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KUMMER STRUCTURES ON A K3 SURFACE - AN OLD QUESTION OF T. SHIODA

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Dedicated to Professor Tetsuji Shioda on the occasion of his sixtieth birthday

Abstract. We apply our earlier results on Fourier-Mukai partners to answer definitively a question about Kummer surface structures, posed by T. Shioda 25 years ago.

0. Introduction

In this note, unless stated otherwise, we shall work in the category of smooth complex projective varieties. For a smooth projective variety X, we denote by $D(X)$ the bounded derived category of coherent sheaves on X. The object $ObD(X)$ contains the set $\{\mathcal{O}_x | x \in X\}$ of structure sheaves of closed points of X. We regard $D(X)$ as a triangulated category in a natural manner. By an equivalence $D(Y) \simeq D(X)$ we always mean an equivalence of triangulated categories.

In [BO], Bondal and Orlov showed that a lot of information can be extracted from the objects $\{\mathcal{O}_x | x \in X\}$ when the canonical divisor K_X is ample or anti-ample. In particular, in this case they further showed that $Y \simeq X$ if $D(Y) \simeq D(X)$. Quite recently, Kawamata [Ka] obtained a generalization of this result. He showed that Y is birationally equivalent to X if $D(X) \simeq D(Y)$ and if K_X or $-K_X$ is big.

On the other hand, Mukai [Mu1] showed, about 20 years ago, that the Poincaré bundle $\mathcal{P}$ on $A \times \hat{A}$, where A is an abelian variety and $\hat{A} := \mathrm{Pic}^0 A$, induces an equivalence $\Phi^{\mathcal{P}}_{\hat{A}\to A} : D(\hat{A}) \to D(A)$. The functor $\Phi^{\mathcal{P}}_{\hat{A}\to A}$ is the so-called Fourier-Mukai transform defined by

$$\Phi^{\mathcal{P}}_{\hat{A}\to A}(\mathcal{X}) = \mathbf{R}\pi_{A*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi^*_{\hat{A}}\mathcal{X}) \tag{0.1}$$

where $\pi_{\hat{A}} : \hat{A} \times A \to \hat{A}$ and $\pi_A : \hat{A} \times A \to A$ are the natural projections. In this equivalence, the structure sheaf $\mathcal{O}_{\hat{a}}$ of the point $\hat{a} \in \hat{A}$ is mapped to the invertible sheaf $\mathcal{P}_{\hat{a}}$ on A corresponding to $\hat{a}$. Therefore the derived category of an abelian variety does not characterize the structure sheaves of points anymore. Indeed, $\hat{A}$ is not (even birationally) isomorphic to A in general even when $D(\hat{A}) \simeq D(A)$.

This example suggests that when K_X is trivial, $D(X)$ no longer has enough information to reconstruct (the birationally equivalence class of) the variety X, and that a new and interesting relationship arises between varieties X, Y with $D(X) \simeq D(Y)$.

The aim of this note is to examplify this idea by studying one old problem in concrete geometry, namely the following problem posed by T. Shioda 25 years ago, from the view point of Fourier-Mukai partners:

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Problem. ([Sh1, Question 5]) Does the Kummer variety $\mathrm{Km}\, A$ uniquely determine the abelian variety A up to isomorphism, i.e. $\mathrm{Km}\, B \simeq \mathrm{Km}\, A \Rightarrow B \simeq A$?

As shown by Shioda, the answer is affirmative in dimension ≥ 3, and it is also affirmative if $\dim \mathrm{Km}\, A = 2$ and $\rho(\mathrm{Km}\, A) = 20$, i.e. $\rho(A) = 4$, the maximal possible case [Sh1],[SM, Theorem 5.1]. (See also Section 2.) He then expected an affirmative answer to the Problem in dimension two.

However, as it is expected from Shioda's Torelli Theorem [Sh2] for abelian surfaces, which he found after he posed the problem above, that one has $\mathrm{Km}\, \hat{A} \simeq \mathrm{Km}\, A$. This was first noticed by [GH, Theorem 1.5 and Remark in Sect.I]. See also [HS, Sect.III.3], and Section 1 for another explanation. Therefore, it is natural and interesting to consider the following modified:

Problem. Let A be an abelian surface. Then, $\mathrm{Km}\, B \simeq \mathrm{Km}\, A \Rightarrow B \simeq A$ or $B \simeq \hat{A}$?

