Maximal Unipotent Monodromy for Complete Intersection CY Manifolds

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Abstract

The computations that are suggested by String Theory in the B model requires the existence of degenerations of CY manifolds with maximum unipotent monodromy. In String Theory such a point in the moduli space is called a large radius limit (or large complex structure limit). In this paper we are going to construct one parameter families of n dimensional Calabi-Yau manifolds, which are complete intersections in toric varieties and which have a monodromy operator T such that $(\mathrm{T}^N - id)^{n+1} = 0$ but $(\mathrm{T}^N - id)^n \neq 0$, i.e the monodromy operator is maximal unipotent.

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1 Introduction

1.1 General Remarks

One of the most important problems in algebraic geometry is the study of families of algebraic varieties parameterized by a variety. Of special interest is the subvariety in the parameter space that parameterizes the singular fibers. This subvariety is called the discriminant locus. One of the main invariants of the discriminant locus is the so called monodromy group. The monodromy group is defined by the action of fundamental group of the complement of the discriminant locus on the cohomology ring of a fixed non-singular fiber. Of special interest is the action of the monodromy group on the middle cohomology group. The structure of such actions is of profound importance. In number theory, the counterpart is the action of Galois group on ètale cohomology.

In this paper we are going to study the simplest case of the above described setting, namely we are going to study families of algebraic manifolds over the unit disk $\mathcal{D}$. We will assume that the only singular fiber is over the center of the disk, i.e. over $0 \in \mathcal{D}$. From the discussion above it follows that we obtain a representation of the fundamental group $\pi_1(\mathcal{D} \setminus 0) = \mathbb{Z}$ in the middle homology, i.e. in $H_n(X_t, \mathbb{Z})/Tor$. The finite dimensional representations of $\mathbb{Z}$ are classified by the Jordan blocks of the linear operator corresponding to $1 \in \mathbb{Z}$. We will give a complete answer to the structure of the monodromy operator in terms of the topology of the singular fiber. Our method of proof is based on Clemens' theory of monodromy and the theory of mixed Hodge structures. We find a simple criterium for the monodromy operator to have a Jordan block of maximal rank. This criterium is based on Leray's theory of residues. We will apply this simple criterium to very concrete examples of complete intersections in $\mathbb{CP}^N$ and toric varieties.

The existence of such degenerations is of prime importance in mirror geometry and in string theory. The computations that were suggested by String Theory in the B model required the existence of degenerations of CY manifolds with maximal unipotent monodromy. In String Theory such a point in the moduli space is called a large radius (complex structure) limit. The case of hypersurfaces in toric varieties was treated in [10], where the construction of a point of maximal degeneracy is done by studying the GKZ hypergeometric system governing periods of the hypersurfaces.

For recent important developments in mirror geometry see [11], [12] and references therein. The results of this paper are closely related to Strominger-Yau-Zaslow conjecture.

1.2 Description of the Paper

In **Section 2** we introduce the basic notions and review some results stated in [8].

In **Section 3** we review the generalization of Lefschetz's theory of vanishing cycles due to Clemens. We describe Clemens' method for constructing Jordan blocks in the monodromy operator.

In **Section 4** we prove a general formula for the number of Jordan blocks in the monodromy operator in terms of some invariants of the singular fiber of one parameter family of Kähler manifolds.

In **Section 5** we prove a simple criterium for the existence of a Jordan block of maximum size in the monodromy operator in terms of Leray's residue calculus. We also construct families of complete intersections of CY manifolds in $\mathbb{CP}^N$ whose monodromy operators contain Jordan block of maximum size.

In **Section 6** we review some basic facts in toric geometry and construct families of complete intersections of CY manifolds in toric varieties whose monodromy operators contain Jordan block of maximum size generalizing the construction in section 5.

In **Section 7** we briefly discuss the connection between the present approach and a previous approach which uses hypergeometric functions. We also discuss an interesting relationship between maximal unipotent monodromy and the SYZ conjecture, and illustrate this in the case of polarized K3 surfaces.

