

The S^1 fixed points in Quot-schemes and mirror principle computations

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Abstract

We describe the S^1 -action on the Quot-scheme $Quot(\mathcal{E}^n)$ associated to the trivial bundle $\mathcal{E}^n = \mathbb{C}P^1 \times \mathbb{C}^n$. In particular, the topology of the S^1 -fixed-point components in $Quot(\mathcal{E}^n)$ and the S^1 -weights of the normal bundle of these components are worked out. Mirror Principle, as developed by three of the current authors in the series of work [L-L-Y1, I, II, III, IV], is a method for studying certain intersection numbers on a stable map moduli space. As an application, in Mirror Principle III, Sec 5.4, an outline was given in the case of genus zero with target a flag manifold. The results on S^1 fixed points in this paper are used here to do explicit Mirror Principle computations in the case of Grassmannian manifolds. In fact, Mirror Principle computations involve only a certain distinguished subcollection of the S^1 -fixed-point components. These components are identified and are labelled by Young tableaux. The S^1 -equivariant Euler class e_{S^1} of the normal bundle to these components is computed. A diagrammatic rule that allows one to write down e_{S^1} directly from the Young tableau is given. From this, the aforementioned intersection numbers on the moduli space of stable maps can be worked out. Two examples are given to illustrate our method. Using our method, the A-model for Calabi-Yau complete intersections in a Grassmannian manifold can now also be computed explicitly. This work is motivated by the intention to provide further details of mirror principle and to understand the relation of mirror principle to physical theory. Some related questions are listed for further study.

Key words: mirror principle, Grassmannian manifold, Quot-scheme, Young tableau, equivariant Euler class, homogeneous bundle.

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0. Introduction and outline.

Introduction.

In Mirror Principle III, Sec. 5.4, in the series of work [L-L-Y1, I, II, III, IV] developed by three of the current authors, they outlined how Mirror Principle can be used to study certain intersection numbers on a stable map moduli space for flag manifolds. In this article, we carry out this computation explicitly in the case of Grassmannian manifolds. This is our main motivation for studying the S^1 -action on Quot-schemes. The latter is, of course, of independent interests from the viewpoint of group actions on manifolds, regardless of Mirror Principle. Two of our main results are the topology of the S^1 -fixed-point components in $Quot(\mathcal{E}^n)$ (Theorem 2.1.9), and the S^1 -weights of the normal bundle to these components (Theorem 2.2.1). Mirror Principle computations involve only a certain distinguished subcollection of the S^1 -fixed-point components. These components are identified and are labelled by Young tableaux. The S^1 -equivariant Euler class e_{S^1} of the normal bundle of each of these components is computed (Theorem 3.3.3). A diagrammatic rule that allows one to write down e_{S^1} directly from the Young tableau is given. From this, the intersection numbers of the moduli space of stable maps can be easily worked out. Two sample calculations are given to illustrate the method (Sec. 4). The answers are self-consistent and is the same as the result computed via the method of Mirror Principle I in a special case. Using our method, the A-model for Calabi-Yau complete intersections in a Grassmannian manifold can now also be computed explicitly. This work is motivated by the intention to provide further details of mirror principle and to understand the relation of mirror principle to physical theory. Some related questions are listed in the end for further pursuit.

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1 Essential backgrounds and notations for physicists.

Essential backgrounds or their main references used in this article and notations for objects involved are collected in this section for the convenience of readers.

• **Schemes, coherent sheaves, and Hilbert polynomials.** (See Eisenbud-Harris [E-H], Hartshorne [Ha: Chapter II], and Friedman [Fri: Chapter 2]; also Kempf [Ke] and Mumford [Mu2].) Let X be a projective variety with a fixed very ample line bundle $\mathcal{O}(1)$, then the Hilbert polynomial of coherent sheaves on X are additive with respect to short exact sequences. In other words, if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of coherent sheaves on X , then $P_{\mathcal{O}(1)}(\mathcal{F}) = P_{\mathcal{O}(1)}(\mathcal{F}') + P_{\mathcal{O}(1)}(\mathcal{F}'')$, where $P_{\mathcal{O}(1)}(\cdot)$ is the Hilbert polynomial of \cdot . (Cf. [Ha: Ex. III.5.1]).

• **Coherent sheaves on a curve and their Hilbert polynomial.**

- A coherent sheaf \mathcal{F} on a smooth curve C fits into a *split exact sequence* of \mathcal{O}_C -modules: $0 \rightarrow \mathcal{F}_{\text{torsion}} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{\vee\vee} \rightarrow 0$, where $\mathcal{F}_{\text{torsion}}$ is the torsion subsheaf of \mathcal{F} and the double dual $\mathcal{F}^{\vee\vee}$ of \mathcal{F} is locally-free. In case C is nodal, then an exact sequence from the normalization \tilde{C} of C can be used to understand coherent sheaves on C as well.

- Let \mathcal{F} be a coherent sheaf on a smooth curve C , then

$$\deg \mathcal{F} = c_1(\mathcal{F}) = c_1(\mathcal{F}_{\text{torsion}}) + c_1(\mathcal{F}^{\vee\vee}) = \dim_{\mathbb{C}} \Gamma(C, \mathcal{F}_{\text{torsion}}) + c_1(\mathcal{F}^{\vee\vee}).$$

- Fix a very ample line bundle $\mathcal{O}(1)$ on C , let k be the rank of \mathcal{F} and g be the arithmetic genus of C . Then the Hilbert polynomial for \mathcal{F} is given by

$$P(\mathcal{F}, t) = (k \deg C) t + \deg \mathcal{F} + k(1 - g).$$

For $C = \mathbb{CP}^1$ with $\mathcal{O}_{\mathbb{CP}^1}(1)$, this is $P(\mathcal{F}, t) = kt + (c_1(\mathcal{F}) + k)$. For \mathcal{F} a torsion sheaf, $r = 0$ and the polynomial becomes $P(\mathcal{F}, t) = c_1(\mathcal{F}) = \dim_{\mathbb{C}} \Gamma(C, \mathcal{F})$.

Cf. [H-L], [H-M], [Ke], and [LP].

• **Quot-scheme.** (See Huybrechts-Lehn [H-L: Chapter 2]; also Grothendieck [Gr3], Kollár [Kol: Sec. I.1], and Mumford [Mu1].) Let S be a projective variety S with a fixed ample line bundle, and \mathcal{F} be a coherent sheaf on S . Then the Quot-scheme $Quot_P(\mathcal{E}^n)$ of Grothendieck is the fine moduli space that parameterizes the set of quotients $\mathcal{F} \rightarrow \mathcal{F}/\mathcal{Y}$ with $P(\mathcal{F}/\mathcal{Y}, t)$ a given polynomial $P = P(t)$. It is the scheme that represents the *Quot*-functor of Grothendieck, cf [Gr3].

• **Quot-scheme compactification of $Hom(\mathbb{CP}^1, Gr_r(\mathbb{C}^n))$.** (Cf. [Str].) Let $C = \mathbb{CP}^1$ with the very ample line bundle $\mathcal{O}_{\mathbb{CP}^1}(1)$, \mathcal{E}^n be a trivialized trivial bundle of rank n over C , $Gr_r(\mathbb{C}^n)$ be the Grassmannian manifold that parameterizes r -planes in \mathbb{C}^n , and

$Hom(\mathbb{CP}^1, Gr_r(\mathbb{C}))$ be the space of morphisms from \mathbb{CP}^1 to $Gr_r(\mathbb{C}^n)$. Then an element $(f : \mathbb{CP}^1 \rightarrow Gr_r(\mathbb{C}^n))$ in $Hom(\mathbb{CP}^1, Gr_r(\mathbb{C}))$ determines a unique rank- r subbundle \mathcal{V} in \mathcal{E}^n , which corresponds in turn to the element $\mathcal{E}^n \rightarrow \mathcal{E}^n/\mathcal{V}$ in $Quot(\mathcal{E}^n)$. This gives a natural embedding of $Hom(\mathbb{CP}^1, Gr_r(\mathbb{C}^n))$ in $Quot(\mathcal{E}^n)$. The component of $Hom(\mathbb{CP}^1, Gr_r(\mathbb{C}^n))$ that contains degree- d image curves in $Gr_r(\mathbb{C}^n)$ is embedded in $Quot_P(\mathcal{E}^n)$ with the Hilbert polynomial $P = P(t) = (n-r)t + d + (n-r)$. This gives a compactification of $Hom(\mathbb{CP}^1, Gr_r(\mathbb{C}^n))$ via Quot-schemes, other than the moduli space of stable maps. Recall also that $Quot_P(\mathcal{E}^n)$ is a smooth, irreducible, rational projective variety of dimension $dn + (n-r)r$, cf. [Ch] and [Kim]. The S^1 -action on \mathbb{CP}^1 induces S^1 -actions on $Hom(\mathbb{CP}^1, Gr_r(\mathbb{C}^n))$ and $Quot(\mathcal{E}^n)$ respectively. The two actions coincide under the natural embedding of $Hom(\mathbb{CP}^1, Gr_r(\mathbb{C}^n))$ in $Quot(\mathcal{E}^n)$.

• **Mirror principle for Grassmannian manifolds.** For the details of Mirror Principle, readers are referred to [L-L-Y1: I, II, III, IV]. Some survey is given in [L-L-Y2]. To avoid digressing too far away, here we shall take [L-L-Y1, III: Sec. 5.4] as our starting point and restrict to the case that the target manifold of stable maps is $X = Gr_r(\mathbb{C}^n)$. Recall first the Plücker embedding $\tau : X = Gr_r(\mathbb{C}^n) \rightarrow Y = \mathbb{CP}^{\binom{n}{r}}$, which induces an isomorphism between the divisor class groups $\tau^* : A^1(Y) \xrightarrow{\sim} A^1(X)$.

Recall next the setup of Mirror Principle for $X = Gr_r(\mathbb{C}^n)$. The geometric objects involved are contained in the following diagram :

$$\begin{array}{ccccccc}
V & & U_d & & V_d & & \mathcal{U}_d \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{ev} & M_{0,1}(d, X) & \xrightarrow{\rho} & M_{0,0}(d, X) & \xleftarrow{\pi} & M_d \xrightarrow{\varphi} W_d \xleftarrow{\psi} Quot_{(d)} \\
& & & & \cup & & \cup \\
& & & & F_0 & \xrightarrow{ev^Y} & Y_0 (\supset X_0 = X) \xleftarrow{g} E_0 = \cup_s E_{0s}, \\
& & & & \downarrow & & \downarrow \\
& & & & X & \xrightarrow{\tau} & Y
\end{array}$$

where

- (1) *Moduli spaces:* $M_{0,0}(d, X)$ is the moduli space of genus-0 stable maps of degree d into X , $M_{0,1}(d, X)$ is the moduli space of genus-0, 1-pointed stable maps of degree d into X , $M_d = M_{0,0}(\mathbb{CP}^1 \times X, (1, d))$, W_d is the linear sigma model at degree d , which can be chosen to be the projective space $\mathbb{P}(H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(d)) \otimes \Lambda^r \mathbb{C}^n)$ for $X = Gr_r(\mathbb{C}^n)$, and $Quot_{(d)} = Quot_P(\mathcal{E}^n)$ with $P = P(t) = (n-r)t + d + (n-r)$;
- (2) *Group actions:* there are \mathbb{C}^\times -actions on M_d , W_d , and $Quot_{(d)}$ respectively that are compatible with the morphisms among these moduli spaces; these \mathbb{C}^\times -actions induce S^1 -actions on these moduli spaces by taking the subgroup $U(1) \subset \mathbb{C}^\times$;
- (3) *Morphisms:* ev is the evaluation map, ρ is the forgetful map, π is the contracting morphism, φ is the collapsing morphism, and ψ is an S^1 -equivariant resolution of singularities of $\varphi(M_d)$, which will be discussed in detail in Sec. 3.1;

- (4) *Bundles*: V is a vector bundle over X , $V_d = \rho_! ev^* V$, $U_d = \rho^* V_d$, and $\mathcal{U}_d = \pi^* V_d$;
- (5) *Special S^1 -fixed-point locus*: $F_0 \simeq M_{0,1}(d, X)$ is the special S^1 -fixed-point component in M_d that corresponds to gluing stable maps (C', f', x') to $\mathbb{C}P^1$ at $x' \in C'$ and $\infty \in \mathbb{C}P^1$, Y_0 is the special S^1 -fixed-point component in W_d such that $\varphi^{-1}(Y_0) = F_0$, and E_0 is the S^1 -fixed-point locus in $\psi^{-1}(Y_0)$ and is called the distinguished S^1 -fixed-point locus or components in $Quot_P(\mathcal{E}^n)$.

Associated to each (V, b) , where b is a multiplicative characteristic class, is the Euler series $A(t) \in A^*(X)(\alpha)[t]$:

$$\begin{aligned} A(t) &= A^{V,b} = e^{-H \cdot t / \alpha} \sum_d A_d e^{d \cdot t}, \\ A_d &= i_0^* b(\mathcal{U}_d) := ev_*^X \left(\frac{\rho^* b(V_d) \cap [M_{0,1}(d, X)]}{e_{\mathbb{C}^\times}(F_0/M_d)} \right) = \frac{(i_{X_0}^* \varphi_* b(\mathcal{U}_d)) \cap [X_0]}{e_{\mathbb{C}^\times}(X_0/W_d)}, \text{ denoted } \frac{\Theta_d}{e_{\mathbb{C}^\times}(X_0/W_d)}, \\ &= g_* \left(\sum_s \frac{(i_{E_{0s}}^* g^* i_{X_0}^* \varphi_* b(\mathcal{U}_d)) \cap [E_{0s}]}{e_{\mathbb{C}^\times}(E_{0s}/Quot(d))} \right), \text{ denoted } g_* \left(\sum_s \frac{\Xi_{d,s}}{e_{\mathbb{C}^\times}(E_{0s}/Quot(d))} \right), \end{aligned}$$

where $\alpha = c_1(\mathcal{O}_{\mathbb{C}P^\infty})(1)$ is the generator for $H_{\mathbb{C}^\times}^*(pt)$. On the other hand, one has the intersection numbers and their generating function

$$\begin{aligned} K_d &= K_d^{V,b} = \int_{M_{0,0}(d,X)} b(V_d), \\ \Phi &= \Phi^{V,b} = \sum_d K_d e^{d \cdot t}. \end{aligned}$$

In the good cases, K_d and Φ can be obtained from A_d and $A(t)$ by appropriate integrals of the form $\int_X e^{-H \cdot t / \alpha} A_d$, where H is the hyperplane class on Y restricted to X , e.g. [L-L-Y1, III: Theorem 3.12]. This integral can be turned into an integral on E_0 :

$$\int_X \tau^* e^{H \cdot t} \cap A_d = \int_{Y_0} e^{H \cdot t} \cap g_* \left(\sum_s \frac{\Xi_{d,s}}{e_{\mathbb{C}^\times}(E_{0s}/Quot(d))} \right) = \sum_s \int_{E_{0s}} \frac{g^* e^{H \cdot t} \cap \widehat{\Xi}_{d,s}}{e_{\mathbb{C}^\times}(E_{0s}/Quot(d))},$$

where $\widehat{\Xi}_{d,s}$ is the Poincaré dual of $\Xi_{d,s}$ with respect to $[E_{0s}]$. As will be discussed in Sec. 3.1, E_{0s} is a flag manifold fibred over X and, hence, $g^* e^{H \cdot t}$ can be read off from the natural fibration of flag manifolds $E_{0s} \rightarrow X$.

Following [L-L-Y1, III: Sec. 5.4], in the case that $b = 1$ the above integral is reduced to the integral

$$\sum_s \int_{E_{0s}} \frac{g^* \psi^* e^{\kappa \cdot \zeta}}{e_{\mathbb{C}^\times}(E_{0s}/Quot(d))},$$

where κ is the hyperplane class in W_d . In this article, we work out all the equivariant Euler classes $e_{\mathbb{C}^\times}(E_{0s}/Quot(d))$ and hence this integral.

