i_1} c^{i_2} \cdots$, $i_1 < i_2 < \cdots$, we can define a map $\Lambda(U) \to \Gamma \otimes B$, $f_I c^I \mapsto f_I(z) \otimes c^{i_1}(z) c^{i_2}(z) \cdots$, $\beta^i \mapsto \beta^i(z) \otimes 1$, $b^i \mapsto 1 \otimes b^i(z)$. Since the image of $\Lambda(U)$ satisfies the expected OPE of the current algebra $O(\Lambda(U), 0) \equiv N(\Lambda(U), 0)$, by the universal property, we have a surjective map $N(\Lambda(U), 0) \to \Gamma \otimes B$. It is easy to check that this map factors through the ideal of relations $\mathcal{I}(U)$, hence we have a surjective map $\mathcal{Q}(U) \to \Gamma \otimes B$. Likewise, we have maps $\Gamma \to \mathcal{Q}(U), B \to \mathcal{Q}(U)$ whose images commute in $\mathcal{Q}(U)$. This yields a map $\Gamma \otimes B \to \mathcal{Q}(U)$. Then we verify that this map is the inverse of $\mathcal{Q}(U) \to \Gamma \otimes B$ above.

We would like to write down a basis for Γ. For this, we will construct Γ in a different way. Define a $\mathbf{Z}$-graded Heisenberg Lie algebra $\mathfrak{g}$ by the relations

$$[\beta_p, \gamma_q] = \delta_{p+q,0} \beta(\gamma)\ \mathbb{1}$$

where the generators β_p are linear in $\beta \in C_0$, and the γ_q are linear in $\gamma \in Hom(\mathbf{R}^n, \mathbf{R})$. Let $\mathfrak{g}_\geq \subset \mathfrak{g}$ be the subalgebra generated by $\mathbb{1}$ and the β_p, γ_p with $p \geq 0$. We make $C^\infty(U)$ a $\mathfrak{g}_\geq$-module by

$$\beta_p \cdot f = \gamma_p \cdot f = 0, \qquad p > 0$$
$$\beta_0 \cdot f = \beta(f), \quad \gamma_0 \cdot f = \gamma f, \quad \mathbb{1} \cdot f = f.$$

Put

$$\Gamma' := \mathfrak{U}\mathfrak{g} \otimes_{\mathfrak{g}_\geq} C^\infty(U).$$

The commutation relations of the operators β_p, γ_q acting on this $\mathfrak{g}$-module translate into the equivalent relations

$$[\beta(z), \beta'(w)] = [\gamma(z), \gamma'(w)] = 0,$$
$$[\beta(z)_+, \gamma(w)] = \beta(\gamma)\ (z-w)^{-1}, \quad [\beta(z)_-, \gamma(w)] = \beta(\gamma)\ (w-z)^{-1}.$$

for $\beta, \beta' \in C_0$ and $\gamma, \gamma' \in Hom(\mathbf{R}^n, \mathbf{R})$, where $\beta(w) = \sum \beta_n w^{-n-1}$, $\gamma(w) = \sum \gamma_n w^{-n}$. By Lemma 2.4, it follows that the operators $\beta(z), \gamma(z) \in QO(\Gamma')$ generate a vertex algebra.

Note that $\mathfrak{g}$ acts on the abelian Lie algebra $C^\infty(U)[t, t^{-1}]$ by derivations defined by

$$\beta_p \cdot f t^q = \beta(f) t^{p+q}, \quad \gamma_p \cdot f t^q = \mathbb{1} \cdot f t^p = 0.$$

We will extend the $\mathfrak{g}$ action on Γ' to an action of the semi-direct product algebra $\mathfrak{g} \triangleright C^\infty(U)[t, t^{-1}]$ on Γ' as follows. *Having this action is the main point of constructing the module Γ'.*

By using the existence of a PBW basis of $\mathfrak{U}\mathfrak{g}$, we will first define a map $f(k) : \mathfrak{U}\mathfrak{g} \otimes C^\infty(U) \to \Gamma'$ for each $ft^k \in C^\infty(U)[t, t^{-1}]$ inductively, and then show that $f(k)$ descends to an operator $f(k) : \Gamma' \to \Gamma'$. Given $f, g \in C^\infty(U)$, we define $f(k)(1 \otimes g) = \delta_{k,-1}(1 \otimes fg)$ for $k \geq -1$. For $k < -1$, we put

$$f(k)(1 \otimes g) = \frac{1}{k+1} \sum_{p<0} p \ \gamma^i_p \ \frac{\partial f}{\partial \gamma^i}(k-p)(1 \otimes g).$$

Note that this definition is recursive. The formula comes from solving for the Fourier modes $f(k)$ using the anticipated relation (from our earlier construction of Γ)

$$\frac{d}{dw} f(w)(1 \otimes g) = \frac{d}{dw}\gamma^i(w) \ \frac{\partial f}{\partial \gamma^i}(w)(1 \otimes g).$$

The sought-after vertex operator $f(w) = \sum f(k) w^{-k-1} \in QO(\Gamma')$ will eventually play the role of the $f(w) \in QO(\Gamma)$ earlier. Now suppose $f(k)(\omega \otimes g)$ is defined for all $f \in C^\infty(U)$ and all $k \in \mathbf{Z}$. We define

$$f(k)(\gamma_p \omega \otimes g) = \gamma_p f(k)(\omega \otimes g)$$
$$f(k)(\beta_p \omega \otimes g) = \beta_p f(k)(\omega \otimes g) - \beta(f)(k+p)(\omega \otimes g).$$

This completes the definition of the $f(k) : \mathfrak{U}\mathfrak{g} \otimes C^\infty(U) \to \Gamma'$. Note that when $f \in C^\infty(U)$ is the restriction of a given linear function $\gamma \in Hom(\mathbf{R}^n, \mathbf{R})$, then $f(k) = \gamma_{k+1}$, i.e. $f(w) = \gamma(w)$ in this case.

Using the recursive definition above, it is straightforward but tedious to check that $f(k)$ descends to an operator $f(k) : \Gamma' \to \Gamma'$, and that $f(w) = \sum f(k) w^{-k-1} \in QO(\Gamma')$ satisfies the relations

$$[\gamma(z), f(w)] = 0, \ \ [\beta(z)_+, f(w)] = \beta(f)(w) \ (z-w)^{-1}, \ \ [\beta(z)_-, f(w)] = \beta(f)(w) \ (w-z)^{-1}. \tag{2.9}$$

To see that Γ' has a module structure over the Lie algebra $\mathfrak{g} \triangleright C^\infty(U)[t, t^{-1}]$, it remains to show that $[f(z), g(w)] = 0$ for all $f, g \in C^\infty(U)$. This follows from the next lemma.

Lemma 2.20. *For $f, g \in C^\infty(U)$, we have*

$$f(z)g(w) = g(w)f(z), \quad (fg)(w) = f(w)g(w), \quad 1(z) = id, \quad \frac{d}{dw} f(w) = \frac{d}{dw}\gamma^i(w) \frac{\partial f}{\partial \gamma^i}(w).$$

In particular Γ' *has a module structure over the Lie algebra* $\mathfrak{g} \triangleright C^\infty(U)[t,t^{-1}]$.

Proof: It follows from (2.9) that the γ_p commute with the commutator $[f(z),g(w)]$ for any $\gamma \in Hom(\mathbf{R}^n,\mathbf{R})$. It follows from the recursive definition of the $f(k)$ that $[f(z),g(w)]1\otimes h=0$ for any $h\in C^\infty(U)$. For $\beta\in C_0$, (2.9) implies that

$$[\beta_p,[f(z),g(w)]] = z^p[\beta(f)(z),g(w)] + w^p[f(z),\beta(g)(w)], \qquad [\gamma_p,[f(z),g(w)]]=0.$$

Using these commutator relations and the existence of a PBW basis of $\mathfrak{U}\mathfrak{g}$, we find by induction that $[f(z),g(w)]$ must be identically zero on Γ'. This proves the first asserted equation. The argument for each of the remaining three equations is analogous. $\square$

Corollary 2.21. *The circle algebra* $G'\subset QO(\Gamma')$ *generated by the operators* $\{\beta(w),f(w)|\beta\in C_0,\ f\in C^\infty(U)\}$ *is a vertex algebra. It is linearly isomorphic to* Γ'.

Proof: By the first equation of the preceding lemma together with (2.9), it follows that $(z-w)[\beta(z),f(w)]=0$ and $[f(z),g(w)]=0$. This implies that G' is a vertex algebra. Moreover we have $\beta_k 1_{\Gamma'} = f(k)1_{\Gamma'}=0$ for $k\geq 0$, where $1_{\Gamma'}:=1\otimes 1\in \mathfrak{U}\mathfrak{g}\otimes_{\mathfrak{g}_\geq} C^\infty(U)$. It follows that the creation map

$$G'\to\Gamma', \quad a(w)\mapsto \lim_{w\to 0} a(w)1_{\Gamma'} = a(-1)1_{\Gamma'}$$

is a well-defined linear map.

We claim that this is a linear isomorphism. In fact, it follows from the preceding lemma, that G' is spanned by the vertex operators, each having the shape

$$:\beta^{I_0}\frac{d\beta}{dw}^{I_1}\frac{d^2\beta}{dw^2}^{I_2}\cdots\frac{d\gamma}{dw}^{J_1}\frac{d^2\gamma}{dw^2}^{J_2}\cdots f_\alpha(w) :\in G'.$$

Its image under the creation map is a nonzero scalar (given by products of factorials) times the vector

$$\beta(-1)^{I_0}\beta(-2)^{I_1}\cdots\gamma(-2)^{J_1}\gamma(-3)^{J_2}\cdots\otimes f_\alpha\in\Gamma'.$$

Here $\{f_\alpha\}$ is a given basis of $C^\infty(U)$, and $\beta(k)=\beta_k$, $\gamma(k)=\gamma_{k+1}$; $\frac{d^k\beta}{dw^k}^I$ means the usual $\frac{d^k}{dw^k}\beta^{i_1}(w)\frac{d^k}{dw^k}\beta^{i_2}(w)\cdots$ for any given finite list $I=\{i_1,i_2,...\}$ of indices ranging over $\{1,..,n\}$; likewise for other multi-index monomials. By the PBW theorem, these vectors form a basis of Γ' indexed by $(I_0,I_1,..,J_1,J_2,...\alpha)$. This implies that $G'\to\Gamma'$ is a linear isomorphism. $\square$

From now on, we identify G' *with* Γ' *via this isomorphism.*

Corollary 2.22. *$\Gamma' \otimes B$ is canonically a $\Lambda(U)[t,t^{-1}]$-module such that $xt^p \cdot 1_{\Gamma'} \otimes 1_B = 0$ for $x \in \Lambda(U)$, $p \geq 0$.*

Proof: An element $x \in \Lambda(U) = C \triangleright \Omega(U)$ can be uniquely written as $x = \beta + b + \sum f_I c^I$ where $\beta \in C_0$, $b = \lambda_i b^i \in C_1$, $f_I \in C^\infty(U)$, $c^I = c^{i_1} c^{i_2} \cdots$, $i_1 < i_2 < \cdots$, as before. We define the linear operator $x(p)$ representing xt^p on $\Gamma' \otimes B$ to be the p-th Fourier mode of the vertex operator

$$x(w) = \beta(w) + b(w) + \sum f_I(w) c(w)^I$$

Here it is understood that $b(w) = \lambda_i b^i(w)$ and the $c^i(w)$ act on the factor B while $\beta(w)$ and the $f_I(w)$ act on the factor Γ'. Taking a second element $x' = \beta' + b' + \sum f'_{I'} c^{I'} \in \Lambda(U)$, and using (2.9) together with the first equation of the preceding lemma, we find that

$$x(z)x'(z') \sim [x,x'](z')(z-z')^{-1}$$

where $[x,x']$ is the Lie bracket in $\Lambda(U)$. This shows that $xt^p \mapsto x(p)$ defines a $\Lambda(U)[t,t^{-1}]$-module structure on $\Gamma' \otimes B$. Finally, by construction, the $\beta(w), b(w), f_I(w), c^i(w)$ are vertex operators whose p-th Fourier modes annihilates $1'_\Gamma \otimes 1_B$ for $p \geq 0$. It follows that the Fourier modes $x(p)$ of $x(z)$, which represent the xt^p, have the same property. $\square$

Corollary 2.23. *We have a vertex algebra isomorphism $\mathcal{Q}(U) = \Gamma \otimes B \to \Gamma' \otimes B = G' \otimes B$ which sends $x(z)$ to $x(z)$ for $x \in \Lambda(U)$.*

Proof: By the universal property of the induced module $\mathcal{V}(U)$, and by the preceding corollary, there is a unique $\Lambda(U)[t,t^{-1}]$-module homomorphism sending $1 \otimes 1 \in \mathcal{V}(U)$ to $1_{\Gamma'} \otimes 1_B \in \Gamma' \otimes B$. Since we identify $\mathcal{V}(U)$ with the vertex algebra generated by the $x(z) \in QO(\mathcal{V}(U))$, this map sends $x(z) \in QO(\mathcal{V}(U))$ to $x(z) \in G' \otimes B'$. In particular, it is a vertex algebra homomorphism. By the preceding lemma, this homomorphism factors through the ideal $\mathcal{I}(U) \subset \mathcal{V}(U)$, hence it descends to $\mathcal{Q}(U) \to G' \otimes B$. By construction it is obvious that $\mathcal{Q}(U) = \Gamma \otimes B$ is spanned by vertex operators of the shape

$$: \beta^{I_0} \frac{d\beta}{dw}^{I_1} \frac{d^2\beta}{dw^2}^{I_2} \cdots \frac{d\gamma}{dw}^{J_1} \frac{d^2\gamma}{dw^2}^{J_2} \cdots f_\alpha(w) : \otimes : b^{K_0} \frac{db}{dw}^{K_1} \frac{d^2 b}{dw^2}^{K_2} \cdots c^{L_0} \frac{dc}{dw}^{L_1} \frac{d^2 c}{dw^2}^{L_2} \cdots :$$

But their images form a basis of $G' \otimes B$. It follows that $\mathcal{Q}(U) \to G' \otimes B$ is an isomorphism. $\square$

From now on, we identify $\mathcal{Q}(U)$ with $G' \otimes B$ via the isomorphism. But it will be convenient to use both points of view.

Corollary 2.24. *There is a canonical map $\Omega(U) \hookrightarrow \mathcal{Q}(U)$ such that $fg \mapsto: f(w)g(w) :=$ $f(w)g(w)$.*

Proof: The map is defined by $f = \sum f_I c^I \mapsto f(w) = \sum f_I(w)c(w)^I \in G' \otimes B$. It is clear that this is independent of basis. By Lemma 2.20, it has the desired multiplicative property. If $f(w) = 0$, then $f_I(w) = 0$ for all I, since the $c(w)^I = c^{i_1}(w)c^{i_2}(w)\cdots$ are independent in B. In particular $f_I(-1)1_{\Gamma'} = 1 \otimes f_I = 0$ in Γ'. It follows that $f_I = 0$ for all I. Thus the map $f \mapsto f(w)$ is injective. $\square$

2.6. MSV chiral de Rham complex of a smooth manifold

Following [29], the idea is to first construct a sheaf of vertex algebras on $\mathbf{R}^n$, and then transfer it onto any given smooth manifold M in a coordinate independent way.

Lemma 2.25. *Any inclusion of open sets $U \overset{\iota}{\subset} U'$ induces a canonical vertex algebra homomorphism $\mathcal{Q}(U') \overset{\mathcal{Q}(\iota)}{\to} \mathcal{Q}(U)$. Moreover, this defines a sheaf of vertex algebras on $\mathbf{R}^n$*

Proof: Given an inclusion $\iota : U \subset U'$, clearly we have a Lie algebra homomorphism $\Lambda(U') \to \Lambda(U)$ induced by restrictions of functions $\Omega(U') \to \Omega(U)$. By the functoriality of the current algebra construction Example 2.13, we get a vertex algebra homomorphism $\mathcal{V}(U') \to \mathcal{V}(U)$. The ideal $\mathcal{I}(U')$ is mapped into $\mathcal{I}(U)$ because $\mathcal{I}(U')$ is generated by vertex operators constructed from the circle products $\circ_{-1}$, $\circ_{-2}$ and the vertex operators $f(w)$, $f \in \Omega(U')$. Thus we have a vertex algebra homomorphism $\mathcal{Q}(U') \to \mathcal{Q}(U)$, which we denote by $\mathcal{Q}(\iota)$. A similar argument shows that given inclusions of open sets $U_1 \overset{\iota_1}{\subset} U_2 \overset{\iota_2}{\subset} U_3$, we get $\mathcal{Q}(\iota_2 \circ \iota_1) = \mathcal{Q}(\iota_1) \circ \mathcal{Q}(\iota_2)$. This shows that the assignment $U \rightsquigarrow \mathcal{Q}(U)$ is a presheaf.

To see that $\mathcal{Q}$ is a sheaf, suppose that $U_i \overset{\iota_i}{\subset} U$ form a covering of U we need to show that the sequence

$$0 \to \mathcal{Q}(U) \to \prod_i \mathcal{Q}(U_i) \rightrightarrows \prod_{i,j} \mathcal{Q}(U_i \cap U_j)$$

is exact. By the PBW theorem applied to $\mathcal{Q}(U) = \Gamma' \otimes B$, each element $a \in \mathcal{Q}(U)$ can be uniquely represented in the shape $a = \sum_P P f_P$, where $f_P \in C^\infty(U)$, and $\{P\}$ a basis of the graded polynomial space

$$\mathbf{C}[\beta^i(k), \gamma^i(k-1), b^i(k), c^i(k) \ : \ k \le -1, \ 1 \le i \le n].$$

Given an inclusion $\iota : U \subset U'$, the restriction map $\mathcal{Q}(\iota) : \mathcal{Q}(U') \to \mathcal{Q}(U)$ sends $\sum P f_P$ to $\sum P f_P|U$. Now our assertion follows from the exact sequence

$$0 \to C^\infty(U) \to \prod_i C^\infty(U_i) \rightrightarrows \prod_{i,j} C^\infty(U_i \cap U_j)$$

This completes the proof. $\square$

In order to transfer the sheaf $\mathcal{Q}$ from $\mathbf{R}^n$ to an arbitrary smooth manifold, we must be able to compare the vertex algebras $\mathcal{Q}(U)$ under diffeomorphisms $U \to U'$ of open sets. For this it is convenient to enlarge the category $Open(\mathbf{R}^n)$ of open sets by allowing any open embedding $U \hookrightarrow U'$ of open sets to be a morphism. We shall denote this new category by $(Open(\mathbf{R}^n), \hookrightarrow)$. This category is a special example of a Grothendieck topology. Namely, if $U_i \overset{\psi_i}{\hookrightarrow} U$, $i = 1, 2$, are two morphisms then we declare the fiber product to be $U_1 \times_U U_2 := \psi_1(U_1) \cap \psi_2(U_2)$. We also declare any collection of morphisms $U_i \overset{\psi_i}{\hookrightarrow} U$ to be a covering if $\cup_i \psi_i(U_i) = U$. Note that any morphism $U \overset{\psi}{\to} U'$ can be factorized as a diffeomorphism followed by an inclusion $U \overset{\varphi}{\to} \psi(U) \overset{\iota}{\subset} U'$. But there may be another open set $W \supset U$ and a diffeomorphism $W \overset{\rho}{\to} U'$ such that ψ is factorized as $U \subset W \overset{\rho}{\to} U'$.

Lemma 2.26. *Any diffeomorphism of open sets $U \overset{\varphi}{\to} U'$ induces a canonical vertex algebra isomorphism $\mathcal{Q}(U') \overset{\mathcal{Q}(\varphi)}{\to} \mathcal{Q}(U)$. Moreover, given diffeomorphisms of open sets $U_1 \overset{\varphi_1}{\to} U_2 \overset{\varphi_2}{\to} U_3$, we get $\mathcal{Q}(\varphi_2 \circ \varphi_1) = \mathcal{Q}(\varphi_1) \circ \mathcal{Q}(\varphi_2)$.*

Proof: The diffeomorphism φ induces the pull-back isomorphism $\varphi^* : \Omega(U') \to \Omega(U)$ on forms. This induces a vertex algebra isomorphism from the subalgebra $\langle f(z) | f \in \Omega(U') \rangle \subset \mathcal{Q}(U')$ onto $\langle f(z) | f \in \Omega(U) \rangle \subset \mathcal{Q}(U)$ by $f(z) \mapsto \varphi^*(f)(z)$. We would like to extend this to $\mathcal{Q}(U') \to \mathcal{Q}(U)$. First note that φ^* does not extend to a Lie algebra isomorphism between $\Lambda(U')$ and $\Lambda(U)$ in general. However it extends to a Lie algebra isomorphism between two larger Lie algebras

$$\varphi^* : Vect(U') \triangleright \Omega(U') \to Vect(U) \triangleright \Omega(U)$$

where $Vect(U') = \Omega(U')\{\frac{\partial}{\partial \gamma^i}, \frac{\partial}{\partial c^i} : 1 \leq i \leq n\} = \Omega(U') \otimes C$ is the Lie algebra of all smooth super derivations on $\Omega(U')$. Since the constant vector fields C can be regarded as a subalgebra of $Vect(U')$, φ^* maps $\Lambda(U') = C \triangleright \Omega(U')$ to a Lie subalgebra of $Vect(U) \triangleright \Omega(U)$. In particular, one can express each of the pull-backs $\varphi^* \frac{\partial}{\partial \gamma^i}, \varphi^* \frac{\partial}{\partial c^i}, \varphi^* c^i, \varphi^* \gamma^i$, uniquely in terms of $\frac{\partial}{\partial \gamma^i}, \frac{\partial}{\partial c^i}, c^i, \gamma^i \in \Omega(U)$, and the coordinates φ^i of φ. In fact, we have

$$\varphi^* \gamma^i = \gamma^i \circ \varphi = \varphi^i, \qquad \varphi^* c^i = \varphi^* d\gamma^i = \frac{\partial \varphi^i}{\partial \gamma^j} c^j$$

and for any constant vector field $X \in C$, the vector field $\varphi^* X \in Vect(U)$ is determined by the condition that

$$(\varphi^* X)(\varphi^* f) = [\varphi^* X, \varphi^* f] = \varphi^*[X, f] = \varphi^*(X(f)), \quad f \in \Omega(U').$$

These pull-back expressions make sense as vertex operators if one *formally* replaces the $\frac{\partial}{\partial \gamma^i}, \frac{\partial}{\partial c^i}, c^i, \gamma^i, \varphi^i \in \Omega(U)$, by their vertex operator counterparts $\beta^i(z), b^i(z), c^i(z), \gamma^i(z), \varphi^i(z) \in \mathcal{Q}(U)$. Equivalently, φ^* determines an injective linear map $\Phi : \Lambda(U') \to \mathcal{Q}(U)$ with the property that $\Phi(f) = (\varphi^* f)(z)$ for $f \in \Omega(U')$. In particular, by Lemma 2.20, $\Phi(fg) = \Phi(f)\Phi(g)$ for $f, g \in \Omega(U')$.

