AUTOEQUIVALENCES OF DERIVED CATEGORY OF A K3 SURFACE AND MONODROMY TRANSFORMATIONS

Shinobu Hosono, Bong H. Lian, Keiji Oguiso, Shing-Tung Yau

Abstract. We consider autoequivalences of the bounded derived category of coherent sheaves on a K3 surface. We prove that the image of the autoequivalences has index at most two in the group of the Hodge isometries of the Mukai lattice. Motivated by homological mirror symmetry we also consider the mirror counterpart, i.e. symplectic version of it. In the case of $\rho(X) = 1$, we find an explicit formula which reproduces the number of Fourier-Mukai (FM) partners from the monodromy problem of the mirror K3 family. We present an explicit example in which a monodromy action does not come from an autoequivalence of the mirror side.

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§0. Introduction – Motivation and Backgrounds

Our main results are Theorems 1.6, 1.9, 1.17 and Proposition 5.8.

In this paper we study the bounded derived category of K3 surfaces motivated by homological mirror symmetry of K3 surfaces.

Homological mirror symmetry due to Kontsevich [Ko] is based on homological and algebraic aspects of manifolds where one considers certain derived categories of manifolds, i.e. the bounded derived category $D(X)$ of a projective variety X, and the bounded derived category $DFuk(\check{X}, \beta)$ of the Fukaya category $Fuk(\check{X}, \beta)$ for the mirror $\check{X}$ with its symplectic structure β. (See [FO3], [Fu] and reference therein.) The homological mirror symmetry conjecture says that *when X and $\check{X}$ form a mirror pair, there should be a (not necessarily canonical) exact equivalence*

$$D(X) \cong DFuk(\check{X}, \beta),$$

where β is a generic symplectic structure and a generic complex structure is assumed for the left hand side. Despite its homological and algebraic nature, homological mirror symmetry bears a close relationship with geometric mirror symmetry due to [SYZ] which is based on the symplectic geometry of the underlying C^∞-manifolds. See also [Mo2] and reference therein for details.

The following mathematical problems are important for understanding fully the homological mirror symmetry:

a) determine the group of autoequivalences $D(X) \to D(X)$,
b) determine the set of varieties Y s.t. $D(Y) \cong D(X)$, i.e. FM partners of X.

Recent results by Orlov[Or2] solve both problems when X is an abelian variety. For K3 surfaces, the problem b) has been studied in detail [Or2],[BM],[Og2].

In this paper, we first consider the natural map from the autoequivalences to the Hodge isometries of the Mukai lattice and prove that the image is a subgroup of index at most two (**Theorem 1.6**). Next we consider the mirror counterpart of the problem a), i.e. the group of autoequivalences of $DFuk(\check{X}, \beta)$. However at this moment not much is known about this side. So we consider the group $\mathrm{Symp}(\check{X}, \beta)$ of the (cohomological) symplectic diffeomorphisms, more precisely the (cohomological) symplectic mapping class group $\pi_0 Symp(\check{X}, \beta) = Symp(\check{X}, \beta)/Symp^0(\check{X}, \beta)$, as a natural substitute for $\mathrm{Auteq} DFuk(\check{X}, \beta)$ (Definition 1.7). Under this substitution we prove that the pullback of the differential forms induces a surjective map from $\mathrm{Symp}(\check{X}, \beta)$ to an orthogonal group $O^+(T(\check{X}))^*$ of the transcendental lattice (**Theorem 1.9**).

Second, we consider a mirror family in the sense of Dolgachev [Do] and define a monodromy group $\mathcal{M}(\check{X})$ of the family. Then using the surjective map in Theorem 1.9 we define a monodromy representation of the group $\pi_0\mathrm{Symp}(\check{X}, \beta)$ and consider its image $\mathcal{MS}(\check{X}) = O^+(T(\check{X}))^*$ in $\mathcal{M}(\check{X})$. Surprisingly, for K3 surfaces with $\rho(X) = 1$, we find that the group index $[\mathcal{M}(\check{X}) : \mathcal{MS}(\check{X})]$ coincides with the number of FM (Fourier-Mukai) partners obtained in [Og2] (**Theorem 1.17**).

Finally, we present explicit calculations of the group index of the monodromy representation in the first non-trivial case X of $\deg(X) = 12$. That the monodromy of the mirror family are realized by autoequivalences in mirror symmetry was first suggested by Kontsevich (see e.g. [Mo2]) and this has been studied further for example in [Hor],[ST],[ACHY]. Our example shows that this need not be the case when we have non-trivial FM partner (**Proposition 5.8**). Our example has appeared in [LY1][LY2], and also studied in detail in completely different context [PS][BP] (before the mirror symmetry).

After we have posted the preliminary version of this paper, B. Szendröi kindly pointed out to us a mistake in the preliminary version. We are also grateful to him for informing us of his work in mirror symmetry of K3 surfaces. (See section 5 of [Sz].)

We also thank D. Huybrechts for informing us of his student, D. Ploog's work on the map $\mathrm{Auteq}D(X) \to O_{Hodge}(\tilde{H}(X, \mathbf{Z}))$.

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§1. Statements of main results

(1-1) Autoequivalences and $\mathrm{Im}(\mathrm{Auteq}D(X) \to O_{Hodge}(\tilde{H}(X, \mathbf{Z})))$

Let X be a K3 surface, a smooth projective surface over $\mathbf{C}$ with $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$. We denote by $(*, **)$ the symmetric bilinear form on $H^2(X, \mathbf{Z})$ given by the cup product. Then $(H^2(X, \mathbf{Z}), (*, **))$ is an even unimodular lattice of signature $(3, 19)$. This lattice is isomorphic to the K3 lattice $\Lambda_{\mathrm{K3}} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ where U is the hyperbolic lattice (an even unimodular lattice of signature (1,1)). An isomorphism $\tau : H^2(X, \mathbf{Z}) \to \Lambda_{\mathrm{K3}}$ is called a marking and a pair (X, τ) is called a marked K3 surface. We denote by $NS(X) = \mathrm{Pic}(X)$ the Néron-Severi lattice of X and by $\rho(X)$ the Picard number, i.e. the rank of $NS(X)$. The lattice $NS(X)$ is primitive in $H^2(X, \mathbf{Z})$ and has signature $(1, \rho(X) - 1)$. We call the orthogonal lattice $T(X) := NS(X)^{\perp}$ in $H^2(X, \mathbf{Z})$ the transcendental lattice. $T(X)$ is primitive in $H^2(X, \mathbf{Z})$ and has signature $(2, 20 - \rho(X))$. We denote by ω_X a nowhere vanishing holomorphic two form. Then the natural inclusion

$$\mathbf{C}\omega_X \oplus \mathbf{C}\bar{\omega}_X \subset T(X) \otimes \mathbf{C}$$

defines a Hodge structure of weight 2 on $T(X)$.

Based on the original work by Mukai[Mu2], Orlov[Or1] showed that Fourier-Mukai transform on the bounded derived category of coherent sheaves induces a Hodge isometry of the Mukai lattice. Let us summarize basic definitions.

Definition 1.1. *For a K3 surface X, we define the Mukai lattice to be a lattice*

$$\tilde{H}(X, \mathbf{Z}) := \left(H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \oplus H^4(X, \mathbf{Z}), \langle\ ,\ \rangle\right) \ ,$$

with its non-degenerate bilinear form $\langle (a, b, c), (a', b', c') \rangle := -(a, c') - (c, a') + (b, b')$ using the cup product $(\ ,\ )$ of the cohomology ring $\oplus H^i(X, \mathbf{Z}) = H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \oplus H^4(X, \mathbf{Z})$. Then there is an isomorphism as lattice $\tilde{H}(X, \mathbf{Z}) \cong U \oplus \Lambda_{K3}$, where U is the hyperbolic lattice and $\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.

Definition 1.2. *An isometry of the Mukai lattices $\varphi : \tilde{H}(X, \mathbf{Z}) \to \tilde{H}(Y, \mathbf{Z})$ is called Hodge isometry if it satisfies $\varphi(\mathbf{C}\omega_X) = \mathbf{C}\omega_Y$, and is denoted by*

$$\varphi : (\tilde{H}(X, \mathbf{Z}), \mathbf{C}\omega_X) \to (\tilde{H}(Y, \mathbf{Z}), \mathbf{C}\omega_Y).$$

For $Y = X$, we define the group of Hodge isometries

$$O_{Hodge}(\tilde{H}(X, \mathbf{Z})) := \{\varphi : (\tilde{H}(X, \mathbf{Z}), \mathbf{C}\omega_X) \to (\tilde{H}(X, \mathbf{Z}), \mathbf{C}\omega_X)\}.$$

For a smooth projective variety X we denote by $D(X) := D^b_{coh}(X)$ the bounded derived category of coherent sheaves on X, which hereafter will be called simply the derived category of X. $ObD(X)$ then consists of the bounded complexes of coherent sheaves on X. $D(X)$ is naturally regarded as a triangulated category. (See [GM] for details.) In this paper, by a functor $F : D(X) \to D(Y)$ we always mean a functor as a triangulated category, i.e. a functor which commutes with the shift functor and preserves the distinguished triangles.

A functor $F : D(X) \to D(Y)$ is called an *equivalence* if there exists a functor $G : D(Y) \to D(X)$ for which we have the relations $G \circ F \cong \mathrm{id}_{D(X)}$ and $F \circ G \cong \mathrm{id}_{D(Y)}$ as functors. G is called *quasi-inverse* of F. The isomorphism class of G (as a functor) is uniquely determined by the isomorphism class of F, although G itself is not uniquely determined. The group of the isomorphism classes of (self-)equivalences $F : D(X) \to D(X)$ is called an *autoequivalence* of $D(X)$ and will be denoted by $\mathrm{Auteq}D(X)$. Note that $\mathrm{Auteq}D(X)$ is a set and hence a group because of the finiteness condition on the coherent sheaves.

For smooth projective varieties X, Y and $\mathcal{E} \in D(X \times Y)$, we consider a functor $\Phi^{\mathcal{E}}_{X\to Y} : D(X) \to D(Y)$ defined by

$$\Phi^{\mathcal{E}}_{X\to Y}(\mathcal{X}) = \mathbf{R}\pi_{Y*}(\mathcal{E} \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi_X^*\mathcal{X}) \tag{1.1}$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the natural projections; $\pi_X(a, b) = a, \pi_Y(a, b) = b$. In this paper we abbreviate the functor (1.1) as $\Phi^{\mathcal{E}}_{X\to Y}(\mathcal{X}) = \pi_{Y*}(\mathcal{E} \otimes \pi_X^*\mathcal{X})$ and also write $\Phi^{\mathcal{E}}$ if $X = Y$.

In what follows we will be mainly concerned with the case where the functor $\Phi^{\mathcal{E}}_{X\to Y}$ gives an equivalence. In this case, this functor is called a *Fourier-Mukai(FM) transform* and $\mathcal{E} \in D(X \times Y)$ is called its *kernel*. The following fundamental result by Orlov allows us to represent an equivalence in the form of FM transform:

Theorem 1.3. ([Or1]) *For smooth projective varieties X, Y and an equivalence $\Phi_{X\to Y} : D(X) \to D(Y)$, there exists $\mathcal{E} \in D(X \times Y)$ such that $\Phi_{X\to Y} = \Phi^{\mathcal{E}}_{X\to Y}$. Moreover $\mathcal{E}$ is determined by Φ unique up to isomorphism as an object in $D(X \times Y)$.* □

For an element $\mathcal{F} = (\mathcal{F}^\bullet) \in ObD(X)$ we define the chern character $ch(\mathcal{F}) = \sum_i (-1)^i\, ch(\mathcal{F}^i) \in \tilde{H}(X, \mathbf{Q}) := \oplus_k H^{2k}(X, \mathbf{Q})$. This is well-defined by the definition of $D(X)$. Furthermore by the uniqueness of the kernel $\mathcal{E} \in D(X \times Y)$ up to isomorphism in Theorem 1.3, the chern character $ch(\mathcal{E})$ does not depend on the choice of the kernel representing an equivalence Φ.

Now let us restrict our attention to K3 surfaces. The following is a fundamental Theorem essentially due to Mukai:

Theorem 1.4. *Let X be a K3 surface and Y, Z smooth projective manifolds.*
1) If $\Phi : D(X) \to D(Y)$ is an equivalence, then Y is a K3 surface.
2) For each $\mathcal{X} \in D(X)$, $ch(\mathcal{X}) \in \tilde{H}(X, \mathbf{Z})$.
3) An equivalence $\Phi^{\mathcal{E}}_{X\to Y} : D(X) \to D(Y)$ induces a Hodge isometry

$$f^{\mathcal{E}}_{X\to Y} : \tilde{H}(X, \mathbf{Z}) \to \tilde{H}(Y, \mathbf{Z})$$

defined by $f^{\mathcal{E}}_{X\to Y}(x) = \pi_{Y*}(\mathcal{Z} \cdot \pi_X^* x)$, where

$$\mathcal{Z} := \pi_X^*(\sqrt{td_X})ch(\mathcal{E})\pi_Y^*(\sqrt{td_Y}) \in \tilde{H}(X \times Y, \mathbf{Z}) \ .$$

4) If both $\Phi^{\mathcal{E}}_{X\to Y}: D(X) \to D(Y)$ *and* $\Phi^{\mathcal{E}'}_{Y\to Z}: D(Y) \to D(Z)$ *are equivalence, then* $f^{\mathcal{E}''}_{X\to Z} = f^{\mathcal{E}'}_{Y\to Z} \circ f^{\mathcal{E}}_{X\to Y}$ *holds as an isometry* $\tilde{H}(X, \mathbf{Z}) \to \tilde{H}(Z, \mathbf{Z})$*, where* $\mathcal{E}'' \in D(X \times Z)$ *is a kernel representing the equivalence* $\Phi^{\mathcal{E}'}_{Y\to Z} \circ \Phi^{\mathcal{E}}_{X\to Y} : D(X) \to D(Z)$*.*

Property 1) is well-known (see for example [Mu1],[BM]). Property 2) and the fact that $\mathcal{Z} \in \tilde{H}(X \times Y, \mathbf{Z})$ in 3) are due to the even property of $H^2(X, \mathbf{Z})$ for K3 surfaces. Property 3) follows from the following facts: the result by Mukai (Theorem 4.9,[Mu2]) shows that $f^{\mathcal{E}}_{X\to Y}$ is an isomorphism, and further it preserves the bilinear form $\langle\ ,\ \rangle$ (Lemma 4.7,[Mu2]). Since the kernel $\mathcal{E} \in D(X \times Y)$ is algebraic, this map preserves the Hodge decomposition and therefore $\mathbf{C}\omega_X$ is mapped to $\mathbf{C}\omega_Y$. The property 4) follows from the projection formula and the Grothendieck-Riemann-Roch theorem. (See an argument [Mu2, pp. 382–383].)