• Conventions and notation.

- (1) All the dimensions are *complex* dimensions unless otherwise noted.

- (2) The S^1 -actions involved in this article are induced from \mathbb{C}^\times -actions and both have the same fixed-point locus. In many places, it is more convenient to phrase things in term of \mathbb{C}^\times -action and we will not distinguish the two actions when this ambiguity causes no harm.
- (3) A locally free sheaf and its associated vector bundle are denoted the same.
- (4) An $I \times J$ matrix whose (i, j) -entry is a_{ij} is denoted by $(a_{ij})_{i,j}$ when the position of an entry is emphasized and by $[a_{ij}]_{I \times J}$ when the size of the matrix is emphasized.
- (5) From Section 2 on, the smooth curve C will be \mathbb{CP}^1 unless other noted.

2 The S^1 -action on Quot-schemes.

Let \mathcal{E}^n be a trivialized trivial bundle of rank n over C . The S^1 -action on the Quot-schemes $Quot(\mathcal{E}^n)$, the topology of the S^1 -fixed-point components, and the S^1 -weights of the normal bundle to these components are studied in this section.

2.1 The S^1 -fixed-point components.

We recall first two basic facts that will be needed in the discussion.

Fact 2.1.1 [modules over P.I.D.]. (Cf. [Ja].)

- (1) *Let D be a principal ideal domain and $D^{\oplus k}$ be a free module of rank k over D . Then any submodule of $D^{\oplus k}$ is free with basis of $\leq k$ elements.*
- (2) *If $A \in M_{k \times k}(D)$ be an $k \times k$ matrix with entries in D , then A is equivalent to a diagonal matrix $Diag\{d_1, \dots, d_s, 0, \dots, 0\}$ for some s , where $d_i \neq 0$ and $d_i | d_j$ if $i \leq j$. (Recall that $A_1, A_2 \in M_{k \times k}(D)$ are called equivalent if $A_2 = PA_1Q$ for some invertible $P, Q \in M_{k \times k}(D)$.)*

Recall the embedding $S^1 = U(1) \hookrightarrow \mathbb{C}^\times$, which acts on $C = \mathbb{C} \cup \{\infty\}$ via $z \rightarrow tz$. This lifts to an S^1 -action (i.e. a linearization) on the trivialized trivial bundle $\mathcal{E}^n \simeq \mathcal{O}_C \otimes \mathbb{C}^n$ given by $(z, v) \mapsto (t \cdot z, v)$. This induces then an S^1 -action $\mathcal{S} \mapsto t \cdot \mathcal{S}$ on the set of coherent subsheaves \mathcal{S} in \mathcal{E} by pulling back local sections: $(t \cdot s)(z) = s(t^{-1}z)$, where $s \in \mathcal{S}(U)$ and $t \cdot s \in (t \cdot \mathcal{S})(t \cdot U)$ with U an open subset in C . Since each subsheaf in \mathcal{E}^n corresponds to a point in the Quot-scheme $Quot(\mathcal{E}^n)$, this gives an S^1 -action $Quot(\mathcal{E}^n)$. (Cf. [Ak], [Ch], and [Str].)

When restricted to the set of rank- r subbundles in \mathcal{E}^n , each holomorphic subbundle in \mathcal{E}^n corresponds to a holomorphic map f from C to a Grassmannian manifold $Gr_r(\mathbb{C}^n)$ and

the above S^1 -action is the S^1 -action on $\text{Hom}(C, \text{Gr}_r(\mathbb{C}^n))$ given by $f \mapsto t \cdot f := f \circ t^{-1}$, cf. [Ak].

In the following, we first characterize the S^1 -fixed-point in $\text{Quot}(\mathcal{E}^n)$ and then give a description of the topology of the S^1 -fixed-point components in $\text{Quot}(\mathcal{E}^n)$.

Lemma 2.1.2 [coherent subsheaf of \mathcal{E}^n]. *Any coherent subsheaf \mathcal{V} of \mathcal{E}^n is locally free.*

Proof. Since any torsion section of \mathcal{V} is supported on a divisor, that support must be contained in an affine chart of the form $C - \{pt\} = \text{Spec } \mathbb{C}[u]$. Since \mathcal{E}^n is globally trivial, $\mathcal{E}^n|_U$ is the sheaf associated to a free $\mathbb{C}[u]$ -module M_U of rank n . Thus, $\mathcal{V}|_U$ is the sheaf associated to a submodule M'_U of M_U . Since $\mathbb{C}[u]$ is a principal ideal domain, M'_U must be free also. This shows that $\mathcal{V}|_U = (M'_U)^\sim$, and hence \mathcal{V} is torsion-free. Since a torsion-free coherent sheaf on a smooth curve must be locally free, this concludes the lemma. \square

Lemma 2.1.3 [S^1 -fixed-point = \mathbb{C}^\times -fixed-point]. *A coherent subsheaf \mathcal{V} of \mathcal{E}^n on C is S^1 -invariant if and only if it is \mathbb{C}^\times -invariant.*

Proof. Only need to show the only-if part. Let \mathcal{V} be a rank- r S^1 -invariant subsheaf in \mathcal{E}^n . Then \mathcal{V} is locally free from Lemma 2.1.2 and hence there exists an S^1 -invariant open dense subset $U \subset C - \{0, \infty\}$ such that $\mathcal{V}|_U$ is realized as a holomorphic rank- r subbundle of $\mathcal{E}^n|_U$ and hence as a holomorphic map from U into a Grassmannian manifold $\text{Gr}_r(\mathbb{C}^n)$. Since $\mathcal{V}|_U$ is also S^1 -invariant, this map factors via $U \rightarrow U/S^1 \rightarrow \text{Gr}_r(\mathbb{C}^n)$. Since U/S^1 is a union of open real line segments, holomorphicity implies then that any such map must be a constant map. This implies that $\mathcal{V}|_{C - \{0, \infty\}}$ is indeed a constant subsheaf in $\mathcal{E}^n|_{C - \{0, \infty\}}$ with respect to the trivialization of \mathcal{E}^n . This shows that \mathcal{V} is also \mathbb{C}^\times -invariant. \square

The following lemma strengthens Statement (2) of Fact 2.1.1 in the case of \mathbb{C}^\times -invariant submodules in $\mathbb{C}[z]^{\oplus k}$.

Lemma 2.1.4 [\mathbb{C}^\times -invariant submodule]. *Let $D = \mathbb{C}[z]$, $A = A(z) \in \text{GL}(l, \mathbb{C}[z])$ be an invertible $l \times l$ -matrix with entries in $\mathbb{C}[z]$, in Fact 2.1.1. If, furthermore, the column vectors of $A(tz)$ generate the same $\mathbb{C}[z]$ -module for all $t \in \mathbb{C}^\times$, then there exist invertible $P \in \text{GL}(l, \mathbb{C})$ and $Q(z) \in \text{GL}(l, \mathbb{C}[z])$ such that $d_i = z^{\alpha_i}$ in Fact 2.1.1 and*

$$A(z) = P \text{Diag}\{z^{\alpha_1}, \dots, z^{\alpha_l}\} Q(z),$$

where $0 \leq \alpha_1 \leq \dots \leq \alpha_l$.

Proof. By a sequence of elementary column transformations (e.g. [Ja]), which correspond to multiplications from the right by a sequence of elementary matrices in $\text{GL}(l, \mathbb{C}[z])$, together with permutations of rows, which corresponds to a multiplication from the left by a sequence of matrices in $\text{GL}(l, \mathbb{C})$, one can render $A(z)$ into a lower triangular form $B(z) = (b_{ij}(z))_{i,j}$ such that

- (1) $b_{ij}(z) = 0$ for all $i < j$,
- (2) $\deg b_{ii}(z) \leq \deg b_{i+1,i+1}(z)$ for all i , and
- (3) $\deg b_{ij}(z) < \deg b_{ii}(z)$ for all $i > j$,

where $\deg(\cdot)$ is the degree of the polynomial (\cdot) with respect to the variable z and $\deg(0) = -\infty$ by convention.

The assumption that the column vectors of $A(tz)$ generate the same $\mathbb{C}[z]$ -module for all $t \in \mathbb{C}^\times$ implies that the column vectors of $B(tz)$ generate the same $\mathbb{C}[z]$ -module as the module generated by the column vectors of $B(z)$ for all $t \in \mathbb{C}^\times$. In terms of matrices, this is equivalent to the existence of $\widehat{Q}(z, t) \in GL(k, \mathbb{C}[z])$ such that $B(tz) = B(z)\widehat{Q}(z, t)$, $t \in \mathbb{C}^\times$. The fact that both $B(tz)$ and $B(z)$ are lower triangular implies that $\widehat{Q}(z, t)$ is also lower triangular.

On the other hand, $\deg b_{ij}(tz) = \deg b_{ij}(z)$ for all i, j . This puts a strong constraint in the form of $B(z)$ in order that $B(tz) = B(z)\widehat{Q}(z, t)$ always admits a solution for $\widehat{Q}(z, t)$ in $GL(l, \mathbb{C}[z])$. Together with the Inequality (3) above: $\deg b_{ij}(tz) < \deg b_{ii}(z)$ for all $i > j$, and a tedious yet straightforward induction argument, one can show that $B(z)$ must be of the form

$$B(z) = B(1) \text{Diag}\{z^{\alpha_1}, \dots, z^{\alpha_l}\}$$

with $0 \leq \alpha_1 \leq \dots \leq \alpha_l$ and $B(1)_{ij} = 0$ if $i < j$ or $\alpha_i = \alpha_j$.

This proves the lemma. □

Proposition 2.1.5 [S^1 -fixed coherent subsheaf]. *Let \mathcal{V} be a rank r coherent subsheaf of \mathcal{E}^n on C . Then \mathcal{V} is a locally free \mathcal{O}_C -module. When \mathcal{V} is in addition S^1 -invariant, then \mathcal{V} determines a unique enlarged sheaf $\widehat{\mathcal{V}}$ such that*

- (1) $\widehat{\mathcal{V}}$ is a constant subsheaf in the globally trivialized \mathcal{E}^n of the same rank r as \mathcal{V} , (thus $\widehat{\mathcal{V}} \simeq \mathcal{O}_C^{\oplus r}$).
- (2) \mathcal{V} is a subsheaf of $\widehat{\mathcal{V}}$.
- (3) Let $\{U_0 = C - \{\infty\} = \text{Spec } \mathbb{C}[z], U_\infty = C - \{0\} = \text{Spec } \mathbb{C}[w]\}$ be an atlas of affine charts on C . Then there exists a constant re-trivialization

$$\widehat{\mathcal{V}}|_{U_0} = \mathcal{O}|_{U_0}^{\oplus r} = (\mathbb{C}[z]^{\oplus r})^\sim$$

such that

$$\mathcal{V}|_{U_0} = (\mathbb{C}[z]z^{\alpha_1} \oplus \dots \oplus \mathbb{C}[z]z^{\alpha_r})^\sim \quad \text{with } 0 \leq \alpha_1 \leq \dots \leq \alpha_r$$

with respect to this new trivialization, where $(\cdot)^\sim$ is the sheaf of modules over the affine scheme $U = \text{Spec } R$ in question associated to the R -module (\cdot) , cf. [Ha]. Similarly for $\widehat{\mathcal{V}}|_{U_\infty}$ and $\mathcal{V}|_{U_\infty}$. (Corresponding to $0 \leq \beta_1 \leq \dots, \leq \beta_r$.)

Remark 2.1.6. In other words, the local diagonal form of \mathcal{V} on an affine chart can be made compatible with the fixed trivialization of \mathcal{E}^n . The sheaf \mathcal{V} can be thought of as obtained by gluing the two independent pieces, $\mathcal{V}|_{U_0}$ and $\mathcal{V}|_{U_\infty}$, on affine charts via an isomorphism

$$(\mathcal{V}|_{U_0})|_{U_0 \cap U_\infty} \simeq (\mathbb{C}[z, z^{-1}]^{\oplus r}) \sim \xrightarrow{z \leftrightarrow w^{-1}} (\mathbb{C}[w^{-1}, w]^{\oplus r}) \simeq (\mathcal{V}|_{U_\infty})|_{U_0 \cap U_\infty}.$$

Proof of Proposition 2.1.5. For Claim (1) and Claim (2). Since \mathcal{V} is an S^1 -fixed subsheaf in \mathcal{E}^n , $\mathcal{V}|_{C-\{0, \infty\}}$ admits a unique trivial extension to a subsheaf of \mathcal{E}^n on the whole C . By construction, it has the same rank as \mathcal{V} . We shall choose $\widehat{\mathcal{V}}$ to be this extension sheaf of $\mathcal{V}|_{C-\{0, \infty\}}$. If \mathcal{V} is not contained in $\widehat{\mathcal{V}}$ as a subsheaf, then there exists an affine chart U of C such that $\mathcal{V}|_U$ has a section s not contained in $\widehat{\mathcal{V}}$. Since $\widehat{\mathcal{V}}|_U = \mathcal{V}|_U$, this implies that s must restrict to the zero-section when localized to $U - \{0, \infty\}$. In other words, it is a torsion section. This contradicts with Lemma 2.1.2, which says that \mathcal{V} must be torsion-free. Consequently, \mathcal{V} must be a subsheaf of $\widehat{\mathcal{V}}$ as well.

For Claim (3). Recall Lemma 2.1.4, with l replaced by r . Since the right multiplication of $A(z)$ by matrices in $GL(r, \mathbb{C}[z])$ does not change the $\mathbb{C}[z]$ -module generated by the column vectors of $A(z)$, while the left multiplication by a constant matrix in $GL(r, \mathbb{C})$ can be interpreted as a change of coordinates without changing the notion of being a constant section in the associated sheaf, this concludes Claim (3) and hence the proposition. \square

Remark 2.1.7. Note the above proposition says that both $\mathcal{V}|_{U_0}$ and $\mathcal{V}|_{U_\infty}$ admit diagonalizations by constant global sections in \mathcal{E}^n , but in general these two sets of diagonalizing constant sections are different. This is all right. Indeed for any two such trivializations, one over U_0 and the other over U_∞ , the localizations of both to $C - \{0, \infty\}$ are isomorphic to the free $\mathbb{C}[z, z^{-1}]$ -module of rank r and hence they glue together to form an \mathbb{C}^\times -fixed coherent sheaf on C .

The remaining problem is to decide when two diagonalized forms of \mathcal{O}_{U_0} -modules (resp. \mathcal{O}_{U_∞} -modules) of rank r determine the same submodule in $\mathcal{V}|_{U_0}$ (resp. $\mathcal{V}|_{U_\infty}$). To determine this, let two diagonal forms of $\mathbb{C}[z]$ -modules be given by

$$B_1(z) = B_1(1) \text{Diag}\{z^{\alpha_1}, \dots, z^{\alpha_r}\} \quad \text{and} \quad B_2(z) = B_2(1) \text{Diag}\{z^{\alpha_1}, \dots, z^{\alpha_r}\}$$

respectively. Then $B_1(z)$ and $B_2(z)$ determine the same $\mathbb{C}[z]$ -module if and only if there exists a $Q(z) \in GL(r, \mathbb{C}[z])$ such that $B_1(z)Q(z) = B_2(z)$. From this, one obtains that

$$\begin{aligned} Q(z) &= \text{Diag}\{z^{-\alpha_1}, \dots, z^{-\alpha_r}\} B_1(1)^{-1} B_2(1) \text{Diag}\{z^{\alpha_1}, \dots, z^{\alpha_r}\} \\ &= \text{Diag}\{z^{-\alpha_1}, \dots, z^{-\alpha_r}\} B \text{Diag}\{z^{\alpha_1}, \dots, z^{\alpha_r}\} \\ &= (z^{-\alpha_i + \alpha_j} b_{ij})_{i,j} \in GL(r, \mathbb{C}[z]), \end{aligned}$$

where $B = B_1(1)^{-1} B_2(1) = (b_{ij})_{i,j}$. This implies that $b_{ij} = 0$ if $\alpha_i > \alpha_j$. Consequently, $Q(z)$ is a *block upper triangular matrix*, whose block form is determined by the multiplicity

of elements in $(\alpha_1, \dots, \alpha_r)$. (For example, if this sequence is $(1, 1, 4, 4, 4, 7)$, then the corresponding block upper triangular matrix will have in the diagonal 2×2 -, 3×3 -, and 1×1 -blocks.) Rephrased in a more geometric way, $B_1(z)$ and $B_2(z)$ determine the same submodule if and only if they correspond to the same flag. Explicitly, the flag associated to $B(z) = B(1)\text{Diag}\{z^{\alpha_1}, \dots, z^{\alpha_r}\}$ is given as follows.

