c=2 Rational Toroidal Conformal Field Theories via the Gauss Product

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Abstract. We find a concise relation between the moduli τ, ρ of a rational Narain lattice $\Gamma(\tau, \rho)$ and the corresponding momentum lattices of left and right chiral algebras via the Gauss product. As a byproduct, we find an identity which counts the cardinality of a certain double coset space defined for isometries between the discriminant forms of rank two lattices.

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§0. Introduction and main results

(0-1) Introduction – some background – Since the 80's, string compactifications on real d-dimensional tori T^d have been a source of several important ideas, such as orbifold [DHVW], T-duality [KY], etc., which later have been successfully generalized to string compactifications on more non-trivial geometries of Calabi-Yau manifolds (see, e.g. [GSW][GY][Po] for references). One thing we learned from these important developments is that certain properties of string theory (conformal field theory) often translate into deep and interesting geometrical insights when interpreted in the language of σ-models (geometry). Gepner's correspondence between N=2 superconformal field theory (SCFT) and σ-models on Calabi-Yau manifolds is one such well-known example. Mirror Symmetry is yet another. In this paper, we will study an important property, known as *rationality*, of conformal string compactifications on T^d. That rationality translates into interesting questions on the geometry side was brought to light in recent works of [Mo][GV]. For example it was found in [Mo] that rationality of a CFT on an elliptic curve E_τ ($\cong T^2$) implies that E_τ has non-trivial endomorphisms, i.e. E_τ is of CM (complex multiplication) type. Several other deep questions on rationality of the string compactification on general Calabi-Yau manifolds have also been raised in [Mo][GV], although their answers are still conjectural.

In this paper we will restrict our attention to CFTs on T^2. In this case, as shown by G. Moore [Mo], one has the following characterization of rationality: a CFT on $T^2(=E_\tau)$ is rational if and only if the parameters τ, ρ are elements of the imaginary quadratic field $\mathbf{Q}(\sqrt{D})$ for some $D<0$, where τ is the complex structure and $\rho = B + \sqrt{-1}Vol(E_\tau)$ is the complexified Kähler modulus. This allows one, in principle, to parameterize rational toroidal CFTs by such pairs τ, ρ. In this paper, we further the investigation along this line by studying precisely *how* this parameterization can be realized. As a result, we find a precise correspondence between a Narain lattice $\Gamma(\tau,\rho)$ and a RCFT triple $(\Gamma_l, \Gamma_r, \varphi)$, which consists of a pair of momentum lattices for chiral algebras plus a gluing map φ. In our view, this is one step toward understanding the geometry of rational CFTs on T^2, and more generally, on Calabi-Yau manifolds. The correspondence above is realized precisely by means of the composition law of Gauss on primitive binary quadratic forms *plus* an important extension to the non-primitive forms. The latter is crucial if one wishes to allow arbitrary discriminants D.

Our result is clearly relevant to the classification problem of rational CFTs with $c=2$ (see for example [DW] and references therein for developments on this problem), which generalizes the old and well-known classification of $c=1$ rational CFTs given in [Gi][Ki1]. However we restrict ourselves in this paper to rational CFTs on T^2, which are among the best known and well-studied examples in string theory (for which we refer the readers to [HMV][GSW][Po][Ki2] and references therein).

(0-2) Main results. Our main results are:

Theorem 4.10, **Theorem 4.11**, **Theorem 5.7** and **Theorem 5.8**,

where we obtain a complete description of rational toroidal conformal field theories in terms of the Gauss product on the classes of binary quadratic forms. The general results are summarized and restated a little differently in **Summary 5.12.**

In section 1, we briefly summarize the classical theory of binary quadratic forms and the Gauss product. The basics of the Gauss product are summarized in Appendix A, where we extend the product to non-primitive quadratic forms. In section 2, we introduce Narain lattices and their moduli space. We also describe the so-called T-duality group. This section is meant for setting up notations and reviewing some well-known results (see e.g. [GSW][Po][Ki2] and references therein for the original works). In section 3, we define RCFTs on T^2 (rational toroidal CFT) and summarize their characterizations given in [HMV] (see also [Mo][Wa][GV] for a more recent perspective). We then state our classification problem (after Proposition 3.4), and study the first part of this problem using some results of [Ni]. These results were first used in [Mo] to study of rational toroidal CFTs and rational CFTs on singular K3 surfaces, i.e. K3 surfaces of maximal Picard number 20 [SI]. In section 4, we discuss our classification in the case when the relevant lattices are primitive (which correspond to primitive quadratic forms). In section 5, we extend it to the general non-primitive case. In subsection (5-4), we will see that the diagonal RCFTs obtained in [GV] fit into our list of RCFTs in a natural way. A summary of the classification is given in **Summary 5.12.**

In our classification, the classical theory of the binary quadratic forms of Dirichlet and Gauss will come into play in an interesting and essential way (**Lemma 4.6** and **Lemma 5.1**). The reader can get a quick glance of this in examples given in subsection (0-3) below (primitive case), and also in (5-5) (non-primitive case).

(0-3) Example ($D = -39$). It will be helpful to see how our classification works in this example now with details given later. Here we present two tables: One is the product table of the class group $Cl(D)$ (cf. Theorem 1.1) of discriminant $D = -39$, and the other is the table listing the RCFT data $(\Gamma_l, \Gamma_r, \varphi)$ against the Narain lattices $\Gamma(\tau_{\mathcal{C}_i}, \rho_{\mathcal{C}_j})$.

To make the first table, let us note that the class group $Cl(D)$ here consists of the following four $SL_2\mathbf{Z}$ equivalence classes of binary quadratic forms:

$$\mathcal{C}_1 = [Q(1,1,10)] \ , \ \mathcal{C}_2 = [Q(2,1,5)] \ , \ \mathcal{C}_3 = [Q(2,-1,5)] \ , \ \mathcal{C}_4 = [Q(3,3,4)] \ .$$

The notation $Q(a,b,c)$ abbreviates the quadratic form $f(x,y) = ax^2 + bxy + cy^2$. A quadratic form can be identified with a lattice with a chosen ordered basis in which the bilinear form is given by the matrix $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$. Under this identification, an $SL_2\mathbf{Z}$ equivalence class $\mathcal{C}$ of quadratic forms is nothing but an isomorphism class of lattices equipped with orientations, while a $GL_2\mathbf{Z}$ equivalence class $\bar{\mathcal{C}}$ of quadratic forms is nothing but an isomorphism class of lattices without orientations. Now we write the product table of $Cl(D)$:

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
$\mathcal{C}_1$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
$\mathcal{C}_2$	$\mathcal{C}_2$	$\mathcal{C}_4$	$\mathcal{C}_1$	$\mathcal{C}_3$
$\mathcal{C}_3$	$\mathcal{C}_3$	$\mathcal{C}_1$	$\mathcal{C}_4$	$\mathcal{C}_2$
$\mathcal{C}_4$	$\mathcal{C}_4$	$\mathcal{C}_3$	$\mathcal{C}_2$	$\mathcal{C}_1$

Table 1. Table of Gauss product ($D = -39$).

Table 2 lists the data for the RCFTs on T^2. The data that determines an RCFT consists of its momentum lattices Γ_l, Γ_r with determinant $-D$, together with an isometry of their discriminant groups φ. This isometry "glues" together the left and right sector of the RCFT. Equivalently, the data $\Gamma_l, \Gamma_r, \varphi$ can also be described in terms of a Narain lattice $\Gamma(\tau, \rho)$ which contains Γ_l, Γ_r. The correspondence between triples $(\Gamma_l, \Gamma_r, \varphi)$ and Narain lattices $\Gamma(\tau, \rho)$, $\tau, \rho \in \mathbf{Q}(\sqrt{D})$, are shown in Table 2. The key observation here is that we have $\mathcal{C} = C_i * C_j^{-1}, \mathcal{C}' = \mathcal{C}_i * \mathcal{C}_j$ for the triple $(\Gamma_{\bar{\mathcal{C}}}, \Gamma_{\bar{\mathcal{C}}'}, \varphi)$ that corresponds to a Narain Lattice $\Gamma(\tau_{\mathcal{C}_i}, \rho_{\mathcal{C}_j})$.

To describe the correspondence more precisely, let us associate to each quadratic form $Q(a,b,c)$ the complex number $\tau_{Q(a,b,c)} = \rho_{Q(a,b,c)} = \frac{b+\sqrt{D}}{2a}$ ($D = b^2 - 4ac$). Note that since $D < 0$, these complex numbers lie in the upper half plane $\mathbf{H}_+$. Given an $SL_2\mathbf{Z}$ equivalence class of quadratic forms $\mathcal{C} = [Q(a,b,c)]$, let $\tau_{\mathcal{C}} = \rho_{\mathcal{C}}$ be the $SL_2\mathbf{Z}$ orbit of $\tau_{Q(a,b,c)}$. Given a $GL_2\mathbf{Z}$ equivalence class $\bar{\mathcal{C}}$, we denote by $\Gamma_{\bar{\mathcal{C}}}$ a lattice in the corresponding isomorphism class $\bar{\mathcal{C}}$ of lattices. Then Table 2 also describes the data of the RCFTs in terms of Narain lattices $\Gamma(\tau_{\mathcal{C}}, \rho_{\mathcal{C}'})$.

	$\rho_{\mathcal{C}_1}$	$\rho_{\mathcal{C}_2}$	$\rho_{\mathcal{C}_3}$	$\rho_{\mathcal{C}_4}$
$\tau_{\mathcal{C}_1}$	$(\Gamma_{\bar{\mathcal{C}}_1}, \Gamma_{\bar{\mathcal{C}}_1}, id)$	$(\Gamma_{\bar{\mathcal{C}}_3}, \Gamma_{\bar{\mathcal{C}}_2}, id)$	$(\Gamma_{\bar{\mathcal{C}}_2}, \Gamma_{\bar{\mathcal{C}}_3}, id)$	$(\Gamma_{\bar{\mathcal{C}}_4}, \Gamma_{\bar{\mathcal{C}}_4}, id)$
$\tau_{\mathcal{C}_2}$	$(\Gamma_{\bar{\mathcal{C}}_2}, \Gamma_{\bar{\mathcal{C}}_2}, id)$	$(\Gamma_{\bar{\mathcal{C}}_1}, \Gamma_{\bar{\mathcal{C}}_4}, \varphi_1)$	$(\Gamma_{\bar{\mathcal{C}}_4}, \Gamma_{\bar{\mathcal{C}}_1}, \varphi_1^{-1})$	$(\Gamma_{\bar{\mathcal{C}}_3}, \Gamma_{\bar{\mathcal{C}}_3}, \varphi_2)$
$\tau_{\mathcal{C}_3}$	$(\Gamma_{\bar{\mathcal{C}}_3}, \Gamma_{\bar{\mathcal{C}}_3}, id)$	$(\Gamma_{\bar{\mathcal{C}}_4}, \Gamma_{\bar{\mathcal{C}}_1}, \varphi_1^{-1})$	$(\Gamma_{\bar{\mathcal{C}}_1}, \Gamma_{\bar{\mathcal{C}}_4}, \varphi_1)$	$(\Gamma_{\bar{\mathcal{C}}_2}, \Gamma_{\bar{\mathcal{C}}_2}, \varphi_2)$
$\tau_{\mathcal{C}_4}$	$(\Gamma_{\bar{\mathcal{C}}_4}, \Gamma_{\bar{\mathcal{C}}_4}, id)$	$(\Gamma_{\bar{\mathcal{C}}_2}, \Gamma_{\bar{\mathcal{C}}_3}, \varphi_2)$	$(\Gamma_{\bar{\mathcal{C}}_3}, \Gamma_{\bar{\mathcal{C}}_2}, \varphi_2)$	$(\Gamma_{\bar{\mathcal{C}}_1}, \Gamma_{\bar{\mathcal{C}}_1}, \varphi_3)$

Table 2. Table of RCFT data. RCFT data $(\Gamma_l, \Gamma_r, \varphi)$ are listed against the Narain lattices $\Gamma(\tau_{\mathcal{C}_i}, \rho_{\mathcal{C}_j})$. Boxed entries define the same RCFT up to worldsheet parity involution.

The lattices $\Gamma_{\bar{\mathcal{C}}_i}$ in Table 2 are not all inequivalent. In fact, it is easy to verify that there are only three equivalence classes $\bar{\mathcal{C}}_1, \bar{\mathcal{C}}_2 = \bar{\mathcal{C}}_3, \bar{\mathcal{C}}_4$. For brevity, we do not describe here the gluing data $\varphi_1, \varphi_2, \varphi_3$ explicitly, but we will discuss their general construction later.

Note that the first column (or row) corresponds to the so-called diagonal modular invariants of RCFT, whose characterization has been obtained recently in [GV](see section (5.4)). We observe that this fits naturally into our general classification (see Proposition 5.7).

(0-4) Average one formula. As a corollary to our Main Theorems, we obtain the following "average one" formula for definite lattices. This formula connects, in an interesting way, lattice problems arising from RCFTs to the class group of binary quadratic forms (see Corollary 4.13).

Theorem 0.1. *Let $\mathcal{L}^p(D)$ be the set of isomorphism classes of primitive, definite, even, integral lattices of determinant $-D$ and rank 2. Also let $A_\Gamma = (\Gamma^*/\Gamma, q_\Gamma)$ be the discriminant group Γ^*/Γ equipped with the quadratic form $q_\Gamma : \Gamma^*/\Gamma \to \mathbf{Q}/2\mathbf{Z}$. Then the following formula holds:*

$$\frac{1}{|Sym^2\mathcal{L}^p(D)|} \sum_{(\Gamma,\Gamma')\in \mathrm{Sym}^2\mathcal{L}^p(D)} |O(\Gamma) \setminus Isom(A_\Gamma, A_{\Gamma'})/O(\Gamma')| = 1 \ , \qquad (0.1)$$

where $Isom(A_\Gamma, A_{\Gamma'})$ is the set of isometries $\varphi : A_\Gamma \xrightarrow{\sim} A_{\Gamma'}$ and the double quotient is defined by the natural actions $\varphi \mapsto \bar{h} \cdot \varphi \cdot \bar{g}^{-1}$ of lattice isometries $g \in O(\Gamma)$, $h \in O(\Gamma')$.

Remark. Here $\mathrm{Sym}^2\mathcal{L}^p(D)$ denotes the set of symmetric pairs of elements of $\mathcal{L}^p(D)$. The set $\mathcal{L}^p(D)$ is identified with the set $\widetilde{Cl}(D)$ of $GL_2\mathbf{Z}$ equivalence classes of the quadratic forms. When the determinant $-D(>2)$ is a prime, the elements of $\widetilde{Cl}(D) \cong \mathcal{L}^p(D)$ are all isogeneous (see e.g. [Za, §12]), and we have $\mathrm{Isom}(A_\Gamma, A_{\Gamma'}) = \{\pm 1\}$. Therefore $|O(\Gamma) \setminus \mathrm{Isom}(A_\Gamma, A_{\Gamma'})/O(\Gamma')| = 1$ for all $(\Gamma, \Gamma') \in Sym^2\mathcal{L}^p(D)$. That verifies (0.1) immediately in this special case. In general $\widetilde{Cl}(D) \cong \mathcal{L}^p(D)$ contains more than one isogeny classes, in which case the "average one formula" is a very interesting generalization. Note that $\mathrm{Isom}(A_\Gamma, A_{\Gamma'})$ is empty if Γ and Γ' are not isogeneous (see [Ni]). The average one formula (0.1) for indefinite lattices $\mathcal{L}^p(D)\,(D > 0)$ also follows from the counting problem of general V*-rational* Narain lattices (see [HLOY2]).

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§1. Classical results on quadratic forms

In this section we summarize some classical results on (positive definite) binary quadratic forms following [Za][Ca]. These results turn out to be crucial for our classification problem.

(1-1) Quadratic forms. Let us denote by $Q(a,b,c)$ an integral quadratic form in two variables;

$$Q(a,b,c) : f(x,y) = ax^2 + bxy + cy^2 \quad (a,b,c \in \mathbf{Z}).$$

Unless stated otherwise, a quadratic form will mean a positive definite, integral, quadratic form in two variables. A quadratic form $Q(a,b,c)$ is called *primitive* if $gcd(a,b,c) = 1$. The group $SL_2\mathbf{Z}$ acts on quadratic forms $A : Q(a,b,c) \mapsto Q(a',b',c')$, $(A \in SL_2\mathbf{Z})$, by

$$^{t}A \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} A = \begin{pmatrix} 2a' & b' \\ b' & 2c' \end{pmatrix} \quad . \tag{1.1}$$

Since $A = \pm id$ acts trivially, only the group $PSL_2\mathbf{Z}$ acts effectively. The action leaves the discriminant $D := b^2 - 4ac (< 0)$ unchanged. Two quadratic forms are said to be *properly equivalent* if they are in the same $SL_2\mathbf{Z}$ orbit. We consider the set of properly equivalent classes and denote it by

$$Cl(D) := \{\ Q(a,b,c) : \text{primitive quadratic form}, D = b^2 - 4ac < 0,\ a > 0\ \}/\sim_{SL_2\mathbf{Z}} .$$

It has been known, since Gauss, that $Cl(D)$ is a finite set. Its cardinality is called a *class number*, denoted by $h(D) = |Cl(D)|$. We often write the set of classes by

$$Cl(D) = \{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_{h(D)}\} \ .$$

Similarly the group $GL_2\mathbf{Z}$ acts on quadratic forms by the same formula as (1.1), and two quadratic forms are said to be *improperly equivalent* if they are in the same $GL_2\mathbf{Z}$ orbit. We consider the set of improperly equivalent classes and denote it by

$$\widetilde{Cl}(D) := \{\ Q(a,b,c);\ \text{primitive quadratic form}, D = b^2 - 4ac < 0,\ a > 0\ \}/\sim_{GL_2\mathbf{Z}} .$$

There is obviously a natural surjection $q : Cl(D) \to \widetilde{Cl}(D)$, $\mathcal{C} \mapsto \bar{\mathcal{C}}$, and $q^{-1}(\bar{\mathcal{C}})$ has either one or two classes.

(1-2) Even lattices of rank two. An abstract (integral) lattice $\Gamma = (\Gamma, (*,*))$ is a $\mathbf{Z}$-module equipped with non-degenerate bilinear form $(*,*) : \Gamma \times \Gamma \to \mathbf{Z}$. Let Γ be a lattice. It is called *even* (respectively positive definite) if $(x,x) \in 2\mathbf{Z}$ $(x \neq 0 \Rightarrow (x,x) > 0)$ for $x \in \Gamma$. We denote by $\Gamma(n)$ the lattice whose bilinear form is given by n-times the bilinear form of Γ, i.e. $\Gamma(n) = (\Gamma, n(*,*))$. We say that Γ is *primitive* if $\Gamma = \Gamma'(n)$ for some even lattice Γ' implies that $n = \pm 1$. We denote the dual lattice $\Gamma^*(:= \mathrm{Hom}(\Gamma, \mathbf{Z}))$ and have $|\det \Gamma| = |\Gamma^*/\Gamma|$, which is called the determinant of the lattice Γ. We consider the following set of isomorphism classes of lattices;

$$\mathcal{L}^p(D) := \{\ \Gamma : \text{even, primitive, positive definite lattices of rank two} \\ \text{and } D = -\det\Gamma\ \}/GL_2\mathbf{Z}$$

In this paper, both $\mathbf{\Gamma}$, $[\Gamma]$ denote the isomorphism class of a lattice Γ.

For an even lattice Γ of determinant $-D$ and rank two, choosing a basis u_1, u_2, we may associate a symmetric matrix,

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} := \begin{pmatrix} (u_1,u_1) & (u_1,u_2) \\ (u_2,u_1) & (u_2,u_2) \end{pmatrix} \quad , \tag{1.2}$$

with $a,b,c \in \mathbf{Z}$. If Γ is positive definite, then $4ac - b^2 = -D > 0$ and $a > 0$. Moreover if Γ is primitive, then $gcd(a,b,c) = 1$. Therefore for an even, positive definite, primitive lattice Γ, we may associate a primitive quadratic form $Q(a,b,c)$ by choosing a basis of Γ. It is clear that changing the basis u_1, u_2 results in a quadratic form which is $GL_2\mathbf{Z}$ equivalent to $Q(a,b,c)$. Similarly, since isomorphic lattices have the same $GL_2\mathbf{Z}$ orbit of the matrix (1.2), they also correspond to the same $GL_2\mathbf{Z}$ equivalence class of quadratic forms. Conversely, it is also clear that a $GL_2\mathbf{Z}$ equivalence class $\bar{\mathcal{C}} = \overline{[Q(a,b,c)]} \in \widetilde{Cl}(D)$ defines a unique isomorphism class of lattices $\Gamma = \mathbf{Z}u_1 \oplus \mathbf{Z}u_2$ by (1.2). Therefore we can identify, and will do so hereafter, the set $\mathcal{L}^p(D)$ with the set $\widetilde{Cl}(D)$ under the natural one to one correspondence

$$\bar{\mathcal{C}} = \overline{[Q(a,b,c)]} \leftrightarrow \mathbf{\Gamma} = [\Gamma] \text{ such that (1.2) holds.} \tag{1.3}$$

By this correspondence we often write explicitly,

$$\widetilde{Cl}(D) = \{\mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \cdots, \mathbf{\Gamma}_{\widetilde{h}(D)}\} \qquad (\widetilde{h}(D) = |\widetilde{Cl}(D)|).$$

Also it will be useful to write $\widetilde{Cl}(D) = \{\, q(\mathcal{C}) | \mathcal{C} \in Cl(D)\,\}$, in terms of the natural surjective map $q : Cl(D) \to \widetilde{Cl}(D)$.

(1-3) Class group. We summarize the following nice property of the set $Cl(D)$:

Theorem 1.1. (Gauss) *The set $Cl(D)$ has a commutative, associative, composition law (Gauss product) which makes $Cl(D)$ an abelian group with the unit being the class represented by $Q(1,0,-\frac{D}{4}) : x^2 - \frac{D}{4}y^2$ for $D \equiv 0 mod\ 4$ and $Q(1,1,\frac{1-D}{4}) : x^2 + xy + \frac{1-D}{4}y^2$ for $D \equiv 1 mod\ 4$.*

Note that since D has the shape $b^2 - 4ac$, we have $D \equiv 0$ or 1 mod 4. The precise definition of the Gauss product is summarized in Appendix A. As it is explained there (Proposition A.7), for any given class $\mathcal{C} \in Cl(D)$, we have

$$q^{-1}(q(\mathcal{C})) = \{\, \mathcal{C}, \mathcal{C}^{-1}\,\} \ . \tag{1.4}$$

Hence if $\mathcal{C}^{-1} \neq \mathcal{C}$, then $\mathcal{C}^{-1}$ and $\mathcal{C}$ are improperly equivalent, and thus we have the following relation:

$$\widetilde{h}(D) = \frac{1}{2}\left(h(D) + \#\{\mathcal{C} \in Cl(D) \mid \mathcal{C}^{-1} = \mathcal{C}\ \}\right) \ . \tag{1.5}$$

(1-4) Non-primitive quadratic forms. In our classification problem, lattices which are not necessarily primitive will appear. We define the following sets of equivalence classes:

$$CL(D) := \{ \; Q(a,b,c) : \text{quadratic form}, D = b^2 - 4ac < 0, \; a > 0 \; \}/ \sim_{SL_2\mathbf{Z}} \quad ,$$

and

$$\widetilde{CL}(D) := \{ \; Q(a,b,c) : \text{quadratic form}, D = b^2 - 4ac < 0, \; a > 0 \; \}/ \sim_{GL_2\mathbf{Z}} \quad .$$

Obviously these two sets respectively include $Cl(D)$ and $\widetilde{Cl}(D)$. If $f(x,y)$ is a quadratic form of discriminant D, then $f(x,y) = \lambda g(x,y)$ for some positive integer λ, and some primitive quadratic form $g(x,y)$ of discriminant D/λ^2. From finiteness of $Cl(D/\lambda^2)$, it follows that $CL(D)$ is finite. As before, we have a natural correspondence between improper equivalence classes of quadratic forms and isomorphism classes of lattices. Under this correspondence, we identify the set $\widetilde{CL}(D)$ with

$$\mathcal{L}(D) := \{ \; \Gamma : \text{even, positive definite lattice of rank two and } -D = |\Gamma^*/\Gamma| \; \}/\text{isom.}.$$

There is also a natural map $q : CL(D) \to \widetilde{CL}(D)$, and we have

$$q^{-1}(q(C)) = \{ \, C, \sigma C \, \} \quad ,$$

where $\sigma \in GL_2\mathbf{Z}$ is an involution which is not in $SL_2\mathbf{Z}$. In this paper, we fix the involution σ so that it acts on quadratic forms by

$$\sigma : Q(a,b,c) \mapsto Q(a,-b,c) \quad . \tag{1.6}$$

Note that if $C \in Cl(D)$, then $\sigma C = C^{-1}$ (see Proposition A.7 in Appendix A). As in (1.5), we have:

$$|\widetilde{CL}(D)| = \frac{1}{2} \left(|CL(D)| + \#\{ C \in CL(D) \, | \, \sigma C = C \, \} \right) \quad . \tag{1.7}$$

(1-5) Reduced forms. Each class $\mathcal{C} \in CL(D)$ has a special representative, which is called *the reduced form.* To describe its construction, let us consider an element

$$S_n = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix} \in SL_2\mathbf{Z}.$$

By (1.1) we have

$$S_0 : Q(a,b,c) \mapsto Q(c,-b,a) \quad , \quad S_n S_0 : Q(a,b,c) \mapsto Q(a, b - 2na, c - nb + n^2 a) \quad .$$

Using these two operations, we see that any given class $C \in CL(D)$ contains a quadratic form $Q(a,b,c)$ with $a \le c$ and $-a < b \le a$, and for which we have $-D = 4ac - b^2 \ge 4a^2 - a^2 = 3a^2$, i.e.

$$0 < a \le \sqrt{-D/3} \quad .$$

Moreover, if $a = c$, we can make $0 \le b \le a = c$. To summarize we have

Proposition 1.2. *The set $CL(D)$ is finite. Each class $C \in CL(D)$ can be represented by a unique form $Q(a,b,c)$, called the reduced form, satisfying*
1) $-a < b \le a$, $0 < a \le \sqrt{-D/3}$, $c = \frac{b^2-D}{4a} \in \mathbf{Z}$, $a < c$, *or*
2) $0 \le b \le a$, $0 < a \le \sqrt{-D/3}$, $a = c = \frac{b^2-D}{4a}$.