Corollary 1.5. *Let X be a K3 surface. An autoequivalence* $\Phi^{\mathcal{E}} := \Phi^{\mathcal{E}}_{X\to X} : D(X) \to D(X)$ *induces a Hodge isometry* $f^{\mathcal{E}} := f^{\mathcal{E}}_{X\to X} : \tilde{H}(X, \mathbf{Z}) \to \tilde{H}(X, \mathbf{Z})$ *which makes the following diagram commutative:*

$$\begin{array}{ccc} D(X) & \xrightarrow{\Phi^{\mathcal{E}}} & D(X) \\ {\scriptstyle ch(\cdot)\sqrt{td_X}}\Big\downarrow & & \Big\downarrow{\scriptstyle ch(\cdot)\sqrt{td_X}} \\ \tilde{H}(X, \mathbf{Z}) & \xrightarrow{f^{\mathcal{E}}} & \tilde{H}(X, \mathbf{Z}) \end{array}$$

We write $f^{\mathcal{E}} = ch(\Phi^{\mathcal{E}})$*. Then we have* $ch(\Phi^{\mathcal{E}_1} \circ \Phi^{\mathcal{E}_2}) = ch(\Phi^{\mathcal{E}_1}) \circ ch(\Phi^{\mathcal{E}_2})$*, i.e. we have a group homomorphism:*

$$ch : AuteqD(X) \to O_{Hodge}(\tilde{H}(X, \mathbf{Z})) \tag{1.2}$$

Theorem 1.6. (Main Theorem 1) *Let ι_2 be the involution which maps $(a, b, c) \in \tilde{H}(X, \mathbf{Z})$ to $(a, -b, c)$. Then*

$$\langle ch(AuteqD(X)), \iota_2 \rangle = O_{Hodge}(\tilde{H}(X, \mathbf{Z})).$$

In particular, the subgroup $ch(AuteqD(X))$ has index at most two in $O_{Hodge}(\tilde{H}(X, \mathbf{Z}))$.

Our proof of Theorem 1.6 is an easy combination of the arguments in [Mu2],[Or],[BM] together with the reflection functors by spherical objects defined in [ST]. We present its full details in the next section.

(1-2) Symplectic diffeomorphisms and surjectivity to $O^+(T(Y))^*$

Let Y be a K3 surface and $A(Y) \subset NS(Y) \otimes \mathbf{R}$ the ample cone, i.e. an open convex cone in $NS(Y) \otimes \mathbf{R}$ generated by the ample classes. We note that due to [Ya] and the real Nakai-Moishezon criterion [CP](, see also [To]), any class in $A(Y)$ is represented by a unique Ricci-flat Kähler form on Y.

Definition 1.7. *Let $\kappa_Y \in A(Y)$, and $Diff(Y)$ be the diffeomorphism group of the underlying real C^∞-four manifold Y.*

1) The pair (Y, κ_Y) or simply κ_Y is called a (cohomological) symplectic structure of the underlying real C^∞-four manifold of Y.

2) The group $Symp(Y, \kappa_Y) := \{g \in Diff(Y) \mid g^*\kappa_Y = \kappa_Y\}$ is called the (cohomological) symplectic diffeomorphism group of (Y, κ_Y), where $g^* : H^2(Y, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ is the isometry induced by g.

3) The group $\pi_0 Symp(Y, \kappa_Y) := Symp(Y, \kappa_Y)/Symp^0(Y, \kappa_Y)$ is called the symplectic mapping class group of (Y, κ_Y), where $Symp^0(Y, \kappa_Y)$ is the connected component of the identity.

4) A symplectic structure (Y, κ_Y) is said to be generic if every primitive lattice $P \subset H^2(Y, \mathbf{Z})$ with $\kappa_Y \in P \otimes \mathbf{R}$ contains $NS(Y)$. In other words, (Y, κ_Y) is generic if it characterizes $NS(Y)$ as the minimal primitive sublattice in $H^2(Y, \mathbf{Z})$ which contains κ_Y after the tensor product $\otimes\mathbf{R}$. $\square$

Remark. 1) If we choose an integral basis $e_1, \cdots, e_\rho$ of $NS(Y)$ and represent $\kappa_Y = \sum \alpha_i e_i$ ($\alpha_i \in \mathbf{R}$), then the condition of κ_Y generic is satisfied if and only if $\alpha_1, \cdots, \alpha_\rho$ are linearly independent over $\mathbf{Q}$. Therefore the set of generic symplectic structures κ_Y is the complement of countably many proper hyperplanes in $A(Y)$.
2) Our definition of $Symp(Y, \kappa_Y)$ is *cohomological* in nature. It need not preserve a symplectic form representing the class κ_Y. Since the mirror map relates complex structures with positive classes (cf. [Do]), this group seems more natural than the group of symplectomorphisms at least in our context. $\square$

Let L be a lattice of signature (p, q) and consider its isometry group $O(L)$. We denote by $O^+(L)$ the subgroup consisting those isometries which preserve the orientation of the maximal positive definite subspaces in $L \otimes \mathbf{R}$. More precisely, $g \in O^+(L)$ if and only if $\pi \circ g$ induces an orientation preserving isomorphism for every positive p-space H in $L \otimes \mathbf{R}$, where $\pi : L \otimes \mathbf{R} \to H$ is the orthogonal projection. For the transcendental lattice $T(Y)$, we consider the group of isometries $O(T(Y))$ and the natural orthogonal representation of $O(T(Y))$ on its discriminant group $A_{T(Y)} := T(Y)^*/T(Y)$ with its ($\mathbf{Q}/\mathbf{Z}$-valued) bilinear form naturally induced from that of $T(Y)$. Then we define

$$O(T(Y))^* := \mathrm{Ker}(O(T(Y)) \to O(A_{T(Y)})) \tag{1.3}$$

and set $O^+(T(Y))^* := O(T(Y))^* \cap O^+(T(Y))$.

Theorem 1.8. ([Don, Section VI]) *For a K3 surface Y, we have:*
1) If $g \in Diff(Y)$, then $g^*|_{H^2(Y,\mathbf{Z})} \in O^+(H^2(Y, \mathbf{Z}))$.
2) The natural map $Diff(Y) \to O^+(H^2(Y, \mathbf{Z}))$ given by 1) *is surjective.* $\square$

Using this, we prove in section 3 the following symplectic analogue of Theorem 1.6:

Theorem 1.9. (Main Theorem 2) *Let (Y, κ_Y) be a generic symplectic structure on Y. Then:*
1) If $g \in Symp(Y, \kappa_Y)$, then $g^*(NS(Y)) = NS(Y)$, $g^*(T(Y)) = T(Y)$ and $g^*|_{NS(Y)} = id_{NS(Y)}$, $g^*|_{T(Y)} \in O^+(T(Y))^*$.
2) The natural map $\pi_0 Symp(Y, \kappa_Y) \to O^+(T(Y))^*$ induced by 1) is surjective.

Remark. The kernel $O(T(Y))^*$ is a subgroup of $O(T(Y))$ consisting those elements $\phi_{T(Y)}$ such that $(\phi_{T(Y)}, \mathrm{id}_{\widetilde{NS}(Y)}) \in O(T(Y)) \times O(\widetilde{NS}(Y))$ extend to isometries in $O(\tilde{H}(Y, \mathbf{Z}))$. In the context of the mirror symmetry, we will identify the K3 surface

Y with the mirror $\check{X}$ of a K3 surface X. There the group $O^+(T(Y))^*$ will be identified with a subgroup of $O_{Hodge}(\tilde{H}(X,\mathbf{Z}))$. $\square$

(1-3) Mirror symmetry of marked M-polarized K3 surfaces

Our results (Theorem 1.6 and Theorem 1.9) have clear interpretations in terms of homological mirror symmetry of K3 surfaces. In order to describe them, we discuss mirror symmetry of marked M-polarized K3 surfaces following Dolgachev[Do].

Let us consider a lattice M of signature $(1,t)$ and assume a primitive embedding $\iota_M : M \hookrightarrow \Lambda_{\mathrm{K3}}$. We fix this embedding ι_M and identify M and $\iota_M(M)$ in Λ_{K3}. Then a pair (X,τ) of a K3 surface X and a marking $\tau : H^2(X,\mathbf{Z}) \to \Lambda_{\mathrm{K3}}$ is called a *marked M-polarized K3 surface* if $\tau^{-1}(M) \subset NS(X)$. We call a K3 surface X *M-polarizable* if there is a marking $\tau : H^2(X,\mathbf{Z}) \to \Lambda_{\mathrm{K3}}$ such that (X,τ) is a marked M-polarized K3 surface. Two marked M-polarized K3 surfaces (X,τ) and (X',τ') are said isomorphic if there exists an isomorphism $\varphi : X \to X'$ such that $\tau' = \tau \circ \varphi^*$.

Let (X,τ) be a marked M-polarized K3 surface and ω_X be a nowhere vanishing holomorphic two form. Since $NS(X) = H^{1,1}(X) \cap H^2(X,\mathbf{Z}) = (\mathbf{C}\omega_X)^\perp \cap H^2(X,\mathbf{Z})$, the line $\tau(\mathbf{C}\omega_X)$ is always orthogonal to $M \otimes \mathbf{C}$. Also since $(\omega_X,\omega_X) = 0$, $(\omega_X,\bar{\omega}_X) > 0$, the line $\tau(\mathbf{C}\omega_X)$ lies in the period domain

$$\Omega(M^\perp) := \{ \mathbf{C}\omega \in \mathbf{P}(M^\perp \otimes \mathbf{C}) \mid (\omega,\omega) = 0, (\omega,\bar{\omega}) > 0 \}$$

for the orthogonal lattice $M^\perp$ in Λ_{K3}. The point $\tau(\mathbf{C}\omega_X)$ is called *period point* of (X,τ). A local family of marked M-polarized K3 surfaces is a family $f : \mathfrak{X} \to \mathcal{B}$ of K3 surfaces together with a trivialization (i.e. a marking) $\tau = \cup_{t\in\mathcal{B}} \tau_t : R^2 f_* \mathbf{Z}_{\mathfrak{X}} \xrightarrow{\sim} \Lambda_{\mathrm{K3}} \times \mathcal{B}$ such that $\tau^{-1}(M \times \{t\}) \subset NS(\mathfrak{X}_t)$. Then the period map, which sends each $\mathfrak{X}_t (t \in \mathcal{B})$ to its period points in $\Omega(M^\perp)$, defines a holomorphic map from $\mathcal{B}$ to $\Omega(M^\perp)$. Due to the surjectivity of the period map (see e.g. [BPV]), the period domain $\Omega(M^\perp)$ parameterizes the marked M-polarized K3 surfaces, and a generic point $t \in \Omega(M^\perp)$ comes from a marked M-polarized K3 surface (X,τ) such that $\tau^{-1}(M) = NS(X)$.

Mirror symmetry of K3 surfaces is well described for marked M-polarized K3 surfaces especially when the orthogonal lattice $M^\perp$ contains a hyperbolic lattice, i.e. $M^\perp = U \oplus \check{M}$. In this case we have the following embedding

$$M \oplus U \oplus \check{M} \subset \Lambda_{\mathrm{K3}}, \tag{1.4}$$

where M and $\check{M}$ are of signature $(1,t)$ and $(1,18-t)$ respectively. In this paper we always assume the above property (1.4) for the lattice M. And we say that the family of marked M-polarized K3 surfaces is mirror symmetric to the family of marked $\check{M}$-polarized K3 surfaces, following [Do].

For the description of mirror symmetry, it will turn out that the Mukai lattice is more natural than the K3 lattice Λ_{K3}. Corresponding to the K3 lattice, let us define *abstract Mukai lattice* $(\tilde{\Lambda}, \langle\ ,\ \rangle)$ by

$$\tilde{\Lambda} = \Lambda^0 \oplus \Lambda_{\mathrm{K3}} \oplus \Lambda^4,$$

where $\Lambda^0 := \mathbf{Z}e, \Lambda^4 := \mathbf{Z}f$ and we naturally extend the bilinear form of Λ_{K3} to $\tilde{\Lambda}$ by setting $\langle e,e\rangle = \langle f,f\rangle = 0, \langle e,f\rangle = -1$, $\langle e,\Lambda_{\mathrm{K3}}\rangle = \langle f,\Lambda_{\mathrm{K3}}\rangle = 0$. Thus $\tilde{H}(X,\mathbf{Z})$

is isomorphic to $\tilde{\Lambda}$. We call an isomorphism of the lattice $\tilde{\tau} : \tilde{H}(X, \mathbf{Z}) \to \tilde{\Lambda}$ a *Mukai marking*, and a Mukai marking $\tilde{\tau}$ satisfying $\tilde{\tau}(H^0(X, \mathbf{Z})) = \Lambda^0, \tilde{\tau}(H^4(X, \mathbf{Z})) = \Lambda^4, \tilde{\tau}(H^2(X, \mathbf{Z})) = \Lambda_{\mathrm{K3}}$ a *graded Mukai marking*. Note that a Mukai marking $\tilde{\tau} : \tilde{H}(X, \mathbf{Z}) \to \tilde{\Lambda}$ is not graded in general. Also note that a marking $\tau : H^2(X, \mathbf{Z}) \to \Lambda_{\mathrm{K3}}$ naturally extends to a graded Mukai marking $\tilde{\tau}$ (up to $\mathrm{Aut}(\Lambda^0) = \mathrm{Aut}(\Lambda^4) = \{\pm 1\}$). Now we can state mirror symmetry of the marked K3 surfaces in terms of the Mukai lattice.