In the Appendix, we give a complete description of Clemen's cell complex for hypersurfaces in a toric variety.

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2 Basic Definitions and Notations

2.1 Mumford's Semi-Stable Reduction Theorem

In this article we study one-parameter families of n dimensional Kähler manifolds

$$\pi : \mathcal{X} \rightarrow \mathcal{D}$$

over a disk $\mathcal{D}$. We assume that $\mathcal{X}$ is a smooth algebraic manifold and that for each $t \neq 0$ $\pi^{-1}(t) = X_t$ is a non-singular n dimensional Kähler manifold. From Hironaka's theorem on the resolution of singularities, we may assume that the singular fibre $X_0 = \pi^{-1}(0)$ is a divisor with normal crossings $X_0 = \cup C_i$.

We will choose local coordinates in the following manner. Let $x_{n+1} \in C_{i_1} \cap ... \cap C_{i_{n+1}} \subset X_0$. Let $U_{x_{n+1}}$ be an open polycylinder in $\mathcal{X}$ containing the point x_{n+1}. Let $z_{i_j} = 0$ be the defining equation of the divisor C_{i_j} in $U_{x_{n+1}}$. It is easy to see that after shrinking the disk $\mathcal{D}$, the fibers of the map $\pi : \mathcal{X} \cap U_{x_{n+1}} \rightarrow \mathcal{D}$ are locally given by

$$z_{i_1}^{m_1} ... z_{i_{n+1}}^{m_{n+1}} = t, \; m_j \geq 1 \;\&\; m_j \in \mathbb{Z}.$$

In the same manner, let $x_k \in X_0$ and let $C_{i_1}, .., C_{i_k}$ be those components of X_0 containing x_k. Let U_{x_k} be an open polycylinder in $\mathcal{X}$ containing x_k but not intersecting C_j with $j \neq i_1, .., i_k$. Then the fibers of the map $\pi : \mathcal{X} \cap U_{x_k} \rightarrow \mathcal{D}$ are locally given by

$$z_{i_1}^{m_1} ... z_{i_k}^{m_k} = t, \; m_j \geq 1 \;\&\; m_j \in \mathbb{Z}.$$

In [8] Mumford proved that after taking a finite covering of the disk $\mathcal{D}$, lifting the family and resolving the singularities, we may assume that the the fibres of the map $\pi{:}\mathcal{X} \rightarrow \mathcal{D}$ are given locally by $z_1^{k_1} ... z_{n+1}^{k_{n+1}} = t$, where k_j is either 0 or 1. From now on we are going to assume that we are in the above setting justified by Mumford's theorem.

2.2 Geometric and Homological Monodromy (Basic Properties)

If we restrict our family $\pi : \mathcal{X} \rightarrow \mathcal{D}$ to the circle $S^1 := \partial\mathcal{D}$ then we get a representation of $\pi_1(S^1) = \mathbb{Z}$ in the group of diffeomorphisms of X_t. Indeed if we remove a point from S^1,and restrict our family to $S^1 \setminus s$, $\pi_1 : \mathcal{X}|_{S^1} \to S^1$ where $s \in S^1$, we will get a trivial C^∞ family $\left(S^1 \backslash s\right) \times X_t \to S^1 \setminus s$. Our family $\pi_1 : \mathcal{X}|_{S^1} \to S^1$ is obtained from the trivial family $\left(S^1 \backslash s\right) \times X_t \to S^1 \setminus s$ by "gluing" it by $\phi \in Diff(X_t)$. The diffeomorphism ϕ is the generator of $\pi_1(S^1) = \mathbb{Z}$. We will call ϕ the geometric monodromy. The induced action of ϕ on $H_n(X_t, \mathbb{Z})$ will be called the monodromy operator and will be denoted by T. The main result about the operator T is that we have always:

$$\left(T^N - id\right)^{n+1} = 0$$

for some positive integer N. Here n is the complex dimension of X_t. This theorem was proved by many mathematicians including Griffiths, N. Katz, Clemens, Landesman, Deligne and so on.

