FOURIER-MUKAI PARTNERS OF A K3 SURFACE OF PICARD NUMBER ONE

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Abstract. We shall give a complete geometrical description of the FM partners of a K3 surface of Picard number 1 and its applications.

Introduction

In this note, we shall study the Fourier-Mukai (FM) partners of a complex projective K3 surface X when X is generic, i.e. when $\rho(X) = 1$.

More concretely, we shall describe a complete list of FM partners of such a K3 surface in terms of the moduli of stable sheaves (Theorem 2.1)[1] and then give two applications; one is a moduli-theoretic interpretation of the set of FM partners of such a K3 surface (Corollary 3.4) and the other is a geometric descriptions of FM partners of an arbitrary projective K3 surface (Proposition 4.1).

Corollary 3.4, the first application, allows us to view naturally the set of FM partners of X with $\rho(X) = 1$ and $deg\, X = 2n$ as a principal homogeneous space of the 2-elementary group $(\mathbf{Z}/2)^{\oplus(\tau(n)-1)}$. Here $\tau(n)$ is the number of prime factors of n.

Proposition 4.1, the second application, is a refinement of the forth-description of FM partners in Theorem 1.2 by Mukai and Orlov (See also Theorem 1.3). The new part is the primitivity of the second factor, i.e. the Néron-Severi factor, of the Mukai vector in the description.

In Section 1, we recall some basic facts about FM partners of K3 surfaces. In Section 2, we shall determine the set of FM partners of a K3 surface of Picard number 1 (Theorem 2.1). The first application (Corollary 3.4) is stated and proved in Section 3 and the second (Proposition 4.1) in Section 4.

Acknowledgement

The main part of this note has been written during the first and third named authors' visit to Harvard University and the second author's visit to the University of Tokyo. They would like to thank Harvard University and the Education Ministry

2000 *Mathematics Subject Classification*. 14J28.

[1]We were informed that Paolo Stellari[St] also obtained this result using [Og1].

of Japan for financial support. The second author also wish to thank the Mathematics Department at the University of Tokyo for hospitality during his visit. The third named author would like to thank organizers of the Conference on *Hilbert Schemes, Vector Bundles and Their Interplay with Representation Theory* at the University of Missouri for an invitation to speak and for their financial support. A talk there is based partly on this note. The second named author is supported by NSF grant DMS-0072158.

§1. A review of FM partners of K3 surfaces

We shall work in the category of smooth projective varieties over $\mathbf{C}$.

Let X be a smooth projective variety. Let $D(X)$ be the bounded derived category of coherent sheaves on X [GM].

Definition 1.1. *Two smooth projective varieties X, Y are said to be Fourier-Mukai (FM) partners of one another if there is an equivalence of triangulated categories $D(X) \simeq D(Y)$. The set of isomorphism classes of FM partners of X is denoted by $FM(X)$.*

In this note, we will mainly be interested in the FM partners of a K3 surface.

Let X be a K3 surface. By $NS(X)$, $T(X)$, $\tilde{H}(X, \mathbf{Z})$, we denote the Néron-Severi lattice, the transcendental lattice and the Mukai lattice, of X respectively. We also denote by ω_X a non-zero holomorphic two form on X.

The following is a fundamental Theorem of Mukai [Mu1,2] and Orlov [Or]:

Fundamental Theorem 1.2. *Let X be a K3 surface. If $Y \in FM(X)$ then Y is also a K3 surface. Moreover the following four statements are equivalent:*

(1) $Y \in FM(X)$.

(2) There is a Hodge isometry $\varphi : (T(Y), \mathbf{C}\omega_Y) \simeq (T(X), \mathbf{C}\omega_X)$.

(3) There is a Hodge isometry $\tilde{\varphi} : (\tilde{H}(Y, \mathbf{Z}), \mathbf{C}\omega_Y) \simeq (\tilde{H}(X, \mathbf{Z}), \mathbf{C}\omega_X)$.

(4) Y is isomorphic to a 2-dimensional fine compact moduli space of stable sheaves on X with respect to some polarization of X.

This theorem connects three of the fundamental aspects of K3 surfaces. Statement (1) is *categorical*, (2) and (3) are *arithmetical*, and (4) is *algebro-geometrical*.

Next we recall a few facts concerning the characterization (4).

We call a primitive sublattice $\widetilde{NS}(X) := H^0(X, \mathbf{Z}) \oplus NS(X) \oplus H^4(X, \mathbf{Z})$ of the Mukai lattice $\tilde{H}(X, \mathbf{Z})$ the extended Néron-Severi lattice of X.

For $v = (r, H, s) \in \widetilde{NS}(X)$ and an ample class $A \in NS(X)$, we denote by $M_A(v)$ (resp. $\overline{M}_A(v)$) the coarse moduli space of stable sheaves (resp. the coarse moduli space of S-equivalence classes of semi-stable sheaves) $\mathcal{F}$ with respect to the polarization A with $\mu(\mathcal{F}) = v$. Here $\mu(\mathcal{F})$ is the Mukai vector of $\mathcal{F}$ defined by

$$\mu(\mathcal{F}) := \mathrm{ch}(\mathcal{F})\sqrt{td_X} = (\mathrm{rk}(\mathcal{F}), c_1(\mathcal{F}), \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) + \mathrm{rk}(\mathcal{F})\ ) \quad .$$

The space $\overline{M}_A(v)$ is a projective compactification of $M_A(v)$ and $\overline{M}_A(v) = M_A(v)$ if all the semi-stable sheaf $\mathcal{F}$ with $v(\mathcal{F}) = v$ is stable, for instance if $(r, s) = 1$.