A remarkable result of [29] says that

$$\Phi(b^i)(z)\ \Phi(c^j)(w) \sim \delta_{ij}(z-w)^{-1}, \quad \Phi(\beta)(z)\ \Phi(f)(w) \sim \Phi(\beta(f))(w)\ (z-w)^{-1} \tag{2.10}$$

for $\beta \in C_0$, $f \in C^\infty(U')$, and all other OPE are trivial. Since each element $x \in \Lambda(U')$ can be uniquely expressed as $x = \beta + b + \sum f_I c^I$, it follows that $\Phi(x) = \Phi(\beta) + \Phi(b) + \sum \Phi(f_I)\Phi(c^I)$. Taking a second element $x' \in \Lambda(U')$, it follows from (2.10) that

$$\Phi(x)(z)\ \Phi(x')(z') \sim \Phi([x, x'])(z')\ (z-z')^{-1}.$$

By the universal property of the current algebra construction of $\mathcal{V}(U')$, the map Φ extends to a vertex algebra homomorphism $\mathcal{V}(U') \to \mathcal{Q}(U)$. Note that Φ maps functions $\Omega(U') \subset \Lambda(U') \hookrightarrow \mathcal{V}(U')$ to functions $\Omega(U) \hookrightarrow \mathcal{Q}(U)$. Since Φ preserves the circle products, it follows that Φ maps the ideal $\mathcal{I}(U') \subset \mathcal{V}(U')$ to $\mathcal{I}(U)$ which is zero in $\mathcal{Q}(U)$. Thus we obtain a homomorphism $\mathcal{Q}(U') \to \mathcal{Q}(U)$ which we denote by $\mathcal{Q}(\varphi)$.

Now consider two diffeomorphisms $U_1 \xrightarrow{\varphi_1} U_2 \xrightarrow{\varphi_2} U_3$. It is clear that

$$(\varphi_2 \circ \varphi_1)^* : Vect(U_3) \triangleright \Omega(U_3) \to Vect(U_1) \triangleright \Omega(U_1)$$

coincides with $\varphi_1^* \circ \varphi_2^*$; this is functoriality of pull-back. However, because the definition of $\mathcal{Q}(\varphi_i)$ involves formal substitutions of non-commuting vertex operators, it is not a priori clear the same should hold for the $\mathcal{Q}(\varphi_i)$. But [29] showed (in the case when all three U_i are the same by direct calculation using generators; but the proof works here), that $\mathcal{Q}(\varphi_2 \circ \varphi_1) = \mathcal{Q}(\varphi_1) \circ \mathcal{Q}(\varphi_2)$ indeed holds; this result requires a certain anomaly cancellation. In any case, specializing this to $U \xrightarrow{\varphi} U' \xrightarrow{\varphi^{-1}} U$, we conclude that $\mathcal{Q}(\varphi)$ is an isomorphism. $\square$

Actually, there is a slight ambiguity in the choice of the isomorphism Φ above. When the classical objects $\frac{\partial}{\partial \gamma^i}, \frac{\partial}{\partial c^i}, c^i, \gamma^i, \varphi^i \in \Omega(U)$ are formally replaced by their vertex operator counterparts $\beta^i(z), b^i(z), c^i(z), \gamma^i(z), \varphi^i(z) \in \mathcal{Q}(U)$, care must be taken when deciding in which order the vertex operators should appear because they do not commute in general. This ambiguity occurs at just one place, namely the β^i. The classical transformation law for the derivation $\frac{\partial}{\partial \gamma^i}$, under coordinate transformations $\tilde{\gamma}^i = g^i(\gamma)$, $\gamma^i = f(\tilde{\gamma})$, is

$$\frac{\partial}{\partial \tilde{\gamma}^i} = \frac{\partial f^j}{\partial \tilde{\gamma}^i} \frac{\partial}{\partial \gamma^i} + \frac{\partial^2 f^k}{\partial \tilde{\gamma}^i \partial \tilde{\gamma}^j} \frac{\partial g^j}{\partial \gamma^l} c^l \frac{\partial}{\partial c^k}$$

There is no ambiguity in the second term on the right side under the formal replacement. But for the first term, we can have the vertex operators $:\frac{\partial f^j}{\partial \tilde{\gamma}^i} \beta^j :$ or $: \beta^j \frac{\partial f^j}{\partial \tilde{\gamma}^i} :$, which are not equal in general. Both choices give the right OPE for $\tilde{\gamma}^i, \tilde{\beta}^j$, but only the second choice guarantees the composition property in the preceding lemma.

Lemma 2.27. *Consider the following commutative diagram in* $(Open(\mathbf{R}^n), \hookrightarrow)$*:*

$$\begin{array}{ccc} V & \overset{\iota}{\subset} & U \\ \varphi' \downarrow & & \varphi \downarrow \\ V' & \overset{\iota'}{\subset} & U' \end{array}$$

where the vertical arrows are diffeomorphisms. This induces a commutative diagram under $\mathcal{Q}$*, i.e.*

$$\mathcal{Q}(\varphi') \circ \mathcal{Q}(\iota') = \mathcal{Q}(\iota) \circ \mathcal{Q}(\varphi).$$

Proof: The commutative diagram says that φ' is the restriction of φ to V. The $\mathcal{Q}(\iota')$, $\mathcal{Q}(\iota)$ are defined by restrictions of (arbitrary) smooth functions; these operations are clearly compatible with formal substitutions, hence with $\mathcal{Q}(\varphi)$. $\square$

Definition 2.28. *For any open embedding* $U \overset{\psi}{\hookrightarrow} U'$*, we define* $\mathcal{Q}(\psi) : \mathcal{Q}(U') \to \mathcal{Q}(U)$ *by* $\mathcal{Q}(\psi) = \mathcal{Q}(\varphi) \circ \mathcal{Q}(\iota)$ *where* $U \overset{\varphi}{\to} \psi(U) \overset{\iota}{\subset} U'$ *is the factorization of* ψ *into a diffeomorphism followed by an inclusion.*

Lemma 2.29. *The assigment* $\mathcal{Q} : (Open(\mathbf{R}^n), \hookrightarrow) \rightsquigarrow \mathcal{VA}$*, defines a sheaf of vertex algebras in the Grothendieck topology on* $\mathbf{R}^n$*.*

Proof: The proof has little to do with vertex algebras. Suppose that one has a sheaf $\mathcal{F}$ in the ordinary topology $(Open(\mathbf{R}^n), \subset)$ on $\mathbf{R}^n$, and that $\mathcal{F}$ further assigns to each diffeomorphism $U \overset{\varphi}{\to} U'$ an isomorphism $\mathcal{F}(\varphi) : \mathcal{F}(U') \to \mathcal{F}(U)$ in a way that Lemmas 2.26 and

2.27 hold for $\mathcal{F}$. Then $\mathcal{F}$ defines a sheaf in the Grothendieck topology $(Open(\mathbf{R}^n), \hookrightarrow)$. Namely, one keeps the same assignment of objects $\mathcal{F}(U)$ for open sets U, and assign to each open embedding ψ a morphism $\mathcal{F}(\psi)$ as in the preceding definition. Then one finds, by straightforward checking, that

•$\mathcal{F} : (Open(\mathbf{R}^n), \hookrightarrow) \rightsquigarrow \mathcal{VA}$ is a functor;

•if $U_i \overset{\psi_i}{\hookrightarrow} U$ is a covering, then the sequence

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is exact.

Now apply this to $\mathcal{Q}$. □

Lemma 2.30. *Given any sheaf of vertex algebras $\mathcal{F} : (Open(\mathbf{R}^n), \hookrightarrow) \rightsquigarrow \mathcal{VA}$ in the Grothendieck topology on $\mathbf{R}^n$, we can attach, to every smooth manifold M^n, a sheaf of vertex algebras $\mathcal{F}_M$ in the ordinary topology of M.*

Proof: This again has little to do with vertex algebras.

Let $\mathcal{B}$ be the set of all coordinate open subsets of M, i.e. $O \in \mathcal{B}$ iff there is a chart $\psi : O \to \mathbf{R}^n$. Let $\mathcal{C}_O$ be the set of such charts, and let $\mathcal{G}_O$ be the groupoid consisting of objects $\mathcal{F}(\psi(O))$, $\psi \in \mathcal{C}_O$, and morphisms

$$\mathcal{F}(\psi(O)) \overset{g_{\psi\varphi} := \mathcal{F}(\psi \circ \varphi^{-1})}{\longrightarrow} \mathcal{F}(\varphi(O)), \quad \psi, \varphi \in \mathcal{C}_O.$$

By definition this groupoid has just one morphism $g_{\psi\varphi}$ between any pair of objects. In particular, these morphisms satisfy the "cocycle condition" that

$$g_{\psi\varphi} g_{\rho\psi} = g_{\rho\varphi}.$$

We define the "average" of $\mathcal{G}_O$ by

$$\bar{\mathcal{G}}_O := \{(v_\varphi) \in \prod_{\psi \in \mathcal{C}_O} \mathcal{F}(\psi(O)) | v_\varphi = g_{\psi\varphi} v_\psi \ \forall \psi, \varphi\}.$$

Note that each projection $\pi_\varphi : \prod_\psi \mathcal{F}(\psi(O)) \to \mathcal{F}(\varphi(O))$, restricts to an isomorphism $\pi_\varphi : \bar{\mathcal{G}}_O \to \mathcal{F}(\varphi(O))$. Hence each tuple $(v_\varphi) \in \bar{\mathcal{G}}_O$ is determined by any one of its entries

v_φ. Conversely, for any $\varphi \in \mathcal{C}_O$ and any $x \in \mathcal{F}(\varphi(O))$, there exists a unique tuple in $\bar{\mathcal{G}}_O$, which we denote by $\bar{x}$, such that $\pi_\varphi(\bar{x}) = x$.

Since a sheaf on M is determined by its values on a collection of open sets forming a basis of M, it suffices to define $\mathcal{F}_M(O)$ for $O \in \mathcal{B}$. We define

$$\mathcal{F}_M(O) = \bar{\mathcal{G}}_O \quad O \in \mathcal{B},$$

and for $O \subset P$, we define the restriction map

$$res_{PO} : \bar{\mathcal{G}}_P \to \bar{\mathcal{G}}_O, \quad v \mapsto \overline{\mathcal{F}(\iota)\pi_\psi(v)}.$$

Here $\psi : P \to \mathbf{R}^n$ is any given chart and $\iota : \psi(O) \subset \psi(P)$ is the inclusion. It is straightforward to check that

•res_{PO} is well-defined, i.e. independent of the choice of ψ;

•$\mathcal{F}_M$ defines a sheaf on M. In other words, we have that

•if $O \subset P \subset Q \in \mathcal{B}$, then

$$res_{PO}\ res_{QP} = res_{QO};$$

•if $O_i \subset O \in \mathcal{B}$ form a covering by open sets, then the sequence

$$0 \to \mathcal{F}_M(O) \to \prod_i \mathcal{F}_M(O_i) \rightrightarrows \prod_{i,j} \mathcal{F}_M(O_i \cap O_j)$$

is exact. $\square$

Applying this to the sheaf $\mathcal{Q}$, we obtain a sheaf $\mathcal{Q}_M$ of vertex (super) algebras for every smooth manifold M. This is the chiral de Rham sheaf. Since $\Omega(U) \hookrightarrow \mathcal{Q}(U)$ by Corollary 2.24, it follows that $\mathcal{Q}_M$ contains the de Rham sheaf Ω_M. To summarize:

Theorem 2.31. *[29] For every smooth manifold M, we have a sheaf $\mathcal{Q}_M$ of vertex algebras which contains Ω_M as a subsheaf of vector spaces.*

When M is fixed and no confusion arises, we shall denote the chiral de Rham sheaf by $\mathcal{Q}$ without the subscript. It was shown further in [29] that for any M, the vertex algebra

$\mathcal{Q}(M)$ of global sections contains a Virasoro element with central charge 0 given in local coordinates by:

$$\omega_{\mathcal{Q}}(z) = \omega_{bos}(z) + \omega_{fer}(z), \quad \omega_{bos}(z) = - : b^i(z)\partial c^i(z) :, \quad \omega_{fer}(z) =: \beta^i(z)\partial\gamma^i(z) :,$$

(cf. [15]). $\mathcal{Q}(M)$ contains another vertex operator $g(z)$, which will also be useful to us. It is defined locally by

$$g(z) =: b^i(z)\partial\gamma^i(z) : .$$

When M is Calabi-Yau, $\mathcal{Q}(M)$ contains a topological vertex algebra (TVA), as in Example 2.19, where $\omega_{\mathcal{Q}}(z), g(z)$ play the roles of $L(z), G(z)$, respectively. The differential $d_{\mathcal{Q}}$ of this TVA still makes sense when M is not Calabi-Yau, and the equation $[d_{\mathcal{Q}}, g(z)] = \omega_{\mathcal{Q}}$ still holds. $d_{\mathcal{Q}}$ is given by the zeroth Fourier mode of the vertex operator

$$: \beta^i(z)c^i(z) : .$$

Moreover $(\mathcal{Q}, d_{\mathcal{Q}})$ has the structure of a complex of sheaves containing the de Rham complex of sheaves (Ω, d_{dR}). In particular $(\Omega(M), d_{dR})$ is a subcomplex of $(\mathcal{Q}(M), d_{\mathcal{Q}})$. $\mathcal{Q}(M)$ is a $\mathbf{Z}_{\geq}$-graded module over the Virasoro algebra, where the grading is given by the eigenvalues of the Fourier mode $\omega_{\mathcal{Q}}(1)$ of the Virasoro element. Moreover the eigenspace of zero eigenvalue is $\Omega(M)$. Since $[d_{\mathcal{Q}}, g(1)] = \omega_{\mathcal{Q}}(1)$, it follows that the Fourier mode $g(1)$ is a contracting homotopy for $d_{\mathcal{Q}}$ in every eigenspace of nonzero eigenvalue. This means that the chiral de Rham cohomology, i.e. the cohomology of $(\mathcal{Q}(M), d_{\mathcal{Q}})$, coincides with the classical de Rham cohomology.

3. From Vector Fields on M to Global Sections of $\mathcal{Q}_M$

Recall that to any given Lie algebra $\mathfrak{g}$, we can attach a Lie (super) algebra as follows. Let $\mathfrak{g}_{-1}$ be the adjoint module of $\mathfrak{g}$, but declared to be an odd vector space. We can then form the semi-direct product Lie algebra $\mathfrak{sg} := \mathfrak{g} \triangleright \mathfrak{g}_{-1}$. Define a linear map $d : \mathfrak{sg} \to \mathfrak{sg}$, $(\xi, \eta) \mapsto (\eta, 0)$, which is a square-zero odd derivation. The result is an example of a *differential (graded) Lie algebra* $(\mathfrak{sg}, d)$, i.e. a Lie algebra equipped with a square-zero derivation.

Assign to $\mathfrak{sg}$ the zero bilinear form, and consider the current algebra $O(\mathfrak{sg}, 0)$ defined in Example 2.13. Then the Lie algebra derivation $d : \mathfrak{sg} \to \mathfrak{sg}$ induces a vertex algebra derivation

$$\mathbf{d} : O(\mathfrak{sg}, 0) \to O(\mathfrak{sg}, 0)$$

such that $(\xi,\eta)(z) \mapsto (\eta,0)(z)$. Note that $(\mathfrak{sg}, d)$ is a $\mathbf{Z}$-graded Lie algebra where $\mathfrak{g}, \mathfrak{g}_{-1}, d$ have degrees 0,-1,1 respectively. This makes $O(\mathfrak{sg}) \stackrel{def}{=} (O(\mathfrak{sg},0), \mathbf{d})$ a degree graded differential vertex algebra where $(\xi,0)(z), (0,\xi)(z)$ have degrees $0,-1$ respectively. Note also that $O(\mathfrak{sg})$ is also weight graded where $(\xi,\eta)(z)$ has weight 1.

Definition 3.1. *An $O(\mathfrak{sg})$-algebra is a degree-weight graded differential vertex algebra $(\mathcal{A}^*,\delta)$ equipped with a homomorphism $\Phi_{\mathcal{A}} : O(\mathfrak{sg}) \to (\mathcal{A},\delta)$. We shall often denote the $O(\mathfrak{sg})$-structure on $\mathcal{A}$ simply by the map $\mathfrak{sg} \to \mathcal{A}$, $(\xi,\eta) \mapsto L_\xi(z) + \iota_\eta(z)$. An $O(\mathfrak{sg})$-algebra homomorphism is a differential vertex algebra homomorphism $f : (\mathcal{A},\delta) \to (\mathcal{A}',\delta')$ such that $f \circ \Phi_{\mathcal{A}} = \Phi_{\mathcal{A}'}$. Likewise we have the categorical notions of $O(\mathfrak{sg})$-modules and $O(\mathfrak{sg})$-module homomorphisms.*

3.1. From vector fields to $O(\mathfrak{s}\mathfrak{X})$-algebras

Consider $\mathfrak{X} = \mathfrak{X}(M)$, the Lie algebra of vector fields on M. For $X \in \mathfrak{X}$ let

$$L_X : \Omega^k(M) \to \Omega^k(M), \quad \iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$$

respectively be the Lie derivative and the interior multiplication by X. They have the familiar (super) commutators:

$$[L_X, L_Y] = L_{[X,Y]}, \quad [L_X, \iota_Y] = \iota_{[X,Y]}, \quad [\iota_X, \iota_Y] = 0.$$

Thus the map $\phi : \mathfrak{s}\mathfrak{X} = \mathfrak{X} \triangleright \mathfrak{X}_{-1} \to Der\ \Omega(M)$, $(X,Y) \mapsto L_X + \iota_Y$ defines an injective Lie algebra homomorphism. Since $\phi \circ d = d_{dR} \circ \phi$, it follows that $(\mathfrak{s}\mathfrak{X}, d) \cong (\phi(\mathfrak{s}\mathfrak{X}), d_{dR})$ as differential Lie algebras. We shall identify these two algebras.

As before, we can consider the corresponding differential vertex algebra $O(\mathfrak{s}\mathfrak{X}) = (O(\mathfrak{s}\mathfrak{X},0), \mathbf{d})$. Since the invariant bilinear form we have chosen for $\mathfrak{s}\mathfrak{X}$ is zero, the diagonal map $\mathfrak{s}\mathfrak{X} \to \mathfrak{s}\mathfrak{X} \oplus \mathfrak{s}\mathfrak{X}$ induces a Lie algebra homomorphism of the corresponding loop algebras, and ultimately a differential vertex algebra homomorphism $O(\mathfrak{s}\mathfrak{X}) \to O(\mathfrak{s}\mathfrak{X}) \otimes O(\mathfrak{s}\mathfrak{X})$. This makes the tensor product of any two $O(\mathfrak{s}\mathfrak{X})$-algebras also an $O(\mathfrak{s}\mathfrak{X})$-algebra; likewise for modules.

Let $X \in \mathfrak{X}$. Write $X = f_i \frac{\partial}{\partial \gamma^i}$ in some local coordinates $\psi : O \to \mathbf{R}^n$. Then $\iota_X = f_i \frac{\partial}{\partial c^i}$ as a derivation on $\Omega(M)$.