For our statement, we need a few preparations. First, we set

$$FM(X) := \{Y | D(Y) \simeq D(X)\}/\mathrm{isom}\,.$$

An element of $FM(X)$ is called a Fourier-Mukai (FM) partners of X. Recall that if X is an abelian (resp. K3) surface then so are its FM partners Y, and that

$$FM(X) = \{Y | (T(Y), \mathbf{C}\omega_Y) \simeq (T(X), \mathbf{C}\omega_X)\}/\mathrm{isom}\,.$$

Here $T(X)$ denotes the transcendental lattice of X, i.e. $T(X) := NS(X)^{\perp}$ in $H^2(X, \mathbf{Z})$, ω_X denotes a nowhere vanishing holomorphic two form on X, and the isomorphism $(T(Y), \mathbf{C}\omega_Y) \simeq (T(X), \mathbf{C}\omega_X)$ stands for a Hodge isometry. These properties are due to Mukai and Orlov ([Mu2], [Or]) and are also well treated in [BM, Theorem 5.1].

Next, for a K3 surface X, we define

$$\mathcal{K}(X) := \{\, B \mid B \text{ is an abelian surface s.t.} \, \mathrm{Km}\, B \simeq X \,\}/\mathrm{isom}\,.$$

We call $\mathcal{K}(X)$ the set of Kummer structures on X. This set $\mathcal{K}(X)$ measures how many different Kummer surface structures X has. Note that $\mathcal{K}(X) = \emptyset$ unless X is a Kummer surface.

Our main result is now stated as follows:

Theorem 0.1. (Main Theorem) *Let A be an abelian surface and assume that $X = Km\, A$. Then:*

1. *$\mathcal{K}(X) = FM(A)$. In particular, $|\mathcal{K}(X)| < \infty$ and $\hat{A} \in \mathcal{K}(X)$.*
2. *If $\rho(A) = 3$ and $\det NS(A)$ is square free or if $\rho(A) = 4$, then $\mathcal{K}(X) = \{A, \hat{A}\}$.*
3. *For an aribitrarily given natural number N, there is an abelian surface A such that $\mathcal{K}(X)$, where $X = Km\, A$, contains N abelian surfaces A_i ($1 \leq i \leq N$) with $NS(A_i) \not\simeq NS(A_j)$, hence $A_i \not\simeq A_j$, for all $1 \leq i \neq j \leq N$.*

Assertion 1 makes an interesting connection between two important notions: Kummer surface structures and Fourier-Mukai partners. Assertion 2 is probably not surprising in light of Shioda's earlier result. However, Assertion 3 shows that K3 surfaces can have arbitrarily high number of non-isomorphic Kummer surface structures! This is quite unexpected since it runs counter to Shioda's original observation. It is also surprising in the light of the theory of lattices. (See also

Theorem 3.1 and Remark in §3.) Assertions 1, 2, and 3 will be proved respectively in Sections 1, 2, and 3.

In Appendix, we remark that $\hat{A} \notin \mathcal{K}(\mathrm{Km}\, A)$ while $\hat{A} \in FM(A)$, whence $\mathcal{K}(\mathrm{Km}\, A) \neq FM(A)$ in general, if a complex 2-torus A is not projective. This is pointed out to us by Yoshinori Namikawa. See also [Na] for relevant work on generalized Kummer varieties and differences between dimension 2 and dimension ≥ 4.

Throughout this note, an abelian surface means a projective complex 2-torus, while a complex 2-torus is not assumed to be projective.

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1. **FM partners and Kummer structures**

In this section, we shall prove assertion 1 of Main Theorem.

Proof of assertion 1. Let A and B be abelian surfaces. By [Ni1] (see also [Mo, Lemma 3.1]), the canonical rational map $\pi_A : A \dashrightarrow \mathrm{Km}\, A$ gives the Hodge isometry

$$\pi_{A,*} : (T(A)(2), \mathbf{C}\omega_A) \simeq (T(\mathrm{Km}\, A), \mathbf{C}\omega_{\mathrm{Km}\, A})\,,$$

and likewise for B and $\mathrm{Km}\, B$. Here, for a lattice $L := (L, (*, **))$ and for a non-zero integer m, we define the lattice $L(m)$ by $L(m) := (L, m(*, **))$.