The following theorem is essentially due to Mukai [Mu2]:

Theorem 1.3. *If $Y \in FM(X)$, then $Y \simeq M_H((r, H, s))$ where H is ample, $r > 0$ and s are integers such that $(r, s) = 1$ and $2rs = (H^2)$, and vice versa. (See also Proposition 4.1 for a refinement.)*

Throughout this note, we will frequently use the following simple:

Lemma 1.4. *Let $\pi : (\mathcal{X}, \mathcal{H}) \to \mathcal{B}$ be a smooth projective family of K3 surfaces. Let $f : \overline{\mathcal{M}} \to \mathcal{B}$ be a relative moduli space of the S-equivalence classes of semi-stable sheaves of π with respect to the polarization $\mathcal{H}$ with Mukai vectors $(r, \mathcal{H}_t, s)$. If $r > 0$, $(r, s) = 1$ and $2rs = (\mathcal{H}_t^2)$, then f is projective and gives a relative FM family of π, i.e. $\overline{\mathcal{M}}_t \in FM(\mathcal{X}_t)$ for all $t \in \mathcal{B}$.*

Proof. The existence of f and its projectivity over $\mathcal{B}$ are shown by Maruyama [Ma2, Corollary 5.9.1]. Note that the assumption about the boundedness there is verfied for a family of surfaces by [Ma1, Corollary 2.5.1]. (See also [HL, Chapter 3].) By the definition, we have $\overline{\mathcal{M}}_t = \overline{\mathcal{M}}_{\mathcal{H}_t}((r, \mathcal{H}_t, s))$. Since $(r, s) = 1$, $2rs = (\mathcal{H}_t^2)$ and $\mathcal{H}_t$ is ample on $\mathcal{X}_t$, it follows that $\overline{\mathcal{M}}_t = \mathcal{M}_{\mathcal{H}_t}((r, \mathcal{H}_t, s)) \in FM(\mathcal{X}_t)$ by Theorem 1.3. $\square$

We also notice here the following direct consequence of the fundamental theorem:

Lemma 1.5. *Let X be a K3 surface of $\rho(X) = 1$ and of degree $2n$, i.e. $NS(X) = \mathbf{Z}H$ and $(H^2) = 2n$. Let $Y \in FM(X)$. Then Y is also of $\rho(X) = 1$ and of degree $2n$.*

Proof. Recall that $T(Y)$ is isometric to $T(X)$ by the fundamental theorem. Combinng this with $NS(X) = T(X)^{\perp}$ in $H^2(X, \mathbf{Z})$ and likewise for Y, we have $\rho(Y) = \rho(X) = 1$ and $\deg Y = \det NS(Y) = \det NS(X) = \deg X$. $\square$

Finally we recall the Counting Formula from [HLOY].

Notation. Let S be an even lattice with a nondegenerate pairing $\langle a, b\rangle$, and signature $sgn\ S$. We write $S^* := \mathrm{Hom}\,(S, \mathbf{Z})$, $O(S)$, $A_S := S^*/S$, and $O(A_S)$, to denote respectively the dual lattice of S, the isometry group of S, the discriminant group of S, and the isometry group of A_S with respect to the natural quadratic form $q_S : A_S \to \mathbf{Q}/2\mathbf{Z}$, $x \ mod\ S \mapsto \langle x, x\rangle \ mod\ 2\mathbf{Z}$. The natural homomorphism $O(S) \to O(A_S)$ defines a left and a right group action of $O(S)$ on $O(A_S)$.

Recall that two lattices S, S' are said to be in the same genus[2] if

$$S \otimes \mathbf{R} \simeq S' \otimes \mathbf{R}, \qquad S \otimes \mathbf{Z}_p \simeq S' \otimes \mathbf{Z}_p \ \forall p \text{ prime}.$$

¿From now on, we will consider only even lattices. By a theorem of Nikulin [Ni], this is equivalent to the conditions that

$$sgn\ S = sgn\ S', \qquad (A_S, q_S) \simeq (A_{S'}, q_{S'}).$$

[2]The condition when $p = 2$ shows that S is even iff S' is even.

Note also that $|A_S| = |det\ S|$. We denote by $\mathcal{G}(S)$ the set of the isomorphism classes of lattices in the genus of S. It is known that $\mathcal{G}(S)$ is finite (See eg. [Ca]). We fix representatives of the classes in $\mathcal{G}(S)$ and simply write

$$\mathcal{G}(S) = \{S_1 = S, ..., S_m\}.$$

Hence we have, for each i, an isomorphism $(A_{S_i}, q_{S_i}) \simeq (A_S, q_S)$.

Let $f : S \hookrightarrow L$ be a primitive embedding (i.e. $L/f(S)$ is free) of an even lattice S into an even unimodular lattice L. Put $T := S^{\perp}$ in L. Then we have

$$(A_T, q_T) \simeq (A_S, -q_S),$$

which induces a natural isomorphism $O(A_T) \simeq O(A_S)$. Via the natural homomorphism $O(T) \to O(A_T)$, this induces a group action of $O(T)$ on $O(A_S)$, and hence on each $O(A_{S_i}) \simeq O(A_S)$ as well.

Specializing this to the case

$$L = H^2(X, \mathbf{Z}),\ S = NS(X),\ T = T(X),$$

where X is a K3 surface, we get a group action of $O_{Hodge}(T(X), \mathbf{C}\omega_X) \subset O(T(X))$ on each $O(A_{S_i})$. Here $O_{Hodge}(T(X), \mathbf{C}\omega_X)$ is the group of Hodge isometries of $(T(X), \mathbf{C}\omega_X)$. Now let $O(S_i)$ and $O_{Hodge}(T(X), \mathbf{C}\omega_X)$ act on $O(A_{S_i})$ respectively from the left and the right, and denote the orbit space of this action by

$$O(S_i)\backslash O(A_{S_i})/O_{Hodge}(T(X), \mathbf{C}\omega_X).$$

Observe that the one-sided quotients $O(A_{S_i})/O_{Hodge}(T(X), \mathbf{C}\omega_X)$ are all isomorphic to $O(A_S)/O_{Hodge}(T(X), \mathbf{C}\omega_X)$, while the double quotients above may depend on the action of $O(S_i)$ on each one-sided quotient.