Note that the uniqueness of the reduced form $Q(a,b,c) \in \mathcal{C}$ for a given class $\mathcal{C} \in CL(D)$ is clear from the fact that $SL_2\mathbf{Z}$ is generated by S_0 and S_1.

§2. Narain lattices and toroidal compactifications

(2-1) Narain lattice. Let $\mathbf{R}^{d,d}$ be the vector space $\mathbf{R}^{2d}$ equipped with the bilinear form $\langle x,x\rangle_{\mathbf{R}^{d,d}} = x_1^2+\cdots+x_d^2-x_{d+1}^2-\cdots-x_{2d}^2$, of signature (d,d). Let $\mathbf{R}^{d,0} \subset \mathbf{R}^{d,d}$ be the subspace of vectors x with $x_{d+1} = \cdots = x_{2d} = 0$. Likewise for $\mathbf{R}^{0,d}$. Then we have an orthogonal decomposition with respect to $\langle\ ,\ \rangle_{\mathbf{R}^{d,d}}$:

$$\mathbf{R}^{d,d} = \mathbf{R}^{d,0} \oplus \mathbf{R}^{0,d} \ . \tag{2.2}$$

Let U be the rank two lattice $\mathbf{Z}e \oplus \mathbf{Z}f$ with bilinear form given by $\langle e,e\rangle = \langle f,f\rangle = 0$, $\langle e,f\rangle = 1$.

Definition 2.2. (Narain lattice)
1) A Narain lattice is a subgroup $\Gamma \subset \mathbf{R}^{d,d}$ of rank $2d$ such that $\langle,\rangle|_\Gamma$ is even unimodular. A Narain embedding is an isometric embedding

$$\Phi : U^{\oplus d} \hookrightarrow \mathbf{R}^{d,d} \ .$$

2) Two Narain lattices Γ, Γ' are said to be equivalent if $\Gamma' = g\Gamma$ for some $g \in O(d,d;\mathbf{R})$ preserving the decomposition (2.2).

Since every abstract even unimodular lattice of signature (d,d) is isomorphic to $U^{\oplus d}$, every Narain lattice is the image of a Narain embedding.

Definition 2.3. (Partition function of CFT on T^d) *Given a Narain lattice Γ, we define its partition function*

$$Z^\Gamma(q,\bar q) = \frac{1}{\eta(q)^d \bar\eta(q)^d} \sum_{p=p_l+p_r\in\Gamma} q^{\frac12|p_l^2|}\bar q^{\frac12|p_r^2|} \ ,$$

and its parity invariant form

$$\tilde Z^\Gamma(q,\bar q) = \frac{1}{\eta(q)^d \bar\eta(q)^d} \sum_{p=p_l+p_r\in\Gamma} \frac12\left(q^{\frac12|p_l^2|}\bar q^{\frac12|p_r^2|} + \bar q^{\frac12|p_l^2|} q^{\frac12|p_r^2|}\right) \ ,$$

where $p = p_l + p_r$ is the decomposition according to (2.2) and $|p_l^2| = |\langle p_l,p_l\rangle_{\mathbf{R}^{2,2}}| = |\langle p_l,p_l\rangle_{\mathbf{R}^{2,0}}|$, $|p_r^2| = |\langle p_r,p_r\rangle_{\mathbf{R}^{2,2}}| = |\langle p_r,p_r\rangle_{\mathbf{R}^{0,2}}|$. $\eta(q) = q^{\frac{1}{24}}\prod_{n>0}(1-q^n)$ ($q = e^{2\pi\sqrt{-1}t}$ with t in the upper half plane) is the Dedekind eta function.

Remark. 1) If $g \in O(d,d;\mathbf{R})$ preserves the decomposition (2.2), then it is obvious that $\Gamma, g\Gamma$ have the same partition function, for any Narain lattice Γ. In other words, equivalent Narain lattices have the same partition function.

2) In [Na], the partition function is computed by means of a path integral in a sigma model with target space a flat torus T^d. A Narain lattice Γ plays the role of the momentum-winding lattice. The partition function is modular invariant, i.e. invariant under $t \to t+1$ and $t \to -1/t$, as a result of that Γ is even and unimodular. See, e.g., [Po, §8.4, p.252] for a proof of modular invariance. We will return to modular invariance later.
3) We introduce the parity invariant form $\tilde{Z}^{\Gamma}(q,\bar{q})$ since this fits well in our diagram given in Theorem 4.10, and also the bijection that we will prove in Proposition 4.12. The parity invariance refers to the invariance under the worldsheet parity involution (which is equivalent to the involution $\pi_2 : (\tau,\rho) \mapsto (\tau, -\bar{\rho})$, cf. Lemma 4.9).

(2-2) Moduli space of Narain lattices. Here we summarize some known facts about the space which parametrizes the equivalence classes of Narain lattices. This space is known as *a Narain moduli space*.

Let us define the group $O(d,d;\mathbf{R})$ explicitly by

$$O(d,d;\mathbf{R}) = \left\{ X \in Mat(2d,\mathbf{R}) \mid {}^tX \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} X = \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \right\}, \tag{2.3}$$

where $\mathbf{1}_d$ represents the unit matrix of size d. Let $O(d;\mathbf{R}) \times O(d;\mathbf{R}) \subset O(d,d;\mathbf{R})$ denote the subgroup preserving the decomposition (2.2). Put

$$\mathbf{E} := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_d & \mathbf{1}_d \\ \mathbf{1}_d & -\mathbf{1}_d \end{pmatrix}.$$

Let Γ_0 be the $\mathbf{Z}$-span of the column vectors of $\mathbf{E}$. Then we may verify that Γ_0 is even unimodular of rank $2d$ in $\mathbf{R}^{d,d}$, hence a Narain lattice. Let $O'(d,d;\mathbf{Z}) \subset O(d,d;\mathbf{R})$ denote the subgroup preserving Γ_0. Note that $\mathbf{E}^{-1} = \mathbf{E}$. Then, it is straightforward to determine the subgroup to be

$$O'(d,d;\mathbf{Z}) = \mathbf{E}O(d,d;\mathbf{Z})\mathbf{E}\ ,$$

with

$$O(d,d;\mathbf{Z}) := \left\{ Y \in Mat(2d,\mathbf{Z}) \mid {}^tY \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix} Y = \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix} \right\}.$$

Proposition 2.4. (Narain moduli space) *The equivalence classes of Narain lattices are parametrized by the coset space*

$$\mathcal{M}_{d,d} := O(d;\mathbf{R}) \times O(d;\mathbf{R}) \setminus O(d,d;\mathbf{R}) / O'(d,d;\mathbf{Z})\ . \tag{2.4}$$

Proof. Given a Narain lattice $\Gamma \subset \mathbf{R}^{d,d}$, there exists an isomorphism $g : \Gamma_0 \to \Gamma$, since even, unimodular indefinite lattices are unique up to isomorphisms. This extends uniquely to an isometry of $\mathbf{R}^{d,d}$ which we denote by the same $g \in O(d,d;\mathbf{R})$. We associate to Γ, the left coset $gO'(d,d;\mathbf{Z}) \in O(d,d;\mathbf{R})/O'(d,d;\mathbf{Z})$. If $g' : \Gamma_0 \to \Gamma$ is another isomorphism, then $g^{-1} \cdot g'$ is an isometry of Γ_0, i.e. an element of $O'(d,d;\mathbf{Z})$. It follows that $g'O'(d,d;\mathbf{Z}) = gO'(d,d;\mathbf{Z})$. This shows that the correspondence $\Gamma \mapsto gO'(d,d;\mathbf{Z})$ is well-defined.

This correspondence is 1-1: if $\Gamma \mapsto gO'(d,d;\mathbf{Z})$ and $\Gamma' \mapsto g'O'(d,d;\mathbf{Z})$ for some $g, g' \in O(d,d;\mathbf{R})$ with $g'O'(d,d;\mathbf{Z}) = gO'(d,d;\mathbf{Z})$, then $g^{-1} \cdot g' \in O'(d,d;\mathbf{Z})$, i.e. $g^{-1} \cdot g'\Gamma_0 = \Gamma_0$. It follows that $\Gamma' = g'\Gamma_0 = g\Gamma_0 = \Gamma$. The correspondence is also onto: if $g \in O(d,d;\mathbf{R})$, then $\Gamma = g\Gamma_0$ is a Narain lattice with $\Gamma \mapsto gO'(d,d;\mathbf{Z})$. This shows that $O(d,d;\mathbf{R})/O'(d,d;\mathbf{Z})$ parameterizes all Narain lattices.

Let Γ be a Narain lattice, and let $\Gamma \mapsto gO'(d,d;\mathbf{Z})$. If $h \in O(d;\mathbf{R}) \times O(d;\mathbf{R})$, then $hg : \Gamma_0 \to h\Gamma$ is an isomorphism, and so $h\Gamma \mapsto hgO'(d,d;\mathbf{Z})$. This shows that the correspondence $\Gamma \mapsto gO'(d,d;\mathbf{Z})$ is compatible with the left action of $O(d;\mathbf{R}) \times O(d;\mathbf{R})$. This induces a 1-1 correspondence between equivalence classes of Narain lattices and $\mathcal{M}_{d,d}$. □

Remark. We introduce some notations here.
1) Introduce the following conjugate of $O(d,d;\mathbf{R})$,

$$O'(d,d;\mathbf{R}) := \mathbf{E}\, O(d,d;\mathbf{R})\, \mathbf{E} \ . \tag{2.5}$$

2) Put $e_i = (0,..,0,e,0,..,0)$, $f_i = (0,..,0,f,0,..,0) \in U^{\oplus d}$, where e, f lies in the i-th slot. Given a Narain embedding $\Phi : U^{\oplus d} \hookrightarrow \mathbf{R}^{d,d}$, we define the following matrix:

$$\begin{aligned} W(\Phi) &:= \frac{1}{\sqrt{2}} \left(\Phi(e_1+f_1) \ \cdots \ \Phi(e_d+f_d) \ \Phi(e_1-f_1) \ \cdots \ \Phi(e_d-f_d)\right) \\ &= \left(\Phi(e_1)\cdots\Phi(e_d) \ \Phi(f_1)\cdots\Phi(f_d)\right) \ \mathbf{E}. \end{aligned} \tag{2.6}$$

It is easy to check that $W(\Phi) \in O(d,d;\mathbf{R})$. Conversely, for any $X \in O(d,d;\mathbf{R})$, there is a unique Narain embedding Φ such that $W(\Phi) = X$. In other words, $O(d,d;\mathbf{R})$ parameterizes all Narain embeddings by the identification $\Phi \equiv W(\Phi)$.
3) Likewise, $O(d,d;\mathbf{R})/O'(d,d;\mathbf{Z})$ parameterizes all Narain lattices $\Gamma = \operatorname{Im}\Phi$ by the identification $\Gamma \equiv W(\Phi)O'(d,d;\mathbf{Z})$. □

Example $(d=1)$**.** We have

$$O'(1,1;\mathbf{R}) = \left\{ \begin{pmatrix} R & 0 \\ 0 & 1/R \end{pmatrix} \mid R \neq 0 \right\} \sqcup \left\{ \begin{pmatrix} 0 & R \\ 1/R & 0 \end{pmatrix} \mid R \neq 0 \right\}, \tag{2.7}$$

where the each factor consists of two connected components. After the conjugation by (2.5), the subgroup $O(1;\mathbf{R}) \times O(1;\mathbf{R}) \subset O(1,1;\mathbf{R})$ becomes a subgroup $G \times G$ in $O'(1,1;\mathbf{R})$ consisting of the four elements: $\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\pm\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then we get

$$G \times G \setminus O'(1,1;\mathbf{R}) = \left\{ G \times G \cdot \begin{pmatrix} R & 0 \\ 0 & 1/R \end{pmatrix} \mid R > 0 \right\}. \tag{2.8}$$

Thus the four components of $O'(1,1;\mathbf{R})$ are collapsed to one. The subgroup $O(1,1;\mathbf{Z}) \subset O'(1,1;\mathbf{R})$ consists of four elements obtained from (2.7) by setting $R = \pm 1$. It is easy to check that each of them acts on (2.8) either trivially or by $R \to 1/R$, which is a well-known duality transformation. This shows that the Narain moduli space $\mathcal{M}_{1,1}$ has the cross section $\{\mathbf{E}\left(\begin{smallmatrix} R & 0 \\ 0 & 1/R \end{smallmatrix}\right)\mathbf{E} \mid 1 \geq R > 0\} \subset O(1,1;\mathbf{R})$. By Proposition 2.4, we get all the (inequivalent) Narain lattices Γ by letting this set act on the fixed Narain lattice Γ_0. Since Γ_0 is spanned by the columns of the matrix $\mathbf{E}$, it follows that such a Γ is spanned by the columns of the matrix

$$\mathbf{E}\begin{pmatrix} R & 0 \\ 0 & 1/R \end{pmatrix}\mathbf{E}\cdot\mathbf{E} = \frac{1}{\sqrt{2}}\begin{pmatrix} R & 1/R \\ R & -1/R \end{pmatrix}.$$

By (2.2), each vector $p \in \Gamma$ decomposes over $\mathbf{R}$ as

$$p = \frac{1}{\sqrt{2}}\begin{pmatrix} mR + n/R \\ mR - n/R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} mR + n/R \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ mR - n/R \end{pmatrix},$$

where $m, n \in \mathbf{Z}$. Finally, the partition function of Γ is

$$Z^{\Gamma}(q,\bar{q}) = \frac{1}{\eta(q)\overline{\eta(q)}} \sum_{m,n\in\mathbf{Z}} q^{\frac{1}{4}(\frac{m}{R}+nR)^2}\bar{q}^{\frac{1}{4}(\frac{m}{R}-nR)^2} \quad .$$

This is the well-known partition function of toroidal compactification on $T^1 = S^1$ with radius R. □

(2-3) Another parameterization of Narain lattices. There is a convenient parameterization of the homogeneous space $O(d;\mathbf{R})\times O(d;\mathbf{R})\backslash O(d,d;\mathbf{R})$ (precisely its conjugate by $\mathbf{E}$, see, e.g. [Na][Ki2][NW]). To summarize this, let us define a matrix in $O'(d,d;\mathbf{R})$ of the form

$$W'(\Lambda, B) := \begin{pmatrix} {}^t\Lambda^{-1} & 0 \\ 0 & \Lambda \end{pmatrix}\begin{pmatrix} \mathbf{1}_d & -B \\ 0 & \mathbf{1}_d \end{pmatrix} \quad ,$$

where $\Lambda \in GL(d,\mathbf{R})$, and $B \in Mat(d,\mathbf{R})$ is an antisymmetric matrix, i.e. ${}^tB = -B$. In the following, we denote by $A(d,\mathbf{R})$ the set of all antisymmetric real matrices.

Proposition 2.5. *The coset space $G\times G\backslash O'(d,d;\mathbf{R})$, where $G\times G := \mathbf{E}\,(O(d,\mathbf{R})\times O(d,\mathbf{R})\,)\mathbf{E}$, can be represented by;*

$$G\times G \setminus O'(d,d;\mathbf{R}) = \{G_{diag}\cdot W'(\Lambda,B) \mid \Lambda \in GL(d,\mathbf{R}),\ B \in A(d,\mathbf{R})\}\ ,$$

where

$$G_{diag}\cdot W'(\Lambda,B) := \left\{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} W'(\Lambda,B) \mid g \in O(d,\mathbf{R}) \right\} . \tag{2.9}$$

Moreover, when $G_{diag}\cdot W'(\Lambda,B) = G_{diag}\cdot W'(\Lambda',B')$, we have $O(d,\mathbf{R})\cdot\Lambda = O(d,\mathbf{R})\cdot\Lambda'$, $B = B'$. So, if we fix a "gauge" for the $O(d,\mathbf{R})$ action on $GL(d,\mathbf{R})$ (i.e. a cross section for the orbit space $GL(d,\mathbf{R})/O(d,\mathbf{R})$), then Λ and B are uniquely determined for each orbit in the coset space $G\times G\setminus O'(d,d;\mathbf{R})$.

An elementary proof of this is given in Appendix C. Here we point out that the $\Lambda \in GL(d,\mathbf{R})$ may be identified with the lattice of the target torus, i.e. $T^d = \mathbf{R}^d/L(\Lambda)$ with $L(\Lambda)$ being the lattice generated by the column vectors of Λ. Under this identification, the matrix ${}^t\Lambda^{-1}$ defines the dual torus $(T^d)^\vee$. Also the antisymmetric matrix B represents the so-called B-field, which generalizes the electro-magnetic field in particle theory to string theory. By Proposition 2.5, the coset $O(d,\mathbf{R})\times O(d,\mathbf{R})\setminus O'(d,d;\mathbf{R})$ is in one-to-one correspondence with the set $(\,GL(d,\mathbf{R})/O(d,\mathbf{R})\,)\times A(d,\mathbf{R})$.

For $d=2$, the parametrization (2.9) can be made even more explicit. We fix a "gauge" for $GL(2,\mathbf{R})/O(2,\mathbf{R})$ by choosing a special cross section;

$$\Lambda_0=\sqrt{\frac{\rho_2}{\tau_2}}\begin{pmatrix}1&\tau_1\\0&\tau_2\end{pmatrix}\in GL(2,\mathbf{R})\ \ (\rho_2,\tau_2>0)\ \ .$$

Note that $\det\Lambda_0=\rho_2$ is the volume of the torus defined by the lattice $L(\Lambda_0)$. Letting the B-field be $B_{12}=:\rho_1$, we arrive at the following parametrization

$$W'(\tau,\rho):=\frac{1}{\sqrt{\rho_2\tau_2}}\begin{pmatrix}\tau_2&0&0&0\\-\tau_1&1&0&0\\0&0&\rho_2&\tau_1\rho_2\\0&0&0&\tau_2\rho_2\end{pmatrix}\begin{pmatrix}1&0&0&-\rho_1\\0&1&\rho_1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\ \ , \tag{2.10}$$

where $\tau=\tau_1+\sqrt{-1}\tau_2,\rho=\rho_1+\sqrt{-1}\rho_2$ $(\tau_2,\rho_2>0)$. It is clear that τ describes the complex structure of the torus $T^2=\mathbf{R}^2/L(\Lambda_0)(=:E_\tau)$. The parameter $\rho=\rho_1+\sqrt{-1}\rho_2=B_{12}+\sqrt{-1}Vol(E_\tau)$ is the so-called complexified Kähler modulus of E_τ. It is also clear that both parameters τ and ρ can take arbitrary values in the upper half plane $\mathbf{H}_+$, i.e. (τ,ρ) ranges over all of $\mathbf{H}_+\times\mathbf{H}_+$.

The right action $O(2,2;\mathbf{Z})$ on the coset space $G\times G\setminus O'(2,2;\mathbf{R})$ is known in the physics literatures as the T-duality group.