From Mumford's result, we can assume that the following conditions hold for the family $\pi : \mathcal{X} \to \mathcal{D}$:

1. $\pi^{-1}(0) = X_0 = \cup_{i=0}^m C_i$ is a divisor of normal crossings and for $k = 0, .., n$, $C_{i_0} \cap .. \cap C_{i_k}, i_0 < ... < i_k$, is a non-singular irreducible subvariety of $\mathrm{Sing}_{n-k}(X_0)$ if non-empty.

2. The fibers of the map $\pi : \mathcal{X} \to \mathcal{D}$ are locally given in the open policylinders $\{U\}$ defined above by $z_{i_1}^{k_1} ... z_{i_{n+1}}^{k_{n+1}} = t$, where k_j is either 0 or 1.

Definition 1 *If the family $\pi : \mathcal{X} \to \mathcal{D}$ satisfies the above conditions we will say that it is in normal form.*

From now on we will consider only families $\pi : \mathcal{X} \to \mathcal{D}$ in normal form.

Notation 2 *Given $X_0 = C_0 \cup ... \cup C_m$ we shall use the following notations:*

$$\begin{gathered} I = \{i_0, ...i_k\} \text{ is an index set with } i_0 < ... < i_k \\ C_I = C_{i_0} \cap ... \cap C_{i_k},\ |I| = k+1; \text{ and} \\ C^{[k]} = \bigcup_{|I|=k+1} C_I. \end{gathered}$$

3 Review of Clemens' Theory of Geometric Monodromy

3.1 Construction of Clemens' retraction map

Let $\pi : \mathcal{X} \to \mathcal{D}$ be a family of Kähler manifolds as defined in Definition 1. We will construct a contraction map:

$$\mathcal{C} : \mathcal{X} \to X_0.$$

3.1.1 Construction of Clemens' Vector Field

The local description of our family $\pi : \mathcal{X} \to \mathcal{D}$ is given in $\mathbb{C}^{n+1}$ by the equation:

$$z_1...z_k = t$$

For some $k = 1, .., n+1$. Without loss of generality we may assume that t is a real number.

Let $z_i = r_i e^{2\pi\sqrt{-1}\phi_i}$,then the equation $z_1...z_k = t$ for $k = 1, .., n+1$ is equivalent to the equations:

$$r_1...r_k = t \text{ and } \sum_{i=1}^{k} \phi_i = 0 \text{ for } k = 1, .., n+1.$$

We will construct first a local vector field which will define the retraction map in this local situation and then using partition of unity we will construct a global vector field and thus the retraction map. It is easy to see that it is enough to construct a retraction of hyperbola in $\mathbb{R}_+^{n+1}$ given by

$$r_1...r_{n+1} = t$$

to the union of coordinate hyperplanes $r_1 = 0, ..., r_{n+1} = 0$, where $r_i > 0$ and $t > 0$. Let us suppose that we will consider only hyperbolas $r_1...r_{n+1} = t$ for $0 < t < \frac{1}{2}$. We will construct special vector field in $\mathbb{R}_+^{n+1}$. We will need to define some notions.

We will denote by $\mathcal{Q}$ the following set in $\mathbb{R}_+^{n+1}$

$$\mathcal{Q}:=\{r = (r_1, .., r_{n+1}) \,|\, r_i \geq 0 \text{ and there exists } 1 \leq i \leq n+1 \text{ such that } 0<\mathrm{r}_i < 1\}.$$

$\mathcal{Q}(1)$ will be the unit cube in $\mathbb{R}_+^{n+1}$, i.e.

$$\mathcal{Q}(1) := \{r = (r_1, .., r_{n+1}) \,|0 \leq r_i \leq 1 \text{ for all } 1 \leq i \leq n+1\}$$

It is clear that for $0 < t < \frac{1}{2}$ the hyperbolas $r_1...r_{n+1} = t$ are contained in $\mathcal{Q}$.