Lemma 3.2. *For $X \in \mathfrak{X}(M)$, there exists a global section $\iota_X(z)$ of the chiral de Rham sheaf such that on a local coordinate open set we have $\iota_X(z) =: f_i(z)b^i(z) :$.*

Proof: Note that $: f_i(z)b^i(z) := f_i(z)b^i(z)$ because the $f_i(z), b^i(z)$ commute. It is enough to show that on two overlapping coordinate open sets $O, \tilde{O} \subset M$, the two local expressions for $\iota_X(z)$ agree on $O \cap \tilde{O}$. Denote the two local expressions for X by $f_i \frac{\partial}{\partial \gamma^i}$ and $\tilde{f}_i \frac{\partial}{\partial \tilde{\gamma}^i}$ respectively. Since X and ι_X are both globally defined on M, on the overlap we have the relations $\tilde{f}_i = f_j \frac{\partial \tilde{\gamma}^i}{\partial \gamma^j}$, $\frac{\partial}{\partial \tilde{c}^i} = \frac{\partial}{\partial c^j} \frac{\partial \gamma^j}{\partial \tilde{\gamma}^i}$. This means that $f_i(z), b^i(z) \in \mathcal{Q}(O)$ and $\tilde{f}_i(z), \tilde{b}^i(z) \in \mathcal{Q}(\tilde{O})$, when restricted to $O \cap \tilde{O}$, are related by $\tilde{f}_i(z) = f_j(z) \frac{\partial \tilde{\gamma}^i}{\partial \gamma^j}(z)$, $\tilde{b}^i(z) = b^j(z) \frac{\partial \gamma^j}{\partial \tilde{\gamma}^i}(z)$. It follows that

$$\tilde{f}_i(z)\tilde{b}^i(z) = f_j(z)b^j(z).$$

Here we have used the identity (a relation coming from the ideal $\mathcal{I}(U)$) $\frac{\partial \gamma^j}{\partial \tilde{\gamma}^i}(z) \frac{\partial \tilde{\gamma}^i}{\partial \gamma^k}(z) = (\frac{\partial \gamma^j}{\partial \tilde{\gamma}^i} \frac{\partial \tilde{\gamma}^i}{\partial \gamma^k})(z) = \delta_{j,k} id$. This completes the proof. $\square$

Remark 3.3. *Even though $X = f_i \frac{\partial}{\partial \gamma^i}$ is globally defined as a vector field, the formal substitution $: f_i(z)\beta^i(z) :$ does not give a well-defined global section of $\mathcal{Q}(M)$. There are two reasons. The first is that as a derivation on $\Omega(O)$, $\frac{\partial}{\partial \gamma^i}$ does not transform like a vector. The second is that $f_i(z), \beta^i(z)$ do not commute as vertex operators in $\mathcal{Q}(O)$. Both of these make $: f_i(z)\beta^i(z) :$ transform in a complicated way and fail to be globally well-defined.*

Recall that the vertex algebra $\mathcal{Q}(M)$ of global sections of the chiral de Rham sheaf has a well-defined differential

$$d_{\mathcal{Q}} : \mathcal{Q}(M) \to \mathcal{Q}(M), \quad a(z) \mapsto (\beta^i(z)c^i(z)) \circ_0 a(z).$$

Note that the last expression is also the commutator of the zeroth Fourier mode of $\beta^i(z)c^i(z)$ with $a(z)$. Thus we obtain a global section

$$L_X(z) \stackrel{def}{=} d_{\mathcal{Q}}\ \iota_X(z)$$

in $\mathcal{Q}(M)$. In local coordinates, $L_X(z)$ is given by

$$L_X(z) =: \beta^i(z)f^i(z) : + : \frac{\partial f^j}{\partial \gamma^i}(z)\ c^i(z)\ b^j(z) : .$$

Lemma 3.4. *Both $\iota_X(z), L_X(z)$ are primary vertex operators of conformal weight 1.*

Proof: This is a local calculation. Recall that the Virasoro element $\omega_{\mathcal{Q}}(z) \in \mathcal{Q}(M)$ is characterized by the fact that locally the vertex operators $b(z), c(z), \beta(z), \gamma(z)$ are primary of conformal weights 1,0,1,0 respectively [15]. In particular, $f(z)$ is primary of conformal weight 0 for any $f \in C^\infty(M)$. That makes $\iota_X(z) =: f_i(z)b^i(z) :$ primary of conformal weight 1. Since $\omega_{\mathcal{Q}}(z)$ is $d_{\mathcal{Q}}$-exact it commutes with $d_{\mathcal{Q}}$. It follows that $L_X(z) = d_{\mathcal{Q}}\ \iota_X(z)$ is also primary of conformal weight 1. $\square$

Theorem 3.5. *The differential vertex algebra $(\mathcal{Q}(M), d_{\mathcal{Q}})$ is an $O(\mathfrak{s}\mathfrak{X})$-algebra.*

Proof: Using the local formulas for $L_X(z), \iota_Y(w) \in \mathcal{Q}(M)$, we get easily

$$L_X(z)\iota_Y(w) \sim \iota_{[X,Y]}(w)\ (z-w)^{-1}.$$

Taking commutators on both sides with the zeroth Fourier mode $D(0)$ of $D(z) = \beta^i(z)c^i(z)$, and recalling that $L_Y(w) = d_{\mathcal{Q}}\ \iota_Y(w) = [D(0), \iota_Y(w)]$, we get

$$L_X(z)L_Y(w) \sim L_{[X,Y]}(w)\ (z-w)^{-1}.$$

We also have $\iota_X(z)\iota_Y(w) \sim 0$. By the universal property of a current algebra (Example 2.13), there is a unique vertex algebra homomorphism $\Phi_{\mathcal{Q}} : O(\mathfrak{s}\mathfrak{X}, 0) \to \mathcal{Q}(M)$ such that, for any $(X,Y) \in \mathfrak{s}\mathfrak{X}$, we have $(X,Y)(z) \mapsto L_X(z) + \iota_Y(z)$. By definition, the differential for $O(\mathfrak{s}\mathfrak{X})$ is given by $\mathbf{d} : (X,Y)(z) \mapsto (Y,0)(z)$, while the differential for $\mathcal{Q}(M)$ is $d_{\mathcal{Q}} : L_X(z) + \iota_Y(z) \mapsto L_Y(z)$. This shows that $\Phi_{\mathcal{Q}}$ intertwines $\mathbf{d}$ and $d_{\mathcal{Q}}$, hence we have a differential vertex algebra homomorphism $\Phi_{\mathcal{Q}} : O(\mathfrak{s}\mathfrak{X}) \to (\mathcal{Q}(M), d_{\mathcal{Q}})$. $\square$

The formula for $L_X(z)$ and the statement that $\mathcal{Q}(M)$ is a module over the current algebra $O(\mathfrak{X}, 0)$ via $X \mapsto L_X(z)$, also appear in [30].

Remark 3.6. *Consider any given Lie subalgebra $\mathfrak{g} \subset \mathfrak{X}$. Then we have a differential Lie subalgebra $\mathfrak{s}\mathfrak{g} \subset \mathfrak{s}\mathfrak{X}$. By Example 2.13, this induces an inclusion of differential vertex algebras $O(\mathfrak{s}\mathfrak{g}) \subset O(\mathfrak{s}\mathfrak{X})$. This makes every $O(\mathfrak{s}\mathfrak{X})$-algebra canonically an $O(\mathfrak{s}\mathfrak{g})$-algebra, and likewise for modules.*

3.2. From group actions to global sections

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and M a smooth G-manifold, i.e. M is equipped with an effective G-action. Then the group homomorphism $G \to Diff(M)$ induces an injective Lie algebra homomorphism given by

$$\mathfrak{g} \to \mathfrak{X}(M), \quad \xi \mapsto X_\xi, \qquad X_\xi(x) = \frac{d}{dt}e^{t\xi} \cdot x|_{t=0}.$$

Thus $\mathfrak{g}$ can be viewed as a Lie subalgebra of $\mathfrak{X} = \mathfrak{X}(M)$. We shall denote L_{X_ξ} simply by L_ξ, and likewise for ι_ξ. Now it follows immediately from the preceding remark that we have

Theorem 3.7. *Let G be Lie group with Lie algebra $\mathfrak{g}$, and M be a G-manifold. Then $(\mathcal{Q}(M), d_{\mathcal{Q}})$ is canonically an $O(\mathfrak{sg})$-algebra.*

In particular, each $\xi \in \mathfrak{g}$ gives rise to two vertex operators $L_\xi(z), \iota_\xi(z) \in \mathcal{Q}(M)$.

4. Classical Equivariant Cohomology Theory

In this section we summarize the theory of classical equivariant cohomology from the de Rham theoretic point of view. All material in this section is taken from the book of Guillemin-Sternberg [18]. This summary will be used as a guide for formulating the vertex algebra analogue of the theory.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and M a topological space equipped with an action of G by homeomorphisms. The equivariant cohomology of M, denoted by $H^*_G(M)$, is defined to be the ordinary cohomology of the quotient $(M \times \mathcal{E})/G$ where $\mathcal{E}$ is any contractible topological space on which G acts freely. It is well-known that this is independent of the choice of $\mathcal{E}$. Furthermore, if the G action on M is free then $H^*_G(M) = H^*(M/G)$. If the action is not free, the quotient space M/G may be pathological, and $H^*_G(M)$ is the appropriate substitute for $H^*(M/G)$. From now on, we let M be a finite-dimensional manifold on which G acts by diffeomorphisms. Then there is a de Rham model for $H^*_G(M)$ together with an equivariant version of the de Rham theorem which asserts the equivalence between this model and the topological definition.

The G-action on M induces a group homomorphism $\rho : G \to Aut\ \Omega(M)$ and a differential Lie algebra homomorphism $\mathfrak{sg} \to Der\ \Omega(M)$, $(\xi, \eta) \mapsto L_\xi + \iota_\eta$, making $\Omega(M)$ a $(\mathfrak{sg}, d)$-module such that the following identities hold

$$\begin{aligned}
[d, \iota_\xi] &= L_\xi \\
\frac{d}{dt}\rho(e^{t\xi})|_{t=0} &= L_\xi \\
\rho(a)L_\xi\rho(a^{-1}) &= L_{Ad_a(\xi)} \\
\rho(a)\iota_\xi\rho(a^{-1}) &= \iota_{Ad_a(\xi)} \\
\rho(a)d\rho(a^{-1}) &= d.
\end{aligned} \tag{4.1}$$

Here $d = d_{dR}$. Note that $\Omega(M)$ is $\mathbf{Z}$-graded by the form degree and that operators L_ξ, ι_ξ, d have degrees $0, -1, 1$ respectively. In the terminology of [18], $(\Omega(M), d_{dR})$ is an example of

Definition 4.1. *A G^*-algebra is a $\mathbf{Z}$-graded differential commutative superalgebra (A, d_A) on which G acts by automorphisms $\rho : G \to Aut(A)$ and $\mathfrak{sg}$ acts by (super) derivations $\mathfrak{sg} \to Der\ A$, $(\xi, \eta) \mapsto L_\xi + \iota_\eta$, such that (4.1) hold with $d = d_A$. A G^*-algebra morphism is an algebra homomorphism which preserves the above structures in an obvious way.*

To make sense of (4.1) in this generality, one can either restrict the G-action to G-finite vectors, or give an appropriate notion of differentiation along a curve in A.

We often denote (A, d_A) simply by A.

Now suppose M is a principal G-bundle. Then the vector fields X_ξ, $\xi \in \mathfrak{g}$, generate a G-invariant trivial subbundle $V \subset TM$. Choose a G-invariant splitting $TM = V \oplus H$; we get $T^*M = V^* \oplus H^*$. Corresponding to the choice of H is a canonical map

$$\begin{aligned} &\mathfrak{g}^* \hookrightarrow \Omega^1(M), \quad \xi' \mapsto \theta^{\xi'} \\ &\theta^{\xi'}|H = 0, \quad \iota_\xi \theta^{\xi'} = \theta^{\xi'}(X_\xi) = \langle \xi', \xi \rangle. \end{aligned} \tag{4.2}$$

Using the G-invariance of the splitting $T^*M = V^* \oplus H^*$, it is easy to see that:

$$L_\xi \theta^{\xi'} = \theta^{ad^*(\xi)\xi'}.$$

The $\theta^{\xi'}$ are called connection one-forms, and the two-forms $\mu^{\xi'} = d\theta^{\xi'} + \frac{1}{2}\theta^{ad^*(\xi_i)\xi'}\theta^{\xi_i'}$ are called the curvature forms of the $\theta^{\xi'}$, where $ad^* : \mathfrak{g} \to End\ \mathfrak{g}^*$ is the coadjoint module.

Definition 4.2. *A G^*-algebra A is said to be of type C if there is an inclusion $\mathfrak{g}^* \hookrightarrow A_1$, $\xi' \mapsto \theta^{\xi'}$ such that $\iota_\xi \theta^{\xi'} = \langle \xi', \xi \rangle$ and the image $C \subset A_1$ is G-invariant.*

Every one-form $\omega \in \Omega^1(M)$ such that $\iota_\xi \omega = 0$ for all ξ can be thought of as a section of H^*, hence it is called a horizontal one-form. Likewise, any form $\omega \in \Omega(M)$ satisfying the same condition is called horizontal.

Definition 4.3. *Let A be a G^*-algebra. An element $\omega \in A$ is said to be horizontal if $\iota_\xi \omega = 0$ for all $\xi \in \mathfrak{g}$. It is called basic if it is horizontal and $\rho(a)\omega = \omega$ for all $a \in G$. We* *denote by A_{hor} and A_{bas}, respectively, the subalgebras of horizontal and basic elements in A.*

If G is connected then the condition that $\rho(a)\omega = \omega$ can be replaced by the equivalent condition that $L_\xi \omega = 0$. In particular, A_{bas} is the $\mathfrak{sg}$-invariant subalgebra of A in this case.

It is easy to see that $d(A_{bas}) \subset A_{bas}$. Thus A_{bas} is a subcomplex of A and its cohomology $H^*_{bas}(A)$ is well-defined. Moreover, if $\phi : A \to B$ is a morphism of G^*-algebras, then $\phi(A_{bas}) \subset B_{bas}$, and so ϕ induces a map

$$\phi_{bas} : H^*_{bas}(A) \to H^*_{bas}(B).$$

Guillemin-Sternberg define the equivariant cohomology of a G^*-algebra A, in a way that is analogous to the topological definition.

Definition 4.4. *Let E be any acyclic G^*-algebra of type C. For any G^*-algebra A, its equivariant cohomology ring $H^*_G(A)$ is defined to be $H^*_{bas}(A \otimes E)$. The usual rule of graded tensor product of two superalgebras applies here.*

Thus in the category of G^*-algebras, an acyclic G^*-algebra E plays the role of a contractible space $\mathcal{E}$ with free G-action. This definition is shown to be independent of the choice of E. Moreover, the equivariant de Rham theorem asserts that for any G-manifold M,

$$H^*_G(M) = H^*_G(\Omega(M)),$$

where the right side is defined by taking $A = \Omega(M)$.

4.1. Weil model for $H^*_G(A)$

There is a natural choice for the acyclic G^*-algebra of type C, namely, the Koszul algebra

$$W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*).$$

This algebra is $\mathbf{Z}_{\geq}$-graded where the generators $c^{\xi'} = \xi' \otimes 1$, $\xi' \in \mathfrak{g}^*$, of the exterior algebra $\Lambda(\mathfrak{g}^*)$, and the generators $z^{\xi'} = 1 \otimes \xi'$ of the symmetric algebra $S(\mathfrak{g}^*)$, are assigned degrees 1 and 2 respectively. The G action $\rho : G \to Aut\ W(\mathfrak{g})$ is the action induced by the coadjoint action on $\mathfrak{g}^*$, and the $(\mathfrak{sg}, d)$-structure on $W(\mathfrak{g})$ is defined on generators by the formulas

$$L_\xi c^{\xi'} = c^{ad^*(\xi)\xi'}, \quad L_\xi z^{\xi'} = z^{ad^*(\xi)\xi'}$$
$$\iota_\xi c^{\xi'} = \langle \xi', \xi \rangle, \quad \iota_\xi z^{\xi'} = c^{ad^*(\xi)\xi'}$$
$$d_W c^{\xi'} = z^{\xi'}, \quad d_W z^{\xi'} = 0.$$

We have an inclusion $\mathfrak{g}^* \hookrightarrow W(\mathfrak{g})$, $\xi' \mapsto c^{\xi'}$. This defines a G^*-algebra structure of type C on $W(\mathfrak{g})$. This G^*-algebra is acyclic because d_W has a contracting homotopy Q given by $Qz^{\xi'} = c^{\xi'}$, $Qc^{\xi'} = 0$. In fact, we have

$$[d_W, Q] = c^{\xi_i'} \frac{\partial}{\partial c^{\xi_i'}} + z^{\xi_i'} \frac{\partial}{\partial z^{\xi_i'}}$$

operating on the polynomial super algebra $W(\mathfrak{g}) = \mathbf{C}[c^{\xi_i'}, z^{\xi_i'} | i = 1, .., dim\ \mathfrak{g}]$, clearly diagonalizably. This implies that the only d_W-cohomology occurs in degree 0 and is one-dimensional. Here the ξ_i' form the dual basis of a given basis ξ_i of $\mathfrak{g}$. Hence $H^*_{bas}(A \otimes W(\mathfrak{g}))$ provides a model for the equivariant cohomology of A, called the Weil model of $H^*_G(A)$.

There is very useful (and crucial for us) change of variables we can perform on $W(\mathfrak{g})$. Note that the $c^{\xi'}$ play the role of connection one-forms, and the corresponding elements playing the role of the curvature two-forms are

$$\gamma^{\xi'} = z^{\xi'} + \frac{1}{2} c^{ad^*(\xi_i)\xi'} c^{\xi_i'}.$$

Note that they are homogeneous of degree 2 in $W(\mathfrak{g})$. We can view $W(\mathfrak{g})$ as an algebra generated by the $c^{\xi'}, \gamma^{\xi'}$. The defining relations of the G^*-algebra structure now become

$$\begin{aligned} L_\xi c^{\xi'} &= c^{ad^*(\xi)\xi'}, \quad L_\xi \gamma^{\xi'} = \gamma^{ad^*(\xi)\xi'} \\ \iota_\xi c^{\xi'} &= \langle \xi', \xi \rangle, \quad \iota_\xi \gamma^{\xi'} = 0 \\ d_W c^{\xi'} &= -\frac{1}{2} c^{ad^*(\xi_i)\xi'} c^{\xi_i'} + \gamma^{\xi'}, \quad d_W \gamma^{\xi'} = \gamma^{ad^*(\xi_i)\xi'} c^{\xi_i'} \end{aligned} \tag{4.3}$$

The differential d_W can be written as a sum $d_{CE} + d_K$ where d_{CE} is the Chevalley-Eilenberg differential of the Lie algebra cohomology complex of $\mathfrak{g}$ with coefficients in the module $S(\mathfrak{g}^*)$, and d_K has the shape of a Koszul differential:

$$d_{CE} = -c^{\xi_i'} \gamma^{\xi_j'} \beta^{[\xi_i, \xi_j]} - \frac{1}{2} c^{\xi_i'} c^{\xi_j'} b^{[\xi_i, \xi_j]}, \quad d_K = \gamma^{\xi_i'} b^{\xi_i}. \tag{4.4}$$

where the b^ξ is an odd derivation on $W(\mathfrak{g})$ defined by $b^\xi c^{\xi'} = \langle \xi', \xi \rangle$, $b^\xi \gamma^{\xi'} = 0$, and the β^ξ is an even derivation on $W(\mathfrak{g})$ defined by $\beta^\xi c^{\xi'} = 0$, $\beta^\xi \gamma^{\xi'} = \langle \xi', \xi \rangle$. The contracting homotopy Q of d_W in the new variables is given by $Qc^{\xi'} = 0$, $Q\gamma^{\xi'} = c^{\xi'}$.

The $(\mathfrak{sg}, d)$-module structure on $W(\mathfrak{g})$ can be given in a more instructive way by rewriting (4.3)-(4.4) as follows. Introduce the Clifford-Weyl algebra

$$Clifford(\mathfrak{g}) \otimes Weyl(\mathfrak{g})$$

to be the associative $\mathbf{C}$-superalgebra with odd generators $c^{\xi'}, b^{\xi}$ and even generators $\gamma^{\xi'}, \beta^{\xi}$, linear in $\xi' \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$, subject to the commutation relations

$$[b^{\xi}, c^{\xi'}] = \langle \xi', \xi \rangle = [\beta^{\xi}, \gamma^{\xi'}].$$

Note that $d_{CE} + d_K$ given by the formulas (4.4) can be thought of as an element in this algebra. Moreover $W(\mathfrak{g})$ becomes a module over this algebra where the $c^{\xi'}, \gamma^{\xi'}$ act by left multiplications, and the b^{ξ}, β^{ξ} act by derivations, as defined above. There is a canonical Lie algebra homomorphism (with commutator as the Lie bracket in the target)

$$(\mathfrak{sg}, d) \hookrightarrow Clifford(\mathfrak{g}) \otimes Weyl(\mathfrak{g}) \tag{4.5}$$

defined by

$$(\xi, \eta) \mapsto \Theta^{\xi}_W + b^{\eta}, \qquad d \mapsto d_{CE} + d_K$$

$$\Theta^{\xi}_W := \Theta^{\xi}_{\Lambda} + \Theta^{\xi}_S, \qquad \Theta^{\xi}_{\Lambda} = -c^{\xi'_i} b^{[\xi,\xi_i]}, \quad \Theta^{\xi}_S = -\gamma^{\xi'_i} \beta^{[\xi,\xi_i]}.$$

Then, here is the main observation: the $(\mathfrak{sg}, d)$-module structure $(\mathfrak{sg}, d) \to End\ W(\mathfrak{g})$ factors through the Clifford-Weyl algebra via the map (4.5). In other words, the operators L_{ξ}, ι_{ξ} defining the G^*-algebra structure on $W(\mathfrak{g})$ can be explicitly represented by Θ^{ξ}_W, b^{ξ}, regarded as operators on $W(\mathfrak{g})$. The vertex algebra analogue of this structure will be crucial later.

From (4.3), we find that $W(\mathfrak{g})_{bas} = S(\mathfrak{g}^*)^G$, the space of G-invariants in $S(\mathfrak{g}^*)$. From (4.4), we find that $d_W = 0$ on $W(\mathfrak{g})_{bas}$. It follows that

$$H^*_{bas}(W(g)) = H^*_G(\mathbf{C}) = S(\mathfrak{g}^*)^G.$$

Since $W(\mathfrak{g})$ is freely generated as an algebra by the variables $c^{\xi'_i}$ and $d_W c^{\xi'_i}$, $W(\mathfrak{g})$ is easily seen to be an initial object in the category of G^*-algebras of type C.

Theorem 4.5. *Let A be any G^*-algebra of type C. Then there exists a G^*-algebra morphism $\rho : W(\mathfrak{g}) \to A$. Furthermore, any two such morphisms are chain homotopic and hence induce the same map from $S(\mathfrak{g}^*)^G \to H^*_{bas}(A)$.*

In particular, the notion of a G^*-algebra of type C is equivalent to the notion of a G^*-algebra A which admits a G^*-algebra morphism $\rho : W(\mathfrak{g}) \to A$. From now on, we will refer to such algebras as $W(\mathfrak{g})$-algebras (not to be confused with the term W^*-algebra, which has a different meaning in [18]).