Therefore, one has

$$(T(B), \mathbf{C}\omega_B) \simeq (T(A), \mathbf{C}\omega_A) \iff (T(\mathrm{Km}\, B), \mathbf{C}\omega_{\mathrm{Km}\, B}) \simeq (T(\mathrm{Km}\, A), \mathbf{C}\omega_{\mathrm{Km}\, A})\,.$$

Hence, by the characterization of the FM partners of K3 and abelian surfaces, one obtains

$$B \in FM(A) \iff \mathrm{Km}\, B \in FM(\mathrm{Km}\, A)\,.$$

On the other hand, we know that $FM(\mathrm{Km}A) = \{\mathrm{Km}\, A\}$ by [Mu2] or by the Counting Formula [HLOY, §3]. Then we have

$$\mathrm{Km}\, B \in FM(\mathrm{Km}\, A) \iff \mathrm{Km}\, B \simeq \mathrm{Km}\, A\,,$$

and therefore

$$B \in FM(A) \iff B \in \mathcal{K}(X)\,.$$

Since $|FM(A)| < \infty$ by [BM, Proposition 5.3] and $\mathcal{K}(X) = FM(A)$, we have now $|\mathcal{K}(A)| < \infty$. See also Section 2 for a more explicit estimate of $|FM(A)|$. As it is remarked in Introduction, $\hat{A} \in FM(A)$. Thus, by $\mathcal{K}(X) = FM(A)$, we have $\hat{A} \in \mathcal{K}(X)$ as well. □

2. **Kummer structures in larger Picard numbers**

In this section, we shall prove assertion 2 of Main Theorem. For this purpose, we shall first give an effective estimate of $|FM(A)|$. This is based on the argument of [HLOY, §3] and the following Theorem due to Shioda [Sh2] (See also [Mo]):

Theorem 2.1. (1) *Let A and B be complex 2-tori. Then*

$$(H^2(B,\mathbf{Z}),\mathbf{C}\omega_B)\simeq(H^2(A,\mathbf{Z}),\mathbf{C}\omega_A)\iff B\simeq A\,\text{or}\,B\simeq\hat{A}\,.$$

(2) *For any given weight two Hodge structure $(U^{\oplus 3},\mathbf{C}\omega)$, there is a marked complex 2-torus (A,τ_A) such that τ_A gives a Hodge isometry*

$$\tau_A:(H^2(A,\mathbf{Z}),\mathbf{C}\omega_A)\simeq(U^{\oplus 3},\mathbf{C}\omega)\,.$$

For the second statement, we recall that U is the even unimodular hyperbolic lattice of rank 2 and the second cohomology lattice $H^2(A,\mathbf{Z})$ of a complex 2-torus A is always isometric to the lattice $U^{\oplus 3}$.

Let A be an abelian surface. Denote by $G=O_{Hodge}(T(A),\mathbf{C}\omega_A)$ the group of Hodge isometries of $(T(A),\mathbf{C}\omega_A)$. We call two primitive embeddings $\iota:T(A)\to U^{\oplus 3}$, $\iota':T(A)\to U^{\oplus 3}$ *G-equivalent* if there exist $\Phi\in O(U^{\oplus 3})$ and $g\in G$ such that $\iota'\circ g=\Phi\circ\iota$, i.e. the following diagram commutes;

$$\begin{array}{ccc}\iota:T(A) & \longrightarrow & U^{\oplus 3}\\ \Big\downarrow{\scriptstyle\exists g} & & \Big\downarrow{\scriptstyle\exists\Phi}\\ \iota':T(A) & \longrightarrow & U^{\oplus 3}\end{array}$$

We denote by $\mathcal{P}^{G\text{-eq}}(T(A),U^{\oplus 3})$ the set of G-equivalence classes of primitive embeddings of $T(A)$ into $U^{\oplus 3}$:

$$\mathcal{P}^{G\text{-eq}}(T(A),U^{\oplus 3}):=\{\text{primitive embedding }\iota:T(A)\to U^{\oplus 3}\}/G\text{-equiv. }\,.$$

Claim. *There is a surjection,*

$$\xi:FM(A)\longrightarrow\mathcal{P}^{G\text{-eq}}(T(A),U^{\oplus 3})\,,\ B\mapsto[\iota_B]\,,$$

such that $\xi^{-1}([\iota_B])=\{B,\hat{B}\}$.