Proposition 2.6. (Duality transformations)
1) The following elements generate the group $O(2,2;\mathbf{Z})$*:*

$$S_1=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\ T_1=\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&-1\\0&0&0&1\end{pmatrix},\ R_1=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1\end{pmatrix},$$
$$S_2=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix},\ T_2=\begin{pmatrix}1&0&0&1\\0&1&-1&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ R_2=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}.$$

2) The above generators act on the orbits $(G\times G)\cdot W'(\tau,\rho)$ *from the right by*

$$S_1:(\tau,\rho)\to(-1/\tau,\rho)\ \ ,\ \ T_1:(\tau,\rho)\to(\tau+1,\rho)\ \ ,\ \ R_1:(\tau,\rho)\to(\rho,\tau)$$
$$S_2:(\tau,\rho)\to(\tau,-1/\rho)\ \ ,\ \ T_2:(\tau,\rho)\to(\tau,\rho+1)\ \ ,\ \ R_2:(\tau,\rho)\to(-\bar\tau,-\bar\rho).$$

3) There is a group isomorphism,

$$O(2,2;\mathbf{Z})\cong P(SL_2\mathbf{Z}\times SL_2\mathbf{Z})\rtimes(\mathbf{Z}_2\times\mathbf{Z}_2)\ ,$$

where $P(SL_2\mathbf{Z}\times SL_2\mathbf{Z})$ *represents the quotient group of* $SL_2\mathbf{Z}\times SL_2\mathbf{Z}$ *by the involution:* $(g,h)\mapsto(-g,-h)$*, and* $\rtimes$ *denotes the semi-direct product.*

Sketch of Proof. We will give a proof of 1) in Appendix C. By Proposition 2.5 and the parametrization of the orbit (2.10), assertion 2) is derived by straightforward calculations. For a proof of 3), we use an explicit surjective group homomorphism, which is constructed in Appendix C, $\phi_{\mathbf{Z}}:SL_2\mathbf{Z}\times SL_2\mathbf{Z}\to O(2,2;\mathbf{Z})\cap$

$O_0'(2,2;\mathbf{R})$ with $Ker(\phi_{\mathbf{Z}}) = \{(\mathbf{1}_2,\mathbf{1}_2),(-\mathbf{1}_2,-\mathbf{1}_2)\}$, where $O_0'(2,2;\mathbf{R})$ is the connected component of the identify (see Proposition C.2). Thus we find a subgroup $P(SL_2\mathbf{Z}\times SL_2\mathbf{Z})$ in $O(2,2;\mathbf{Z})$. As is clear from the argument for the decomposition (C.5), this subgroup is a normal subgroup with $P(SL_2\mathbf{Z}\times SL_2\mathbf{Z}) \setminus O(2,2;\mathbf{Z}) = \{\mathbf{1}_4, R_1, R_2, R_1R_2\} \cong \mathbf{Z}_2\times\mathbf{Z}_2$. □

(2-4) Narain lattice ($d = 2$). We now use the explicit parameterization $W'(\tau,\rho)$ in (2.10) of the coset space $G\times G\setminus O'(2,2;\mathbf{R})$, to produce a convenient parameterization $\Gamma(\tau,\rho)$ of Narain lattices. By (2.6), the element $W'(\tau,\rho)\in O'(2,2;\mathbf{R})$ determines a unique Narain embedding $\Phi_{\tau,\rho} : U^{\oplus 2} \hookrightarrow \mathbf{R}^{2,2}$ via $W(\Phi_{\tau,\rho}) = \mathbf{E}W'(\tau,\rho)\mathbf{E}$. We will write down $\Phi_{\tau,\rho}$ explicitly. Consider

$$W(\Phi_{\tau,\rho})\mathbf{E} = \mathbf{E}W'(\tau,\rho) =: (\mathbf{e}_1(\tau,\rho)\ \mathbf{e}_2(\tau,\rho)\ \mathbf{e}_3(\tau,\rho)\ \mathbf{e}_4(\tau,\rho)). \tag{2.11}$$

Then we have $\langle \mathbf{e}_i(\tau,\rho), \mathbf{e}_{j+2}(\tau,\rho)\rangle_{\mathbf{R}^{2,2}} = \delta_{ij}$, $\langle \mathbf{e}_i(\tau,\rho), \mathbf{e}_j(\tau,\rho)\rangle_{\mathbf{R}^{2,2}} = \langle \mathbf{e}_{i+2}(\tau,\rho), \mathbf{e}_{j+2}(\tau,\rho)\rangle_{\mathbf{R}^{2,2}} = 0$ $(1\le i,j\le 2)$. By (2.6), the Narain embedding $\Phi_{\tau,\rho}$ is given by by $\Phi_{\tau,\rho}(e_i) = \mathbf{e}_i(\tau,\rho)$, $\Phi_{\tau,\rho}(f_j) = \mathbf{e}_{j+2}(\tau,\rho)$, and its image is the Narain lattice

$$\Gamma(\tau,\rho) := \mathbf{Z}\,\mathbf{e}_1(\tau,\rho)\oplus\mathbf{Z}\,\mathbf{e}_2(\tau,\rho)\oplus\mathbf{Z}\,\mathbf{e}_3(\tau,\rho)\oplus\mathbf{Z}\,\mathbf{e}_4(\tau,\rho)\subset\mathbf{R}^{2,2}. \tag{2.12}$$

Explicitly the $e_i(\tau,\rho)$ are

$$\frac{1}{\sqrt{2\tau_2\rho_2}}\begin{pmatrix}\tau_2\\-\tau_1\\\tau_2\\-\tau_1\end{pmatrix},\ \frac{1}{\sqrt{2\tau_2\rho_2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\ \frac{1}{\sqrt{2\tau_2\rho_2}}\begin{pmatrix}\rho_2\\\rho_1\\-\rho_2\\\rho_1\end{pmatrix},\ \frac{1}{\sqrt{2\tau_2\rho_2}}\begin{pmatrix}-\rho_1\tau_2+\tau_1\rho_2\\\rho_1\tau_1+\tau_2\rho_2\\-\rho_1\tau_2-\tau_1\rho_2\\\rho_1\tau_1-\tau_2\rho_2\end{pmatrix}. \tag{2.13}$$

The following properties of the Narain lattice $\Gamma(\tau,\rho)$ are immediate from Propositions 2.5, 2.6:

Proposition 2.7.
1) $\Gamma(\tau,\rho), \Gamma(\tau',\rho')$ are equivalent if and only if (τ,ρ) and (τ',ρ') are related by a duality transformation on $\mathbf{H}_+^2$.
2) Every Narain lattice $\Gamma\subset\mathbf{R}^{2,2}$ is equivalent to $\Gamma(\tau,\rho)$ for some $\tau,\rho\in\mathbf{H}_+$.

§3. Rational conformal field theory

(3-1) Algebraic CFT. Algebraically, a conformal field theory (CFT) is described by its so-called *chiral algebras* and their representations. For a mathematical exposition see [FLM][LZ][Kac][MN]. For a physical exposition see [GSW][Po]. We give a rough schematic description of this theory here, but will be more precise when we come to CFTs on T^2. Examples of chiral algebras $\mathcal{A}$ often come from infinite dimensional Lie algebras and certain generalizations such as the W-algebras. The basic setup of a CFT includes (1) two chiral algebras $\mathcal{A}_L,\mathcal{A}_R$, which are called respectively the *left* and the *right* chiral algebras; (2) a class of representations $\mathcal{H}_{L,j}$ $(j\in\Lambda_L)$ and $\mathcal{H}_{R,k}$ $(k\in\Lambda_R)$, where $\Lambda_{L,R}$ are some index sets; (3) the characters of the representations $ch_{L,j}(q) = Tr_{\mathcal{H}_{L,j}}q^{d_L}$, $\overline{ch_{R,j}(q)} = Tr_{\mathcal{H}_{R,j}}\bar{q}^{d_L}$, where d_L, d_R are the scaling operators in the chiral algebras and $q = e^{2\pi\sqrt{-1}t}, \bar{q} = e^{-2\pi\sqrt{-1}\bar{t}}$; (4) (*partition function*), a real analytic modular invariant (i.e. invariant under the transformations $t \to t+1, t \to -1/t$) function of the shape

$$Z(q,\bar{q}) = \sum_{i\in\Lambda_L, j\in\Lambda_R} N_{ij}\, ch_{L,i}(q)\overline{ch_{R,j}(q)} \;,$$

where N_{ij} are some positive integers. A CFT is called *rational* iff its chiral algebras have only finitely many irreducible representations, in which case the index sets Λ_L, Λ_R are the finite lists of representations.

We now consider CFTs on T^2. This classes of CFTs are parameterized by equivalence classes of Narain lattices. For a generic Narain lattice Γ, the chiral algebras are generated by vertex operators of a Heisenberg algebra (also known as "$U(1)$ currents" in physics). The chiral algebras also contain the $c = 2$ Virasoro algebra as a subalgebra. The lattice Γ plays the role of the momentum-winding lattice. Each momentum-winding vector $p = p_l + p_r \in \Gamma$ corresponds to a pair of irreducible representations labeled by p_l, p_r ("$U(1)$ charges" of the $U(1)$ currents). The partition function is then given by $Z^\Gamma(q,\bar{q})$, as in Definition (2.3). When the CFT becomes rational, something very interesting happens. First, the chiral algebras become significantly enlarged. Second, the infinite list of representations (of the small chiral algebras), indexed by $p \in \Gamma$, reconstitute themselves, and then break up into finitely many irreducible representation of the enlarged chiral algebras.

The main task of this paper is to describe Narain lattices Γ which yield rational CFT in terms of the momentum lattices of the left and right handed chiral algebras, and count them.

(3-2) Discriminant A_Γ. Let Γ be an even, positive definite lattice. The quotient Γ^*/Γ is a group called the *discriminant group* of Γ (, see [Ni] for details). Let us fix a basis u_1, u_2 of Γ, and denote its dual basis by u_1^*, u_2^*. Then the intersection form (1.2) relates these two bases by

$$(u_1^*\; u_2^*) = (u_1\; u_2)\;\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}^{-1} \;. \tag{3.3}$$

The discriminant group is the abelian group generated by u_1^*, u_2^* modulo the lattice $\Gamma \subset \Gamma^*$. The integral bilinear form on Γ extends to a rational bilinear form $(\,,\,)$ on Γ^*. Since Γ is even, we have a natural quadratic form $q_\Gamma : \Gamma^*/\Gamma \to \mathbf{Q}/2\mathbf{Z}$, called the *discriminant form* of Γ,

$$q_\Gamma(v \bmod \Gamma) := (v,v) \bmod 2\mathbf{Z} \;\;.$$

If $v = m_1u_1^* + m_2u_2^*$, then (v,v) can evaluated using the linear relation (3.3). Associated to the quadratic form q_Γ, we have a $\mathbf{Q}/\mathbf{Z}$-valued bilinear form $(w,v)_q := \frac{1}{2}(q_\Gamma(w+v) - q_\Gamma(w) - q_\Gamma(v))$. We denote the pair $(\Gamma^*/\Gamma, q_\Gamma)$ by A_Γ, and call it the *discriminant* of Γ.

Consider the discriminants $A_\Gamma, A_{\Gamma'}$ of two lattices Γ, Γ'. The group isomorphisms which preserve the discriminant form will be called *isometries* of the discriminants. Clearly an isomorphism of lattices induces an isometry of their discriminants. (But the converse is not true.) We denote the set of isometries by

$$\mathrm{Isom}(A_\Gamma, A_{\Gamma'}) := \{\varphi : \Gamma^*/\Gamma \xrightarrow{\sim} {\Gamma'}^*/\Gamma' \mid q_{\Gamma'}(\varphi(v)) \equiv q_\Gamma(v) \bmod 2\mathbf{Z}\;\}$$

Clearly, the orthogonal group $O(\Gamma)$ acts naturally on the set $\mathrm{Isom}(A_\Gamma, A_{\Gamma'})$ from the left, and $O(\Gamma')$ from the right.

Remark. The set $\mathrm{Isom}(A_\Gamma, A_{\Gamma'})$ is nonempty if and only if the two discriminants are isomorphic. This is the case exactly when the discriminant groups are isomorphic, $\Gamma^*/\Gamma \cong \Gamma'^*/\Gamma' \cong \mathbf{Z}_{d_1} \oplus \mathbf{Z}_{d_2}$ $(d_1|d_2)$, and the isomorphism preserves the discriminant forms. To spell this out explicitly, let us fix the isomorphisms; $\Gamma^*/\Gamma \cong \mathbf{Z}_{d_1} \oplus \mathbf{Z}_{d_2}$ (respectively $\Gamma'^*/\Gamma' \cong \mathbf{Z}_{d_1} \oplus \mathbf{Z}_{d_2}$) by choosing a basis w_1, w_2 (respectively w_1', w_2'). With respect to these bases, we represent an isomorphism $\varphi : \Gamma^*/\Gamma \xrightarrow{\sim} \Gamma'^*/\Gamma'$ by a matrix: $(\varphi(w_1)\,\varphi(w_2)) = (w_1'\,w_2')\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ where α, β (respectively γ, δ) are integers considered mod d_1 (d_2). Let $(\ ,\ )_q$, $(\ ,\ )_{q'}$ denote the bilinear forms of $A_\Gamma, A_{\Gamma'}$. That φ is an isometry means that $(\varphi(w_i), \varphi(w_j))_{q'} = (w_i, w_j)_q$, i.e.

$$ {}^t\!\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \begin{pmatrix}(w_1', w_1')_{q'} & (w_1', w_2')_{q'} \\ (w_2', w_1')_{q'} & (w_2', w_2')_{q'}\end{pmatrix} \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}(w_1, w_1)_q & (w_1, w_2)_q \\ (w_2, w_1)_q & (w_2, w_2)_q\end{pmatrix} . \tag{3.4} $$

□

Discriminants of even, positive definite lattices of rank two and the isometries between them are central objects in our description of rational conformal field theory on T^2.

(3-3) $c = 2$ **RCFT.** If Γ_l, Γ_r are even, positive definite lattices of determinant $-D$ and $\varphi : A_{\Gamma_l} \to A_{\Gamma_r}$ is an isometry, then we call $(\Gamma_l, \Gamma_r, \varphi)$ *a triple* of determinant $-D$. We say that the triple is primitive if Γ_l, Γ_r are primitive lattices. We define an equivalence relation on triples as follows:

$$ \begin{aligned} &(\Gamma_l, \Gamma_r, \varphi) \sim (\Gamma_l', \Gamma_r', \varphi') \\ &\Leftrightarrow \exists \text{ isomorphisms } g : \Gamma_l \to \Gamma_l',\ \ h : \Gamma_r \to \Gamma_r' \text{ such that } \varphi' = \bar{h} \cdot \varphi \cdot \bar{g}^{-1}. \end{aligned} $$

Here $\bar{g} : A_{\Gamma_l} \to A_{\Gamma_l'}$ is the isometry induced by g and similarly for $\bar{h}$. We call an equivalence class of triples $[(\Gamma_l, \Gamma_r, \varphi)]$ an *RCFT data*. For our description of the RCFT data, we define the following sets of equivalence classes:

$$ \begin{aligned} RCFT_D &:= \{ \text{ triples of determinant } -D\}/\sim\ , \\ RCFT_D^p &:= \{ \text{ primitive triples of determinant } -D\}/\sim\ , \end{aligned} $$

and also

$$ \begin{aligned} RCFT_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) := \{ &\text{ triples } (\Gamma_l', \Gamma_r', \varphi') \text{ of determinant } -D \\ &\text{with } [\Gamma_l'] = \mathbf{\Gamma}_l,\ [\Gamma_r'] = \mathbf{\Gamma}_r\}/\sim\ , \\ RCFT_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) := \{ &\text{ primitive triples } (\Gamma_l', \Gamma_r', \varphi') \text{ of determinant } -D \\ &\text{with } [\Gamma_l'] = \mathbf{\Gamma}_l,\ [\Gamma_r'] = \mathbf{\Gamma}_r\}/\sim . \end{aligned} $$

Note that we have the obvious disjoint union:

$$ RCFT_D = \bigsqcup RCFT_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) , $$

where the $\mathbf{\Gamma}_l, \mathbf{\Gamma}_r$ range over all classes in $\widetilde{CL}(D) \equiv \mathcal{L}(D)$. Likewise, we have a similar decomposition for $RCFT_D^p$ with $\widetilde{Cl(D)} \equiv \mathcal{L}^p(D)$. Note that $RCFT_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)$ is nonempty if and only if Γ_l, Γ_r are in the same genus [Ni].

Lemma 3.3. *Fix lattices Γ_l, Γ_r in respective classes $\mathbf{\Gamma}_l, \mathbf{\Gamma}_r \in \mathcal{L}(D)$. Then the following map is well-defined and bijective:*

$$\begin{array}{llll} B_{\Gamma_l,\Gamma_r}: & RCFT_D(\mathbf{\Gamma}_l,\mathbf{\Gamma}_r) & \rightarrow & O(\Gamma_l)\setminus Isom(A_{\Gamma_l},A_{\Gamma_r})/O(\Gamma_r) \\ & [(\Gamma_l,\Gamma_r,\varphi)] & \mapsto & [\varphi] \end{array} \tag{3.5}$$

Proof. By definition, an arbitrary class in $RCFT_D(\mathbf{\Gamma}_l,\mathbf{\Gamma}_r)$ may be represented by a triple $(\Gamma_l,\Gamma_r,\varphi)$ with some $\varphi \in Isom(A_{\Gamma_l},A_{\Gamma_r})$. The triples having this shape $(\Gamma_l,\Gamma_r,*)$ are preserved exactly by the group $O(\Gamma_l)\times O(\Gamma_r)$ acting on triples by

$$(g\Gamma_l, h\Gamma_r, \bar{h}\varphi\bar{g}^{-1}) = (\Gamma_l,\Gamma_r,\bar{h}\varphi\bar{g}^{-1}) .$$

This shows that the map is well-defined. Now the bijectivity of the map is also clear. □

This map will be used in the next section.

Definition 3.4. ($c = 2$ RCFT) *Given a triple $(\Gamma_l,\Gamma_r,\varphi)$ of determinant $-D$, we define the partition function*

$$Z^{\Gamma_l,\Gamma_r,\varphi}(q,\bar{q}) = \frac{1}{|\eta(q)|^4}\sum_{a\in\Gamma_l^*/\Gamma_l}\theta_a^{\Gamma_l}(q)\overline{\theta_{\varphi(a)}^{\Gamma_r}(q)} ,$$

and its parity invariant form

$$\tilde{Z}^{\Gamma_l,\Gamma_r,\varphi}(q,\bar{q}) = \frac{1}{|\eta(q)|^4}\sum_{a\in\Gamma_l^*/\Gamma_l}\frac{1}{2}\left(\theta_a^{\Gamma_l}(q)\overline{\theta_{\varphi(a)}^{\Gamma_r}(q)} + \overline{\theta_a^{\Gamma_l}(q)}\theta_{\varphi(a)}^{\Gamma_r}(q)\right) ,$$

where $\theta_a^{\Gamma}(q) := \sum_{v\in\Gamma} q^{\frac{1}{2}(v+a)^2}$, for $a\in\Gamma^*/\Gamma$, is a theta function of the lattice Γ.

Remark. 1) It is clear that equivalent triples have the same partition function, and thus the partition function is defined for a class $[(\Gamma_l,\Gamma_r,\varphi)] \in RCFT_D$.
2) The involution $(\Gamma_l,\Gamma_r,\varphi)\mapsto(\Gamma_r,\Gamma_l,\varphi^{-1})$ on triples is obviously compatible with the equivalence relation on triples. In particular, the involution acts on the sets $RCFT_D, RCFT_D^p$. The parity invariant form $\tilde{Z}^{\Gamma_l,\Gamma_r,\varphi}(q,\bar{q})$ is defined so that it is invariant under this involution.
3) Note that $Z^{\Gamma_l,\Gamma_r,\varphi}(q,\bar{q})$ is nothing but a finite linear combination of products $ch_{L,a}(q)\overline{ch_{R,b}(q)}$, where

$$ch_{L,a}(q) = \frac{\theta_a^{\Gamma_l}(q)}{\eta(q)^2} \quad , \quad ch_{R,b}(q) = \frac{\theta_b^{\Gamma_r}(q)}{\eta(q)^2} \quad ,$$

with $a\in\Gamma_l^*/\Gamma_l, b\in\Gamma_r^*/\Gamma_r$, are characters of representations of certain chiral algebras $\mathcal{A}_L$ and $\mathcal{A}_R$. They can also be viewed as characters of certain unitary reducible representations of the $c=2$ Virasoro algebra (see e.g. [KP]). □

Proposition 3.5. *The partition function for a triple is modular invariant.*

Proof. The theta function of an even lattice Γ ($s := \mathrm{rk}\,\Gamma$) has the transformation property, $\theta_a^{\Gamma}(q)|_{t\to t+1} = e^{\pi\sqrt{-1}(a,a)}\theta_a^{\Gamma}(q)$. Also, by the Poisson resummation formula, we have (see e.g. [KP, Proposition 3.4]);

$$\theta_a^{\Gamma}(q)\big|_{t\to -1/t} = \frac{1}{|\Gamma^*/\Gamma|^{\frac{1}{2}}}(-\sqrt{-1}t)^{\frac{s}{2}}\sum_{b\in\Gamma^*/\Gamma} e^{-2\pi\sqrt{-1}(a,b)}\theta_b^{\Gamma}(q),$$

where $(a, b) \in \mathbf{Q}/\mathbf{Z}$ is the bilinear form on the discriminant group Γ^*/Γ (see section (3-2)). Using these relations and also $\frac{1}{|\Gamma^*/\Gamma|}\sum_{a\in\Gamma^*/\Gamma} e^{2\pi\sqrt{-1}(a,b-b')} = \delta_{bb'}$, it is straightforward to verify that $Z^{\Gamma_l,\Gamma_r,\varphi}(q,\bar{q})$ is invariant under $t \to t+1, t \to -1/t$. In the calculation, we use the fact that φ is an isometry and thus we have $(\varphi(a), b) = (a, \varphi^{-1}(b))$ for $a \in \Gamma_l^*/\Gamma_l, b \in \Gamma_r^*/\Gamma_r$. □

We now state our problem precisely as follows:

Classification Problem. *(1) Formulate a correspondence between the triples $(\Gamma_l, \Gamma_r, \varphi)$ and the Narain lattices $\Gamma(\tau, \rho)$. (2) Describe the values $\tau, \rho \in \mathbf{H}_+$ for the equivalence classes of Narain lattices $\Gamma(\tau, \rho)$ which correspond to the classes of triples in $RCFT_D$, and count them.*

Prop. 1.6.1 of Nikulin [Ni] gives a correspondence between primitive embeddings in a fixed unimodular lattice and isometries of discriminant groups by means of over-lattices. An equivalent approach using *gluing theory* can be found in [CS, Chapter 4] and references therein. This correspondence will be needed to do (1) (see below) where we seek a correspondence between abstract triples and the parameters τ, ρ. The importance of discriminant groups in the study of toroidal RCFTs was first observed in [Mo]. In the next two sections we will carry out (2). The precise counting of both Narain lattices and triples will involve taking into account a certain natural involution, $(\tau, \rho) \mapsto (\tau, -\bar{\rho})$, which represents the worldsheet parity involution. We will find that the Gauss product on the class group $Cl(D)$ and its extension to $CL(D)$ play a central role.

(3-4) Over-lattices. Our aim here is to define a map from $RCFT_D$ to the Narain moduli space $\mathcal{M}_{2,2}$. The main idea is to look at *over-lattices* of $\Gamma_l \oplus \Gamma_r(-1)$.

Given a triple $(\Gamma_l, \Gamma_r, \varphi)$, put

$$\Gamma^\varphi := \{\ (x, y) \in \Gamma_l^* \oplus \Gamma_r^*(-1) \,|\, \varphi(x \ mod\ \Gamma_l) = y \ mod\ \Gamma_r(-1)\ \}. \tag{3.6}$$

It is easy to check that this is a sublattice of $\Gamma_l^* \oplus \Gamma_r^*(-1)$ containing $\Gamma_l \oplus \Gamma_r(-1)$, such that

$$\Gamma^\varphi/(\Gamma_l \oplus \Gamma_r(-1)) = \{\ a \oplus \varphi(a) \,|\, a \in \Gamma_l^*/\Gamma_l\ \}\ , \tag{3.7}$$

or equivalently,

$$\Gamma^\varphi = \bigcup_{a\in\Gamma_l^*/\Gamma_l} (a \oplus \varphi(a) + \Gamma_l \oplus \Gamma_r(-1))\ . \tag{3.8}$$

Fix a pair of isometric embeddings $\iota_1 : \Gamma_l \to \mathbf{R}^{2,0}$, $\iota_2 : \Gamma_r(-1) \to \mathbf{R}^{0,2}$. Extend ι_1, ι_2 over $\mathbf{R}$, and denote their extensions by ι_1, ι_2 also.

Proposition 3.6. *Let $(\Gamma_l, \Gamma_r, \varphi)$ be a triple, and ι_1, ι_2 be isometric embeddings as above. Then*

1) Γ^φ is an even, unimodular, integral lattice of signature $(2, 2)$. Hence the image $(\iota_1 \oplus \iota_2)\Gamma^\varphi \subset \mathbf{R}^{2,2}$ is a Narain lattice.

2) The equivalence class $[(\iota_1 \oplus \iota_2)\Gamma^\varphi]$ of Narain lattices is independent of the choices of ι_1, ι_2. Moreover $[(\iota_1 \oplus \iota_2)\Gamma^\varphi]$ depends only on the equivalence class of the triple $(\Gamma_l, \Gamma_r, \varphi)$.