3.1.2 Construction of the Vector Field in $\mathbb{R}_+^{n+1}$

Let us suppose that $r = (r_1, .., r_{n+1}) \in \mathcal{Q}$ is a point such that $0 \leq r_{i_1} \leq 1, ..., 0 \leq r_{i_k} \leq 1$. To the point $r \in \mathcal{Q}$ we will assign a point $r(i_1, .., i_k) = (r_1, ..., r_{n+1}) \in \mathbb{R}_+^{n+1}$ $r_{i_1} = ... = r_{i_k} = 1$ and the rest of the coordinates of $r(i_1, .., i_k)$ are the same as the point $r \in \mathcal{Q}$. Let $l(r)$ be the line that joints the point $r \in \mathcal{Q} \subset\mathbb{R}_+^{n+1} r(i_1, .., i_k) \in \mathbb{R}_+^{n+1}$ with the point $r(i_1, .., i_k) \in \mathbb{R}_+^{n+1}$. In this way we define a vector field in $\mathcal{Q} \subset\mathbb{R}_+^{n+1}$. Using a partition of unity and using the vector field in $\mathcal{Q}$ that we have just defined, we obtain a global vector field on $\pi : \mathcal{X} \to \mathcal{D}$.

When we integrate this vector field, we obtain the Clemens' contraction map $\mathcal{C} : \mathcal{X} \to X_0$. Thus we obtain a map for each $t \in \mathcal{D} \setminus 0 : \mathcal{C}_t : X_t \to X_0$.

Definition 3 *The map $\mathcal{C}_t$ as defined above will be called Clemens map.*

For details of the construction see [4].

3.2 Properties of the Clemens' map

It is easy to prove the $\mathcal{C}_t$ has the following properties:

Lemma 4 *The Clemens' map $\mathcal{C}_t$ has the following properties:* ***(i)*** *Suppose $z \in C_I$, then $\mathcal{C}_t^{-1}(z) = \left(S^1\right)^k$ is a k dimensional real torus and* ***(ii)*** *$\mathcal{C}_t$ defines a diffeomorphism between $X_t \setminus \mathcal{C}_t^{-1}(\mathrm{Sing}(X_0))$ and $X_0 \setminus Sing(X_0)$.*

Proof of Lemma 4: For the proof of Lemma 4 see [4]. ■.

Definition 5 *Let $\mathcal{T}(i_1, .., i_k)$ be the tubular neighborhood of $C_{i_0} \cap .. \cap C_{i_k}$ in $C_{i_1} \cap .. \cap C_{i_k}$. We will denote by $p(i_1, .., i_k) : \mathcal{T}(i_1, .., i_k) \to C_{i_0} \cap C_{i_1} \cap ... \cap C_{i_k}$ the projection maps for any $i_1, .., i_k$.*

Definition 6 *Given an $(n-k)$-cycle $\gamma \in H_{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$, we define an n-cycle $p_k^{-1}(\gamma) \in H_n(X_0 \setminus Sing(X_0))$ as follows. Let*

$$p_1^{-1}(\gamma) := \partial(p(i_1, .., i_k)^{-1}(\gamma)).$$

This is the boundary of $p(i_1, .., i_k)^{-1}(\gamma)$, hence it is cycle of dimension $n-k+1$ in $C_{i_1} \cap ... \cap C_{i_k} \setminus C_{i_0} \cap ... \cap C_{i_k}$. Let

$$p_2^{-1}(\gamma) = \partial(p(i_2, .., i_k)^{-1} p_1^{-1}(\gamma)).$$

This is a cycle of dimension $n-k+2$ in $C_{i_2} \cap ... \cap C_{i_k} \setminus C_{i_1} \cap ... \cap C_{i_k}$. By continuing this way, we define at the end

$$p_k^{-1}(\gamma) = \partial(p(i_k)^{-1} p_{k-1}^{-1}(\gamma)).$$

This is a cycle of dimension n in $C_{i_k} \setminus C_{i_{k-1}} \cap C_{i_k}$. We denote by $\pi_k : p_k^{-1}(\gamma) \to \gamma$ the natural projection.

