FOURIER-MUKAI NUMBER OF A K3 SURFACE

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Abstract. We shall give a Counting Formula for the number of Fourier-Mukai partners of a K3 surface and consider three applications.

0. Introduction

In his papers [Mu 1,2,3], Mukai discovered the fundamental importance of manifolds with equivalent bounded derived categories of coherent sheaves. Such manifolds are now known as Fourier-Mukai (FM) partners, which have been the focus of attention in several contexts of mathematics.

The aim of this note is to derive an explicit counting formula for the FM number, i.e. the cardinality of the set $FM(X)$ of FM partners, for a projective K3 surface X in terms of the Néron-Severi lattice $NS(X)$ and the Hodge structure $(T(X), \mathbf{C}\omega_X)$ of the transcendental lattice $T(X)$ (Counting Formula 2.3), and to give then three applications (Corollary 2.7, Theorems 3.3 and 4.1).

The Counting Formula 2.3 is inspired by an earlier work of Bridgeland, Maciocia [BM]. Our proof here is based on the fundamental result of Mukai [Mu3] and Orlov [Or] (see also [BM], [HLOY1] and Theorem 2.2 for summary), and the theory of primitive embeddings of lattices due to Nikulin [Ni1]. What we actually need here is a sort of equivariant version of the theory in [Ni1]. Namely, we introduce the notion of G-equivalence classes of primitive embeddings of an even non-degenerate lattice T into an even unimodular indefinite lattice Λ, where G is a prescribed subgroup of the orthogonal group $O(T)$ (Definition 1.1), and we find a formula for the cardinality of the set $\mathcal{P}^{G\text{-eq}}(T, \Lambda)$ of G-equivalence classes of the primitive embeddings $\iota : T \to \Lambda$ (Theorem 1.4).

We give a precise formula for $|\mathcal{P}^{G\text{-eq}}(T, \Lambda)|$ in §1. Such a natural and useful formula like Theorem 1.4 is clearly not unexpected. However, we are unable to find this general formula anywhere in the literature, and so we give a proof in Appendix A. We remark that in the special case $G = \{\mathrm{id}\}$, the formula is given explicitly in [MM].

We deduce the Counting Formula 2.3 in §2.

We shall then give three applications.

First, as an immediate consequence of the Counting Formula, we see that a K3 surface X such that $\rho(X) \geq 3$ and $det\, NS(X)$ is square free, has no FM partner other than X itself (Corollary 2.7 3)). This is in sharp contrast to the cases of $\rho = 1, 2$ described in [Og1] (See also Corollary 2.7 4) and Theorem 3.3). We shall also show how some of known cases are derived from our Counting Formula (Corollary 2.7 1) and 4)).

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Next, as an arithmetical application, we recast two famous questions on class numbers, after Gauss, circa 1800, into geometric terms of FM partners of K3 surfaces of Picard number 2 (Theorem 3.3, Questions I' and II' in §3). Here we recall the two questions on the class number $h(p)$ of the real quadratic field $\mathbf{Q}(\sqrt{p})$ for a prime number $p \equiv 1 \ mod \ 4$:

Question I. *Are there infinitely many primes p such that $h(p) = 1$?*

Question II. *Is there a sequence of primes $p_1, p_2, ...$ such that $h(p_k) \to \infty$?*

Our reformulations (Questions I' and II') of these arithmetical questions might afford us a new geometrical approach to studying them via moduli of stable sheaves on a family of K3 surfaces (cf. Theorem 2.2 and an argument in §4).

Finally, for a more geometrical application, we shall construct explicitly a pair of smooth projective families of K3 surfaces, $f : \mathcal{X} \to \Delta$ and $g : \mathcal{Y} \to \Delta$, with a sequence $\{t_k\}_{k=1}^{\infty} \subset \Delta$ such that $\lim_{k\to\infty} t_k = 0$ and $\mathcal{X}_{t_k} \simeq \mathcal{Y}_{t_k}$, but $\mathcal{X}_0 \not\simeq \mathcal{Y}_0$ (Theorem 4.1). Here Δ is a unit disk. This explicitly shows the non-Hausdorff nature of the moduli space of unpolarized K3 surfaces. We carry out the construction in §4.

We remark that, aside from the three applications here, the Counting Formula 2.3 has also been applied to a study of Kummer structures on a K3 surface [HLOY2].

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1. G-equivalence classes of primitive embeddings

The goal of this section is Theorem 1.4. Here we shall a need a G-equivariant version of the notion of primitive embeddings [Ni1].

Unless otherwise stated, by a lattice $L := (L, (*, **))$, we mean a pair of a free $\mathbf{Z}$-module L of finite rank and its non-degenerate symmetric integral-valued bilinear form $(*, **) : L \times L \to \mathbf{Z}$. We set $L^* = \mathrm{Hom}(L, \mathbf{Z})$.

Let $\Lambda = (\Lambda, (*, **))$ be an even, unimodular, indefinite lattice. Consider an even lattice T which admits at least one primitive embedding $\iota_0 : T \hookrightarrow \Lambda$, and a subgroup $G \subset O(T)$. The group G is not necessarily a finite group. Let S be a lattice isomorphic to the orthogonal lattice $\iota_0(T)^\perp$ in Λ and choose an isomorphism $\iota_0^S : S \simeq \iota_0(T)^\perp$. Throughout §1, the objects Λ, T, ι_0, G, S and ι_0^S are fixed.

Definition 1.1. Two primitive embeddings $\iota : T \hookrightarrow \Lambda$, $\iota' : T \hookrightarrow \Lambda$ are called G-equivalent if there exist $\Phi \in O(\Lambda)$ and $g \in G$ such that the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{\Phi} & \Lambda \\ \uparrow \iota & & \uparrow \iota' \\ T & \xrightarrow{g} & T \end{array} .$$

Our main interest in this section is the set of all primitive embeddings:

$$\mathcal{P}(T, \Lambda) := \{\iota : T \hookrightarrow \Lambda; \text{primitive embedding}\} ,$$

and the set of G-equivalence classes, i.e. the quotient set

$$\mathcal{P}^{G\text{-eq}}(T, \Lambda) := \mathcal{P}(T, \Lambda)/G\text{-equivalence} .$$

We denote the restriction of the bilinear form $(\ , \ )$ on Λ to S and T, respectively, by $(\ , \ )_S$, $(\ , \ )_T$. These coincide with the original bilinear forms of S and T. (Here and hereafter, we often identify T and S with the primitive sublattices $\iota_0(T)$ and $\iota_0(T)^\perp = \iota_0^S(S)$ of Λ *via the fixed isomorphisms* ι_0 and ι_0^S.) Since Λ is unimodular and S and T are primitive in Λ, we have surjective maps $\pi_S : \Lambda = \Lambda^* \to S^*$, $\pi_T : \Lambda = \Lambda^* \to T^*$ defined by $\pi_S(l) = (l, *)|_S$, $\pi_T(l) = (l, *)|_T$, and natural isomorphisms induced by π_S and π_T ([Ni1, Corollary 1.6.2]):

$$S^*/S \overset{\bar{\pi}_S}{\underset{\sim}{\leftarrow}} \Lambda/(S \oplus T) \overset{\bar{\pi}_T}{\underset{\sim}{\rightarrow}} T^*/T \quad .$$

Here the last isomorphisms are written explicitly as $\bar{\pi}_S(l \bmod S \oplus T) = l_S \bmod S$ and $\bar{\pi}_T(l \bmod S \oplus T) = l_T \bmod T$, where l_S, l_T are defined by the orthogonal decomposition $l = l_S + l_T$ made in $\Lambda \otimes \mathbf{Q} = (S \oplus T) \otimes \mathbf{Q}$. We make the identifications $S \subset S^* \subset S \otimes \mathbf{Q}$ and $T \subset T^* \subset T \otimes \mathbf{Q}$ via the respective nondegenerate forms $(\ , \ )_S$ and $(\ , \ )_T$.

As it is defined by [Ni1, §1], the discriminant group (A_S, q_S) of the lattice $(L, (\ , \ )_L)$ is the pair of the abelian group $A_L := L^*/L$ and a natural quadratic form $q_L : A_L \to \mathbf{Q}/2\mathbf{Z}$ defined by $q_L(x \bmod L) := (x, x)_{L\otimes\mathbf{Q}} \bmod 2\mathbf{Z}$. Here $(\ , \ )_{L\otimes\mathbf{Q}}$ is the natural $\mathbf{Q}$-linear extension of the bilinear form $(\ , \ )_L$ of L.