Proof. 1) This follows from Proposition 1.6.1 in [Ni].
2) Suppose we have an equivalence $(\Gamma_l, \Gamma_r, \varphi) \sim (\Gamma'_l, \Gamma'_r, \varphi')$ given by isomorphisms $g : \Gamma_l \to \Gamma'_l$, $h : \Gamma_r \to \Gamma'_r$. As above, we fix choices of isometric embeddings ι_1, ι_2 and ι'_1, ι'_2 for the two triples. Extend g, h over $\mathbf{R}$, and denote their extensions by g, h also. By (3.6), we have $(g \oplus h)\Gamma^\varphi = \Gamma^{\varphi'}$. Clearly, the isometry $f := (\iota'_1 \oplus \iota'_2) \circ (g \oplus h) \circ (\iota_1 \oplus \iota_2)^{-1} = (\iota'_1 \cdot g \cdot \iota_1^{-1}) \oplus (\iota'_2 \cdot h \cdot \iota_2^{-1}) : \mathbf{R}^{2,2} \to \mathbf{R}^{2,2}$ preserves the decomposition $\mathbf{R}^{2,0} \oplus \mathbf{R}^{0,2}$, hence is an element of $O(d;\mathbf{R}) \times O(d;\mathbf{R})$. But $f \circ (\iota_1 \oplus \iota_2)\Gamma^\varphi = (\iota'_1 \oplus \iota'_2)\Gamma^{\varphi'}$. It follows that $[(\iota_1 \oplus \iota_2)\Gamma^\varphi] = [(\iota'_1 \oplus \iota'_2)\Gamma^{\varphi'}]$. This proves 2). □

Now by Propositions 3.6 and 2.4, we have a well-defined map:

$$F : \bigcup_D RCFT_D \longrightarrow \mathcal{M}_{2,2}, \quad [(\Gamma_l, \Gamma_r, \varphi)] \longmapsto [(\iota_1 \oplus \iota_2)\Gamma^\varphi] \ . \tag{3.9}$$

Remark. 1) In the construction of Γ^φ in the proof, if we replace Γ_l, Γ_r by their images under the chosen embeddings ι_1, ι_2, then the resulting Γ^φ actually sits inside $\mathbf{R}^{2,2}$, rather than being just an abstract lattice.
2) Using the relation (3.8), it is a simple exercise to show that the partition function of the Narain lattice $\Gamma = (\iota_1 \oplus \iota_2)\Gamma^\varphi$ (Definition 2.3)) is given by

$$Z^\Gamma(q, \bar{q}) = Z^{\Gamma_l, \Gamma_r, \varphi}(q, \bar{q}). \tag{3.10}$$

(3-5) RCFTs in $\mathcal{M}_{2,2}$. Consider the Narain lattice $\Gamma(\tau, \rho)$ and its equivalence class $[\Gamma(\tau, \rho)] \in \mathcal{M}_{2,2}$. For generic $\tau, \rho \in \mathbf{H}_+$, the class $[\Gamma(\tau, \rho)]$ does not correspond to an RCFT, i.e. is not in the image of $RCFT_D$ under the map (3.9) for any D.

Definition 3.7. *We call a Narain lattice $\Gamma(\tau, \rho)$ rational if its class $[\Gamma(\tau, \rho)]$ is in the image of the "over-lattice map" F given by (3.9).*

To study the rationality condition for $\Gamma(\tau, \rho)$, we consider

$$\Pi_l := \Gamma(\tau, \rho) \cap \mathbf{R}^{2,0} \ \ , \ \ \Pi_r := \Gamma(\tau, \rho) \cap \mathbf{R}^{0,2} \ \ . \tag{3.11}$$

We call them *the momentum lattices of (left and right chiral algebras corresponding to)* $\Gamma(\tau, \rho)$. Note that: 1) The lattices Π_l, Π_r are sublattices of $\Gamma(\tau, \rho)$, which are zero for generic τ, ρ. 2) The natural embedding $\Pi_l \hookrightarrow \Gamma(\tau, \rho)$ is a *primitive embedding*, i.e. the natural map $\Gamma(\tau, \rho)^* = \Gamma(\tau, \rho) \to \Pi_l^*$ is surjective, where $x \in \Gamma(\tau, \rho)^*$ is mapped to $\langle x, * \rangle : \Pi_l \to \mathbf{Z}$ under the natural map. Likewise for Π_r. The following properties (Propositions 3.8 and 3.10) are well-known in the literatures [DHMV][Mo][Wa].

Proposition 3.8. *The Narain lattice $\Gamma(\tau, \rho)$ is rational if and only if*

$$rk\Pi_l = rk\Pi_r = 2 \ . \tag{3.12}$$

Proof. ⇒) Suppose $\Gamma(\tau, \rho)$ is rational, i.e. $[\Gamma(\tau, \rho)] = [\Gamma^\varphi]$ for some triple $(\Gamma_l, \Gamma_r, \varphi)$ with $\Gamma_l \subset \mathbf{R}^{2,0}$, and $\Gamma_r(-1) \subset \mathbf{R}^{0,2}$. This means that $\Gamma^\varphi = f\Gamma(\tau, \rho)$ for some

$f \in O(d;\mathbf{R}) \times O(d;\mathbf{R})$. So we have $\Pi_l := \Gamma(\tau,\rho) \cap \mathbf{R}^{2,0} = f\Gamma(\tau,\rho) \cap \mathbf{R}^{2,0} = \Gamma^\varphi \cap \mathbf{R}^{2,0} = \Gamma_l$, which has rank 2. Likewise for Π_r.

$\Leftarrow$) When the both sublattices Π_l and Π_r have rank 2, they are orthogonal complements of each other in $\Gamma(\tau,\rho)$ (since Π_l, Π_r are primitive in $\Gamma(\tau,\rho)$). Thus, for $x \in \Gamma(\tau,\rho)$, we may consider the orthogonal decomposition $x = x_l + x_r$ ($x_l \in \Pi_l \otimes \mathbf{Q}, x_r \in \Pi_r \otimes \mathbf{Q}$) over $\mathbf{Q}$. The natural (surjective) map $\Gamma(\tau,\rho) \to \Pi_l^*$ may be given by $x \mapsto \langle x, * \rangle = \langle x_l, * \rangle$, and has the kernel exactly equal to Π_r. Similarly we have the natural surjective map $\Gamma(\tau,\rho) \to \Pi_r^*$ with the kernel Π_l. From this, as in [Ni], we obtain

$$\Pi_l^*/\Pi_l \xleftarrow[\iota_L]{\sim} \Gamma(\tau,\rho)/(\Pi_l \oplus \Pi_r) \xrightarrow[\iota_R]{\sim} \Pi_r^*/\Pi_r \quad . \tag{3.13}$$

We have now $\iota_R \circ \iota_L^{-1} : \Pi_l^*/\Pi_l \xrightarrow{\sim} \Pi_r^*/\Pi_r$, which implies that Π_l and Π_r have the same determinant $|\Pi_l^*/\Pi_l| = |\Pi_r^*/\Pi_r| =: -D'$. Also, over the discriminant groups Π_l^*/Π_l and Π_r^*/Π_r, we have natural quadratic forms q_l and q_r given by

$$q_l(x_l \bmod \Pi_l) = \langle x_l, x_l \rangle \bmod 2\mathbf{Z} \ \ , \ \ q_r(x_r \bmod \Pi_r) = \langle x_r, x_r \rangle \bmod 2\mathbf{Z},$$

taking values in $\mathbf{Q}/2\mathbf{Z}$. Since $\Gamma(\tau,\rho)$ is even, integral, we have

$$q_l(x_l) + q_r(x_r) \equiv \langle x_l, x_l \rangle + \langle x_r, x_r \rangle = \langle x_l + x_r, x_l + x_r \rangle = \langle x, x \rangle \equiv 0 \bmod 2\mathbf{Z}.$$

Therefore the isomorphisms in (3.13) give rise to an isometry of the quadratic forms: $\iota_R \circ \iota_L^{-1} : (\Pi_l^*/\Pi_l, q_l) \to (\Pi_r^*/\Pi_r, -q_r)$. By construction, the over-lattice determined by the triple $(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})$ is $\Gamma(\tau,\rho)$ (see Proposition 1.6.1 in [Ni]). Therefore $[\Gamma(\tau,\rho)]$ is in the image of the map F, i.e. $\Gamma(\tau,\rho)$ is rational. □

For a rational Narain lattice $\Gamma(\tau,\rho)$, the correspondence

$$[\Gamma(\tau,\rho)] \mapsto [(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})] \tag{3.14}$$

defines the inverse of the map

$$F : \bigcup_D RCFT_D \to \mathcal{N} := \{ [\Gamma(\tau,\rho)] \,|\, \Gamma(\tau,\rho) : \text{rational} \} \ .$$

Proposition 3.9. *The "over-lattice map" given by (3.9) defines a bijection*

$$F : \bigcup_D RCFT_D \ \to \ \mathcal{N} \ \ . \tag{3.15}$$

For later use, we present equivalent characterizations given in the literatures [DHMV][Mo][Wa]:

Proposition 3.10. *Consider (3.11). The following statements are equivalent;*
1) $rk\,\Pi_l = 2$ *or* $rk\,\Pi_r = 2$,
2) $rk\,\Pi_l = rk\,\Pi_r = 2$,
3) $\tau, \rho \in \mathbf{Q}(\sqrt{D})$ *for some integer* $D < 0$.

Proof. 1) $\Rightarrow$ 2): By symmetry, we consider only the case $\mathrm{rk}\,\Pi_l = 2$. By definition of Π_l and Π_r, they are sublattices of the Narain lattice $\Gamma(\tau,\rho)$. Furthermore we have

$\Pi_r \subset \Pi_l^\perp$, where $\Pi_l^\perp$ is the orthogonal complement of Π_l in $\Gamma(\tau,\rho)$. Let $x \in \Pi_l^\perp$. Then we have the unique decomposition $x = x_l + x_r$ over $\mathbf{R}$ with some $x_l \in \mathbf{R}^{2,0}$ and $x_r \in \mathbf{R}^{0,2}$. Since Π_l has rank 2, we see that $x_l \in \Pi_l \otimes \mathbf{R} = \mathbf{R}^{2,0}$. It follows that $\langle x_l, x\rangle = \langle x_l, x_l\rangle = 0$, so that $x_l = 0$. Therefore $x = x_r \in \Pi_r$. This shows that $\Pi_r \supset \Pi_l^\perp$, hence $\Pi_r = \Pi_l^\perp$ has rank 2.
2) $\Rightarrow$ 1) is obvious.
1) $\Leftrightarrow$ 3): Fix a basis of $\Gamma(\tau,\rho)$ as in (2.12). Then an arbitrary element of $\Gamma(\tau,\rho)$ may be represented by

$$p(m,n) = m_1\mathbf{e}_1(\tau,\rho) + m_2\mathbf{e}_2(\tau,\rho) + n_1\mathbf{e}_3(\tau,\rho) + n_1\mathbf{e}_4(\tau,\rho) \ . \tag{3.16}$$

By (2.2) we have a unique decomposition $p(m,n) = p_l(m,n) + p_r(m,n)$ over $\mathbf{R}$ where $p_l(m,n) = {}^t(*,*,0,0)$ and $p_r(m,n) = {}^t(0,0,*,*)$. We have

$$p(m,n) \in \Pi_l = \Gamma(\tau,\rho) \cap \mathbf{R}^{2,0} \ \Leftrightarrow\ p_r(m,n) = 0.$$

This is equivalent to

$$\begin{cases} \tau_2 m_1 \qquad\quad - \rho_2 n_1 - (\rho_1\tau_2 + \tau_1\rho_2)n_2 &= 0 \\ -\tau_1 m_1 + m_2 + \rho_1 n_1 + (\rho_1\tau_1 - \tau_2\rho_2)n_2 &= 0 \ . \end{cases} \tag{3.17}$$

The integral solutions to these linear equations determine the lattice Π_l. The lattice of integral solutions ($\cong \Pi_r$) is of maximal rank, i.e. 2, if and only if they can be solved over $\mathbf{Q}$, or in other words, the normal vectors of the hyperplanes (3.17) are in $\mathbf{Q}^4$. Therefore we have

$$\mathrm{rk}\ \Pi_l = 2 \ \Leftrightarrow\ \tau_1, \rho_1, \rho_1\tau_1 - \tau_2\rho_2, \frac{\rho_2}{\tau_2} \in \mathbf{Q} \ \Leftrightarrow\ \tau_1, \tau_2^2, \rho_1, \rho_2^2, \frac{\rho_2}{\tau_2} \in \mathbf{Q}. \tag{3.18}$$

The last condition says that $\tau = \tau_1 + \sqrt{-1}\tau_2$ satisfy a quadratic equation $a\tau^2 + b\tau + c = 0$ for some integers a, b, c ($b^2 - 4ac < 0$) and that the same holds for $\rho = \rho_1 + \sqrt{-1}\rho_2$. By the condition $\frac{\rho_2}{\tau_2} \in \mathbf{Q}$, we see that τ, ρ must be in the same quadratic imaginary fields $\mathbf{Q}(\sqrt{D})$. Conversely, it is straightforward to see that for $\tau, \rho \in \mathbf{Q}(\sqrt{D})$ the last condition of (3.18) holds. Thus we obtain the equivalence of 1) and 3). □

Remark. Condition 3) above was found by Moore [Mo] to characterize rationality of CFTs on T^2. It is also pointed out by him that the condition that $\tau \in \mathbf{Q}(\sqrt{D})$ is equivalent to that the elliptic curve E_τ is of CM type, namely E_τ has non-trivial endmorphisms (see e.g. [Ha,IV.4]). A generalization to higher dimensional tori $T^{2d} (d \geq 1)$ has been done in [Wa, Theorem 4.5.5], where it was shown that rationality implies that T^{2d} is isogeneus to a product $E_{\tau_1} \times E_{\tau_2} \times \cdots \times E_{\tau_d}$ of elliptic curves of CM types. (More precisely, it is shown that rationality is equivalent to that T^{2d} is isogeneous to a product as above with each τ_i, ρ_i in $\mathbf{Q}(\sqrt{D_i})$ for some $D_i < 0$ $(i = 1, \cdots, d)$.) As shown in [Mo][Wa], these points corresponding to RCFTs are dense in the Narain moduli space. It is argued in [Mo] that a similar density property holds for string compactifications on K3 surfaces, where RCFTs correspond to σ-models on singular K3 surfaces, i.e. K3 surfaces with $\rho(X) = 20$ (see e.g. [SI]).

In case of Calabi-Yau compactifications in dimension three, the connection to CFTs is less clear except at some special points such as the Gepner points, where one has a precise dictionary comparing CFTs and the geometry of Calabi-Yau threefolds (see e.g. [Gr]). Recently, Gukov-Vafa [GV] have proposed a criterion for rationality of sigma models on Calabi-Yau threefolds X. They conjecture that RCFT occurs if and only if both X and its mirror manifold $X^\vee$ are of CM type. Roughly, a Calabi-Yau threefold is called of CM type if its (Weil and Griffiths) intermediate Jacobians are of CM type. See [Bo, Theorem 2.3], for more precise definitions and several other equivalent definitions of CM type Calabi-Yau threefolds.

§4. Classification of $c = 2$ RCFT — primitive case —

In this section, we classify the $c = 2$ RCFT data $[(\Gamma_l, \Gamma_r, \varphi)]$ of primitive triples. More precisely we describe the image of the set of $RCFT_D^p$ under the "over-lattice map" F in (3.15), and determine its cardinality (up to an involution).

(4-1) Narain lattices parameterized by $Cl(D)$. Given a positive definite quadratic form $Q(a,b,c) : f(x,y) = ax^2 + bxy + cy^2$, we can write

$$f(x,y) = a|x + \tau_{Q(a,b,c)} y|^2 \ \ , \ \ \tau_{Q(a,b,c)} := \frac{b + \sqrt{b^2 - 4ac}}{2a},$$

where $D = b^2 - 4ac < 0$. So we have a map

$$Q(a,b,c) \mapsto \tau_{Q(a,b,c)} \in \mathbf{H}_+ := \{x + \sqrt{-1}y \,|\, y > 0\}. \tag{4.4}$$

It is easy to verify that the $SL_2\mathbf{Z}$ action on quadratic forms is compatible with the $PSL_2\mathbf{Z}$ actions on $\mathbf{H}_+$ under this map. This shows that the $PSL_2\mathbf{Z}$ orbit of $\tau_{Q(a,b,c)} \in \mathbf{H}_+$ depends only on the class $C = [Q(a,b,c)] \in CL(D)$. We denote the orbit by τ_C. We also write $\rho_{Q(a,b,c)} := \tau_{Q(a,b,c)}$, $\rho_C := \tau_C$, and put

$$[\Gamma(\tau_C, \rho_{C'})] := [\Gamma(\tau_Q, \rho_{Q'})]$$

for any $Q \in C$, $Q' \in C'$. This makes sense since the equivalence class $[\Gamma(\tau,\rho)]$ is invariant under $PSL_2\mathbf{Z} \times PSL_2\mathbf{Z}$ acting on τ, ρ, by Proposition 2.7 1). Note that if $Q(a_1,b_1,c_1), Q(a_2,b_2,c_2)$ belong to the same class $C \in CL(D)$, then $Q(a_1,-b_1,c_1)$, $Q(a_2,-b_2,c_2)$ also belong to the same class. Since $-\bar{\rho}_{Q(a,b,c)} = \rho_{Q(a,-b,c)}$, it follows that the involution $(\tau,\rho) \mapsto (\tau, -\bar{\rho})$ on $\mathbf{H}_+ \times \mathbf{H}_+$, induces an involution on the classes $[\Gamma(\tau_C, \rho_{C'})]$. If C, C' are primitive, then $[\Gamma(\tau_C,\rho_{C'})] \mapsto [\Gamma(\tau_C, \rho_{C'^{-1}})]$ under this involution, because $[Q(a,-b,c)] = [Q(a,b,c)]^{-1}$ in the group $Cl(D)$ (see Proposition A.7 in Appendix A).

Definition 4.4. *For negative integers $D < 0$, we put*

$$\mathcal{N}_D^p := \{\ [\Gamma(\tau_{\mathcal{C}}, \rho_{\mathcal{C}'})] \ \mid\ \mathcal{C}, \mathcal{C}' \in Cl(D)\ \} \ \ , \ \ \widetilde{\mathcal{N}}_D^p := \mathcal{N}_D^p / \mathbf{Z}_2\,, \tag{4.5}$$

where $\mathbf{Z}_2$ denotes the involution on $\widetilde{\mathcal{N}}_D^p$ induced by $(\tau,\rho) \mapsto (\tau, -\bar{\rho})$ on $\mathbf{H}_+ \times \mathbf{H}_+$.

Proposition 4.5. *We have a canonical bijection* $\widetilde{\mathcal{N}}_D^p \cong Sym^2\widetilde{Cl}(D)$. *In particular, we have*

$$|\widetilde{\mathcal{N}}_D^p| = \frac{1}{2}\tilde{h}(D)(\tilde{h}(D)+1). \tag{4.6}$$

Proof. Obviously, we have a surjective map

$$Cl(D)\times Cl(D) \to \widetilde{\mathcal{N}}_D^p, \qquad (C,C') \mapsto [\Gamma(\tau_C,\rho_{C'})]\, mod\, \mathbf{Z}_2.$$

Since $\mathbf{Z}_2 : [\Gamma(\tau_C,\rho_{C'})] \mapsto [\Gamma(\tau_C,\rho_{C'^{-1}})]$, the pairs (C,C') and $(C,{C'}^{-1})$ have the same image. Since $[\Gamma(\tau,\rho)]$ is invariant under duality transformations, we have $[\Gamma(\tau,\rho)] = [\Gamma(\rho,\tau)] = [\Gamma(-\bar\rho,-\bar\tau)]$, by Proposition 2.6. It follows that the pairs $(C,C'),(C',C),({C'}^{-1},C^{-1})$ also have the same image under the map. Note that $\widetilde{Cl}(D) = Cl(D)/\approx$, where $\approx$ is the relation $C \approx C^{-1}$. This shows that our map descends to $Sym^2\widetilde{Cl}(D) \to \widetilde{\mathcal{N}}_D^p$. It remains to show that this is injective.

Let $(C_1,C_1'),(C_2,C_2') \in Cl(D)\times Cl(D)$ be two pairs having the same image. Then $[\Gamma(\tau_{Q_1},\rho_{Q_1'})] = [\Gamma(\tau_{Q_2},\rho_{Q_2'})]$ or $[\Gamma(\tau_{Q_2},-\bar\rho_{Q_2'})]$, where $Q_i \in C_i$, $Q_i' \in C_i'$. By Proposition 2.7 1), it follows that $(\tau_{Q_1},\rho_{Q_1'})$ transforms, by a duality transformation on $\mathbf{H}_+\times\mathbf{H}_+$, to either $(\tau_{Q_2},\rho_{Q_2'})$ or $(\tau_{Q_2},-\bar\rho_{Q_2'})$. By Proposition 2.6, $(\tau_{Q_1},\rho_{Q_1'})$ transforms, by some $g \in P(SL_2\mathbf{Z}\times SL_2\mathbf{Z}) \rtimes (\mathbf{Z}_2\times\mathbf{Z}_2)$, to one of the following:

$$(\tau_{Q_2},\rho_{Q_2'}),\ (\rho_{Q_2'},\tau_{Q_2}),\ (-\bar\rho_{Q_2'},-\bar\tau_{Q_2}),\ (\tau_{Q_2},-\bar\rho_{Q_2'}),\ (-\bar\rho_{Q_2'},\tau_{Q_2}),\ (\rho_{Q_2'},-\bar\tau_{Q_2}).$$

It follows that (C_1,C_1') is equal to one of the following:

$$(C_2,C_2'),\ (C_2',C_2),\ ({C_2'}^{-1},C_2^{-1}),\ (C_2,{C_2'}^{-1}),\ ({C_2'}^{-1},C_2),\ (C_2',C_2^{-1}).$$

Therefore (C_1,C_1') and (C_2,C_2') represent the same element in $Sym^2\widetilde{Cl}(D)$. This shows that the map $Sym^2\widetilde{Cl}(D) \to \widetilde{\mathcal{N}}_D^p$ is injective. □

(4-2) Key lemma. If $\Gamma(\tau,\rho)$, $\Gamma(\tau',\rho')$ are equivalent Narain lattices, then their left momentum lattices (3.11) are isomorphic; likewise for their right momentum lattices. By Proposition (2.7) 1), this happens if (τ,ρ) transforms to (τ',ρ') by a duality transformation on $\mathbf{H}_+\times\mathbf{H}_+$. In particular, the isomorphism class of the momentum lattices of $\Gamma(\tau_Q,\rho_{Q'})$ depends only on the equivalence classes $[Q] = \mathcal{C}, [Q'] = \mathcal{C}' \in Cl(D)$. We denote the isomorphism classes of the left and right momentum lattices by

$$\Gamma(\tau_{\mathcal{C}},\rho_{\mathcal{C}'})\cap\mathbf{R}^{2,0} \ \ ,\ \ \Gamma(\tau_{\mathcal{C}},\rho_{\mathcal{C}'})\cap\mathbf{R}^{0,2} \quad . \tag{4.7}$$

Lemma 4.6. (Key lemma) *For* $\mathcal{C}_1,\mathcal{C}_2 \in Cl(D)$, *we have*

$$\Gamma(\tau_{\mathcal{C}_1},\rho_{\mathcal{C}_2})\cap\mathbf{R}^{2,0} = q(\mathcal{C}_1 * \mathcal{C}_2^{-1}) \ \ ,\ \ \Gamma(\tau_{\mathcal{C}_1},\rho_{\mathcal{C}_2})\cap\mathbf{R}^{0,2} = q(\mathcal{C}_1*\mathcal{C}_2)(-1) \ \ , \tag{4.8}$$

where $*$ *is the Gauss product on* $Cl(D)$ *and* q *is the natural map* $Cl(D)\to\widetilde{Cl}(D)$.

Proof. To evaluate the isomorphism classes of the left and right momentum lattices (4.7), we choose forms $Q(a,b,c)\in\mathcal{C}_1$, $Q(a',b',c')\in\mathcal{C}_2$ which are *concordant*. In fact, by Lemma A.3 we can arrange that

$$(i)\ aa'\neq 0,\ (a,a')=1 \qquad (ii)\ b=b' \qquad (iii)\ \frac{b^2-D}{4aa'}\in\mathbf{Z}\ , \tag{4.9}$$

where $D = b^2 - 4ac = {b'}^2 - 4a'c'$. By definition, the Gauss product $\mathcal{C}_1 * \mathcal{C}_2$ is

$$[Q(a,b,c)] * [Q(a',b',c')] := [Q(aa', b, \frac{b^2 - D}{4aa'})] \ .$$

(See Appendix A for details.) We will compute a $\mathbf{Z}$-basis for the right momentum lattice Π_r of $\Gamma(\tau,\rho)$, $\tau := \tau_{Q(a,b,c)}, \rho := \rho_{Q(a',b',c')}$, and compare the resulting quadratic form of Π_r with $Q(aa', b, \frac{b^2-D}{4aa'})$. Likewise for Π_l.