Proof. Let $B\in FM(A)$. Choose a marking $\tau_B:H^2(B,\mathbf{Z})\to U^{\oplus 3}$ and a Hodge isometry $\gamma:(T(A),\mathbf{C}\omega_A)\simeq(T(B),\mathbf{C}\omega_B)$. Define $\xi:[B]\mapsto[\iota_B]$, where $\iota_B=\tau_B\circ\gamma$. This is a well-defined surjection with $\xi^{-1}(\xi(A))=\{A,\hat{A}\}$. The proof of this fact is exactly the same as the argument for Theorem 3.3 of [HLOY], except that Theorem 2.1 here plays the corresponding role of the Torelli Theorem for K3 surfaces there. Note that ξ may fail to be injective because B and $\hat{B}$ need not be isomorphic in general. □

Recall Theorem 1.4 of [HLOY]:

$$|\mathcal{P}^{G\text{-eq}}(T(A),U^{\oplus 3})|=\sum_{j=1}^{m}|O(S_j)\setminus O(A_{S_j})/G|\ \ . \tag{2.1}$$

Here $\mathcal{G}(NS(A)):=\{S_1,S_2,\cdots,S_m\}$ ($S_1\simeq NS(A)$) is the genus of the Néron-Severi lattice $NS(A)$ and $O(A_{S_i})$ is the orthogonal group of the discriminant group $A_{S_i}=S_i^*/S_i$ with respect to the natural $\mathbf{Q}/2\mathbf{Z}$-valued quadratic form q_{S_i} and the set $O(S_j)\setminus O(A_{S_j})/G$ is the orbit space of the natural action of $O(S_j)\times G$ on A_{S_j}. Note that $NS(A)$ is even and hyperbolic. We also recall that the genus of $NS(A)$ is nothing but the set of isomorphism classes of even lattices S such that $(NS(A),q_{NS(A)})\simeq(S,q_S)$, by [Ni2, Corollary 1.9.4].

We can now prove assertion 2 of Main Theorem.

If $\det NS(A)$ is square free, it follows that $A_{NS(A)}$ is cyclic, i.e. $l(A_{NS(A)}) = 1$. Here $l(M)$ denote the minimal number of generators of a finite abelian group M. Hence, if, in addition, $\rho(A) = 3$, then one has

$$\text{rank } NS(A) = 3 \geq 2 + l(A_{NS(A)})\,.$$

Similarly, if $\rho(A) = 4$, then one has

$$\text{rank } NS(A) = 4 \geq 2 + l(A_{NS(A)})\,.$$

Indeed, if $\rho(A) = 4$, then $\text{rank}\, T(A) = 2$. Since $(A_{NS(A)}, q_{NS(A)}) \simeq (A_{T(A)}, -q_{T(A)})$, we have $l(A_{NS(A)}) = l(A_{T(A)}) \leq \text{rank}\, T(A) = 2$.

Hence, by [Ni2, Theorem 1.14.2], one has that $\mathcal{G}(NS(A)) = \{NS(A)\}$ and that the natural map $O(NS(A)) \to O(A_{NS(A)})$ is surjective in each case. Now it follows from the equation (2.1) that $|\mathcal{P}^{G\text{-eq}}(T(A), U^{\oplus 3})| = 1$. Hence, by the preceding claim, we have $FM(A) = \{A, \hat{A}\}$. Combining this with assertion 1 of Main Theorem, we obtain the desired equality $\mathcal{K}(X) = \{A, \hat{A}\}$. $\square$

Remark. In [SM], it is shown that an abelian surface A with $\rho(A) = 4$ is necessarily a direct product of two elliptic curves, namely, $A \simeq E \times F$. Since $\hat{E} \simeq E$ and $\hat{F} \simeq F$, it follows that $\hat{A} \simeq A$ if $A \simeq E \times F$. Combining this with assertion 2, one has $\mathcal{K}(X) = \{A\}$ if $\rho(A) = 4$. This gives an alternative explanation for the result [SM, Theorem 5.1] quoted in Introduction. $\square$

3. K3 surfaces with many Kummer structures

In this section, we shall show assertion 3 of Main Theorem.