Proposition 1.2. ([Ni1, Corollary 1.6.2]) *Put* $\sigma_0 = \bar{\pi}_S \circ \bar{\pi}_T^{-1} : A_T \tilde{\to} A_S$. *Then* σ_0 *is an isometry:* $\sigma_0 : (A_T, -q_{A_T}) \cong (A_S, q_{A_S})$.

As it is well-known, primitive embedding of T into Λ are closely related to the so-called genus of the lattice S. We shall briefly recall this here.

Definition-Theorem 1.3. Let M be an even lattice. The set of isomorphism classes of the lattices M' satisfying the following equivalent conditions 1) and 2), are called the *genus* of M, and will be denoted by $\mathcal{G}(M)$:

1) $M' \otimes \mathbf{Z}_p \cong M \otimes \mathbf{Z}_p$ for all primes p, and $M' \otimes \mathbf{R} \cong M \otimes \mathbf{R}$.
2) $(A_{M'}, q_{M'}) \cong (A_M, q_M)$, and $\operatorname{sgn} M' = \operatorname{sgn} M$ where $\operatorname{sgn} M$ is the signature of the lattice M.

The equivalence of 1) and 2) is due to [Ni1, Corollary 1.9.4]. From the condition 2), we have $\det M' = \det M$ and $\operatorname{rank} M' = \operatorname{rank} M$ if $M' \in \mathcal{G}(M)$. It is well-known that the genus $\mathcal{G}(M)$ is a finite set (see e.g. [Cs, Page 128, Theorem 1.1]).

Consider the genus $\mathcal{G}(S)$ of S and put: $\mathcal{G}(S) = \{S_1, S_2, \cdots, S_m\}$ $(S_1 := S)$. By definition, we have an isomorphism $(A_S, q_S) \cong (A_{S_j}, q_{S_j})$ for each $S_j \in \mathcal{G}(S)$. Since $\sigma_0 : (A_T, -q_T) \cong (A_S, q_S)$ by Proposition 1.2, we can then choose an isomorphism for each $S_j \in \mathcal{G}(S)$: $\varphi_j : (A_T, -q_T) \tilde{\to} (A_{S_j}, q_{S_j})$.

Throughout §1, these isomorphisms φ_j $(1 \le j \le m)$ are fixed. Now consider an arbitrarily primitive embedding $\iota : T \hookrightarrow \Lambda$. We have, by Proposition 1.2,

$$(A_{\iota(T)^\perp}, q_{\iota(T)^\perp}) \cong (A_T, -q_T) \cong (A_S, q_S)$$

and also $\operatorname{sgn} \iota(T)^\perp = \operatorname{sgn} \Lambda - \operatorname{sgn} T$. Therefore there exists a unique $j = j(\iota) \in \{1, 2, \cdots, m\}$ and an isomorphism $S_j \tilde{\to} \iota(T)^\perp$. If two primitive embeddings $\iota : T \hookrightarrow \Lambda$ and $\iota' : T \hookrightarrow \Lambda$ are G-equivalent, there exist elements $\Phi \in O(\Lambda)$ and $g \in G$ such that $\Phi \circ \iota = \iota' \circ g$. Since the restriction $\Phi|_{\iota(T)}$ is then an isometry from $\iota(T)$ to $\iota'(T)$, we have $S_{j(\iota)} \cong \iota(T)^\perp \cong \iota'(T)^\perp \cong S_{j(\iota')}$, which implies that $j(\iota) = j(\iota')$, i.e. $S_{j(\iota)} = S_{j(\iota')}$ in $\mathcal{G}(S)$. Therefore setting

$$\mathcal{P}_j(T,\Lambda) := \{\iota : T \hookrightarrow \Lambda \in \mathcal{P}(T,\Lambda) \mid \iota(T)^\perp \cong S_j\ \},$$

$$\mathcal{P}_j^{G\text{-eq}}(T,\Lambda) := \{[\iota : T \hookrightarrow \Lambda] \in \mathcal{P}^{G\text{-eq}}(T,\Lambda) \mid \iota(T)^\perp \cong S_j\ \},$$

we have the following well-defined disjoint unions:

$$\mathcal{P}(T,\Lambda) = \bigcup_{j=1}^m \mathcal{P}_j(T,\Lambda)\ \ ,\ \ \mathcal{P}^{G\text{-eq}}(T,\Lambda) = \bigcup_{j=1}^m \mathcal{P}_j^{G\text{-eq}}(T,\Lambda)\ \ .$$

The following is a generalization of [MM], which will be proved in Appendix A:

Theorem 1.4. *For each j, we have $|\mathcal{P}_j^{G\text{-}eq}(T,\Lambda)| = |O(S_j) \setminus O(A_{S_j})/G|$. In particular,*

$$|\mathcal{P}^{G\text{-}eq}(T,\Lambda)| = \sum_{j=1}^m |O(S_j) \setminus O(A_{S_j})/G|\,.$$

Here $O(S_j)$ acts on $O(A_{S_j})$ through the natural map $O(S_j) \to O(A_{S_j})$, $h \mapsto \bar{h}$. G acts on $O(A_{S_j})$ through the composite of the natural map $G \subset O(T) \to O(A_T)$ and the adjoint map $ad(\varphi_j) : O(A_T) \to O(A_{S_j})$.

2. FM number of a K3 surface

Let $D(X)$ be the derived category of bounded complexes of coherent sheaves on a smooth projective variety X. A smooth projective variety Y is called a *Fourier-Mukai (FM) partner* of X if there is an equivalence $D(Y) \cong D(X)$ as triangulated categories. We denote by $FM(X)$ the set of isomorphism classes of FM partners of X.

Definition 2.1. We call the cardinality $|FM(X)|$ of the set $FM(X)$ the Fourier-Mukai (FM) number of X.

A FM partner of a K3 surface is again a K3 surface by a result of Mukai [Mu3] and Orlov [Or] (See also [BM]).

By a K3 surface, we mean a smooth *projective* surface X over $\mathbf{C}$ with $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$. We denote by $(*, **)$ the symmetric bilinear form on $H^2(X, \mathbf{Z})$ given by the cup product. Then $(H^2(X,\mathbf{Z}), (*, **))$ is an even unimodular lattice of signature $(3, 19)$. This lattice is isomorphic to the K3 lattice $\Lambda_{\mathrm{K3}} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ where U is the hyperbolic lattice (an even unimodular lattice of signature (1,1)). An isomorphism $\tau : H^2(X,\mathbf{Z}) \to \Lambda_{\mathrm{K3}}$ is called a marking and a pair (X, τ) is called a marked K3 surface. We denote by $NS(X)$ the Néron-Severi lattice of X and by $\rho(X)$ the Picard number, i.e. the rank of $NS(X)$. The lattice $NS(X)$ is primitive in $H^2(X,\mathbf{Z})$ and has signature $(1, \rho(X) - 1)$. We call the orthogonal lattice $T(X) := NS(X)^\perp$ in $H^2(X,\mathbf{Z})$ the transcendental lattice. $T(X)$ is primitive in $H^2(X,\mathbf{Z})$ and has signature $(2, 20-\rho(X))$. We denote by ω_X a nowhere vanishing holomorphic two form on X. Then one has a natural inclusion

$$\mathbf{C}\omega_X \oplus \mathbf{C}\bar{\omega}_X \subset T(X) \otimes \mathbf{C}.$$

This defines a Hodge structure $(T(X), \mathbf{C}\omega_X)$ of weight 2 on $T(X)$. Note that $T(X)$ is the minimal primitive sublattice of $H^2(X,\mathbf{Z})$ such that $T(X) \otimes \mathbf{C}$ contains $\mathbf{C}\omega_X$.

The following is a fundamental theorem due to Mukai and Orlov. This connects three aspects of FM partners of K3 surfaces. Statement 1) is categorical, 2) is arithmetical, and 3) is geometrical.

Theorem 2.2. ([Mu3],[Or]) *Let X and Y be K3 surfaces. Then the following statements are equivalent:*

1) $D(Y) \cong D(X)$;
2) There exists a Hodge isometry $g : (T(Y), \mathbf{C}\omega_Y) \tilde{\to} (T(X), \mathbf{C}\omega_X))$;
3) Y is a two dimensional fine compact moduli space of stable sheaves on X with respect to some polarization on X.