Proposition 7 *For any point $z \in \gamma \in H_{n-k}(C_{i_0} \cap .. \cap C_{i_k}, \mathbb{Q})$ we have $\pi_k^{-1}(z) = \left(S^1\right)^k$, where $\pi_k : p_k^{-1}(\gamma) \to \gamma$ is defined above. In other words the preimage of a point is a k dimensional real torus.*

Proof of Proposition 7**:** Proposition 7 follows directly from Definition 6 of the cycle $p_k^{-1}(\gamma) \in H_n(X_0 \setminus \mathrm{Sing}(X_0))$. ■.

Definition 8 *Given $\gamma \in H_{n-k}(C_{i_0} \cap .. \cap C_{i_k}, \mathbb{Q})$, we denote $\gamma_t := \mathcal{C}_t^{-1}(\gamma)$ and $\gamma_t(j) := \mathcal{C}_t^{-1}\left(p_j^{-1}(\gamma)\right)$, which are n-cycles representing elements in $H_n(X_t, \mathbb{Q})$.*

Lemma 9 *The cycles γ_t and $\gamma_t(j)$ above are homological to each other for 1$\leq j \leq k$.*

Proof of Lemma 9: We will prove first that $\gamma_t(j)$ and $\gamma_t(j+1)$ are homological to each other for $1 \leq j \leq k$, i.e. there exists a $n+1$ chains $\Gamma_0(j)$ in $C_{i_{j+1}} \cap .. \cap C_{i_k} \backslash C_{i_j} \cap .. \cap C_{i_k} \subset X_0$ as follows using Definitions 5 and 6: $\Gamma_0(j) := p(i_j, .., i_k)^{-1}(p_j^{-1}(\gamma)) \backslash (p_j^{-1}(\gamma))$. Let us define the $n+1$ dimensional chain $\Gamma_{t,\gamma}(j)$ in X_t as follows: $\Gamma_{t,\gamma}(j) := \mathcal{C}_t^{-1}(\Gamma_0(j))$. Lemma 4 implies directly that

$$\partial\left(\Gamma_{t,\gamma}(j)\right) = \mathcal{C}_t^{-1}(p_j^{-1}(\gamma)) - \mathcal{C}_t^{-1}(p_{j-1}^{-1}(\gamma)) = \gamma_t(j) - \gamma_t(j-1).$$

Lemma 9 is proved. ■.

In [4] the following result was proved:

Theorem 10 *Let* $\gamma \in H_{n-k}(Sing_k(X_0), \mathbb{Z})$ *be such that* $\mathcal{C}_t^{-1}(\gamma) = \gamma_t \in H_n(X_t, \mathbb{Z})$ *is a non-zero. Then there exists cycles* $\alpha_1, .., \alpha_k \in H_n(X_t, \mathbb{Z})$ *such that* $T(\alpha_j) = \gamma_t + \sum_{i=1}^{j} \alpha_i$ *for* $1 \leq j \leq k$.

We sketch the construction here. (See [4] for details.)