We begin by the following result first found in [Og2]:

Theorem 3.1. *For any given natural number N, there is an even hyperbolic lattice S such that $\text{rank}\, S = 2$, $\det S = -pq$ and $|\mathcal{G}(S)| \geq N$. Here $\mathcal{G}(S)$ is the genus of the lattice S, and p and q are some distinct prime numbers (including 1).*

Remark. Such phenomena seem quite rare and are perhaps impossible if $\text{rank}\, S \geq 3$ (cf. [Cs, Chapters 9 and 10] and [CL, §8 - §10]). In this sense, as it is pointed out to us by Professor B. Gross, this Theorem is a sort of miracle which happens probably only in rank 2. $\square$

Proof. Here we shall sketch the argument. Our argument requires some basic properties about class groups. These are found in the book [Za]. (See also [HLOY, Section 4] for summary.) Let P be the set of prime numbers including 1. By [Iw, Main Theorem], the set

$$\mathcal{N} := \{(n, p, q) \in \mathbf{N} \times P^2 | 4n^2 + 1 = pq\}$$

is an infinite set. Note that $pq \equiv 1 \bmod 4$ and $p \neq q$ for $(n, p, q) \in \mathcal{N}$. Therefore pq is a so-called fundamental discriminant. Let $H(pq)$ be the narrow class group of (the ring of integers $O(pq)$ of) the real quadratic field $\mathbf{Q}(\sqrt{pq})$. We denote by $h(pq)$ the class number of $\mathbf{Q}(\sqrt{pq})$, i.e. $|H(pq)|$. Recall that there is a natural bijective correspondence between $H(pq)$ and the set $\mathcal{L}(pq)$ of the proper, i.e. $\text{SL}(2, \mathbf{Z})$, isomorhism classes of even hyperbolic rank 2 lattices S with $\det S = -pq$. Recall also that $\mathcal{L}(pq)$ is decomposed into at most two sets of the same cardinality:

$$\mathcal{L}(pq) = \tilde{\mathcal{G}}(S_1) \cup \tilde{\mathcal{G}}(S_2)\,.$$

Here $\tilde{\mathcal{G}}(S_i)$ is the set of proper, i.e. $\mathrm{SL}(2, \mathbf{Z})$, isomorphism classes of lattices in the same genus as S_i. Combining these two facts with $\mathrm{GL}(2, \mathbf{Z})/\mathrm{SL}(2, \mathbf{Z}) \simeq \mathbf{Z}/2$, we have

$$|\mathcal{G}(S_1)| \geq \frac{|\tilde{\mathcal{G}}(S_1)|}{2} \geq \frac{|\mathcal{L}(pq)|}{4} = \frac{h(pq)}{4} \, .$$

Therefore, it is enough to show that $h(pq) \to \infty$ when $pq \to \infty$. We note that there is a sequence $\{(n_k, p_k, q_k)\}_{k=1}^{\infty} \in \mathcal{N}$ such that $p_k q_k \to \infty$ when $k \to \infty$. This is because $|\mathcal{N}| = \infty$.

Let us show that $h(pq) \to \infty$ when $pq \to \infty$. Since $(2n + \sqrt{pq})(-2n + \sqrt{pq}) = 1$ by $4n^2 + 1 = pq$, the element $2n + \sqrt{pq} > 1$ is a unit of $O(pq)$. Therefore, the fundamental unit $\epsilon(pq)$ of $O(pq)$ satisfies $1 < \epsilon(pq) < pq$. Now, combining this with the Siegel-Brauer formula

$$\lim_{pq \to \infty} \frac{\log(h(pq) \log \epsilon(pq))}{\log pq} = \frac{1}{2} \, ,$$

one has $h(pq) \to \infty$ when $pq \to \infty$. $\square$

Let us return to the proof of assertion 3 of Main Theorem.

Let S be the lattice in Theorem (4.1) and take N non-isomorphic elements

$$S_1 := S, S_2, \cdots, S_N \in \mathcal{G}(S) \, .$$

Then S_i are even hyperbolic lattices of $\operatorname{rank} S_i = 2$ and satisfy $(A_{S_i}, q_{S_i}) \simeq (A_S, q_S)$.