Proposition 1.10. *Let (X, τ) and $(\check{X}, \check{\tau})$, respectively, be generic marked M-polarized K3 surfaces and $\check{M}$-polarized K3 surfaces. Then we have the following identifications in the abstract Mukai lattice,*

$$\underbrace{\overbrace{U \;\oplus\; M}^{\tilde{\tau}(\widetilde{NS}(X))}}_{\tilde{\check{\tau}}(T(\check{X}))} \;\oplus\; \underbrace{\overbrace{U \;\oplus\; \check{M}}^{\tilde{\tau}(T(X))}}_{\tilde{\check{\tau}}(\widetilde{NS}(\check{X}))} \subset \tilde{\Lambda} \tag{1.5}$$

where $\widetilde{NS} := H^0 \oplus NS \oplus H^4 \cong U \oplus NS$. $\square$

Hereafter we fix the basis e, f for the first hyperbolic lattice (the basis we have introduced for the abstract Mukai lattice) and the corresponding basis $\check{e}, \check{f}$ for the second hyperbolic lattice U in (1.5). We assume the intersections to be $\langle \check{e}, \check{e} \rangle = \langle \check{f}, \check{f} \rangle = 0$ and $\langle \check{e}, \check{f} \rangle = -1$.

Remark. For the lattice $U \oplus \check{M}$ in the above embedding (1.5), we can associate two different, but isomorphic, domains. One is the period domain $\Omega(U \oplus \check{M})$ which describes the complex structure deformation space of (X, τ), and the other is a tube domain,

$$T_{\check{M}} := \check{M} \otimes \mathbf{R} + iV(\check{M} \otimes \mathbf{R})$$

where $V(\check{M} \otimes \mathbf{R}) := \{x \in \check{M} \otimes \mathbf{R} \mid (x, x) > 0\}$. This tube domain is understood as a covering of the complexified Kähler moduli space of $(\check{X}, \check{\tau})$. One can see the mirror correspondence explicitly in the map $\mu : T_{\check{M}} \to \Omega(U \oplus \check{M})$ defined by

$$\mu(x) = \mathbf{C}(\frac{1}{2}\langle x, x \rangle \check{f} + x + \check{e}) \;, \quad (x = \check{B} + i\check{K} \in T_{\check{M}}), \tag{1.6}$$

which is called the *mirror map*. We can verify easily that this map is bijective using a property $\langle \omega, \check{f} \rangle \neq 0$ for $\mathbf{C}\omega \in \Omega(U \oplus \check{M})$(see [Do], Lemma (4.1)). $\square$

Let $\tilde{\tau} : \tilde{H}(X, \mathbf{Z}) \to \tilde{\Lambda}$ be a Mukai marking. We consider the group

$$O_{Hodge}(\tilde{\Lambda}, \tilde{\tau}(\mathbf{C}\omega_X)) := \{g \in O(\tilde{\Lambda}) | g(\tilde{\tau}(\mathbf{C}\omega_X)) = \tilde{\tau}(\mathbf{C}\omega_X)\}.$$

Then $O_{Hodge}(\tilde{\Lambda}, \tau(\mathbf{C}\omega_X)) \cong O_{Hodge}(\tilde{H}(X, \mathbf{Z}))$ (cf. Definition 1.2).

Now we can argue two different pictures for the same group $O_{Hodge}(\tilde{\Lambda}, \tilde{\tau}(\mathbf{C}\omega_X))$ based on the mirror relation (1.5).

The first is to understand this group as the Hodge isometries of marked M-polarized K3 surface (X, τ) and its extension $(X, \tilde{\tau})$ to a graded Mukai marking. By our Theorem 1.6, this group contains the image $ch(\mathrm{Auteq} D(X))$ as a subgroup of index at most two.

The second is the mirror picture to the first and valid for the mirror $(\check{X}, \check{\tau})$ and its extension $(\check{X}, \tilde{\check{\tau}})$ to a graded Mukai marking. To describe this let us recall the mirror relation $\tilde{\tau}(T(X)) = U \oplus \check{M} = \tilde{\check{\tau}}(\widetilde{NS}(\check{X}))$ for generic $(X, \tilde{\tau})$ and $(\check{X}, \tilde{\check{\tau}})$. Among the Hodge isometries in $O_{Hodge}(\tilde{\Lambda}, \tilde{\tau}(\mathbf{C}\omega_X))$, let us focus on the isometries of $\tilde{\Lambda}$ which stabilize $\tilde{\tau}(T(X))$ and act as identity on $\tilde{\tau}(T(X))$. We denote the subgroup consisting of these isometries by

$$O^1_{Hodge}(X, \tilde{\tau}) := \{\varphi \in O_{Hodge}(\tilde{\Lambda}, \tilde{\tau}(\mathbf{C}\omega_X)) \mid \varphi|_{U\oplus\check{M}} = \mathrm{id}_{U\oplus\check{M}}\}. \tag{1.7}$$

Now consider an index two subgroup $O^{1,+}_{Hodge}(X, \tilde{\tau})$ of $O^1_{Hodge}(X, \tilde{\tau})$ which preserves the orientations of the positive two planes in $(U \oplus M) \otimes \mathbf{R}$. Recall the isomorphism $\mu : T_{\check{M}} \tilde{\rightarrow} \Omega(U \oplus \check{M})$ (1.6) and observe that elements in $O^{1,+}_{Hodge}(X, \tilde{\tau})$ preserve each complexified Kähler class of the mirror $(\check{X}, \tilde{\check{\tau}})$ defined by $\check{B} + i\check{K} := \mu^{-1}(\mathbf{C}\omega_X)$. (On the mirror side, presumably $ch(\mathrm{Auteq}D(X)) = O^+_{Hodge}(\tilde{H}(X, \mathbf{Z}))$ should hold, where $O^+_{Hodge}(\tilde{H}(X, \mathbf{Z}))$ is the index two subgroup which preserves the orientations of the positive four planes in $\tilde{H}(X, \mathbf{Z}) \otimes \mathbf{R}$. (See [Sz, Conjecture 5.4].) Note that the tube domain $T_{\check{M}}$ and the period domain $\Omega(U \oplus \check{M})$ both have two connected components. This also seems closely related to such index-two phenomena in the present paper.)

Now recall that the kernel $O(T(\check{X}))^* \cong O(\tilde{\check{\tau}}(T(\check{X})))^*$ is a subgroup of $O(T(\check{X}))$ consisting of those elements $\phi_{T(\check{X})}$ such that $(\phi_{T(\check{X})}, \mathrm{id}_{\widetilde{NS}(\check{X})}) \in O(T(\check{X})) \times O(\widetilde{NS}(\check{X}))$ extend to elements of $O(\tilde{H}(\check{X}, \mathbf{Z})) \cong O(\tilde{\Lambda})$ (cf. Remark after Theorem 1.9). Using this extension property, we can identify the subgroup $O^1_{Hodge}(X, \tilde{\tau})$ with $O(T(\check{X}))^*$ and further its index two subgroup $O^{1,+}_{Hodge}(X, \tilde{\tau})$ with $O^+(T(\check{X}))^*$.

Now we can apply Theorem 1.9 for $Y = \check{X}$ to see that all elements in $O^{1,+}_{Hodge}(X, \tilde{\tau})$ come from the symplectic mapping class group of $(\check{X}, \check{\tau})$ with respect to its Kähler class $\mathrm{Im}(\mu^{-1}(\mathbf{C}\omega_X))$. This is the mirror interpretation of the subgroup $O^{1,+}_{Hodge}(X, \tilde{\tau})$ $\subset O_{Hodge}(\tilde{\Lambda}, \tilde{\tau}(\mathbf{C}\omega_X))$. Here we have arrived at a subgroup $O^{1,+}_{Hodge}(X, \tilde{\tau})$ which has the index greater than two in $O_{Hodge}(\tilde{\Lambda}, \tilde{\tau}(\mathbf{C}\omega_X))$. This should be attributed to the fact that we have replaced the desired autoequivalence group of "$DFuk(\check{X})$" by the well-known but possibly smaller symplectic mapping class group. Note, for example, that the shift functor contained in $\mathrm{Auteq}D(X)$ does not have its counter part in the symplectic mapping class group but should have in "$\mathrm{Auteq}DFuk(\check{X})$".

(1-4) Fourier-Mukai partners and monodromy of the mirror family

When a derived category $D(X)$ is given, we can ask for the varieties Y which admit the equivalence $\Phi : D(X) \cong D(Y)$. The smooth projective varieties with this property are called *Fourier-Mukai(FM) partners* of X. If X has ample canonical or anticanonical bundle, Bondal and Orlov[BO] proved that X itself is the only FM partner. For K3 surfaces, however, X has in general finitely many FM partners:

Proposition 1.11. **([BM, Proposition 5.3]**, *see also* **[Og2, Proposition (1.9)])** *For a given K3 surface X, there are only finitely many FM partners.* □

For a generic K3 surface, or more precisely, for a K3 surfaces of $\rho(X) = 1$, the number of FM partners has been determined as follows;

Proposition 1.12. **([Og2, Proposition (1.10)])** *Let X be a K3 surface with $NS(X) = \mathbf{Z}h$. Set $deg(X) = (h^2) = 2n$. Then the number of FM partners of X is given by $2^{p(n)-1}$, where $p(1) = 1$ and $p(n)$ $(n \geq 2)$ is the number of prime numbers p (≥ 2) such that $p|n$.* $\square$

We will arrive at the same number studying the monodromy representation of the mirror family $\check{X}$. To define the mirror family let us first remark that a lattice of rank one admits a unique primitive embedding into the K3 lattice Λ_{K3}. Now let us consider a rank one lattice $M_n = \langle 2n \rangle$, i.e. a lattice $M_n = \mathbf{Z}v$ with its bilinear form determined by $(v, v) = 2n$. Then because of the uniqueness (up to isomorphism) of the primitive embedding we have the following decomposition;

$$M_n \oplus U \oplus \check{M}_n \subset \Lambda_{\mathrm{K3}},$$

where $\check{M}_n := \langle -2n \rangle \oplus U \oplus E_8(-1)^{\oplus 2}$. A K3 surface with $NS(X) = \mathbf{Z}h$ and $\deg(X) = 2n$ may be regarded as a generic member of the family of the marked M_n-polarized K3 surfaces. The mirror family defined in (1-3) is the marked $\check{M}_n$-polarized K3 surfaces $(\check{X}, \check{\tau})$, which are parametrized by the period domain $\Omega(T(\check{X})) = \Omega(U \oplus M_n)$. The generic member $(\check{X}, \check{\tau})$ of the family has its transcendental lattice $T(\check{X}) \cong U \oplus M_n$ and *classified* by $\Omega^0(U \oplus M_n)$, a complement of countable union of proper closed subsets of $\Omega(U \oplus M_n)$. $(\Omega(U \oplus M_n) \setminus \Omega^0(U \oplus M_n)$ is however dense in $\Omega(U \oplus M_n)$, see e.g. [Og2]). For generic elements of the marked $\check{M}$-polarized K3 surfaces, we have:

Lemma 1.13. *Let $(\check{X}_1, \check{\tau}_1)$ and $(\check{X}_2, \check{\tau}_2)$ be marked $\check{M}_n$-polarized K3 surfaces parametrized by $\Omega^0(U \oplus M_n)$. Then $\check{X}_1 \cong \check{X}_2$ if and only if there exists $g \in O(U \oplus M_n)$ such that $g(\mathbf{C}\tau_1(\omega_{X_1})) = \mathbf{C}\tau_2(\omega_{X_2})$.*

The proof of this lemma will be given in the section 4. Using the natural action $\iota : O(U \oplus M_n) \to \mathrm{Aut}\Omega^0(U \oplus M_n)$, we consider the quotient space $\Omega^0(U \oplus M_n)/O(U \oplus M_n) \cong \Omega^{0,+}(U \oplus M_n)/O^+(U \oplus M_n)$, where $\Omega^{0,+}(U \oplus M_n) = \Omega^0(U \oplus M_n) \cap \Omega^+(U \oplus M_n)$ and $\Omega^+(U \oplus M_n)$ is one of the two connected components of $\Omega(U \oplus M_n)$. This space is the classifying space of the generic $\check{M}_n$-polarizable K3 surface by Lemma 1.13. The closure of $\Omega^{0,+}(U \oplus M_n)/O^+(U \oplus M_n)$ is $\Omega^+(U \oplus M_n)/O^+(U \oplus M_n)$. Note that $\pm\mathrm{id}_{U \oplus M_n}$ acts trivially on the period domain, i.e. $\mathrm{Ker}(\iota) = \{\pm\mathrm{id}_{U \oplus M_n}\}$. This leads to the following definition:

Definition 1.14. *We call the group $O^+(U \oplus M_n)/\{\pm id_{U \oplus M_n}\}$ the monodromy group of the $\check{M}_n$-polarizable K3 surfaces $\check{X}$, and denote it by $\mathcal{M}_n(\check{X})$.* $\square$

Now consider the definition (1.3) for $T(Y) = T(\check{X}) \cong U \oplus M_n$, and see the following composition of natural maps;

$$O^+(U \oplus M_n)^* \to O^+(U \oplus M_n) \to O^+(U \oplus M_n)/\{\pm\mathrm{id}_{U \oplus M_n}\} =: \mathcal{M}_n(\check{X}). \quad (1.8)$$

Note that $-\mathrm{id}_{U \oplus M_n}$ is contained in $O^+(U \oplus M_n)^*$ only for $n = 1$ since $A((U \oplus M_n)^*/(U \oplus M_n)) = A(M_n^*/M_n) = \langle v/2n \rangle \cong \mathbf{Z}/2n$. Therefore we see that

Lemma 1.15. *The composition map $O^+(U \oplus M_n)^* \to \mathcal{M}_n(\check{X})$ defined in (1.8) is injective for $n \geq 2$ and has the kernel $\{\pm id_{U \oplus M_n}\}$ for $n = 1$.* $\square$

Applying our Theorem 1.9 to $Y = \check{X}$ and using $T(Y) \cong U \oplus M_n$ for a generic $\check{M}_n$-polarizable K3 surface, we have

$$O^+(U \oplus M_n)^* = \mathrm{Im}(\pi_0\mathrm{Symp}(\check{X}, \kappa_{\check{X}}) \to O(U \oplus M_n)).$$

Based on this relation and Lemma 1.13, we define:

Definition 1.16. *For a generic symplectic structure $(\check{X}, \kappa_{\check{X}})$ of a generic $\check{M}_n$-polarizable K3 surface $\check{X}$, we call the group $O^+(U \oplus M_n)^*$ for $n \geq 2$ (respectively, the group $O^+(U \oplus M_n)^*/\{\pm id_{U \oplus M_n}\}$ for $n = 1$) the monodromy representation of the symplectic mapping class group of $(\check{X}, \kappa_{\check{X}})$. We denote this group by $\mathcal{MS}_n(\check{X})$.* $\square$

Now we can state our theorem:

Theorem 1.17. (Main Theorem 3) *Let X be a K3 surface of $\rho(X) = 1$ and $deg(X) = 2n$. Then the number of FM partners of X is given by the index $[\mathcal{M}_n(\check{X}) : \mathcal{MS}_n(\check{X})]$.*

Recall that the number of FM partners is $2^{p(n)-1}$ by Proposition 1.12. Our theorem above derives the same numbers from the monodromy property of the mirror family $\check{X}$. A proof of Theorem 1.17 will be given in section 4.