For the rest of this section, X will be a K3 surface. Our Counting Formula 2.3 follows from specializing Theorem 1.4 to the case

$$\Lambda = H^2(X, \mathbf{Z}),\ (T, \mathbf{C}\omega) = (T(X), \mathbf{C}\omega_X),\ S = NS(X),\ G = O_{Hodge}(T(X), \mathbf{C}\omega_X).$$

Note that in this case the group G is a finite cyclic group of order $2I$ such that $\varphi(2I) | \mathrm{rk}\, T(X)$. (See Appendix B for proof.)

The aim of this section is to prove the following theorem:

Theorem 2.3. (The Counting Formula) *Set $\mathcal{G}(NS(X)) = \{S_1 = S, S_2, \cdots, S_m\}$ where $m = |\mathcal{G}(NS(X))|$. Then,*

$$|FM(X)| = \sum_{j=1}^{m} |O(S_j) \setminus O(A_{S_j}) / O_{Hodge}(T(X), \mathbf{C}\omega_X)|.$$

Moreover the j-th summand here coincides with the number of FM partners $Y \in FM(X)$ with $NS(Y) \simeq S_j$.

Proof. By Theorem 1.4, it suffices to show the following:

Theorem 2.4. *There is a natural bijective correspondence*

$$\mathcal{P}^{G-eq}(T(X), H^2(X, \mathbf{Z})) \leftrightarrow FM(X).$$

Moreover, under this bijection a primitive embedding ι with $\iota(T)^{\perp} \simeq S_j$ corresponds to a FM partner Y of X with $NS(Y) \simeq S_j$.

Our proof of Theorem 2.4 is based on Theorem 2.2 and the two most fundamental facts about K3 surfaces, namely, the global Torelli theorem and the surjectivity of period map (See e.g. [BPV, Chapter VIII] and the references therein).

In what follows, for convenience, we identify $H^2(X, \mathbf{Z}) = \Lambda_{K3}$ through a fixed marking τ_X and set $(T, \mathbf{C}\omega) = (T(X), \mathbf{C}\omega_X) \subset H^2(X, \mathbf{Z})$ under the natural inclusion. We proceed in five steps.

Step 0. For an arbitrary primitive embedding $\iota : T \hookrightarrow \Lambda_{\mathrm{K3}}$, there exists a marked K3 surface (Y, τ_Y) with the commutative diagram:

$$\begin{array}{cccc} \tau_Y : & H^2(Y, \mathbf{Z}) & \tilde{\to} & \Lambda_{\mathrm{K3}} \\ & \cup & & \uparrow \iota \\ \iota^{-1} \circ \tau_Y|_{T(Y)} : & (T(Y), \mathbf{C}\omega_Y) & \to & (T, \mathbf{C}\omega) \end{array}$$

Proof. This follows immediately from the surjectivity of the period map and the minimality of T. □

Step 1. There is a map $\tilde{c} : \mathcal{P}(T, \Lambda_{\mathrm{K3}}) \to FM(X)$, i.e. there exists a FM partner $Y(\iota)$ for each primitive embedding $\iota : T \hookrightarrow \Lambda_{\mathrm{K3}}$.

Proof. Let $\iota : T \hookrightarrow \Lambda_{\mathrm{K3}}$ be a primitive embedding, then by Step 0. we have a marked K3 surface (Y, τ_Y) with the commutative diagram in Step 0. Then, by Theorem 2.2, we have $Y \in FM(X)$. Assume there exists another marked K3 surface $(Y', \tau_{Y'})$ with the commutative diagram in Step 0 (Y replaced by Y'). Then

$$\tau_Y^{-1} \circ \tau_{Y'} : (H^2(Y', \mathbf{Z}), \mathbf{C}\omega_{Y'}) \to (H^2(Y, \mathbf{Z}), \mathbf{C}\omega_Y)$$

is a Hodge isometry, which means that $Y' \cong Y$ by the Torelli theorem (See e.g. [BPV]). Therefore there exists a well-defined map $\tilde{c} : \mathcal{P}(T, \Lambda_{\mathrm{K3}}) \to FM(X)$. $\square$

Step 2. The map $\tilde{c}$ is surjective.

Proof. Let $Y \in FM(X)$ and fix a marking $\tau_Y : H^2(Y, \mathbf{Z}) \to \Lambda_{\mathrm{K3}}$. Then by Theorem 2.2, we have a Hodge isometry $g : (T(Y), \mathbf{C}\omega_Y) \xrightarrow{\sim} (T(X), \mathbf{C}\omega_X)$ and the following commutative diagram in which $T' = \tau_Y(T(Y)), \omega' = \tau_Y(\omega_Y)$:

$$\begin{array}{ccccccc}
\Lambda_{\mathrm{K3}} & \underset{\tau_Y}{\overset{\sim}{\longleftarrow}} & H^2(Y, \mathbf{Z}) & & H^2(X, \mathbf{Z}) & \underset{\tau_X}{\overset{\sim}{\longrightarrow}} & \Lambda_{\mathrm{K3}} \\
\cup\, \iota_Y & & \cup & & \cup & & \cup\, \iota_X \\
(T', \mathbf{C}\omega') & \overset{\sim}{\longleftarrow} & (T(Y), \mathbf{C}\omega_Y) & \underset{g}{\overset{\sim}{\longrightarrow}} & (T(X), \mathbf{C}\omega_X) & \overset{\sim}{\longrightarrow} & (T, \mathbf{C}\omega)
\end{array}$$

The map $\iota := \iota_Y \circ \tau_Y|_{T(Y)} \circ g^{-1} \circ \tau_X^{-1}|_T : T \hookrightarrow \Lambda_{\mathrm{K3}}$ is then a primitive embedding and $Y = \tilde{c}(\iota)$. $\square$

Step 3. If two primitive embeddings $\iota, \iota' : (T, \mathbf{C}\omega) \hookrightarrow \Lambda_{\mathrm{K3}}$ are G-equivalent, then $\tilde{c}(\iota) = \tilde{c}(\iota')$. This together with the surjectivity in Step 2 entails that the map $\tilde{c}$ descends to a surjective map $c : \mathcal{P}^{G\text{-eq}}(T, \Lambda_{\mathrm{K3}}) \to FM(X)$.

Proof. Suppose $\iota, \iota' : (T, \mathbf{C}\omega) \hookrightarrow \Lambda_{\mathrm{K3}}$ are G-equivalent, and denote $Y = \tilde{c}(\iota), Y' = \tilde{c}(\iota')$. We also fix markings $\tau_Y, \tau_{Y'}$ of Y and Y'. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
H^2(Y, \mathbf{Z}) & \underset{\tau_Y}{\overset{\sim}{\longrightarrow}} & \Lambda_{\mathrm{K3}} & \underset{\exists\Phi}{\overset{\sim}{\longrightarrow}} & \Lambda_{\mathrm{K3}} & \underset{\tau_{Y'}}{\overset{\sim}{\longleftarrow}} & H^2(Y', \mathbf{Z}) \\
\cup & & \uparrow \iota & & \uparrow \iota' & & \cup \\
(T(Y), \mathbf{C}\omega_Y) & \overset{\sim}{\longrightarrow} & (T, \mathbf{C}\omega) & \underset{\exists g}{\overset{\sim}{\longrightarrow}} & (T, \mathbf{C}\omega) & \overset{\sim}{\longleftarrow} & (T(Y'), \mathbf{C}\omega_{Y'})
\end{array}$$

where $\Phi \in O(\Lambda_{\mathrm{K3}})$ and $g \in G = O_{Hodge}(T, \mathbf{C}\omega)$. From this diagram, we have a Hodge isometry $\tau_{Y'}^{-1} \circ \Phi \circ \tau_Y : (H^2(Y, \mathbf{Z}), \mathbf{C}\omega_Y) \to (H^2(Y', \mathbf{Z}), \mathbf{C}\omega_{Y'})$, which implies $Y \cong Y'$ by the global Torelli theorem. $\square$

Step 4. The induced map $c : \mathcal{P}^{G\text{-eq}}(T, \Lambda_{\mathrm{K3}}) \to FM(X)$ is injective.