• $\mathbf{Z}$-basis for Π_r. A vector $p \in \Gamma(\tau,\rho)$ lies in Π_r iff $p_l = 0$, where $p = p_l + p_r$ is its decomposition under (2.2). But it is difficult to find a $\mathbf{Z}$-basis by solving $p_l = 0$ directly. A better way is to first find a $\mathbf{Q}$-basis, as follows. Note that $(\Pi_r)^\perp = \Pi_l$ in $\Gamma(\tau,\rho)$. Since Π_l is a rank two sublattice in a rank four lattice, it must be defined by two independent linear integral equations whose coefficients necessarily form a $\mathbf{Q}$-basis of Π_r. Thus to find a $\mathbf{Q}$-basis of Π_r, we can write down defining equations for Π_l and read off the coefficients. As in the proof of Proposition 3.8, we denote an arbitrary vector in $\Gamma(\tau,\rho)$ by $p(m,n) = m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + n_1\mathbf{e}_3 + n_2\mathbf{e}_4$. Then

$$\begin{aligned} p(m,n) \in \Pi_l &\Leftrightarrow p_r(m,n) = 0 \\ &\Leftrightarrow \left\{ \begin{array}{ll} 2a'm_1 \qquad\qquad - 2a\ n_1 \qquad - 2bn_2 & = 0 \\ -2a'bm_1 + 4aa'm_2 + 2abn_1 + (b^2 + D)n_2 & = 0 \ . \end{array} \right. \end{aligned} \tag{4.10}$$

The last two equations read $\langle w_1, p(m,n)\rangle = \langle w_2, p(m,n)\rangle = 0$, where

$$w_1 = -a\,\mathbf{e}_1 - b\,\mathbf{e}_2 + a'\,\mathbf{e}_3 \ \ , \ \ w_2 = ab\,\mathbf{e}_1 + (b^2 - 2ac)\,\mathbf{e}_2 - a'b\,\mathbf{e}_3 + 2aa'\,\mathbf{e}_4.$$

So w_1, w_2 form a $\mathbf{Q}$-basis of $\Pi_r \otimes \mathbf{Q}$. Consider the new $\mathbf{Q}$-basis

$$w_1' = w_1 = -a\,\mathbf{e}_1 - b\,\mathbf{e}_2 + a'\,\mathbf{e}_3 \ \ , \ \ w_2' = \frac{1}{2aa'}(w_2 + bw_1) = -\frac{c}{a'}\,\mathbf{e}_2 + \mathbf{e}_4 \ \ .$$

Note that w_1' is an integral primitive vector because $(a,b) = (a,a') = 1$ and so is w_2' by $\frac{c}{a'} = \frac{b^2-D}{4aa'} \in \mathbf{Z}$. By looking at the coefficient of $\mathbf{e}_4$, we may claim that they form a $\mathbf{Z}$-basis of Π_r.

Computing the quadratic form, we obtain

$$\begin{pmatrix} (w_1', w_1') & (w_1', w_2') \\ (w_2', w_1') & (w_2', w_2') \end{pmatrix} = \begin{pmatrix} -2aa' & -b \\ -b & -\frac{2c}{a'} \end{pmatrix} \ \ ,$$

which coincides with $-Q(aa', b, \frac{c}{a'}) \in \mathcal{C}_1 * \mathcal{C}_2$. It follows that the isomorphism class of the lattice $\Pi_r(-1)$ coincides with the improper equivalence class of $\mathcal{C}_1 * \mathcal{C}_2$, i.e. $[\Pi_r(-1)] = q(\mathcal{C}_1 * \mathcal{C}_2)$.

• $\mathbf{Z}$-basis for Π_l. As before, first we find a $\mathbf{Q}$-basis of $\Pi_l \otimes \mathbf{Q}$:

$$u_1 = 2a\,\mathbf{e}_1 + 2a'\,\mathbf{e}_3 \ \ , \ \ u_2 = 2ab\,\mathbf{e}_1 + (b^2 - D)\,\mathbf{e}_2 - 2a'b\,\mathbf{e}_3 + 4aa'\,\mathbf{e}_4 \ .$$

Put

$$u_1' = \frac{1}{2}u_1 = a\,\mathbf{e}_1 + a'\,\mathbf{e}_3 \ \ , \ \ u_2' = \frac{1}{4a}(u_2 + bu_1) = b\,\mathbf{e}_1 + c\,\mathbf{e}_2 + a'\,\mathbf{e}_4 \ .$$

Since $(a,a')=1$, we have integers k,l satisfying $ka+la'=1$. Now do a further change of basis to

$$u_1''=a\,\mathbf{e}_1+a'\,\mathbf{e}_3\ \ ,\ \ u_2''=\frac{1}{a'}(u_2'-kbu_1')=lb\,\mathbf{e}_1+\frac{c}{a'}\,\mathbf{e}_2-kb\,\mathbf{e}_3+\,\mathbf{e}_4\ .$$

As before, the coefficients of u_1'',u_2'' allow us to conclude that they form an integral basis for Π_l. Computing its quadratic form, we obtain

$$\begin{pmatrix}(u_1'',u_1'') & (u_1'',u_2'')\\ (u_2'',u_1'') & (u_2'',u_2'')\end{pmatrix}=\begin{pmatrix}2aa' & -b+2lba'\\ -b+2lba' & -2klb^2+\frac{2c}{a'}\end{pmatrix}\ \ , \tag{4.11}$$

which coincides with $Q(aa',-b+2lba',-klb^2+\frac{c}{a'})$.

Now note that

$$\begin{aligned}&Q(a,b,c)\ \underset{SL_2\mathbf{Z}}{\sim}\ Q(a,b-2(kb)a,c_k)=Q(a,-b+2lba',c_k)\\ &Q(a',b',c')=Q(a',b,c')\ \underset{GL_2\mathbf{Z}}{\sim}\ Q(a',-b,c')\ \underset{SL_2\mathbf{Z}}{\sim}\ Q(a',-b+2(lb)a',c_l')\ \ ,\end{aligned}$$

where $c_k:=c-(kb)b+(kb)^2a$ and $c_l':=c'+(lb)(-b)+(lb)^2a'$. It is easy to verify that $Q(a,-b+2lba',*)$ and $Q(a',-b+2(lb)a',*)$ are concordant forms whose Gauss product coincides with (4.11). Notice that we have used the improper equivalence once in the equivalence calculation above. So we conclude that $[\Pi_l]=q(\mathcal{C}_1*\mathcal{C}_2^{-1})$. □

Motivated by the key lemma, we introduce

Definition 4.7. *For classes $\mathbf{\Gamma}_l,\mathbf{\Gamma}_r\in\mathcal{L}^p(D)\equiv\widetilde{Cl}(D)$, we set*

$$\mathcal{G}_D^p(\mathbf{\Gamma}_l,\mathbf{\Gamma}_r):=\{\ (\mathcal{C}_1,\mathcal{C}_2)\in Sym^2Cl(D)\mid q(\mathcal{C}_1*\mathcal{C}_2^{-1})=\mathbf{\Gamma}_l,\ q(\mathcal{C}_1*\mathcal{C}_2)=\mathbf{\Gamma}_r\ \}$$

$$\mathcal{G}_D^p:=\bigsqcup_{\mathbf{\Gamma}_l,\mathbf{\Gamma}_r\in\mathcal{L}^p(D)}\mathcal{G}_D^p(\mathbf{\Gamma}_l,\mathbf{\Gamma}_r)\ \ (=Sym^2Cl(D))\ \ . \tag{4.12}$$

Let $\widetilde{\mathcal{G}}_D^p(\mathbf{\Gamma}_l,\mathbf{\Gamma}_r)$ and $\widetilde{\mathcal{G}}_D^p$, respectively, be the images under the natural map $Sym^2Cl(D)\to Sym^2\widetilde{Cl}(D)$ induced by $q:Cl(D)\to\widetilde{Cl}(D)$.

Note that $\mathcal{G}_D^p(\mathbf{\Gamma}_l,\mathbf{\Gamma}_r)$ is well-defined since $q(\mathcal{C}_1*\mathcal{C}_2)=q(\mathcal{C}_2*\mathcal{C}_1)$ and also $q(\mathcal{C}_1*\mathcal{C}_2^{-1})=q(\mathcal{C}_2*\mathcal{C}_1^{-1})$. Note also that $\widetilde{\mathcal{G}}_D^p(\mathbf{\Gamma}_l,\mathbf{\Gamma}_r)=\widetilde{\mathcal{G}}_D^p(\mathbf{\Gamma}_r,\mathbf{\Gamma}_l)$, and that $\tilde{\mathcal{G}}_D^p(\mathbf{\Gamma}_l,\mathbf{\Gamma}_r)=\tilde{\mathcal{G}}_D^p(\mathbf{\Gamma}_l',\mathbf{\Gamma}_r')$ holds if and only if $\{\mathbf{\Gamma}_l,\mathbf{\Gamma}_r\}=\{\mathbf{\Gamma}_l',\mathbf{\Gamma}_r'\}$. From this, we have:

$$\widetilde{\mathcal{G}}_D^p=\bigsqcup_{(\mathbf{\Gamma}_l,\mathbf{\Gamma}_r)\in\mathrm{Sym}^2\widetilde{Cl}(D)}\tilde{\mathcal{G}}_D^p(\mathbf{\Gamma}_r,\mathbf{\Gamma}_l)\ \ (=\mathrm{Sym}^2\widetilde{Cl}(D))\ \ . \tag{4.13}$$

(4-3) Classification. Now we complete our classification.

Proposition 4.8. *The inverse of the "over-lattice map" $F^{-1} : \mathcal{N} \to \cup_D RCFT_D$ (see (3.15)) restricts to an injective map $f : \mathcal{N}_D^p \to RCFT_D^p$, i.e.*

$$\begin{array}{cccc} F^{-1}: & \mathcal{N} & \to & \cup_D RCFT_D \\ & \cup & & \cup \\ f: & \mathcal{N}_D^p & \to & RCFT_D^p \end{array}$$

Proof. By (3.14), we have $F^{-1} : [\Gamma(\tau,\rho)] \mapsto [(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})]$. Let $[\Gamma(\tau,\rho)] \in \mathcal{N}_D^p$, i.e. $\tau = \tau_{Q_1}, \rho = \rho_{Q_2}$ for some forms Q_i of discriminant D. Then $[\Pi_l] = q([Q_1] * [Q_2]^{-1}), [\Pi_r] = q([Q_1] * [Q_2])$, by Lemma 4.6. It follows immediately that Π_l, Π_r are both primitive lattices of determinant $-D$. This shows that $F^{-1}(\mathcal{N}_D^p) \subset RCFT_D^p$. □

Consider the quotient

$$\widetilde{RCFT}_D^p := RCFT_D^p / (\pi_2 : [(\Gamma_l, \Gamma_r, \varphi)] \mapsto [(\Gamma_r, \Gamma_l, \varphi^{-1})]),$$

and the decomposition

$$\widetilde{RCFT}_D^p = \bigsqcup_{(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \in \mathrm{Sym}^2 \widetilde{Cl}(D)} \widetilde{RCFT}_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \quad , \tag{4.14}$$

with

$$\widetilde{RCFT}_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) := \{ \ [(\Gamma_l, \Gamma_r, \varphi)] \, mod \, (\pi_2) \mid [\Gamma_l] = \mathbf{\Gamma}_l, \ [\Gamma_r] = \mathbf{\Gamma}_r \ \}.$$

where $[(\Gamma_l, \Gamma_r, \varphi)] \, mod \, (\pi_2)$ represents the $\mathbf{Z}_2$-orbit; $[(\Gamma_l, \Gamma_r, \varphi)] \sim [(\Gamma_r, \Gamma_l, \varphi^{-1})]$.

Lemma 4.9. *The involutions $\pi_1 : (\tau,\rho) \mapsto (\tau, -\bar\rho)$ and $\pi_2 : [(\Gamma_l, \Gamma_r, \varphi)] \mapsto [(\Gamma_r, \Gamma_l, \varphi^{-1})]$ are compatible with the map $f : \mathcal{N}_D^p \to RCFT_D^p$, i.e. we have the diagram*

$$\begin{array}{ccc} [\Gamma(\tau,\rho)] & \xrightarrow{f} & [(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})] \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ [\Gamma(\tau,-\bar\rho)] & \xrightarrow{f} & [(\Pi_r(-1), \Pi_l, \iota_L \circ \iota_R^{-1})] \end{array} \quad . \tag{4.15}$$

Proof. Consider the explicit basis $\mathbf{e}_1(\tau,\rho), \cdots, \mathbf{e}_4(\tau,\rho)$ of $\Gamma(\tau,\rho)$ in (2.12). The involution $\pi_2 : (\tau,\rho) \mapsto (\tau,-\bar\rho)$ exchanges the upper two components and the lower two components of each $\mathbf{e}_j(\tau,\rho)$ (up to signs). Let $\pi_0 : \mathbf{R}^{2,2} \to \mathbf{R}^{2,2}$ be the linear map defined by ${}^t(1,0,0,0) \leftrightarrow {}^t(0,0,1,0)$, ${}^t(0,1,0,0) \leftrightarrow {}^t(0,0,0,1)$. Then we have $\pi_0 \Gamma(\tau,\rho) = \Gamma(\tau,-\bar\rho)$. Clearly π_0 is an involutive anti-isometry of $\mathbf{R}^{2,2}$, i.e. $\langle \pi_0(x), \pi_0(x) \rangle = -\langle x, x \rangle$, which exchanges the two subspaces $\mathbf{R}^{2,0}$ and $\mathbf{R}^{0,2}$. Let Π_l, Π_r be the left and the right momentum lattices, and ι_L, ι_R the isomorphisms (3.13) for $\Gamma(\tau,\rho)$. Then we have

$$\pi_0 \Pi_l = \Gamma(\tau,-\bar\rho) \cap \mathbf{R}^{0,2} =: \Pi_r', \qquad \pi_0 \Pi_r = \Gamma(\tau,-\bar\rho) \cap \mathbf{R}^{2,0} =: \Pi_l',$$

where Π_l', Π_r' are the left and the right momentum lattices of $\Gamma(\tau,-\bar\rho)$. This shows that $\pi_0 : \Pi_l \to \Pi_r'(-1)$, $\pi_0 : \Pi_r(-1) \to \Pi_l'$, define lattice isomorphisms. Let ι_L', ι_R' be the isomorphisms (3.13) for $\Gamma(\tau,-\bar\rho)$. Then it is easy to check that $\iota_R' \circ {\iota_L'}^{-1} = \bar\pi_0 \circ \iota_L \circ \iota_R^{-1} \circ \bar\pi_0$. It follows that the triples $(\Pi_l', \Pi_r'(-1), \iota_R' \circ {\iota_L'}^{-1})$, $(\Pi_r(-1), \Pi_l, \iota_L \circ \iota_R^{-1})$ are equivalent. This shows that $f : [\Gamma(\tau,-\bar\rho)] \mapsto [(\Pi_r(-1), \Pi_l, \iota_L \circ \iota_R^{-1})]$. □

Now we can state our main theorems:

Theorem 4.10. *Consider the map $g : \mathcal{G}_D^p \to \mathcal{N}_D^p$ defined by $(\mathcal{C}_1, \mathcal{C}_2) \mapsto [\Gamma(\tau_{\mathcal{C}_1}, \rho_{\mathcal{C}_2})]$, and the injective map $f : \mathcal{N}_D^p \to RCFT_D^p$ in Proposition 4.8. Then there exist corresponding induced maps $\tilde{g} : \widetilde{\mathcal{G}}_D^p \to \tilde{\mathcal{N}}_D^p$ and $\tilde{f} : \tilde{\mathcal{N}}_D^p \to \widetilde{RCFT}_D^p$ such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{G}_D^p & \xrightarrow{\ g\ } & \mathcal{N}_D^p & \xrightarrow{\ f\ } & RCFT_D^p \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \widetilde{\mathcal{G}}_D^p & \xrightarrow{\ \tilde{g}\ } & \tilde{\mathcal{N}}_D^p & \xrightarrow{\ \tilde{f}\ } & \widetilde{RCFT}_D^p \end{array}$$

Moreover $\tilde{g}$ is bijective and $\tilde{f}$ is injective.

Proof. In Proposition 4.5, we saw that the map

$$Cl(D) \times Cl(D) \to \mathcal{N}_D^p, \qquad (C_1, C_2) \mapsto [\Gamma(\tau_{C_1}, \rho_{C_2})],$$

induces the bijection $\widetilde{\mathcal{G}}_D^p = \mathrm{Sym}^2 \widetilde{Cl}(D) \to \tilde{\mathcal{N}}_D^p$ (as well as the surjection $g : \mathcal{G}_D^p = \mathrm{Sym}^2 Cl(D) \to \mathcal{N}_D^p$). This is the bijection $\tilde{g}$ we seek. The commutative diagram (4.15) implies that f induces a map $\tilde{f}$, as required. That $\tilde{f}$ is injective follows immediately from the fact that f is injective. □

Theorem 4.11.
1) The map $f : \mathcal{N}_D^p \to RCFT_D^p$ is bijective. Hence $\tilde{f}$ is also bijective.
2) The composition $\tilde{f} \circ \tilde{g}$ is a bijection with

$$\tilde{f} \circ \tilde{g} \left(\widetilde{\mathcal{G}}_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \right) = \widetilde{RCFT}_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \ . \tag{4.16}$$

Proof. 1) By the preceding theorem, both f and $\tilde{f}$ are injective. Surjectivity of f will be proved in section (5-2). Thus $\tilde{f}$ is also surjective by (4.15).
2) That $\tilde{f} \circ \tilde{g}$ is bijective follows from 1) and the preceding theorem. Let $(C_1, C_2) \in \mathcal{G}_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)$, i.e. $q(C_1 * C_2) = \mathbf{\Gamma}_l$ and $q(C_1 * C_2^{-1}) = \mathbf{\Gamma}_r$. By the key lemma (Lemma 4.6), we have $f \circ g((C_1, C_2)) = f([\Gamma(\tau_{C_1}, \rho_{C_2})]) \in RCFT_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)$. This shows that

$$\tilde{f} \circ \tilde{g} \left(\widetilde{\mathcal{G}}_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \right) \subset \widetilde{RCFT}_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r).$$

The reverse inclusion follows from the fact that $\tilde{f} \circ \tilde{g}$ is a bijection. □

Proposition 4.12. *For primitive classes $\mathbf{\Gamma}_l, \mathbf{\Gamma}_r \in \tilde{C}l(D)$, the bijection B_{Γ_l, Γ_r} : $RCFT_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \to O(\Gamma_l) \setminus Isom(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma_r)$ in (3.5) induces a bijection*

$$\widetilde{RCFT}_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \ \leftrightarrow \ O(\Gamma_l) \setminus Isom(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma_r) \ .$$

Proof. We only need to show that the surjective map $RCFT_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \to \widetilde{RCFT}_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)$, $[(\Gamma_l, \Gamma_r, \varphi)] \mapsto [(\Gamma_l, \Gamma_r, \varphi)] mod(\pi_2)$ is also injective. This is a restriction of the map $\pi_2 : RCFT_D^p \to \widetilde{RCFT}_D^p$. The preimage of a given $[(\Gamma_l, \Gamma_r, \varphi)] mod(\pi_2)$ consists of the two RCFT data $[(\Gamma_l, \Gamma_r, \varphi)], [(\Gamma_r, \Gamma_l, \varphi^{-1})]$. If $\mathbf{\Gamma}_l \neq \mathbf{\Gamma}_r$, then only the first RCFT data lie in $RCFT_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)$. If $\mathbf{\Gamma}_l = \mathbf{\Gamma}_r$, then the two RCFT data are of the shape $[(\Gamma, \Gamma, \psi)]$, $[(\Gamma, \Gamma, \psi^{-1})]$ for some $\psi \in O(A_\Gamma)$. In Appendix B, we show that $\psi = \psi^{-1}$. Thus, in either case, the preimage of $[(\Gamma_l, \Gamma_r, \varphi)] mod(\pi_2)$ in $RCFT_D^p(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)$ contains just one RCFT data. □

As a corollary to Theorem 4.11 and Proposition 4.12, we have the following equality.

Corollary 4.13. *For $\mathbf{\Gamma}_l, \mathbf{\Gamma}_r \in \widetilde{Cl}(D)$, and for arbitrary choices of lattices $\Gamma_l \in \mathbf{\Gamma}_l$ and $\Gamma_r \in \mathbf{\Gamma}_r$, we have*

$$|\widetilde{\mathcal{G}}^p_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)| = |O(\Gamma_l) \setminus Isom(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma_r)| \quad . \tag{4.17}$$

Remark. 1) It is interesting to note that this equality connects the double coset space, which is group theoretical and arithmetical in nature, to the Gauss product, which is algebraic.
2) The average one formula in Theorem 0.1 follows immediately from the above equality and (4.13).
3) As pointed out earlier, if Γ_l is not isogeneous to Γ_r, then the set $\mathrm{Isom}(A_{\Gamma_l}, A_{\Gamma_r})$ is empty (see e.g. [Ni, Corollary 1.9.4]), which implies $|\widetilde{\mathcal{G}}^p_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)| = 0$. This can also be shown directly from the properties of Gauss product (see [Ca]).

§5. Classification of $c = 2$ RCFT — non-primitive case —

We now consider the general case of even, positive definite, not necessarily primitive lattices $\mathbf{\Gamma}_l, \mathbf{\Gamma}_r \in \mathcal{L}(D)$ of fixed determinant $-D$. As one might expect, most of the arguments in the preceding section apply to this case, once the Gauss product is extended to $CL(D)$. This extension is done in Appendix A.

(5-1) Refinement of key lemma. Let us look more closely at the proof of the key lemma (Lemma 4.6). The proof relies only on the existence of concordant forms satisfying properties (4.9). Using those properties, we found a $\mathbf{Z}$-basis for the left and right momentum lattices $\Pi_l = \Gamma(\tau_{Q_1}, \rho_{Q_2}) \cap \mathbf{R}^{2,0}$, $\Pi_r = \Gamma(\tau_{Q_1}, \rho_{Q_2}) \cap \mathbf{R}^{0,2}$ for $Q_1 \in \mathcal{C}_1, Q_2 \in \mathcal{C}_2$, and identified their isomorphism classes $[\Pi_l]$ and $[\Pi_r]$ with the Gauss products $q(\mathcal{C}_1 * \mathcal{C}_2^{-1})$ and $q(\mathcal{C}_1 * \mathcal{C}_2)$, respectively. Thus in the case of non-primitive forms, if (4.9) holds for $C_1, C_2 \in CL(D)$ and the composition $C_1 * C_2$ is defined, then we expect that the same lemma holds for C_1, C_2, as we now show.

We recall some elementary notions introduced in Appendix A. We define the gcd $\lambda := gcd(a, b, c)$ of a class $[Q(a, b, c)] \in CL(D)$, and the scalar multiple of a class in an obvious manner. Clearly a class is primitive iff its gcd is 1. In particular, if λ is the gcd of $C \in CL(D)$, then $\frac{1}{\lambda}C$ is primitive. Two classes are *coprime* if their gcd's are coprime. It is shown in Appendix A that there is a composition law $C_1 * C_2$ for coprime classes $C_1, C_2 \in CL(D)$, that generalizes the Gauss product for primitive classes.

Lemma 5.1. (Refinement of the key lemma) *For coprime quadratic forms $C_1, C_2 \in CL(D)$, the following relations hold;*

$$\Gamma(\tau_{C_1}, \rho_{C_2}) \cap \mathbf{R}^{2,0} = q(C_1 * \sigma C_2) \quad , \quad \Gamma(\tau_{C_1}, \rho_{C_2}) \cap \mathbf{R}^{0,2} = q(C_1 * C_2)(-1) \quad , \tag{5.1}$$

where $*$ is the composition law for coprime classes in $CL(D)$, and q is the natural map $q : CL(D) \to \widetilde{CL}(D) \equiv \mathcal{L}(D)$.

Proof. As before, we can choose quadratic forms $Q(\alpha, \beta, \gamma) \in C_1$, $Q(\alpha', \beta', \gamma') \in C_2$ to evaluate the left hand sides of the equations (5.1). By Lemma A.3, we can arrange that

$$(i)\ \ \alpha\alpha' \neq 0,\ \ (\alpha, \alpha') = 1 \qquad (ii)\ \ \beta = \beta' \qquad (iii)\ \ \frac{\beta^2 - D}{4\alpha\alpha'} \in \mathbf{Z}\ . \tag{5.2}$$

Now we see that the proof of Lemma 4.6 is valid verbatim, by respectively replacing $a, b, c; a', b', c'$ there by $\alpha, \beta, \gamma; \alpha', \beta', \gamma'$ here, and C^{-1} there by σC here (cf. Proposition A.7, 4)). □

For the composition law on $CL(D)$, the following property will be useful (see Appendix A, Remark after Definition A.6 for a proof):

Lemma 5.2. *For coprime classes $C_1, C_2 \in CL(D)$, if $C_1 * C_2 \in Cl(D)$ then $C_1, C_2 \in Cl(D)$.*

(5-2) Proof of surjectivity in Theorem 4.11. We now proceed to proving the surjectivity part of Theorem 4.11.