In section 5, we will also study in details the first non-trivial case of $n = 6$. We will present the monodromy calculations explicitly following [LY2][PS][BP] and show how the monodromy property is connected to the numbers of FM partners in this particular case.

§2 Autoequivalences and Proof of Theorem 1.6

(2-1) Various autoequivalences.

For our proof of Theorem 1.6 let us recall basic autoequivalences in order.
1) *Shift functor* $[n]$*:* $D(X) \to D(X) (n \in \mathbf{Z})$ defined by $K^\bullet \to L^\bullet = K^{\bullet + n}$, i.e. the shift by n to the left, is an autoequivalence. This functor does not change the complex except its order-preserved numbering. However we should note that $f^{[n]} (:= ch([n])) = -\mathrm{id}$ if $n \equiv 1$ (2) and $f^{[n]} = \mathrm{id}$ if $n \equiv 0$ (2) by the definition $ch(K^\bullet) = \sum_i (-1)^i ch(K^i)$.
2) *Aut*(X): An automorphism $g \in \mathrm{Aut}(X)$ gives rise to an autoequivalence $g : D(X) \to D(X)$ by sending $K^\bullet$ to $L^\bullet$ with $L^i = g^* K^i$. Since the quasi-inverse to g is given by g^{-1}, g is an autoequivalence. g is represented by a kernel $\mathcal{O}_{\Gamma(g)} \in D(X \times X)$, i.e. $g = \Phi^{\mathcal{O}_{\Gamma(g)}}$ as an element of Auteq$D(X)$, where $\Gamma(g) = \{(x, g(x)) | x \in X\} \subset X \times X$ is the graph of g and $\mathcal{O}_{\Gamma(g)}$ is the structure sheaf of the reduced closed subscheme $\Gamma(g) \subset X \times X$ (which we identify the pushforward $\iota_* \mathcal{O}_{\Gamma(g)}$ under $\iota : \Gamma(g) \hookrightarrow X \times X$). The induced action $f^{\mathcal{O}_{\Gamma(g)}}$ on $\tilde{H}(X, \mathbf{Z})$ is the pullback by g^*.
3) *Tensoring by line bundles:* Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle (invertible sheaf) on X. Then we may associate to it an autoequivalence $\Phi^{\mathcal{L}} := \Phi^{\pi_2^* \mathcal{L}} : D(X) \to D(X)$ by $\mathcal{X} \mapsto \pi_{2*}(\pi_2^* \mathcal{L} \otimes \pi_1^* \mathcal{X}) = \mathcal{L} \otimes \pi_{2*}(\pi_1^* \mathcal{X})$, where π_1 and π_2 are the natural projections $\pi_{1,2} : X \times X \to X$ to the first and the second X, respectively. The quasi-inverse of this is simply given by $\Phi^{\mathcal{L}^{-1}}$. The induced action $f^{\mathcal{L}} := f^{\pi_2^* \mathcal{L}}$ on $\tilde{H}(X, \mathbf{Z})$ is the multiplication by the chern character $ch(\mathcal{L}) = (1, c_1(\mathcal{L}), \frac{1}{2} c_1(\mathcal{L})^2)$ in the graded ring $\tilde{H}(X, \mathbf{Z})$.

These three functors 1),2),3) above form a subgroup of Auteq$D(X)$ which is isomorphic to $(\mathbf{Z} \times \mathrm{Pic}X) \rtimes Aut(X)$.
4) *Twistings by spherical objects [ST]:* Let C be a smooth rational curve in a K3 surface X, then $(C^2) = -2$. In this paper, we mean by smooth (-2) curve a smooth rational curve. Now consider the structure sheaf $\mathcal{O}_C (:= \iota_* \mathcal{O}_C)$. Using the exact sequence $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ and evaluating $\mathrm{Hom}(*, \mathcal{O}_C)$, we have $\mathrm{Ext}^i(\mathcal{O}_C, \mathcal{O}_C) = \mathbf{C}$ for $i = 0, 2$ and $\mathrm{Ext}^i(\mathcal{O}_C, \mathcal{O}_C) = 0$ $(i \neq 0, 2)$. This a

simple example what is called a *spherical object* in the derived category[ST]. Here we simply summarize relevant results restricting our attentions to the case of K3 surface X.

Theorem 2.1. ([ST, Theorem 1.2]) *Let X be a K3 surface. $\mathcal{E} \in D(X)$ is called spherical if it satisfies*

$$Hom^i_{D(X)}(\mathcal{E},\mathcal{E}) = \begin{cases} \mathbf{C} & i=0,2 \\ 0 & i \neq 0,2 \end{cases} .$$

(Note that $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$.) For a spherical object $\mathcal{E}$ we consider the mapping cone $\mathcal{C} := Cone(\pi_1^*\mathcal{E}^\vee \otimes \pi_2^*\mathcal{E} \to \mathcal{O}_\Delta)$ of the natural evaluation $\pi_1^*\mathcal{E}^\vee \otimes \pi_2^*\mathcal{E} \to \mathcal{O}_\Delta$, where $\Delta \subset X \times X$ is the diagonal and $\mathcal{O}_\Delta$ is at the zeroth position of the complex $\mathcal{C}$. $\mathcal{E}^\vee = \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_{X\times X})$ is the derived dual of $\mathcal{E}$, and π_1 and π_2 are the natural projections $\pi_{1,2} : X \times X \to X$ to the first and the second, respectively. Then the functor $T_\mathcal{E} := \Phi^\mathcal{C} : D(X) \to D(X)$ defines an equivalence. This functor is called a twist functor. The corresponding map $t_\mathcal{E} := f^\mathcal{C}$ on $\tilde{H}(X,\mathbf{Z})$ is given by

$$t_\mathcal{E}(x) = x + \langle ch(\mathcal{E})\sqrt{td_X}, x\rangle ch(\mathcal{E})\sqrt{td_X}. \tag{2.1}$$

Lemma 2.2. *Let $T_\mathcal{E}$ ($\mathcal{E} = \mathcal{O}_C$) be the twist functor with respect to a smooth (-2) curve C in a K3 surface X and consider an autoequivalence $\Phi^{\mathcal{O}_X(C)} \circ T_\mathcal{E} : D(X) \to D(X)$. Then the corresponding action $f^{\mathcal{O}_X(C)} \circ t_\mathcal{E}$ in $\tilde{H}(X,\mathbf{Z})$ is a Hodge isometry of $(\tilde{H}(X,\mathbf{Z}), \langle\ ,\ \rangle)$ and coincides with the reflection by the curve C,*

$$f^{\mathcal{O}_X(C)} \circ t_\mathcal{E}(x) = x + \langle x, C\rangle C.$$

Proof. From the exact sequence $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$, we have $ch(\mathcal{O}_C) = ch(\mathcal{O}_X) - ch(\mathcal{O}_X(-C)) = (1,0,0) - (1,-C,\frac{1}{2}(-C)^2) = (0,C,1)$. Now for $x = (\alpha,\beta,\gamma) \in \tilde{H}(X,\mathbf{Z}) = H^0(X,\mathbf{Z}) \oplus H^2(X,\mathbf{Z}) \oplus H^4(X,\mathbf{Z})$ we have

$$\begin{aligned} t_\mathcal{E}(x) &= x + \langle ch(\mathcal{O}_C)\sqrt{td_X}, x\rangle ch(\mathcal{O}_C)\sqrt{td_X} \\ &= x + \langle (0,C,1)\cdot(1,0,1), (\alpha,\beta,\gamma)\cdot(1,0,1)\rangle (0,0,1) \\ &= (\alpha,\beta,\gamma) + (-\alpha + (\beta,C)) \times (0,C,1) \\ &= (\alpha, \beta + (-\alpha+(\beta,C))C, \gamma - \alpha + (\beta,C)) \end{aligned}$$

Now we note that $f^{\mathcal{O}_X(C)}(x) = ch(\mathcal{O}_X(C))x$, i.e. simply the multiplication by $ch(\mathcal{O}_X(C)) = (1,C,\frac{1}{2}C^2)$. Composing these two actions we obtain the desired result,

$$f^{\mathcal{O}_X(C)} \circ t_\mathcal{E}(x) = (\alpha, \beta + (\beta,C)C, \gamma) = x + \langle x, C\rangle C \quad \square$$

5) *Switching functor (a special case of the Fourier-Mukai transform for fine moduli spaces of stable sheaves):* Let us first recall the following result due to Mukai.

Theorem 2.3. ([Mu2] *see also* **[BM, Cor.2.8])** *Let X be a K3 surface with a fixed polarization. Consider a smooth fine compact two dimensional moduli space Y of stable sheaves on X and denote by $\mathcal{P}$ the universal sheaf on $X \times Y$ (for which we have for each $y \in Y$ the stable sheaf $\mathcal{P}_y = \pi_Y^*\mathcal{O}_y \otimes \mathcal{P}$ on $X \times \{y\} \cong X$,* *which represents the point y in the moduli space Y). Then Y is smooth, hence a K3 surface and $\Phi^{\mathcal{P}}_{Y\to X}: D(Y)\to D(X)$ is a FM transform and satisfies $f^{\mathcal{P}}(ch(\mathcal{O}_y)) = ch(\pi_{X*}\mathcal{P}_y)\sqrt{td_X}$. (Note that $ch(\mathcal{O}_y)\sqrt{td_X} = ch(\mathcal{O}_y) = (0,0,1)$.)* $\square$

For our purpose we apply this theorem to a special case. Let $\Delta \subset X\times X$ be the diagonal and $\mathcal{I}_\Delta$ be its ideal sheaf in $X\times X$. We can regard X as the fine moduli space of the ideal sheaves of points $\mathcal{I}_x (x\in X)$, which are certainly stable. Therefore $\mathcal{I}_\Delta$ is the universal sheaf on $X\times X$. By the above theorem we have the corresponding FM transform (autoequivalence) $\Phi^{\mathcal{I}_\Delta}: D(X)\to D(X)$ for which we have $f^{\mathcal{I}_\Delta}(\alpha,\beta,\gamma) = (\gamma,-\beta,\alpha)$. The last equation for $f^{\mathcal{I}_\Delta}$ follows from $ch(\mathcal{I}_\Delta) = ch(\mathcal{O}_{X\times X}) - ch(\mathcal{O}_\Delta)$ and $f^{\mathcal{I}_\Delta}(x) = -x - \langle ch(\mathcal{O}_X)\sqrt{td_X}, x\rangle ch(\mathcal{O}_X)\sqrt{td_X}$, i.e.

$$f^{\mathcal{I}_\Delta}(\alpha,\beta,\gamma) = -(\alpha,\beta,\gamma) - \langle(\alpha,\beta,\gamma),(1,0,1)\rangle(1,0,1) = (\gamma,-\beta,\alpha).$$

We call this autoequivalence $\Phi^{\mathcal{I}_\Delta}$ a *switching functor*.

(2-2) FM transforms on a K3 surface.

Here we recall a theorem of Mukai and Orlov on FM transforms on a K3 surface:

Theorem 2.4. ([Mu1,2], [Or1], *see also* **[BM])** *Let X and Y be K3 surfaces. Then the following statements are equivalent;*
1) there exists a FM transform $\Phi: D(Y)\to D(X)$
2) there exists a Hodge isometry $f_T: (T(Y),\mathbf{C}\omega_Y)\to (T(X),\omega_X)$
3) there exists a Hodge isometry $f: (\tilde H(Y,\mathbf{Z}),\mathbf{C}\omega_Y)\to(\tilde H(X,\mathbf{Z}),\mathbf{C}\omega_X)$
4) Y is isomorphic to a two dimensional fine moduli space of stable sheaves on X.