Clemens' construction of Jordan block by Picard-Lefschetz Duality. Let $\gamma \in H_{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Z})$ be a cycle such that $\mathcal{C}_t^{-1}(\gamma)$ be a non-zero element in $H_n(X_t, \mathbb{Z})$. In Definition 6 we defined a cycle $p_1^{-1}(\gamma)$ in $C_{i_2} \cap ... \cap C_{i_k} \backslash C_{i_1} \cap ... \cap C_{i_k}$. It is easy to see by using the fact that $\mathcal{C}_t^{-1}(\gamma) \in H_n(X_t, \mathbb{Z})$ is nonzero and Lemma 9 that

$$p_1^{-1}(\gamma) \in H_{n-k+1}(C_{i_2} \cap ... \cap C_{i_k} \backslash C_{i_1} \cap ... \cap C_{i_k}, \mathbb{Z}) \ \& \ p_1^{-1}(\gamma) \neq 0.$$

Let us denote by $\gamma_1 \in H_{n-k+1}(C_{i_2} \cap ... \cap C_{i_k}; C_{i_1} \cap ... \cap C_{i_k}, \mathbb{Z})$ the Picard Lefschetz dual cycle of $p_1^{-1}(\gamma)$. Let $T_{i_3,...,i_k}(\overline{\gamma_1})$ be the tubular neighborhood of the closure of $\overline{\gamma_1}$ in $C_{i_3} \cap ... \cap C_{i_k} \backslash C_{i_2} \cap ... \cap C_{i_k}$. Let us denote by $p_2^{-1}(\overline{\gamma_1})$ the boundary of $T_{i_3,...,i_k}(\overline{\gamma_1})$, i.e.

$$p_2^{-1}(\overline{\gamma_1}) = \partial T_{i_3,...,i_k}(\overline{\gamma_1})$$

It is easy to see that

$$p_2^{-1}(\overline{\gamma_1}) \in H_{n-k+2}(C_{i_3,...,i_k} \backslash C_{i_2,...,i_k}, \mathbb{Z}).$$

Let us denote by $\gamma_3 \in H_{n-k+2}(C_{i_3,...,i_k}; C_{i_2,...,i_k}, \mathbb{Z})$ the Picard-Lefschetz dual to $p_2^{-1}(\overline{\gamma_1})$. We can continue this process and thus we will define cycles $\gamma, \gamma_1, ..., \gamma_k$, where $\gamma_j \in H_{n-k+2}(C_{i_j,...,i_k}; C_{i_{j-1},...,i_k}, \mathbb{Z})$. Clemens proved in [4] that the monodromy operator T acts as follows on $\mathcal{C}_t^{-1}(\gamma), \mathcal{C}_t^{-1}(\gamma_1), ..., \mathcal{C}_t^{-1}(\gamma_k)$:

$$\mathrm{T}(\mathcal{C}_t^{-1}(\gamma)) = \mathcal{C}_t^{-1}(\gamma), ..., \mathrm{T}(\mathcal{C}_t^{-1}(\gamma_k)) = \mathcal{C}_t^{-1}(\gamma) + \textstyle\sum_{j=1}^{k} \mathcal{C}_t^{-1}(\gamma_j).$$

Corollary 11 *Let* $\pi : \mathcal{X} \to \mathcal{D}$ *be a family of Kähler manifolds over the disk such that:*

1. *For* $t \neq 0$, $\pi^{-1}(t) := X_t$ *is a non singular variety of complex dimension* $n \geq 1$.

2. $\pi^{-1}(0) = X_0 = \cup_{i=0}^{m} C_i$ *is a divisor of normal crossing and* π *is locally given by* $z_1^{n_1} \times ... \times z_k^{n_k} = t$, *where* n_i *are positive integers.*

3. *Suppose* $C_0 \cap .. \cap C_n$ *is a point and* $\mathcal{C}_t^{-1}(C_0 \cap .. \cap C_n) = \gamma_t$ *is a non zero cycle in* $H_n(X_t, \mathbb{Q})$,

then the monodromy operator of the family $\pi : \mathcal{X} \to \mathcal{D}$ *has a Jordan block of size* $n+1$. (See [4].)

4 The Jordan Normal Form of the Monodromy Operator

We begin by introducing the following combinatorial invariant of a family of algebraic varieties $\pi : \mathcal{X} \to \mathcal{D}$ put in a Mumford form. This will be needed later.