Let $B(1) = (u_1, \dots, u_r)$ be the column vectors of $B(1)$ and suppose that

$$\alpha_1 = \dots = \alpha_{j_1} < \alpha_{j_1+1} = \dots = \alpha_{j_2} < \dots < \alpha_{j_s+1} = \dots = \alpha_r,$$

then the flag associated to $B(z)$ is given by

$$(\text{Span}\{u_1, \dots, u_{j_1}\} \subset \text{Span}\{u_1, \dots, u_{j_2}\} \subset \dots \subset \text{Span}\{u_1, \dots, u_{j_s}\} \subset \mathbb{C}^r) \in \text{Fl}_{j_1, \dots, j_s}(\mathbb{C}^r),$$

where note that the last \mathbb{C}^r should be identified with $\widehat{\mathcal{V}} \in \text{Gr}_r(\mathbb{C}^n)$.

Definition 2.1.8 [admissible pair of sequences]. Recall the Hilbert polynomial $P = P(t) = (n-r)t + d + (n-r)$. Then $(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r)$ is called an *admissible pair of sequences* with respect to $P(t)$ if it satisfies

- (1) $0 \leq \alpha_1 \leq \dots \leq \alpha_r$, $0 \leq \beta_1 \leq \dots \leq \beta_r$, and
- (2) $(\alpha_1 + \dots + \alpha_r) + (\beta_1 + \dots + \beta_r) = d$.

From the above discussions and the fact that, for any element in $\text{Fl}_{j_1, \dots, j_s}(\mathbb{C}^r)$, one can always construct a $B(z)$ in the above form such that $B(1)$ is mapped to that flag by the above correspondence, one concludes the following proposition.

Theorem 2.1.9 [topology of S^1 -fixed-point locus]. Let $(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r)$ be an admissible pair of sequences of non-negative integers, $\text{Fl}_{j_1, \dots, j_s, r}(\mathbb{C}^n)$ and $\text{Fl}_{j'_1, \dots, j'_s, r}(\mathbb{C}^n)$ be the flag manifold associated to the multiplicity of elements in $(\alpha_1, \dots, \alpha_r)$ and $(\beta_1, \dots, \beta_r)$ respectively, as discussed above. Let

$$\text{Fl}_{j_1, \dots, j_s, r}(\mathbb{C}^n) \rightarrow \text{Gr}_r(\mathbb{C}^n) \quad \text{and} \quad \text{Fl}_{j'_1, \dots, j'_s, r}(\mathbb{C}^n) \rightarrow \text{Gr}_r(\mathbb{C}^n)$$

be the natural projections. Then the subset $F_{\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r}$ of the S^1 -fixed-point locus that is associated to $(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r)$ is connected and is given by the fiber product

$$\text{Fl}_{j_1, \dots, j_s, r}(\mathbb{C}^n) \times_{\text{Gr}_r(\mathbb{C}^n)} \text{Fl}_{j'_1, \dots, j'_s, r}(\mathbb{C}^n).$$

Remark 2.1.10.

- (1) The base $\text{Gr}_r(\mathbb{C}^n)$ corresponds to the choices of $\widehat{\mathcal{V}}$. The fiber over a point in the base is the product of two flag manifolds that gives all possible S^1 -fixed subsheaves \mathcal{V} of $\widehat{\mathcal{V}}$ that have the specified Hilbert polynomial of $\mathcal{E}^n/\mathcal{V}$ associated to $(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r)$.

- (2) The dimension of components of S^1 -fixed-point locus varies from component to component. When the S^1 -fixed-point locus is non-empty, the dimension of each component is bounded below by the dimension of the target Grassmannian manifold $Gr_r(\mathbb{C}^n)$ that one starts with. The above expression implies that the only case that a fixed-point component has the dimension the same as that of $Gr_r(\mathbb{C}^n)$ is when that component itself is homeomorphic to $Gr_r(\mathbb{C}^n)$. This happens exactly when $\alpha_1 = \cdots = \alpha_r$ and $\beta_1 = \cdots = \beta_r$. Such $(\alpha_1, \cdots, \alpha_r ; \beta_1, \cdots, \beta_r)$ is admissible only for special Hilbert polynomials.

2.2 The S^1 -weight system of the tangent space of Quot-scheme at an S^1 -fixed-point component.

After recalling some related facts in the preparatory remarks, we compute the S^1 -weights and their multiplicities of the tangent space of the Quot-scheme $Quot(\mathcal{E}^n)$ at an S^1 -fixed-point.

Preparatory remarks.

Recall (cf. [Ch], [H-L], and [Kol]) that the tangent space of Quot-scheme at a point is given by

$$T_{(\mathcal{E}^n \rightarrow \mathcal{E}^n/\mathcal{V})} Quot_P(\mathcal{E}^n) \simeq Hom_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V}).$$

When $(\mathcal{E}^n \rightarrow \mathcal{E}^n/\mathcal{V})$ is an S^1 -fixed-point, S^1 acts both on \mathcal{V} and $\mathcal{E}^n/\mathcal{V}$. The S^1 -action on $T_{(\mathcal{E}^n \rightarrow \mathcal{E}^n/\mathcal{V})} Quot_P(\mathcal{E}^n)$ is translated to the S^1 -action on $Hom_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V})$ by conjugations: $f \mapsto t \cdot f \cdot t^{-1}$ for $t \in S^1$.

Recall the inclusion of S^1 -invariant subsheaves $\mathcal{V} \subset \widehat{\mathcal{V}}$ in \mathcal{E}^n . One thus has a natural morphism $\mathcal{E}^n/\mathcal{V} \rightarrow \mathcal{E}^n/\widehat{\mathcal{V}}$. Since $\widehat{\mathcal{V}}$ is a constant rank- r subbundle in \mathcal{E}^n , $\mathcal{E}^n/\widehat{\mathcal{V}}$ is a rank- $(n-r)$ trivial bundle on C . Since $(\mathcal{E}^n/\mathcal{V})|_{C-\{0,\infty\}} \xrightarrow{\sim} (\mathcal{E}^n/\widehat{\mathcal{V}})|_{C-\{0,\infty\}}$ from the restriction of the above morphism and the restriction to the stalks

$$(\mathcal{E}^n/\mathcal{V})_0 \rightarrow (\mathcal{E}^n/\widehat{\mathcal{V}})_0 \quad (\text{resp. } (\mathcal{E}^n/\widehat{\mathcal{V}})_\infty \rightarrow (\mathcal{E}^n/\mathcal{V})_\infty)$$

at 0 (resp. ∞) given by

$$(\mathbb{C}[z]^{\oplus n}/(\mathbb{C}[z]z^{\alpha_1} \oplus \cdots \oplus \mathbb{C}[z]z^{\alpha_r})) \otimes_{\mathcal{O}_C(U_0)} \mathcal{O}_{C,0} \rightarrow \mathbb{C}[z]^{\oplus(n-r)} \otimes_{\mathcal{O}_C(U_0)} \mathcal{O}_{C,0}$$

(resp.

$$(\mathbb{C}[z]^{\oplus n}/(\mathbb{C}[z]z^{\beta_1} \oplus \cdots \oplus \mathbb{C}[z]z^{\beta_r})) \otimes_{\mathcal{O}_C(U_\infty)} \mathcal{O}_{C,\infty} \rightarrow \mathbb{C}[z]^{\oplus(n-r)} \otimes_{\mathcal{O}_C(U_\infty)} \mathcal{O}_{C,\infty})$$

are surjective, the morphism $\mathcal{E}^n/\mathcal{V} \rightarrow \mathcal{E}^n/\widehat{\mathcal{V}}$ is surjective and one has the following split exact sequence of torsion-part/locally-free-part decomposition

$$0 \rightarrow \widehat{\mathcal{V}}/\mathcal{V} \rightarrow \mathcal{E}^n/\mathcal{V} \rightarrow \mathcal{E}^n/\widehat{\mathcal{V}} \rightarrow 0.$$

Since any constant rank- $(n-r)$ subbundle in \mathcal{E}^n that is transverse to $\widehat{\mathcal{V}}$ is S^1 -invariant and is mapped isomorphically to $\mathcal{E}^n/\widehat{\mathcal{V}}$, the above decomposition is also S^1 -equivariant.

The S^1 -action on $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V})$ when $(\mathcal{E}^n \rightarrow \mathcal{E}^n/\mathcal{V})$ is an S^1 -fixed-point.

The above discussion gives an S^1 -invariant decomposition of the tangent space to the Quot-scheme at an S^1 -fixed-point :

$$\begin{aligned} \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V}) &= \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \widehat{\mathcal{V}}/\mathcal{V} \oplus \mathcal{E}^n/\widehat{\mathcal{V}}) \\ &= \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{F}_0) \oplus \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{F}_\infty) \oplus \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\widehat{\mathcal{V}}), \end{aligned}$$

where \mathcal{F}_0 (resp. \mathcal{F}_∞) is the torsion subsheaf of $\mathcal{E}^n/\mathcal{V}$ supported at 0 (resp. ∞). We shall now study the three summands in the decomposition and their S^1 -weight system, denoted by Wt_1 , Wt_2 , and Wt_3 respectively. Due to the tediousness of the discussion, we itemize the argument below.

• *The summands $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{F}_0)$ and $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{F}_\infty)$:*

- (1) These two components can be calculated via the restriction of the former to U_0 and the latter to U_∞ . The problem is reduced then to the study of the group of homomorphisms of $\mathbb{C}[z]$ -modules and the S^1 -action on it. Explicitly,

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{F}_0) = \text{Hom}_{\mathbb{C}[z]}(\oplus_{j=1}^r \mathbb{C}[z] \cdot z^{\alpha_j}, \oplus_{i=1}^r \mathbb{C}[z] \cdot \bar{e}_{0i}),$$

where $z^{\alpha_i} \cdot \bar{e}_{0i} = 0$ for $i = 1, \dots, r$, and

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{F}_\infty) = \text{Hom}_{\mathbb{C}[w]}(\oplus_{j=1}^r \mathbb{C}[w] \cdot w^{\beta_j}, \oplus_{i=1}^r \mathbb{C}[w] \cdot \bar{e}_{\infty i}),$$

where $w^{\beta_i} \cdot \bar{e}_{\infty i} = 0$ for $i = 1, \dots, r$.

- (2) *Computation of the weight systems Wt_1 and Wt_2 :*

- (2.1) Realize an element in $\oplus_{i=1}^r \mathbb{C}[z] \cdot \bar{e}_{0i}$ as a column vector and let

$$f(z) = (f_{ij}(z))_{i,j} \in \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{F}_0)$$

with respect to the local bases $(z^{\alpha_1}, \dots, z^{\alpha_r})$ and $(\bar{e}_{01}, \dots, \bar{e}_{0r})$ for \mathcal{V} and \mathcal{F}_0 respectively. Then $\deg f_{ij}(z) < \alpha_i$ and (cf. the similar computation for the weight system Wt_3 below),

$$(t \cdot f)(z) = (t^{\alpha_j} f_{ij}(t^{-1}z))_{i,j}, \quad t \in S^1.$$

Thus, the rank-1 S^1 -eigen-spaces in $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{F}_0)$ can be chosen to be generated by

$$\varepsilon_{ij}[\mu_{ij}^0] = \varepsilon_{ij}[\mu_{ij}^0](z) := (f_{kl}(z))_{k,l}, \quad \text{where } f_{kl}(z) = \begin{cases} 0 & \text{if } (k,l) \neq (i,j), \\ z^{\alpha_j - \mu_{ij}^0} & \text{if } (k,l) = (i,j). \end{cases}$$

whose S^1 -weight is μ_{ij}^0 that satisfies

$$\alpha_j - \alpha_i < \mu_{ij}^0 \leq \alpha_j.$$

From this, one has

$$Wt_1 = \bigsqcup_{i,j=1}^r ((\alpha_j - \alpha_i, \alpha_j] \cap \mathbb{Z})$$

with the multiplicity of a given integer in the set being the number of times it appears in the disjoint union.

(2.2) Rewrite $(\alpha_1, \dots, \alpha_r)$ as

$$0 \leq \underbrace{a_1 (= \alpha_1)}_{m_1} < \dots < \underbrace{a_k (= \alpha_r)}_{m_k},$$

with the multiplicity indicated. For an interval $I \subset \mathbb{R}$, let χ_I be the characteristic function $\chi_I(x) = 1$, if $x \in I$, and $= 0$, otherwise. Let $\chi^A = \sum_{i=1}^k m_i \chi_{(-a_i, 0]}$ and define χ_m^A by $\chi_m^A(x) = \sum_{j=1}^k m_j \chi^A(x - a_j)$. Then the multiplicity for $\mu \in Wt_1$ is given by $\chi_m^A(\mu)$.

(2.3) Realize an element in $\oplus_{i=1}^r \mathbb{C}[w] \cdot \bar{e}_{\infty i}$ as a column vector and let

$$g(w) = (g_{ij}(w))_{i,j} \in \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{F}_\infty).$$

Then $\deg g_{ij}(w) < \beta_i$ and (cf. the similar computation for the weight system Wt_3 below),

$$(t \cdot g)(w) = (t^{-\beta_j} g_{ij}(tw))_{i,j}, \quad t \in S^1.$$

Thus, the rank-1 S^1 -eigen-spaces in $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{F}_\infty)$ can be chosen to be generated by

$$\varepsilon_{ij}[\mu_{ij}^\infty] = \varepsilon_{ij}[\mu_{ij}^\infty](z) := (g_{kl}(z))_{k,l}, \quad \text{where } g_{kl}(w) = \begin{cases} 0 & \text{if } (k,l) \neq (i,j), \\ w^{\beta_j + \mu_{ij}^\infty} & \text{if } (k,l) = (i,j). \end{cases}$$

whose S^1 -weight is μ_{ij}^∞ that satisfies

$$-\beta_j \leq \mu_{ij}^\infty < \beta_i - \beta_j.$$

From this, one has

$$Wt_2 = \bigsqcup_{i,j=1}^r ([-\beta_j, \beta_i - \beta_j) \cap \mathbb{Z})$$

with the same rule of counting multiplicity as for Wt_1 .

(2.4) Rewrite $(\beta_1, \dots, \beta_r)$ as

$$0 \leq \underset{n_1}{b_1 (= \beta_1)} < \dots < \underset{n_l}{b_l (= \beta_r)}$$

with the multiplicity indicated. Let $\chi^B = \sum_{i=1}^l n_i \chi_{[0, \beta_i]}$ and define χ_m^B by $\chi_m^B(x) = \sum_{j=1}^l n_j \chi^A(x + b_j)$. Then the multiplicity for $\mu \in Wt_2$ is given by $\chi_m^B(\mu)$.