In the following arguments, since we need not only the results of Theorem 2.4 but also the arguments, we sketch proof here along [BM]:
1)⇒2) Writing the FM transform $\Phi = \Phi^{\mathcal{E}}_{Y\to X}$ by a kernel $\mathcal{E}$, one has a Hodge isometry $f^{\mathcal{E}}_{Y\to X}: (\tilde H(Y,\mathbf{Z}),\mathbf{C}\omega_Y)\to(\tilde H(X,\mathbf{Z}),\mathbf{C}\omega_X)$ (Theorem 1.4). Recall that the transcendental lattice is the minimal primitive sublattice of $\tilde H(X,\mathbf{Z})$ which satisfies $\mathbf{C}\omega_X\subset T(X)\otimes\mathbf{C}$. Therefore the restriction $f^{\mathcal{E}}_{Y\to X}$ to $T(Y)$ gives the desired Hodge isometry $f_T := f^{\mathcal{E}}_{Y\to X}|_{T_Y}: (T(Y),\mathbf{C}\omega_Y)\to(T(X),\mathbf{C}\omega_X)$.
2)⇒3) By Nikulin's theorem of primitive embedding of lattices [Ni, Theorem (1.14.4)] $f_T: (T(Y),\mathbf{C}\omega_Y)\to(T(X),\mathbf{C}\omega_X)$ extends to $f: (\tilde H(Y,\mathbf{Z}),\mathbf{C}\omega_Y)\to(\tilde H(X,\mathbf{Z}),\mathbf{C}\omega_X)$ satisfying $f|_{T(Y)} = f_T$.
3)⇒4) Since this part is crucial for our purpose, we present this in detail. Let $f: (\tilde H(Y,\mathbf{Z}),\mathbf{C}\omega_Y)\to(\tilde H(X,\mathbf{Z}),\mathbf{C}\omega_X)$ be a Hodge isometry and set $v = f((0,0,1))$. Let us first show that, *if necessary composing f with a suitable $f^{\mathcal{E}}_X = ch(\Phi^{\mathcal{E}}_X)$ of $\Phi^{\mathcal{E}}_X\in AuteqD(X)$, one may assume for $v=(r,l,s)$ that $r>1$, l is ample and s is coprime to r.* First of all let us note that, since l in $(r,l,s) = f((0,0,1))$ is algebraic, we may assume $r>1$ considering a composition $f^{\mathcal{L}}\circ f$ with a suitable line bundle and further with the Hodge isometries $f^{\mathcal{I}_\Delta}, f^{[1]} = -\mathrm{id}$ associated to the switching functor $\Phi^{\mathcal{I}_\Delta}$ and the shift functor [1], respectively. We redefine (r,l,s) to be the image of the vector $(0,0,1)$ under the compositions to ensure $r>1$. Now we show that one can satisfy the condition $(r,s)=1$ further compositions of suitable Hodge isometries. For this purpose let us write $u=(a,b,c)=f((1,0,0))$. Since f is a Hodge isometry we have $\langle u,v\rangle = \langle(1,0,0),(0,0,1)\rangle = -1$. On the other hand, by the definition of $\langle\ ,\ \rangle$, we calculate $\langle u,v\rangle = -as-cr+(b,l)$. Therefore we

have $-1 = -as - cr + (b, l)$, which means $s, r, (b, l)$ are coprime. The last condition ensures that there exists an integer n for which r and $s + n(b, l)$ are coprime. Now consider a composition $f^{\mathcal{O}_X(nb)} \circ f$ for which we have

$$f^{\mathcal{O}_X(nb)} \circ f((0,0,1)) = f^{\mathcal{O}_X(nb)}(v) = (r, l + rnb, s + n(b,l) + \frac{n^2 r}{2}(b,b)).$$

Since r and $s + n(b, l)$ are coprime, redefining (r, l, s) to be $f^{\mathcal{O}_X(b)} \circ f((0, 0, 1))$ we have $(r, s) = 1$ and $r > 1$. Finally consider a composition by $f^{\mathcal{O}_X(rA)}$ with A being sufficiently ample. Then we have

$$f^{\mathcal{O}_X(rA)}(r, l, s) = (1, rA, \frac{1}{2}(A, A))(r, l, s) = (r, r^2 A + l, \frac{1}{2} r^3 (A, A) + s),$$

i.e. we obtain an ample class for the second factor preserving the other conditions.

Since all the functors we used so far are Hodge isometries, we may assume that $v = f((0, 0, 1)) = (r, l, s)$ has the desired properties from the beginning.

Now since $(0, 0, 1)$ is algebraic and f is a Hodge isometry, $v = f((0, 0, 1))$ is also algebraic and lies in $H^0(X, \mathbf{Z}) \oplus NS(X) \oplus H^4(X, \mathbf{Z})$, which is perpendicular to $T(X)$. Therefore one can consider the moduli space Y^+ of stable sheaves on X whose Mukai vector is v, i.e. stable sheaves $\mathcal{E}$ with $ch(\mathcal{E})\sqrt{td_X} = v$ with respect to the ample polarization l. Recall that since v is primitive Y^+ is fine [Mu2, Theorem A.6] and $\langle v, v\rangle = \langle (1, 0, 0), (1, 0, 0)\rangle = 0$ implies $\dim Y^+ = 2$. Moreover Y^+ is smooth, non-empty and compact. This smoothness follows form the Main Theorem in [Mu1], non-emptiness follows from Theorem 5.4 in [Mu2] and compactness follows from Proposition 4.1 in [Mu2] and $(r, s) = 1$. Then by the main Theorem of Bridgeland [Br, Theorems 5.1 and 5.3] (see also [Or1] for another argument) one has an equivalence $\Phi^{\mathcal{E}}_{Y^+\to X} : D(Y^+) \to D(X)$ for which $f^{\mathcal{E}}_{Y^+\to X}((0, 0, 1)) = v$. Then, using 1)⇒3), $g := (f^{\mathcal{E}})^{-1} \circ f : \tilde{H}(Y, \mathbf{Z}) \to \tilde{H}(Y^+, \mathbf{Z})$ is a Hodge isometry satisfying $g((0, 0, 1)) = (0, 0, 1)$. This Hodge isometry g of the Mukai lattices reduces to that of the second cohomologies H^2, i.e. $g : H^2(Y, \mathbf{Z}) = (0, 0, 1)^{\perp}/\mathbf{Z}(0, 0, 1) \xrightarrow{\sim} H^2(Y^+, \mathbf{Z}) = (0, 0, 1)^{\perp}/\mathbf{Z}(0, 0, 1)$. Therefore by the Torelli Theorem, we conclude $Y \cong Y^+$.
4)⇒1) This is a special case of the Theorem 2.3. □

(2-3) Proofs of Theorem 1.6

Proof of Theorem 1.6: Now we come to a proof of our main Theorem. Let us consider a K3 surface X with a fixed *graded* Mukai markings $\tilde{\tau}_X : \tilde{H}(X, \mathbf{Z}) \to \tilde{\Lambda}$. We set $\omega'_X := \tilde{\tau}_X(\omega_X)$. Then let $g \in O_{Hodge}(\tilde{H}(X, \mathbf{Z})) = O_{Hodge}(\tilde{\Lambda}, \mathbf{C}\omega'_X)$ and consider $g((0, 0, 1))$. As in the proof of Theorem 2.4 we consider compositions of g with the Hodge isometries $f^{\mathcal{L}}_X, f^{\mathcal{I}_\Delta}_X$, which correspond to autoequivalences $\Phi^{\mathcal{L}}_X$ and $\Phi^{\mathcal{I}_\Delta}_X$, respectively. Choosing a suitable composition, we may assume

$$F(f^{\mathcal{L}_1}_X, f^{\mathcal{L}_2}_X, f^{\mathcal{I}_\Delta}_X) \circ g((0, 0, 1)) = (a, b, c)$$

with a and c coprime and b ample, i.e. $b = \tilde{\tau}_X(B)$ for an ample line bundle on X, where $F(f^{\mathcal{L}_1}_X, \cdots)$ represents the composition we chose. Then, since the marking $\tilde{\tau}_X$ is graded, and by the argument for 3)⇒ 4) of Theorem 2.4, there is a K3 surface Y and $\mathcal{E} \in D(Y \times X)$ such that $\Phi^{\mathcal{E}}_{Y\to X} : D(Y) \to D(X)$ is an equivalence and $f^{\mathcal{E}}_{Y\to X}((0, 0, 1)) = (a, b, c)$. Here the Hodge isometry $f^{\mathcal{E}}_{Y\to X} : (\tilde{H}(Y, \mathbf{Z}), \mathbf{C}\omega_Y) \to$

$(\tilde{H}(X,\mathbf{Z}), \mathbf{C}\omega_X)$ is identified with the Hodge isometry $\tilde{\tau}_X \circ f^{\mathcal{E}}_{Y\to X} \circ \tilde{\tau}_Y^{-1} : (\tilde{\Lambda}, \mathbf{C}\omega'_Y) \to (\tilde{\Lambda}, \mathbf{C}\omega'_X)$ under the *graded* Mukai marking $\tilde{\tau}_Y : \tilde{H}(Y,\mathbf{Z}) \to \tilde{\Lambda}$ with $\omega'_Y = \tilde{\tau}_Y(\omega_Y)$. Now consider a Hodge isometry under this identification

$$h := (F(f_X^{\mathcal{L}_1}, f_X^{\mathcal{L}_2}, f_X^{\mathcal{I}_\Delta}) \circ g)^{-1} \circ f^{\mathcal{E}}_{Y\to X} : (\tilde{\Lambda}, \mathbf{C}\omega'_Y) \to (\tilde{\Lambda}, \mathbf{C}\omega'_X). \tag{2.2}$$

Then this has the property $h((0,0,1)) = (0,0,1)$. Therefore h induces a Hodge isometry in the K3 lattice $\Lambda^2 = (0,0,1)^\perp/\mathbf{Z}(0,0,1)$;

$$\bar{h} : ((0,0,1)^\perp/\mathbf{Z}(0,0,1), \mathbf{C}\omega'_Y) \tilde{\to} ((0,0,1)^\perp/\mathbf{Z}(0,0,1), \mathbf{C}\omega'_X).$$

Since the Mukai marking $\tilde{\tau}_X$ and $\tilde{\tau}_Y$ are graded, $\tilde{\tau}_X \circ \bar{h} \circ \tilde{\tau}^{-1}|_{H^2(Y,\mathbf{Z})}$ is a Hodge isometry from $(H^2(Y,\mathbf{Z}), \mathbf{C}\omega_Y)$ to $(H^2(X,\mathbf{Z}), \mathbf{C}\omega_X)$. This implies that $Y \cong X$ and that the map h in (2.2) is a Hodge isometry from $(\tilde{\Lambda}, \mathbf{C}\omega'_X)$ to $(\tilde{\Lambda}, \mathbf{C}\omega'_X)$. Now set $h((1,0,0)) = (a,b,c)$. Then since h is an isometry, one has

$$\begin{aligned} b^2 - 2ac &= \langle (a,b,c), (a,b,c)\rangle = \langle (1,0,0), (1,0,0)\rangle = 0 \\ -a &= \langle (a,b,c), (0,0,1)\rangle = \langle (1,0,0), (0,0,1)\rangle = -1. \end{aligned}$$

Therefore one knows $h((1,0,0)) = (a,b,c) = (1,b,\frac{1}{2}b^2)$. Since $(1,0,0)$ is algebraic and h is a Hodge isometry, $(1,b,\frac{1}{2}b^2)$ is algebraic, i.e. b comes from a line bundle, say, B. Let $f_X^{(-B)}$ be the Hodge isometry corresponding to the autoequivalence $\Phi_X^{(-B)}$. Then we have a Hodge isometry

$$k := f_X^{(-B)} \circ h : (\tilde{\Lambda}, \mathbf{C}\omega'_X) \to (\tilde{\Lambda}, \mathbf{C}\omega'_X)$$

which satisfies $k((1,0,0)) = (1,0,0)$ and $k((0,0,1)) = (0,0,1)\cdot(1,-b,\frac{1}{2}b^2) = (0,0,1)$. In particular we have $k|_{\Lambda^2} : (\Lambda^2, \mathbf{C}\omega'_X) \to (\Lambda^2, \mathbf{C}\omega'_X)$, where $\Lambda^2 = \Lambda_{\mathrm{K3}}$.

Recall that the reflections $r_{C_1}, \cdots, r_{C_m}$ for smooth (-2) curves $C_1, \cdots, C_m$ come from the twisting functors $\Phi_X^{\mathcal{O}_X(C_1)} \circ T_{C_1}, \cdots, \Phi_X^{\mathcal{O}_X(C_m)} \circ T_{C_m}$. These reflections act as identity on Λ^0, Λ^4 and act on Λ^2. The composition

$$l := r_{C_1} \circ \cdots \circ r_{C_m} \circ k : (\tilde{\Lambda}, \mathbf{C}\omega'_X) \to (\tilde{\Lambda}, \mathbf{C}\omega'_X)$$

satisfies $l((0,0,1)) = (0,0,1), l((1,0,0)) = (1,0,0)$ and $l : (\Lambda^2, \mathbf{C}\omega'_X) \to (\Lambda^2, \mathbf{C}\omega'_X)$. Now define the positive cone $\mathcal{P}^+$ to be one of the two connected components of $\{x \in \tau_X(NS(X)) \otimes \mathbf{R} | (x,x) > 0\}$ which contains the ample class. Using the Nakai-Moishezon criterion, the Hodge index Theorem, and the fact that any irreducible curve $C \subset X$ satisfies $C^2 \geq 0$ unless $C \cong \mathbf{P}^1$, we may describe the ample cone $A(X)$ by

$$A(X) = \{x \in \mathcal{P}^+ \mid (x, C) > 0, \forall C \cong \mathbf{P}^1\}.$$

This means $A(X)$ is the fundamental domain of the reflection group $\langle r_C | C \cong \mathbf{P}^1, C \subset X\rangle$ acting on $\mathcal{P}^+$. Therefore composing suitable reflections, and ι_2 if necessary[1], we can assume that l preserves the ample $A(X)$. (This is the only place where we may need the Hodge isometry ι_2.) Therefore there is an automorphism

[1]We thank B. Szendröi for pointing this out to us.