Proof. Take two G-equivalence classes $[\iota], [\iota'] \in \mathcal{P}^{G\text{-eq}}(T, \Lambda_{\mathrm{K3}})$ and denote $Y = c([\iota]) = \tilde{c}(\iota)$, $Y' = c([\iota']) = \tilde{c}(\iota')$. We fix markings $\tau_Y, \tau_{Y'}$ for Y and Y', respectively. To see the injectivity we show that $f : Y' \cong Y$ implies $[\iota'] = [\iota]$. Denote the Hodge isometry $f^* : (H^2(Y, \mathbf{Z}), \mathbf{C}\omega_Y) \to (H^2(Y', \mathbf{Z}), \mathbf{C}\omega_{Y'})$ associated with the isomorphism $f : Y' \cong Y$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
\Lambda_{\mathrm{K3}} & \underset{\tau_Y}{\overset{\sim}{\longleftarrow}} & H^2(Y, \mathbf{Z}) & \underset{f^*}{\overset{\sim}{\longrightarrow}} & H^2(Y', \mathbf{Z}) & \underset{\tau_{Y'}}{\overset{\sim}{\longrightarrow}} & \Lambda_{\mathrm{K3}} \\
\iota\uparrow & & \cup & & \cup & & \uparrow \iota' \\
(T, \mathbf{C}\omega) & \overset{\sim}{\longleftarrow} & (T(Y), \mathbf{C}\omega_Y) & \overset{\sim}{\longrightarrow} & (T(Y'), \omega_{Y'}) & \overset{\sim}{\longrightarrow} & (T, \mathbf{C}\omega)
\end{array}$$

From this diagram we obtain;

$$\begin{array}{ccc}
\Lambda_{\mathrm{K3}} & \underset{\exists\Phi}{\overset{\sim}{\longrightarrow}} & \Lambda_{\mathrm{K3}} \\
\iota\uparrow & & \uparrow \iota' \\
(T, \mathbf{C}\omega) & \underset{\exists g}{\overset{\sim}{\longrightarrow}} & (T, \mathbf{C}\omega)
\end{array}$$

where $g \in O_{Hodge}(T, \mathbf{C}\omega)$. We then conclude $[\iota] = [\iota']$ in $\mathcal{P}^{G\text{-eq}}(T, \Lambda_{\mathrm{K3}})$. $\square$

It now follows that c is bijective. The second assertion in the theorem is clear from the construction of c. $\square$

In an actual application of the Counting Formula 2.3, we need to determine the genus of $NS(X)$. The following theorem due to [Ni1, Theorem 1.14.2] is very useful when $\rho(X)$ is large:

Theorem 2.5. *Let S be an even indefinite non-degenerate lattice. If $rk\, S \geq 2 + l(S)$, then $\mathcal{G}(S) = \{S\}$ and the natural map $O(S) \to O(A_S)$ is surjective. Here $l(S)$ is the minimal number of generators of A_S, e.g. $l(A_S) = 1$ when A_S is cyclic.*

Corollary 2.6. *If $\rho(X) \geq l(NS(X)) + 2$, then $FM(X) = \{X\}$.*

Proof. By Theorem 2.5, we have $\mathcal{G}(NS(X)) = \{NS(X)\}$. Then, by the Counting Formula, we have $|FM(X)| = |O(NS(X)) \setminus O(A_{NS(X)})/O_{Hodge}(T(X), \mathbf{C}\omega_X)|$. Since the natural map $O(NS(X)) \to O(A_{NS(X)})$ is surjective again by Theorem 2.5, we have $|FM(X)| = 1$. $\square$

Corollary 2.7. *1)* (cf. [Mu1, Proposition 6.2]) *If $\rho(X) \geq 12$, then $FM(X) = \{X\}$. In particular, $FM(Km\, A) = \{Km\, A\}$.*
2) If $\rho(X) \geq 3$ and $detNS(X)$ is square-free, then $FM(X) = \{X\}$.
3) If $X \to \mathbf{P}^1$ is a Jacobian K3 surface, i.e. an elliptic K3 surface with a section, then $FM(X) = \{X\}$.
4) (cf. [Og1, Proposition 1.10]) *Assume that $\rho(X) = 1$ and $NS(X) = \mathbf{Z}H$ with $(H^2) = 2n$. Then $|FM(X)| = 2^{\tau(n)-1}$. Here $\tau(1) = 1$, and $\tau(n)$ is the number of prime factors of $n \geq 2$, e.g. $\tau(4) = \tau(2) = 1$, $\tau(6) = 2$.*

Proof.

1): This follows from Corollary 2.6 and the following calculation: $l(NS(X)) = l(T_X) \leq rk\, T(X) \leq 10 \leq \rho(X) - 2$.

2): Since $\det NS(X)$ is square-free, it follows that $A_{NS(X)}$ is cyclic. Then $l(NS(X)) = 1$. Since $\rho(X) \geq 3$, the result follows from Corollary 2.6.

3): The unimodular hyperbolic lattice U is a direct summand of $NS(X)$, say $NS(X) = U \oplus N$, by the assumption. Thus, $l(NS(X)) = l(N) \leq rk\, N = \rho(X) - 2$. Therefore, the result follows from Corollary 2.6.

4): We assume $n \geq 2$. The case $n = 1$ is similar and easier. Since $NS(X) = \mathbf{Z}H$ with $(H^2) = 2n$, we have $\mathcal{G}(NS(X)) = \{NS(X)\}$, $O(NS(X)) \simeq O_{Hodge}(T(X), \mathbf{C}\omega_X)) \simeq \mathbf{Z}/2$ and $(A_{NS(X)}, q_{NS(X)}) \simeq (\mathbf{Z}/2n, q_{2n})$. Here q_N on $\mathbf{Z}/N$ is defined by $q_N(1) = 1/N$. Therefore, $|FM(X)| = |O(\mathbf{Z}/2n, q_{2n})|/2$ by the Counting Formula. By a straightforward calculation (cf. [Og1]), we have also $|O(\mathbf{Z}/2n, q_{2n})| = 2^{\tau(n)}$. $\square$

3. Class Numbers and Fourier-Mukai Numbers

In this section, we focus on K3 surfaces of $\rho(X) = 2$ (Theorem 3.3) and recast the two class number problems, Questions I and II in Introduction, in corresponding geometrical terms (Questions I' and II').

We begin by recalling the following fact which is well-known to experts (see [Ni 1, 2] and [Mo]). This proposition shows that every question about the class numbers of real quadratic fields can, in principle, be recast in terms of K3 surfaces.

Proposition 3.1. *For any even hyperbolic lattice with $rk\ S \le 10$, there is a projective K3 surface X with $NS(X) \simeq S$. In particular, for every rank 2 even hyperbolic lattice S, there is a projective K3 surface X with $NS(X) \simeq S$.*

Next we briefly recall some basic background on class numbers from [Cs] and [Za]. Let D be an odd fundamental discriminant, i.e. a rational integer D such that $D \equiv 1\ mod\ 4$ and D square-free. Put $K := \mathbf{Q}(\sqrt{D})$, and let $\mathcal{O}_K$ denote its ring of integers. Throughout this section, this D is fixed. Let $H(D)$ denote the narrow ideal class group of $\mathcal{O}_K$, i.e. the quotient of the group of fractional ideals of $\mathcal{O}_K$, by the subgroup of principal fractional ideals with *positive* norm. A theorem of Hurwitz says that $H(D)$ is finite. The number $h(D) = |H(D)|$ is *the class number* of K.

Theorem 3.2. ([Za], [Cs]) *The following three sets are in natural 1-1 correspondence:*

1) The class group $H(D)$.
2) The set $\mathcal{B}(D)$ of proper, i.e. $SL(2,\mathbf{Z})$-equivalence classes of integral binary quadratic forms $ax^2 + bxy + cy^2$ with discriminant $b^2 - 4ac = D$.
3) The set $\mathcal{L}(D)$ of proper, i.e. orientation-preserving, isomorphism classes of even hyperbolic rank 2 lattices S with $det\ S = -D$.

Note that every class in $\mathcal{L}(D)$ can be represented by a pair $\left(\mathbf{Z}^2, \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\right)$, where the matrix represents the bilinear form with respect to the standard base of $\mathbf{Z}^2$. Two such pairs are properly isomorphic if and only if their matrices are conjugate under $SL(2,\mathbf{Z})$.