Definition 12 *We will define Clemens' simplicial complex of the family* $\pi : \mathcal{X} \to \mathcal{D}$ *as follows: To each divisor* C_i *we attached a point* p_i *in* $\mathbb{R}^d$, *where* d *is a large enough integer. We will assume that the points* p_i *are in general position, i.e. they do not lie in a hyperplane. If* C_i *intersects* C_j *then we attach to the points* p_i *and* p_j *one dimensional simplex. If* C_i, C_j *and* C_k *intersect then we attached a two dimensional simplex on* p_i, p_j *and* p_k. *We continue in that manner and we obtain a simplicial complex* $\Pi(X_0)$ *which we will call Clemens' simplicial complex.*

4.1 Definition of the Gysin Map

Definition 13 *Let* X *be a compact complex manifold, and* C *be a divisor of normal crossing. The map* $G_k : H_{k+2}(X, \mathbb{Z}) \to H_k(C, \mathbb{Z})$ *defined by* $G_k(\gamma) := \gamma \cap [C]$, *where* $\gamma \cap [C]$ *means intersection of class of cohomology in* X, *is called the Gysin map.*

Remark 14 *We will use the Gysin map in case* $C_{i_0} \cap ... \cap C_{i_k} \subset C_{i_1} \cap .. \cap C_{i_k}$ *and will denote by*

$$G_k : \bigoplus_{i_1,..,i_k} H_{n-k+2}(C_{i_1} \cap .. \cap C_{i_k}, \mathbb{Q}) \to H_{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$$

where $G_k(\gamma)$ *is the image of the cycle* $[\gamma \cap [C_{i_0} \cap .. \cap C_{i_k}]]$ *in* $H_{n-k}(Sing_{n-1}(X_0), \mathbb{Q}))$.

Definition 15 *The dual* G_k^* *of the Gysin map using Poincare duality is defined as follows for the pair* (X, C)

$$G_k^* : H^k(C, \mathbb{Z}) \to H^{k+2}(X, \mathbb{Z}),$$

where $G_k^*(\alpha) = \alpha \wedge c_1[C]$ *and* $c_1[C]$ *is the first Chern class of the line bundle defined by the normal crossing divisor* C.

4.2 Review of Deligne's Theory of Mixed Hodge Structures

The cohomology of $X_0 \setminus \mathrm{Sing}(X_0)$ can be computed as the cohomology of the de Rham log complex $\mathcal{A}^*(X_0, \log <\mathrm{Sing(X_0)}>)$.

Definition 16 *We will say that a form ω on one of the components C_i of X_0 had a logarithmic singularities if for each point $z \in X_0$ and some open neighborhood $U \subset C_i$ of the point z we have*

$$\omega|_U = \alpha \tfrac{dz^1}{z^1} \wedge ... \wedge \tfrac{dz^k}{z^k},$$

where α is a C^∞ form in U e and locally on X_0 is defined by the equations $z^1 \times ... \times z^k = 0$.

Definition 17 *We define the de Rham log complex as follows:*

$$\begin{gathered}\mathcal{A}^*(X_0, \log <\mathrm{Sing(X_0)}>) \\ = \{\omega \in C^\infty\left(X_0 \backslash \mathrm{Sing(X_0)}, \Omega^*\right) | \omega \text{ and } d\omega \text{ are } \mathrm{C}^\infty \text{ forms on } X_0 \backslash \mathrm{Sing(X_0)} \\ \text{which have log singularities on } \mathrm{Sing(X_0)}\}.\end{gathered}$$

Remark 18 *It is easy to see that if $\omega \in \mathcal{A}^m(X_0, \log <Sing(X_0)>)$ then ω locally around a point $z \in U \subset C_i$ in each of the components of X_0 and $z \in$Sing(X_0) is given by*

$$\omega|_U = \omega_1 \wedge \tfrac{dz_{i_1}}{z_{i_1}} \wedge ... \wedge \tfrac{dz_{i_k}}{z_{i_k}},$$

where ω_1 is a C^∞ a $(m-k)$ form on $U \subset C_i$; and Sing(X_0) $\cap\, U$ is given by $z_{i_1} \cdots z_{i_k} = 0$ in U.