$\varphi \in \mathrm{Aut}(X)$ such that $l = \varphi^*(= f_X^{\mathcal{O}_{\Gamma(\varphi)}})$ by the global Torelli theorem. Here we note that φ^* is identity on Λ^0 and Λ^4. To summarize, we have obtained an identity

$$r_{C_1} \circ \cdots \circ r_{C_m} \circ (\iota_2)^s \circ f_X^{(-B)} \circ (F(f_X^{\mathcal{L}_1}, f_X^{\mathcal{L}_2}, f_X^{\mathcal{I}_\Delta}) \circ g)^{-1} \circ f_{X \to X}^{\mathcal{E}} = \varphi^*$$

as an element of $O_{Hodge}((\tilde{\Lambda}, \mathbf{C}\omega'_X)) = O_{Hodge}(\tilde{H}(X, \mathbf{Z}))$. In the above equation all, except possibly g, are in the group generated by the image of the homomorphism

$$ch : \mathrm{Auteq}D(X) \to O_{Hodge}(\tilde{H}(X, \mathbf{Z})) = O_{Hodge}(\tilde{\Lambda}, \mathbf{C}\omega'_X),$$

and ι_2. Therefore we conclude that $g \in \langle \mathrm{Im}(ch), \iota_2 \rangle$ as well. $\square$

§3 Symplectic mapping class group and Proof of Theorem 1.9

In this section we prove the surjectivity of the map $Symp(Y, \kappa_Y) \to O^+(T(Y))^*$ which implies Theorem 1.9. (Note that $Symp^0(Y, \kappa_Y)$ acts on $H^2(Y, \mathbf{Z})$ trivially.)

Let Y be a K3 surface and $A(Y) \subset NS(Y) \otimes \mathbf{R}$ be the ample cone. As described in Definition 1.7, for $\kappa_Y \in A(Y)$ we consider a symplectic structure (Y, κ_Y) and the group of symplectic diffeomorphisms $\mathrm{Symp}(Y, \kappa_Y)$. We assume this symplectic structure (Y, κ_Y) is generic. Under this assumption we consider the induced action g^* on $H^2(Y, \mathbf{Z})$ of $g \in \mathrm{Symp}(Y, \kappa_Y)$.

Lemma 3.1. *Let (Y, κ_Y) be a generic symplectic structure in the sense of Definition 1.7, 4) and $g \in Symp(Y, \kappa_Y)$. Then we have*
1) $g^*(NS(Y)) = NS(Y)$, in fact $g^*|_{NS(Y)} = id_{NS(Y)}$,
2) $g^*(T(Y)) = T(Y)$.

Proof. Let us first recall that if κ_Y is generic, then $NS(Y)$ is characterized as the minimal primitive sublattice of $H^2(Y, \mathbf{Z})$ which contains κ_Y after tensoring with $\mathbf{R}$. (See the definition and Remark given after Definition 1.7.) This implies $g^*(NS(Y)) = NS(Y)$, since $g^*(\kappa_Y) = \kappa_Y$ and $g^*(H^2(Y, \mathbf{Z})) = H^2(Y, \mathbf{Z})$. (Note that a diffeomorphism $g \in Diff(Y)$ does not preserve the lattice $NS(Y)$ in general.) The induced action g^* of $g \in Diff(Y)$ on $H^2(Y, \mathbf{Z})$ is an isometry. Therefore we may conclude $g^*(T(Y)) = T(Y)$ as well from $g^*(NS(Y)) = NS(Y)$ and $T(Y) = NS(Y)^\perp$ in $H^2(Y, \mathbf{Z})$. Moreover, one can show $g^*|_{NS(Y)} = \mathrm{id}_{NS(Y)}$ as follows: Let $x \in NS(Y)$. Then $(\kappa_Y, g^*x) = (g^*\kappa_Y, g^*x) = (\kappa_Y, x)$, i.e. $(\kappa_Y, g^*x - x) = 0$ and $g^*x - x \in NS(Y)$. If $g^*x - x \neq 0$, then one has $\kappa_Y \in (g^*x - x)^\perp \otimes \mathbf{R}$, where $(g^*x - x)^\perp$ is the orthogonal lattice in $NS(Y)$. This is a contradiction to the minimality of $NS(Y)$ for generic κ_Y. This proves $g^*x = x$ for all $x \in NS(Y)$. $\square$

Recall the definition of the kernel subgroup $O(T(Y))^*$ and its index two subgroup $O^+(T(Y))^* = O(T(Y))^* \cap O^+(T(Y))$. We can now prove our Theorem 1.9 using the Theorem 1.8.

Proof of Theorem 1.9: Let $g \in \mathrm{Symp}(Y, \kappa_Y)$. Consider the induced action on the discriminant groups $A_{NS(Y)} = NS(Y)^*/NS(Y)$ and $A_{T(Y)} = T(Y)^*/T(Y)$. Then by Lemma 3.1 we see that $NS(Y)$ and $T(Y)$ are stable under the induced action g^*, and $g^*|_{A_{NS(Y)}} = \mathrm{id}_{A_{NS(Y)}}$. This implies $g^*|_{A_{T(Y)}} = \mathrm{id}_{A_{T(Y)}}$, and $g^*|_{T(Y)} \in O(T(Y))^*$. Since $g \in \mathrm{Symp}(Y, \kappa_Y) \subset Diff(Y)$, we have $g^* \in O^+(H^2(Y, \mathbf{Z}))$ by Theorem 1.8 and $g^*(\kappa_Y) = \kappa_Y$ by the assumption. Putting all together, we conclude that $g^*|_{T(Y)} \in O^+(T(Y))$ and further $g^*|_{T(Y)} \in O^+(T(Y))^*$. Other statements in Theorem 1.9, 1) are included in Lemma 3.1.

To show the surjectivity of the map $\mathrm{Symp}(Y, \kappa_Y) \to O^+(T(Y))^*$, let us take $\varphi \in O^+(T(Y))^*$. Then $(\mathrm{id}, \varphi) \in O(NS(Y)) \times O(T(Y))$ extends to $\tilde{\varphi} \in O(H^2(Y, \mathbf{Z}))$ since $\mathrm{id}|_{A_{NS(Y)}} = \mathrm{id}_{A_{NS(Y)}}$ and $\varphi|_{A_{T(Y)}} = \mathrm{id}_{A_{T(Y)}}$. This extension $\tilde{\varphi}$ is an element in $O^+(H^2(Y, \mathbf{Z}))$ since $\varphi \in O^+(T(Y))$ and $\mathrm{id}(\kappa_Y) = \kappa_Y$. Then Theorem 1.8 ensures the existence $g \in Diff(Y)$ such that $g^* = \tilde{\varphi}$. By construction of $\tilde{\varphi}$, we have $g \in \mathrm{Symp}(Y, \kappa_Y)$ and $g^*|_{T(Y)} = \varphi$. This shows the surjectivity. □

§4 Monodromy group, FM partners, and Proof of Theorem 1.17

Here we present Proof of Lemma 1.13 and Proof of Theorem 1.17.

Proof of Lemma 1.13: Recall that for generic $(\check{X}_1, \tau_1)$ and $(\check{X}_2, \tau_2)$ we have $\tau_i(T(\check{X}_1)) = U \oplus M_n$ for $i = 1, 2$. If $\check{X}_1 \cong \check{X}_2$, then there exists a Hodge isometry $\varphi^* : H^2(\check{X}_2, \mathbf{Z}) \to H^2(\check{X}_1, \mathbf{Z})$. Since the Hodge isometry maps the transcendental lattice to the transcendental lattice, i.e. $\varphi^* : T(\check{X}_2) \to T(\check{X}_1)$, the restriction of $\tau_2 \circ (\varphi^*)^{-1} \circ \tau_1^{-1}$ to $U \oplus M_n$ gives a desired isometry g of $U \oplus M_n$.

Conversely, assume there exists $g \in O(U \oplus M_n)$ with the property $g(\mathbf{C}\tau_1(\omega_{\check{X}_1})) = \mathbf{C}\tau_2(\omega_{\check{X}_2})$. Since the primitive embedding $U \oplus M_n \hookrightarrow \Lambda_{\mathrm{K3}}$ is unique up to isomorphism [Ni,(1.14.4)], there is an isometry $\bar{g} : \Lambda_{\mathrm{K3}} \to \Lambda_{\mathrm{K3}}$ such that $\bar{g}|_{U \oplus M_n} = g$. Then $\tau_2^{-1} \circ \bar{g} \circ \tau_1$ is a Hodge isometry from $H^2(\check{X}_1, \mathbf{Z})$ to $H^2(\check{X}_2, \mathbf{Z})$. Therefore $\check{X}_1 \cong \check{X}_2$ by the Torelli Theorem. □

We can now proceed to Proof of Theorem 1.17. However we state one general result whose special case will be used in the proof.

Lemma 4.1. *Let M be a lattice of signature $(1, t)$ and U be the hyperbolic lattice. Consider the lattice $N = U \oplus M$. Denote the discriminant by $A_N = N^*/N = M^*/M$ and consider the orthogonal group $O(A_N)$ with respect to the natural induced form on A_N. Then*

$$[O(N) : O(N)^*] = |O(A_N)|,$$

where $O(N)^* = Ker(O(N) \to O(A_N))$.

Proof. Since $N = U \oplus M$, we have $l(A_N) = l(A_M)$ and $\mathrm{rk} N \geq l(A_N) + 2$. Here $l(A)$ denotes the minimal number of generators of a finite abelian group A. Therefore by [Ni,Theorem 1.14.2], the natural map $O(N) \to O(A_N)$ is surjective. Then the result follows immediately by the definition of the kernel $O(N)^*$. □

Proof of Theorem 1.17: Since the case $n = 1$ is easy, we consider only the case $n \geq 2$. For the lattice $M_n = \langle 2n \rangle$ the group $O(A_{U \oplus M_n})$ is easily described by the Chinese remainder theorem as

$$O(A_{U \oplus M_n}) = O(M_n^*/M_n) = O(\mathbf{Z}/2n) \cong (\mathbf{Z}/2)^{p(n)},$$

where $p(n)$ is the number of the primes dividing n (cf. [Sc, lemma 3.6.1] and the proof there). Using the Lemma 4.1 we have

$$[O(U \oplus M_n) : O(U \oplus M_n)^*] = |O(A_{U \oplus M_n})| = 2^{p(n)}.$$

Since $O(U \oplus M_n)^* \to O(U \oplus M_n)/\{\pm \mathrm{id}\}$ is injective (Lemma 1.15), we have

$$[O(U \oplus M_n)/\{\pm \mathrm{id}\} : O(U \oplus M_n)^*] = 2^{p(n)-1}.$$

By definition we have $[O(U \oplus M_n)^* : O^+(U \oplus M_n)^*] \le 2$, and we find an element $(-\mathrm{id}_U, \mathrm{id}_{M_n})$ in $O(U \oplus M_n)^* \setminus O^+(U \oplus M_n)^*$, thus we can conclude $[O(U \oplus M_n)^* : O^+(U \oplus M_n)^*] = 2$. We have $[O(U \oplus M_n)/\{\pm\mathrm{id}\} : O^+(U \oplus M_n)/\{\pm\mathrm{id}\}] = 2$ as well. Therefore we finally conclude

$$[O^+(U \oplus M_n)/\{\pm\mathrm{id}\} : O^+(U \oplus M_n)^*] = 2^{p(n)-1},$$

i.e. $[\mathcal{M}_n(\check{X}) : \mathcal{MS}_n(\check{X})] = 2^{p(n)-1}$ $\square$

§5 Mirror family of a K3 surface with $\deg(X) = 12$

1) The mirror family: Consider the following one-parameter family of a surface given by three quadrics and a hyperplane in $\mathbf{P}^6$;

$$Q(\psi) : \begin{cases} U_1 + U_2 + U_3 + U_4 + U_5 + U_6 - \psi U_0 = 0 \quad (\psi \in \mathbf{C} \setminus \{\pm 2, \pm 6\}) \\ U_1 U_2 = U_0^2, \quad U_3 U_4 = U_0^2, \quad U_5 U_6 = U_0^2, \end{cases}$$

where $U_0, \cdots, U_6$ are homogeneous coordinates of $\mathbf{P}^6$. $Q(\psi)$ has 12 double points at $U_0 = 0$, which may be blown up to

$$\tilde{Q}(\psi) : \begin{cases} U_1 + U_2 + U_3 + U_4 + U_5 + U_6 - \psi U_0 = 0 \qquad (\psi \neq \pm 2, \pm 6) \\ U_1 V_1 = V_2 U_0, \quad U_3 V_3 = V_4 U_0, \quad U_5 V_5 = V_6 U_0 \\ U_2 V_2 = V_1 U_0, \quad U_4 V_4 = V_3 U_0, \quad U_6 V_6 = V_5 U_0, \end{cases}$$

where $([U_0, \cdots, U_6], [V_1, V_2], [V_3, V_4], [V_5, V_6]) \in \mathbf{P}^6 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. The natural map $\tilde{Q}(\psi) \to Q(\psi)$ is a crepant resolution and we see that $\tilde{Q}(\psi)$ is a smooth K3 surface. Also we may verify that the singularities of $\tilde{Q}(\psi)$ at $\psi = \pm 2, \pm 6$ are ordinary double points.