We recall a few more facts about the sets $H(D), \mathcal{B}(D), \mathcal{L}(D)$. For details, see [Za, §12] and [Cs, Chap. 14 §3]. Consider the exact sequence

$$0 \to \mathcal{J} \to H(D) \xrightarrow{sq} H(D)^2 \to 0$$

where $sq(x) = x^2$ and $\mathcal{J}$ is the kernel of sq. It is known that $|\mathcal{J}| = |H(D)/H(D)^2| = 2^{n-1}$, where n is the number of prime factors of D. Given a rank 2 even lattice L with $det\ L = -D$, let $\tilde{\mathcal{G}}(L)$ be the set of *proper* isomorphism classes of lattices in the same genus as L. The set of genera $\{\tilde{\mathcal{G}}(L) | [L] \in \mathcal{L}(D)\}$ is a principal homegeneous space of the group $H(D)/H(D)^2$. Therefore, there are $[L_1], ..., [L_{2^{n-1}}] \in \mathcal{L}(D)$, and a decomposition:

$$\mathcal{L}(D) = \coprod_{k=1}^{2^{n-1}} \tilde{\mathcal{G}}(L_k)$$

where the genera $\tilde{\mathcal{G}}(L_k)$ all have the same cardinality. Now the subset $\mathcal{J} \subset H(D)$ consists of the ideal classes of the so-called 2-sided ideals. This means that, under the correspondence $H(D) \leftrightarrow \mathcal{L}(D)$, each element in $\mathcal{J}$ corresponds to a class in $\mathcal{L}(D)$ representable as a pair $\left(\mathbf{Z}^2, \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\right)$ such that $\left(\mathbf{Z}^2, \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\right) \simeq \left(\mathbf{Z}^2, \begin{pmatrix} 2a & -b \\ -b & 2c \end{pmatrix}\right)$ under $SL(2,\mathbf{Z})$. In particular the number of such classes in $\mathcal{L}(D)$ is $|\mathcal{J}| = 2^{n-1}$.

Now we have:

Theorem 3.3. *If $D = p$ is a prime and X is a K3 surface with $\rho(X) = 2$ and $det\ NS(X) = -p$, then $|FM(X)| = (h(p) + 1)/2$.*

Remark. For each prime number $p \equiv 1 \ mod\ 4$, there is a K3 surface X such that $NS(X) \simeq S_0$ by Proposition 3.1. Here $S_0 = \left(\mathbf{Z}^2, \begin{pmatrix} 2 & 1 \\ 1 & \frac{1-p}{2} \end{pmatrix}\right)$. Note that S_0 is an even hyperbolic lattice of rank 2 and of $det\, S_0 = -p$. $\square$

Proof. Put $S = NS(X)$. Since $D = p$, i.e. $n = 1$, it follows that $\tilde{\mathcal{G}}(S) = \mathcal{L}(D)$, hence $h(p) = |\tilde{\mathcal{G}}(S)|$. Since $\mathcal{G}(S)$ consists of isomorphism (not necessarily proper) classes of lattices, it follows that $\mathcal{G}(S)$ is the orbit space of the group $\langle\sigma\rangle = GL(2, \mathbf{Z})/SL(2, \mathbf{Z})(\simeq C_2)$ acting on $\tilde{\mathcal{G}}(S)$. Since $|\mathcal{J}| = 1$ because $n = 1$, it follows that there is just one class $[L] \in \tilde{\mathcal{G}}(S)$ (in fact, $[L] = [S_0]$) which is fixed by σ acting on $\tilde{\mathcal{G}}(S)$. Since σ is an involution, it follows that $|\mathcal{G}(S)| = (|\tilde{\mathcal{G}}(S)| + 1)/2$. On the other hand, the group A_S has p elements. Since p is prime, it follows that $A_S \simeq \mathbf{Z}/p$, and that $O(A_S) = \{\pm id\}$. Since the group $O_{Hodge}(T(X), \mathbf{Z}\omega_X)$ obviously contains $\pm id$, it follows that each of the quotients $O(A_{S_i})/O_{Hodge}(T(X), \mathbf{C}\omega_X)$ in the Counting Formula for $|FM(X)|$ has just 1 element. Therefore $|FM(X)| = |\mathcal{G}(S)|$. Now we are done. $\square$

From a table [Fl], for a prime 1297 ($\equiv 1 \bmod 4$), we have $h(1297) = 11$. Then for any K3 surface X with $\rho(X) = 2$ and $det\, NS(X) = -1297$, we have $|FM(X)| = 6$. In this case the value $|FM(X)|$ is not a power of 2, either. This example may be contrasted to the case $\rho(X) = 1$, where $|FM(X)|$ is always a power of 2 (Corollary 2.7 4).)

p	$h(p)$	$\lvert FM(X)\rvert$
229	3	2
257	3	2
401	5	3
577	7	4
733	3	2
761	3	2
1009	7	4
1093	5	3
1129	9	5
1229	3	2
1297	11	6
1373	3	2
1429	5	3
1489	3	2

By Theorem 3.3, Gauss' questions can now be stated in the following geometrical form:

Question I'. *Are there infinitely many K3 surfaces X, with each X having $\rho(X) = 2$ and $FM(X) = \{X\}$, such that the numbers $|det\ NS(X)|$ are distinct primes?*

Question II'. *Is there a sequence of K3 surfaces $X_1, X_2, \ldots$ with $\rho(X_k) = 2$ and $|FM(X_k)| \to \infty$ such that the numbers $|det\ NS(X_k)|$ are primes?*

4. Non-Hausdorff properties of the moduli of K3 surfaces

In this section, we shall show the following:

Theorem 4.1. *Let Δ be a unit disk in $\mathbf{C}$. Then there is a pair of smooth projective families of K3 surfaces $\mathcal{X} \to \Delta$, $\mathcal{Y} \to \Delta$ and a sequence $\{t_k\} \subset \Delta$ such that*

$$\lim_{k\to\infty} t_k = 0\,,\ \mathcal{X}_{t_k} \simeq \mathcal{Y}_{t_k} \text{ but } \mathcal{X}_0 \not\simeq \mathcal{Y}_0\,.$$

In our construction, we shall make use of the 3-rd characterization of FM partners of a K3 surface in Theorem 2.2. We begin by recalling a few facts about the 3-rd characterization, i.e. 2-dimensional fine compact moduli spaces of stable sheaves on a K3 surface X.

Set $\widetilde{NS}(X) := H^0(X,\mathbf{Z}) \oplus NS(X) \oplus H^4(X,\mathbf{Z})$. We call $\widetilde{NS}(X)$ the extended Néron-Severi lattice of X. For $v = (r,H,s) \in \widetilde{NS}(X)$ and an ample class $A \in NS(X)$, we denote by $M_A(v)$ (resp. $\overline{M}_A(v)$) the coarse moduli space of stable sheaves (resp. the coarse moduli space of S-equivalence classes of semi-stable sheaves) $\mathcal{F}$ with respect to the polarization A with $\mu(\mathcal{F}) = v$. Here $\mu(\mathcal{F})$ is the so-called Mukai vector of $\mathcal{F}$, which is defined by $\mu(\mathcal{F}) := \mathrm{ch}(\mathcal{F})\sqrt{td_X}$. By [Ma1, 2] (see also [Mu3]), the space $\overline{M}_A(v)$ is a projective compactification of $M_A(v)$, and $\overline{M}_A(v) = M_A(v)$ if all the semi-stable sheaf $\mathcal{F}$ with $\mu(\mathcal{F}) = v$ is stable, for instance if $(r,s) = 1$.

The following theorem essentially due to Mukai [Mu3] is proved in [Or] in the course of the proof of Theorem 2.2:

Theorem 4.2. *If $Y \in FM(X)$, then $Y \simeq M_H((r,H,s))$ where H is ample, $r > 0$ and s are integers such that $(r,s) = 1$ and $2rs = (H^2)$, and vice versa.*

In our proof, we need is the following relative version:

Lemma 4.3. *Let $\pi : (\mathcal{X},\mathcal{H}) \to \mathcal{B}$ be a smooth projective family of K3 surfaces. Let $f : \overline{\mathcal{M}} \to \mathcal{B}$ be a relative moduli space of the S-equivalence classes of semi-stable sheaves of π with respect to the polarization $\mathcal{H}$ with Mukai vectors $(r,\mathcal{H}_t,s)$. If $r > 0$, $(r,s) = 1$ and $2rs = (\mathcal{H}_t^2)$, then f is projective and gives a relative FM family of π, i.e. $\overline{\mathcal{M}}_t \in FM(\mathcal{X}_t)$ for all $t \in \mathcal{B}$.*

Proof. The existence of f and its projectivity over $\mathcal{B}$ are shown by Maruyama [Ma1] and [Ma2]. By the definition, we have $\overline{\mathcal{M}}_t = \overline{\mathcal{M}}_{\mathcal{H}_t}((r,\mathcal{H}_t,s))$. Since $(r,s) = 1$, $2rs = (\mathcal{H}_t^2)$ and $\mathcal{H}_t$ is ample on $\mathcal{X}_t$, it follows that $\overline{\mathcal{M}}_t = \mathcal{M}_{H_t}((r,\mathcal{H}_t,s)) \in FM(\mathcal{X}_t)$ by Theorem 4.2. $\square$

Proof of Theorem 4.1. Consider the number

$$m := \max\{\rho(X) | X \text{ is a K3 surface with } |FM(X)| \geq 2\}\,.$$

Such a number m exists and satisfies $m \leq 11$ by Corollary 2.7 1) and 4). Take one such X and choose then $Y \in FM(X)$ such that $Y \not\simeq X$. Write $Y = M_H(r,H,s)$ as in Theorem 4.2.