Let us choose a primitive embedding

$$\varphi_i : S_i \to U^{\oplus 3}$$

for each i. For instance, if $S_i = \mathbf{Z}\langle v_{i1}, v_{i2} \rangle$ with

$$((v_{i1}, v_{i2})) = \begin{pmatrix} 2a_i & b_i \\ b_i & 2c_i \end{pmatrix} ,$$

then we can define φ_i by $\varphi_i(v_{i1}) = e_1 + a_i f_1$ and $\varphi_i(v_{i2}) = b_i f_1 + e_2 + c_i f_2$, where e_j and f_j are the standard basis of the j-th factor U of $U^{\oplus 3}$.

Set $T_i := \varphi_i(S_i)^{\perp}$ in $U^{\oplus 3}$ and $T := T_1$. Then

$$(A_{T_i}, q_{T_i}) \simeq (A_{S_i}, -q_{S_i}) \simeq (A_S, -q_S)$$

and the signature of T_i is $(2, 2)$ for each i. Therefore $T_i \in \mathcal{G}(T)$.

Since pq is square free, we have $A_T \simeq A_S \simeq \mathbf{Z}/pq$. Thus,

$$\operatorname{rank} T = 4 \geq 2 + 1 = 2 + l(A_T) \, .$$

In addition, T is indefinite. Hence $\mathcal{G}(T) = \{T\}$ by [Ni2, Theorem 1.14.2]. Therefore $T_i \simeq T$ for each i. Let us choose an isometory

$$\sigma_i : T \; \simeq \; T_i$$

for each i. We set $\sigma_1 := id$.

Choose a maximal weight two Hodge structure $(T, \mathbf{C}\omega)$ on T. Here the term *maximal* means that T satisfies the following property: if $T' \subset T$ is a primitive sublattice of T such that $\mathbf{C}\omega \in T' \otimes \mathbf{C}$ then $T' = T$. This is equivalent to that $\langle \operatorname{Re} \omega, \operatorname{Im} \omega \rangle^{\perp} \cap T = \{0\}$ in $T \otimes \mathbf{R}$. As it is easily shown, there certainly exists a maximal weight two Hodge structure on T. Note that the Hodge structure $(T(V), \mathbf{C}\omega_V)$ on $T(V)$ of an abelian surface V is maximal in this sense.

Set $\omega_i := \sigma_i(\omega)$. Then $(T_i, \mathbf{C}\omega_i)$ is a maximal weight two Hodge structure on T_i such that $\sigma_i : (T, \mathbf{C}\omega) \simeq (T_i, \mathbf{C}\omega_i)$. Then, by Theorem 2.1, there is a marked complex 2-torus (A_i, τ_i) such that

$$\tau_i : (H^2(A_i, \mathbf{Z}), \mathbf{C}\omega_{A_i}) \simeq (U^{\oplus 3}, \mathbf{C}\omega_i)\,.$$

Then, by the maximality of $T(A_i)$ and T_i, one has the Hodge isometry

$$\tau_i|_{T(A_i)} : (T(A_i), \mathbf{C}\omega_{A_i}) \simeq (T_i, \mathbf{C}\omega_i)$$

and whence an isometry of the orthogonal lattices

$$\tau_i|_{NS(A_i)} : NS(A_i) \simeq S_i$$

for each i. Since S_i is hyperbolic, the complex 2-tori A_i are actually projective, i.e. abelian surfaces. Since $S_i \not\simeq S_j$ $(i \neq j)$ by the choice of S_i, one has $NS(A_i) \not\simeq NS(A_j)$ $(i \neq j)$ as well.

Set $A := A_1$. Then, composing the Hodge isometries $(\tau|_{T(A)}^{-1})$, σ_i^{-1} and $\tau_i|_{T(A_i)}$, one has a Hodge isometry

$$(T(A_i), \mathbf{C}\omega_{A_i}) \simeq (T(A), \mathbf{C}\omega_A)\,.$$

Therefore $A_i \in FM(A)$ by the characterization of $FM(A)$. Thus, $A_i \in \mathcal{K}(\mathrm{Km}\, A)$ by assertion 1 of Main Theorem. Hence, these N abelian surfaces A_i $(1 \leq i \leq N)$ satisfy the requirement in assertion 3 of Main Theorem. □

Appendix A.