Theorem 5.1. ([PS, Theorem 1]) *For generic $\psi \in \mathbf{C} \setminus \{\pm 2, \pm 6\}$, one has*

$$NS(\tilde{Q}(\psi)) \cong E_8(-1)^{\oplus 2} \oplus U \oplus \langle -12 \rangle.$$

Sketch of Proof. Blowing up the 12 double points introduces the following 12 lines; i.e. a line

$$l_{0,+1,+1} := \{([0,0,0,1,0,-1,0],[V_1,V_2],[0,1],[0,1]) | (V_1,V_2) \in \mathbf{P}^1\}$$

and its obvious permutations of the coordinates, which appear 12 middle points of the cube in Fig.1. As we see in Fig.1, there are 8 lines of different type. These are represented by lines (corresponding to the corners of the cube in Fig.1), e.g.

$$l_{+1,+1,+1} := \{([0,U_1,0,U_3,0,U_5,0],[0,1],[0,1],[0,1]) | U_1 + U_3 + U_5 = 0\}.$$

These two types of lines are independent of ψ and altogether generate a sublattice of rank 17 in $NS(\tilde{Q}(\psi))$. What is interesting, and even crucial for the following analysis, is that we have 12 more lines which *dependents* on ψ, and which are typically represented by

$$m_{1,+1,+1}(\beta) := \{([U_0, \beta U_0, \beta^{-1} U_0, U_3, U_4, -U_3, -U_4],[1,\beta],[V_3,V_4],[V_3,-V_4]) \mid \\ U_3 V_3 = V_4 U_0, U_4 V_4 = V_3 U_0\}$$

with β satisfying $\beta+\beta^{-1}=\psi$. Suitable permutations of the coordinates generate 12 lines of this type. These lines contribute two more linearly independent classes in $NS(\tilde{Q}(\psi))$, and in total we see $\mathrm{rk}\tilde{Q}(\psi)\geq 19$. On the other hand, 19 is the maximal Picard number for a non-constant family of K3 surfaces (see e.g. [Og1, Main Theorem]). Therefore $\mathrm{rk}\tilde{Q}(\psi)=19$ for generic $\psi\in\mathbf{C}\setminus\{\pm 2,\pm 6\}$. Working out the intersection numbers in detail, we obtain the desired result. □

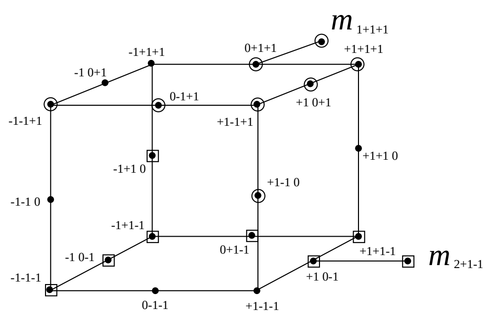

FIG.1. CUBE $[-1,1]^3$ REPRESENTING THE LINES. The notation $+1+1-1$, for example, represents the line $l_{+1,+1,-1}$. Two additional lines, $m_{1,+1,+1}, m_{2,+1,-1}$, are also written to show the generators of two $E_8(-1)$ lattices.

Remark. $\tilde{Q}(\psi)$ may be written in the affine chart $U_0\neq 0$ by a single equation,

$$f(X,Y,Z)=X+\frac{1}{X}+Y+\frac{1}{Y}+Z+\frac{1}{Z}-\psi, \tag{5.1}$$

where $X=U_1/U_0, Y=U_3/U_0, Z=U_5/U_0$. The Newton polytope of this defining equation is simply given by the convex hull $\mathrm{Conv.}\big(\{(\pm1,0,0),(0,\pm1,0),(0,0,\pm1)\}\big)$. Then the polar dual of this polytope is given by $[-1,1]^3$, which is the cube depicted in Fig.1. In fact, the polytope $[-1,1]^3$ is so-called *reflexive polytope* considered by Batyrev [Ba1] to explain the mirror symmetry of Calabi-Yau hypersurfaces in toric varieties. If we follow his construction, we obtain a defining equation

$$f(X,Y,Z)=\psi_1(X+\frac{1}{X})+\psi_2(Y+\frac{1}{Y})+\psi_3(Z+\frac{1}{Z})+1$$

for a generic hypersurface in a toric (Fano) variety whose crepant resolution may be read from the toric diagram (the cube) in Fig.1. When we fix a crepant resolution of the ambient toric variety, the generic hypersurface defines a family of smooth K3 surfaces with the Néron-Severi lattice of rank 17, which is generated by the lines appeared in the vertices of the cube. Our K3 surface $\tilde{Q}(\psi)$ is a specialization of this generic family to $\psi_1=\psi_2=\psi_3=-\frac{1}{\psi}$ (see [LY2]). □

Proposition 5.2. $\{\tilde{Q}(\psi)\}_{\psi\in\mathbf{C}\setminus\{\pm2,\pm6\}}$ *is a (covering of the) family mirror to* $\langle 12\rangle$*-polarizable K3 surface.*

Proof. For the lattice $M_n=\langle 2n\rangle$, the orthogonal lattice $M_n^\perp$ in the K3 lattice Λ_{K3} has the form $M_n^\perp=U\oplus\check{M}_n$ with

$$\check{M}_n=\langle -2n\rangle\oplus U\oplus E_8(-1)^{\oplus 2}.$$

Then combining this with Theorem 5.1, we conclude the statement for $n = 6$. $\square$

Remark. Note the isomorphism $\tilde{Q}(\psi) \cong \tilde{Q}(-\psi)$. Indeed, in the formula (5.1) one sees a birational map $\tilde{Q}(\psi) \to \tilde{Q}(-\psi) : (X, Y, Z) \to (-X, -Y, -Z)$. Since any birational map between K3 surfaces is biregular (isomorphism), the claim follows. $\square$

2) Period integrals, PF equation and monodromy: For $|\psi| > 6$ consider the integral;

$$\Pi(\psi) = \frac{1}{(2\pi i)^3} \int_{|X|=|Y|=|Z|=1} \frac{-\psi}{f(X,Y,Z)} \frac{dX}{X} \frac{dY}{Y} \frac{dZ}{Z}$$

for the defining equation $f(X, Y, Z)$ (5.1) of $\tilde{Q}(\psi)$ in the affine chart $U_0 \neq 0$. This integral represents a period integral for the family $\tilde{Q}(\psi)$ (see [PS],[Ba2]).

Proposition 5.3. ([Ba2], [LY2]) *When $|\psi| > 6$, the integral can be evaluated to be a convergent power series*

$$\Pi(x) = \sum_{k,l,m \geq 0} c(k,l,m) x^{k+l+m} := \sum_{k,l,m \geq 0} \frac{(2(k+l+m))!}{(k!)^2 (l!)^2 (m!)^2} x^{k+l+m} \quad (x := \frac{1}{\psi^2}),$$

and satisfies the following differential equation:

$$\{\theta^3 + 36x^2(\theta+1)(2\theta+1)(2\theta+3) - 2x(2\theta+1)(10\theta^2+10\theta+3)\}\Pi(x) = 0, \quad (5.2)$$

where $\theta = x\frac{d}{dx}$. $\square$

The differential equation is defined on $\mathbf{P}^1$ and has regular singularities at $x = 0, \frac{1}{36}, \frac{1}{4}, \infty$. Thus the affine space $\mathbf{C}$ is naturally compactified to $\mathbf{P}^1$ considering the boundary point $\psi = \infty$ ($x = \frac{1}{\psi^2} = 0$). (Note that $\pm\psi$ are naturally identified in accord with the isomorphism $\tilde{Q}(\psi) \cong \tilde{Q}(-\psi)$.) The boundary point $x = 0$ is the so-called *maximally unipotent monodromy point* [Mo1], and plays important roles in the applications of mirror symmetry. As we see in (5.2), about this boundary point all indices of the local solutions degenerate to zero, and we may apply the Frobenius method to generate all solutions (see, e.g. [HLY] and references therein for details). Now we define a ratio of two solutions

$$t := \frac{1}{2\pi i} \frac{\partial}{\partial \rho} \log \Pi(x,\rho)|_{\rho=0} = \frac{1}{2\pi i} \frac{1}{\Pi(x)} \frac{\partial}{\partial \rho} \Pi(x,\rho)|_{\rho=0}, \quad (5.3)$$

where $\Pi(x,\rho) := \sum_{k,l,m\geq 0} c(k+\rho, l+\rho, m+\rho) x^{k+l+m+3\rho}$ is the formal extension of the series to apply the Frobenius method. The ratio (5.3) may be regarded as a projective coordinate of the period point in $\Omega(U \oplus M_n)$ (cf. [Do, Section 7]). The inverse relation to (5.3), i.e. $x = x(t)$ is called the *mirror map*. In [LY1], nice modular properties related to the genus zero modular groups have been observed for many examples of the mirror family of M_n-polarized K3 surfaces. In our case of $\tilde{Q}(\psi)$, it may be summarized as follows:

Proposition 5.4. ([LY2]) *The mirror map $x = x(t)$ is the Hauptmodul of the genus zero modular group $\Gamma_0(6)_+$ and satisfies the following Schwarzian equation:*

$$\{t, x\} = \frac{1 - 52x + 1500x^2 - 6048x^3 + 15552x^4}{2(1-36x)^2(1-4x)^2x^2},$$

where $\{t, x\} = \frac{t'''}{t'} - \frac{3}{2}(\frac{t''}{t'})^2$ with $' := \frac{d}{dx}$. $\square$

Remark. 1) We follow the notation in [CN] for the extension of the group $\Gamma_0(n)$ by the involutory normalizers, the Fricke involution and the Atkin-Lehner involutions. The Fricke involution is a normalizer in $PSL(2,\mathbf{R})$ of the modular subgroup,

$$\Gamma_0(n) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbf{Z}) \mid c \equiv 0 \bmod n \},$$

and it is represented by a coset $W_n := \begin{pmatrix} 0 & -\frac{1}{\sqrt{6}} \\ \sqrt{6} & 0 \end{pmatrix} \Gamma_0(n)$. The Atkin-Lehner involutions W_r are generalizations of the Fricke involution ($r = n$) where $r \geq 1$ is a divisor of n such that r and n/r are coprime. ($W_1 = \Gamma_0(n)$.) Then the notation $\Gamma_0(n)_+$ is used to represent the group obtained from $\Gamma_0(n)$ by adjoining all possible Atkin-Lehner involutions, while the $\Gamma_0(n)_{+n}$ represents the group joined the Fricke involution only. In our case we may work out explicit generators of the groups as follows;

$$\begin{aligned} \Gamma_0(6)_+ &= \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{6}} \\ \sqrt{6} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} \\ 2\sqrt{3} & \sqrt{3} \end{pmatrix} \rangle, \\ \Gamma_0(6)_{+6} &= \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{6}} \\ \sqrt{6} & 0 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix} \rangle. \end{aligned} \tag{5.4}$$

2) In [BP] a different family of K3 surfaces and therefore a different Picard-Fuchs equation is studied. The family in [BP] is a different covering of the mirror family of $\langle 12 \rangle$-polarizable K3 surfaces. The definitions of the Period integral and the mirror map $x = x(t)$ are parallel to our family $\tilde{Q}(\psi)$. In this case, we obtain for the mirror map $x = x(t)$ the Hauptmodul of the genus zero modular group $\Gamma_0(6)_{+6}$, which is an index two subgroup of $\Gamma_0(6)_+$. The difference of the modular group comes from the difference of the covering. In fact, the precise relation between these two parametrizations may be found in [PS, Remark4].

3) The Schwartzian equation in Theorem 5.4 has four singularities at $0, \frac{1}{36}, \frac{1}{4}, \infty$. To see these singularities in one affine chart, it is convenient to apply a fractional linear transformation $z = \frac{48x}{12x+1}$ which sends the four singular points $(0, \frac{1}{36}, \frac{1}{4}, \infty)$ to $(a_0, a_1, a_2, a_3) = (0, 1, 3, 4)$. Then the Schwartzian equation may be written in the standard form (see [Hi, Chapter 10] for example):

$$\{t, z\} = \sum_{i=0}^{3} \left(\frac{1}{2} \frac{1 - \alpha_i^2}{(z - a_i)^2} + \frac{\beta_i}{z - a_i} \right),$$

with $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(\beta_0, \beta_1, \beta_2, \beta_3) = (\frac{13}{24}, -\frac{3}{16}, \frac{1}{48}, -\frac{3}{8})$. The α_i's determine the local form of the mapping $t(z) = c_0(z - a_i)^{\alpha_i} + \cdots$ near the singularities. From this we can determine the image of the complex x plane to the upper half plane parametrized by t. In fact, since the group $\Gamma_0(6)_+$ is a genus zero group, the image coincides with a fundamental domain of the group. In Fig. 2, we have depicted the fundamental domain of $\Gamma_0(6)_+$ and also that of $\Gamma_0(6)_{+6}$. $\square$

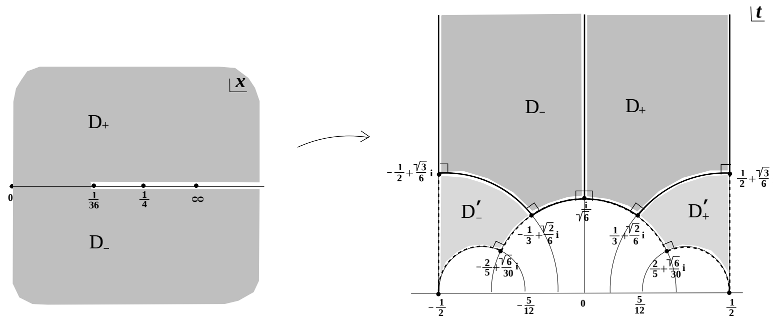

Fig.2. Fundamental domain of $\Gamma_0(6)_+$. The fundamental domain of $\Gamma_0(6)_+$ is shown as an image of a complex x-plane by $t = t(x)$. See also Table 1 for the correspondence of the (regular) singular points and the points of non-trivial stabilizer on the t-plane. The union $D_+ \cup D_- \cup D'_+ \cup D'_-$ represents the fundamental domain of $\Gamma_0(6)_{+6}$.

x	t	stabilizer	FM transform
0	$i\infty$	$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\Phi_0 = \Phi_X^{\pi_2^*\mathcal{O}(1)}$
$\frac{1}{36}$	$\frac{i}{\sqrt{6}}$	$S_1 = \begin{pmatrix} 0 & \frac{-1}{\sqrt{6}} \\ \sqrt{6} & 0 \end{pmatrix}$	$\Phi_1 = T_{\mathcal{O}_X}$
$\frac{1}{4}$	$\frac{1}{3} + \frac{\sqrt{2}}{6}i$	$S_2 = \begin{pmatrix} -\sqrt{2} & \frac{1}{\sqrt{2}} \\ -3\sqrt{2} & \sqrt{2} \end{pmatrix}$	$-$
∞	$\frac{1}{2} + \frac{\sqrt{3}}{6}i$	$S_2 S_1 T^{-1}$	$\Phi_0^{-1} \circ \Phi_1 \circ$ "$(-)$"

Table 1. Monodromy matrices and FM transforms. For the regular singular points on the x-plane, we list the corresponding monodromy matrices as an element in $\Gamma_0(6)_+$ and FM transforms. The blank (-) in the third (and the fourth) line indicates that this does not come from Auteq$D(X)$ (see Proposition 5.8 and Remark after that).