Lemma 5.3. *Let $(\Gamma_l, \Gamma_r, \varphi)$ be a (not necessarily primitive) triple of determinant $-D$ with $\Gamma_l \subset \mathbf{R}^{2,0}$ and $\Gamma_r(-1) \subset \mathbf{R}^{0,2}$, and consider the corresponding over-lattice Γ^φ. There exist coprime quadratic forms Q_1, Q_2 of discriminant D such that Γ^φ is equivalent to $\Gamma(\tau_{Q_1}, \rho_{Q_2})$.*

Proof. By construction, as in section (3-4), $\Gamma^\varphi \subset \mathbf{R}^{2,2}$ is a Narain lattice. By Proposition 2.7 2), $\Gamma^\varphi = h\Gamma(\tau, \rho)$ for some $\tau, \rho \in \mathbf{H}_+$ and $h \in O(2, \mathbf{R}) \times O(2, \mathbf{R})$. So we have

$$\Gamma_l = \Gamma^\varphi \cap \mathbf{R}^{2,0} = h\Pi_l \ \ , \ \ \Gamma_r(-1) = \Gamma^\varphi \cap \mathbf{R}^{0,2} = h\Pi_r \ \ , \tag{5.3}$$

where Π_l, Π_r are the momentum lattices of the left and right chiral algebras of $\Gamma(\tau, \rho)$. This shows that rk Π_l = rk $\Pi_r = 2$. By Proposition 3.10, we have $\tau, \rho \in \mathbf{Q}(\sqrt{D'})$ for some $D' < 0$. So explicitly, τ and ρ may be written

$$\tau = \frac{b + c\sqrt{D'}}{2a} = \frac{bc' + \sqrt{(cc')^2 D'}}{2ac'} \ \ , \ \ \rho = \frac{b' + c'\sqrt{D'}}{2a'} = \frac{b'c + \sqrt{(cc')^2 D'}}{2a'c} \ \ ,$$

where $(a, b, c) = (a', b', c') = 1$ and $a, a', c, c' > 0$. Now we express these values in terms of the following quadratic forms of discriminant $(Kcc')^2 D'$, i.e. we have $\tau = \tau_{Q_1}, \rho = \rho_{Q_2}$ with

$$Q_1 = Q(Kac', Kbc', K\frac{(bc')^2 - (cc')^2 D'}{4ac'}) \ \ ,$$
$$Q_2 = Q(Ka'c, Kb'c, K\frac{(b'c)^2 - (cc')^2 D'}{4a'c}) \ \ ,$$

where a positive integer K is chosen so that both Q_1 and Q_2 are integral. Clearly, we can choose such an integer K so that Q_1 and Q_2 are coprime as well. Thus the composition $[Q_1] * [Q_2]$ makes sense. Now by Lemma 5.1, we find that $\det \Pi_l = \det \Pi_r = (Kcc')^2 D'$. It follows that $D = (Kcc')^2 D'$, since we have $\det \Pi_l = \det \Gamma_l = D$ by (5.3). Since $\Gamma^\varphi = h\Gamma(\tau, \rho) = h\Gamma(\tau_{Q_1}, \rho_{Q_2})$, this completes the proof. □

Proof of the surjectivity in Theorem 4.11. We want to show that the map $f : \mathcal{N}_D^p \to RCFT_D^p$, $[\Gamma(\tau, \rho)] \mapsto [(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})]$ is surjective. Recall that every element in $RCFT_D^p$ has the shape $[(\Gamma_l, \Gamma_r, \varphi)]$, where $\Gamma_l \subset \mathbf{R}^{2,0}$ and $\Gamma_r(-1) \subset \mathbf{R}^{0,2}$ are of determinant $-D$ and also have primitive bilinear forms. By the preceding lemma, we have $[\Gamma^\varphi] = [\Gamma(\tau,\rho)]$, where $\tau = \tau_{Q_1}$, $\rho = \rho_{Q_2}$, for some coprime forms Q_1, Q_2 of discriminant D. In particular, we have

$$\Pi_l = \Gamma(\tau,\rho) \cap \mathbf{R}^{2,0} = h\Gamma^\varphi \cap \mathbf{R}^{2,0} = h\Gamma_l$$

for some $h \in O(2;\mathbf{R}) \times O(2;\mathbf{R})$. This shows that $\Pi_l \cong \Gamma_l$; likewise $\Pi_r \cong \Gamma_r(-1)$. By Lemma 5.1, we have

$$[\Gamma_l] = [\Pi_l] = q([Q_1] * \sigma[Q_2]) \ \ , \ \ [\Gamma_r(-1)] = [\Pi_r] = q([Q_1] * [Q_2]) \ .$$

Since the bilinear forms of Γ_l, Γ_r are primitive, Lemma 5.2 implies $[Q_1], [Q_2]$ are also primitive, i.e. lie in $Cl(D)$. This shows that $[\Gamma(\tau,\rho)] \in \mathcal{N}_D^p$. Hence we have $f : [\Gamma(\tau,\rho)] \mapsto [(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})]$. It remains to show that $[(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})] = [(\Gamma_l, \Gamma_r, \varphi)]$. It is straightforward to check that the isomorphism $h : \Gamma^\varphi \to \Gamma(\tau,\rho)$ induces an equivalence of triples $(\Gamma_l, \Gamma_r, \varphi) \sim (\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})$. □

(5-3) Classification. Using the refined key lemma (Lemma 5.1), it is easy to generalize Theorems 4.10, 4.11, and Proposition 4.12 to the non-primitive case, with only minor modifications. The following definitions are, respectively, extensions of Definitions 4.7 and 4.4:

Definition 5.4. *For $\mathbf{\Gamma}_l, \mathbf{\Gamma}_r \in \mathcal{L}(D) \equiv \widetilde{CL}(D)$, we set*

$$\mathcal{G}_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) := \{\ (C_1, C_2) \in PSym^2 CL(D) \mid q(C_1 * \sigma C_2) = \mathbf{\Gamma}_l,\ q(C_1 * C_2) = \mathbf{\Gamma}_r\ \},$$

$$\mathcal{G}_D := \bigsqcup_{\mathbf{\Gamma}_l, \mathbf{\Gamma}_r \in CL(D)} \mathcal{G}_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \ \ (= PSym^2 CL(D)) \ , \tag{5.4}$$

where $PSym^2 CL(D) := \{(C, C') \in Sym^2 CL(D) \mid C, C' : \text{coprime}\ \}$. Also we define $\widetilde{\mathcal{G}}_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)$ and $\widetilde{\mathcal{G}}_D$ to be the image under the natural map $PSym^2 CL(D) \to PSym^2 \widetilde{CL}(D)$ induced by $q : CL(D) \to \widetilde{CL}(D)$.

As before, we have the following decomposition:

$$\widetilde{\mathcal{G}}_D = \bigsqcup_{(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \in \mathrm{PSym}^2 \widetilde{CL}(D)} \tilde{\mathcal{G}}_D(\mathbf{\Gamma}_r, \mathbf{\Gamma}_l) \ \ (= \mathrm{PSym}^2 \widetilde{CL}(D)) \ \ . \tag{5.5}$$

Definition 5.5. *We set:*

$$\mathcal{N}_D := \{\ [\Gamma(\tau_{C_1}, \rho_{C_2})] \mid C_1, C_2 \in CL(D) : \ \text{coprime}\ \} \ \ ,$$

and $\widetilde{\mathcal{N}}_D$ be the image under $q : CL(D) \to \widetilde{CL}(D)$.

Let $\widetilde{RCFT}_D$ be the quotient of $RCFT_D$ by the involution π_2 (see Lemma 4.9). Then we have the decomposition

$$\widetilde{RCFT}_D = \bigsqcup_{(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \in \mathrm{Sym}^2 \widetilde{CL}(D)} \widetilde{RCFT}_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \ \ , \tag{5.6}$$

as in (4.14).

Proposition 5.6. *The inverse of the "over-lattice map" $F^{-1} : \mathcal{N} \to \cup_D RCFT_D$ (see (3.15)) restricts to an injective map $f : \mathcal{N}_D \to RCFT_D$. Moreover f is compatible with the involutions $\pi_1 : (\tau, \rho) \mapsto (\tau, -\bar{\rho})$ and $\pi_2 : [(\Gamma_l, \Gamma_r, \varphi)] \mapsto [(\Gamma_r, \Gamma_l, \varphi^{-1})]$ (see the diagram (4.15)).*

Proof. If $F^{-1} : [\Gamma(\tau_{C_1}, \rho_{C_2})] \in \mathcal{N}_D \mapsto [(\Pi_l, \Pi_r(-1), \varphi)]$, then $[\Pi_l] = q(C_1 * \sigma C_2)$, $[\Pi_r] = q(C_1 * C_2)$ by Lemma 5.1. From these relations, we conclude $\det \Pi_l = \det \Pi_l = D$ and hence $[(\Pi_l, \Pi_r(-1), \varphi)] \in RCFT_D$. The compatibility with the involutions π_1, π_2 follows from the same argument as in Lemma 4.9. □

Now Theorem 4.10 generalizes to

Theorem 5.7. *Let g be the map $\mathcal{G}_D \to \mathcal{N}_D$ defined by $(C_1, C_2) \mapsto [\Gamma(\tau_{C_1}, \rho_{C_2})]$, and f be the injective map $\mathcal{N}_D \to RCFT_D$ given in Proposition 5.6. Then there exist corresponding induced maps $\tilde{g} : \widetilde{\mathcal{G}}_D \to \tilde{\mathcal{N}}_D$ and $\tilde{f} : \tilde{\mathcal{N}}_D \to \widetilde{RCFT}_D$ such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{G}_D & \xrightarrow{g} & \mathcal{N}_D & \xrightarrow{f} & RCFT_D \\ \downarrow q & & \downarrow \pi_1 & & \downarrow \pi_2 \\ \widetilde{\mathcal{G}}_D & \xrightarrow{\tilde{g}} & \tilde{\mathcal{N}}_D & \xrightarrow{\tilde{f}} & \widetilde{RCFT}_D \end{array}$$

Moreover $\tilde{g}$ is bijective and $\tilde{f}$ is injective.

Since the proof of Theorem 4.10 carries over with straightforward modifications, we do not repeat it here. Theorem 4.11 now generalizes to

Theorem 5.8.
1) The map $f : \mathcal{N}_D \to RCFT_D$ is bijective. Hence $\tilde{f}$ is also bijective.
2) The composition $\tilde{f} \circ \tilde{g}$ is a bijection with

$$\tilde{f} \circ \tilde{g} \left(\widetilde{\mathcal{G}}_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \right) = \widetilde{RCFT}_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \ . \tag{5.7}$$

Proof. 1) By Proposition 5.6, f is injective. The proof of surjectivity of f is similar to that of Theorem 4.11 in section (5-2). Since f is bijective, so is $\tilde{f}$ by Proposition 5.6.
2) Again, the proof of Theorem 4.11 2) carries over to the non-primitive case. □

Proposition 4.12 and its corollary, however, do not generalize immediately to non-primitive forms. This is because the relation $\varphi^2 = id$ for $\varphi \in O(A_\Gamma)$ no longer holds in general for a non-primitive lattice Γ.

Proposition 5.9.
1) If $\mathbf{\Gamma}_l \neq \mathbf{\Gamma}_r$ ($\mathbf{\Gamma}_l, \mathbf{\Gamma}_r \in \widetilde{CL}(D)$), the bijective map B_{Γ_l, Γ_r} in (3.5) defines a natural one-to-one correspondence

$$\widetilde{RCFT}_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r) \ \leftrightarrow \ O(\Gamma_l) \setminus Isom(A_{\Gamma_l}, A_{\Gamma_r}) / O(\Gamma_r) \ .$$

2) If $\mathbf{\Gamma}_l = \mathbf{\Gamma}_r =: \mathbf{\Gamma}$, there is a one-to-one correspondence

$$\widetilde{RCFT}_D(\mathbf{\Gamma}, \mathbf{\Gamma}) \ \leftrightarrow \ (O(\Gamma) \setminus Isom(A_\Gamma, A_\Gamma) / O(\Gamma)) \, / \sim ,$$

where $\sim$ represents the identification of $[\varphi]$ with $[\varphi^{-1}]$. The correspondence is given by mapping the class $[(\Gamma, \Gamma, \varphi)] \sim [(\Gamma, \Gamma, \varphi^{-1})]$ to the class $[\varphi] \sim [\varphi^{-1}]$.

The proof of Proposition 4.12 now carries over with some slight modifications in part 2), although we omit the details here. As a corollary, we have

Corollary 5.10. *For $\boldsymbol{\Gamma}_l, \boldsymbol{\Gamma}_r \in \widetilde{CL}(D)$, and for any choice of lattices $\Gamma_l \in \boldsymbol{\Gamma}_l$ and $\Gamma_r \in \boldsymbol{\Gamma}_r$, the following equality holds:*

$$|\widetilde{\mathcal{G}}_D(\boldsymbol{\Gamma}_l, \boldsymbol{\Gamma}_r)| = \begin{cases} |O(\Gamma_l) \setminus Isom(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma_r)| \ , & \boldsymbol{\Gamma}_l \neq \boldsymbol{\Gamma}_r \\ \frac{1}{2}|O(\Gamma) \setminus Isom(A_\Gamma, A_\Gamma)/O(\Gamma)| + \frac{1}{2}n_\Gamma & \boldsymbol{\Gamma}_l = \boldsymbol{\Gamma}_r =: \boldsymbol{\Gamma} \end{cases} \tag{5.8}$$

where $n_\Gamma := \#\{[\varphi] \in O(\Gamma) \setminus Isom(A_\Gamma, A_\Gamma)/O(\Gamma) \mid [\varphi^{-1}] = [\varphi]\,\}$.

Example. Here for the reader's convenience, we present an example of a non-primitive lattice Γ which has an isometry $[\varphi] \neq [\varphi^{-1}]$. The example has determinant $-D = 236$ and bilinear form

$$\begin{pmatrix} 12 & 2 \\ 2 & 20 \end{pmatrix} \text{ with } \Gamma = \mathbf{Z}v_1 \oplus \mathbf{Z}v_2.$$

The discriminant group Γ^*/Γ is isomorphic to $\mathbf{Z}_{d_1} \oplus \mathbf{Z}_{d_2} =: \langle u_{d_1}\rangle \oplus \langle u_{d_2}\rangle$ with $d_1 = 2$, $d_2 = 118$. The generators may be chosen explicitly as $u_{d_1} = \frac{1}{2}v_2$, $u_{d_2} = \frac{1}{118}(v_1 - 6v_2)$. The discriminant form q_Γ may be evaluated by using these expression for u_{d_1}, u_{d_2}. Now let us make the following orthogonal decomposition of the discriminant form (cf. Appendix B):

$$(\Gamma^*/\Gamma, q_\Gamma) = (\,(\mathbf{Z}_2 \oplus \mathbf{Z}_2) \oplus \mathbf{Z}_{59}, q_1 \oplus q_2\,)\ ,$$

where q_1 represents the discriminant form on the component $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. If we write the generators for each factor of $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_{59}$, respectively, by $u, v, v^\perp$, they are given by $u = \frac{1}{2}v_2$, $v = \frac{1}{2}v_1$, $v^\perp = \frac{1}{59}(v_1 - 6v_2)$. With respect to these generators, the discriminant form q_Γ may be represented by

$$q_\Gamma = q_1 \oplus q_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \oplus (\frac{12}{59})\ .$$

It is rather straightforward to determine all isometries in $O(A_\Gamma)$. The results are:

$$\begin{array}{l} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus (\pm 1)\ ,\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (\pm 1)\ ,\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus (\pm 1)\ , \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \oplus (\pm 1)\ ,\ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \oplus (\pm 1)\ ,\ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \oplus (\pm 1)\ . \end{array} \tag{5.9}$$

As for the group $O(\Gamma)$, we see that it is trivial, i.e. $O(\Gamma) = \{\pm id\}$. Looking at the induced action on Γ^*/Γ, we see that $O(\Gamma)$ acts on $O(A_\Gamma)$ by $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \oplus (\pm 1)\}$. Hence each of the six pairs in (5.9) represents the class $O(\Gamma) \setminus O(A_\Gamma)/O(\Gamma)$. Now for the classes $[\varphi_1] = \{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \oplus (\pm 1)\}$, $[\varphi_2] = \{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus (\pm 1)\}$, we see $[\varphi_1^{-1}] = [\varphi_2]$, $[\varphi_1] = [\varphi_2^{-1}]$. This should be contrasted to the fact that we have $\varphi^2 = id$ (and hence $[\varphi^{-1}] = [\varphi]$) for all $\varphi \in O(A_\Gamma)$ if Γ is a primitive lattice (see Appendix B). □

(5-4) Diagonal RCFTs. In their recent work [GV], Gukov and Vafa obtained a characterization of diagonal rational conformal field theories. We summarize their result as follows:

(Diagonal RCFTs) Let Π *be a lattice with quadratic form* $\lambda\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ *such that* $gcd(a,b,c)=1$ *and* $D=\lambda^2(b^2-4ac)$*. Then a necessary and sufficient condition for a class of Narain lattices* $[\Gamma(\tau,\rho)]\in\mathcal{N}_D$ *to satisfy* $f([\Gamma(\tau,\rho)])=[(\Pi,\Pi,id)]$ *is that*

$$\tau=\frac{b+\sqrt{D/\lambda^2}}{2a}\ \ ,\ \ \rho=\lambda a\tau\ \ , \tag{5.10}$$

where the equalities are understood to be up to $PSL_2\mathbf{Z}$ *transformations.*

This characterization can be seen as a special case of our general classification scheme as follows:

Proposition 5.11. (Diagonal triples) *Under the bijection* $f:\mathcal{N}_D\to RCFT_D$*, the 'diagonal' triples* $[(\Pi,\Pi,id)]$ *correspond to Narain lattices of the shape* $[\Gamma(\tau_C,\rho_{C_e})]$*, where* $[\Pi]=q(C)$*, and* $C_e\in CL(D)$ *is the unit class (see Proposition A.7 in Appendix A). Moreover, we have* $\rho_{C_e}=\lambda a\tau_C$ *if* $C=[\lambda Q'(a,b,c)]$*.*

Proof. Put $Q:=\lambda Q'(a,b,c)$, and let $(\Pi_l,\Pi_r,\iota_R\circ\iota_L^{-1})$ be the triple corresponding to the Narain lattice $\Gamma(\tau_Q,\rho_{Q_e})$, where $Q_e\in C_e$ is the reduced form. Then $f([\Gamma(\tau_C,\rho_{C_e})])=[(\Pi_l,\Pi_r,\iota_R\circ\iota_L^{-1})]$. By Lemma 5.1, we have $[\Pi_l]=[\Pi_r]=q(C*C_e)=q(C)$. In fact, by computing Π_l,Π_r explicitly, as in the proof of Lemma 4.6, one finds that $\iota_R\circ\iota_L^{-1}$ is induced by an isomorphism $\iota:\Pi_l\to\Pi_r$ (cf. [GV]). It follows that

$$f([\Gamma(\tau_C,\rho_{C_e})])=[(\Pi,\Pi,id)] \tag{5.11}$$

where $\Pi=\Pi_l$. Conversely, given any rank two lattice Π, if $C\in CL(D)$ with $\Pi=q(C)$, then (5.11) holds.

Note that ρ_{C_e} is the $PSL_2\mathbf{Z}$ orbit of $\sqrt{D}/2$ if $D\equiv 0 \bmod 4$, or of $(1+\sqrt{D})/2$ if $D\equiv 1 \bmod 4$. Also we have $\tau_{\lambda Q(a,b,c)}=((\lambda b)^2+\sqrt{D})/(2\lambda a)$, and $D\equiv(\lambda b)^2 \bmod 4$. It is easy to see that $\lambda a\,\tau_{\lambda Q(a,b,c)}=((\lambda b)^2+\sqrt{D})/2$ is in the $PSL_2\mathbf{Z}$ orbit of $\sqrt{D}/2$ if $D\equiv 0 \bmod 4$, or of $(1+\sqrt{D})/2$ if $D\equiv 1 \bmod 4$. It follows that $\rho_{C_e}=\lambda a\tau_C$. □

(5-5) Summary and an example ($D=-144$).

We summarize the main results of this paper a bit differently, and illustrate them in an example as follows.

Summary 5.12. *Any two coprime classes of positive definite quadratic forms* $A,B\in CL(D)$ *yield an RCFT (i.e. an element in* $RCFT_D$*) with the momentum lattices of the left and right chiral algebras* $q(A*\sigma B)$*,* $q(A*B)\in\widetilde{CL}(D)$*. Conversely, all RCFTs arise this way. Moreover, the RCFT arising from* $A,B\in CL(D)$ *in this way corresponds to the class of Narain lattices* $[\Gamma(\tau_A,\rho_B)]$*. Two pairs* (A',B') *and* (A,B) *yield the same RCFT iff* $[\Gamma(\tau_{A'},\rho_{B'})]=[\Gamma(\tau_A,\rho_B)]$*.*

We will describe the set $\widetilde{RCFT}_D$ for $D=-144$, in terms of the Gauss product on quadratic forms. By Proposition 1.2, we find 8 classes C_i in $CL(D)$:

$$C_1=[Q(1,0,36)]\ ,\ C_2=[Q(4,0,9)]\ ,\ C_3=[Q(5,4,8)]\ ,\ C_4=[Q(5,-4,8)]\ ,$$
$$C_5=[Q(2,0,18)]\ ,\ C_6=[Q(3,0,12)]\ ,\ C_7=[Q(4,4,10)]\ ,\ C_8=[Q(6,0,6)]\ ,$$

where $C_4=\sigma C_3$ and C_5 to C_8 are not primitive. Their compositions are given in the following table:

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
C_1	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
C_2	C_2	C_1	C_4	C_3	C_5	C_6	C_7	C_8
C_3	C_3	C_4	C_2	C_1	C_7	C_6	C_5	C_8
C_4	C_4	C_3	C_1	C_2	C_7	C_6	C_5	C_8
C_5	C_5	C_5	C_7	C_7	$-$	C_8	$-$	$-$
C_6	C_6	C_6	C_6	C_6	C_8	$-$	C_8	$-$
C_7	C_7	C_7	C_5	C_5	$-$	C_8	$-$	$-$
C_8	C_8	C_8	C_8	C_8	$-$	$-$	$-$	$-$

Table 3. Table of Gauss product on $CL(-144)$. The blanks "$-$" mean that the product is not defined.