The Counting Formula 1.6. [HLOY] *For a given K3 surface X, set $S = NS(X)$ and write $\mathcal{G}(S) = \{S_1, ..., S_m\}$. Then we have*

$$|FM(X)| = \sum_{i=1}^{m} |O(S_i)\backslash O(A_{S_i})/O_{Hodge}(T(X), \mathbf{C}\omega_X)|.$$

Moreover the i-th summand here coincides with the number of FM partners $Y \in FM(X)$ with $NS(Y) \simeq S_i$.

Remark. (Appendix of [HLOY]) The group $O_{Hodge}(T(X), \mathbf{C}\omega_X)$ turns out to be always a finite cyclic group of even order $2I$ such that $\varphi(2I)|rk\ T(X)$, where $\varphi(J) := |(\mathbf{Z}/J)^{\times}|$ is the Euler function. Note that $\varphi(2I)$ is even unless $I = 1$. □

One can derive the following important consequence from the Counting Formula:

Corollary 1.7.

(1) [Mu] *$FM(X) = \{X\}$ if $\rho(X) \geq 12$. In particular, $FM(Km\,A) = \{Km\,A\}$.*

(2) [Og1] *Let X be a K3 surface with $NS(X) = \mathbf{Z}H$ and $(H^2) = 2n$. Then*

$$|FM(X)| = 2^{\tau(n)-1}\,,$$

where $\tau(n)$ is the number of prime factors of n, e.g. $\tau(12) = \tau(6) = 2$, $\tau(8) = \tau(2) = 1$.

(3) [HLOY] *Let X be a K3 surface such that $\rho(X) = 2$ and $\det NS(X) = -p$, where p is a prime number. Let $h(p)$ be the class number of $\mathbf{Q}(\sqrt{p})$. Then*

$$|FM(X)| = \frac{h(p)+1}{2}\,.$$

§2. $FM(X)$ for K3 surfaces of Picard number 1

In this section, we prove the following theorem which describes the set of the FM partners of X geometrically:

Theorem 2.1. *Let X be a K3 surface such that $NS(X) = \mathbf{Z}H$ and $(H^2) = 2n$. Then*

$$FM(X) = \{M_H((r, H, s)) \mid rs = n,\ (r, s) = 1,\ r \geq s\ \}.$$

Proof. Note that

$$|\{(r, s)|rs = n, (r, s) = 1, r \geq s > 0\}| = 2^{\tau(n)-1} = |FM(X)|\,.$$

Here the first equality is elementary and the second equality is due to Corollary 1.6(2). Then, Theorem 2.1 follows from the next:

Proposition 2.2. *Let r (and respectively, r') be a positive integer satisfying $r|n$, $(r, \frac{n}{r}) = 1$ (respectively $r'|n$, $(r', \frac{n}{r'}) = 1$). Set $s = \frac{n}{r}$ and $s' = \frac{n}{r'}$. Then*

(1) $M_H((r, H, s)) \in FM(X)$.

(2) *If $M_H((r', H, s')) \cong M_H((r, H, s))$ as abstract variety, then $r' = r$ or $r' = s$.*

In what follows, we shall show this proposition.

Proof of (1). This is a special case of Theorem 1.3. □

Proof of (2). In what follows, we set $v(r) := (r, H, s)$. Let us consider the universal sheaf $\mathcal{E} \in \mathrm{Coh}(M_H(v(r)) \times X)$, and the associated FM transform:

$$\Phi^{\mathcal{E}}_{M_H(v(r))\to X} : D(M_H(v(r))) \to D(X)\ \ .$$

Then corresponding to this functor we have a Hodge isometry:

$$f_r := f^{\mathcal{E}}_{M_H(v(r))\to X} : (\tilde{H}(M_H(v(r)), \mathbf{Z}), \mathbf{C}\omega_{M_H(v(r))}) \ \overset{\sim}{\to}\ (\tilde{H}(X, \mathbf{Z}), \mathbf{C}\omega_X) \tag{2.1}$$

with property $f_r((0, 0, 1)) = v(r)$. Likewise

$$f_{r'} : (\tilde{H}(M_H(v(r')), \mathbf{Z}), \mathbf{C}\omega_{M_H(v(r'))}) \ \overset{\sim}{\to}\ (\tilde{H}(X, \mathbf{Z}), \mathbf{C}\omega_X) \tag{2.2}$$

with $f_{r'}((0,0,1)) = v(r')$.

Let $g : M_H(v(r')) \tilde{\to} M_H(v(r))$ be an isomorphism. This g induces a Hodge isometry

$$g^* : (H^2(M_H(v(r)), \mathbf{Z}), \mathbf{C}\omega_{M_H(v(r))}) \tilde{\to} (H^2(M_H(v(r')), \mathbf{Z}), \mathbf{C}\omega_{M_H(v(r'))}) \ .$$

Identifying the zero-th and four-th cohomology of the two varieties with $\mathbf{Z}$, we have a Hodge isometry, $\tilde{g} := (\mathrm{id}, g^*, \mathrm{id})$,

$$\tilde{g} : (\tilde{H}(M_H(v(r)), \mathbf{Z}), \mathbf{C}\omega_{M_H(v(r))}) \ \tilde{\to} \ (\tilde{H}(M_H(v(r')), \mathbf{Z}), \mathbf{C}\omega_{M_H(v(r'))}) \ .$$

Define

$$f_r((-1,0,0)) =: u(r) \ \ , \ \ f_{r'}((-1,0,0)) =: u(r')$$

for the Hodge isometries f_r and $f_{r'}$ above. Then

$$\begin{aligned} \langle v(r)^2 \rangle = \langle u(r)^2 \rangle = 0 \ \ , \ \ \langle u(r), v(r) \rangle = 1 \ \ , \\ \langle v(r')^2 \rangle = \langle u(r')^2 \rangle = 0 \ \ , \ \ \langle u(r'), v(r') \rangle = 1 \ \ . \end{aligned} \tag{2.3}$$