The relation between the modular group and the monodromy group of the period integrals is described by the well-known relation;

$$PSL(2, \mathbf{R}) \cong SO^+(2, 1; \mathbf{R}). \tag{5.5}$$

Let us describe this isomorphism explicitly for the Minkowski space $\mathbf{R}^{2,1} = (U \oplus M_n) \otimes \mathbf{R}$ fixing a basis e, v, f for the lattice $U \oplus M_n$ (precisely $U = \mathbf{Z}e \oplus \mathbf{Z}f$ and $M_n = \mathbf{Z}v$ with $(v, v) = 2n$). Then the orthogonal group is defined by

$$SO(2, 1; \mathbf{R}) := \{g \mid {}^t g \Sigma g = \Sigma, \det(g) = 1\}\ ,\ \Sigma = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2n & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

$SO^+(2,1;\mathbf{R})$ is an index two subgroup consisting of the elements which preserve orientations of the positive two planes. We can write the isomorphism (5.5) (anti-homomorphism) explicitly;

$$R : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ac & \frac{c^2}{n} \\ ab & ad+bc & \frac{cd}{n} \\ nb^2 & 2nbd & d^2 \end{pmatrix}. \tag{5.6}$$

We may also consider the natural map $O(U \oplus M_n) \to O(2,1;\mathbf{R})$ and define a homomorphism $F : O^+(U \oplus M_n) \to SO^+(2,1;\mathbf{R})$ by $F(g) = \det(g)g$. Then we have $\mathrm{Ker}F = \{\pm \mathrm{id}_{U\oplus M_n}\}$ and thus the induced map

$$\bar{F} : \mathcal{M}_n(\check{X}) = O^+(U \oplus M_n)/\{\pm \mathrm{id}_{U\oplus M_n}\} \to SO^+(2,1;\mathbf{R}) \cong PSL(2,\mathbf{R})$$

is injective. For a subgroup $G \subset O^+(U \oplus M_n)$, consider the following composition map:

$$q : G \hookrightarrow O^+(U \oplus M_n) \to O^+(U \oplus M_n)/\{\pm \mathrm{id}_{U\oplus M_n}\}.$$

Then, up to the kernel $\mathrm{Ker}(q)$, we can identify G with the image $R^{-1} \circ \bar{F} \circ q(G)$ in $PSL(2,\mathbf{R})$. For example, using the (injective) map (1.8) we may regard the group $O^+(U \oplus M_n)^*$ $(n \geq 2)$ as a subgroup of $PSL(2,\mathbf{R})$.

The following group isomorphisms are derived in purely arithmetic manner in [Do], and will be utilized in the following.

Theorem 5.5. **([Do, Theorem (7.1), Remark (7.2)])** *We have:*
1) $O^+(U \oplus M_n)^* \cong \Gamma_0(n)_{+n}$ $(n \geq 2)$, $O^+(U \oplus M_1)^*/\{\pm id_{U\oplus M_1}\} \cong PSL(2,\mathbf{Z})$
2) $O^+(U \oplus M_n)/\{\pm id_{U\oplus M_n}\} \cong \Gamma_0(n)_+$ □

Remark. When $\mathrm{Ker}(q)$ is non-trivial, we encounter the sign ambiguity to determine $G \subset O^+(U \oplus M_n)$ from its image in $PSL(2,\mathbf{R})$. However the different choice of the sign can be understood as coming from the Hodge isometry associated to the shift functor [1] or its counterpart in $\mathrm{Auteq}DFuk(\check{X})$. Because of this, we do not have to pay much attention to this sign problem. This is a reason we define the monodromy group $\mathcal{M}_n(\check{X})$ by the quotient $O^+(U \oplus M_n)/\{\pm \mathrm{id}_{U\oplus M_n}\}$ instead of the conventional definition of the monodromy group of the period integrals. □

3) Monodromy S_2 and FM partner: For each stabilizer $T, S_1, S_2 \in \Gamma_0(6)_+$ defined in Table 1, consider the matrices $R(T), R(S_1), R(S_2) \in SO^+(2,1;\mathbf{R})$. We can verify these matrices are in the image of the map $O^+(U \oplus M_n)/\{\pm \mathrm{id}_{U\oplus M_n}\} \to SO^+(2,1;\mathbf{R})$. Though the preimages $\bar{T}, \bar{S}_1, \bar{S}_2 \in O^+(U \oplus M_n)$ of $R(T), R(S_1), R(S_2)$ are determined only up to signs as matrices, we fix these signs so that $\bar{T} = R(T)$, $\bar{S}_1 = -R(S_1)$, $\bar{S}_2 = -R(S_2)$, i.e.

$$\bar{T} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 6 & 12 & 1 \end{pmatrix}, \bar{S}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \bar{S}_2 = \begin{pmatrix} -2 & -12 & -3 \\ 1 & 5 & 1 \\ -3 & -12 & -2 \end{pmatrix}.$$

These matrices are understood as the monodromy around the regular singular points in the x-plane since the regular singularities of the Picard-Fuchs equation (5.2) are mapped to the corresponding points with non-trivial stabilizers under the map $t = t(x)$. The choices of the signs above are made so that they are consistent to the local properties of the PF equation, for example, $\det(\bar{S}_1) = \det(\bar{S}_2) = -1$ which come from the Picard-Lefshetz formula around the corresponding double points of $\tilde{Q}(\pm 6)$ and $\tilde{Q}(\pm 2)$ respectively.

Proposition 5.6. *The matrices $R(T), R(S_1), R(S_2)$ generate the monodromy group $\mathcal{M}_6(\check{X}) = O^+(U \oplus M_6)/\{\pm id_{U\oplus M_6}\}$ defined in Definition 1.16 ($n = 6$).*

Proof. Note that the stabilizer S_2 may be related to the generators of $\Gamma_0(6)_+$ by $S_2 = \begin{pmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} \\ 2\sqrt{3} & \sqrt{3} \end{pmatrix} S_1$. Then by Theorem 5.5, 2) the claim follows. □

Proposition 5.7. *Consider $M_6 = \langle 12 \rangle$, then one has:*
1) $\bar{T}, \bar{S}_1 \in O^+(U \oplus M_6)^*$.
2) $\pm\bar{S}_2 \in O^+(U \oplus M_6) \setminus O^+(U \oplus M_6)^*$, i.e. $\pm\bar{S}_2$ does not come from the symplectic mapping class group of $\check{X}$.
3) The monodromy representation of the symplectic diffeomorphisms $\mathcal{MS}_6(\check{X})(= O^+(U \oplus M_6)^*)$ is generated by $\bar{T}, \bar{S}_1, (\bar{S}_1\bar{S}_2)^2$.

Proof. The discriminant is given by $A_{U\oplus M_6} \cong M_6^*/M_6 = \frac{\mathbf{Z}}{12}v/\mathbf{Z}v$. Since both $\bar{T}$ and $\bar{S}_1$ act on the basis v by $v \mapsto v \bmod U$, the induced action on the discriminant is the identity $\mathrm{id}_{A_{U\oplus M_6}}$. This shows 1). On the other hand $\bar{S}_2$ acts on v non-trivially, $v \mapsto 5v \bmod U$. Therefore $\pm\bar{S}_2 \in O^+(U \oplus M_6)$ does not belong to the kernel $O^+(U \oplus M_6)^*$. By Theorem 1.9 we conclude 2).

For 3), let us evaluate $(\bar{S}_1\bar{S}_2)^2 = (R(S_1)R(S_2))^2 = (R(S_2S_1))^2 = R((S_2S_1)^2)$ (R is an anti-homomorphism) and note that

$$S_2S_1 = \begin{pmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} \\ 2\sqrt{3} & \sqrt{3} \end{pmatrix}^2 = \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix}.$$

Therefore we obtain three generators T, S_1, S_2S_1 of $\Gamma_0(6)_{+6}$ which is isomorphic to $O^+(U \oplus M_6)^*$ by Theorem 5.5, 1). □

Remark. The $\bar{S}_2$ represents the monodromy around $x = \frac{1}{4}$ and has a simple geometric interpretation. Let us recall the explicit constructions of the algebraic cycles (lines) in $\tilde{Q}(\psi)(= \check{X})$. As we have sketched in the proof of Theorem 5.1, we may take 17 linearly independent lines in $NS(\tilde{Q}(\psi))$ which are independent of ψ (coming from the toric divisors of the ambient space). The full lattice $NS(\tilde{Q}(\psi))$ is obtained by considering the lines, $m(\beta)$-lines, which are dependent of generic ψ ($\beta + \beta^{-1} = \psi$). Explicitly we have

$$\beta_\pm = \frac{\psi \pm \sqrt{\psi^2 - 4}}{2} = \frac{1 \pm \sqrt{1 - 4x}}{2\sqrt{x}} \qquad (x = \frac{1}{\psi^2}).$$

Namely at $x = \frac{1}{4}$ two lines $m(\beta_+)$ and $m(\beta_-)$ coincide, and these are exchanged under the monodromy operation around $x = \frac{1}{4}$. (The monodromy around $x = 0$ is fictitious because of the isomorphism $\tilde{Q}(\psi) = \tilde{Q}(-\psi)$.) By Lemma 3.1, 1) the induced actions of the symplectic diffeomorphisms must be trivial on the lattice $NS(\check{X})$ if the symplectic structure is generic. Therefore the non-trivial monodromy behavior of the $m(\beta)$-lines explains that the monodromy $\bar{S}_2$ does not come from the symplectic diffeomorphism for a generic symplectic structure on $\check{X}$. We may argue similarly the monodromy around $x = \infty$. □

Combining Proposition 5.6 and Proposition 5.7, we evaluate directly the group index, $[\mathcal{M}_n(\check{X}) : \mathcal{MS}_n(\check{X})] = 2$ ($n = 6$), which reproduce the number $p(6) = 2$ of FM partners of the mirror X (Proposition 1.12) from completely different picture. Finally let us prove that the monodromy $\pm\bar{S}_2$ in Proposition 5.7, 2) does not come from the autoequivalence of $D(X)$.

Proposition 5.8. *Let X be a generic $M_6 = \langle 12 \rangle$-polarizable K3 surface, i.e. the mirror of $\tilde{Q}(\psi)$, then the monodromy $\pm \bar{S}_2 \in O^+(U \oplus M_6) = O^+(\widetilde{NS}(X))$ does not come from $AuteqD(X)$.*

Proof. Since $\pm \bar{S}_2 \notin O^+(U \oplus M_n)^*$, the element $(\pm \bar{S}_2, \mathrm{id}_{U \oplus \check{M}_n}) \in O(U \oplus M_n) \times O(U \oplus \check{M}_n)$ does not extend to an isometry $O(\tilde{\Lambda})$ (where $U \oplus M_n \oplus U \oplus \check{M}_n \subset \tilde{\Lambda}$). Note that for (X, τ) generic, we have $\tau(T(X)) = U \oplus \check{M}_n$. Note also that when $\rho(X) = 1$, the restriction $f|_{T(X)}$ of a Hodge isometry $f \in O_{Hodge}(\tilde{H}(X, \mathbf{Z}), \mathbf{C}\omega_X)$ is trivial, i.e. $f|_{T(X)} = \pm \mathrm{id}_{T(X)}$ (see for example [Og2,(4.1)]). Therefore $\bar{S}_2$ does not extend to a Hodge isometry in $O(\tilde{\Lambda}, \mathbf{C}\omega_1)$ ($\omega_1 = \tau(\omega_X)$). By Theorem 1.6, we obtain the desired result. $\square$

Remark. As explained in the previous Remark (after Proposition 5.7), $\bar{S}_2 \in O^+(U \oplus M_6)$ comes from the monodromy about $x = \frac{1}{4}$, and the monodromy acts non-trivially on $NS(\check{X})(= \check{M}_6)$ of $\check{X} = \tilde{Q}(\psi)$. Recall that the lines depicted in the toric diagram (the cube) in Fig.1 are independent of ψ, and generate a sublattice of rank 17 in $NS(\check{X})$. This sublattice has signature $(1, 16)$ ([PS,Proposition 1]). Therefore the monodromy around $x = \frac{1}{4}$ fixes this sublattice of signature $(1, 16)$. From this argument, we see that the monodromy around $x = \frac{1}{4}$ preserves the orientations of the positive three planes in $U \oplus M_6 \oplus \check{M}_6 \subset \Lambda_{\mathrm{K3}} \cong H^2(\check{X}, \mathbf{Z})$. By the surjectivity 2) in Theorem 1.8, we may conclude that the monodromy does come from $Diff(\check{X})$ although it does not come from $Symp(\check{X}, \kappa_{\check{X}})$ (Proposition 5.7,2)). Now the mirror counterpart to this statement is clear. We expect that: *Let X be a generic $M_6 = \langle 12 \rangle$-polarizable K3 surface and $Y \not\cong X$ be the FM partner of X. Then for the monodromy $\bar{S}_2$ there exists a kernel $\mathcal{E} \in D(X \times Y)$ such that $ch(\Phi^{\mathcal{E}}_{X \to Y})|_{U \oplus M_6} = \bar{S}_2$ in $O^+(U \oplus M_6)$.* Unfortunately, this does not follow directly from Theorem 1.6 (Main Theorem 1). $\square$

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Shinobu Hosono†
Department of mathematics
Harvard University
Cambridge, MA 02138
Email: hosono@math.harvard.edu
hosono@ms.u-tokyo.ac.jp

Bong H. Lian
Department of mathematics
Brandeis University
Waltham, MA 02154
Email: lian@brandeis.edu

Keiji Oguiso†
Department of mathematics
Harvard University
Cambridge, MA 02138
Email: keiji@math.harvard.edu
oguiso@ms.u-tokyo.ac.jp

Shing-Tung Yau
Department of mathematics
Harvard University
Cambridge, MA 02138
Email: yau@math.harvard.edu

† On leave of absence from Graduate School of Mathematical Sciences, University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, JAPAN