Let us consider the Kuranishi family $u : \mathcal{U} \to \mathcal{K}$ of X. The base space $\mathcal{K}$ is assumed to be a small polydisk in $H^1(X,T_X) \simeq \mathbf{C}^{20}$. Choosing a marking $\tau : R^2u_*\mathbf{Z}_{\mathcal{U}} \simeq \Lambda \times \mathcal{K}$, where Λ is the K3 lattice, we consider the period map

$$\pi : \mathcal{K} \to \mathcal{P} = \{[\omega] \in \mathbf{P}(\Lambda\otimes\mathbf{C}) | (\omega.\omega) = 0, (\omega.\overline{\omega}) > 0\} \subset \mathbf{P}(\Lambda\otimes\mathbf{C}) \simeq \mathbf{P}^{21}\,.$$

By the local Torelli theorem (see e.g. [BPV, Chapter VIII]), $\mathcal{K}$ is isomorphic, via π, to an open neighborhood in $\mathcal{P}$, which we again denote by $\mathcal{P}$. In what follows, we identify $\mathcal{K}$ with $\mathcal{P}$.

Set $\Lambda_0 := \tau_0(NS(X))$. Let us consider the subset $\mathcal{K}^0$ of $\mathcal{K} = \mathcal{P}$ defined by the linear equations $([\omega], \Lambda_0) = 0$. This $\mathcal{K}^0$ parametrizes K3 surfaces $\mathcal{U}_t$ such that $\Lambda_0 \subset \tau_t(NS(\mathcal{U}_t))$, or in other words, $\mathcal{K}^0$ is the subspace of $\mathcal{K}$ in which $NS(X)$ is kept to be algebraic. Since $rank\ \Lambda_0 = \rho(X) = m$, we have $\dim \mathcal{K}^0 = 20 - m \geq 1$. Let us take a generic small one-dimensional disk Δ such that $0 \in \Delta \subset \mathcal{K}^0$ and consider the family $\varphi : \mathcal{X} \to \Delta$ induced from $u : \mathcal{U} \to \mathcal{K}$. By definition, there is an invertible sheaf $\mathcal{H}$ on $\mathcal{X}$ such that $\mathcal{H}_0 = H$. Since ampleness is an open condition, $\mathcal{H}$ is φ-ample. Therefore, the morphism φ projective. By [Og2, Main Theorem], we see that $NS(\mathcal{X}_t) \simeq \Lambda_0 \simeq NS(X)$ for generic $t \in \Delta$ and that the set $\mathcal{S} := \{s \in \Delta | \rho(\mathcal{X}_s) > m\}$ is everywhere dense in Δ with respect to the Euclidean topology.

Let us consider the family of FM partners $f : \mathcal{Y} \to \Delta$ of $\varphi : \mathcal{X} \to \Delta$ with Mukai vectors $(r, \mathcal{H}_t, s)$. This exists by Lemma 4.3. By the definition of m, $\mathcal{S}$ and f, one has $\mathcal{Y}_0 \simeq Y \not\simeq X \simeq \mathcal{X}_0$ and $\mathcal{Y}_s \simeq \mathcal{X}_s$ for $s \in \mathcal{S}$. Then, any sequence $\{t_k\} \subset \mathcal{S}$ with $\lim_{k\to\infty} t_k = 0$ satisfies the desired property. □

Appendix A. Proof of Theorem 1.4

In this appendix, for completeness, we shall give a sketch of a proof of Theorem 1.4. Throughout this appendix, we use the same notation as in §1.

Let T and S be the even lattices defined at the beginning of Section 1 and fix $S_j \in \mathcal{G}(S)$. The goal is to show the following equality:

$$|\mathcal{P}_j^{G\text{-eq}}(T, \Lambda)| = |O(S_j) \setminus O(A_{S_j})/G|\,,$$

which clearly implies the first equality in Theorem 1.4. This equality is a consequence of Propostions A.7 and A.9 below.

Definition A.1. ([Ni1, §1]) An over-lattice $L = (L, (*, **)_L)$ of $S_j \oplus T$ is a pair consisting of a $\mathbf{Z}$-module such that $S_j \oplus T \subset L \subset S_j^* \oplus T^*$ and the bilinear form $(\ ,\ )_L$ given by the restriction of $(\ ,\ )_{S_j^*} \oplus (\ ,\ )_{T^*}$ to L. An over-lattice L is called *integral* if its bilinear form $(\ ,\ )_L$ is $\mathbf{Z}$-valued. An integral over-lattice is a lattice in our sense.

Definition A.2. We denote by $\mathcal{U}_j := \mathcal{U}(S_j \oplus T)$ the set of over-lattices $L = (L, (\ ,\ )_L)$ of $S_j \oplus T$ such that L is integral, unimodular and such that the inclusion $T \subset L$ is primitive. We also denote by $\mathcal{E}_j$ the subset of $\mathcal{U}_j$ which consists of even lattices.

Lemma A.3. ([Ni1, §1]) *Let $L \in \mathcal{U}_j$. Then:*

1) The orthogonal lattice $T^\perp$ in L coincides with S_j, and the inclusion $S_j \subset L$ is primitive.

2) The natural projections $\bar{\pi}_{L,S_j}$ and $\bar{\pi}_{L,T}$ (see §1 for definition) are group isomorphisms.

3) L is even, i.e. $L \in \mathcal{E}_j$ if and only if $q_{S_j}(\varphi_L(x)) = -q_T(x) \quad (\forall x \in A_T)$, where $\varphi_L := \bar{\pi}_{L,S_j} \circ \bar{\pi}_{L,T}^{-1} : A_T \tilde{\to} A_{S_j}$.

Proof. This follows directly from Proposition 1.2. See also [Ni1, §1]. □

Remark. 1) The relation $q_{S_j}(\varphi_L(x)) = -q_T(x) \quad (\forall x \in A_T)$ is nothing but that the isomorphism φ_L is in fact an isometry of the discriminant lattices

$$\varphi_L : (A_T, -q_T) \tilde{\to} (A_{S_j}, q_{S_j})\,.$$

2) The isometry $\varphi_L : (A_T, -q_T) \tilde{\to} (A_{S_j}, q_{S_j})$ associated to $L \in \mathcal{E}_j$ recovers L explicitly as: $L/(S_j \oplus T) = \{\varphi_L(a) \oplus a | a \in A_T\} \subset S_j^*/S_j \oplus T^*/T$. □

Lemma A.4. *For an arbitrary isometry $f : (A_T, -q_T) \tilde{\to} (A_{S_j}, q_{S_j})$, we define the over-lattice L^f of $S_j \oplus T$ by: $L^f/(S_j \oplus T) = \{f(a) \oplus a | a \in A_T\}$. Then L^f is an element of $\mathcal{E}_j$, and the associated isometry $\varphi_{L^f} (= \bar{\pi}_{L^f,S_j} \circ \bar{\pi}^{-1}_{L^f,T})$ coincides with f.*

Proof. This is an old result known as gluing methods, see [CoS, Chap. 4] for example, and also follows immediately from Proposition 1.2. □

By Lemma A.3, 3), each $L \in \mathcal{E}_j$ gives rise to an isometry $\varphi_L : (A_T, -q_T) \tilde{\to} (A_{S_j}, q_{S_j})$. Combininig this with Proposition 1.2, we obtain the following bijective correspondence such that $L \mapsto \varphi_L, f \mapsto L^f$ are inverses of each other:

$$\mathcal{E}_j = \mathcal{E}(S_j \oplus T) \leftrightarrow \{\varphi : (A_T, -q_T) \tilde{\to} (A_{S_j}, q_{S_j})\}\,.$$