Now consider the Hodge isometry,

$$\tilde{\varphi} := f_{r'} \circ \tilde{g} \circ f_r^{-1} : (\tilde{H}(X, \mathbf{Z}), \mathbf{C}\omega_X) \ \tilde{\to} \ (\tilde{H}(X, \mathbf{Z}), \mathbf{C}\omega_X).$$

We have $\tilde{\varphi}(v(r)) = v(r')$ and $\tilde{\varphi}(u(r)) = u(r')$. Hereafter we consider the following restrictions:

$$\begin{aligned} \varphi := \tilde{\varphi}|_{\widetilde{NS}(X)} : \widetilde{NS}(X) \tilde{\to} \widetilde{NS}(X) \ \ , \\ \varphi_T := \tilde{\varphi}|_{T(X)} : (T(X), \mathbf{C}\omega_X) \tilde{\to} (T(X), \mathbf{C}\omega_X) \ \ . \end{aligned}$$

We also fix a basis for $\widetilde{NS}(X)$;

$$\widetilde{NS}(X) = \mathbf{Z}e \oplus \mathbf{Z}H \oplus \mathbf{Z}f \cong U \oplus \langle 2n \rangle \ \ ,$$

where e and f represent the (fundamental) classes of H^0 and H^4 respectively. Also we set

$$\begin{aligned} v(r) = re + H + sf \ , \ u(r) = le + kH + mf \ ; \\ v(r') = r'e + H + s'f \ , \ u(r') = l'e + k'H + m'f \ . \end{aligned}$$

Lemma 2.3. *Define $U_r := \mathbf{Z}u(r) \oplus \mathbf{Z}v(r)$, and denote by $\pi(r)$ a generator of $U_r^\perp$ in $\widetilde{NS}(X)$. Then up to sign, we have*

$$\pi(r) = 2n(-l + rk)e + 2n(m - sk)f + (rm - ls)H \ \ ,$$

and $\widetilde{NS}(X) = U_r \perp \mathbf{Z}\pi(r)$ (orthogonal direct sum). Similarly for $U_{r'} := \mathbf{Z}u(r') \oplus \mathbf{Z}v(r')$, we have $\widetilde{NS}(X) = U_{r'} \perp \mathbf{Z}\pi(r')$ and, up to sign,

$$\pi(r') = 2n(-l' + r'k')e + 2n(m' - s'k')f + (r'm' - l's')H \ \ .$$

Proof. We have $\mathrm{rk}\, U_r^\perp = 1$, which implies that the generator of $U_r^\perp$ is unique up to sign. Moreover, since $\det \widetilde{NS}(X) = -2n$ and $U_r \cong U$ is unimodular, one concludes that $\widetilde{NS}(X) = U_r \perp \mathbf{Z}\Pi(r)$ for some $\Pi(r)$ such that $\langle \Pi(r)^2 \rangle = 2n$. By using the equations (2.3), one can verify directly that $\pi(r)$ has the required properties for $\Pi(r)$,

$$\langle \Pi(r), u(r) \rangle = \langle \Pi(r), v(r) \rangle = 0 \ , \ \ \langle \Pi(r)^2 \rangle = 2n \ \ ,$$

from which we may conclude $\Pi(r) = \pm\pi(r)$. The same argument applies to $U_{r'}$ and $\pi(r')$. $\square$

Lemma 2.4.
$$r'm' - l's' \equiv \pm(rm - ls) \ \ mod\, 2n \ \ .$$

Proof. Since $\mathrm{rk}\, T(X) = 19$, we see that $O_{Hodge}(T(X), \mathbf{C}\omega_X) = \langle \pm \mathrm{id} \rangle$, which implies $\varphi_{T(X)^*/T(X)} = \pm\mathrm{id}$. Since $T(X)$ and $\widetilde{NS}(X)$ are primitive and orthogonal to each other in the unimodular lattice $\tilde{H}(X, \mathbf{Z})$, we have $\varphi|_{\widetilde{NS}(X)^*/\widetilde{NS}(X)} = \pm\mathrm{id}$. Now, using the explicit form of $\pi(r)$, we have

$$\widetilde{NS}(X)^*/\widetilde{NS}(X) = \langle \frac{\pi(r)}{2n} \bmod \widetilde{NS}(X) \rangle = \big\langle (rm - ls)\frac{H}{2n} \bmod \widetilde{NS}(X) \big\rangle,$$

and similarly for $\pi(r')$,

$$\widetilde{NS}(X)^*/\widetilde{NS}(X) = \langle \frac{\pi(r')}{2n} \bmod \widetilde{NS}(X) \rangle = \big\langle (r'm' - l's')\frac{H}{2n} \bmod \widetilde{NS}(X) \big\rangle.$$

Moreover, since $\varphi : \widetilde{NS}(X) \xrightarrow{\sim} \widetilde{NS}(X)$ is an isometry and $\varphi(U_r) = U_{r'}$, we have

$$\varphi(\pi(r)) = \pm \pi(r') \ \ .$$

In particular, in $\widetilde{NS}(X)^*/\widetilde{NS}(X)$, we have

$$\varphi((rm - ls)\frac{H}{2n}) = \pm (r'm' - l's')\frac{H}{2n} \ \ .$$

On the other hand, by $\varphi|_{\widetilde{NS}(X)^*/\widetilde{NS}(X)} = \pm\mathrm{id}$, we have

$$\varphi((rm - ls)\frac{H}{2n}) = \pm (rm - ls)\frac{H}{2n} \ \ .$$

Since $\frac{H}{2n}$ is a generator of $\widetilde{NS}(X)^*/\widetilde{NS}(X) \simeq \mathbf{Z}/2n$, we finally obtain

$$r'm' - l's' \equiv \pm(rm - ls) \ \bmod 2n \ \ . \quad \square$$

Here for convenience we list the explicit equations which follow from the relations in (2.3);

$$rs = n \ \ , \ \ lm = nk^2 \ \ , \ \ -mr - ls + 2nk = 1 \tag{2.4}$$

$$r's' = n \ \ , \ \ l'm' = n(k')^2 \ \ , \ \ -m'r' - l's' + 2nk' = 1 \tag{2.5}$$

Now our proof of (2) in Proposition 2.2 will be completed by the following purely arithmetical lemma:

Lemma 2.5. *Let $r, s, l, m, k, r', s', l', m', k'$ be natural numbers satisfying (2.4) and (2.5). Then the following hold:*

(1) If $m'r' - l's' \equiv mr - ls \ mod\ 2n$, then $r' = r$.