Let us denote by $\bar{C}_i = q(C_i)$ the $GL_2\mathbf{Z}$ equivalence classes of C_i. Then the set $\widetilde{CL}(D)$ consists 7 classes, $\bar{C}_1, \cdots, \bar{C}_3 = \bar{C}_4, \cdots, \bar{C}_8$. From Table 3, it is easy to determine the set $\mathcal{G}_D(\bar{C}_i, \bar{C}_j)$ defined in (5.4). For example, we have,

$$\begin{aligned}
&\mathcal{G}_D(\bar{C}_1, \bar{C}_2) = \{(C_3, C_3), (C_4, C_4)\} \ , \\
&\mathcal{G}_D(\bar{C}_3, \bar{C}_3) = \{(C_1, C_3), (C_1, C_4), (C_2, C_3), (C_2, C_4)\} \ , \\
&\mathcal{G}_D(\bar{C}_8, \bar{C}_8) = \{(C_1, C_8), (C_2, C_8), (C_3, C_8), (C_4, C_8), (C_5, C_6), (C_6, C_7)\} \ .
\end{aligned}$$

We also have

$$\begin{aligned}
&\widetilde{\mathcal{G}}_D(\bar{C}_1, \bar{C}_2) = \{(\bar{C}_3, \bar{C}_3)\} \ , \\
&\widetilde{\mathcal{G}}_D(\bar{C}_3, \bar{C}_3) = \{(\bar{C}_1, \bar{C}_3), (\bar{C}_2, \bar{C}_3)\} \\
&\widetilde{\mathcal{G}}_D(\bar{C}_8, \bar{C}_8) = \{(\bar{C}_1, \bar{C}_8), (\bar{C}_2, \bar{C}_8), (\bar{C}_3, \bar{C}_8), (\bar{C}_5, \bar{C}_6), (\bar{C}_6, \bar{C}_7)\} \ .
\end{aligned}$$

By Theorem 5.8, the set $\widetilde{RCFT}_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)$ is in one to one correspondence to the set $\tilde{g}(\widetilde{\mathcal{G}}_D(\mathbf{\Gamma}_l, \mathbf{\Gamma}_r)) \subset \tilde{\mathcal{N}}_D$, i.e. Narain lattices up to the parity involution. For the above examples, we have, respectively

$$\begin{aligned}
&\{[\Gamma(\tau_{C_3}, \rho_{C_3})]_{\mathbf{Z}_2}\} \ , \ \{[\Gamma(\tau_{C_1}, \rho_{C_3})]_{\mathbf{Z}_2}, [\Gamma(\tau_{C_2}, \rho_{C_3})]_{\mathbf{Z}_2}\} \ , \\
&\{[\Gamma(\tau_{C_1}, \rho_{C_8})]_{\mathbf{Z}_2}, [\Gamma(\tau_{C_2}, \rho_{C_8})]_{\mathbf{Z}_2}, [\Gamma(\tau_{C_3}, \rho_{C_8})]_{\mathbf{Z}_2}, [\Gamma(\tau_{C_5}, \rho_{C_6})]_{\mathbf{Z}_2}, [\Gamma(\tau_{C_6}, \rho_{C_7})]_{\mathbf{Z}_2}\},
\end{aligned}$$

where $[\Gamma(\tau, \rho)]_{\mathbf{Z}_2}$ represents the $\mathbf{Z}_2$-orbit: $[\Gamma(\tau, \rho)] \sim [\Gamma(\tau, -\bar{\rho})]$. The diagonal triples correspond to the Narain lattices $[\Gamma(\tau_{C_1}, \rho_{C_k})]$ for $k = 1, \cdots, 8$.

A direct computation of the double coset spaces is not difficult, and it is a good exercise to verify the equality obtained in Corollary 5.10 in this case.

Appendix A. Gauss product on $CL(D)$

In this appendix, we extend the Gauss product on $Cl(D)$ to a composition law $C_1 * C_2$ on $CL(D)$, which includes non-primitive forms, but defined only when the classes $C_1, C_2 \in CL(D)$ are coprime. The composition is commutative and associative whenever defined. As seen in Lemmas 4.6, and 5.1, this extended composition law does arise naturally in our description of RCFTs.

The observation here is that the classical composition law given in terms of primitive concordant forms is valid verbatim for non-primitive forms. We say that two forms $Q(a_1, b_1, c_1)$ and $Q(a_2, b_2, c_2)$ are *concordant* if they satisfy the following conditions;

$$(1) \quad a_1 a_2 \neq 0 \ , \qquad (2) \quad b_1 = b_2 (=: b) \ , \qquad (3) \quad \frac{b^2 - D}{4 a_1 a_2} \in \mathbf{Z} \ . \qquad (A.1)$$

Note that in case of $D < 0$, the first condition (1) is void since $a_1 \neq 0, a_2 \neq 0$ for $D = b_1^2 - 4a_1c_1 = b_2^2 - 4a_2c_2 < 0$. We say that the forms are coprime if $gcd(a_1, b_1, c_1, a_2, b_2, c_2) = 1$. It is clear that if $Q(a_1, b_1, c_1)$ is coprime with one quadratic form in a class $C \in CL(D)$, then it is coprime with all quadratic forms in C. Thus it makes sense to speak of coprime classes. We write $Q(a, b, *)$ to denote $Q(a, b, \frac{b^2 - D}{4a})$. We call the number $gcd(a, b, c)$ the gcd of the form $Q(a, b, c)$. Since equivalent forms have the same gcd, we can speak of the gcd of a class.

The following construction parallels to that given in [Ca] in the primitive case, and is valid for both $D < 0$ and $D > 0$. For the modern definition using fractional ideals, see for example [Za].

Lemma A.1. *Consider a primitive class $\mathcal{C} \in Cl(D)$. For an arbitrary nonzero integer M, there is a quadratic form $Q(a, b, c) \in \mathcal{C}$ such that $(a, M) = 1$.*

Proof. Take an quadratic form $Q(a', b', c') : f(x, y) = a'x^2 + b'xy + c'y^2 \in \mathcal{C}$. Then f represents an integer which is coprime to M. To see this, let us first consider the case where M has only one prime factor, say $M = p^e$. In this case the claimed integer $f(x, y)$ may be found by considering the following four cases:

(1) When $p \nmid a'$. Take x, y so that $p \nmid x, p | y$, then we have $(f(x, y), p) = 1$.
(2) When $p | a'$, $p \nmid c'$. Take x, y so that $p | x, p \nmid y$, then we have $(f(x, y), p) = 1$.
(3) When $p | a', p | c'$. In this case $p \nmid b'$ by the condition $gcd(a', b', c') = 1$ ($Q(a', b', c')$ is primitive). If we take $p \nmid x, p \nmid y$, then we have $(f(x, y), p) = 1$.

In the general case, let $M = p_0^{e_0} p_1^{e_1} \cdots p_k^{e_k}$ be the prime factorization of M. Put

$$S_1 = \{ p_i \mid p_i \nmid a' \} \ , \ S_2 = \{ p_i \mid p_i | a', \ p_i \nmid c' \} \ , \ S_3 = \{ p_i \mid p_i | a', \ p_i | c' \} \ ,$$

and define

$$x = (\prod_{p \in S_2} p) \bar{x} \ \ , \ \ y = (\prod_{q \in S_1} q) \bar{y} \ \ ,$$

with some integers $\bar{x}$ and $\bar{y}$ satisfying

$$p \nmid \bar{x} \ \ (p \in S_1 \cup S_3) \ \text{ and } \ p \nmid \bar{y} \ \ (p \in S_2 \cup S_3).$$

Then it is clear that for the all prime factors $p_0, \cdots, p_k$ of M, if $p_i \in S_r$ $(r = 1, 2, 3)$, then x and y have the properties (r), and thus we have $(f(x, y), M) = 1$.

In this way, we find $(x, y) = (n_1, n_2)$ such that $f(n_1, n_2)$ is coprime to M. We may assume that $(x, y) = (n_1, n_2)$ is primitive in $\mathbf{Z}^2$, since otherwise we may set $(x, y) = (\frac{n_1}{m}, \frac{n_2}{m}) \in \mathbf{Z}^2$, with $m = gcd(n_1, n_2)$, preserving the property $(f(\frac{n_1}{m}, \frac{n_2}{m}), M) = 1$. When (n_1, n_2) is primitive in $\mathbf{Z}^2$, there is an $SL_2\mathbf{Z}$ transformation $g : (n_1, n_2) \mapsto (1, 0)$. Then the quadratic form $Q(a, b, c) = g \cdot Q(a', b', c')$ has the desired property. □

Lemma A.2. *Assume two primitive quadratic forms are equivalent:* $Q(a_1, b_1, c_1) \sim Q(a_2, b_2, c_2)$ *and* $b_1 = b_2 =: b$. *If there is an integer* $l \in \mathbf{Z}$ *such that* $l|c_1, l|c_2$ *and* $(a_1, a_2, l) = 1$, *then*

$$Q(la_1, b, c_2/l) \;\sim\; Q(la_2, b, c_2/l) \;\; .$$

Proof. Suppose $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right) \in SL_2\mathbf{Z}$ transforms $Q(a_1, b_1, c_1)$ to $Q(a_2, b_2, c_2)$, i.e.

$$ {}^t\!\begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} 2a_1 & b \\ b & 2c_1 \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} 2a_2 & b \\ b & 2c_2 \end{pmatrix} \;\; , \qquad (A.2)$$

Eliminating r, u from the resulting equations, we obtain $a_2 s + c_1 t = a_1 s + c_2 t = 0$. From these relations and our assumptions, we see that $l|s$. Then the matrix $\left(\begin{smallmatrix} r & s/l \\ lt & u \end{smallmatrix}\right)$ transforms $Q(la_1, b, c_2/l)$ to $Q(la_2, b, c_2/l)$. □

Lemma A.3. *Let* C_1, C_2 *be coprime classes in* $CL(D)$, *and* λ_1, λ_2 *be their respective gcd's. For an arbitrary nonzero integer* M, *there exist quadratic forms* $Q_1 \in C_1, Q_2 \in C_2$ *such that*

$$Q_1 = Q(\lambda_1 a_1, \beta_1, \gamma_1) \;\;,\;\; Q_2 = Q(\lambda_2 a_2, \beta_2, \gamma_2) \;\;,$$

with $(a_1, a_2) = (a_1, M) = (a_2, M) = 1$ *and also* $\beta_1 = \beta_2 =: \beta$. *Furthermore* Q_1 *and* Q_2 *are concordant.*

Proof. Applying Lemma A.1 to the primitive class $\frac{1}{\lambda_1} C_1$, we obtain a quadratic form $Q(a_1, b_1, c_1) \in \frac{1}{\lambda_1}\, C_1$ with $(a_1, \lambda_2 M) = 1$. Likewise, we have $Q(a_2, b_2, c_2) \in \frac{1}{\lambda_2}\, C_2$ with $(a_2, \lambda_1 a_1 M) = 1$. Let

$$Q_1 := Q(\lambda_1 a_1, \beta_1, \gamma_1) \;\;,\;\; Q_2 := Q(\lambda_2 a_2, \beta_2, \gamma_2) \;\;,$$

with $\beta_i := \lambda_i b_i, \gamma_i := \lambda_i c_i$. Since λ_1 and λ_2 are coprime, we have $(\lambda_1 a_1, \lambda_2 a_2) = 1$, hence there exist integers A_1, A_2 satisfying $\lambda_1 a_1 A_1 + \lambda_2 a_2 A_2 = 1$. Then we have the equivalence under $SL_2\mathbf{Z}$

$$Q_1 \sim Q(\lambda_1 a_1, \beta_1 - 2\lambda_1 a_1 \frac{\beta_1 - \beta_2}{2} A_1, *) \,,\;\; Q_2 \sim Q(\lambda_2 a_2, \beta_2 - 2\lambda_2 a_2 \frac{\beta_2 - \beta_1}{2} A_2, *) \;\; .$$

Note that $\beta_1^2 - \beta_2^2 \equiv 0 \bmod 4$, so that $(\beta_1 - \beta_2)/2$ is an integer. We see that the above two quadratic forms are of the form; $Q_1 \sim Q(\lambda_1 a_1, B, C_1)\,,\, Q_2 \sim Q(\lambda_2 a_2, B, C_2)\,,$ with $B := \lambda_2 a_2 A_2 \beta_1 + \lambda_1 a_1 A_1 \beta_2$, and $C_1 = (B^2 - D)/(4\lambda_1 a_1)$ and $C_2 = (B^2 - D)/(4\lambda_2 a_2)$. Since $(\lambda_1 a_1, \lambda_2 a_2) = 1$, the quadratic forms $Q(\lambda_1 a_1, B, C_1)$ and $Q(\lambda_2 a_2, B, C_2)$ are concordant forms which have the asserted properties. □

Definition A.4. *For any concordant forms $Q_1 = Q(\alpha_1, \beta, \gamma_1), Q_2 = Q(\alpha_2, \beta, \gamma_2)$ of a same discriminant D, we set the composition of Q_1, Q_2 by*

$$Q_1 * Q_2 := Q(\alpha_1\alpha_2, \beta, \frac{\beta^2 - D}{4\alpha_1\alpha_2}) \ . \tag{A.3}$$

Proposition A.5. *Assume $C_1, C_2 \in CL(D)$ are coprime. The class $C_3 \in CL(D)$ of the composition $Q_1 * Q_2$ is independent of the choices of concordant forms $Q_1 \in C_1$, $Q_2 \in C_2$.*

Proof. Suppose that $Q'_1 = Q(\alpha'_1, \beta', \gamma'_1)$, $Q'_2 = Q(\alpha'_2, \beta', \gamma'_2)$ are concordant, that $Q''_1 = Q(\alpha''_1, \beta'', \gamma''_1)$, $Q''_2 = Q(\alpha''_2, \beta'', \gamma''_2)$ are also concordant, and that $Q'_1 \sim Q''_1, Q'_2 \sim Q''_2$. We will show that $Q'_1 * Q'_2 \sim Q''_1 * Q''_2$.

Let λ_i be the gcd of Q'_i, Q''_i. Put $M = a'_1a'_2a''_1a''_2$ $(\alpha_i := \lambda_i a'_i, \alpha''_i := \lambda_i a''_i)$. Then by Lemma A.3, there exists another pair of concordant forms,

$$Q_1 = Q(\lambda_1 a_1, \beta, \gamma_1) \in C_1 \ \ , \ \ Q_2 = Q(\lambda_2 a_2, \beta, \gamma_2) \in C_2 \ \ ,$$

satisfying $(a_1, a_2) = (a_1, M) = (a_2, M) = 1$. Since $(\lambda_1, \lambda_2) = 1$, $\lambda_1|\beta, \lambda_2|\beta$, we have $\beta = \lambda_1\lambda_2 b$ for some integer b; likewise $\beta' = \lambda_1\lambda_2 b'$, and $D = \lambda_1^2\lambda_2^2 D_0$. Therefore from (A.3), we have

$$Q_1 * Q_2 = \lambda_1\lambda_2 Q(a_1a_2, b, \frac{b^2 - D_0}{4a_1a_2}) \ , \ \ Q'_1 * Q'_2 = \lambda_1\lambda_2 Q(a'_1a'_2, b', \frac{b'^2 - D_0}{4a'_1a'_2}) \ .$$

Since $(a_1a_2, a'_1a'_2) = 1$, there exist integers such that $a'_1a'_2m'_{12} + a_1a_2m_{12} = 1$. From this, we get

$$\begin{aligned} Q_1 * Q_2 &= \lambda_1\lambda_2\, Q(a_1a_2, b, *) \sim \lambda_1\lambda_2 Q(a_1a_2, B, *) \ \ , \\ Q'_1 * Q'_2 &= \lambda_1\lambda_2\, Q(a'_1a'_2, b', *) \sim \lambda_1\lambda_2 Q(a'_1a'_2, B, *) \ \ , \end{aligned} \tag{A.4}$$

where $B := b - 2a_1a_2\frac{b-b'}{2}m_{12} = b' - 2a'_1a'_2\frac{b'-b}{2}m'_{12}$.

Now from the shape of B, we find

$$Q(a_1, \lambda_2 B, c_1) \sim Q(a_1, \lambda_2 b, *) \sim Q(a'_1, \lambda_2 b', *) \sim Q(a'_1, \lambda_2 B, c'_1) \ \ ,$$

where $c_1 = \lambda_2^2(B^2 - D_0)/4a_1$, $c'_1 = \lambda_2^2(B^2 - D_0)/4a'_1$ with $D = \lambda_1^2\lambda_2^2 D_0$. Then we see that $\lambda_2 a'_2|c_1$ and $\lambda_2 a'_2|c'_1$ since we have $(B^2 - D_0)/4a_1a_2, (B^2 - D_0)/4a'_1a'_2 \in \mathbf{Z}$ from the equivalences in (A.4) and $(a'_1a'_2, a_1a_2) = 1$. By Lemma A.2, we have

$$\lambda_2 Q(a_1a'_2, B, c_1/\lambda_2^2a'_2) \sim \lambda_2 Q(a'_1a'_2, B, c'_1/\lambda_2^2a'_2) \ \ .$$

In a similar way, starting from $Q(a_2, \lambda_1 B, c_2) \sim Q(a_2, \lambda_1 b, *) \sim Q(a'_2, \lambda_1 b', *) \sim Q(a'_2, \lambda_1 B, c'_2)$ with $c_2 = \lambda_1^2(B^2 - D_0)/4a_2$ and $c'_2 = \lambda_1^2(B^2 - D_0)/2a'_2$, we obtain

$$\lambda_1 Q(a_1a_2, B, c_2/\lambda_1^2a_1) \sim \lambda_1 Q(a_1a'_2, B, c'_2/\lambda_1^2a_1) \ \ .$$

Combining these two, and using $c_1/\lambda_2^2a'_2 = c'_2/\lambda_1^2a_1$, we obtain

$$\lambda_1\lambda_2 Q(a_1a_2, B, c_2/\lambda_1^2a_1) \sim \lambda_1\lambda_2 Q(a'_1a'_2, B, c'_1/\lambda_2^2a'_2) \ \ . \tag{A.5}$$

Now from (A.4) and (A.5), we see that $Q_1 * Q_2 \sim Q'_1 * Q'_2$. Likewise, we get $Q_1 * Q_2 \sim Q''_1 * Q''_2$. It follows that $Q'_1 * Q'_2 \sim Q''_1 * Q''_2$. □

By the proposition above, the composition of coprime classes in $CL(D)$ now makes sense:

Definition A.6. *For coprime classes $C_1, C_2 \in CL(D)$, we define*

$$C_1 * C_2 = [Q_1 * Q_2] \in CL(D) \ ,$$

for any choice of concordant forms $Q_1 \in C_1, Q_2 \in C_2$.

Remark 1) If both classes C_1, C_2 are primitive, i.e. in $Cl(D)$, then the composition is nothing but the Gauss product.
2) If $C_1, C_2 \in CL(D)$ are coprime classes and λ_1, λ_2 their respective gcd's, then we have

$$\frac{1}{\lambda_1\lambda_2} \times C_1 * C_2 \in Cl(D/(\lambda_1\lambda_2)^2) \ ,$$

i.e., it is a primitive class. This follows from definition (A.3) and the fact that $\lambda_1\lambda_2|\beta$ and $(\lambda_1\lambda_2)^2|(\beta^2 - D)$ hold because $(\lambda_1, \lambda_2) = 1$.
3) It follows from 2) that if either C_1 or C_2 is not primitive, then the composition $C_1 * C_2$ is not primitive. This is an important fact, used in our proof of surjectivity in Theorem 4.11. □

The following properties of the composition law generalize the classical properties (Theorem 1.1) of Gauss' group law on $(Cl(D), *)$.

Proposition A.7. *For pairwise coprime classes $C_1, C_2, C_3 \in CL(D)$, the following properties hold:*
1) $C_1 * C_2 = C_2 * C_1$,
2) $(C_1 * C_2) * C_3 = C_1 * (C_2 * C_3)$
3) Let C_e be the class containing $Q(1, 0, -\frac{D}{4})$ for $D \equiv 0$ mod 4 or $Q(1, 1, \frac{1-D}{4})$ for $D \equiv 1$ mod 4, then $C_e * C = C$ for any $C \in CL(D)$.
4) If $C_1 \in Cl(D)$, i.e. primitive class, then $\sigma C_1 * C_1 = C_e$, where σ is the involution defined by $Q(a, b, c) \mapsto Q(a, -b, c)$ as in (1.6).

Proof. 1) This follows from Definition A.6 based on Proposition A.5 and (A.3).
2) Let λ_i be the gcd of C_i. By Lemma A.3, we have concordant forms $Q_1 = Q(\lambda_1 a_1, \beta_1, \gamma_1)$, $Q_2 = Q(\lambda_2 a_2, \beta_2, \gamma_2)$ with $(a_1, a_2) = 1$ and $\beta_1 = \beta_2 =: \beta$. Then by Proposition A.5, we have $Q_1 * Q_2 = Q(\lambda_1\lambda_2 a_1 a_2, \beta, *) \in C_1 * C_2$. Since $C_1 * C_2$ is a multiple by $\lambda_1\lambda_2$ of a primitive class (see Remark 2)) and $(\lambda_1\lambda_2, \lambda_3) = 1$ by assumption, we may apply Lemma A.3 to coprime classes $C_1 * C_2$ and C_3 with $M = a_1 a_2$. By this we see that there exist a quadratic form $Q(\lambda_3 a_3, \beta_3, *) \in C_3$ with $(a_1 a_2, a_3) = 1$. Now we note the following equivalences;

$$Q(\lambda_1\lambda_2 a_1 a_2, \beta, *) \sim Q(\lambda_1\lambda_2 a_1 a_2, b, *) \ , \ \ Q(\lambda_3 a_3, \beta_3, *) \sim Q(\lambda_3 a_3, b, *) \ ,$$

where $b = \beta - 2\lambda_1\lambda_2 a_1 a_2 \frac{\beta - \beta_3}{2} m_{12} = \beta_3 - 2\lambda_3 a_3 \frac{\beta_3 - \beta}{2} m_3$ with integers m_{12}, m_3 satisfying $a_1 a_2 m_{12} + a_3 m_3 = 1$. From the shape of b, we find

$$Q_1 \sim Q_1' = Q(\lambda_1 a_1, b, *) \ , \ Q_2 \sim Q_2' = Q(\lambda_2 a_2, b, *) \ , \ Q_3 \sim Q_3' = Q(\lambda_3 a_3, b, *) \ ,$$

with conditions $(a_1, a_2) = (a_1, a_3) = (a_2, a_3) = 1$. It follows that

$$(Q_1' * Q_2') * Q_3' = Q(\lambda_1\lambda_2\lambda_3 a_1 a_2 a_3, b, *) = Q_1' * (Q_2' * Q_3') \ .$$

Here we have used the fact that each pair of quadratic forms being composed are concordant. This yields the associativity on classes by Proposition A.5.

3) Any class $C \in CL(D)$ is coprime to the class C_e, and so $C * C_e$ is defined. For any quadratic form $Q(\lambda a, \beta, \gamma) \in C$, we can find a quadratic form $Q(1, \beta, *) \in C_e$ (by using the equivalence $Q(1, \beta', *) \sim Q(1, \beta' - 2n, *)$). Then the composition becomes $Q(\lambda a, \beta, \gamma) * Q(1, \beta, *) = Q(\lambda a, \beta, \gamma)$, which implies $C * C_e = C$.
4) Let $Q(a, b, c)$ with $ac \neq 0$ be a quadratic form in a primitive class C. (It is not hard to see that we can choose $Q(a, b, c)$ with $ac \neq 0$. In fact, this is obvious if $D < 0$.) Then, since $\sigma\, Q(a, b, c) = Q(a, -b, c) \sim Q(c, b, a)$, we have $[\sigma\, Q(a, b, c) * Q(a, b, c)] = [Q(c, b, a) * Q(a, b, c)] = [Q(ac, b, 1)]$. Depending on the congruence of D mod 4, $Q(ac, b, 1) \sim Q(1, -b, ac)$ is equivalent either to $Q(1, 0, -D/4)$ or to $Q(1, 1, (1 - D)/4)$. This implies $\sigma\, C * C = C_e$. $\square$

Appendix B. $O(A_\Gamma)$ for a primitive lattice Γ

For a primitive lattice $\Gamma \in \mathcal{L}^p(D)$, the group $O(A_\Gamma)$, by definition, consists of all isometries of the discriminant A_Γ. Here we prove the following property:

Proposition B.1. *If $\Gamma \in \mathcal{L}^p(D)$, then any element $\varphi \in O(A_\Gamma)$ satisfies $\varphi^2 = id$.*

Proof. First note that, for a primitive lattice Γ, the discriminant group Γ^*/Γ is isomorphic to either $\mathbf{Z}_{|D|}$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_{2d}$ with $2d = |D|/2$. In the first case, the claim holds since we have the decomposition

$$O(A_{\mathbf{Z}_{|D|}}) = O(A_{\mathbf{Z}_{p_1^{e_1}}}) \times \cdots \times O(A_{\mathbf{Z}_{p_k^{e_k}}}) \ , \ \ (|D| = p_1^{e_1} \cdots p_k^{e_k}),$$

and $O(A_{\mathbf{Z}_{p_i^{e_i}}}) = \{\pm 1\}$ for each prime factor p_i of D. In the second case, we have a group decomposition

$$\Gamma^*/\Gamma \cong \mathbf{Z}_2 \oplus \mathbf{Z}_{2d} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_{2^l} \oplus \mathbf{Z}_{2s+1} \quad ,$$

by writing $d = 2^{l-1}(2s+1)\,(l \geq 1)$. More precisely, if we denote $\mathbf{Z}_2 = \langle u_2 \rangle$ and $\mathbf{Z}_{2d} = \langle u_{2d} \rangle$, then $u := u_2$, $v := (2s+1)u_{2d}$, $v' := 2^l u_{2d}$ generate each group, $\mathbf{Z}_2$, $\mathbf{Z}_{2^l}$, $\mathbf{Z}_{2s+1}$, respectively. Since $2^l(x,y) \equiv (2s+1)(x,y) \equiv 0 \bmod \mathbf{Z}$ holds for $x \in \mathbf{Z}_2 \oplus \mathbf{Z}_{2^l}$, $y \in \mathbf{Z}_{2s+1}$ and $(2^l,(2s+1)) = 1$, we see $(x,y) \equiv 0 \bmod \mathbf{Z}$ and have the following orthogonal decomposition:

$$(\Gamma^*/\Gamma, q_\Gamma) \cong (\mathbf{Z}_2 \oplus \mathbf{Z}_{2^l} \oplus \mathbf{Z}_{2s+1}, q_{\mathbf{Z}_2 \oplus \mathbf{Z}_{2^l}} \oplus q_{\mathbf{Z}_{2s+1}}) \ , \qquad (B.1)$$

hence $O(A_\Gamma) \cong O(A_{\mathbf{Z}_2 \oplus \mathbf{Z}_{2^l}}) \times O(A_{\mathbf{Z}_{2s+1}})$. Thus we only need to consider $O(A_{\mathbf{Z}_2 \oplus \mathbf{Z}_{2^l}})$.