Lemma A.5. *Fix an isometry $\varphi_j : (A_T, -q_T) \tilde{\to} (A_{S_j}, q_{S_j})$. Then the following map is bijective: $\mu_j : O(A_{S_j}) \to \mathcal{E}_j\,;\, \sigma \mapsto L^{\sigma\circ\varphi_j}$.*

Proof. The assignment $\sigma \mapsto \sigma \circ \varphi_j$ gives a bijection between $O(A_{S_j})$ and the set $\{\varphi : (A_T, -q_T) \tilde{\to} (A_{S_j}, q_{S_j})\}$. Combining this bijection with Lemma A.3, we obtain the result. □

Definition A.6. 1) Two over-lattices $L, L' \in \mathcal{E}_j$ are called G-equivalent if there exist isometries $\Phi : L \tilde{\to} L'$ and $g : T \tilde{\to} T$ such that $g \in G$ and $\Phi|_T = g$:

$$\begin{array}{cccc} \Phi: & L & \tilde{\to} & L' \\ & \cup & & \cup \\ g: & T & \tilde{\to} & T \end{array}$$

2) We denote the set of G-equivalence classes of the elements of $\mathcal{E}_j$ by $\mathcal{E}_j^{G\text{-eq}}$.

Remark. By Lemma A.3, 1), the orthogonal lattice $T^\perp$ in L coincides with S_j for all $L \in \mathcal{U}_j$. Therefore when L and L' are G-equivalent, the isometry $\Phi : L \tilde{\to} L'$ determines $f := \Phi_{S_j} \in O(S_j)$ as well as $g = \Phi|_T \in G$. Conversely any pair $(f, g) \in O(S_j) \times G$ induces an isometry $f \oplus g : S_j^* \oplus T^* \tilde{\to} S_j^* \oplus T^*$. Moreover, this isometry sends an over-lattice $L \subset S_j^* \oplus T^*$ to an over-lattice $L' = (f \oplus g)(L) \subset S_j^* \oplus T^*$. Clearly L and L' are G-equivalent. □

Proposition A.7. *The bijective map $\mu_j : O(A_{S_j}) \to \mathcal{E}_j$ in Lemma A.5 descends to the bijective map: $\bar{\mu}_j : O(S_j) \setminus O(A_{S_j})/G \to \mathcal{E}_j^{G\text{-eq}}$, where the double coset is defined as in Theorem 1.4.*

Proof. Let $\sigma_1, \sigma_2 \in O(A_{S_j})$. Assume $L^{\sigma_1\circ\varphi_j}$ and $L^{\sigma_2\circ\varphi_j}$ are G-equivalent over-lattices of $S_j \oplus T$, where $\varphi_j : (A_T, -q_T) \tilde{\to} (A_{S_j}, q_{S_j})$ is a fixed isometry. By definition of G-equivalence, we have isometries $\Phi : L^{\sigma_1\circ\varphi_j} \tilde{\to} L^{\sigma_2\circ\varphi_j}$ and $g : T \tilde{\to} T$ such that $\Phi|_T = g \in G$. By Lemma A.3, 1), $S_j = T^\perp$ in both $L^{\sigma_1\circ\varphi_j}$ and $L^{\sigma_2\circ\varphi_j}$, and thus we have $\Phi(S_j \oplus T) = S_j \oplus T$ and $\Phi_{S_j} := \Phi|_{S_j} \in O(S_j)$, $\Phi_T := \Phi|_T = g \in G$. Now we have $L^{\sigma_2\circ\varphi_j}/(S_j \oplus T) = \{\sigma_2 \circ \varphi_j(a) \oplus a \mid a \in A_T\}$, and

$$\begin{aligned} \Phi(L^{\sigma_1\circ\varphi_j}/(S_j \oplus T)) &= \{\bar{\Phi}_{S_j} \circ \sigma_1 \circ \varphi_j(a) \oplus \bar{g}(a) \mid a \in A_T\} \\ &= \{\bar{\Phi}_{S_j} \circ \sigma_1 \circ \varphi_j \circ \bar{g}^{-1}(a) \oplus a \mid a \in A_T\}. \end{aligned}$$

Comparing these two equations, we obtain $\sigma_2 = \bar{\Phi}_{S_j} \circ \sigma_1 \circ (\varphi_j \circ \bar{g}^{-1} \circ \varphi_j^{-1})$, which implies $[\sigma_1] = [\sigma_2]$ in $O(S_j) \setminus O(A_{S_j})/G$.

To see that the map μ_j is well-defined on the double coset $O(S_j) \setminus O(A_{S_j})/G$, let us assume $[\sigma_1] = [\sigma_2]$ in the double coset. Then there exist $f \in O(S_j)$ and $g \in G$ such that $\sigma_2 = \bar{f} \circ \sigma_1 \circ (\varphi_j \circ \bar{g} \circ \varphi_j^{-1})$. Now it is clear from the above calculations that for the $\mathbf{Q}$-linear extensions $f \oplus g : S_j^* \oplus T^* \to S_j^* \oplus T^*$ we have $(f \oplus g)(L^{\sigma_1 \circ \varphi_j}) = L^{\sigma_2 \circ \varphi_j}$. Then, by Remark after Definition A.6, $L^{\sigma_1 \circ \varphi_j}$ and $L^{\sigma_2 \circ \varphi_j}$ are G-equivalent.

These two observations show that the map μ_j descends to a well-defined injective map $\bar{\mu}_j$. Since μ_j is surjective, so is $\bar{\mu}_j$. □

Let $[\iota] \in \mathcal{P}_j^{G\text{-eq}}(T, \Lambda)$. Then, by definition, $\iota(T)^\perp \cong S_j$. We choose a representative $\iota : T \hookrightarrow \Lambda$ of this G-equivalence class. We also fix an isometry $\sigma : S_j \tilde{\to} \iota(T)^\perp$. Now consider the isomorphism $\sigma \oplus \iota : S_j \oplus T \tilde{\to} \iota(T)^\perp \oplus \iota(T)$ and its extension to the dual lattices. Then one has the following commutative diagram:

$$\begin{array}{cccc} \sigma \oplus \iota : & S_j^* \oplus T^* & \tilde{\to} & (\iota(T)^\perp)^* \oplus \iota(T)^* \\ & \cup & & \cup \\ & L(\sigma, \iota) & \tilde{\to} & \Lambda \\ & \cup & & \cup \\ \sigma \oplus \iota : & S_j \oplus T & \tilde{\to} & \iota(T)^\perp \oplus \iota(T) \end{array}$$

where we define the over-lattice $L(\sigma, \iota)$ of $S_j \oplus T$ by $L(\sigma, \iota) := (\sigma \oplus \iota)^{-1}(\Lambda)$. Since Λ is even integral unimodular, so is $L(\sigma, \iota)$, i.e. $L(\sigma, \iota) \in \mathcal{E}_j$.

Lemma A.8. *The over-lattices $L(\sigma, \iota)$ above satisfy the following:*

1) If $\sigma : S_j \tilde{\to} \iota(T)^\perp$ and $\sigma' : S_j \tilde{\to} \iota(T)^\perp$, then $[L(\sigma, \iota)] = [L(\sigma', \iota)]$ as an element of $\mathcal{E}_j^{G\text{-eq}}$.

2) If $\iota : T \hookrightarrow \Lambda$ and $\iota' : T \hookrightarrow \Lambda$ are G-equivalent primitive embeddings $\mathcal{P}_j(T, \Lambda)$, then $[\iota] = [\iota']$ in $\mathcal{P}_j^{G\text{-eq}}(T, \Lambda)$ and $[L(\sigma, \iota)] = [L(\sigma, \iota')]$ in $\mathcal{E}_j^{G\text{-eq}}$.

Proof. Both claims are immediate from a diagram similar to the above and the definition of G-equivalence. Details are left for readers. □

Proposition A.9. *1) The following map is well-defined:*

$$\xi : \mathcal{P}_j^{G\text{-eq}}(T, \Lambda) \to \mathcal{E}_j^{G\text{-eq}}, \quad [\iota] \mapsto [L(\sigma, \iota)]$$

2) The map $\xi : \mathcal{P}_j^{G\text{-eq}}(T, \Lambda) \to \mathcal{E}_j^{G\text{-eq}}$ is bijective.