(2) If $m'r' - l's' \equiv -(mr - ls) \ mod\ 2n$, then $r' = s$.

Proof of (1). From the equations (2.4), (2.5), we have

$$mr + ls \equiv -1 \bmod 2n \ , \ \ m'r' + l's' \equiv -1 \bmod 2n \ . \tag{2.6}$$

Then adding these equations with the given one, $m'r' - l's' \equiv mr - ls \bmod 2n$, we may conclude

$$mr \equiv m'r' \bmod n \ , \ \ ls \equiv l's' \bmod n \ . \tag{2.7}$$

Consider the prime decomposition of n, and write

$$rs = r's' = n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \ .$$

Note that from (2.6), we have $(r, s) = 1$. Therefore, if $p_i|r$ then $p_i^{e_i}|r$, and similarly if $p_i|r'$, then $p_i^{e_i}|r'$. Therefore if we show $\{p : \text{prime s.t. } p|r\} = \{p : \text{prime s.t. } p|r'\}$, then the desired relation $r = r'$ follows.

To show that the set of primes divisors of r and r' are equal, let us assume that there is a prime number p such that $p|r$ and $p\not| r'$ (or vice versa, if necessary). Then, since p divides $n = rs = r's'$, we have $p|s'$. Since p divides s' and n, we see $p|ls$ from the second equation of (2.7). Now note that $p\not| s$ by $(r, s) = 1$ and $p|r$. Therefore we have $p|l$. By $p|r$ and $p|l$, we have $p|mr + ls$, whence $p|(-1)$ by $p|n$ and the formula (2.6), a contradiction. □

Proof of (2). ¿From the given equation, $m'r' - l's' \equiv -mr + ls \bmod 2n$, and the third equations in (2.4) and (2.5), we have $-2mr \equiv 1 + m'r' - l's' \equiv -2l's' \bmod 2n$. Similarly, we have $-2m'r' \equiv 1 + mr - ls \equiv -2ls \bmod 2n$. To summarize, we have

$$mr \equiv l's' \bmod n \ , \ \ ls \equiv m'r' \bmod n \ .$$

Now assume $r' \neq s$, there is a prime p such that $p|s$ but $p\not| r'$ (or vice versa if necessary). Then the same argument as in (1) leads us to a contradiction $-1 \equiv m'r' + l's' \equiv 0 \bmod p$. □

Now our proof of Proposition 2.2 is completed. □

§3. First Application

In this section, we describe the set of FM partners of a polarized K3 surface of Picard number 1 and of degree $2n$ in terms of a certain covering of the moduli of polarized K3 surfaces of degree $2n$. Note that if X is a K3 surface of Picard number 1 and of degree $2n$, then so are the FM partners Y of X by Lemma 1.5.

By a quasi-polarized K3 surface of degree $2n$, we mean a pair (X, H) where X is a K3 surface and H is a primitive nef and big line bundle on X with $(H^2) = 2n$. Choose a primitive embedding $\langle 2n \rangle \to \Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ so that the image, denoted by $\mathbf{Z}h$, lies in the first copy of U. An isomorphism $\mu : H^2(X, \mathbf{Z}) \simeq \Lambda_{K3}$ with $\mu(H) = h$ is called an admissible marking, and the triple (X, H, μ) is called a marked quasi-polarized K3 surface of degree $2n$. Let

$$\Lambda_n := \langle 2n \rangle^\perp = \langle -2n \rangle \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \ \subset \Lambda_{K3},$$

$$\mathcal{D}_n := \{[\omega] \in \mathbf{P}(\Lambda_n \otimes \mathbf{C}) | (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\}.$$

The space $\mathcal{D}_n$ is a bounded symmetric domain with two connected components. By the surjectivity of the period mapping and the uniqueness of primitive embedding, $\mathcal{D}_n$ consists of the period points $[\mu(\omega_X)]$ of marked quasi-polarized K3 surfaces (X, H, μ).

Let $\iota : O(\Lambda_n) \to Aut(\mathcal{D}_n)$ be the natural homomorphism induced by $\Lambda_n \to \Lambda_n \otimes \mathbf{C}$. Consider the groups

$$\tilde{\Gamma}_n := Im\ \iota, \quad \Gamma_n := \{g \in \tilde{\Gamma}_n | \exists f \in O(\Lambda_{K3}) \text{ s.t. } f(h) = h,\ g = \iota(f|_{\Lambda_n})\},$$

and the quotient spaces

$$\tilde{\mathcal{P}}_n := \mathcal{D}_n/\tilde{\Gamma}_n\,,\ \mathcal{P}_n := \mathcal{D}_n/\Gamma_n\,.$$

Since $\tilde{\Gamma}_n$ and Γ_n are arithmetic groups acting on the bounded symmetric domain $\mathcal{D}_n$, the quotient spaces $\tilde{\mathcal{P}}_n$ and $\mathcal{P}_n$ are both quasi-projective by [BB]. The second space $\mathcal{P}_n$ is the period domain which parameterizes the isomorphism classes of quasi-polarized K3 surfaces (X, H) of degree $2n$. Here $(X, H) \simeq (X', H')$ if and only if there is $\sigma : X \simeq X'$ such that $\sigma^* H' = H$.