Let u, v be the respective generators of $\mathbf{Z}_2$ and $\mathbf{Z}_{2^l}$ as above. Then an element $\varphi \in O(A_{\mathbf{Z}_2 \oplus \mathbf{Z}_{2^l}})$ can be represented by

$$\varphi : (u,v) \mapsto (u',v') = (u,v) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad .$$

Since $2u' \equiv 0, 2^l v' \equiv 0$, it follows that $\alpha = 0,1$, $\beta = 0,1$, $\gamma = 0, 2^{l-1}$. Also δ must be zero or an odd integer. First, note that if a matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right)$ represents an isometry $\varphi \in O(A_{\mathbf{Z}_2 \oplus \mathbf{Z}_{2^l}})$, then δ must be ± 1. In the following, we condider two cases, $l \geq 2$ and $l = 1$, seperately.

- $l \geq 2$: In this case, there are five possibilities for φ:

$$\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \ , \ \begin{pmatrix} 1 & 0 \\ 2^{l-1} & k \end{pmatrix} \ , \ \begin{pmatrix} 1 & 1 \\ 0 & k \end{pmatrix} \ , \ \begin{pmatrix} 1 & 1 \\ 2^{l-1} & k \end{pmatrix} \ , \begin{pmatrix} 0 & 1 \\ 2^{l-1} & k \end{pmatrix} \ , \qquad (B.2)$$

where k is odd. First of all, we may exclude the last (the fifth) isometry from our consideration. To see this, write the relations between the generators: $u' = 2^{l-1}v$, $v' = u + kv$. Then we have $2^{l-1}v' = k2^{l-1}v = 2^{l-1}v$ and also $u' = ku' = k2^{l-1}v = 2^{l-1}v$. Hence we have $u' = 2^{l-1}v'$, which cannot be for the generators. Thus the fifth isometry will never appear for $l \geq 2$.

For the first four possible isometries, we verify that each one is an involution. This is easy for the first three cases, and proceeds as follows for the fourth case: We have

$$\begin{pmatrix} 1 & 1 \\ 2^{l-1} & k \end{pmatrix}^2 = \begin{pmatrix} 1+2^{l-1} & k+1 \\ 2^{l-1}(k+1) & 2^{l-1}+k^2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 2^{l-1}+k^2 \end{pmatrix} .$$

Since an isometry of the form $\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$ exists only for $m = \pm 1$, we obtain $2^{l-1} + k^2 = \pm 1$. But $2^{l-1} + k^2 \not\equiv -1 \ mod \ 2^l$ for k is odd and $l \geq 3$. This shows that $\varphi^2 = id$ for $l \geq 3$. When $l = 2$, the isometry takes the following form: $\varphi_\pm = \begin{pmatrix} 1 & 1 \\ 2 & \pm 1 \end{pmatrix}$. Now write the discriminant form $q_2 = \begin{pmatrix} \frac{a}{2} & \frac{b}{2} \\ \frac{b}{2} & \frac{c}{4} \end{pmatrix}$ with respect to the generators u, v, and assume that q_2 allows the isometry $\varphi_\pm$. Then we see that the discriminat form q_2 is restricted to be

$$q_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{c}{4} \end{pmatrix} \quad , \quad (\frac{c}{4} = 0, \frac{1}{2}, 1, \frac{3}{2}) .$$

This shows that we have for the generators; $(u, u) \equiv 1$ mod $2\mathbf{Z}$, $(u, v) \equiv \frac{1}{2}$ mod $\mathbf{Z}$, $(v, v) \equiv \frac{c}{4}$ mod $2\mathbf{Z}$. Note that $l_1 := 2u, l_2 := 4v$ are elements in Γ, and in fact, give a basis of Γ. (Precisely, l_1, l_2 are lattice vectors chosen, respectively, from $2(u + \Gamma), 4(v + \Gamma)$.) Then the following symmetric matrix represents the bilinear form of the lattice Γ:

$$\begin{pmatrix} (l_1, l_1) & (l_1, l_2) \\ (l_2, l_1) & (l_2, l_2) \end{pmatrix} = \begin{pmatrix} 4a_0 & 4b_0 \\ 4b_0 & 8c_0 \end{pmatrix}$$

where $a_0, b_0, c_0 \in \mathbf{Z}$. This shows that Γ is not primitive, since the corresponding quadratic form becomes $Q(2a_0, 4b_0, 4c_0)$. Therefore q_2 does not appear from a primitive lattice Γ, and hence $\varphi_\pm$ may be excluded.

• $l = 1$: In this case, there are six possibilities for φ:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} . \quad (B.3)$$

We verify $\varphi^2 = id$ for the first four. For the rest, $\varphi^2 = id$ does not hold, but we may exclude these by a similar argument above done for $l = 2$: As above assume the following possible forms for the discriminant form $q_{\mathbf{Z}_2 \times \mathbf{Z}_2}$:

$$q = \begin{pmatrix} \frac{a}{2} & 0 \\ 0 & \frac{b}{2} \end{pmatrix} \ (a, b = 1, 2, 3) \ , \ q' = \begin{pmatrix} \frac{a}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{b}{2} \end{pmatrix} \ (a, b = 0, 1, 2, 3),$$

with respect to the generators u and v. Then it is straightforward to see that a non-involutive isometry (i.e. the fifth or sixth of (B.3)) is possible only for the discriminant form $q_1 := \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$. We may claim that this discriminant form q_1 never appears from a primitive lattice Γ in the exactly same way as above for $l = 2$.

This completes the proof. □

Note that for non-primitive lattices, the index d_1 in $\Gamma^*/\Gamma \cong \mathbf{Z}_{d_1} \oplus \mathbf{Z}_{d_2}$ $(d_1|d_2)$ can be greater than two in general. Even if we have $d_1 = 2$, the above proof shows that $\varphi_\pm$ for $l = 2$ case or the last two cases of (B.3) for $l = 1$ can be possible for non-primitive lattices. The latter case has appeared in the example presented at the end of section (5-3).

Appendix C. The coset space $O(d,\mathbf{R}) \times O(d,\mathbf{R}) \setminus O(d,d;\mathbf{R})$ and $O(d,d;\mathbf{Z})$

Here we will prove Proposition 2.5, and claim 1) of Proposition 2.6.

Proof of Proposition 2.5. Let $W' \in O'(d,d;\mathbf{R})$, and write the corresponding "half conjugated matrix" $\widetilde{W}$ as

$$\widetilde{W} := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_d & \mathbf{1}_d \\ \mathbf{1}_d & -\mathbf{1}_d \end{pmatrix} W' = \begin{pmatrix} X & X' \\ Y & Y' \end{pmatrix}.$$

Then the condition that $W' \in O'(d,d;\mathbf{R})$ becomes

$$ {}^tXX - {}^tYY = \mathbf{0}_d \ , \quad {}^tX'X - {}^tY'Y = \mathbf{1}_d \ , \quad {}^tX'X' - {}^tY'Y' = \mathbf{0}_d, \qquad (C.1)$$

where $\mathbf{0}_d$ is the $d \times d$ zero matrix. Here we conclude that $\det X \neq 0$. To see this, assume that the real symmetric matrix ${}^tXX = {}^tYY$ has a (real) eigenvector $\mathbf{v}(\neq \mathbf{0})$ with zero eigenvalue. Then we have ${}^tXX\mathbf{v} = {}^tYY\mathbf{v} = \mathbf{0}$, from which we have $||X\mathbf{v}||^2 = ||Y\mathbf{v}||^2 = 0$, i.e. $X\mathbf{v} = Y\mathbf{v} = \mathbf{0}$. However, by the second equation of (C.1), we have $\mathbf{v} = ({}^tX'X - {}^tY'Y)\mathbf{v} = \mathbf{0}$, which is a contradiction. Therefore we conclude that all eigenvalues are not zero, which means $\det X \neq 0$, $\det Y \neq 0$.

Now consider the $O(d,\mathbf{R}) \times O(d,\mathbf{R})$-orbit of the matrix W':

$$\frac{1}{2} \begin{pmatrix} \mathbf{1}_d & \mathbf{1}_d \\ \mathbf{1}_d & -\mathbf{1}_d \end{pmatrix} \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix} \begin{pmatrix} \mathbf{1}_d & \mathbf{1}_d \\ \mathbf{1}_d & -\mathbf{1}_d \end{pmatrix} W' = \frac{1}{\sqrt{2}} \begin{pmatrix} g_LX + g_RY & g_LX' + g_RY' \\ g_LX - g_RY & g_LX' - g_RY' \end{pmatrix}. \qquad (C.2)$$

Since $\det X \neq 0$, g_RYX^{-1} makes sense. It's easy to check that it lies in $O(d;\mathbf{R})$. Thus by choosing $g_L = g_RYX^{-1}$, we get $g_LX - g_RY = \mathbf{0}_d$, and (C.2) becomes

$$\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 2Y & YX^{-1}X' + Y' \\ 0 & YX^{-1}X' - Y' \end{pmatrix},$$

where $g := g_R$. Put $\Lambda := \frac{1}{\sqrt{2}}(YX^{-1}X' - Y')$. By (C.1), we find that $\sqrt{2}Y = {}^t\Lambda^{-1}$ and that ${}^t\Lambda \frac{1}{\sqrt{2}}(YX^{-1}X' + Y') = -\frac{1}{2}({}^tY'YX^{-1}X' - {}^tX'\,{}^tX^{-1}\,{}^tYY') =: -B$ is antisymmetric. This shows that each orbit has the shape $G_{diag} \cdot W'(\Lambda, B)$ as claimed.

Conversely, it is easy to check that each $G_{diag} \cdot W'(\Lambda, B)$ ($\Lambda \in GL(d,\mathbf{R}), B \in A(d,\mathbf{R})$) is a $O(d,\mathbf{R}) \times O(d,\mathbf{R})$-orbit. This proves the first assertion.

The last assertion of Proposition (2.5) is straightforward. □

We now describe generators of the discrete group $O(2,2;\mathbf{Z})$, as in Proposition 2.6. In general, doing the same for $O(d,d;\mathbf{Z})$ ($d \geq 3$) is much harder. However for $d = 2$, we have the following nice description of $\mathbf{R}^{2,2}$. Let $Mat_{2,2}(\mathbf{R})$ be the space of 2×2 real matrices, equipped with the quadratic form $||\ast|| := \det(\ast)$. Its associated bilinear form has the signature $(2,2)$, and therefore $Mat_{2,2}(\mathbf{R}) \cong \mathbf{R}^{2,2}$ as quadratic spaces. Consider an embedding of the hyperbolic lattice $U^{\oplus 2}$ defined by

$$\begin{array}{cccc} \Phi_0 : & U^{\oplus 2} & \hookrightarrow & Mat_{2,2}(\mathbf{R}) \\ & e_1, e_2, f_1, f_2 & \mapsto & E_1, E_2, F_1, F_2 \ , \end{array} \qquad (C.3)$$

where

$$E_1 = \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \ E_2 = \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ F_1 = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ , \ F_2 = \sqrt{2}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \ .$$

The embedding is isometric since $\Phi_0(x_1e_1+x_2e_2+y_1f_1+y_2f_2) = \sqrt{2}\left(\begin{smallmatrix} x_1 & x_2 \\ -y_2 & y_1 \end{smallmatrix}\right)$ has determinant $2x_1y_1 + 2x_2y_2$. We denote by $\Phi_0^{\mathbf{R}} : U^{\oplus 2} \otimes \mathbf{R} = \mathbf{R}^{2,2} \cong M_{2,2}(\mathbf{R})$ the real scalar extension of Φ_0.

Proposition C.1. *1) The linear action $X \mapsto gXh^{-1}$ of $(g,h) \in SL_2\mathbf{R} \times SL_2\mathbf{R}$ on $X \in Mat_{2,2}(\mathbf{R})$ defines a group homomorphism*

$$\phi_{\mathbf{R}} : SL_2\mathbf{R} \times SL_2\mathbf{R} \to O'(2,2;\mathbf{R}) \cong O(2,2;\mathbf{R}) \ .$$

2) $\phi_{\mathbf{R}}$ maps surjectively onto $O'_0(2,2;\mathbf{R})$, the connected component of the identity, and has the kernel $\{(\mathbf{1}_2, \mathbf{1}_2), (-\mathbf{1}_2, -\mathbf{1}_2)\}$.

Sketch of Proof. It is easy to verify claim 1). For 2), we observe that the differential $d\phi_{\mathbf{R}}$ gives an isomorphism of the Lie algebras, and therefore the image of $\phi_{\mathbf{R}}$ is given by the connected component of the identity. □

Proposition C.2. *The restriction of $\phi_{\mathbf{R}}$ to $SL_2\mathbf{Z} \times SL_2\mathbf{Z}$ defines a surjective group homomorphism:*

$$\phi_{\mathbf{Z}} : SL_2\mathbf{Z} \times SL_2\mathbf{Z} \ \to \ O(2,2;\mathbf{Z}) \cap O'_0(2,2;\mathbf{R}) \ ,$$

with $Ker(\phi_{\mathbf{Z}}) = \{(\mathbf{1}_2, \mathbf{1}_2), (-\mathbf{1}_2, -\mathbf{1}_2)\}$.

Proof. It is clear that the image of $\phi_{\mathbf{Z}}$ lies in $O(2,2;\mathbf{Z}) \cap O'_0(2,2;\mathbf{R})$ by Proposition C.1. To prove surjectivity, let us solve for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), h = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in SL_2\mathbf{Z}$ in

$$\phi_{\mathbf{R}}(g,h) = \begin{pmatrix} ad' & -ac' & -bc' & -bd' \\ -ab' & aa' & ba' & bb' \\ -cb' & ca' & da' & db' \\ -cd' & cc' & dc' & dd' \end{pmatrix} = S, \qquad (C.4)$$

for any given $S \in O(2,2;\mathbf{Z}) \cap O'_0(2,2;\mathbf{R})$. By Proposition C.1, there are solutions $g, h \in SL_2\mathbf{R}$. To prove surjectivity, we will show that they are integral. First note that $a^2 = \det S_{12;12}$, $b^2 = \det S_{12;34}$, $c^2 = \det S_{34;12}$, $d^2 = \det S_{34;34}$; ${a'}^2 = \det S_{23;23}$, ${b'}^2 = -\det S_{23;14}$, ${c'}^2 = -\det S_{14;23}$, ${d'}^2 = \det S_{14;14}$, where $S_{ij;kl}$ represents the minor made by the i,j-th rows and the k,l-th columns. From this, we see that the square of each variable $a, b, ...$ is integer. So we can write $a = x_a\sqrt{r_a}$, with some integer x_a and a square free positive integer r_a. Likewise, $b = x_b\sqrt{r_b}, c = x_c\sqrt{r_c}$ and so on. The 16 components of (C.4) are required to be integers. For example, we have $ad' = x_a x_{d'}\sqrt{r_a r_{d'}} \in \mathbf{Z}$. The last expression becomes integer only if $r_a = r_{d'}$. Likewise we have $r_a = r_b = \cdots = r_{d'} =: r$. Therefore we have,

$$a, b, c, d, a', b', c', d' \in \sqrt{r}\mathbf{Z} \ ,$$

for some square free integer r. Since $ad - bc = a'd' - b'c' = 1$, we see that the only possibility is $r = 1$, which means that $g, h \in SL_2\mathbf{Z}$. Thus $\phi_{\mathbf{Z}}$ is surjective. It is easy to determine the kernel of $\phi_{\mathbf{Z}}$. □

Recall that $SL_2\mathbf{Z}$ is generated by $\mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{T} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. It is easy to see that $\phi_{\mathbf{Z}}$ maps $(\mathbf{1}_2, \mathbf{S}), (\mathbf{1}_2, \mathbf{T}), (\mathbf{S}, \mathbf{1}_2), (\mathbf{T}, \mathbf{1}_2)$, respectively, to the matrices S_1, T_1, S_2, T_2 given in 2) of Proposition 2.6.

The two generators R_1 and R_2 are related to the other connected components of $O'(2,2;\mathbf{R})$. In fact, $O'(2,2;\mathbf{R})$ consists of the following four connected components:

$$O'(2,2;\mathbf{R}) = O'_{+,+}(2,2;\mathbf{R}) \sqcup O'_{+,-}(2,2;\mathbf{R}) \sqcup O'_{-,+}(2,2;\mathbf{R}) \sqcup O'_{-,-}(2,2;\mathbf{R}) ,$$

where, for example, $O'_{-,+}(2,2;\mathbf{R})$ consists of those elements which reverse the orientation of a positive definite two plane and preserve that of a negative definite two plane. (Note that if g reverses the orientation of a single positive definite two plane, then it does so for every positive two plane. Similarly for negative definite two planes. Likewise if g preserves the orientation of a two plane.) Obviously, $O'_0(2,2;\mathbf{R}) = O'_{+,+}(2,2;\mathbf{R})$. Now giving the orientations $(e_1+f_1)\wedge(e_2+f_2)$ and $(e_1-f_1)\wedge(e_2-f_2)$, respectively, for positive and negative definite two planes, we see that R_1 belongs to $O'_{-,+}(2,2;\mathbf{R}) \cap O(2,2;\mathbf{Z})$ and R_2 belongs to $O'_{-,-}(2,2;\mathbf{R}) \cap O(2,2;\mathbf{Z})$. It is now clear that we have the following decomposition of $O(2,2;\mathbf{Z})$;

$$O(2,2;\mathbf{Z}) = O_{+,+}(2,2;\mathbf{Z}) \sqcup O_{+,-}(2,2;\mathbf{Z}) \sqcup O_{-,+}(2,2;\mathbf{Z}) \sqcup O_{-,-}(2,2;\mathbf{Z}) , \quad (C.5)$$

with $O_{+,+}(2,2;\mathbf{Z}) = O'_0(2,2;\mathbf{R}) \cap O(2,2;\mathbf{Z})$ and $O_{+,-}(2,2;\mathbf{Z}) = O_{+,+}(2,2;\mathbf{Z})R_1R_2$, $O_{-,+}(2,2;\mathbf{Z}) = O_{+,+}(2,2;\mathbf{Z})R_1$, $O_{+,-}(2,2;\mathbf{Z}) = O_{+,+}(2,2;\mathbf{Z})R_2$. In particular, this yields property 1) of Proposition 2.6.

References

[Bo] C. Borcea, *Calabi-Yau threefolds and complex multiplication*, in "Essays on Mirror Manifolds" ed. S.-T. Yau (1992), 489–502.

[Ca] J. W. S. Cassels, *Rational quadratic forms*, Academic Press (1978).

[CS] J.H. Conway and N.J.Sloan, *Sphere packings, Lattices and Groups*, Springer-Verlag (1988).

[DVV] R. Dijkgraaf, E. Verlinde and H. Verlinde, *On moduli spaces of conformal field theories with* $c \geq 1$ (in Proceedings of 1987 Copenhagen Conference *Perspective in String Theory*).

[DHVW] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds*, Nuclear Physics **B261** (1985), 678 – 686.

[DW] S. Dulat and K. Wendland, *Towards a classification of unitary conformal field theories with central charge* $c = 2$, JHEP **0006** (2000), 012.

[FLM] I. Frenkel, A. Meurman and J. Lepowsky, *Vertex Operator Algebras and the Monster*, Academic Press, Boston (1988).

[Gi] P. Ginsparg, *Curiosities at* $c = 1$, Nuclear Physics **B295** (1988), 153 – 170.

[GSW] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory 1,2*, Cambridge University Press (1987).

[GV] S. Gukov and C. Vafa, *Rational conformal field theories and complex multiplication*, hep-th/0203213.

[Gr] B. Greene, *Constructing mirror manifolds*, in [GY] (1997), 29–69.

[GY] B. Greene and S.-T. Yau, Editors, *Mirror Symmetry II*, AMS and International Press (1996).

[Ha] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag (1977).

[HMV] J. Harvey, G. Moore and C. Vafa, *Quasicrystalline compactification*, Nucl. Phys. **B304** (1988), 269 – 290.

[HLOY1] S. Hosono, B.H. Lian, K. Oguiso, S.T. Yau, *Counting Fourier-Mukai partners and applications,*, math.AG/0202014.

[HLOY2] S. Hosono, B.H. Lian, K. Oguiso, S.T. Yau, *Counting unimodular lattices in* $\mathbf{R}^{r,s}$, math.QA/0301095.

[Kac] V. Kac, *Vertex algebras for beginners*, University Lecture Series **10** (1998), American Mathematical Society, Providence, RI..

[KP] V. Kac and D. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. **53** (1984), 125–264.

[KY] K. Kikkawa and M. Yamasaki, *Casimir effects in superstring theories.*, Phys.Lett. **B149** (1984), 357.

[Ki1] E. Kiritsis, *Proof of the completeness of the classification of rational conformal theories with* $c = 1$, Phys. Lett. B **217** (1989), 427.

[Ki2] E. Kiritsis, *Introduction to superstring theory, hep-th/9709062*, Leuven Univ. Press. (1998).

[LZ] B.H. Lian and G.J. Zuckerman, *Commutative quantum operator algebras*, Jour. Pure Appl. Alg. **100** (1995), 117 – 139, q-alg/9501014.

[MN] A. Matsuo and K. Nagatomo, *Axioms for a vertex algebra and the locality of quantum fields*, MSJ Memoirs **4** (1999), Mathematical Society of Japan, Tokyo.

[Mo] G. Moore, *Arithmetic and attractors*, hep-th/9807087.

[NW] W. Nahm and K. Wendland, *A hiker's guide to K3: aspects of N=(4,4) superconformal field theory with central charge c=6*, Commun. Math. Phys. **216** (2001), 85–138.

[Na] K.S. Narain, *New heterotic string theories in uncompactified dimensions* < 10, Physics Letters **169B** (1986), 41–46.

[Ni] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izv. **14** (1980), 103–167.

[Wa] K. Wendland, *Moduli Spaces of Unitary Conformal Field Theories*, Ph.D. thesis (available at http://www-biblio.physik.uni-bonn.de/dissertationen/2000/doc/index.shtml).

[Po] J. Polchinski, *String Theory, vol.1,2*, Cambridge University Press (1998).

[SI] T. Shioda and H. Inose, *On singular K3 surfaces*, in Complex Analysis and Algebraic Geometry, W.L. Bailey and T. Shioda, eds. (1977), 119–136.

[Za] D. Zagier, *Zetafunktionen und quadratische Körper : eine Einführung in die höhere Zahlentheorie*, Springer-Verlag (1981).

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