In this appendix, we shall show the following:

Proposition A.1. *Consider the complex 2-torus $A = \mathbf{C}^2/L$ defined by a rank 4 discrete lattice L in $\mathbf{C}^2$ which is generated by*

$$e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,,\; e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,,\; e_3 := \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\,,\; e_4 := \begin{pmatrix} \gamma \\ \delta \end{pmatrix}\,.$$

Assume that L is generic in the sense that α, β, γ and δ are algebraically independent over $\mathbf{Q}$. Then:

1. *$NS(A) = \{0\}$ and A is not projective.*
2. *$\hat{A} \not\simeq A$ and $\hat{A} \in FM(A)$.*
3. *$\mathcal{K}(\mathrm{Km}\, A) = \{A\}$.*

In particular, $\hat{A} \notin \mathcal{K}(\mathrm{Km}\, A)$, i.e. $\mathrm{Km}\, \hat{A} \not\simeq \mathrm{Km}\, A$, while $\hat{A} \in FM(A)$.

This proposition shows certain differences between the Kummer structures on a non-projective K3 surface and a projective K3 surface.

Proof of assertion 1. Let z_1 and z_2 be the coordinates of $\mathbf{C}^2$ corresponding to the first and the second projections $\mathbf{C}^2 \to \mathbf{C}$. We identify L with $H_1(A, \mathbf{Z})$. Let $\langle v_i \rangle_{i=1}^4$ be the dual basis of $H^1(A, \mathbf{Z})$ of the basis $\langle e_i \rangle_{i=1}^4$ of $H_1(A, \mathbf{Z})$. Then, by the description of e_i, we have

$$dz_1 = v_1 + \alpha v_3 + \gamma v_4\,,\; dz_2 = v_2 + \beta v_3 + \delta v_4$$

in $H^1(A, \mathbf{C})$. From this, we obtain

$$\omega := dz_1 \wedge dz_2 = v_1 \wedge v_2 + \beta v_1 \wedge v_3 + \delta v_1 \wedge v_4 - \alpha v_2 \wedge v_3 - \gamma v_2 \wedge v_4 + (\alpha\delta - \beta\gamma) v_3 \wedge v_4\,.$$

Note that $NS(A) = \{\eta \in H^2(A, \mathbf{Z}) | (\eta, \omega) = 0\}$. This due to the Lefschetz $(1,1)$ Theorem. Let $\eta \in NS(A)$ and write η as

$$\eta = av_1 \wedge v_2 + bv_1 \wedge v_3 + cv_1 \wedge v_4 + dv_2 \wedge v_3 + ev_2 \wedge v_4 + fv_3 \wedge v_4 \,.$$

Here the coefficients a, b, $\cdots$, f are integers. By $(\eta, \omega) = 0$, one has

$$a(\alpha\delta - \beta\gamma) + b\gamma - c\alpha + d\delta - e\beta + f = 0 \,.$$

Since α, β, γ, δ are algebraically independent over $\mathbf{Q}$, we have then

$$a = b = c = d = e = f = 0 \,, \text{ i.e.} \,, \eta = 0 \,.$$

Hence $NS(A) = \{0\}$. This also implies that A is not projective. $\square$

Proof of assertion 2. The second statement $\hat{A} \in FM(A)$ follows from [Mu1], that the Poincaré bundle induces the equivalence $D(\hat{A}) \simeq D(A)$.

Let us show the first statement. Let $\hat{L}$ be the rank 4 discrete lattice in $\mathbf{C}^2$ generated by

$$f_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,, f_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,, f_3 := \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \,, f_4 := \begin{pmatrix} \beta \\ \delta \end{pmatrix} \,.$$

Then, by the formula [Sh2, (2.13)], we have $\hat{A} = \mathbf{C}^2/\hat{L}$. Assume that $A \simeq \hat{A}$. Then, there is $B \in \mathrm{GL}(2, \mathbf{C})$ such that $Be_1 \,, Be_2 \,, Be_3 \,, Be_4 \,\in \hat{L}$.