The preceding theorem shows that associated to any $W(\mathfrak{g})$-algebra B is a canonical map $\kappa_G : S(\mathfrak{g}^*)^G \to H^*_{bas}(B)$. Since for any G^*-algebra A, $A \otimes W(\mathfrak{g})$ is a $W(\mathfrak{g})$-algebra, we have

$$\kappa_G : S(g^*)^G \to H^*_{bas}(A \otimes W(\mathfrak{g})) \cong H^*_G(A).$$

This is called the Chern-Weil map. Consider the case where $A = \Omega(M)$ and $M = pt$. Using the Weil model, we see that

$$H^*_G(\mathbf{C}) = H^*_{bas}(\Omega(pt) \otimes W(\mathfrak{g})) = H^*_{bas}(W(\mathfrak{g})) = S(\mathfrak{g}^*)^G.$$

Topologically, the map $\kappa_G : S(\mathfrak{g}^*)^G \to H^*_G(M)$ is induced by $M \to pt$.

4.2. Cartan model for $H^*_G(A)$

Let A be a $W(\mathfrak{g})$-algebra and B a G^*-algebra. Define

$$\phi = c^{\xi'_i} \otimes \iota_{\xi_i} \in End(A \otimes B).$$

Since ϕ is a derivation and $\phi^{n+1} = 0$, $n = dim\, G$, it follows that $\Phi = exp(\phi)$ is a well-defined automorphism of the commutative (super) algebra $A \otimes B$, known as the Mathai-Quillen isomorphism.

Theorem 4.6. *The Mathai-Quillen isomorphism satisfies*

$$\begin{aligned} \Phi(L_\xi \otimes 1 + 1 \otimes L_\xi)\Phi^{-1} &= L_\xi \otimes 1 + 1 \otimes L_\xi \\ \Phi(\iota_\xi \otimes 1 + 1 \otimes \iota_\xi)\Phi^{-1} &= \iota_\xi \otimes 1 \\ \Phi d \Phi^{-1} &= d - \gamma^{\xi'_i} \otimes \iota_{\xi_i} + c^{\xi'_i} \otimes L_{\xi_i} \end{aligned}$$

where $d = d_A \otimes 1 + 1 \otimes d_B$.

The second relation shows that $\Phi((A \otimes B)_{hor}) = A_{hor} \otimes B$. Let's specialize to $A = W(\mathfrak{g})$. Since $d_W | W(\mathfrak{g})_{hor} = c^{\xi'_i} L_{\xi_i}$ it follows that

$$\Phi d \Phi^{-1} | W(\mathfrak{g})_{hor} \otimes B = (c^{\xi'_i} \otimes 1)(L_{\xi_i} \otimes 1 + 1 \otimes L_{\xi_i}) + 1 \otimes d_B - \gamma^{\xi'_i} \otimes \iota_{\xi_i}.$$

Since Φ is G-equivariant, we have $\Phi((W(\mathfrak{g}) \otimes B)_{bas}) = (S(\mathfrak{g}^*) \otimes B)^G =: C_G(B)$. On $C_G(B)$, the operator $L_a \otimes 1 + 1 \otimes L_a$ is zero, so

$$\Phi d \Phi^{-1} | C_G(B) = 1 \otimes d_B - \gamma^{\xi'_i} \otimes \iota_{\xi_i} =: d_G.$$

In particular, $H^*_G(B) \cong H^*(C_G(B), d_G)$. The right side is called the Cartan model for $H^*_G(B)$.

5. Chiral Equivariant Cohomology Theory

For simplicity, we shall assume throughout that G is a connected compact Lie group. Under this assumption, the appropriate analogue of a G^*-algebra is the notion of an $O(\mathfrak{sg})$-algebra given by Definition 3.1. Our theory can be easily modified to allow disconnected G by further requiring that an $O(\mathfrak{sg})$-algebra comes equipped with a compatible G-action, as in the classical setting.

Using the classical Weil model of $H_G^*(A)$ as a guide, we will define the chiral equivariant cohomology $\mathbf{H}_G^*(\mathcal{A})$ of an arbitrary $O(\mathfrak{sg})$-algebra $\mathcal{A}$. We have seen that the chiral de Rham complex $\mathcal{Q}(M)$ of a G-manifold M is an example of an $O(\mathfrak{sg})$-algebra. For $\mathcal{A} = \mathcal{Q}(M)$, the chiral equivariant cohomology of $\mathbf{H}_G^*(\mathcal{Q}(M))$ is a vertex algebra analogue of the classical equivariant cohomology of the G^*-algebra $\Omega(M)$. We will see that there is a canonical inclusion $H_G^*(M) \hookrightarrow \mathbf{H}_G^*(\mathcal{Q}(M))$.

Recall that an $O(\mathfrak{sg})$-algebra is a $\mathbf{Z}$-graded differential vertex algebra $(\mathcal{A}^*, d_{\mathcal{A}})$ equipped with a differential vertex algebra homomorphism $\Phi_{\mathcal{A}} : O(\mathfrak{sg}) = (O(\mathfrak{sg}, 0), \mathbf{d}) \to (\mathcal{A}, d_{\mathcal{A}})$, $(\xi, \eta)(z) \mapsto L_\xi(z) + \iota_\eta(z)$.

Definition 5.1. *Let $\mathcal{I} \subset O(\mathfrak{sg}, 0)$ be the vertex subalgebra generated by the odd currents $(0, \xi)(z)$, $\xi \in \mathfrak{g}$. Let $\mathcal{A}$ be a given $O(\mathfrak{sg})$-algebra. We define the horizontal and basic subalgebras of $\mathcal{A}$ to be respectively*

$$\mathcal{A}_{hor} = Com(\Phi_{\mathcal{A}}\mathcal{I}, \mathcal{A}), \qquad \mathcal{A}_{bas} = Com(\Phi_{\mathcal{A}}O(\mathfrak{sg}, 0), \mathcal{A}).$$

Thus $\mathcal{A}_{hor}$ consists of $a(z) \in \mathcal{A}$ which strictly commute with the elements $\iota_\xi(z) \in \mathcal{A}$, and $\mathcal{A}_{bas}$ consists of $a(z) \in \mathcal{A}_{hor}$ which strictly commute with the elements $L_\xi(z) \in \mathcal{A}$.

Since $d_{\mathcal{A}}\ \iota_\xi(z) = L_\xi(z)$, and $d_{\mathcal{A}}$ is a square-zero derivation of all the circle products, it follows that $(\mathcal{A}_{bas}, d_{\mathcal{A}})$ is vertex algebra with a compatible structure of a cochain complex. Its cohomology $\mathbf{H}_{bas}^*(\mathcal{A})$ is therefore a vertex algebra, which we will call the chiral basic cohomology of $\mathcal{A}$. An $O(\mathfrak{sg})$-algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ sends $\mathcal{A}_{bas}$ to $\mathcal{B}_{bas}$, so it induces a vertex algebra homomorphism

$$\phi_{bas} : \mathbf{H}_{bas}^*(\mathcal{A}) \to \mathbf{H}_{bas}^*(\mathcal{B}).$$

Here is a small but important surprise: to get an induced homomorphism on basic cohomology, one needs less than an $O(\mathfrak{sg})$-algebra homomorphism. That is because the notion of basic subalgebra uses only half the $O(\mathfrak{sg})$-algebra structures.

Definition 5.2. *Let $\mathcal{A}, \mathcal{B}$ be $O(\mathfrak{sg})$-algebras. A differential vertex algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is said to be basic if $\phi(\mathcal{A}_{bas}) \subset \mathcal{B}_{bas}$. In particular, a basic homomorphism induces a vertex algebra homomorphism ϕ_{bas} on basic cohomology.*

Example 5.3.

Let $\mathcal{A}, \mathcal{B}$ be $O(\mathfrak{sg})$-algebras and consider the map

$$\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}, \quad a \mapsto a \otimes 1.$$

It is obviously a vertex algebra homomorphism, and it respects the differentials because $(1 \otimes d_{\mathcal{B}})(a \otimes 1) = 0$. Now note that $\mathcal{A}_{bas} \otimes 1 \subset (\mathcal{A} \otimes \mathcal{B})_{bas}$, again because $1 \otimes L_\xi(n)(a \otimes 1) = 1 \otimes \iota_\xi(n)(a \otimes 1) = 0$ for $n \geq 0$. Thus ϕ is a basic homomorphism. But ϕ will not be an $O(\mathfrak{sg})$-algebra homomorphism unless $O(\mathfrak{sg})$ acts trivially on $\mathcal{B}$, in which case $1 \otimes L_\xi(z) = 1 \otimes \iota_\xi(z) = 0$.

5.1. Semi-infinite Weil algebra

We saw that a crucial ingredient in the Weil model of the classical equivariant cohomology is the Koszul algebra $W(\mathfrak{g})$, and that via the Clifford-Weyl algebra, one can write down the G^*-algebra structure on $W(\mathfrak{g})$ very explicitly. It turns out that in the vertex algebra setting, there is a natural algebra that unifies the Koszul and the Clifford-Weyl algebras into a single object. This is the semi-infinite Weil algebra.

$$\mathcal{W} = \mathcal{W}(\mathfrak{g}) = \mathcal{E}(\mathfrak{g}) \otimes \mathcal{S}(\mathfrak{g})$$

which we introduced in Example 2.15.

If we declare that the vertex operators $b^\xi, c^{\xi'}, \beta^\xi, \gamma^{\xi'}$, have the respective degrees $-1, 1, -1, 1$, then this defines a $\mathbf{Z}$-valued bigrading

$$\mathcal{W} = \oplus_{p,q} \mathcal{W}_{p,q}$$

where $\mathcal{W}_{p,q} = \mathcal{E}^p \otimes \mathcal{S}^q$ is the degree (p, q) subspace. This bigrading turns out to come from the following two vertex operators in $\mathcal{E}(\mathfrak{g}), \mathcal{S}(\mathfrak{g})$:

$$j_{bc}(z) = -: b^{\xi_i}(z) c^{\xi'_i}(z) :, \quad j_{\beta\gamma}(z) =: \beta^{\xi_i}(z) \gamma^{\xi'_i}(z) : .$$

Their zeroth Fourier modes are diagonalizable operators on $\mathcal{E}(\mathfrak{g}), \mathcal{S}(\mathfrak{g})$ respectively with integer eigenvalues. The eigenspaces $\mathcal{E}^p, \mathcal{S}^q$ are also called the subspaces of bc-number p and $\beta\gamma$-number q. We put

$$\mathcal{W}^n = \oplus_{n=p+2q}\mathcal{W}_{p,q}.$$

There is yet another bigrading on $\mathcal{W}$ which is compatible with the one above. If we declare that the vertex operators $\beta^\xi, \gamma^{\xi'}, b^\xi, c^{\xi'}$, have the respective weights 1,0,1,0, then this defines a $\mathbf{Z}_{\geq}$-valued bigrading

$$\mathcal{W} = \oplus_{m,n\geq 0}\mathcal{W}[m,n]$$

where $\mathcal{W}[m,n] = \mathcal{E}[m] \otimes \mathcal{S}[n]$ is the weight (m,n) subspace. This bigrading turns out to come from Virasoro elements in the vertex algebras $\mathcal{E}(\mathfrak{g}), \mathcal{S}(\mathfrak{g})$. Put

$$\omega_{\mathcal{W}}(z) = \omega_{\mathcal{E}}(z) + \omega_{\mathcal{S}}(z), \quad \omega_{\mathcal{E}}(z) = - : b^{\xi_i}(z)\partial c^{\xi_i'}(z) :, \quad \omega_{\mathcal{S}}(z) =: \beta^{\xi_i}(z)\partial\gamma^{\xi_i'}(z) : .$$

An OPE calculation by Wick's theorem yields

Lemma 5.4. *The vertex operators $\omega_{\mathcal{E}}(z), \omega_{\mathcal{S}}(z)$ are Virasoro elements of central charges $\mp 2dim\ \mathfrak{g}$ in the respective vertex algebras $\mathcal{E}(\mathfrak{g}), \mathcal{S}(\mathfrak{g})$. Moreover, $\omega_{\mathcal{E}}(z)$ is the unique Virasoro element such that $b^\xi(z), c^{\xi'}(z)$ are primary of conformal weight 1,0 respectively. Likewise $\omega_{\mathcal{S}}(z)$ has a similar characterization in $\mathcal{S}(\mathfrak{g})$. The $\mathcal{E}[m], \mathcal{S}[n]$ are the respective eigenspaces of $\omega_{\mathcal{E}}(1), \omega_{\mathcal{S}}(1)$, of eigenvalues m, n. Moreover, $\omega_{\mathcal{E}}(0)$, $\omega_{\mathcal{S}}(0)$, act respectively on $\mathcal{E}(\mathfrak{g}), \mathcal{S}(\mathfrak{g})$, as the derivation ∂.*

Note that the subspace $\mathcal{E}[0]$ consists of the vertex operators which are polynomial in the (anti-commuting) vertex operators $c^{\xi'}(z)$, and is canonically isomorphic to the classical exterior algebra $\Lambda(\mathfrak{g}^*)$. Likewise $\mathcal{S}[0]$ is canonically isomorphic to the classical symmetric algebra $S(\mathfrak{g}^*)$. It follows that $\mathcal{W}[0,0]$ is nothing but a copy of the classical Koszul algebra $W(\mathfrak{g})$.

Define the vertex operators (suppressing the variable z):

$$\Theta^\xi_{\mathcal{W}} = \Theta^\xi_{\mathcal{E}} + \Theta^\xi_{\mathcal{S}}, \quad \Theta^\xi_{\mathcal{E}} =: b^{[\xi,\xi_i]}c^{\xi_i'} :, \quad \Theta^\xi_{\mathcal{S}} = - : \beta^{[\xi,\xi_i]}\gamma^{\xi_i'} : . \tag{5.1}$$

$$D = J + K, \quad J = - : c^{\xi_i'}\gamma^{\xi_j'}\beta^{[\xi_i,\xi_j]} : -\frac{1}{2} : c^{\xi_i'}c^{\xi_j'}b^{[\xi_i,\xi_j]} :, \quad K =: \gamma^{\xi_i'}b^{\xi_i} : \tag{5.2}$$

The Fourier mode $J(0)$ is called the semi-infinite differential. The next four lemmas follows easily from direct computations by Wick's theorem.

Lemma 5.5. $J(0)^2 = K(0)^2 = D(0)^2 = [K(0), J(0)] = 0$.

Corollary 5.6. *The complex $(\mathcal{W}^*, D(0))$ has the structure of a double complex $(\mathcal{W}^{*,*}, J(0), K(0))$ where $\mathcal{W}^{p,q}$ is defined to be $\mathcal{W}_{p-q,q}$. Thus we have*

$$D(0) : \mathcal{W}^n \to \mathcal{W}^{n+1}, \quad J(0) : \mathcal{W}^{p,q} \to \mathcal{W}^{p+1,q}, \quad K(0) : \mathcal{W}^{p,q} \to \mathcal{W}^{p,q+1}.$$

Lemma 5.7. $D(0) b^\xi(z) = \Theta^\xi_{\mathcal{W}}(z)$.

Lemma 5.8. *The vertex operators $\Theta^\xi_{\mathcal{E}}(z), \Theta^\xi_{\mathcal{S}}(z)$ are characterized in their respective algebras $\mathcal{E}(\mathfrak{g}), \mathcal{S}(\mathfrak{g})$ by the properties that they are the only weight one elements such that*

$$\Theta^\xi_{\mathcal{E}}(z) b^\eta(w) \sim b^{[\xi,\eta]}(w)\ (z-w)^{-1}, \quad \Theta^\xi_{\mathcal{E}}(z) c^{\eta'}(w) \sim c^{ad^*(\xi)\eta'}(w)\ (z-w)^{-1}.$$
$$\Theta^\xi_{\mathcal{S}}(z) \beta^\eta(w) \sim \beta^{[\xi,\eta]}(w)\ (z-w)^{-1}, \quad \Theta^\xi_{\mathcal{S}}(z) \gamma^{\eta'}(w) \sim \gamma^{ad^*(\xi)\eta'}(w)\ (z-w)^{-1}.$$

Lemma 5.9. *The $\Theta^\xi_{\mathcal{E}}$ are primary of conformal weight 1 with respect to $\omega_{\mathcal{E}}$. Likewise for the $\Theta^\xi_{\mathcal{S}}$ with respect to $\omega_{\mathcal{S}}$.*

Lemma 5.10. *There is a vertex algebra homomorphism $O(\mathfrak{g}, \kappa) \to \mathcal{E}(\mathfrak{g})$ such that $\xi(z) \mapsto \Theta^\xi_{\mathcal{E}}(z)$. Likewise we have $O(\mathfrak{g}, -\kappa) \to \mathcal{S}(\mathfrak{g})$. Here $\kappa(\xi, \eta) = Tr(ad(\xi) ad(\eta))$, is the Killing form of $\mathfrak{g}$.*

Proof: We have

$$\Theta^\xi_{\mathcal{E}}(z) \Theta^\eta_{\mathcal{E}}(w) \sim \kappa(\xi, \eta)\ (z-w)^{-2} + \Theta^{[\xi,\eta]}_{\mathcal{E}}(w)\ (z-w)^{-1}.$$

By the universal property of $O(\mathfrak{g}, \kappa)$ given in Example 2.13, we get the first desired homomorphism. The case for $\mathcal{S}(\mathfrak{g})$ is analogous. $\square$

Combining Lemmas 5.5, 5.7, and 5.10, we get

Theorem 5.11. *$O(\mathfrak{sg}) \to \mathcal{W}(\mathfrak{g})$, $(\xi, \eta)(z) \mapsto \Theta^\xi_{\mathcal{W}}(z) + b^\eta(z)$, with $\mathbf{d} \mapsto D(0)$, defines an $O(\mathfrak{sg})$-algebra structure on $\mathcal{W}(\mathfrak{g})$.*

We also have the vertex algebra analogues of the relations (4.3).

Lemma 5.12.

$$\Theta^{\xi}_{\mathcal{W}}(z)c^{\xi'}(w) \sim c^{ad^*(\xi)\xi'}(w)(z-w)^{-1}, \quad \Theta^{\xi}_{\mathcal{W}}(z)\gamma^{\xi'}(w) \sim \gamma^{ad^*(\xi)\xi'}(w)(z-w)^{-1}$$
$$b^{\xi}(z)c^{\xi'}(w) \sim \langle \xi', \xi\rangle (z-w)^{-1}, \quad b^{\xi}(z)\gamma^{\xi'}(w) \sim 0$$
$$D(0)c^{\xi'} = -\frac{1}{2} : c^{ad^*(\xi_i)\xi'} c^{\xi'_i} : + \gamma^{\xi'}, \quad D(0)\gamma^{\xi'} =: \gamma^{ad^*(\xi_i)\xi'} c^{\xi'_i} :$$

Lemma 5.13. *[10][1]* $(\mathcal{W}^*, D(0))$ *is acyclic.*

Proof: Put $h =: \beta^{\xi_i}\partial c^{\xi'_i}:$. Then we find that $J(0)h = 0, \quad K(0)h = \omega_{\mathcal{W}}$. It follows that, $D(0)h(z) = \omega_{\mathcal{W}}(z)$, implying that $[D(0), h(1)] = \omega_{\mathcal{W}}(1)$. Since $\omega_{\mathcal{W}}(1)$ is diagonalizable with nonnegative eigenvalues, it follows that the cohomology of $(\mathcal{W}^*, D(0))$ is the same as the cohomology of the subcomplex $(\mathcal{W}^*[0,0], D(0))$. Recall that $\mathcal{W}^*[0,0]$ is canonically isomorphic to the classical Weil algebra W. From the formulas for the vertex operators J, K, we see that $J(0), K(0)$ restricted to $\mathcal{W}[0,0]$ coincide with their classical counterparts d_{CE}, d_K respectively under the isomorphism. Hence d_W coincides with $D(0)$ restricted to $\mathcal{W}[0,0]$. Thus $(\mathcal{W}[0,0], D(0))$ and (W, d_W) are isomorphic as complexes. But the latter is acyclic. □

Lemma 5.14. $J =: (\Theta^{\xi_i}_{\mathcal{S}} + \frac{1}{2}\Theta^{\xi_i}_{\mathcal{E}})c^{\xi'_i}:$.

Proof: By definition, we have $J =: c^{\xi'_i}\Theta^{\xi_i}_{\mathcal{S}} : + \frac{1}{2} : c^{\xi'_i}\Theta^{\xi_i}_{\mathcal{E}}:$. Since the $\Theta^{\xi}_{\mathcal{S}}$ commute with the $c^{\xi'}$, it suffices to show that

$$: c^{\xi'_i}\Theta^{\xi_i}_{\mathcal{E}} :=: \Theta^{\xi_i}_{\mathcal{E}} c^{\xi'_i} :$$

By Lemma 2.9, we have

$$: \Theta^{\xi_i}_{\mathcal{E}} c^{\xi'_i} := - : (c^{\xi'_j} b^{[\xi_i,\xi_j]})c^{\xi'_i} := - : c^{\xi'_j} b^{[\xi_i,\xi_j]} c^{\xi'_i} : - \partial c^{\xi'_j}\langle [\xi_i, \xi_j], \xi'_i\rangle.$$

The second term on the right vanishes because $ad^*(\xi_i)\xi'_i = 0$, while the first term coincides with $: c^{\xi'_i}\Theta^{\xi_i}_{\mathcal{E}}:$. □

We now define the vertex algebra analogue of a G^*-algebra of type C.

Definition 5.15. *A* $\mathcal{W}(\mathfrak{g})$*-algebra is a differential vertex algebra* $(\mathcal{A}, d_{\mathcal{A}})$ *equipped with a differential vertex algebra homomorphism* $\rho_{\mathcal{A}} : \mathcal{W}(\mathfrak{g}) \to \mathcal{A}$*. We define a homomorphism of* $\mathcal{W}(\mathfrak{g})$*-algebras (and modules) in an obvious way.*

Lemma 5.16. $\mathcal{W}_{hor} = \langle b\rangle \otimes \mathcal{S}(\mathfrak{g})$, $\mathcal{W}_{bas} = \mathcal{W}^{\mathfrak{g}_{\geq}}_{hor}$.