• *The summand $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\widehat{\mathcal{V}})$:*

- (1) Since $\mathcal{E}^n/\widehat{\mathcal{V}}$ is represented by a rank- $(n-r)$ constant subbundle in \mathcal{E}^n transverse to $\widehat{\mathcal{V}}$, it can be further decomposed into a direct sum of constant line subbundles in \mathcal{E}^n . Since all the bundles involved are constant, the decomposition of the quotient $\mathcal{E}^n/\widehat{\mathcal{V}} = \mathcal{O}_C^{\oplus(n-r)}$ is S^1 -invariant. Recall that S^1 acts on \mathcal{E}^n and hence on $\mathcal{E}^n/\widehat{\mathcal{V}}$ via the trivial linearization. With respect to this decomposition, one has

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\widehat{\mathcal{V}}) = \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{O}_C^{\oplus(n-r)}) = H^0(C, \mathcal{V}^\vee)^{\oplus(n-r)}.$$

Remark. A connected component of the fixed-point locus can be stratified by subsets labelled by the isomorphism classes of vector bundles associated to the S^1 -invariant subsheaves \mathcal{V} in \mathcal{E}^n . There can be more than one strata for a connected component.

(2) *Computation of the weight system Wt_3 :*

(2.1) Recall the S^1 -invariant decompositions

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\widehat{\mathcal{V}}) = \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{O}_C^{\oplus(n-r)}) = \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{O}_C)^{\oplus(n-r)}.$$

The existence of such S^1 -invariant decomposition implies that the sought-for S^1 -weight system for $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\widehat{\mathcal{V}})$ consists of $(n-r)$ -many copies of the S^1 -weight system for $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{O}_C)$.

(2.2) Let $f \in \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{O}_C)$. Then, as a morphism of sheaves on C , f is described by a pair

$$\begin{aligned} (f_0, f_\infty) &\in \text{Hom}_{\mathcal{O}_C(U_0)}(\mathcal{V}(U_0), \mathcal{O}_C(U_0)) \times \text{Hom}_{\mathcal{O}_C(U_\infty)}(\mathcal{V}(U_\infty), \mathcal{O}_C(U_\infty)) \\ &= \text{Hom}_{\mathbb{C}[z]}(\oplus_{i=1}^r \mathbb{C}[z] \cdot z^{\alpha_i}, \mathbb{C}[z]) \times \text{Hom}_{\mathbb{C}[w]}(\oplus_{i=1}^r \mathbb{C}[w] \cdot w^{\beta_i}, \mathbb{C}[w]) \end{aligned}$$

such that $f_0|_{U_0 \cap U_\infty} = f_\infty|_{U_0 \cap U_\infty}$.

(2.3) Recall the proof of Theorem 5.3 in [Ko], which says in our case that *the weight system of the tangent bundle at an S^1 -fixed-point depends only on the connected component of the fixed-point locus*. Thus, to compute the weight system one can choose the S^1 -invariant subsheaf \mathcal{V} in \mathcal{E}^n such that the two diagonalized local pieces on affine charts U_0 and U_∞ match (i.e. \mathcal{V} becomes the direct sum of appropriate ideal sheaves in constant line subbundles in \mathcal{E}^n). From the previous discussions, there are many - even continuous families of - such \mathcal{V} . However, as will be clear from the explicit expression that the weight system obtained is indeed independent of which such \mathcal{V} is chosen for the computation, as long as they belong to the same fixed-point component. This gives a consistency check of the method.

(2.4) Let \mathcal{V} be an S^1 -invariant subsheaf of \mathcal{E}^n such that the two local diagonalizations match and suppose that α_i is matched with $\beta_{i'}$, $i = 1, \dots, r$. Then \mathcal{V} is decomposed into a direct sum $\bigoplus_{i=1}^r \mathcal{I}_{\alpha_i, \beta_{i'}}$, where $\mathcal{I}_{\alpha_i, \beta_{i'}}$ is a subsheaf in a constant line subbundle $\simeq \mathcal{O}_C$ in \mathcal{E}^n with the local data as a sheaf of \mathcal{O}_C -module:

$$\begin{array}{ccccccc} \text{on } U_0 & & \text{on } U_0 \cap U_\infty & & \text{on } U_0 \cap U_\infty & & \text{on } U_\infty \\ \mathbb{C}[z] \cdot z^{\alpha_i} & \rightarrow & \mathbb{C}[z, z^{-1}] & \xleftrightarrow{z \leftrightarrow 1/w} & \mathbb{C}[w, w^{-1}] & \leftrightarrow & \mathbb{C}[w] \cdot w^{\beta_{i'}} \end{array} .$$

The vector bundle associated to \mathcal{V} is isomorphic to $\bigoplus_{i=1}^r \mathcal{O}(-\alpha_i - \beta_{i'})$ and $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{O}_C)$ is further decomposed into an S^1 -invariant direct sum

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{O}_C) = \bigoplus_{i=1}^r \text{Hom}_{\mathcal{O}_C}(\mathcal{I}_{\alpha_i, \beta_{i'}}, \mathcal{O}_C) .$$

(2.5) For simplicity of notation, we shall drop temporarily the indices i and i' . At the level of sheaf morphisms, the data that encodes $f \in \text{Hom}_{\mathcal{O}_C}(\mathcal{I}_{\alpha, \beta}, \mathcal{O}_C)$ is given by a pair

$$\begin{aligned} (f_0, f_\infty) &\in \text{Hom}_{\mathbb{C}[z]}(\mathbb{C}[z] \cdot z^\alpha, \mathbb{C}[z]) \times \text{Hom}_{\mathbb{C}[w]}(\mathbb{C}[w] \cdot w^\beta, \mathbb{C}[w]) \quad , \\ & (z^\alpha \xrightarrow{f_0} h_0(z), w^\beta \xrightarrow{f_\infty} h_\infty(w)) \end{aligned}$$

such that the following matching condition holds

$$z^{-\alpha} h_0(z) = w^{-\beta} h_\infty(w) \quad \text{under } z \rightarrow 1/w .$$

Consequently,

$$\begin{aligned} \text{Hom}_{\mathcal{O}_C}(\mathcal{I}_{\alpha, \beta}, \mathcal{O}_C) &= \{ (h_0(z), h_\infty(w)) \mid \deg h_0(z) \leq \alpha + \beta \text{ and } h_\infty(w) = w^{\alpha+\beta} h_0(1/w) \} \\ &\simeq H^0(C, \mathcal{I}^\vee) = H^0(C, \mathcal{O}_C(\alpha + \beta)) . \end{aligned}$$

(2.6) The S^1 -action on $\text{Hom}_{\mathcal{O}_C}(\mathcal{I}_{\alpha, \beta}, \mathcal{O}_C)$ is given by $f \mapsto t \cdot f$, where $t \cdot f$ is the composition of the following conjugation of $f = (h_0(z), h_\infty(w))$:

$$\begin{array}{l} \text{on } U_0 : \\ s_0(z) z^\alpha \xrightarrow{t^{-1}} s_0(tz) (tz)^\alpha = t^\alpha s_0(tz) \cdot z^\alpha \xrightarrow{f} t^\alpha s_0(tz) h_0(z) \xrightarrow{t} t^\alpha s_0(tt^{-1}z) h_0(t^{-1}z) = t^\alpha s_0(z) h_0(t^{-1}z) \\ \text{on } U_\infty : \\ s_\infty(w) w^\beta \xrightarrow{t^{-1}} s_\infty(t^{-1}w) (t^{-1}w)^\beta = t^{-\beta} s_\infty(t^{-1}w) \cdot w^\beta \xrightarrow{f} t^{-\beta} s_\infty(t^{-1}w) h_\infty(w) \xrightarrow{t} t^{-\beta} s_\infty(t^{-1}tw) h_\infty(tw) = t^{-\beta} s_\infty(w) h_\infty(tw) . \end{array}$$

One can check directly that if (f_0, f_∞) satisfies the matching condition, then so does $(t \cdot f)_0, (t \cdot f)_\infty$. Consequently,

$$f = (h_0(z), h_\infty(w)) \xrightarrow{t} t \cdot f = (t^\alpha h_0(t^{-1}z), t^{-\beta} h_\infty(tw)) \text{ on } \text{Hom}_{\mathcal{O}_C}(\mathcal{I}_{\alpha,\beta}, \mathcal{O}_C).$$

If f is an invariant direction of the S^1 -action on $\text{Hom}_{\mathcal{O}_C}(\mathcal{I}_{\alpha,\beta}, \mathcal{O}_C)$, then $t \cdot f = t^\mu f$ for some $\mu \in \mathbb{Z}$. From the above expression, this means that

$$(t^\alpha h_0(t^{-1}z), t^{-\beta} h_\infty(tw)) = (t^\mu h_0(z), t^\mu h_\infty(w)) \text{ for all } t,$$

which implies that

$$f = f_\mu := (h_0(z), h_\infty(w)) = (cz^{\alpha-\mu}, cw^{\beta+\mu}).$$

From this, one concludes that

$$-\beta \leq \mu \leq \alpha, \quad \mu \in \mathbb{Z};$$

with the associated weight subspace spanned by f_μ .

(2.7) Resume the indices (i, i') for $\mathcal{I}_{\alpha_i, \beta_{i'}}$. Then

Lemma [weight subsystem Wt_3].

(1) Let Wt'_3 be the system of weights of the S^1 -action on $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{O}_C)$. Then the weight system Wt_3 for the S^1 -action on

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\widehat{\mathcal{V}})$$

is given by $Wt_3 = (n-r) Wt'_3$, i.e. same set of integers μ as in Wt'_3 but with multiplicity $m_\mu = (n-r)m'_\mu$,

(2) Wt'_3 is given by

$$Wt'_3 = \bigsqcup_{i=1}^r ([-\beta_{i'}, \alpha_i] \cap \mathbb{Z}).$$

Recall $(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r)$ rewritten as

$$\begin{array}{ccccccc} 0 & \leq & a_1 & < & \dots & < & a_k (= \alpha_r); & & 0 & \leq & b_1 & < & \dots & < & b_l (= \beta_r) \\ & & m_1 & & \dots & & m_k & & & & n_1 & & \dots & & n_l \end{array}$$

with the multiplicity indicated. Then any $\nu \in [-\beta_r, \alpha_r] \cap \mathbb{Z}$ is in Wt'_3 . Its multiplicity m'_ν in Wt'_3 is given by

$$m'_\mu = \begin{cases} n_l + \dots + n_j & \text{if } -b_j \leq \mu < -b_{j-1}. \\ r & \text{if } -b_1 \leq \mu \leq a_1, \\ m_k + \dots + m_j & \text{if } a_{j-1} < \mu \leq a_j, \end{cases}$$

From this expression, it is clear that Wt_3 depends only on $(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r)$ and hence only on the connected component of the S^1 -fixed-point locus, as it should.

Proof of Lemma. Consider the two sets of lattice points in $\mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{R}^2$:

$$A = \{(\alpha_i, r - i + 1) \mid i = 1, \dots, r\} \quad \text{and} \quad B = \{(-\beta_i, r - i + 1) \mid i = 1, \dots, r\},$$

and the r -many line segments connecting $(-\beta_{i'}, r - i' + 1)$ and $(\alpha_i, r - i + 1)$. Let π be the projection of \mathbb{R}^2 to the horizontal axis $L \supset \mathbb{Z}$. Then, for an integer $\mu \in \mathbb{Z} \subset L$, the multiplicity m_μ of μ in Wt_3' is the same as the number of the line segments above whose projection into L contain μ . Thus, $m_\mu > 0$ if and only if $\mu \in [-\beta_r, \alpha_r]$. To read off m_μ , one combs the collection of line segments so that each line segment becomes a three-edged-path with the first and the third edge horizontal and the middle one vertical and contained in the vertical axis, cf. FIGURE 2-2-1. From this,

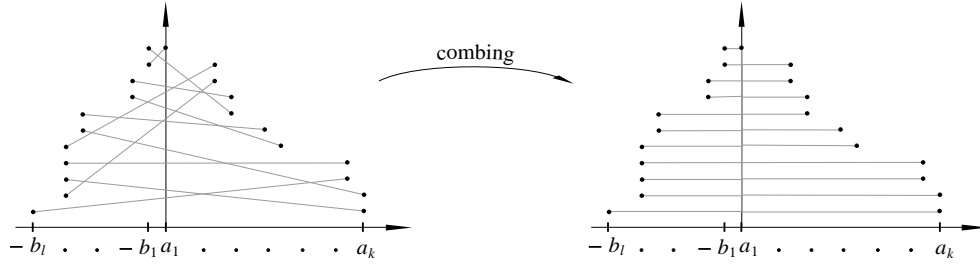


FIGURE 2-2-1. The mutiplicity of μ and the combing of the line segments.

one concludes m_μ as stated in the Lemma. This concludes the proof.

□

To summarize :

Theorem 2.2.1 [S^1 -weight]. *The S^1 -weights on the tangent space*

$$T_{(\mathcal{E}^n \rightarrow \mathcal{E}^n/\mathcal{V})} \text{Quot}_P(\mathcal{E}^n) \simeq \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V})$$

of Quot-scheme $\text{Quot}(\mathcal{E}^n)$ at an S^1 -fixed-point $(\mathcal{E}^n \rightarrow \mathcal{E}^n/\mathcal{V})$ are the disjoint union of Wt_1 , Wt_2 , and Wt_3 , as given above.

This concludes the computation of the weight system. We now turn to the combinatorics of this system.

2.3 Combinatorics of the S^1 -weight system and the multiplicity of 0.

A generating function for the multiplicity of weights in Wt_3 is immediate, following same argument as in the counting of the states at various levels in conformal field theory, e.g. [G-S-W]. An example is given by the following formal function

$$\prod_{j=0}^{\infty} \frac{1}{1 - q_0^{n-r} \cdots q_j^{n-r} s^j t}.$$

It remains unclear to us whether the weight systems Wt_1 and Wt_2 also have elegant generating functions; nevertheless they can be obtained from the following manipulations.

- *The weight subsystem Wt_1 :*

- (1) Consider the formal expansion

$$\left(\prod_{j=0}^{\infty} \frac{1}{1 - q_{-j} \cdots q_0 s^j t v_j} \right) \Big|_{q_0=1} = \sum_{k,l,P} A_{k,l,P}^{(1)}(\mathbf{q}) A_P^{(2)}(\mathbf{v}) s^k t^l,$$

where \mathbf{q} and \mathbf{v} represent collectively the two sets of variables q_i and v_i respectively. Note that both $A_{k,l,P}^{(1)}(\mathbf{q})$ and $A_P^{(2)}(\mathbf{v})$ are monomials.

- (2) Do the substitutions

$$A_P^{(2)}(\mathbf{v}) \xrightarrow{v_j \rightarrow A_{k,l,P}^{(1)}(\mathbf{q}_{i \rightarrow i+j+1})} \widehat{A}_P^{(2)}(\mathbf{q}),$$

where $\mathbf{q}_{i \rightarrow i+j+1}$ means that q_i is replaced by q_{i+j+1} for all i . The result $\widehat{A}_P^{(2)}(\mathbf{q})$ is a monomial in \mathbf{q} and the multiplicity of $j \in \mathbb{Z}$ is n_j if $q_j^{n_j}$ appears as a primary factor of $\widehat{A}_P^{(2)}(\mathbf{q})$.

(Cf. See Example 2.3.2 below.)

- *The weight subsystem Wt_2 :*

- (1) Consider the formal expansion

$$\left(\prod_{j=0}^{\infty} \frac{1}{1 - q_0 \cdots q_j s^j t v_j} \right) \Big|_{q_0=1} = \sum_{k,l,P} B_{k,l,P}^{(1)}(\mathbf{q}) B_P^{(2)}(\mathbf{v}) s^k t^l.$$

(2) Do the substitutions

$$B_P^{(2)}(\mathbf{v}) \xrightarrow{v_j \rightarrow B_{k,l,P}^{(1)}(\mathbf{q}_{i \rightarrow i-j-1})} \widehat{B_P^{(2)}}(\mathbf{q}),$$

where $\mathbf{q}_{i \rightarrow i-j-1}$ means that q_i is replaced by q_{i-j-1} for all i . The result $\widehat{B_P^{(2)}}(\mathbf{q})$ is a monomial in \mathbf{q} and the multiplicity of $j \in \mathbb{Z}$ is n_j if $q_j^{n_j}$ appears as a primary factor of $\widehat{B_P^{(2)}}(\mathbf{q})$.