Proof. The well-definedness of the map ξ is immediate by Lemma A.8. To show surjectivety, let $[K]$ be in $\mathcal{E}_j^{G\text{-eq}}$. Then we have $S_j \oplus T \subset K \subset S_j^* \oplus T^*$, and $T \hookrightarrow K$ is primitive. Let $\iota_K : T \hookrightarrow K$ be an inclusion. Since K is an even unimodular indefinite lattice, there exists an isometry $f : K \tilde{\to} \Lambda$ by the theorem of Milnor (see [Se]). Then for the embedding $\iota := f \circ \iota_K : T \hookrightarrow \Lambda$ and $\sigma := f|_{S_j} : S_j \tilde{\to} \iota(T)^\perp$ we have $K = L(\sigma, \iota)$, that is $[K] = \xi([\iota])$. To show the injectivity of ξ, assume $[L(\sigma, \iota)] = [L(\sigma', \iota')]$ in $\mathcal{E}_j^{G\text{-eq}}$. Then we have:

$$\begin{array}{ccccccc} (\iota(T)^\perp)^* \oplus \iota(T)^* & \overset{\sim}{\longleftarrow} & S_j^* \oplus T^* & \overset{\sim}{\longrightarrow} & S_j^* \oplus T^* & \overset{\sim}{\longrightarrow} & (\iota'(T)^\perp)^* \oplus \iota'(T)^* \\ \cup & & \cup & & \cup & & \cup \\ \Lambda & \overset{\sim}{\longleftarrow} & L(\sigma, \iota) & \underset{\exists \Phi}{\overset{\sim}{\longrightarrow}} & L(\sigma', \iota') & \overset{\sim}{\longrightarrow} & \Lambda \\ \cup & & \cup & & \cup & & \cup \\ \iota(T)^\perp \oplus \iota(T) & \underset{\sigma \oplus \iota}{\overset{\sim}{\longleftarrow}} & S_j \oplus T & \underset{\Phi_{S_j} \oplus \Phi_T}{\overset{\sim}{\longrightarrow}} & S_j \oplus T & \underset{\sigma' \oplus \iota'}{\overset{\sim}{\longrightarrow}} & \iota'(T)^\perp \oplus \iota'(T) \end{array}$$

where $\Phi_{S_j} := \Phi|_{S_j} \in O(S_j)$ and $\Phi_T := \Phi|_T \in G$. If we define $\Phi' = (\sigma' \oplus \iota') \circ \Phi \circ (\sigma \oplus \iota)^{-1} : \Lambda \xrightarrow{\sim} \Lambda$, we see that $\Phi' \circ \iota = \iota' \circ \Phi_T$, which means that $[\iota] = [\iota']$ in $\mathcal{P}_j^{G\text{-eq}}(T, \Lambda)$. This completes the proof. □

Appendix B. The group of Hodge isometries

In this appendix, we shall give a proof of the following Proposition applied in previous sections. This is probably well-known to experts. Our proof here is similar to an argument in [Ni2][St].

Proposition B.1. *Let T be a lattice of signature $(2, rank T - 2)$ and $(T, \mathbf{C}\omega)$ be a weight two Hodge structure on T, i.e. ω is an element of $T \otimes \mathbf{C}$ such that $(\omega, \omega) = 0$ and $(\omega, \overline{\omega}) > 0$. Assume the following minimality condition on T: if $T' \subset T$ is primitive and $\omega \in T' \otimes \mathbf{C}$, then $T' = T$. Then $O_{Hodge}(T, \mathbf{C}\omega)$ is a cyclic group of even order, say, $2I$ such that $\varphi(2I) | rank T$. Here $\varphi(J) = |(\mathbf{Z}/J)^{\times}|$ is the Euler function. Moreover, if g is a generator of $O_{Hodge}(T, \mathbf{C}\omega)$, then $g(\omega) = \zeta_{2I}\omega$, where ζ_{2I} is a primitive $2I$-th root of unity.*

We proceed in five steps.

Step 1. $|O_{Hodge}(T, \mathbf{C}\omega)| < \infty$.

Proof. Put $t := \mathrm{rank}\, T$. Let $P := \langle \mathrm{Re}\,\omega, \mathrm{Im}\,\omega \rangle$. Then P is a positive definite 2-dimensional plane by $(\omega, \omega) = 0$, $(\omega, \overline{\omega}) > 0$. Set $N := P^{\perp}$ in $T \otimes \mathbf{R}$. Since $\mathrm{sgn}\, T = (2, t-2)$, N is a negative definite. Let $g \in O_{Hodge}(T, \mathbf{C}\omega)$. By definition of $O_{Hodge}(T, \mathbf{C}\omega)$, we have $g(P) = P$ and $g(N) = N$. Therefore, via the assignment $g \mapsto (g|P, g|N)$, we have an embedding $O_{Hodge}(T, \mathbf{C}\omega) \subset O(P) \times O(N)$. This should be continuous. Since P and N are both definite, $O(P) \times O(N)$ is compact. Since $O_{Hodge}(T, \mathbf{C}\omega) \subset O(T)$ is discrete, this implies the result. □

Let $g \in O_{Hodge}(T, \mathbf{C}\omega)$. Then $g(\omega) = \alpha(g)\omega$ for some $\alpha(g) \in \mathbf{C}^{\times}$. The assignment $g \mapsto \alpha(g)$ defines a group homomorphism $\alpha : O_{Hodge}(T, \mathbf{C}\omega) \to \mathbf{C}^{\times}$.

Step 2. α is injective.

Proof. Let $g \in \mathrm{Ker}\,\alpha$. Since g is defined over $\mathbf{Z}$ (i.e. g is represented by a matrix of integral entries with respect to integral basis of T), the invarinat set $T^g := \{x \in T | g(x) = x\}$ is a primitive sublattice of T such that $T^g \otimes \mathbf{C} = (T \otimes \mathbf{C})^g \ni \omega$. Thus, by the minimality condition, we have $T^g = T$. This means $g = \mathrm{id}_T$. □

Step 3. $O_{Hodge}(T, \mathbf{C}\omega) \simeq \mathbf{Z}/2I$ for some $I \in \mathbf{N}$.

Proof. By Steps 1 and 2, $O_{Hodge}(T, \mathbf{C}\omega)$ is isomorphic to a finite subgroup of the multiplicative group $\mathbf{C}^{\times}$ of the field $\mathbf{C}$. Such groups are always cyclic. Therefore $O_{Hodge}(T, \mathbf{C}\omega)$ is a finite cyclic group. Since the involution $-id_T$ is an element of $O_{Hodge}(T, \mathbf{C}\omega)$, the order of $O_{Hodge}(T, \mathbf{C}\omega)$ is also even. □

Set $O_{Hodge}(T, \mathbf{C}\omega) = \langle g \rangle$. Then $\mathrm{ord}\,(g) = 2I$.

Step 4. All the eigenvalues of g (on $T \otimes \mathbf{C}$) are primitive $2I$-th roots of unity.

Proof. Since $\mathrm{ord}\,(g) = 2I$, all the eigenvalues of g are $2I$-th roots of unity. Since α is injective, $g(\omega) = \zeta_{2I}\omega$, where ζ_{2I} is a primtive $2I$-th root of unity. Let d be a positive integer such that $d | 2I$ and $d \neq 2I$. Suppose that g admits a primitive

d-th root of unity ζ_d as its eigenvalue. Then g^d has an eigenvalue 1. Since g is defined over $\mathbf{Z}$, there is $x \in T - \{0\}$ such that $g^d(x) = x$. Since $g^d(\omega) = \zeta_e \omega$, where $e = 2I/d > 1$ and ζ_e is a primitive e-th root of unity, we have

$$(x, \omega) = (g^d(x), g^d(\omega)) = (x, \zeta_e \omega)\,.$$

This implies $(x, \omega) = 0$, because $\zeta_e \neq 1$ by $e > 1$. However, we have then $\omega \in (x)^{\perp} \otimes \mathbf{C}$, a contradiction to the minimality condition of T. Therefore all the eigenvalues of g are primitive $2I$-th roots of unity. □

Step 5. $\varphi(2I) | t = \operatorname{rank} T$.

Proof. Since g is defined over $\mathbf{Q}$ and the primitive $2I$-th roots of unity are mutually conjugate over $\mathbf{Q}$, their multiplicities as eigenvalue of g are the same, say m. Combining this with the fact that the number of primitive $2I$-th roots of unity is $\varphi(2I)$, we have $m \cdot \varphi(2I) = t$. This implies $\varphi(2I)|t$. □

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