Lemma 3.1. *The group Γ_n is a normal subgroup of $\tilde{\Gamma}_n$ and*

$$\tilde{\Gamma}_n/\Gamma_n \simeq (\mathbf{Z}/2)^{\oplus(\tau(n)-1)}\,.$$

Proof. By [Ni, Theorem 1.14.2] we have an exact sequence

$$1 \to O(\Lambda_n)^* \to O(\Lambda_n) \to O(A_{\Lambda_n}) \to 1$$

where $O(\Lambda_n)^*$ denotes the kernel of the third arrow. We have

$$O(A_{\Lambda_n}) = O(\Lambda_n^*/\Lambda_n) \simeq O(\mathbf{Z}h^*/\mathbf{Z}h) \simeq (\mathbf{Z}/2)^{\oplus\tau(n)}.$$

Moding out the exact sequence above by $\pm id$ (given that $n > 1$), we get

$$1 \to O(\Lambda_n)^* \to O(\Lambda_n)/\pm id \to (\mathbf{Z}/2)^{\oplus(\tau(n)-1)} \to 1.$$

Now our assertion follows from the natural isomorphisms

$$\tilde{\Gamma}_n \simeq O(\Lambda_n)/\pm id, \quad \Gamma_n \simeq O(\Lambda_n)^*. \quad \square$$

Therefore the natural morphism

$$\varphi : \mathcal{P}_n = \mathcal{D}_n/\Gamma_n \to \tilde{\mathcal{P}}_n = \mathcal{D}_n/\tilde{\Gamma}_n$$

is Galois with Galois group $(\mathbf{Z}/2)^{\oplus(\tau(n)-1)}$.

Let $\mathcal{D}_n^1 \subset \mathcal{D}_n$ denote the subset consisting of the period points $[\mu(\omega_X)]$ of marked polarized K3 surfaces (X, μ) of degree $2n$ and of $\rho(X) = 1$. Note that H is uniquely determined by X and is ample when $\rho(X) = 1$. This subset $\mathcal{D}_n^1$ is the complement of a countable union of of proper closed subsets in $\mathcal{D}_n$ [Og2].

Consider the restriction of φ to the Picard number 1 part

$$\mathcal{P}_n^1 := \mathcal{D}_n^1/\Gamma_n \to \tilde{\mathcal{P}}_n^1 := \mathcal{D}_n^1/\tilde{\Gamma}_n\,.$$

Note that if $[\mu(\omega_X)] \in \mathcal{D}_n^1$ for (X, H, μ), then $\rho(X) = 1$ and $\mu(T(X)) = \Lambda_n$.

Proposition 3.2. *There is a natural 1-1 correspondence between the sets* $\tilde{\mathcal{P}}_n^1$ *and* $\{FM(X) | \rho(X) = 1,\ deg\ X = 2n\}$.

Proof. By Lemma 1.5, if X is of $\rho(X) = 1$ and of degree $2n$, then so are the FM partners Y of X. Given a marked K3 surface (X, μ) with a period point $[\mu(\omega_X)] \in \mathcal{D}_n^1$, we associate to it the set $FM(X)$. Let $g \in \tilde{\Gamma}_n$. Then the period point $g[\mu(\omega_X)]$ represents a second marked K3 surface (X', μ') in $\mathcal{D}_n^1$ with $g[\mu(\omega_X)] = [\mu'(\omega_{X'})]$. But $\mu(T(X)) = \Lambda_n = \mu'(T(X'))$. It follows that $(T(X), \mathbf{C}\omega_X) \simeq (T(X'), \mathbf{C}\omega_{X'})$, hence $FM(X) = FM(X')$. Thus we have a well-defined map

$$\tilde{\mathcal{P}}_n^1 \to \{FM(X) | \rho(X) = 1,\ deg\ X = 2n\},\ [\mu(\omega_X)] \mapsto FM(X).$$

This map is surjective.

We now show that the map is injective. Let (X, μ), (X', μ') be two marked K3 surfaces of degree $2n$ and of Picard number 1. Assume that $FM(X) = FM(X')$. Then, there is a Hodge isometry $f : (T(X), \mathbf{C}\omega_X) \to (T(X'), \mathbf{C}\omega_{X'})$. This means that $g' := \mu' \circ f \circ \mu^{-1} : \Lambda_n \to \Lambda_n$ is an element of $O(\Lambda_n)$ with $\iota(g')[\mu(\omega_X)] = [\mu'(\omega_{X'})]$. Hence $\iota(g') \in \tilde{\Gamma}_n$. This shows that the period points of (X, μ) and (X', μ') are in the same $\tilde{\Gamma}_n$-orbit in $\mathcal{D}_n$. $\square$

Proposition 3.3. *The map* $\varphi : \mathcal{P}_n \to \tilde{\mathcal{P}}_n$ *is unramified over* $\tilde{\mathcal{P}}_n^1$ *and the fiber of* φ *at the point* $\tilde{\Gamma}_n[\mu(\omega_X)] \in \tilde{\mathcal{P}}_n^1$, *corresponding to* $FM(X)$, *is naturally isomorphic to the set* $FM(X)$.

Proof. The Galois group $\tilde{\Gamma}_n/\Gamma_n$ acts transitively on each fiber naturally. Thus each fiber, as a set, has cardinality at most $|\tilde{\Gamma}_n/\Gamma_n| = 2^{\tau(n)-1}$. Now suppose that $\rho(X) = 1$. Define the map $FM(X) \to \varphi^{-1}\tilde{\Gamma}_n[\mu(\omega_X)]$ as follows.

Let $X' \in FM(X)$. One can choose an admissible marking μ' of X'. Then $\mu'(T(X)) = \Lambda_n$. We set $X' \mapsto \Gamma_n[\mu'(\omega_{X'})] \in \mathcal{P}_n$. If $\tilde{\mu}$ is another such marking of X', then we have $f := \mu' \circ \tilde{\mu}^{-1} \in O(\Lambda_{K3})$ with $f(h) = h$. Since $\mu'(T(X')) = \tilde{\mu}(T(X')) = \Lambda_n$, and $[\mu'(\omega_{X'})], [\tilde{\mu}(\omega_{X'})] \in \mathcal{D}_n$, it follows that $g = f|_{\Lambda_n} \in \Gamma_n$. This shows that $\Gamma_n[\tilde{\mu}(\omega_{X'})] = \Gamma_n[\mu'(\omega_{X'})]$. Thus $\Gamma_n[\mu'(\omega_{X'})]$ is independent of the choice of marking μ'.