Proof: Here $\langle b\rangle$ is the vertex algebra generated by the $b^{\xi} \in \mathcal{E}(\mathfrak{g})$, and $(\cdots)^{\mathfrak{g}_{\geq}}$ is the subspace of $(\cdots)$ annihilated by the $\Theta^{\xi}_{\mathcal{W}}(n)$, $n \geq 0$, $\xi \in \mathfrak{g}$. The first equality follows immediately from the fact that a vertex operator in $\mathcal{E}(\mathfrak{g})$ commutes with the b^ξ iff it is in $\langle b\rangle$. The second equality follows from the definition of the basic subalgebra and the fact that $a \in \mathcal{W}_{hor}$ commutes with the $\Theta^\xi_{\mathcal{W}}$ iff it is annihilated by the $\Theta^\xi_{\mathcal{W}}(n)$. $\square$

Here is another small surprise: unlike in the classical case where $d_K|W_{hor} = 0$ and $d_{bas} = d_W|W_{bas} = 0$, neither $K(0)|\mathcal{W}_{hor}$ nor $D(0)|\mathcal{W}_{bas}$ is zero in general.

Clearly $\omega_{\mathcal{W}} \notin \mathcal{W}_{hor}$. Since the vertex operators $\Theta^\xi_{\mathcal{W}}$ are primary of conformal weight 1 with respect to the Virasoro element $\omega_{\mathcal{W}}$, they do not commute with $\omega_{\mathcal{W}}$ unless $\mathfrak{g}$ is abelian, in which case the $\Theta^\xi_{\mathcal{W}}$ are identically zero. However, since the Fourier mode $\omega_{\mathcal{W}}(1)$ acts diagonalizably on $\mathcal{W}$ and the vertex operators $b^\xi, \Theta^\xi_{\mathcal{W}}$ have weight 1, it follows that $\omega_{\mathcal{W}}(1)$ also acts diagonalizably on the basic vertex subalgebra $\mathcal{W}_{bas}$. Again, the subspace of zero eigenvalue is canonically isomorphic to the classical basic subalgebra W_{bas}. Since $[D(0), h(1)] = \omega_{\mathcal{W}}(1)$ and $D(0)^2 = 0$, it follows that $\omega_{\mathcal{W}}(1)$ commutes with $D(0)$, hence its action on $\mathcal{W}_{bas}$ descends to $H^*_{bas}(\mathcal{W})$. Here $h = \beta^{\xi_i}\partial c^{\xi'_i}$. Note that $h \notin \mathcal{W}_{bas}$, so we cannot conclude that $\omega_{\mathcal{W}}(1)$ acts by zero on cohomology. In fact, we will see that $\omega_{\mathcal{W}}(1)$ does not act by zero on cohomology.

Lemma 5.17. *$H^*_{bas}(W(\mathfrak{g}))$ is canonically isomorphic to the eigenspace of zero eigenvalue of $\omega_{\mathcal{W}}(1)$ in $\mathbf{H}^*_{bas}(\mathcal{W}(\mathfrak{g}))$.*

Proof: Recall that (W, d_W) is isomorphic to $(\mathcal{W}[0,0], D(0))$. Restricted to $\mathcal{W}[0,0]$, the basic subalgebra condition reduces to $b^\xi(0)a = \Theta^\xi_{\mathcal{W}}(0)a = 0$, $a \in \mathcal{W}[0,0]$. It is easily seen that this coincides with the basic subalgebra condition on W under the isomorphism above. This shows that $(\mathcal{W}_{bas}[0], D(0))$ is isomorphic to (W_{bas}, d_W), hence $\mathbf{H}^*_{bas}(\mathcal{W}(\mathfrak{g}))[0] \cong H^*_{bas}(W)$. $\square$

5.2. Weil model for $\mathbf{H}^*_G(\mathcal{A})$

Definition 5.18. *For a given $O(\mathfrak{sg})$-algebra $\mathcal{A}$, we define its chiral G-equivariant cohomology to be*

$$\mathbf{H}^*_G(\mathcal{A}) = \mathbf{H}^*_{bas}(\mathcal{A} \otimes \mathcal{W}(\mathfrak{g})).$$

For $\mathcal{A} = \mathbf{C}$, $\mathbf{H}^*_G(\mathbf{C}) = \mathbf{H}^*_{bas}(\mathcal{W}(\mathfrak{g}))$, a vertex algebra which is already interesting and difficult to compute. Consider the map

$$\mathcal{W}(\mathfrak{g}) \to \mathcal{A} \otimes \mathcal{W}(\mathfrak{g}), \quad a \mapsto 1 \otimes a.$$

In Example 5.3, we saw that this is a basic homomorphism but not an $O(\mathfrak{sg})$-algebra homomorphism in general. This induces a vertex algebra homomorphism

$$\kappa_G : \mathbf{H}^*_G(\mathbf{C}) = \mathbf{H}^*_{bas}(\mathcal{W}(\mathfrak{g})) \to \mathbf{H}^*_{bas}(\mathcal{A} \otimes \mathcal{W}(\mathfrak{g})) = \mathbf{H}^*_G(\mathcal{A}).$$

This is our vertex algebra analogue of the Chern-Weil map.

Recall that given a manifold M, the vertex algebra $\mathcal{Q}(M)$ has a Virasoro element (in local form)

$$\omega_{\mathcal{Q}} =: \beta^i \partial\gamma^i : - : b^i \partial c^i :$$

which is $d_{\mathcal{Q}}$-exact in $(\mathcal{Q}(M), d_{\mathcal{Q}})$, and $\omega_{\mathcal{Q}}(1)$ acts diagonalizably with eigenvalues in $\mathbf{Z}_{\geq}$. Since $\omega_{\mathcal{W}}$ is $D(0)$-closed in $(\mathcal{W}(\mathfrak{g}), D(0))$, it follows that

$$\omega_{\mathcal{Q}\otimes\mathcal{W}} = \omega_{\mathcal{Q}} \otimes 1 + 1 \otimes \omega_{\mathcal{W}}$$

is also $d_{\mathcal{Q}\otimes\mathcal{W}}$-closed. In particular, $\omega_{\mathcal{Q}\otimes\mathcal{W}}(1)$ commutes with $d_{\mathcal{Q}\otimes\mathcal{W}}$. Again, since the vertex operators $\iota_\xi \otimes 1 + 1 \otimes b^\xi$, $L_\xi \otimes 1 + 1 \otimes \Theta^\xi_{\mathcal{W}}$ have weight 1, it follows that $\omega_{\mathcal{Q}\otimes\mathcal{W}}(1)$ acts diagonalizably on the basic subalgebra $(\mathcal{Q}(M) \otimes \mathcal{W}(\mathfrak{g}))_{bas}$ and on the basic cohomology $\mathbf{H}^*_G(\mathcal{Q}(M))$. Note, however, that the vertex operator $\omega_{\mathcal{Q}\otimes\mathcal{W}}$ is not a basic element in general.

More generally, consider an $O(\mathfrak{sg})$-algebra $(\mathcal{A}, d_{\mathcal{A}})$ with $O(\mathfrak{sg})$-structure $(\xi, \eta) \mapsto L_\xi + \iota_\eta$. Suppose that $\mathcal{A}$ has no negative weight elements. Then $\mathcal{A}[0]$ is canonically a commutative associative algebra with product $\circ_{-1}$. Moreover, the operators $d_{\mathcal{A}}, L_\xi \circ_0, \iota_\eta \circ_0$ on $\mathcal{A}[0]$ define a G^*-structure on $\mathcal{A}[0]$. If we assume, furthermore, that $\mathcal{A}$ has a $d_{\mathcal{A}}$-closed Virasoro element $\omega_{\mathcal{A}}$ such that L_ξ, ι_η are primary of conformal weight 1 with respective to $\omega_{\mathcal{A}}$, then $\omega_{\mathcal{A}\otimes\mathcal{W}} = \omega_{\mathcal{A}} \otimes 1 + 1 \otimes \omega_{\mathcal{W}}$ is $(d_{\mathcal{A}} + D(0))$-closed Virasoro element in $\mathcal{A} \otimes \mathcal{W}(\mathfrak{g})$.

Lemma 5.19. *Let $\mathcal{A}$ be a vertex algebra and $\omega \in \mathcal{A}$ be a Virasoro element. If $a \in \mathcal{A}$ is primary of conformal weight one with respect to ω, then $\omega(m)$, $m \geq 0$, preserves the subalgebra $Com(\langle a \rangle, \mathcal{A})$.*

Proof: By Lemma 2.4, the OPE $\omega(z)a(w) \sim a(w)(z-w)^{-1} + \partial a(w)(z-w)^{-2}$ translates into

$$[\omega(m), a(n)] = -na(m+n-1).$$

Recall that $b \in Com(\langle a \rangle, \mathcal{A})$ iff $a(n)b = 0$ for all $n \geq 0$. Thus for such an element b, we have

$$a(n)\omega(m)b = na(m+n-1)b = 0$$

for all $n, m \geq 0$. Thus $\omega(m)b \in Com(\langle a\rangle, \mathcal{A})$ for all $m \geq 0$. $\square$

Theorem 5.20. *Let $(\mathcal{A}, d_{\mathcal{A}})$ be an $O(\mathfrak{sg})$-algebra with no negative weight elements. Then the chiral equivariant cohomology $\mathbf{H}^*_G(\mathcal{A})$ is a degree-weight graded vertex algebra with $\mathbf{Z}_{\geq}$-valued weights such that $\mathbf{H}^*_G(\mathcal{A})[0] = H^*_G(\mathcal{A}[0])$. If $\mathcal{A}$ has a $d_{\mathcal{A}}$-closed Virasoro structure $\omega_{\mathcal{A}}$ such that the $O(\mathfrak{sg})$-structure on $\mathcal{A}$ are given by primary operators L_ξ, ι_η of conformal weight 1, then the operators $\omega_{\mathcal{A}\otimes\mathcal{W}}(m)$ induce an action on $\mathbf{H}^*_G(\mathcal{A})$ for all $m \geq 0$.*

Proof: Obviously $\mathcal{A} \otimes \mathcal{W}(\mathfrak{g})$ has no negative weight elements, so that the same holds for its chiral basic cohomology. The weight zero subspace of $\mathcal{A} \otimes \mathcal{W}(\mathfrak{g})$ is the tensor product of weight zero spaces $\mathcal{A}[0] \otimes \mathcal{W}[0,0]$. We saw that $(\mathcal{W}[0,0], D(0)) = (W, d_W)$ is the classical Weil algebra, and that $(\mathcal{A}[0], d_{\mathcal{A}})$ is canonically a G^*-algebra. It is clear that $(\mathcal{A} \otimes \mathcal{W}(\mathfrak{g}))_{bas}[0]$ coincides with the classical basic complex $(\mathcal{A}[0] \otimes W)_{bas}$. This yields the first assertion.

The basic subalgebra $(\mathcal{A} \otimes \mathcal{W}(\mathfrak{g}))_{bas}$ consists of elements annihilated by the $n \geq 0$ Fourier modes of the vertex operators $\iota_\xi \otimes 1 + 1 \otimes b^\xi$ and the $L_\xi \otimes 1 + 1 \otimes \Theta^\xi_{\mathcal{W}}$, each of which is primary of conformal weight one with respect to the Virasoro element $\omega_{\mathcal{A}\otimes\mathcal{W}}$. By the preceding lemma, $\omega_{\mathcal{A}\otimes\mathcal{W}}(m)$, for all $m \geq 0$, acts on the basic subalgebra. Since $\omega_{\mathcal{W}}$ is $D(0)$-exact, and $\omega_{\mathcal{A}}$ is assumed $d_{\mathcal{A}}$-closed, it follows that $d_{\mathcal{A}} + D(0)$ commutes with $\omega_{\mathcal{A}\otimes\mathcal{W}}$. Hence the action of its $m \geq 0$ Fourier modes descends to an action on the basic cohomology $\mathbf{H}^*_G(\mathcal{A})$. $\square$

Remark 5.21. *From this, it is clear that our Chern-Weil map κ_G restricts to the classical Chern-Weil map on the weight zero subspaces.*

5.3. Cartan model for $\mathbf{H}^*_G(\mathcal{A})$

We introduce the vertex algebra analogues of the Mathai-Quillen isomorphism and the Cartan model. We will show that the Cartan model is equivalent to the Weil model in the vertex algebra setting.

Given a vector space V, we call a linear map $\phi \in End(V)$ pronilpotent if the restriction of ϕ to any finite dimensional subspace of V is nilpotent. In this case,

$$e^\phi = 1 + \phi + \frac{1}{2!}\phi^2 + \frac{1}{3!}\phi^3 + \cdots$$

is a well-defined automorphism of V. Let $\mathcal{A}$ be a vertex (super) algebra and $a \in \mathcal{A}$ a homogeneous vertex operator such that the Fourier mode $\hat{a}(0)$ is pronilpotent. Since $\hat{a}(0)$

is a derivation of all circle products, it follows that $e^{\hat{a}(0)}: \mathcal{A} \to \mathcal{A}$ is an automorphism of the vertex algebra. As a reminder, we shall write $a(0)$ instead of $\hat{a}(0)$ for notational simplicity.

Let $(\mathcal{A}, d_{\mathcal{A}})$ be a $\mathcal{W}(\mathfrak{g})$-algebra and $(\mathcal{B}, d_{\mathcal{B}})$ an $O(\mathfrak{sg})$-algebra. Consider the vertex operator

$$\phi(z) = c^{\xi_i'}(z) \otimes \iota_{\xi_i}(z) \in \mathcal{A} \otimes \mathcal{B}.$$

We claim that the zeroth Fourier mode $\phi(0)$ is pronilpotent, as an operator on $\mathcal{A} \otimes \mathcal{B}$. It suffices to show that for any given homogeneous element $u \otimes v \in \mathcal{A} \otimes \mathcal{B}$, we have $\phi(0)^k(u \otimes v) = 0$ for $k >> 0$. First, note that $c^{\xi'}(p)u = 0 = \iota_{\xi}(p)v$ for $p >> 0$. In particular, there is an integer $N > 0$ such that $\phi(0)(u \otimes v) = \sum_{|p|<N} (-1)^{|u|} c^{\xi_i'}(-p-1)u \otimes \iota_{\xi_i}(p)v$. Second, note that the modes $c^{\xi_i'}(p), \iota_{\xi_i}(q)$ are all pairwise anti-commuting, and in particular, each one is square-zero. This shows that $\phi(0)^k(u \otimes v) = 0$ for $k > 2N \; dim \; \mathfrak{g}$.

By analogy with the classical case, we call

$$\Phi = e^{\phi(0)} \equiv e^{\hat{\phi}(0)}$$

the Mathai-Quillen isomorphism of $\mathcal{A} \otimes \mathcal{B}$.

Theorem 5.22. *The Mathai-Quillen isomorphism satisfies*

$$\Phi(L_\xi \otimes 1 + 1 \otimes L_\xi) = L_\xi \otimes 1 + 1 \otimes L_\xi$$
$$\Phi(\iota_\xi \otimes 1 + 1 \otimes \iota_\xi) = \iota_\xi \otimes 1$$
$$\Phi d \Phi^{-1} = d + (-\gamma^{\xi_i'} \otimes \iota_{\xi_i} + c^{\xi_i'} \otimes L_{\xi_i})(0)$$

where $d = d_{\mathcal{A}} \otimes 1 + 1 \otimes d_{\mathcal{B}}$.

Proof: The vertex operators $L_\xi \otimes 1 + 1 \otimes L_\xi$ commute with the vertex operator ϕ because their OPE with ϕ is regular, since the $c^{\xi_i'}$ transform in the coadjoint module of $\mathfrak{g}$ while the ι_{ξ_i} transform in the adjoint module. It follows that

$$\phi(0)(L_\xi \otimes 1 + 1 \otimes L_\xi) = \phi \circ_0 (L_\xi \otimes 1 + 1 \otimes L_\xi) = 0.$$

This proves the first equality.

Next a simple OPE calculation yields

$$\phi \circ_0 (\iota_\xi \otimes 1) = -1 \otimes \iota_\xi$$
$$\phi \circ_0 (1 \otimes \iota_\xi) = 0$$

This gives the second equality.

Finally, as operators on $\mathcal{A}\otimes\mathcal{B}$:

$$[\phi(0),d]=-(d\phi)(0),\quad [\phi(0),[\phi(0),d]]=-(\phi\circ_0 d\phi)(0).$$

Applying Lemma 5.12 and using that $(\mathcal{A},d_{\mathcal{A}})$ is a $\mathcal{W}(\mathfrak{g})$-algebra, we get

$$\begin{aligned} d\phi &= d_{\mathcal{A}}c^{\xi'_j}\otimes\iota_{\xi_j}-c^{\xi'_j}\otimes d_{\mathcal{B}}\iota_{\xi_j}\\ &=(-\frac{1}{2}:c^{ad^*(\xi_i)\xi'_j}c^{\xi'_i}:+\gamma^{\xi'})\otimes\iota_{\xi_j}-c^{\xi'_j}\otimes L_{\xi_j}\end{aligned}$$

Since ϕ has regular OPE with the first term, we find that

$$\phi\circ_0 d\phi=\phi\circ_0(c^{\xi'_j}\otimes L_{\xi_j})=c^{\xi'_i}c^{\xi'_j}\otimes\iota_{[\xi_i,\xi_j]}.$$

The last expression has regular OPE with ϕ, hence applying $[\phi(0),-]$ to d more than twice yields zero. It follows that

$$\Phi d\Phi^{-1}=d-(d\phi)(0)-\frac{1}{2}\phi\circ_0 d\phi$$

yields the third equality. $\square$

It follows that $\Phi(\mathcal{A}\otimes\mathcal{B})_{hor}=\mathcal{A}_{hor}\otimes\mathcal{B}$. Specializing to the case $\mathcal{A}=\mathcal{W}(\mathfrak{g})$, we have $\mathcal{A}_{hor}=\langle b\rangle\otimes\mathcal{S}(\mathfrak{g})$. Since Φ is an $O(\mathfrak{g},0)$-module homomorphism by the preceding theorem, it follows that

$$\Phi(\mathcal{W}\otimes\mathcal{B})_{bas}=(\langle b\rangle\otimes\mathcal{S}(\mathfrak{g})\otimes\mathcal{B})^{\mathfrak{g}_\geq}=:C_G(\mathcal{B}).$$

Put

$$d_G=\Phi d\Phi^{-1}|C_G(\mathcal{B}).$$

For any $O(\mathfrak{sg})$-algebra $\mathcal{B}$, the cohomology of the differential vertex algebra $(C_G(\mathcal{B}),d_G)$ will be called the Cartan model for $\mathbf{H}^*_G(\mathcal{B})$. We have the following vertex algebra analogue of Cartan's fundamental theorem:

Theorem 5.23. *The Mathai-Quillen isomorphism induces the differential vertex algebra homomorphism*

$$((\mathcal{W}(\mathfrak{g})\otimes\mathcal{B})_{bas},d_{\mathcal{W}\otimes\mathcal{B}})\to(C_G(\mathcal{B}),d_G).$$

Hence

$$\mathbf{H}^*_G(\mathcal{B})=H^*((\mathcal{W}(\mathfrak{g})\otimes\mathcal{B})_{bas},d_{\mathcal{W}\otimes\mathcal{B}})\cong H^*(C_G(\mathcal{B}),d_G).$$

6. Abelian Case

We now specialize to the case when $G = T$ is an n dimensional torus. For the trivial $O(\mathfrak{sg})$-algebra $\mathbf{C}$, we give a complete description of the chiral T-equivariant cohomology $\mathbf{H}^*_G(\mathbf{C})$. Recall that $\mathbf{C} = \mathcal{Q}(pt)$, so that $\mathbf{H}^*_G(\mathbf{C})$ is a vertex algebra analogue of $H^*_G(\mathbf{C})$, the classical equivariant cohomology of a point. For general $O(\mathfrak{sg})$-algebras $\mathcal{A}$, we derive the analogue of a well-known spectral sequence that computes the classical T-equivariant cohomology.

6.1. The case $\mathcal{A} = \mathbf{C}$

Our first task is to compute

$$\mathbf{H}^*_T(\mathbf{C}) = \mathbf{H}^*_{bas}(\mathcal{W}(\mathfrak{t})).$$

Since T is abelian, both the adjoint and the coadjoint modules are trivial. In follows that all the vertex operators $\Theta^{\xi}_{\mathcal{W}}$ in (5.1) are identically zero. Likewise the J (5.2) is also zero. The differential $D(0)$ on $\mathcal{W}$ is just

$$K(0) = \sum_{n \in \mathbf{Z}} \gamma^{\xi'_i}(n-1)\ b^{\xi_i}(-n)$$

Let $\langle\gamma\rangle$ be the abelian vertex algebra generated by the $\gamma^{\xi'}$, $\xi' \in \mathfrak{t}^*$, in $\mathcal{W}(\mathfrak{t})$.