Remark 2.3.1.

- (1) The powers k and l and the monomials $A_{k,l,P}^{(1)}(\mathbf{q})$ and $A_P^{(2)}(\mathbf{v})$ are related as follows. $A_P^{(2)}(\mathbf{v})$ is the monomial that encodes the partition P of k into the summation l -many non-negative integers. Corresponding to P is a conjugate partition \widehat{P} . The Young diagram associated to P is conjugate to that associated to \widehat{P} . The monomial $A_{k,l,P}^{(1)}(\mathbf{q})$ is determined by the partition \widehat{P} . Similarly for $B_{k,l,P}^{(1)}(\mathbf{q})$ and $B_P^{(2)}(\mathbf{v})$. (Cf. Example 2.3.2 below.)
- (2) Item (1) above implies that for a distinguished S^1 -fixed-point component $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$, the subweight system Wt_1 is generated completely by the Young diagram associated to $d = \alpha_1 + \dots + \alpha_r$ as a partition of d by a “partial tensor” with the conjugate Young diagram, as illustrated in FIGURE 2-3-1.

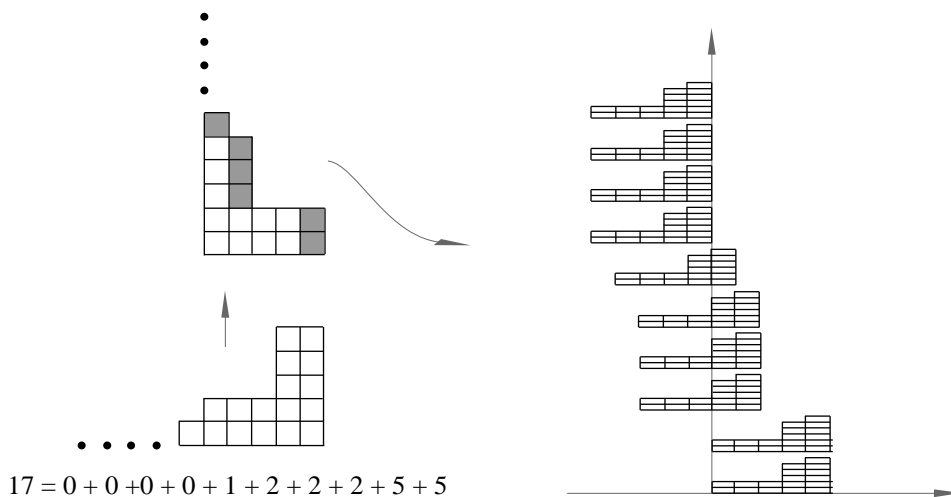


FIGURE 2-3-1. Generation of Wt_1 from a single Young diagram. In the final diagram, the vertical scale is only 1/4 of the horizontal scale.

Example 2.3.2 [weight computation for Wt_1]. Consider the expansion

$$\left(\prod_{j=0}^{\infty} \frac{1}{1 - q_{-j} \cdots q_0 s^j t v_j} \right) \Big|_{q_0=1} = \sum_{k,l,P} A_{k,l,P}^{(1)}(\mathbf{q}) A_P^{(2)}(\mathbf{v}) s^k t^l.$$

Consider, for example, the case $r = 10$ and $d = 17$. Then, to determine the weight subsystem Wt_1 for the normal bundle to the component $F_{0,0,0,0,1,2,2,2,5,5;0,\dots,0}$ in E_0 , one only needs to look at the (unique) term in the expansion with $A_P^{(2)}(\mathbf{v}) = v_0^4 v_1 v_2^3 v_5^2$, corresponding to the partition $P : 17 = 0 + 0 + 0 + 0 + 1 + 2 + 2 + 2 + 5 + 5$:

$$\begin{aligned} & (A_{17,10,P}^{(1)}(\mathbf{q}) A_P^{(2)}(\mathbf{v}) s^{17} t^{10}) \Big|_{q_0=1} \\ &= q_{-1} (q_{-2} q_{-1})^3 (q_{-5} q_{-4} q_{-3} q_{-2} q_{-1})^2 v_0^4 v_1 v_2^3 v_5^2 s^{17} t^{10} \\ &= q_{-5}^2 q_{-4}^2 q_{-3}^2 q_{-2}^5 q_{-1}^6 v_0^4 v_1 v_2^3 v_5^2 s^{17} t^{10}. \end{aligned}$$

Observe that the conjugate partition $\widehat{P} : 17 = 0 + 0 + 0 + 0 + 0 + 2 + 2 + 2 + 5 + 6$ is encoded in the monomial in q 's. Now do the substitution with the rule of shifting the indices as given above:

$$\begin{aligned} & v_0^4 v_1^1 v_2^3 v_5^2 \longrightarrow \\ & (q_{-4}^2 q_{-3}^2 q_{-2}^2 q_{-1}^5 q_0^6)^4 (q_{-3}^2 q_{-2}^2 q_{-1}^2 q_0^5 q_1^6)^1 (q_{-2}^2 q_{-1}^2 q_0^2 q_1^5 q_2^6)^3 (q_1^2 q_2^2 q_3^2 q_4^5 q_5^6)^2 \Big|_{q_0=1} \\ &= q_{-4}^8 q_{-3}^{10} q_{-2}^{16} q_{-1}^{28} q_1^{25} q_2^{22} q_3^4 q_4^{10} q_5^{12}, \end{aligned}$$

where some of the indices are boldfaced to make the pattern manifest. Let α be the generator of $H_{S^1}^*(pt)$. Then the S^1 -weights in Wt_1 for the normal bundle to $F_{0,0,0,0,1,2,2,2,5,5;0,\dots,0}$ is

$$8(-4\alpha), 10(-3\alpha), 16(-2\alpha), 28(-\alpha), 25(\alpha), 22(2\alpha), 4(3\alpha), 10(4\alpha), 12(5\alpha).$$

□

Though it can be obtained also from Sec. 2.2, the following lemma follows immediately from the combinatorics of the weight system discussed in this subsection.

Lemma 2.3.3 [multiplicity of 0]. *The multiplicity of 0 in the S^1 -weight system Wt to the restriction of the tangent bundle $T_* \text{Quot}_P(\mathcal{E}^n)$ to $F_{\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r}$ is equal to $\dim F_{\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r}$. Consequently, the S^1 -weight system of the normal bundle to $F_{\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r}$ is exactly the subsystem of non-zero weights in Wt .*

Corollary 2.3.4 [e_{S^1} invertible]. *Let E be any of the S^1 -fixed-point component in $\text{Quot}_P(\mathcal{E}^n)$. Then the S^1 -weights of the normal bundle $\nu_E(\text{Quot}_P \mathcal{E}^n)$ to E are all nonzero. Consequently $e_{S^1}(\nu/\text{Quot}_P(\mathcal{E}^n))$ is invertible in $A^*(E)(\alpha)$, where α is a generator of the ring $H_{S^1}^*(pt)$.*

Proof of Lemma 2.3.3. Recall the three subsystems $Wt = Wt_1 + Wt_2 + Wt_3$ from Sec. 2.2. The multiplicity of 0 in Wt_3 is $(n - r)r$. For the weight subsystem Wt_1 , from the above

discussion on the Young tableau associated to Wt_1 and also the characteristic function χ_m^A for Wt_1 defined in Sec. 2.2, one has that the multiplicity of 0 in Wt_1 is given by

$$m_{k-1}m_k + m_{k-2}(m_k + m_{k-1}) + \cdots + m_1(m_l + \cdots + m_2).$$

Similarly the multiplicity of 0 in the weight subsystem Wt_2 is given by

$$n_{l-1}n_l + n_{l-2}(n_l + n_{l-1}) + \cdots + n_1(n_l + \cdots + n_2).$$

Consequently the multiplicity of 0 in Wt is given by

$$\begin{aligned} & m_{k-1}m_l + m_{k-2}(m_k + m_{k-1}) + \cdots + m_1(m_l + \cdots + m_2) \\ & + n_{l-1}n_l + n_{l-2}(n_l + n_{l-1}) + \cdots + n_1(n_l + \cdots + n_2) + (n-r)r. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \dim F_{\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta} \\ & = \dim(Fl_{m_1, m_1+m_2, \dots, m_1+\cdots+m_{k-1}, r}(\mathbb{C}^n)) + \dim(Fl_{n_1, n_1+n_2, \dots, n_1+\cdots+n_{l-1}, r}(\mathbb{C}^n)) - \dim(Gr_r(\mathbb{C}^n)) \\ & = (n-r)r + (m_1 + \cdots + m_{k-1})m_k + (m_1 + \cdots + m_{k-2})m_{k-1} + \cdots + (m_1 + m_2)m_3 + m_1m_2 \\ & \quad + (n-r)r + (n_1 + \cdots + n_{l-1})n_l + (n_1 + \cdots + n_{l-2})n_{l-1} + \cdots + (n_1 + n_2)n_3 + n_1n_2 \\ & \quad - (n-r)r, \end{aligned}$$

where we have used that fact that $m_1 + \cdots + m_k = n_1 + \cdots + n_l = r$. By rearrangement of terms, we see that this is the same as the multiplicity of 0 and hence conclude the lemma. \square

3 Mirror principle computation for Grassmannian manifolds.

3.1 The distinguished S^1 -fixed-point components and the hyperplane-induced class.

To make the comparison immediate, here we follow the notations in [L-L-Y1: III, Sec. 5.4]. Recall the following approach ibidem to compute $A(t)$ when there is a commutative diagram :

$$\begin{array}{ccccc} F_0 & \xrightarrow{e^Y} & Y_0 & \xleftarrow{g} & E_0 \\ \downarrow i & & \downarrow j & & \downarrow k \\ M_d & \xrightarrow{\varphi} & W_d & \xleftarrow{\psi} & \mathcal{Q}_d \end{array} ,$$

where \mathcal{Q}_d is an S^1 -manifold, $\psi : \mathcal{Q}_d \rightarrow W_d$ is an S^1 -equivariant resolution of singularities of $\varphi(M_d)$, E_0 is the set of fixed-points in $\psi^{-1}(Y_0)$ and is called the distinguished S^1 -fixed-point component, and $\varphi_*[M_d] = \psi_*[\mathcal{Q}_d]$ in $A_*^{S^1}(W_d)$.

In the current case, X is the Grassmannian manifold $Gr_r(\mathbb{C}^n)$, \mathcal{Q}_d is the Quot-scheme $Quot_{P(t)=(n-r)t+(d+n-r)}(\mathcal{E}^n)$, and the linear sigma model W_d for X is the projective space $\mathbb{P}(H^0(C, \mathcal{O}_C(d)) \otimes \Lambda^r \mathbb{C}^n)$ of $\binom{n}{r}$ -tuple of degree- d homogeneous polynomials on C . This is a linear sigma model for $\mathbb{P}(\Lambda^r \mathbb{C}^n)$ that is turned into a linear sigma model for X via the Plücker embedding $Gr_r(\mathbb{C}^n) \rightarrow \mathbb{P}(\Lambda^r \mathbb{C}^n)$.

An element in W_d can be written as

$$\left[\sum_j c_{1j} z_0^j z_1^{d-j} : \sum_j c_{2j} z_0^j z_1^{d-j} : \cdots \right],$$

where $[z_0 : z_1]$ is the homogeneous coordinates for C and $c_{ij} \in \mathbb{C}$ with $1 \leq i \leq \binom{n}{r}$ and $0 \leq j \leq d$. The group S^1 acts on W_d by

$$\left[\sum_j c_{1j} z_0^j z_1^{d-j} : \cdots \right] \mapsto \left[\sum_j c_{1j} (tz_0)^j z_1^{d-j} : \cdots \right], \quad t \in S^1.$$

There are $(d+1)$ -many S^1 -fixed-point components in W_d , each of which consists of points of the form $[c_{1j} z_0^j z_1^{d-j} : c_{2j} z_0^j z_1^{d-j} : \cdots]$ for $0 \leq j \leq d$ and is isomorphic to $\mathbb{P}(\Lambda^r(\mathbb{C}^n))$. From [L-L-Y1: II, III], the S^1 -fixed-point component F_0 in M_d consists of degree- $(1, d)$ stable maps (C, f) into $C \times Gr_r(\mathbb{C}^n) \subset C \times \mathbb{P}(\Lambda^r \mathbb{C}^n)$ that is obtained by gluing a degree- $(1, 0)$ stable map $(C_1 = \mathbb{CP}^1, f_1, \infty)$ and a degree- $(0, d)$ stable map (C_2, f_2, x) with $f_1(\infty) = f_2(x)$ at their marked point. Regard these as stable maps into the projective space $\mathbb{P}(\Lambda^r \mathbb{C}^n)$, then [L-L-Y1: I, Sec.2, Example 10 and III, Sec. 3] implies that the S^1 -fixed-point component Y_j in W_d consists of point of the form $[c_{1d} z_0^{d-j} z_1^j : c_{2d} z_0^{d-j} z_1^j : \cdots]$. In particular, Y_0 consists of points of the form $[c_{1d} z_0^d : c_{2d} z_0^d : \cdots]$.

The map

$$\psi : \mathcal{Q}_d = Quot_{P(t)}(\mathcal{E}^n) \longrightarrow W_d = \mathbb{P}(H^0(C, \mathcal{O}_C(d)) \otimes \Lambda^r \mathbb{C}^n)$$

is given as follows. Write $C = \text{Proj } \mathbb{C}[z_0, z_1]$, where $\mathbb{C}[z_0, z_1]$ is regarded as a graded ring with grading given by the total degree. Then \mathcal{E}^n is the sheaf associated to the graded $\mathbb{C}[z_0, z_1]$ -module $\mathfrak{M} := \mathbb{C}[z_0, z_1]^{\oplus n}$, whose grade- d piece \mathfrak{M}_d is given by

$$\mathfrak{M}_d = \{ (f_1, \dots, f_n) \mid f_i \text{ homogeneous polynomial of } d \text{ in } z_0, z_1 \}.$$

A point $(\mathcal{E}^n \rightarrow \mathcal{E}^n/\mathcal{V}) \in \mathcal{Q}_d$ is the same as a subsheaf $\mathcal{V} \hookrightarrow \mathcal{E}^n$, which then corresponds to a graded submodule $\mathfrak{N}_{\mathcal{V}}$ in \mathfrak{M} of rank r . Let $e_1, \dots, e_r \in \mathfrak{M}$ be a basis for $\mathfrak{N}_{\mathcal{V}}$. Express each e_i as a column vector with entries in $\mathbb{C}[z_0, z_1]$ and consider the matrix $A_{\mathcal{V}} = [e_1, \dots, e_r]$. When the quotient sheaf $\mathcal{E}^n/\mathcal{V}$ has degree d , all the $r \times r$ -minors of $A_{\mathcal{V}}$, if not zero, must be of degree d as well. The map ψ sends $(\mathcal{E}^n \rightarrow \mathcal{E}^n/\mathcal{V})$ then to the $\binom{n}{r}$ -tuple of $r \times r$ -minors of $A_{\mathcal{V}}$. (Cf. [Ha], [So], [Str], and [S-S].)

Since ψ is S^1 -equivariant, it sends an S^1 -fixed-point component in \mathcal{Q}_d into an S^1 -fixed-point component in W_d . To see which S^1 -fixed-point component in \mathcal{Q}_d is sent to Y_0 , one only needs to check where a single point in $F_{\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r}$ is mapped to.

Lemma 3.1.1. $\psi(F_{\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r}) \subset Y_{\beta_1 + \dots + \beta_r}$.