Since $(T(X), \mathbf{C}\omega_X) \simeq (T(X'), \mathbf{C}\omega_{X'})$, the previous injectivity argument shows that $g[\mu(\omega_X)] = [\mu'(\omega_{X'})]$ for some $g \in \tilde{\Gamma}_n$. Thus $\tilde{\Gamma}_n[\mu(\omega_X)] = \tilde{\Gamma}_n[\mu'(\omega_{X'})]$. Hence for all $X' \in FM(X)$, the orbits $\Gamma_n[\mu'(\omega_{X'})] \in \mathcal{D}_n/\Gamma_n$ are in the same fiber $\varphi^{-1}[\mu(\omega_X)]$. Thus the map $FM(X) \to \varphi^{-1}(\tilde{\Gamma}_n[\mu(\omega_X)])$ is well-defined.

Next we shall show that this map is injective. Suppose that $\Gamma_n[\mu'(\omega_{X'})] = \Gamma_n[\mu(\omega_X)]$, i.e. $[\mu'(\omega_{X'})] = g[\mu(\omega_X)]$ for some $g \in \Gamma_n$. Then $g = \iota(f|_{\Lambda_n})$ for some $f \in O(\Lambda_{K3})$ with $f(h) = h$. This shows that $\mu^{-1} \circ f \circ \mu' : H^2(X', \mathbf{Z}) \to H^2(X, \mathbf{Z})$ is an isomorphism sending $(T(X'), \mathbf{C}\omega_{X'})$, $NS(X') = \mathbf{Z}{\mu'}^{-1}(h)$ to $(T(X), \mathbf{C}\omega_X)$, $NS(X) = \mathbf{Z}\mu^{-1}(h)$ respectively. Hence $X' \simeq X$ by the Torelli Theorem ([PSS], see also [BPV]) . Recall that $|\varphi^{-1}(\tilde{\Gamma}_n[\mu(\omega_X)])| \le 2^{\tau(n)-1}$ by Lemma 3.1 and $|FM(X)| = 2^{\tau(n)-1}$ by Corollary 1.6 (2). So, the map above is surjective as well.

Now we have:

$$|\varphi^{-1}[\mu(\omega_X)]| = 2^{\tau(n)-1} = deg\, \varphi\, .$$

Thus, φ is unramified over $\tilde{\mathcal{P}}_n^1$. $\square$

Combining all together, we obtain the following:

Corollary 3.4. *The morphism $\varphi : \mathcal{P}_n^1 \to \tilde{\mathcal{P}}_n^1$ defines a principal homogeneous space structure of the 2-elementary abelian group $\tilde{\Gamma}_n/\Gamma_n \simeq (\mathbf{Z}/2)^{\oplus(\tau(n)-1)}$ on $FM(X)$ of a K3 surface X with $\rho(X) = 1$ and $\deg X = 2n$.*

As it is remarked before, the quotient spaces $\mathcal{P}_n$ and $\tilde{\mathcal{P}}_n$ are quasi-projective. Thus there exist projective compactifications $\overline{\mathcal{P}_n}$, $\overline{\tilde{\mathcal{P}}_n}$, and a morphism

$$\varphi : \overline{\mathcal{P}_n} \to \overline{\tilde{\mathcal{P}}_n}.$$

Question 3.5. *What is the ramification locus like? Note that this locus is Zariski closed subset which lies in the complement of the dense subset $\tilde{\mathcal{P}}_n^1$.*

§4. Second Application

Let X be a K3 surface. We shall call a Mukai vector $(r, H, s) \in \tilde{NS}(X)$ *special* if H is ample and *primitive*, and that $r > 0$, $(r, s) = 1$, and $(H^2) = 2rs$, hold.

As our second application of Theorem 2.1, we shall show the following refinement of Theorem 1.3:

Proposition 4.1. *Let X be K3 surface and $Y \in FM(X)$. Then there is a special Mukai vector (r, H, s) such that $Y \simeq M_H((r, H, s))$.*

Proof. By Theorem 1.3, we have $Y \simeq M_{H'}((a, H', b))$ where H' is an ample line bundle and a and b are positive integers such that $(a, b) = 1$ and $({H'}^2) = 2ab$. Write $H' = mH$ where H is primitive and m a positive integer. Consider the Kuranishi family of the polarized K3 surface (X, H), namely a smooth projective family

$$\pi : (\mathcal{X}, \mathcal{H}) \to \mathcal{K}$$

over some 19-dimensional polydisk $\mathcal{K}$ with central fiber $(\mathcal{X}_0, \mathcal{H}_0) = (X, H)$. As it is well known, the set

$$\mathcal{K}^1 := \{t \in \mathcal{K} | NS(\mathcal{X}_t) = \mathbf{Z}\mathcal{H}_t\}$$

is everywhere dense in $\mathcal{K}$. (See eg. [Og2]). We fix a marking $R^2\pi_*\mathbf{Z}_{\mathcal{X}} \simeq \Lambda_{K3} \times \mathcal{K}$ so that $\mathcal{H}_t \mapsto (h, t)$, for some constant vector $h \in \Lambda_{K3}$ with $(h^2) = 2n$.

Let

$$f : \mathcal{Y} \to \mathcal{K},$$

be the relative moduli space of semi-stable sheaves with respect to $\mathcal{H}$ with Mukai vectors $(a, m\mathcal{H}_t, b) \in \widetilde{NS}(\mathcal{X}_t)$. This f is projective and $\mathcal{Y}_t = M_{m\mathcal{H}_t}((a, m\mathcal{H}_t, b)) \in FM(\mathcal{X}_t)$ by Lemma 1.4.