Theorem 6.1. *The inclusion $\langle\gamma\rangle \subset \mathcal{W}(\mathfrak{t})$ induces a canonical isomorphism*

$$\mathbf{H}^*_T(\mathbf{C}) = \mathbf{H}^*_{bas}(\mathcal{W}(\mathfrak{t})) \cong \langle\gamma\rangle.$$

Proof: Since the $\Theta^{\xi}_{\mathcal{W}}$ are zero, it follows that $\mathcal{W}_{bas} = \mathcal{W}_{hor} = \langle b\rangle \otimes \mathcal{S}(\mathfrak{t}) = \langle b, \beta, \gamma\rangle$, the vertex subalgebra of $\mathcal{W}(\mathfrak{g})$ generated by the vertex operators $b^{\xi}, \beta^{\xi}, \gamma^{\xi'}$. Since the $b^{\xi}(n)$, $n \geq 0$, act by zero on this space, the basic differential becomes

$$d_{bas} = D(0)|\mathcal{W}_{bas} = \sum_{n>0} \gamma^{\xi'_i}(n-1)\ b^{\xi_i}(-n).$$

We claim that the odd operator

$$F = \sum_{n>0} \beta^{\xi_i}(-n) c^{\xi'_i}(n-1)$$

is the a homotopy inverse for d_{bas}. First, it is obvious that $F\mathcal{W}_{bas} \subset \mathcal{W}_{bas}$. Moreover we have

$$[d_{bas}, F] = \sum_{n>0} \Big(-b^{\xi_i}(-n)c^{\xi'_i}(n-1) + \beta^{\xi_i}(-n)\gamma^{\xi'_i}(n-1)\Big)$$

This acts diagonalizably on $\mathcal{W}_{bas}$ with eigenvalues in $\mathbf{Z}_{\geq}$. This shows that all basic cohomology is concentrated in eigenspace-zero. This is the subspace of $\mathcal{W}_{bas}$ annihilated by the $c^{\xi'}(n-1), \gamma^{\xi'_i}(n-1)$, $n > 0$, which is just the subalgebra $\langle\gamma\rangle$. On the other hand d_{bas} is identically zero on this subalgebra. $\square$

Note that $\langle\gamma\rangle$ is a free polynomial algebra generated by the commuting vertex operators $\partial^k\gamma^{\xi'}$, $k \geq 0$, which are linear in $\xi' \in \mathfrak{t}^*$. Each of these vertex operators has cohomology degree 2 and Virasoro weight k. As expected from Theorem 5.20, the chiral equivariant cohomology $\langle\gamma\rangle$ contains the polynomial subalgebra generated by the weight zero vertex operators $\gamma^{\xi'}$, which is a copy of the classical equivariant cohomology $S(\mathfrak{t}^*)$.

6.2. A spectral sequence for $\mathbf{H}^*_T(\mathcal{A})$

Recall that $\mathcal{W}_{hor} = \langle b, \beta, \gamma\rangle$ has a monomial basis given by interated Wick products of the $b^{\xi_i}, \beta^{\xi_i}, \gamma^{\xi'_i}$ and their derivatives. In particular, there is a $\mathbf{Z}_{\geq}$ valued grading on $\mathcal{W}_{hor}$, which we shall call the $b\#$, which is given by the eigenvalues of the diagonalizable operator on $\mathcal{W}_{hor}$:

$$B = \sum_{n>0} b^{\xi_i}(-n)c^{\xi'_i}(n-1)$$

The idea is that the vertex operators b are non-classical (because they have weight one), and we should first "crop" them from the chiral Cartan complex. Likewise for the β.

Let $(\mathcal{A}, d_{\mathcal{A}})$ be a given $O(\mathfrak{st})$-algebra. Recall that the Cartan model for the chiral T-equivariant cohomology of $\mathcal{A}$ is the cohomology of the complex

$$C_T(\mathcal{A}) = (\mathcal{W}_{hor} \otimes \mathcal{A})^{\mathfrak{t}_{\geq}} = \langle b, \beta, \gamma\rangle \otimes \mathcal{A}^{\mathfrak{t}_{\geq}}.$$

The second equality follows from the important fact that the $O(\mathfrak{t})$-structure on $\mathcal{W}(\mathfrak{t})$ is trivial because T is abelian. The differential is

$$d_T = D(0) \otimes 1 + 1 \otimes d_{\mathcal{Q}} - (\gamma^{\xi'_i} \otimes \iota_{\xi_i})(0) + (c^{\xi'_i} \otimes L_{\xi_i})(0).$$

Again, $D(0) = K(0)$ because T is abelian. Consider the vertex subalgebra

$$\mathcal{C}_T(\mathcal{A}) = \langle\gamma\rangle \otimes \mathcal{A}^{\mathfrak{t}_{\geq}}.$$

We claim that

$$d_T|\mathcal{C}_T(\mathcal{A}) = 1 \otimes d_{\mathcal{A}} - (\gamma^{\xi_i'} \otimes \iota_{\xi_i})(0).$$

By Theorem 6.1, $D(0)|\langle\gamma\rangle = 0$. We have $(c^{\xi_i'} \otimes L_{\xi_i})(0) = \sum_{n\in\mathbf{Z}} c^{\xi_i'}(-n-1)L_{\xi_i}(n)$. Since $L_\xi(n)\mathcal{A}^{\mathfrak{t}_\geq} = 0$ for $n \geq 0$, and $c^{\xi'}(-n-1)\langle\gamma\rangle = 0$ for $n < 0$, we see that d_T reduces to the desired form. Thus we obtain an inclusion of complexes

$$\varphi : (\mathcal{C}_T(\mathcal{A}), d_T) \hookrightarrow (C_T(\mathcal{A}), d_T).$$

We shall call the first complex the *small* chiral Cartan complex of $\mathcal{A}$.

Lemma 6.2. *The map induced by φ on cohomology is surjective.*

Proof: Let $a \in \langle b, \beta, \gamma\rangle \otimes \mathcal{A}^{\mathfrak{t}_\geq}$ be nonzero and d_T-closed. We will show that a is d_T-cohomologous to an element in $\langle\gamma\rangle \otimes \mathcal{A}^{\mathfrak{t}_\geq}$. Let a_{max} be the component of a with the maximum $b\#$. Suppose this $b\#$ is positive. We can write $a_{max} = \sum p_j \otimes \omega_j$ where the $\omega_j \in \mathcal{A}^{\mathfrak{t}_\geq}$ are linearly independent elements and the $p_j \in \langle b, \beta, \gamma\rangle$ have the same maximum $b\#$.

We look at the effects of each of the four terms in d_T on the $b\#$ in $\langle b, \beta, \gamma\rangle \otimes \mathcal{A}^{\mathfrak{t}_\geq}$. We have

$$\begin{array}{cc} \textit{operators} & b\# \\ K(0)\otimes 1 = \sum_{n\geq 0} \gamma^{\xi_i'}(n) b^{\xi_i}(-n-1) & +1 \\ 1\otimes d_{\mathcal{A}} & 0 \\ -(\gamma^{\xi_i'} \otimes \iota_{\xi_i})(0) & 0 \\ (c^{\xi_i'} \otimes L_{\xi_i})(0) = \sum_{n<0} c^{\xi_i'}(-n-1)L_{\xi_i}(n) & -1. \end{array} \tag{6.1}$$

It follows that the term $(K(0)\otimes 1)a_{max}$ is the component in $d_T a$ with the highest $b\#$. Since $d_T a = 0$, this highest term must be zero, hence $\sum K(0)p_j \otimes \omega_j = 0$, implying that $K(0)p_j = 0$. By Theorem 6.1, $p_j = K(0)q_j$ for some q_j with $b\#$ one less than that of p_j. Put

$$a' = a - d_T \sum q_j \otimes \omega_j = a - a_{max} + \cdots$$

where the terms in $\cdots$ have only components with lower $b\#$ than a_{max}. Thus by induction, we see that a is d_T-cohomologous to an element with $b\#$ zero i.e. in $\langle\beta,\gamma\rangle \otimes \mathcal{A}^{\mathfrak{t}_\geq}$. So we may assume that a does not depend on the b.

Next we show that a does not depend on the β either. Since a is independent of the b, it follows that the $b^{\xi_i}(-n-1)a$, $n \geq 0$, $i = 1, .., \mathit{dim}\ \mathfrak{t}$, are linearly independent. Again by (6.1), $(K(0)\otimes 1)a = 0$. It follows that $\gamma^{\xi_i'}(n)b^{\xi_i}(-n-1)a = 0$ for all $n \geq 0$. This shows that a is independent of the β. This completes the proof. $\square$

Lemma 6.3. *The map induced by φ on cohomology is injective.*

Proof: Given an element ω in the small complex $\langle\gamma\rangle \otimes \mathcal{A}^{\mathfrak{t}_\geq}$ which is d_T-exact in the big complex $\langle b, \beta, \gamma\rangle \otimes \mathcal{A}^{\mathfrak{t}_\geq}$, we want to show that ω is d_T-exact in the small complex. Write $\omega = d_T a$. We want to find a' in the small complex so that $d_T a = d_T a'$. Again, let a_{max} be the component of a with the maximum $b\#$. By (6.1), $(K(0) \otimes 1)a_{max}$ is the component of $d_T a$ with the maximum and positive $b\#$. Since $d_T a$ is in the small complex, it does not depend on the b, implying that $(K(0) \otimes 1)a_{max} = 0$. Thus we get the shape $a_{max} = \sum K(0)q_j \otimes \omega_j$ and that

$$a - d_T \sum q_j \otimes \omega_j = a - a_{max} + \cdots$$

where $\cdots$ have only components with lower $b\#$ than a_{max}, as before. Thus we may as well assume that a does not depend on the b. Following the same argument as in the preceding lemma, we see that a does not depend on the β either. $\Box$

Theorem 6.4. *For any $O(\mathfrak{st})$-algebra $(\mathcal{A}, d_\mathcal{A})$, we have $\mathbf{H}^*_T(\mathcal{A}) \cong H^*(\mathcal{C}_T(\mathcal{A}), d_T)$.*

Proof: The two preceding lemmas show that φ induces an isomorphism on cohomology. $\Box$

From now on, we specialize to the case $\mathcal{A} = \mathcal{Q}(M)$ where M is a T-manifold.

Theorem 6.5. *For any T-manifold M, we have $\mathbf{H}^*_T(\mathcal{Q}(M)) \cong H^*(\mathcal{C}_T(\mathcal{Q}(M)), d_T)$.*

Observe that the small complex on the right side is a double complex with the differentials

$$d = 1 \otimes d_\mathcal{Q} = \sum_{n \in \mathbf{Z}} \beta^i(n) c^i(-n-1)$$

$$\delta = -(\gamma^{\xi_i'} \otimes \iota_{\xi_i})(0) = -\sum_{n \geq 0} \gamma^{\xi_i'}(-n-1)\iota_{\xi_i}(n).$$

where the expression for d makes use of a choice of local coordinates on M. This is an analogue of the double complex structure in the classical Cartan model (see Chap. 6 [18]). Note however that in the classical case, the double complex structure is on the Cartan complex itself, whereas in our case, it is on a much smaller subcomplex of the Cartan complex. It is also clear that the weight zero piece of the small complex coincides with the classical Cartan complex.

As usual, associated to the double complex structure on the small Cartan complex, there are two filtrations and two spectral sequences [19]. We shall consider the following one:

$$F_k^n = \oplus_{p+q=n,\ p\geq k}\, \mathcal{C}_T^{p,q}, \qquad \mathcal{C}_T^{p,q} = \langle\gamma\rangle^p \otimes (\mathcal{Q}(M)^{q-p})^{\mathfrak{t}_{\geq}}.$$

Here p denotes the γ-number on $\langle\gamma\rangle$, and $q-p$ is the bc-number on $\mathcal{Q}(M)$. Let (E_r, δ_r) be the spectral sequence associated with this filtration.

Theorem 6.6. *In each weight, the spectral sequence (E_r, δ_r) converges to the graded object associated with $H^*(\mathcal{C}_T(\mathcal{Q}(M)), d+\delta)$. In fact, in each weight, the spectral sequence collapses at E_r for some r.*

Proof: The first statement follows immediately from the fact that both d, δ are operators of weight zero, and the filtering spaces F_k and the terms in the spectral sequence are all graded by the weight. In a given weight m, we have $\mathcal{Q}(M)^{q-p}[m] = 0$ for $|q-p| >> 0$. For if $|q-p|$ is not bounded then either the vertex operators $\partial^k c^i$ or the $\partial^k b^i$ would have to be present on some coordinate open set of M, with unbounded k, because these operators are fermionic. But $wt\ \partial^k c^i = k$ and $wt\ \partial^k b^i = k+1$, violating that m is fixed and that the weights in $\mathcal{C}_T$ are bounded below by zero. This shows that $E_r[m] = E_{r+1}[m] = \cdots$ for all $r >> 0$ (cf. p66 [18]). $\Box$

Note that

$$E_1^{p,q} = H^q(\mathcal{C}_T^{p,*}, d), \qquad \delta_1 = \delta : E_1^{p,q} \to E_1^{p+1,q}.$$

7. Non-Abelian Case

We shall now proceed to construct cohomology classes in the chiral equivariant cohomology of the trivial $O(\mathfrak{sg})$-algebra $\mathbf{C}$. As before, $\mathfrak{g}$ is the complexified Lie algebra of the compact group G. We also choose an orthonormal basis ξ_i with respect to a fixed G-invariant pairing on $\mathfrak{g}$. We shall often identify $\mathfrak{g}, \mathfrak{g}^*$ via this pairing, for convenience

Theorem 7.1. *For any (connected) compact group G, the vertex operator $\gamma^{\xi_i}\partial\gamma^{\xi_i}$ represents a nonzero class in $\mathbf{H}_G^4(\mathbf{C})[1]$.*

Proof: It is straightforward to show that this vertex operator is basic and closed. Since the basic complex $\mathcal{W}_{bas}$ contains no vertex operators involving the c^{ξ_i}, for any given homogeneous element $\mu \in \mathcal{W}_{bas}$, $K(0)\mu = (\gamma^{\xi_i} b^{\xi_i})(0)\mu$ is either zero or it will contain some b^{ξ_i}. Thus if

$$(K(0) + J(0))\mu = \gamma^{\xi_i}\partial\gamma^{\xi_i},$$

then $J(0)\mu$ must have components (in the standard basis) of the form $\gamma^{\xi_i}\partial\gamma^{\xi_i}$. We will show that this leads to a contradiction.

First note that for $B = \beta^{\xi_i}\beta^{\xi_i}$, we have

$$J(z)B(w) \sim (c^{\xi_i}\beta^{[\xi_i,\xi_j]}\beta^{\xi_j})(w)(z-w)^{-1} = 0$$

because $\beta^{[\xi,\xi_i]}\beta^{\xi_i} = 0$. In particular $B(2)$ commutes with $J(0)$. Hence

$$J(0)B(2)\mu = B(2)J(0)\mu = B(2)(\gamma^{\xi_i}\partial\gamma^{\xi_i} - K(0)\mu) = dim\ \mathfrak{g} - B(2)K(0)\mu.$$

But since $K(0)\mu$ is either zero or contains some b^{ξ}, the same is true of the second term in the last expression above. But since that second term has weight zero and b^{ξ} has weight one, this forces that second term to be zero. Since $J(0)$ has $bc\#$ 1 and weight zero, and $dim\ \mathfrak{g}$ has $bc\#$ 0 and weight zero, there must be a component of $B(2)\mu$ having $bc\#$ -1 and weight zero. This is absurd because weight zero elements cannot have negative $bc\#$. $\square$

Remark 7.2. *The vertex operator $B = \beta^{\xi_i}\beta^{\xi_i} \in \mathcal{W}(\mathfrak{g})_{bas}$ turns out to be part of a current algebra $O(sl_2, -\frac{dim(\mathfrak{g})}{8}\kappa)$-structure which plays a fundamental role in the description of the full chiral equivariant cohomology of $\mathbf{C}$. This will be explained in a future follow-up paper.*

7.1. Weight one classes

Theorem 5.20 gives a complete description of $\mathbf{H}^*_G(\mathbf{C})[0]$, i.e. it coincides with the classical equivariant cohomology $H^*_G(\mathbf{C})$. We now give a complete description of the weight one piece.

Notations. We identify $Sym(\mathfrak{g})$ with the algebra $\mathbf{C}[\gamma^{\xi_1}, ..., \gamma^{\xi_n}]$, $n = dim\ \mathfrak{g}$. For $P \in Hom_{\mathfrak{g}}(\mathfrak{g}, Sym(\mathfrak{g}))$, we write $P : \xi \mapsto P_{\xi}$. Throughout this subsection, P shall denote such a map.

Lemma 7.3. *Let $Q : \mathfrak{g} \to Sym(\mathfrak{g})$, $\xi \mapsto Q_{\xi}$, be any linear map such that $Q_{\xi_i}b^{\xi_i} \in \mathcal{W}[1]$ is $\mathfrak{g}$-invariant. Then Q is a $\mathfrak{g}$-module map. Likewise the same is true under the assumption that $Q_{\xi_i}\beta^{\xi_i}$ or $Q_{\xi_i}\partial\gamma^{\xi_i}$ is $\mathfrak{g}$-invariant.*

Proof: We will prove one case, the other two being similar. We have

$$0 = \Theta^{\xi}_{\mathcal{W}}(0)Q_{\xi_i}b^{\xi_i} = (\Theta^{\xi}_{\mathcal{S}}(0)Q_{\xi_i})b^{\xi_i} + Q_{\xi_i}b^{[\xi_i,\xi]}.$$

But the second term on the right is equal to $-Q_{[\xi,\xi_j]}b^{\xi_j}$. By linear independence of the b^{ξ_i}, it follows that $\Theta^{\xi}_{\mathcal{S}}(0)Q_{\xi_i} = Q_{[\xi,\xi_j]}$. This says that Q is a $\mathfrak{g}$-module map. $\square$

Lemma 7.4. *For $\omega \in \mathcal{S}(\mathfrak{g})$, ω is $\mathfrak{g}_{\geq}$ invariant iff $J(0)\omega = 0$.*

Proof: Since ω has no b, c,

$$J(0)\omega = (c^{\xi_i}\Theta^{\xi_i}_{\mathcal{S}})(0)\omega = \sum_{n\geq 0} \frac{1}{n!}\partial^n c^{\xi_i}\Theta^{\xi_i}_{\mathcal{S}}(n)\omega.$$

Since all nonzero terms on the right side are independent, $J(0)\omega = 0$ iff $\Theta^{\xi_i}_{\mathcal{S}}(n)\omega = 0$ for $n \geq 0$. $\square$

Lemma 7.5. *$P_{\xi_i} b^{\xi_i} \in \mathcal{W}_{bas}$.*

Proof: The vertex operator $P_{\xi_i} b^{\xi_i}$ is clearly $\mathfrak{g}$-invariant, i.e. killed by the $\Theta^{\xi}_{\mathcal{W}}(0)$. Since it has weight one, it suffices to show that it is killed by $\Theta^{\xi}_{\mathcal{W}}(1)$. Now $\Theta^{\xi}_{\mathcal{W}}(1)P_{\xi_i} b^{\xi_i}$ is the term with second order pole in the OPE of $\Theta^{\xi}_{\mathcal{W}}(z)(P_{\xi_i} b^{\xi_i})(w)$. Since $\Theta^{\xi}_{\mathcal{W}}(z)$ has the shape $\beta\gamma + bc$, there is no double contraction by Wick's theorem. So there is no second order pole in the OPE. $\square$

Lemma 7.6. *$P_{\xi_i}\partial\gamma^{\xi_i} \in \mathcal{W}_{bas}$, hence it lies in $Ker\ J(0)$.*

Proof: Again, it is clear that $P_{\xi_i}\partial\gamma^{\xi_i}$ is horizontal and $\mathfrak{g}$-invariant. We have

$$\Theta^{\xi}_{\mathcal{W}}(1)P_{\xi_i}\partial\gamma^{\xi_i} = \Theta^{\xi}_{\mathcal{S}}(1)P_{\xi_i}\partial\gamma^{\xi_i} = \gamma^{\xi_i}P_{[\xi_i,\xi]} = \gamma^{\xi_i}(-\gamma^{\xi_j}\beta^{[\xi_i,\xi_j]})(0)P_{\xi} = 0.$$

Since $P_{\xi_i}\partial\gamma^{\xi_i}$ has weight one, this shows that it is $\mathfrak{g}_{\geq}$-invariant, hence basic. $\square$

Lemma 7.7. *$P_{\xi_i}\beta^{\xi_i} \in \mathcal{W}_{bas}$, hence it lies in $Ker\ J(0)$.*

Proof: Recall that $B = \beta^{\xi_j}\beta^{\xi_j}$ commutes with J. Thus by the preceding lemma, we have

$$0 = B(1)J(0)P_{\xi_i}\partial\gamma^{\xi_i} = J(0)B(1)P_{\xi_i}\partial\gamma^{\xi_i} = J(0)(B(1)P_{\xi_i})\partial\gamma^{\xi_i} + 2J(0)P_{\xi_i}\beta^{\xi_i}.$$

For the last equality, we have used the second identity in Lemma 2.9, and the fact that $B(1)\partial\gamma^{\xi_i} = 2\beta^{\xi_i}$ and $(B(0)P_{\xi_i})(0)\partial\gamma^{\xi_i} = 0$, which follow from Wick's theorem. Since B is $\mathfrak{g}$-invariant, it follows that the $B(1)P_{\xi_i}$ transform in the adjoint module, i.e. $\xi \mapsto B(1)P_{\xi}$ defines element of $Hom_{\mathfrak{g}}(\mathfrak{g}, Sym(\mathfrak{g}))$. By the preceding lemma, $J(0)(B(1)P_{\xi_i})\partial\gamma^{\xi_i} = 0$. This shows that $J(0)P_{\xi_i}\beta^{\xi_i} = 0$. $\square$

Lemma 7.8. *$J(0)P_{\xi_i} b^{\xi_i} = 0$.*

Proof: Note that $K(0)P_{\xi_i}\beta^{\xi_i} = P_{\xi_i} b^{\xi_i}$. Since $J(0), K(0)$ commute, it follows from the preceding lemma that $J(0)P_{\xi_i} b^{\xi_i} = 0$. $\square$

Theorem 7.9. *The weight one subspace $\mathbf{H}^*_G(\mathbf{C})[1]$ is canonically isomorphic to $Hom_G(\mathfrak{g}, Sym(\mathfrak{g}))$.*

Proof: Any element $a \in \mathcal{W}_{hor}[1]$ can be uniquely written as

$$a = P_{\xi_i} b^{\xi_i} + Q_{\xi_i} \beta^{\xi_i} + R_{\xi_i} \partial\gamma^{\xi_i} \tag{7.1}$$

where P, Q, R are linear maps $\mathfrak{g} \to Sym(\mathfrak{g})$. Suppose a is $\mathfrak{g}$-invariant. Since the $\beta^\xi, \gamma^\xi, b^\xi$ form three copies of the adjoint module, each of the three terms in a above must be separately $\mathfrak{g}$-invariant. By Lemma 7.3, it follows that P, Q, R are $\mathfrak{g}$-module maps. By the preceding four lemmas, the three terms in (7.1) are separately basic and killed by $J(0)$.