Proof. Recall that, for a fixed-point $(\mathcal{E}^n \rightarrow \mathcal{E}^n/\mathcal{V}) \in F_{\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r}$,

$$\deg \mathcal{E}^n/\mathcal{V} = \alpha_1 + \dots + \alpha_r + \beta_1 + \dots + \beta_r = d.$$

Observe also that the special fixed-points in $F_{\alpha_1, \dots, \beta_r}$, for which the two local diagonalization match with $\alpha_i \rightarrow \beta_{i'}$, corresponds to a subsheaf \mathcal{V} in \mathcal{E}^n is isomorphic to the direct sum $\oplus_i \mathcal{I}_{\alpha_i(0) + \beta_{i'}(\infty)}$ of ideal sheaves $\mathcal{I}_{\alpha_i(0) + \beta_{i'}(\infty)}$ in \mathcal{O}_C associated to the degree- d divisor/subscheme $\alpha_i(0) + \beta_{i'}(\infty)$ in C . Its associated matrix $A_{\mathcal{V}}$ can be written as

$$\begin{pmatrix} z_0^{\alpha_1} z_1^{\beta_{1'}} & & & \\ & \ddots & & \\ & & z_0^{\alpha_r} z_1^{\beta_{r'}} & \\ & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \\ 0 & \cdots & 0 & \end{pmatrix} \quad \text{with zero entries } a_{ij} \text{ for } i \neq j,$$

after a constant re-trivialization of \mathcal{E}^n . The $r \times r$ -minors of this matrix are all zero except the one from the top $r \times r$ -submatrix, whose value is $z_0^{\alpha_1 + \dots + \alpha_r} z_1^{\beta_1 + \dots + \beta_r}$. Thus, ψ maps such point to some

$$[0 : \dots : 0 : z_0^{\alpha_1 + \dots + \alpha_r} z_1^{\beta_1 + \dots + \beta_r} : 0 : \dots : 0],$$

which lies in $Y_{\beta_1 + \dots + \beta_r}$. This proves the lemma. □

Since $0 \leq \beta_1 \leq \dots \leq \beta_r$, one concludes that

Corollary 3.1.2 [distinguished components]. *The distinguished S^1 -fixed-point locus E_0 is given by*

$$E_0 = \coprod_{\substack{0 \leq \alpha_1 \leq \dots \leq \alpha_r \\ \alpha_1 + \dots + \alpha_r = d}} F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0},$$

a disjoint union of flag manifolds determined by the multiplicities of entries in $(\alpha_1, \dots, \alpha_r)$ with $\alpha_1 + \dots + \alpha_r = d$. (Cf. Theorem 2.1.9)

On each distinguished S^1 -fixed-point component $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$, there is the pulled-back hyperplane class $k^* \psi^* \kappa = g^* j^* \kappa$, where κ is the hyperplane class on W_d . To see what it is, recall first the multiplicity numbers m_1, \dots, m_k for $0 \leq \alpha_1 \leq \dots \leq \alpha_r$ and the following fact/definition :

Fact/Definition 3.1.3 [special Schubert cycle]. (Cf. [Fu1], also [Gr2] and [Jo].) Recall that, over the flag manifold $Fl = Fl_{m_1, m_1 + m_2, \dots, r}(\mathbb{C}^n)$, there is a universal flag of

bundles $S_1 \hookrightarrow S_2 \hookrightarrow \dots \hookrightarrow S_{k+1} = Fl \times \mathbb{C}^n$ with $\text{rank } S_i = m_1 + \dots + m_i$. Then the intersection Chow ring $A^*(Fl)$ is generated by the Chern classes of the quotient bundles S_i/S_{i-1} , $1 \leq i \leq k+1$ and $S_0 = 0$, with relations determined by $\prod_{i=1}^{k+1} c(S_i/S_{i-1}) = 1$. The Schubert cycles that represent these special generators are called *special Schubert cycles*.

Since $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0} \simeq Fl_{m_1, m_1+m_2, \dots, r}(\mathbb{C}^n)$, this gives $A^*(F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0})$. Recall also from Sec. 2.1 that points in $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$ can be represented by $n \times r$ -matrices $B(z)$ with coefficients in $\mathbb{C}[z]$. Since the map g is the Plücker embedding and it sends $B(z)$ to the tuple of $r \times r$ -minors of $B(1)$ multiplied by the factor z^d , the image $g(F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0})$ coincides with the image of the Grassmannian manifold $Gr_r(\mathbb{C}^n)$ in Y_0 via Plücker embedding and g is indeed the fibration to the base Grassmannian manifold given in Theorem 2.1.9.

Let $S \hookrightarrow Gr_r(\mathbb{C}^n) \times \mathbb{C}^n$ be the universal rank- r bundle over $Gr_r(\mathbb{C}^n)$. Then the Plücker embedding in the direct bundle language is the section from the projectivization of the tautological bundle map $\wedge^r S = \det S \hookrightarrow Gr_r(\mathbb{C}^n) \times \wedge^r \mathbb{C}^n$ over $Gr_r(\mathbb{C}^n)$ and, hence, the hyperplane class on $\mathbb{C}P^{\binom{n}{r}-1}$ is pulled back to the Chern class $-c_1(S)$ on $Gr_r(\mathbb{C}^n)$ via the Plücker embedding. On the other hand, the embedding of $Y_0 \simeq \mathbb{C}P^{\binom{n}{r}-1}$ in $W_d \simeq \mathbb{C}P^{\binom{n}{r}d + \binom{n}{r}-1}$ has degree 1 from [L-L-Y1: I and II]. Together one concludes that:

Corollary 3.1.4 [pulled-back hyperplane class]. *Let κ be the hyperplane class in W_d . Then, with the notation in Fact/Definition 3.1.3, one has*

$$k^* \psi^* \kappa = g^* j^* \kappa = -c_1(S_k)$$

on the distinguished S^1 -fixed-point component $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$. Since S^1 and \mathbb{C}^\times act on these components trivially, these classes lift naturally as to classes on $(F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0})_{\mathbb{C}^\times}$ and will be denoted by the same notation.

Remark 3.1.5 [pulled-back hyperplane in Chern roots]. In terms of Chern roots to be discussed in Sec. 3.3, this class is represented by $-(y_1 + \dots + y_r) = y_{r+1} + \dots + y_n$.

3.2 The weight subspace decomposition of the normal bundle to the distinguished components.

In this subsection, we work out an ingredient needed for the computation of the \mathbb{C}^\times -equivariant Euler class of the normal bundle to a distinguished S^1 -fixed-point component in Quot-scheme.

Reduction of structure group and the S^1 -weight subspaces in matrix forms.

Note that the notation P in this section is for parabolic subgroups. Recall that the $GL(n, \mathbb{C})$ -action on \mathbb{C}^n induces a $GL(n, \mathbb{C})$ -action on the set of local sections in \mathcal{E}^n . Thus, given a $g \in GL(n, \mathbb{C})$, one has a correspondence $\mathcal{V} \mapsto g \cdot \mathcal{V}$ with a specified isomorphism from \mathcal{V} to $g \cdot \mathcal{V}$. This induces a $GL(n, \mathbb{C})$ -action on $Quot_{P(t)}(\mathcal{E}^n)$, which leaves all the S^1 -fixed-point component invariant. This $GL(n, \mathbb{C})$ -action on $Quot_{P(t)}(\mathcal{E}^n)$ commutes with the S^1 -action discussed earlier. In this way, the normal bundle ν to a S^1 -fixed-point component E is realized as a homogeneous $GL(n, \mathbb{C})$ -bundle and its structure group is the stabilizer P of a point p in that component: $\nu_E \mathcal{Q}_d = GL(n, \mathbb{C}) \times_P \mathbb{C}^R$, where R is the codimension of E in \mathcal{Q}_d , \mathbb{C}^R is identified with the fiber of $\nu_E \mathcal{Q}_d$ at p with the P -action induced from $GL(n, \mathbb{C})$.

The existence of a flag manifold also as a compact quotient implies that one can choose a compact $U(n)$ in $GL(n, \mathbb{C})$ such that each S^1 -fixed-point component is also a $U(n)$ -orbit. Then the new stabilizer at a point becomes

$$P_0 = U(n) \cap P = U(m_1) \times \cdots \times U(m_k) \times U(n-r)$$

and

$$\nu_E \mathcal{Q}_d = GL(n, \mathbb{C}) \times_P \mathbb{C}^R = U(n) \times_{P_0} \mathbb{C}^R.$$

In this way, we have reduced the structure group of $\nu_E \mathcal{Q}_d$ to P_0 that remains compatible with the S^1 -action. Applying this to each of the distinguished S^1 -fixed-point components $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$, we then realize $T_* Quot_{P(t)}(\mathcal{E}^n)|_{F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}}$ as a homogeneous $U(n, \mathbb{C})$ -bundle, determined by a representation of P_0 .

Given $0 \leq \alpha_1 \leq \dots \leq \alpha_r$ rewritten as

$$\begin{array}{ccccccc} 0 & \leq & a_1 & < & \cdots & < & a_k (= \alpha_r) \\ & & m_1 & & \cdots & & m_k \end{array}$$

with the multiplicity indicated, fix a point on $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$ represented by the subsheaf \mathcal{V} in \mathcal{E}^n determined by

$$\mathcal{V}(U_0) = \mathbb{C}[z] \cdot z^{\alpha_1} \oplus \cdots \oplus \mathbb{C}[z] \cdot z^{\alpha_r} \oplus 0^{\oplus(n-r)} \quad \text{and} \quad \mathcal{V}(U_\infty) = \mathbb{C}[w]^{\oplus r} \oplus 0^{\oplus(n-r)}$$

(or equivalently, the graded submodule in \mathfrak{M} generated by $(0, \dots, 0, z_0^{\alpha_i}, 0, \dots, 0)$ for $1 \leq i \leq r$, in the notation of Sec. 2.1). Then P is the subgroup of appropriate block upper triangular matrices in $GL(n, \mathbb{C})$. Fix a Hermitian inner product on \mathbb{C}^n , which renders \mathcal{E}^n a trivialized Hermitian vector bundle, and let $U(n) \hookrightarrow GL(n, \mathbb{C})$ be the subgroup of $GL(n, \mathbb{C})$ with respect to this inner product. Then the induced action of $U(n)$ on $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$ is transitive with $P_0 = P \cap U(n) = U(m_1) \times \cdots \times U(m_k) \times U(n-r)$ being the subgroup of $U(n)$ that consists of $m_1 \times m_1, \dots, m_k \times m_k, (n-r) \times (n-r)$ unitary diagonal blocks.

There is an embedding of *Hom*-groups

$$Hom_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V}) \hookrightarrow Hom_{\mathbb{C}[z]}(\mathbb{C}[z] \cdot z^{\alpha_1} \oplus \cdots \oplus \mathbb{C}[z] \cdot z^{\alpha_r}, \mathbb{C}[z] \cdot \bar{e}_1 \oplus \cdots \oplus \mathbb{C}[z] \cdot \bar{e}_r \oplus \mathbb{C}[z]^{\oplus(n-r)}),$$

where the annihilator $\text{Ann}(\bar{e}_i)$ of \bar{e}_i is the ideal (z^{α_i}) in $\mathbb{C}[z]$. Let m_0 be the multiplicity of 0 in $\alpha_1, \dots, \alpha_r$. Then $\bar{e}_1 = \dots = \bar{e}_{m_0} = 0$ and the image is a submodule of the latter that consists of matrices of polynomials with degree bounds:

$$f = [f_{ij}(z)]_{(n-m_0) \times r},$$

where

$$\deg f_{ij}(z) \leq \begin{cases} \alpha_{m_0+i} - 1 & \text{for } 1 \leq i \leq r - m_0 \text{ and } 1 \leq j \leq r, \\ \alpha_j & \text{for } r - m_0 + 1 \leq i \leq n - m_0 \text{ and } 1 \leq j \leq r. \end{cases}$$

The P_0 -action on $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V})$ is given by

$$f \longmapsto g \diamond f, \quad \text{for } f \in \text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V}) \text{ and } g \in P_0$$

with

$$g \diamond f := g \odot f \odot \text{Diag}\{z^{-\alpha_1}, \dots, z^{-\alpha_r}\} \cdot g^{-1} \cdot \text{Diag}\{z^{\alpha_1}, \dots, z^{\alpha_r}\},$$

where g in the formula is the lower-right $(n - m_0) \times (n - m_0)$ submatrix of the defining matrix of g when acting on \mathbb{C}^n , g^{-1} is the $r \times r$ upper-left submatrix of the inverse of the defining matrix for g , the operation \cdot is the usual matrix multiplication, and the operation \odot is the usual matrix multiplication followed by truncations of terms in an entry that exceeds the degree bound above. This shows explicitly that the P_0 -action and the S^1 -action on $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V})$ commute.

From the previous discussions on the S^1 -weight system, each monomial in an entry (a Laurent polynomial in z) of

$$\tilde{f} := f \odot \text{Diag}\{z^{-\alpha_1}, \dots, z^{-\alpha_r}\}$$

gives an S^1 -invariant subspace in $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V})$. The degree bound for an entry in \tilde{f} is given by

$$\begin{cases} -\alpha_j \leq \deg \tilde{f}_{ij}(z) \leq \alpha_{m_0+i} - \alpha_j - 1 & \text{for } 1 \leq i \leq r - m_0 \text{ and } 1 \leq j \leq r, \\ -\alpha_j \leq \deg \tilde{f}_{ij}(z) \leq 0 & \text{for } r - m_0 + 1 \leq i \leq n - m_0 \text{ and } 1 \leq j \leq r. \end{cases}$$

Thus one has a decomposition of the P_0 -module by the S^1 -weight subspaces, each of which is itself a P_0 -module:

$$\tilde{f} = z^{-\alpha_r} \tilde{f}_{(\alpha_r)} + \dots + z^{-1} \tilde{f}_{(1)} + \tilde{f}_{(0)} + z \tilde{f}_{(-1)} + \dots + z^{\alpha_r - \alpha_1 - 1} \tilde{f}_{(-(\alpha_r - \alpha_1 - 1))},$$

where the S^1 -weight for z^μ -component here is $-\mu$, (cf. the expression $z^{\alpha_j - \mu_{ij}^0}$ in the discussion of the S^1 -weight system Wt_1). (Note that here we are assuming the generic situation, in which $\alpha_1 < \alpha_r$ and hence $\alpha_r - \alpha_1 - 1 \geq 0$. If $\alpha_1 = \dots = \alpha_r$, then $\alpha_r - \alpha_1 - 1 = -1$ and $\tilde{f} = z^{-\alpha_r} \tilde{f}_{(\alpha_r)} + \dots + z^{-1} \tilde{f}_{(1)} + \tilde{f}_{(0)}$.)

The P_0 -module decomposition of S^1 -weight spaces and the P_0 -weight system.

• S^1 -weight-subspace decomposition.