Let $\mathcal{L}$ be an f-ample primitive invertible sheaf on $\mathcal{Y}$. By Lemma 1.5, $NS(\mathcal{Y}_t)$ is isometric to $NS(\mathcal{X}_t) = \mathbf{Z}\mathcal{H}_t$ for $t \in \mathcal{K}^1$. Therefore $\mathcal{L}$ is of degree $2n$.

For $t \in \mathcal{K}^1$, the fiber $\mathcal{X}_t$ has Picard number 1. In this case, we know that

$$\mathcal{Y}_t \simeq \mathcal{M}_{\mathcal{H}_t}((r_t, \mathcal{H}_t, s_t))$$

for some Mukai vector $(r_t, \mathcal{H}_t, s_t)$ which is special, by Theorem 2.1.

Since $2r_t s_t = (\mathcal{H}_t^2) = (H^2) = 2n$ is constant and $\mathcal{K}^1$ is dense in $\mathcal{K}$, there are positive integers r,s and a sequence $\{t_k\}$ in $\mathcal{K}^1$ such that $r_{t_k} = r$ and $s_{t_k} = s$ are constant and $\lim_{k\to\infty} t_k = 0$.

Now consider the moduli space $Y' := M_H((r,H,s)) = \overline{M}_H((r,H,s))$ over $X = \mathcal{X}_0$. Y' is a FM partner of X by Theorem 1.3. Once again, by Lemma 1.4, we can extend this to a relative moduli space of stable sheaves over $\mathcal{K}$:

$$f' : \mathcal{Y}' \to \mathcal{K}$$

which is projective over $\mathcal{K}$, and such that $\mathcal{Y}'_0 = Y'$, that $\mathcal{Y}'_t = M_{\mathcal{H}_t}((r, \mathcal{H}_t, s))$ is a FM partner of $\mathcal{X}_t$, and that there exists an f'-ample primitive invertible sheaf $\mathcal{L}'$ of degree $2n$.

Since $\mathcal{K}$ is a polydisk, we can choose markings

$$R^2 f_* \mathbf{Z}_{\mathcal{Y}} \simeq \Lambda_{K3} \times \mathcal{K}, \qquad R^2 f'_* \mathbf{Z}_{\mathcal{Y}'} \simeq \Lambda_{K3} \times \mathcal{K}$$

so that $\mathcal{L}_t \mapsto (l,t)$ and $\mathcal{L}'_t \mapsto (l,t)$ respectively, for some constant primitive vector $l \in \Lambda_{K3}$ with $(l^2) = 2n$. Here we used the fact that $O(\Lambda_{K3})$ acts on the set of primitive vectors l with $(l^2) = 2n$ transitively. Corresponding to f and f', we have then two period maps:

$$\eta : \mathcal{K} \to \mathcal{P}_n\,, \qquad \eta' : \mathcal{K} \to \mathcal{P}_n\,.$$

Here $\mathcal{P}_n$ is the period domain defined in Section 3. Recall that this space is quasi-projective, hence Hausdorff.

At t_k, we have $\mathcal{Y}_{t_k} \simeq \mathcal{Y}'_{t_k}$. Since both sides have Picard number 1, we also have $(\mathcal{Y}_{t_k}, \mathcal{L}_{t_k}) \simeq (\mathcal{Y}'_{t_k}, \mathcal{L}'_{t_k})$. It follows that $\eta(t_k) = \eta'(t_k)$ for all k.

Since $\mathcal{P}_n$ is Hausdorff, it follows that $\eta(0) = \eta'(0)$. This means $(\mathcal{Y}_0, \mathcal{L}_0) \simeq (\mathcal{Y}'_0, \mathcal{L}'_0)$. Therefore $\mathcal{Y}_0 \simeq \mathcal{Y}'_0$, i.e. $Y \simeq Y' = M_H((r,H,s))$. $\square$

References

[BB] W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Annals of Math. **84** (1966), 442–528.

[BPV] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Springer-Verlag (1984).

[Cs] J. W. S. Cassels, *Rational quadratic forms*, Academic Press (1978).

[GM] S.I. Gelfand, Y.I. Manin, *Methods of homological algebra*, Springer-Verlag (1991).

[HLOY] S. Hosono, B. Lian, K. Oguiso, S.T. Yau, *Counting Fourier-Mukai partners and applications, math.AG/0202014, submitted*.

[HL] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Vieweg (1997).

[Ma1] M. Maruyama, *Stable vector bundles on an algebraic surface*, Nagoya Math. J. **58** (1975), 25–68.

[Ma2] M. Maruyama, *Moduli of stable sheaves, II*, J. Math. Kyoto Univ. **18** (1978), 557–614.

[Mu1] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984), 101–116.

[Mu2] S. Mukai, *On the moduli space of bundles on K3 surfaces I, in Vector bundles on algebraic varieties*, Oxford Univ. Press (1987), 341–413.

[Ni] V. V. Nikulin, *Integral symmetric bilinear forms and some of their geometric applications*, Math. USSR Izv. **14** (1980), 103–167.

[Og1] K. Oguiso, *K3 surfaces via almost-primes, math.AG/0110282*, Math. Res. Lett. **9** (2002), 47–63.

[Og2] K. Oguiso, *Local families of K3 surfaces and applications, (math.AG/0011258 and math.AG/0104049)*, J. Alg. Geom. to appear.

[Or] D. Orlov, *Equivalences of derived categories and K3 surfaces, math.AG/9606006*, J. Math.Sci. **84** (1997), 1361–1381.

[PPS] I. Pjatetskii-Shapiro, I. Shafarevich, *A Torelli theorem for algebraic surfaces of type K3*, Math. USSR-Izv **5** (1971), 547–587.

[St] P. Stellari, *Some remarks about the FM-partners of K3 surfaces with small Picard number*, math.AG/0205126.

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