We have

$$D(0) P_{\xi_i} \beta^{\xi_i} = P_{\xi_i} b^{\xi_i}.$$

It follows that the first term in (7.1) is $D(0)$-exact. Suppose, in addition, that a is $D(0)$-closed, i.e. a represents a class in $\mathbf{H}^*_G(\mathbf{C})[1]$. Then (7.1) represents the same class if we drop the first term, so we may assume that $P = 0$. Then

$$0 = D(0)a = Q_{\xi_i} b^{\xi_i} + D(0) R_{\xi_i} \partial\gamma^{\xi_i}.$$

The second term is zero because $R_{\xi_i} \partial\gamma^{\xi_i}$ is obviously killed by $K(0)$ and is killed by $J(0)$ by Lemma 7.6. It follows that $Q = 0$. This shows that we have a canonical surjective linear map

$$Hom_{\mathfrak{g}}(\mathfrak{g}, Sym(\mathfrak{g})) \to \mathbf{H}^*_G(\mathbf{C})[1], \quad R \mapsto [R_{\xi_i} \partial\gamma^{\xi_i}]. \tag{7.2}$$

Suppose a given R is killed by this map. Then for some $P', Q', R' \in Hom_{\mathfrak{g}}(\mathfrak{g}, Sym(\mathfrak{g}))$, we have

$$R_{\xi_i} \partial\gamma^{\xi_i} = D(0)(P'_{\xi_i} b^{\xi'_i} + Q'_{\xi_i} \beta^{\xi_i} + R'_{\xi_i} \partial\gamma^{\xi_i}) = Q'_{\xi_i} b^{\xi}.$$

This implies that $R = Q' = 0$. This shows that (7.2) is injective. $\square$

7.2. Weight two classes

For simplicity, we shall assume that G is simple throughout this subsection. The Virasoro algebra will be playing a crucial role here.

Lemma 7.10. *Let A be a vertex algebra in which $L_1, L_2 \in A$ are Virasoro elements of central charges c_1, c_2. Suppose that L_2 is quasi-primary of conformal weight 2 with respect to L_1, i.e.*

$$L_1(z)L_2(w) \sim \frac{1}{2} c_3 (z-w)^{-4} + 2L_2(w)(z-w)^{-2} + \partial L_2(w)(z-w)^{-1}$$

for some scalar c_3. Then $L_1 - L_2$ is a Virasoro element of central charge $c_1 + c_2 - 2c_3$.

Proof: This is a well-known trick borrowed from physics [17][33]. By Lemma 2.4 again, we find

$$L_2(z)L_1(w) \sim \frac{1}{2}c_3(z-w)^{-4} + 2L_2(w)(z-w)^{-2} + \partial L_2(w)(z-w)^{-1}.$$

Now combining the four OPEs of $L_i(z)L_j(w)$, $i,j = 1,2$, we get the desired OPE of $L_1 - L_2$ with itself. $\square$

Since $\mathcal{S}(\mathfrak{g})$ is an $O(\mathfrak{g}, -\kappa)$-module, by Lemma 5.10, it has a Virasoro element given by the Sugawara-Sommerfield formula

$$L_{\mathcal{S}} = - : \Theta_{\mathcal{S}}^{\xi_i} \Theta_{\mathcal{S}}^{\xi_i} :$$

where ξ_i is an orthonormal basis of $(\mathfrak{g}, \kappa)$, as in Example 2.18.

Lemma 7.11. *$L_{\mathcal{S}}$ is a quasi-primary of conformal weight 2 with respect to $\omega_{\mathcal{S}}$. In fact,*

$$\omega_{\mathcal{S}}(z)L_{\mathcal{S}}(w) \sim dim\ \mathfrak{g}\ (z-w)^{-4} + 2L_{\mathcal{S}}(w)\ (z-w)^{-2} + \partial L_{\mathcal{S}}(w)(z-w)^{-1}.$$

Proof: By Lemma 2.9, we find that if a, b are primary of conformal weight 1 with respect to a Virasoro element ω, then we have

$$\omega(z)c(w) \sim \sum_{k \geq 0} (a \circ_k b)(w)(z-w)^{-k-3} + 2c(w)(z-w)^{-2} + \partial c(w)(z-w)^{-1}.$$

where $c =: ab :$. By Lemma 5.9, the $\Theta_{\mathcal{S}}^{\xi}$ are primary of conformal weight 1 with respect to $\omega_{\mathcal{S}}$, and so we can apply this to the case $\omega = \omega_{\mathcal{S}}$, $a = -b = \Theta_{\mathcal{S}}^{\xi_i}$, in $\mathcal{S}(\mathfrak{g})$, so that $c = L_{\mathcal{S}}$ when we sum over i. We find that $a \circ_k b = \delta_{k,1}\ \kappa(\xi_i, \xi_i)$. Summing over i, this becomes $\delta_{k,1} dim\ \mathfrak{g}$, which yields the desired OPE of $\omega_{\mathcal{S}}(z)L_{\mathcal{S}}(w)$. $\square$

Corollary 7.12. *$\omega_{\mathcal{S}} - L_{\mathcal{S}}$ is a Virasoro element in $\mathcal{W}(\mathfrak{g})$ of central charge 0.*

Proof: By Lemma 5.4, $\omega_{\mathcal{S}}$ has central charge $2dim\ \mathfrak{g}$. By Example 2.18, $L_{\mathcal{S}}$ has central charge $2dim\ \mathfrak{g}$ also. Now our assertion follows from the two preceding lemmas. $\square$

Lemma 7.13. *$\omega_{\mathcal{S}} - L_{\mathcal{S}}$ commutes with the $\Theta_{\mathcal{S}}^{\xi}$, and lives in the basic subalgebra $\mathcal{W}(\mathfrak{g})_{bas}$.*

Proof: By Lemma 5.9, the $\Theta_{\mathcal{S}}^{\xi}$ are primary of conformal weight 1 with respect to $\omega_{\mathcal{S}}$. By Example 2.18, the same is true with respect to $L_{\mathcal{S}}$. It follows that $(\omega_{\mathcal{S}}(z) - L_{\mathcal{S}}(z))\Theta_{\mathcal{S}}^{\xi}(w) \sim$

0. Now Lemma 2.4 implies our first assertion. Since $\omega_{\mathcal{S}} - L_{\mathcal{S}} \in \mathcal{S}(\mathfrak{g})$, it also commutes with the $b^{\xi} \in \mathcal{E}(\mathfrak{g})$. $\square$

Note however that $\omega_{\mathcal{S}} - L_{\mathcal{S}}$ is not $D(0)$-closed. The idea is to try to "correct" it, so that it becomes $D(0)$-closed without destroying its *basic* property. Since $\omega_{\mathcal{S}} - L_{\mathcal{S}}$ is basic, any correction would need to be basic as well. In particular, it must be G-invariant and horizontal. We examine the simplest nontrivial case first: $G = SU(2)$. Let's ask: what is the simplest possible (say, lowest degree as a polynomial in $\langle b\rangle \otimes \mathcal{S}(\mathfrak{g})$) horizontal G-invariant vertex operator C such that $\omega_{\mathcal{S}} - L_{\mathcal{S}} + C$ is $D(0)$-closed? Choose the standard basis x, h, y for the complexified Lie algebra $\mathfrak{g} = sl_2$.

Lemma 7.14. *$C = -\gamma^{h'} b^x b^y + \frac{1}{2}\gamma^{x'} b^x b^h - \frac{1}{2}\gamma^{h'} b^y b^h$ is the unique homogeneous lowest degree element which is G-invariant and horizontal, and makes $\mathbf{L} = \omega_{\mathcal{S}} - L_{\mathcal{S}} + C$, $D(0)$-closed. Moreover, this $\mathbf{L}$ is also a Virasoro element with central charge 0.*

Proof: Note that all the vertex operators appearing in C commute with each other. It is clear that C is horizontal since it does not depend on the c^{ξ}. Note also that C comes from a cubic trace polynomial in $\mathfrak{g}^* \otimes \mathfrak{g} \otimes \mathfrak{g}$, hence is G-invariant [34]. We have verified the uniqueness assertion and $D(0)$-closed condition by direct computations.

Clearly $C(z)C(w) \sim 0$ and C is primary of conformal weight 0 with respect to $\omega_{\mathcal{S}}$. By Lemmas 5.8, 2.9, we find that

$$-L_{\mathcal{S}}(z)\gamma^{\xi'}(w) = (:\Theta^{\xi_i}_{\mathcal{S}}\Theta^{\xi_i}_{\mathcal{S}}:)(z)\gamma^{\xi'}(w) \sim \gamma^{ad^*(\xi_i)ad^*(\xi_i)\xi'}(w)(z-w)^{-2} + \cdots = \gamma^{\xi'}(w)(z-w)^{-2} + \cdots$$

where $\cdots$ are lower order poles, and the ξ_i form an orthonormal basis of $\mathfrak{g}$. Hence $-L_{\mathcal{S}}(z)C(w) \sim C(w)(z-w)^{-2} + \cdots$. This shows that $(\omega_{\mathcal{S}} - L_{\mathcal{S}})(z)C(w) \sim C(w)(z-w)^{-2} + \cdots$. That $\mathbf{L}$ is Virasoro element with central charge 0 now follows from the next lemma. $\square$

Lemma 7.15. *Let A be a vertex algebra and $L \in A$ be a Virasoro element. Suppose that $a, b \in A$ are vertex operators with the property that*

$$a(z)a(w) \sim 0, \quad L(z)a(w) \sim a(w)(z-w)^{-2} + b(w)(z-w)^{-1}.$$

Then $L + a$ is Virasoro element with the same central charge.

Proof: By (2.4), we find that $a(z)L(w) \sim a(w)(z-w)^{-2} + (\partial a(w) + b(w))(z-w)^{-1}$. It follows that

$$\begin{aligned}(L(z)+a(z))(L(w)+a(w)) \sim \frac{1}{2}c(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1} \\ + a(w)(z-w)^{-2} + (\partial a(w) - b(w))(z-w)^{-1} + a(w)(z-w)^{-2} + b(w)(z-w)^{-1}\end{aligned}$$

This shows that $L+a$ has the OPE of a Virasoro element of the same central charge c. $\square$

Now comes a crucial observation. A computation using Wick's theorem shows that

$$C = K(0)(\Theta_{\mathcal{S}}^{\xi_i} b^{\xi_i}) = (K(0)\Theta_{\mathcal{S}}^{\xi_i})b^{\xi_i}$$

where the ξ_i is an orthonormal basis of $(\mathfrak{g}, \kappa)$. This computation suggests that for a general G, we should consider the vertex operator

$$\mathbf{L} \stackrel{def}{=} \omega_{\mathcal{S}} - L_{\mathcal{S}} + (K(0)\Theta_{\mathcal{S}}^{\xi_i})b^{\xi_i}.$$

From the shape of the vertex operator K and the $\Theta_{\mathcal{S}}^{\xi}$, one finds that the last term of the right side above is a priori a G-invariant vertex operator having the shape $bb\gamma$.

Theorem 7.16. $\mathbf{L}$ *is a Virasoro element with central charge 0, which is basic,* $D(0)$*-closed, and satisfies*

$$\mathbf{L}(0)a = \partial a, \qquad \mathbf{L}(1)a = (wt\ a)a$$

for any homogeneous element $a \in \langle\gamma\rangle \cap \mathcal{W}(\mathfrak{g})_{bas}$*, where* $\langle\gamma\rangle$ *is the vertex algebra generated by the* $\gamma^{\xi'}$*.*

Proof: Applying Wick's theorem, we find that the last term of $\mathbf{L}$ is

$$C = (K(0)\Theta_{\mathcal{S}}^{\xi_i})b^{\xi_i} = b^{\xi_i} b^{\xi_j} \gamma^{ad^*(\xi_i)\xi_j'}.$$

This implies that $C(z)C(w) \sim 0$. To apply the preceding lemma to show the Virasoro property for $\mathbf{L}$, it remains to verify the OPE

$$(\omega_{\mathcal{S}} - L_{\mathcal{S}})(z)\ \gamma^{\xi'}(w) \sim \gamma^{\xi'}(w)(z-w)^{-2} + \cdots$$

where $\cdots$ means lower order poles. This is verbatim as in Lemma 7.14.

To show the basic property, by Lemma 7.13, it suffices to check it for C. Clearly C commutes with the b^{ξ}. Applying Wick's theorem again, we get easily $(\Theta_{\mathcal{S}}^{\xi} + \Theta_{\mathcal{E}}^{\xi})(z)\ C(w) \sim 0$. This shows that C is basic.

Next, we claim that

$$\mathbf{L} = D(0)(\Theta_{\mathcal{S}}^{\xi_i} b^{\xi_i} + \beta^{\xi_i}\partial c^{\xi_i'}),$$

hence it is automatically $D(0)$-closed. By Wick's theorem, we get

$$\begin{aligned}
K(0)(\beta^{\xi_j}\partial c^{\xi'_j}) &= -:b^{\xi_i}\partial c^{\xi'_i}: + :\beta^{\xi_i}\partial\gamma^{\xi'_i}: := \omega_{\mathcal{W}} \\
J(0)(\beta^{\xi_i}\partial c^{\xi'_i}) &= 0 \\
(\Theta_{\mathcal{S}}^{\xi_i} c^{\xi'_i})(0)(\Theta_{\mathcal{S}}^{\xi_j} b^{\xi_j}) &=: (\Theta_{\mathcal{S}}^{\xi_i} c^{\xi'_i} \circ_0 \Theta_{\mathcal{S}}^{\xi_j}) b^{\xi_j} : + : \Theta_{\mathcal{S}}^{\xi_j}(\Theta_{\mathcal{S}}^{\xi_i} c^{\xi'_i} \circ_0 b^{\xi_j}) : \\
&=: (\Theta_{\mathcal{S}}^{[\xi_i,\xi_j]} c^{\xi'_i}) b^{\xi_j} : - : \partial c^{\xi_j} b^{\xi_j} : + : \Theta_{\mathcal{S}}^{\xi_i}\Theta_{\mathcal{S}}^{\xi_i} :, \qquad \textit{Lemma 2.9} \\
&=: \Theta_{\mathcal{S}}^{[\xi_i,\xi_j]} c^{\xi'_i} b^{\xi_j} : + : \Theta_{\mathcal{S}}^{\xi_i}\Theta_{\mathcal{S}}^{\xi_i} : - \omega_{\mathcal{E}} \\
&= - : \Theta_{\mathcal{S}}^{\xi_i}\Theta_{\mathcal{E}}^{\xi_i} : + : \Theta_{\mathcal{S}}^{\xi_i}\Theta_{\mathcal{S}}^{\xi_i} : - \omega_{\mathcal{E}} \\
\frac{1}{2}(: \Theta_{\mathcal{E}}^{\xi_i} c^{\xi'_i} :)(0)(\Theta_{\mathcal{S}}^{\xi_j} b^{\xi_j}) &=: \frac{1}{2} : \Theta_{\mathcal{S}}^{\xi_j}(\Theta_{\mathcal{E}}^{\xi_i} c^{\xi'_i} \circ_0 b^{\xi_j}) :=: \Theta_{\mathcal{S}}^{\xi_i}\Theta_{\mathcal{E}}^{\xi_i} :
\end{aligned}$$

Applying Lemma 5.14, we find that the sum of the four left sides plus $K(0)(\Theta_{\mathcal{S}}^{\xi_i} b^{\xi_i}) = C$ yields $D(0)(\Theta_{\mathcal{S}}^{\xi_i} b^{\xi_i} + \beta^{\xi_i}\partial c^{\xi'_i})$ on the one hand, and $\mathbf{L}$ on the other hand.

Finally, let $a \in \langle\gamma\rangle \cap \mathcal{W}(\mathfrak{g})_{bas}$. Then in terms of the generators of $\mathcal{W}$, a does not depend on the vertex operators $b^{\xi}, c^{\xi'}, \beta^{\xi}$. In particular a commutes with C above and with the $\Theta_{\mathcal{E}}^{\xi}$. Since a is assumed basic, it commutes with the $\Theta_{\mathcal{W}}^{\xi}$, hence with the $\Theta_{\mathcal{S}}^{\xi}$ as well. In particular, a commutes with $L_{\mathcal{S}}$. It follows that

$$\mathbf{L}(z)a(w) \sim \omega_{\mathcal{S}}(z)a(w) \sim \cdots + (wt\ a)a(w)(z-w)^{-2} + \partial a(w)(z-w)^{-1}$$

where $\cdots$ here means terms with higher order poles. Here we have used Lemma 5.4 to get the left side. This completes the proof. $\square$

Corollary 7.17. $\mathbf{L}$ *represents a nontrivial weight two class in* $\mathbf{H}_G^0(\mathbf{C})$.

Proof: It is easy to verify that $\mathbf{L}$ has cohomology degree zero and weight two. Since $a = \gamma^{\xi'_i}\partial\gamma^{\xi_i}$ is a nonzero weight one class by Theorem 7.1, it follows that $\mathbf{L}(1)a = \mathbf{L}\circ_1 a = a$ by the preceding theorem. Since circle products are cohomological operations, it follows that $\mathbf{L}$ cannot represent the zero class. $\square$

Corollary 7.18. $\mathbf{H}_G^*(\mathbf{C})$ *is a non-abelian vertex algebra.*

Proof: Since $\mathbf{L}$ represents a nonzero Virasoro element in the vertex algebra $\mathbf{H}_G^*(\mathbf{C})$, it does not commute with itself. $\square$

Remark 7.19. *This indicates that the departure of the chiral theory from the classical theory in the non-abelian case is quite dramatic.*

Lemma 7.20. *Let A be any vertex algebra and $a \in A$ such that $a(m) = 0$ for some $m < 0$. Then a commutes with A.*

Proof: By Lemma 2.4 we have

$$[a(m), b(q)] = \sum_p \binom{m}{p} (a \circ_p b)(m + q - p).$$

Consider the maximum $m < 0$ such that $a(m) = 0$. Suppose a does not commute with b, so that there exists a largest $N \geq 0$ such that $a \circ_N b \neq 0$. Pick $q = N - m - 1$. Then

$$\begin{aligned} 0 = [a(m), b(q)]\mathbb{1} &= \sum_{p=0}^{N} \binom{m}{p} (a \circ_p b)(m + q - p)\mathbb{1} \\ &= \binom{m}{N} (a \circ_N b)(-1)\mathbb{1} = \binom{m}{N} a \circ_N b \neq 0 \end{aligned}$$

a contradiction. $\square$

Corollary 7.21. *Any positive weight nonzero class $a \in \mathbf{H}^*_G(\mathbf{C})$ with $\mathbf{L}(1)a = (wt\ a)a$ cannot be killed by ∂.*

Proof: If a is killed by ∂, then $a(m) = 0$ for some $m < 0$. The preceding lemma says that a must be in the center of $\mathbf{H}^*_G(\mathbf{C})$. But $\mathbf{L}(1)a = (wt\ a)a$ and $wt\ a > 0$ imply that a does not commute with $\mathbf{L}$, a contradiction. $\square$

7.3. A general spectral sequence in the Cartan model

We now generalize the spectral sequence for computing $\mathbf{H}^*_G(\mathcal{A})$ to non-abelian G in the Cartan model. In a future paper, we will give an example to show that unlike in the classical case, the spectral sequence in the chiral case does not collapse at E_1, in general.

Recall that the chiral Cartan model for a $O(\mathfrak{sg})$-algebra $(\mathcal{A}, d_{\mathcal{A}})$ is

$$C_G(\mathcal{A}) = (\mathcal{W}_{hor} \otimes \mathcal{A})^{\mathfrak{g}_{\geq}}$$

equipped with the chiral Cartan differential

$$d_G = D(0) \otimes 1 + 1 \otimes d_{\mathcal{Q}} - (\gamma^{\xi'_i} \otimes \iota_{\xi_i})(0) + (c^{\xi'_i} \otimes L_{\xi_i})(0).$$

f The key observation here is that d_G can be broken up into two commuting differentials as follows. Write $D(0) = K(0) + J(0)$ as before, and put

$$d = K(0) \otimes 1 + 1 \otimes d_{\mathcal{A}}, \quad \delta = J(0) \otimes 1 - (\gamma^{\xi_i'} \otimes \iota_{\xi_i})(0) + (c^{\xi_i'} \otimes L_{\xi_i})(0)$$

so that $d_G = d + \delta$. As usual, $\mathcal{A}^s$ denotes the subspace of $\mathcal{A}$ consisting of elements a with $deg_{\mathcal{A}} a = s$. Let $C_G(\mathcal{A})^{p,q}$ be the subspace of $C_G(\mathcal{A})$ consisting of elements with

$$\beta\gamma\# + deg_{\mathcal{A}} = q, \quad \beta\gamma\# - b\# = p.$$

Note that the vertex operators $\Theta^{\xi}_{\mathcal{S}} \otimes 1 + 1 \otimes L_{\xi} \in \mathcal{W}_{hor} \otimes \mathcal{A}$ are homogeneous of degrees $(p, q) = (0, 0)$, and the $b^{\xi} \otimes 1$ have degree $(-1, 0)$. It follows that $C_G(\mathcal{A})$ is graded by the number (p, q).