- (1) Recall the multiplicity m_i , $1 \leq i \leq k$, of the sequence $0 \leq \alpha_1 \leq \cdots \leq \alpha_r$ and m_0 the multiplicity of 0 in the sequence. Then the matrices g , g^{-1} , f , \tilde{f} can be put into a block form. For example, the (I, J) -block for f is an $m_I \times m_J$ submatrix if $m_0 = 0$, or an $m_{I+1} \times m_J$ submatrix if $m_0 > 0$, or an $r \times m_J$ submatrix if $m_0 = 0$ and $I = k + 1$ or if $m_0 > 0$ and $I = k$. (Cf. FIGURE 3-2-1.)
- (2) In terms of the block form, the decomposition of \tilde{f} into a summation of matrices with only one non-zero block gives the decomposition of $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V})$ into representations of P_0 . Consequently, each such summand is the representation of the form $\rho_{m_I} \otimes (\rho_{m_J}^{-1})^t = \rho_{m_I} \otimes \bar{\rho}_{m_J}$, where ρ_{m_I} is the defining representation of $U(m_I)$, $(\rho^{-1})^t$ its inverse transpose, which is the same as its complex conjugate $\bar{\rho}$.
- (3) This decomposition is compatible with the S^1 -weight subspace decomposition. In fact, the block form of the S^1 -weight summand $\tilde{f}_{(s)}$, $-\alpha_r \leq s \leq \max\{\alpha_r - \alpha_1 - 1, 0\}$, is determined by the Young diagram corresponding to the partition $d = \alpha_1 + \cdots + \alpha_r$. They are all “sparse-lower-triangular” block matrices, (cf. FIGURE 3-2-1). These block forms are invariant under the conjugation followed by truncations of terms of degree higher than the upper degree bounds

$$\tilde{f}_{(s)} \longmapsto g \odot \tilde{f}_{(s)} \odot g^{-1}$$

and hence this gives a decomposition of the homogeneous bundle into the direct sum of S^1 -weight homogeneous subbundles. In particular, the lower sub-triangular block form of $\tilde{f}_{(0)}$ corresponds to the tangent bundle $T_*F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$. The dimension is consistent with the computation in Lemma 2.3.3.

- (4) These sparse-lower-triangular block matrices are determined by the Young diagram corresponding to the partition $d = \alpha_1 + \cdots + \alpha_r$. The rule from a Young diagram to the sparse-lower-triangular block forms can be summarized in three steps:
 - (4.1) Take the dual of the Young diagram and put the zero-matrix of the same dimension as \tilde{f} into the same block form. Copy these zero-matrices by multiplying by a weight factor z^ν with $-\alpha_r \leq \nu \leq \max\{\alpha_r - \alpha_1 - 1, 0\}$.
 - (4.2) Recall a_I and m_I at the beginning of this subsection. Multiply the dual Young diagram horizontally by the multiplicity m_I and fill the block forms with these multi-strip as indicated in FIGURE 3-2-1, beginning with the block form with z^{-a_I} -factor. This corresponds to the S^1 -weight system Wt_1 .
 - (4.3) Add all these matrices and fill in all the blocks in matrices with negative ν in z^ν such that some block above it is already filled. For the block form with factor z^0 , fill in all the blocks in the rows lower than the last filled row. These additional filling corresponds to the S^1 -weight system Wt_3 .

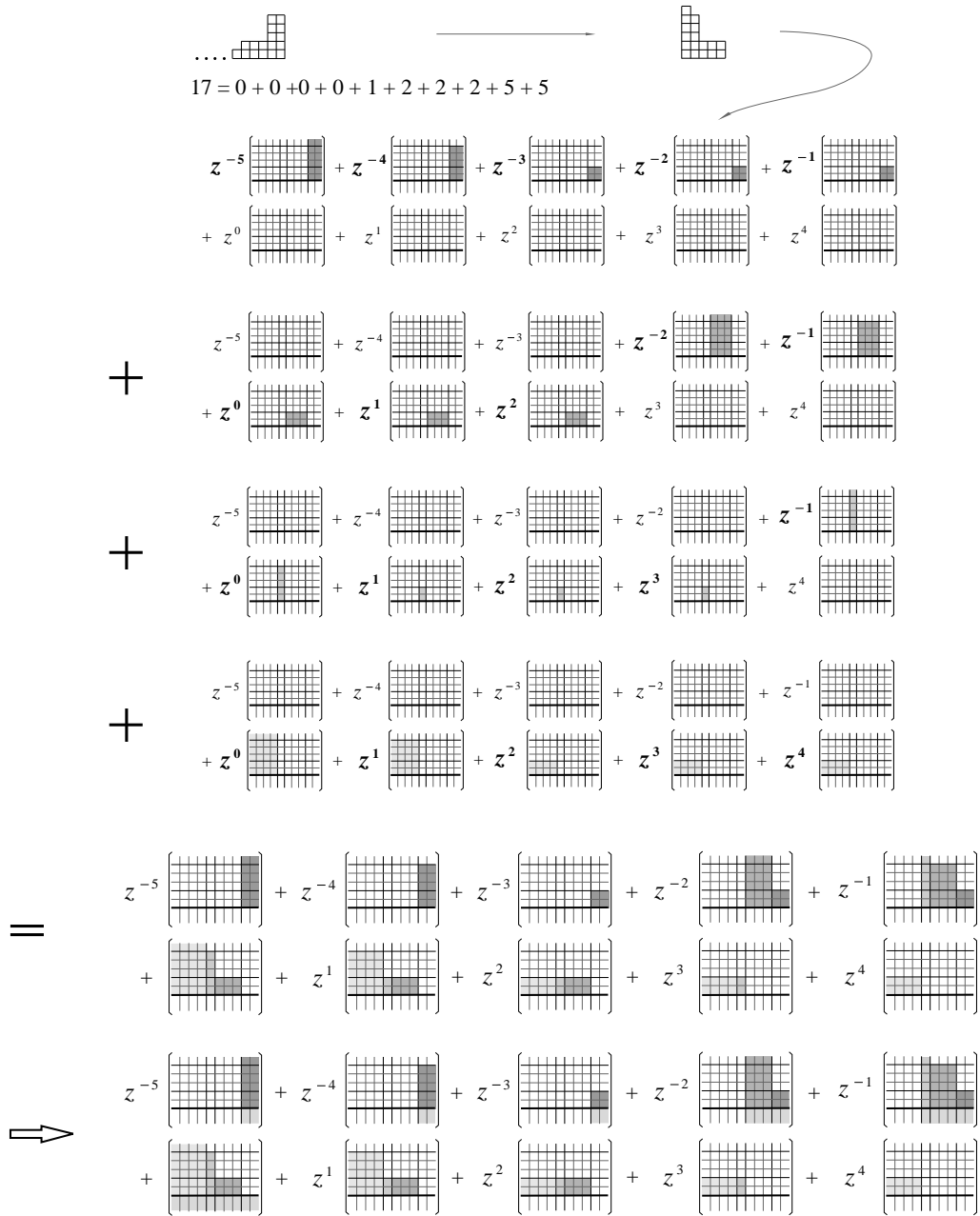


FIGURE 3-2-1. The simultaneous decomposition of $Hom_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V})$ by weight subspaces of S^1 and representations of P_0 . Original entries in the matrix are divided by light lines while blocks are divided by dark lines. The thick dark line divides the upper $(r - m_0)$ rows and the lower $(n - r)$ rows. All the unshaded blocks are zero. Observe how the block forms are all determined by the Young diagram - the conjugate Young diagram is horizontally fattened by the various multiplicities and then distribute into the block forms (cf. the blocks with the same dark shades) -.

(5) The block decomposition of each S^1 -weight subspace into the direct sum of P_0 -modules.

(5.1) *Definition of diagonal and off-diagonal blocks:* When $m_0 = 0$, the diagonal blocks follow the usual definition. When $m_0 > 0$, the diagonal blocks here are the blocks that are above and adjacent to the usual diagonal blocks (i.e. the $(I, I + 1)$ -blocks). All other blocks are called off-diagonal.

- The diagonal blocks corresponds to a representation $\rho_{m_I} \otimes \bar{\rho}_{m_I}$, where ρ_{m_I} is the defining representation of some $U(m_I)$.

- The off-diagonal blocks are irreducible representations $\rho_1 \otimes \rho_2^*$ of the product $U(m_{I_1}) \times U(m_{I_2})$, where ρ_j is the defining representation of $U(m_{I_j})$, $i = 1, 2$.

(5.2) Let $(\lambda_1, \dots, \lambda_{I_1})$ be the weight system of the representation ρ_1 and $(\lambda'_1, \dots, \lambda'_{I_2})$ be the weight system of the representation ρ_2 (with multiple weight repeated correspondingly), then the weight system of $\rho_1 \otimes \bar{\rho}_2$ is given by

$$(\lambda_i - \lambda'_j \mid 1 \leq i \leq I_1, 1 \leq j \leq I_2).$$

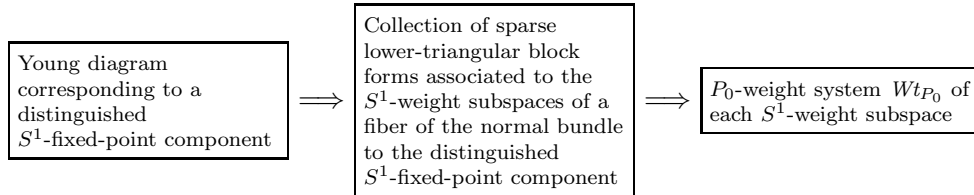
• *The P_0 -weight system Wt_{P_0} of $\text{Hom}_{\mathcal{O}_C}(\mathcal{V}, \mathcal{E}^n/\mathcal{V})$.*

(1) Recall the fixed maximal torus the diagonal subgroup $T = (\mathbb{C}^\times)^n$ in P_0 . Let E_{ij} be a $(n - m_0) \times r$ matrix with 1 in (i, j) -entry and zero elsewhere. Then, every subspace of an S^1 -weight subspace in the lower-triangular block that consists of constant multiples of some E_{ij} is a P_0 -weight subspace of weight $\lambda_{i+m_0} - \lambda_j$. Consequently, the P_0 -weight system can be directly read off from the collection of sparse lower-triangular block forms obtained from the Young diagram corresponding to the distinguished S^1 -fixed-point component. In expression,

$$Wt_{P_0}(\text{Young diagram}) = \bigsqcup_{(\text{triangular block form } \Delta)} \bigsqcup_{(\text{block } \square \in \Delta)} \bigsqcup_{(i,j) \in \square} (\lambda_{i+m_0} - \lambda_j)$$

where the Young diagram is the one corresponding to $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$ (namely the partition $d = \alpha_1 + \dots + \alpha_r$), \sqcup means the disjoint combination with multiplicity allowed, $\square \in \Delta$ means that the block is in the triangular block form Δ , and $(i, j) \in \square$ means that the (i, j) -position in the matrix for \tilde{f} lies in the block \square .

(2) In summary :



Let us now turn to the computation of the \mathbb{C}^\times -equivariant Euler class of the normal bundle to a distinguished component in $\text{Quot}_{P(t)}(\mathcal{E}^n)$.

3.3 Structure of the induced bundle on $BC^\times \times_{\mathbb{C}^\times} F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$ and the \mathbb{C}^\times -equivariant Euler class $e_{\mathbb{C}^\times} \nu_{F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}} \text{Quot}_{P(t)}(\mathcal{E}^n)$.

We compute first the Chern polynomials of the normal bundle $\nu_{F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}} \text{Quot}_{P(t)}(\mathcal{E}^n)$ of $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$ in $\text{Quot}_{P(t)}(\mathcal{E}^n)$ and then use this to express the \mathbb{C}^\times -equivariant Euler class after working out the bundle structure of the induced bundle of $\nu_{F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}}$ over $BC^\times \times F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$.

The following fact is in Borel and Hirzebruch [B-H], with slight modification to fit into our situation :

Fact 3.3.1 [Chern class and representation]. (Cf. [B-H], [B-T], [Fu1], [Hi], [M-S], and [Sp].)

- (1) Let T be a maximal torus of $U(n)$, \mathfrak{h} be the corresponding Cartan subalgebra, and $Fl(n) := U(n)/T$. Then, there are canonical homomorphisms

$$\{ \text{integral linear functional on } \mathfrak{h} \} \simeq H^1(T, \mathbb{Z}) \rightarrow H^2(Fl(n), \mathbb{Z}),$$

where the second homomorphism is surjective and is given by the transgression homomorphism associated to the principal T -bundle $U(n) \rightarrow Fl(n)$ from the quotient map. With respect to the defining representation of $U(n)$ on the Hermitian \mathbb{C}^n , T corresponds to a unique orthonormal basis in \mathbb{C}^n up to permutations. In terms of this basis, T is realized as the group of unitary diagonal matrices. Thus, T comes with a natural product decomposition $T = U(1)^{\times n}$ that is invariant under the Weyl group action and each $U(1)$ -factor of which is canonically oriented. This decomposition specifies then a distinguished basis x_1, \dots, x_n for $H^1(T, \mathbb{Z})$, unique up to permutations. Regard x_i also as elements in the other two groups via the above homomorphism and let $y_i = -x_i$ in $H^2(Fl(n), \mathbb{Z})$. Up to permutations, y_i in $H^2(Fl(n), \mathbb{Z})$ are the first Chern class of the line bundles associated to the tautological flag bundle over $Fl(n)$. These y_i generate $H^2(Fl(n), \mathbb{Z})$ and they satisfy

$$\sigma_k(y_1, \dots, y_n) = 0, \text{ for } k = 1, \dots, n,$$

where σ_k is the elementary symmetric polynomial of degree k for n variables.

- (2) [Chern root]. Let $P_0 = U(m_1) \times \dots \times U(m_k) \times U(m_{k+1}) \subset U(n)$, where $m_1 + \dots + m_k + m_{k+1} = n$, T be a maximal torus of $U(n)$ contained in P_0 , and $\eta : U(n) \rightarrow B = U(n)/P_0$ be the principal P_0 -bundle over B from the quotient map. Then $Fl(n)$ is a split manifold for η . Let $\zeta : Fl(n) \rightarrow B$ be the induced map from η , then $\zeta^* : H^*(B, \mathbb{Z}) \rightarrow H^*(Fl(n), \mathbb{Z})$ is injective and $\zeta^*c(\eta) = \prod_{i=1}^n (1 + y_i)$.
- (3) [naturality of Chern class]. Let ρ be an m -dimensional unitary representation of P_0 with weights $w_j = a_{j1}x_1 + \dots + a_{jn}x_n$, $j = 1, \dots, m$, and $V := U(n) \times_{\rho} \mathbb{C}^m$ be the associated homogeneous vector bundle over $U(n)/P_0$. Then

$$\zeta^*c(V) = \prod_{j=1}^m (1 + w_j) = \prod_{j=1}^m (1 + a_{j1}y_1 + \dots + a_{jn}y_n).$$

This expression is invariant under the Weyl group action P_0 , $Sym_{m_1} \times \cdots \times Sym_{m_{k+1}}$ on the set $\{y_1, \dots, y_{m_1}; y_{m_1+1}, \dots, y_{m_1+m_2}; \cdots; y_{m_1+\cdots+m_{k-1}+1}, \dots, y_{n-r}\}$ by the permutations that Sym_{m_1} permutes the first m_1 letters, Sym_{m_2} the next m_2 letters, and so on. The result is an integral polynomial function of symmetric functions in y_1, \dots, y_{m_1} , in $y_{m_1+1}, \dots, y_{m_1+m_2}$, and so on respectively. Each of these partial symmetric products of Chern roots y_i can be identified with the special Schubert cycles in the flag manifold $U(n)/P_0$.

Recall the P_0 -weight system associated to the Young diagram corresponding to $F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$ and Corollary 2.3.4, which says that all the S^1 -weight of a fiber of the normal bundle are non-zero. Let ν be the normal bundle in consideration. Then the above fact implies that the Chern polynomial $c_\nu(t)$ of ν is given by

$$c_\nu(t) = \prod_{\left(\begin{array}{l} \text{triangular block form } \Delta_w \\ \text{for non-zero } S^1\text{-weight} \end{array} \right)} \prod_{(\text{block } \square \in \Delta_w)} \prod_{(i,j) \in \square} (t + y_{i+m_0} - y_j),$$

where the first product on the right hand side of the equality ranges over all possible non-zero S^1 -weights w . The result is an integral polynomial function of the special Schubert cycles in $A_*(F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0})$, cf. Fact/Definition 3.1.3.

The S^1 -action on $\nu = \nu_{F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}} \text{Quot}_{P(t)}(\mathcal{E}^n)$ induces a bundle

$$\mathcal{T} \rightarrow \mathbb{C}P^\infty \times F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}.$$

Let $\nu = \bigoplus_w \nu_w$ be the decomposition of the normal bundle as a direct sum of S^1 -weight subspace and $\mathcal{T} = \bigoplus_w \mathcal{T}_w$ be the induced decomposition of \mathcal{T} .