By $Be_1 \,, Be_2 \in \hat{L}$, there are integers m_i, n_i such that

$$Be_1 = \begin{pmatrix} m_1 + m_3\alpha + m_4\beta \\ m_2 + m_3\gamma + m_4\delta \end{pmatrix} \,,$$

$$Be_2 = \begin{pmatrix} n_1 + n_3\alpha + n_4\beta \\ n_2 + n_3\gamma + n_4\delta \end{pmatrix} \,.$$

Then, we have

$$B = \begin{pmatrix} m_1 + m_3\alpha + m_4\beta & n_1 + n_3\alpha + n_4\beta \\ m_2 + m_3\gamma + m_4\delta & n_2 + n_3\gamma + n_4\delta \end{pmatrix} \,.$$

Therefore

$$Be_3 = \begin{pmatrix} m_1\alpha + m_3\alpha^2 + m_4\alpha\beta + n_1\beta + n_3\alpha\beta + n_4\beta^2 \\ m_2\alpha + m_3\alpha\gamma + m_4\alpha\delta + n_2\beta + n_3\beta\gamma + n_4\beta\delta \end{pmatrix} \,.$$

Since $Be_3 \in \hat{L}$, there are integers l_i such that

$$\begin{pmatrix} m_1\alpha + m_3\alpha^2 + m_4\alpha\beta + n_1\beta + n_3\alpha\beta + n_4\beta^2 \\ m_2\alpha + m_3\alpha\gamma + m_4\alpha\delta + n_2\beta + n_3\beta\gamma + n_4\beta\delta \end{pmatrix} = \begin{pmatrix} l_1 + l_3\alpha + l_4\beta \\ l_2 + l_3\gamma + l_4\delta \end{pmatrix} \,.$$

Since α, β, γ, and δ are algebraically independent over $\mathbf{Q}$, this implies that

$$m_2 = m_3 = m_4 = n_2 = n_3 = n_4 = 0 \,.$$

However, then B is of the form,

$$B = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,,$$

and therefore $\det B = 0$, a contradiction. Hence $A \not\simeq \hat{A}$. $\square$

Proof of assertion 3. Let B be a complex 2-torus such that $\mathrm{Km}\, B \simeq \mathrm{Km}\, A$. Let us choose an isomorphism $\varphi : \mathrm{Km}\, A \simeq \mathrm{Km}\, B$. Since $NS(A) = \{0\}$, there is no effective curves on A. Therefore $\mathrm{Km}\, A$ contains no effective curves other than the

16 exceptional curves C_i arising from the resolution $\mathrm{Km}\, A \to A/\langle -1 \rangle$. Therefore $\mathrm{Km}\, B$ also contains only 16 $\mathbf{P}^1$s, say D_i, by $\mathrm{Km}\, B \simeq \mathrm{Km}\, A$. These D_i must then coincide with the 16 exceptional curves arising from the resolution $\mathrm{Km}\, B \to B/\langle -1 \rangle$ and satisfy $\varphi(\cup_{i=1}^{16} C_i) = \cup_{i=1}^{16} D_i$. Now recall that A can be reconstructed from $\mathrm{Km}\, A$ as follows: first take the (unique) double cover of $\mathrm{Km}\, A$ branched along the divisor $\sum_{i=1}^{16} C_i$ and then contract the proper transform of $\sum_{i=1}^{16} C_i$. Likewise B from $\mathrm{Km}\, B$ and $\sum_{i=1}^{16} D_i$. Since $\varphi(\sum_{i=1}^{16} C_i) = \sum_{i=1}^{16} D_i$, it follows that φ then lifts to an isomorphism $\tilde{\varphi} : A \simeq B$. This completes the proof. $\square$

Remark. In [Ni1, main result], there is a bijective correspondence between the set of Kummer structures $\mathcal{K}(X)$ on X and $\mathcal{C}(X)/\mathrm{Aut}(X)$, where $\mathcal{C}(X)$ is the set of the configurations of 16 disjoint $\mathbf{P}^1$s on X. However $\mathcal{C}(X)$ is an infinite set and the natural action of $\mathrm{Aut}(X)$ on $\mathcal{C}(X)$ is not transitive in general, when X is a projective K3 surface. It seems much harder to prove our main theorem by such a geometric method using this bijective correspondence.

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