Lemma 7.22. *We have that*

a. d, δ are $O(\mathfrak{sg})$-invariant;

b. d, δ preserve $C_G(\mathcal{A})$;

c. $d^2 = \delta^2 = [d, \delta] = 0$;

d. $d : C_G(\mathcal{A})^{p,q} \to C_G(\mathcal{A})^{p,q+1}$, $\delta : C_G(\mathcal{A})^{p,q} \to C_G(\mathcal{A})^{p+1,q}$.

Proof: a. Recall that $[J(0), b^{\xi}] = \Theta^{\xi}_{\mathcal{W}}$. Since $J(0)^2 = 0$, it follows that $[J(0), \Theta^{\xi}_{\mathcal{W}}] = 0$. Likewise $[D(0), \Theta^{\xi}_{\mathcal{W}}] = 0$, hence $[K(0), \Theta^{\xi}_{\mathcal{W}}] = 0$. Likewise $[d_{\mathcal{A}}, L_{\xi}] = 0$. It follows that $[d, \Theta^{\xi}_{\mathcal{W}} \otimes 1 + 1 \otimes L_{\xi}] = 0$. Since $d_G = d + \delta$ is $O(\mathfrak{sg})$-invariant, so is δ.

b. $K = \gamma^{\xi_i'} b^{\xi_i}$ obviously preserves $\mathcal{W}_{hor} = \langle b, \beta, \gamma \rangle$. Since $K(0), d_{\mathcal{A}}$ are both $O(\mathfrak{sg})$-invariant, it follows that they both, and hence d too, preserve $C_G(\mathcal{A})$.

c. Since $d_G^2 = d^2 + [d, \delta] + \delta^2$, assertion c. follows from d., which we show next.

d. Recall that $J =: c^{\xi_i'}(\Theta^{\xi_i}_{\mathcal{S}} + \frac{1}{2}\Theta^{\xi_i}_{\mathcal{E}})$:. Thus we can further break up $J(0)$ into two terms, the first being $(c^{\xi_i'}\Theta^{\xi_i}_{\mathcal{S}})(0)$. If we add to this the term $(c^{\xi_i'} \otimes L_{\xi_i})(0)$ appearing in δ, the sum acting on $C_G(\mathcal{A})$ takes the form

$$(c^{\xi_i} L^{tot}_{\xi})(0) = \sum_{n \geq 0} c^{\xi_i'}(n) L^{tot}_{\xi}(-n-1)$$

where $L^{tot}_{\xi} = \Theta^{\xi}_{\mathcal{S}} \otimes 1 + 1 \otimes L_{\xi}$, because $C_G(\mathcal{A})$ is $\mathfrak{g}_{\geq}$-invariant. (This is also consistent with the fact that $C_G(\mathcal{A})$ has no $c^{\xi'}$.) We now list the effects of all terms appearing in d_G on the various gradings on $C_G(\mathcal{A})$:

$$\begin{array}{cccc} operators & b\# & \beta\gamma\# & deg_{\mathcal{A}} \\ K(0) \otimes 1 = \sum_{n \geq 0} \gamma^{\xi_i'}(n) b^{\xi_i}(-n-1) & +1 & +1 & 0 \\ 1 \otimes d_{\mathcal{A}} & 0 & 0 & +1 \\ -(\gamma^{\xi_i'} \otimes \iota_{\xi_i})(0) & 0 & +1 & -1 \\ (c^{\xi_i'} \otimes L^{tot}_{\xi_i})(0) & -1 & 0 & 0 \\ \frac{1}{2}(c^{\xi_i'}\Theta^{\xi_i}_{\mathcal{E}})(0) & -1 & 0 & 0. \end{array} \tag{7.3}$$

The first two operators add up to d and the rest add up to δ. From the table, it follows that d, δ have the right effects on $C_G(\mathcal{A})^{p,q}$ as claimed. $\square$

It is also clear that the weight zero piece of the complex coincides with the classical Cartan complex with differentials reduced to $d = 1 \otimes d_{\mathcal{A}}$ and $\delta = -\gamma^{\xi'_i}(-1) \otimes \iota_{\xi_i}(0)$. In particular in weight zero, the double complex structure above agrees with the classical one.

As usual, associated to the double complex structure on the chiral Cartan complex, there are two filtrations and two spectral sequences. We shall consider the following one:

$$F_k^n = \oplus_{p+q=n,\ p\geq k}\ C_G(\mathcal{A})^{p,q}.$$

Let (E_r, δ_r) be the spectral sequence associated with this filtration. Let's specialize to the case

$$\mathcal{A} = \mathcal{Q}(M).$$

Theorem 7.23. *In each weight, the spectral sequence (E_r, δ_r) converges to the graded object associated with $H^*(C_G(\mathcal{Q}(M)), d+\delta)$. In fact, in each weight, the spectral sequence collapses at E_r for some r.*

Proof: The first statement follows immediately from the fact that both d, δ are operators of weight zero, and the filtering spaces F_k and the terms in the spectral sequence are all graded by the weight. In a given weight m, we claim that there are no nonzero elements in $C_G^{p,q}$ for $|q-p| >> 0$. Note that $|q-p| = |deg_{\mathcal{Q}} + b\#|$. So if $|q-p|$ is not bounded then either the vertex operators $\partial^k b^{\xi_i}$, or the $\partial^k c^i$, or the $\partial^k b^i$ would have to be present on some coordinate open set of M, with unbounded k, because these operators are fermionic. But $wt\ \partial^k b^{\xi_i} = wt\ \partial^k c^i = k$ and $wt\ \partial^k b^i = k+1$, violating that m is fixed and that the weights in $C_G(\mathcal{Q}(M))$ are bounded below by zero. This shows that for a given weight m, $E_r[m] = E_{r+1}[m] = \cdots$ for all $r >> 0$. $\square$

Note that

$$E_1^{p,q} = H^q(C_G(\mathcal{Q}(M))^{p,*}, d), \qquad \delta_1 = \delta : E_1^{p,q} \to E_1^{p+1,q}.$$

7.4. A general spectral sequence in the Weil model

Let $(\mathcal{A}, d_{\mathcal{A}})$ be a $O(\mathfrak{sg})$-algebra. The Weil model for the chiral equivariant cohomology of $(\mathcal{A}, d_{\mathcal{A}})$ is the cohomology of the complex

$$\mathcal{D}_G(\mathcal{A}) = ((\mathcal{W}(\mathfrak{g}) \otimes \mathcal{A})_{bas},\ K(0)\otimes 1 + J(0) \otimes 1 + 1 \otimes d_{\mathcal{A}}).$$

The three terms in the differential have the following gradings:

$$\begin{array}{cccc} operators & bc\# & \beta\gamma\# & deg_{\mathcal{A}} \\ K(0)\otimes 1 & -1 & +1 & 0 \\ J(0)\otimes 1 & +1 & 0 & 0 \\ 1\otimes d_{\mathcal{A}} & 0 & 0 & +1. \end{array} \tag{7.4}$$

Let $\mathcal{D}_G(\mathcal{A})^{p,q}$ be the subspace of $\mathcal{D}_G(\mathcal{A})$ of homogeneous degree

$$p = bc\# + \beta\gamma\# + deg_{\mathcal{A}}, \quad q = \beta\gamma\#.$$

The vertex operators $\Theta^{\xi}_{\mathcal{W}}\otimes 1 + 1\otimes L_{\xi}$ are homogeneous of degrees $(p,q)=(0,0)$, and the $b^{\xi}\otimes 1 + 1\otimes \iota_{\xi}$ have degrees $(p,q)=(-1,0)$. The operators

$$d = K(0)\otimes 1, \quad \delta = J(0)\otimes 1 + 1\otimes d_{\mathcal{A}}$$

have degrees $(p,q)=(0,1)$ and $(p,q)=(1,0)$ respectively.

Lemma 7.24. *We have that*

a. *d,δ preserve $\mathcal{D}_G(\mathcal{A})$;*

b. *$d^2=\delta^2=[d,\delta]=0$;*

c. *$d:\mathcal{D}_G(\mathcal{A})^{p,q}\to\mathcal{D}_G(\mathcal{A})^{p,q+1}$, $\delta:\mathcal{D}_G(\mathcal{A})^{p,q}\to\mathcal{D}_G(\mathcal{A})^{p+1,q}$.*

Proof: a. We have seen that $[K(0),\Theta^{\xi}_{\mathcal{W}}]=0$, implying that d commutes with $\Theta^{\xi}_{\mathcal{W}}\otimes 1 + 1\otimes L_{\xi}$. We also have that $K(0)$ commute with $b^{\xi}\otimes 1+1\otimes\iota_{\xi}$. Thus $K(0)$ preserves $\mathcal{D}_G(\mathcal{A})$. Since $d+\delta$ preserves $\mathcal{D}_G(\mathcal{A})$, so does δ.

b. Since $0=(d+\delta)^2=d^2+[d,\delta]+\delta^2$, assertion b. follows from c., which we have shown above. $\square$

As in the Cartan model, we have the filtration

$$F^n_k = \oplus_{p+q=n,\ p\geq k}\ \mathcal{D}_G(\mathcal{A})^{p,q}.$$

Let (E_r,δ_r) be the spectral sequence associated with this filtration. Let's specialize to the case

$$\mathcal{A}=\mathcal{Q}(M).$$

Theorem 7.25. *In each weight, the spectral sequence (E_r,δ_r) converges to the graded object associated with $H^*(\mathcal{D}_G(\mathcal{Q}(M)),d+\delta)$. In fact, in each weight, the spectral sequence collapses at E_r for some r.*

Proof: The argument is similar to the case of the Cartan model above. The only difference is that we now have $|q-p|=|deg_{\mathcal{Q}}+bc\#|$. So if $|q-p|$ is not bounded then either the vertex operators $\partial^k b^{\xi_i}$, or the $\partial^k c^{\xi'_i}$, or the $\partial^k c^i$, or the $\partial^k b^i$ would have to be present on some coordinate open set of M, with unbounded k, because these operators are fermionic. The rest of the argument is verbatim. $\square$

Note that

$$E_1^{p,q} = H^q(\mathcal{D}_G(\mathcal{Q}(M))^{p,*}, d), \qquad \delta_1 = \delta : E_1^{p,q} \to E_1^{p+1,q}.$$

7.5. Abelianization?

Recall that classically if T is a closed subgroup of G then $\mathfrak{g}^* \to \mathfrak{t}^*$ induces a map $W(\mathfrak{g}^*) \to W(\mathfrak{t}^*)$. Since every G^*-algebra is canonically a T^*-algebra, it follows that for any given G^*-algebra A one has a canonical map

$$A \otimes W(\mathfrak{g}^*) \to A \otimes W(\mathfrak{t}^*).$$

This induces on cohomology a map $H^*_G(A) \to H^*_T(A)$. In fact, when T is a maximal torus of G, then a spectral sequence argument shows that this map yields an isomorphism

$$H^*_G(A) \cong H^*_T(A)^W$$

where $W = N(T)/T$ is the Weyl group of G. See Chap. 6 [18].

One might expect that there would be a similar construction in the vertex algebra setting. Unfortunately, this cannot be expected to go through, at least not in a naive way. Here is why.

Lemma 7.26. *$\mathcal{W}(\mathfrak{g})$ is simple. In other words, it has no nontrivial ideal.*

Proof: As before, we can regard $\mathcal{W}(\mathfrak{g})$ as a polynomial (super) algebra with generators given by the negative Fourier modes $b^\xi(n), c^{\xi'}(n)$ (odd) and $\beta^\xi(n), \gamma^{\xi'}(n)$ (even), $n < 0$, which are linear in $\xi \in \mathfrak{g}$ and $\xi' \in \mathfrak{g}^*$. In this polynomial representation, each of the non-negative Fourier modes act by formal differentiation; for example $b^\xi(m)c^{\xi'}(-n) = \langle \xi', \xi \rangle \delta_{m-n+1,0}$ for $m \geq 0, n > 0$. From this, it is clear that any nonzero polynomial in $\mathcal{W}(\mathfrak{g})$ can be reduced to a nonzero scalar by a suitable repeated application of these derivations. Translated into vertex algebra operations, it says that any nonzero vertex operator in $\mathcal{W}(\mathfrak{g})$ can be reduced to a nonzero multiple of 1 by taking suitable repeated circle products with the generators of $\mathcal{W}(\mathfrak{g})$. In other words, any nonzero ideal of the circle algebra $\mathcal{W}(\mathfrak{g})$ contains 1. $\square$

Warning. It is not true that a positive Fourier mode of a generator of $\mathcal{W}$ acts as a derivation of the circle products in $\mathcal{W}$. For example $\beta^{\xi}(1) : \mathcal{W} \to \mathcal{W}$ is not a derivation of circle products. But if we represent $\mathcal{W}$ as a polynomial space, then this is a derivation with respect to the usual polynomial products. This follows immediately from the construction of $\mathcal{E}, \mathcal{S}$ as induced modules over a Lie algebra. See Example 2.15.

The preceding lemma shows that there exists a vertex algebra homomorphism $\mathcal{W}(\mathfrak{g}) \to \mathcal{W}(\mathfrak{t})$ extending the classical map $W(\mathfrak{g}^*) \to W(\mathfrak{t}^*)$ only if $\mathfrak{t} = \mathfrak{g}$. Next, suppose $T \subset G$ is a maximal torus. *Is there a vertex algebra homomorphism $\mathbf{H}^*_G(\mathbf{C}) \to \mathbf{H}^*_T(\mathbf{C})$ that extends the classical map $H^*_G(\mathbf{C}) \to H^*_T(\mathbf{C})$?*

Theorem 7.27. *The answer is negative.*

Proof: Suppose there were such a map $f : \mathbf{H}^*_G(\mathbf{C}) \to \mathbf{H}^*_T(\mathbf{C})$. Since $\mathbf{H}^*_T(\mathbf{C})$ is an abelian vertex algebra by Theorem 6.1, it follows that f must kill $\mathbf{L} \in \mathbf{H}^*_G(\mathbf{C})$, hence the ideal generated by $\mathbf{L}$. In particular by Theorem 7.16, the image of ∂ on $\mathbf{H}^*_G(\mathbf{C})[0] = H^*_G(\mathbf{C}) = S(\mathfrak{g}^*)^{\mathfrak{g}} \subset \langle\gamma\rangle$ must also be killed. On the other hand, since ∂ is a vertex algebra operation, it follows that

$$0 = f(\partial a) = \partial f(a). \tag{7.5}$$

By assumption, the restriction $f : H^*_G(\mathbf{C}) \to H^*_T(\mathbf{C})$ is the classical map. Consider for example $a = \gamma^{\xi'_i}\gamma^{\xi'_i} \in H^*_G(\mathbf{C})$ which is obviously nonzero, so that $f(a)$ is nonzero. In fact $f(a)$ has the same shape as a but we sum over only an orthonormal basis of $\mathfrak{t}$. But according to the description of $\mathbf{H}^*_T(\mathbf{C})$ given by Theorem 6.1, we have that $\partial f(a) \neq 0$, contradicting (7.5). $\square$

This shows that one cannot hope to get new information via an abelianization that extends the classical isomorphism $H^*_G(\mathbf{C}) \cong H^*_T(\mathbf{C})^W$, at least in the case of a point. This suggests that in the new theory, the chiral equivariant cohomology for non-abelian groups may be far more interesting that in the classical case.

8. Concluding Remarks

We have constructed a cohomology theory $\mathbf{H}^*_G(\mathcal{A})$ for $O(\mathfrak{sg})$-algebras $\mathcal{A}$, which is the vertex algebra analogue of the classical equivariant cohomology of G^*-algebras. A principal example we give is when $\mathcal{A} = \mathcal{Q}(M)$, the chiral de Rham complex of a G-manifold M. It turns out that there are other similar differential vertex algebras associated to a G-manifold which give rise to interesting chiral equivariant cohomology. For example, we can consider the vertex subalgebra $\mathcal{Q}'(M)$ generated by the weight zero subspace $\Omega(M)$ of $\mathcal{Q}(M)$. It turns out that $\mathcal{Q}'(M)$ is an abelian differential vertex algebra that belongs to an appropriate category on which the functor $\mathbf{H}^*_G$ is defined. Moreover, $\mathbf{H}^*_G(\mathcal{Q}'(M))$ is also a degree-weight graded vertex algebra containing the classical equivariant cohomology $H^*_G(M)$ as the weight zero subspace. We can prove the following

Theorem 8.1. *If the G-action on M has a fixed point, then the chiral Chern-Weil map $\kappa_G : \mathbf{H}^*_G(\mathbf{C}) \to \mathbf{H}^*_G(\mathcal{Q}'(M))$ is injective. If, furthermore, G is simple, then $\mathbf{H}^*_G(\mathcal{Q}'(M))$ is a conformal vertex algebra with the Virasoro element $\kappa_G(\mathbf{L})$.*

Details of this and other related results will appear in a forthcoming paper.

References

[1] F. Akman, The Semi-infinite Weil Complex of a Graded Lie Algebra, PhD Thesis, Yale University, 1993.

[2] F. A. Berezin, Introduction to Superanalysis, Math. Phys. and Appl. Math., 9 D. Reidel Publishing Co., Dordrecht-Boston (1987).

[3] R. Borcherds, Vertex operator algebras, Kac-Moody algebras and the Monster, Proc. Nat. Acad. Sci. USA 83 (1986) 3068-3071.

[4] R. Dijkgraaf, H. Verlinde and E. Verlinde, Notes on topological string theory and 2d quantum gravity, in String Theory and Quantum Gravity (Trieste 1990), World Sci. Publishing, River Edge, NJ (1991) 91-156.

[5] H. Cartan, Notions d'algèbre différentielle; application aux groupes de Lie et aux variétiés où opère un groupe de Lie, Colloque de Topologie, C.B.R.M., Bruxelles 15-27 (1950).

[6] H. Cartan, La Transgression dans un groupe de Lie et dans un espace fibré principal, Colloque de Topologie, C.B.R.M., Bruxelles 57-71 (1950).

[7] T. Eguchi and S.K. Yang, N=2 superconformal models as topological field theories, Mod. Phys. Lett. A5 (1990) 1693-1701.

[8] M. Duflo, S. Kumar, and M. Vergne, Sur la Cohomologie Équivariante des Variétés Différentiables, Astérisque 215 (1993).

[9] B. Feigin, The Semi-Infinite Homology of Lie, Kac-Moody and Virasoro Algebras, Russian Math. Survey 39 (1984) no. 2 195-196.

[10] B. Feigin and E. Frenkel, Semi-Infinite Weil Complex and the Virasoro Algebra, Comm. Math. Phys. 137 (1991), 617-639.

[11] I.B. Frenkel, H. Garland, G.J. Zuckerman, Semi-Infinite Cohomology and String Theory , Proc. Natl. Acad. Sci. USA Vol. 83, No. 22 (1986) 8442-8446.

[12] I.B. Frenkel, Y.Z. Huang, J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem. Amer. Math. Soc. 104 (1993), no. 494, viii+64.

[13] I.B. Frenkel, J. Lepowsky, A. Meurman, Vertex Operator Algebras and the Monster, Academic Press, New York, 1988.

[14] I.B. Frenkel, and Y.C. Zhu, Vertex Operator Algebras Associated to Representations of Affine and Virasoro Algebras , Duke Mathematical Journal, Vol. 66, No. 1, (1992), 123-168.

[15] D. Friedan, E. Martinec, S. Shenker, Conformal Invariance, Supersymmetry and String Theory , Nucl. Phys. B271 (1986) 93-165.

[16] E. Getzler, Two-dimensional topological gravity and equivariant cohomology, Commun. Math. Phys. 163 (1994) 473-489.

[17] P. Goddard, A. Kent, and D. Olive, Virasoro Algebras and Coset Space Models, Phys. Lett B 152 (1985) 88-93.

[18] V. Guillemin and S. Sternberg, Supersymmetry and Equivariant de Rham Theory, Springer, 1999.

[19] P. Hilton, and U. Stammbach, A Course in Homological Algebra, 2nd ed., Springer-Verlag, New York 1997.

[20] V. Kac, Vertex Algebras for Beginners, AMS Univ. Lecture Series Vol. 10, 2nd corrected ed., 2001.

[21] V. Kac, and D. Peterson, Infinite-dimensional Lie algebras, theta functions and modular forms, Adv. Math. 53 (1984) 125-264.

[22] H. Li, Local systems of vertex operators, vertex superalgebras and modules, hep-th/9406185.

[23] J. Kalkman, A BRST Model Applied to Symplectic Geometry, Thesis, Utrecht (1993).

[24] B. Kostant, Graded manifolds, graded Lie theory and pre-quantization, Lecture Notes in Math. 570, Springer-Verlag, Berlin (1977) 177-306.

[25] B. Lian, G.J. Zuckerman, New perspectives on the BRST-algebraic structure of string theory, Commun. Math. Phys. 154 (1993) 613-646.

[26] B. Lian, G.J. Zuckerman, Commutative Quantum Operator Algebras , J. Pure Appl. Algebra 100 (1995) no. 1-3, 117-139.

[27] B. Lian, On the classification of simple vertex operator algebras, Comm. Math. Phys. 163 (1994) No. 2, 307-357.

[28] A. Linshaw, Vertex Algebras and Invariant Theory, Ph.D. Thesis, Brandeis University, 2005.

[29] F. Malikov, V. Schectman, and A. Vaintrob, Chiral de Rham Complex, Comm. Math. Phys, 204, (1999) 439-473.

[30] F. Malikov, V. Schectman, Chiral de Rham Complex II, arXiv:math/AG9901065 v4.

[31] V. Mathai, D. Quillen, Superconnections, Thom classes and equivariant differential forms, Topology 25 (1986), no. 1, 85-110.

[32] G. Moore, and N. Seiberg, Classical and quantum conformal field theory, Comm. Math. Phys. 123 (1989) 177-254.

[33] J. Polchinski, String Theory, Vol 2, Cambridge Press 1998, Chap. 11.

[34] C. Procesi, The Invariant Theory of $n \times n$ Matrices, Adv. Math. 19, (1976) 306-381.

[35] D. Quillen, Superconnections and the Chern character, Topology 24, (1985), no. 1, 89-95.

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