Lemma 3.3.2 [induced bundle of S^1 -weight summand]. *Let*

$$\mathbb{C}P^\infty \xleftarrow{\text{pr}_1} \mathbb{C}P^\infty \times F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0} \xrightarrow{\text{pr}_2} F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$$

be the projection maps. Then

$$\mathcal{T}_w = \text{pr}_1^* \mathcal{O}_{\mathbb{C}P^\infty}(-w) \otimes \text{pr}_2^* \nu_w.$$

Proof. Let $E = EC^\times \rightarrow BC^\times = \mathbb{C}P^\infty$ be the universal principal \mathbb{C}^\times -bundle. First notice that the associated line bundle of EC to the representation of \mathbb{C}^\times on \mathbb{C} by $v \mapsto tv$ for $t \in \mathbb{C}^\times$, $v \in \mathbb{C}$ (i.e. the $w = 1$ representation) is $\mathcal{O}_{\mathbb{C}P^\infty}(-1)$. Since \mathbb{C}^\times acts on ν_w by a single weight w , the induced action of \mathbb{C}^\times on the projectivization $\mathbb{P}\nu_w$ of ν_w is trivial. Thus, as bundles over $\mathbb{C}P^\infty \times F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$,

$$\mathbb{P}\mathcal{T}_w = \mathbb{P}(E \times_{\mathbb{C}^\times} \nu_w) = E \times_{\mathbb{C}^\times} \mathbb{P}\nu_w = \mathbb{C}P^\infty \times \mathbb{P}\nu_w = \mathbb{P}\text{pr}_2^* \nu_w.$$

Since $1 \rightarrow \mathbb{C}^\times \rightarrow GL(\mathbb{C}) \rightarrow PGL(\mathbb{C}) \rightarrow 1$ is a central extension, the above isomorphism of projective bundles implies that $\mathcal{T}_w = \mathcal{L} \otimes \text{pr}_2^* \nu_w$ for some line bundle \mathcal{L} over $\mathbb{C}P^\infty \times F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$. By construction,

$$\mathcal{T}_w|_{\mathbb{C}P^\infty \times * } \simeq \text{pr}_1^* \mathcal{O}(-w) \otimes \mathbb{C}^R \quad \text{and} \quad \mathcal{T}_w|_{* \times F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}} \simeq \text{pr}_2^* \nu_w,$$

where R is the rank of ν_w . Since line bundles over flag manifolds are determined by their first Chern class and the second cohomology of flag manifolds are torsion-free, by the multiplicativity of the Chern character under tensor products and a comparison of first Chern classes, one concludes that

$$\mathcal{L}|_{\mathbb{C}P^\infty \times * } \simeq \mathcal{O}_{\mathbb{C}P^\infty}(-w) \quad \text{and} \quad \mathcal{L}|_{* \times F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}} \simeq \mathcal{O}_{F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}}.$$

Consider now a finite model $\mathbb{C}P^N$ for $\mathbb{C}P^\infty$ with N very large. Then, since

$$X_N := \mathbb{C}P^N \times F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}$$

is simply-connected and Kähler, from the long exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^1(X_N, \mathbb{Z}) & \longrightarrow & H^1(X_N, \mathcal{O}_{X_N}) & \longrightarrow & H^1(X_N, \mathcal{O}_{X_N}^*) & \xrightarrow{c_1} & H^2(X_N, \mathbb{Z}) & \longrightarrow & \dots \\ & & \parallel & & \parallel & & \parallel & & \parallel & & \\ & & 0 & & H^0_{\bar{\partial}}(X_N) = 0 & & \text{Pic}(X_N) & & H^2(\mathbb{C}P^N, \mathbb{Z}) \oplus H^2(F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}, \mathbb{Z}) & & \end{array}$$

associated to the exponential sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_{X_N} \rightarrow \mathcal{O}_{X_N}^* \rightarrow 0$, one concludes that the Picard variety $\text{Pic}(X_N)$ is contained in $\text{Pic}(\mathbb{C}P^N) \times \text{Pic}(F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0})$ and hence that every line bundle on X_N is of the form $\text{pr}_1^* \mathcal{L}_1 \otimes \text{pr}_2^* \mathcal{L}_2$. Together with the earlier discussion in the proof, one has in particular that $\mathcal{L} = \text{pr}_1^* \mathcal{O}_{\mathbb{C}P^N}(-w)$ over X_N for all large N . Let $N \rightarrow \infty$, one then concludes the lemma. \square

Let $R(w)$ be the rank of ν_w . By the multiplicativity of Euler class and the rule under the tensor with a line bundle (cf. [Fu1]), we conclude that

Theorem 3.3.3 [Euler class]. *The S^1 equivariant Euler class of the normal bundle $\nu = \nu_{F_{\alpha_1, \dots, \alpha_r; 0, \dots, 0}} \text{Quot}_{P(t)}(\mathcal{E}^n)$ is given by*

$$\begin{aligned} e_{\mathbb{C}^\times} \nu &= \prod_w e_{\mathbb{C}^\times} \nu_w = \prod_w c_{\nu_w}(-w\alpha) \\ &= \left(\begin{array}{c} \text{triangular block form } \Delta_w \\ \text{associated to nonzero } w \end{array} \right) \prod_{(\text{block } \square \in \Delta_w)} \prod_{(i,j) \in \square} (-w\alpha + y_{i+m_0} - y_j), \end{aligned}$$

where $\alpha = c_1(\mathcal{O}_{\mathbb{C}P^\infty}(1))$, $c_{\nu_w}(t) = t^{R(w)} + c_1(\nu_w)t^{R(w)-1} + \dots$ is the Chern polynomial of ν_w . The triangular block forms Δ_w , $-\alpha_r \leq w \leq \max\{\alpha_r - \alpha_1 - 1, 0\}$, associated with the Young diagram corresponding to the partition $d = \alpha_1 + \dots + \alpha_r$ are defined by Item (4) in the subheading “ S^1 -weight-subspace decomposition” of the heading “The P_0 -module decomposition of S^1 -weight spaces and the P_0 -weight system” in Sec. 3.2.

Here pr_1^* , pr_2^* in the formula are omitted for simplicity of notations.

4 Illustrations by two examples.

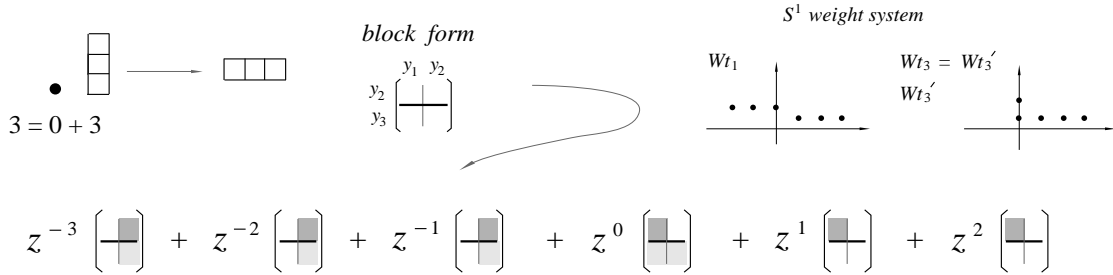
In this section, we present two simple examples of the Mirror Principle computation that are computable by hand to illustrate the discussions in this article. In these examples, the distinguished S^1 -fixed-point components in the related components of Quot-schemes are either Grassmannian manifolds or complete flag manifolds. The Schubert calculus of these follow from Fulton in [Fu1] and Monk in [Mo]. In particular, for the complete flag manifold $Fl(3) := Fl_{1,2}(\mathbb{C}^3)$, the cohomology ring $H^*(Fl(3), \mathbb{Z})$ is generated by $y_1, y_2, (y_3 = -(y_1 + y_2))$, where y_i are the first Chern class the graded line bundles on $Fl(3)$ associated to the flag of universal rank-1 and rank-2 subbundles over $Fl(3)$, cf. Fact 3.3.1. The intergral of the top classes are given by

$$\int_{Fl(3)} y_1^3 = \int_{Fl(3)} y_2^3 = 0 \quad \text{and} \quad \int_{Fl(3)} y_1^2 y_2 = - \int_{Fl(3)} y_1 y_2^2 = -1,$$

following [Mo].

Example 4.1 [$Gr_2(\mathbb{C}^3)$, **degree 3**]. In this case, $n = 3, r = 2, n - r = 1$, and $d = 3$. There are two distinguished S^1 -fixed-point components in the related component $Quot_{P(t)=t+4}(\mathcal{E}^3)$ of Quot-scheme:

- $F_{0,3;0,0} \simeq Fl_{1,2}(\mathbb{C}^3)$, $dim = 3$.



– Grouping of Chern roots: $\{y_1 ; y_2 ; y_3\}$.

– \mathbb{C}^\times -equivariant Euler class of normal bundle:

$$e_{\mathbb{C}^\times}(\nu) = (-3\alpha)(-3\alpha + y_3 - y_2)(-2\alpha)(-2\alpha + y_3 - y_2) \cdot (-\alpha)(-\alpha + y_3 - y_2)(\alpha + y_2 - y_1)(2\alpha + y_2 - y_1).$$

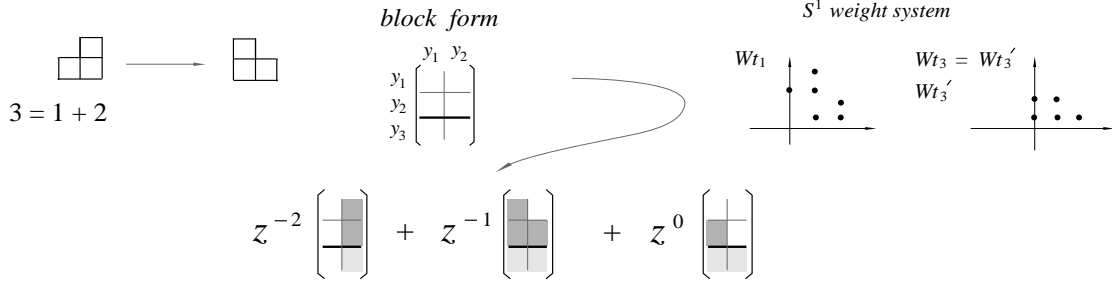
– Pulled-back hyperplane class:

$$k^* \psi^* \kappa = g^* j^* \kappa = -(y_1 + y_2).$$

– The integral over the component :

$$\int_E \frac{k^* \psi^* e^{\kappa \cdot \zeta}}{e_{\mathbb{C}^\times}(E/\mathcal{Q}_d)} = -\frac{103}{1296} \frac{1}{\alpha^{11}} - \frac{23}{108} \frac{\zeta}{\alpha^{10}} - \frac{29}{864} \frac{\zeta^2}{\alpha^9}.$$

• $F_{1,2;0,0} \simeq Fl_{1,2}(\mathbb{C}^3)$, $dim = 3$.



– Grouping of Chern roots : $\{y_1 ; y_2 ; y_3\}$.

– \mathbb{C}^\times -equivariant Euler class of normal bundle :

$$e_{\mathbb{C}^\times}(\nu) = (-2\alpha + y_1 - y_2)(-2\alpha)(-2\alpha + y_3 - y_2) \cdot (-\alpha)(-\alpha + y_2 - y_1)(-\alpha + y_3 - y_1)(-\alpha)(-\alpha + y_3 - y_2).$$

– Pulled-back hyperplane class :

$$k^* \psi^* \kappa = g^* j^* \kappa = -(y_1 + y_2).$$

– The integral over the component :

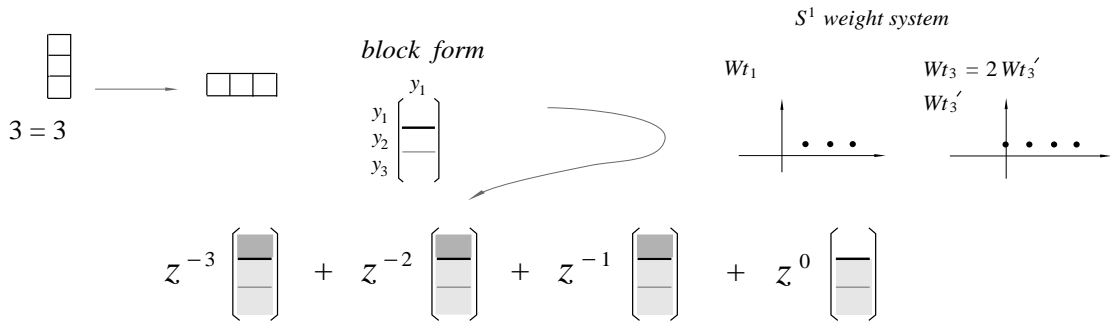
$$\int_E \frac{k^* \psi^* e^{\kappa \cdot \zeta}}{e_{\mathbb{C}^\times}(E/\mathcal{Q}_d)} = \frac{3}{16} \frac{\zeta}{\alpha^{10}} + \frac{1}{32} \frac{\zeta^2}{\alpha^9}.$$

$$\text{Total integral} = -\frac{103}{1296} \frac{1}{\alpha^{11}} - \frac{11}{432} \frac{\zeta}{\alpha^{10}} - \frac{1}{432} \frac{\zeta^2}{\alpha^9}.$$

□

Example 4.2 [$Gr_1(\mathbb{C}^3)$, degree 3]. In this case, $n = 3$, $r = 1$, $n - r = 2$, and $d = 3$. There is one distinguished S^1 -fixed-point component in the related component $Quot_{P(t)=2t+5}(\mathcal{E}^3)$ of Quot-scheme:

– $F_{3;0} \simeq Gr_1(\mathbb{C}^3)$, $dim = 2$.



– Grouping of Chern roots: $\{y_1 ; y_2, y_3\}$.

– \mathbb{C}^\times -equivariant Euler class of normal bundle:

$$e_{\mathbb{C}^\times}(\nu) = (-3\alpha)(-3\alpha + y_2 - y_1)(-3\alpha + y_3 - y_1)(-2\alpha)(-2\alpha + y_2 - y_1) \cdot (-2\alpha + y_3 - y_1)(-\alpha)(-\alpha + y_2 - y_1)(-\alpha + y_3 - y_1).$$

– Pulled-back hyperplane class:

$$k^*\psi^*\kappa = g^*j^*\kappa = -y_1.$$

– The integral over the component:

$$\int_E \frac{k^*\psi^*e^{\kappa\zeta}}{e_{\mathbb{C}^\times}(E/\mathcal{Q}_d)} = -\frac{103}{1296} \frac{1}{\alpha^{11}} - \frac{11}{432} \frac{\zeta}{\alpha^{10}} - \frac{1}{432} \frac{\zeta^2}{\alpha^9},$$

which is the same as the total integral in Example 4.1, as it should be since $Gr_1(\mathbb{C}^3) = Gr_2(\mathbb{C}^3)$.

□

Remark 4.3. One can check that the integral values are correct, using the result in [L-L-Y1, I] for the computation for $\mathbb{C}P^2$. Simple examples as they are, one observes that the intermediate details in the computation do depend on the presentation of a Grassmannian manifold and these details are in general very different. The fact that either presentation gives an identical answer provides thus a computational check of the theory developed.

Remark 4.4. Now that we can compute the integral that is related to the intersection numbers on the moduli space of rational stable maps into Grassmannian manifolds, the A-model for Calabi-Yau complete intersections in a Grassmannian manifold can also be computed explicitly.

Remark 4.5. Two immediate questions follow from the current work:

- (1) the automatization of the calculations via a computer code, following the diagrammatic rules discussed, and the computation for more examples and
- (2) generalization of the discussion to flag manifolds, which involves hyper-Quot schemes.

The study of them will be reported in another work.

With these remarks, we conclude this paper.

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