Deligne proved that there exists a mixed Hodge structure on $X_0 \backslash \mathrm{Sing}(X_0)$. The existence of Mixed Hodge Structure is based on the following filtration:

Definition 19 *On the complex $\mathcal{A}^*(X_0, \log <Sing(X_0)>)$ we define the weight filtration W_l to be those forms ϕ that locally around Sing (X_0)*

$$\phi \in \mathcal{A}^*(U)\left\{\tfrac{dz_{i_1}}{z_{i_1}} \wedge ... \wedge \tfrac{dz_{i_k}}{i_k}\right\}.$$

Definition 20 *The Poincare Residue Operator $R^{[k]} : W_k \to \mathcal{A}^{*-k}(C^{[k]})$ is defined by*

$$R^{[k]}\left(\alpha \wedge \tfrac{dz_{i_1}}{z_{i_1}} \wedge ... \wedge \tfrac{dz_{i_k}}{i_k}\right) = \alpha|_{C_I}.$$

Definition 21 *Let us consider the decreasing filtration... $\supset W^{-l} \supset W^{-l+1} \supset$ ...where $W^{-l} = W_l$. Accordingly there is a spectral sequence $\{E_r\}$ such that E_∞ is the associated graded to the weight filtration in $H^*(X_0 \backslash Sing(X_0), \mathbb{C})$.*

The filtration was reversed by Deligne so that we can form a spectral sequence of the filtered de Rham logarithmic complex. By using the Poincare residue map Deligne proved the following Theorems:

Theorem 22 *The cohomology of $X_0\backslash Sing(X_0)$ are equal to the cohomology of the De Rham log complex $\mathcal{A}^*(X_0, \log < Sing(X_0) >)$.*

(For the proof of Theorem 22 see [6].)

Theorem 23 ***i.*** *The spectral sequence defined as above degenerates at the second step.* ***ii.*** *$E_1 \backsimeq \oplus H^*(C_I)$ and the mapping $d_1 : E_1 \to E_1$ is a morphism of Hodge structures given by the Gysin map (See Definition 13.)*

$$H^*(C_{i_0} \cap ... \cap C_{i_l}) \to H^*(C_{i_1} \cap ... \cap C_{i_l}).$$

4.3 Jordan Normal Form of the Monodromy Operator

In this section we will prove the following Theorem:

Theorem 24 ***i.*** *The number of Jordan blocks of rank $k \leq n$ is equal to the rank of the group*

$$H_{n-k}(\mathrm{Si}\, ng_k(X_0), \mathbb{Q})/\mathrm{Im}\,(G_k)\,,$$

where G_k is the Gysin map.

ii. *The number of Jordan blocks of rank $n+1$ is equal to $\dim H_n(\Pi(X_0), \mathbb{Q})$, where $\Pi(X_0)$ is the Clemens' polyhedra defined in Definition 12.*

Proof of Theorem 24: The proof of both part is based on Corollary 11. Part **i** of Theorem 24 follows directly from the following three Lemmas and Theorem 10:

Lemma 25 . *Let $\gamma \in \mathrm{Im}\, G_k \subseteq H_{n-k}(Sing_k(X_0), \mathbb{Q})$, then $\gamma_t = \mathcal{C}_t^{-1}(\gamma)$ is homological to zero in X_t.*

Lemma 26 *Suppose that $\gamma \in H_{n-k}(Sing_k(X_0), \mathbb{Q})/\mathrm{Im}\, G_k$ and $\gamma \notin \mathrm{Im}\, G_k$, then there exists a non zero class of cohomology $\widetilde{\omega} \in H^n(X_0\backslash Sing(X_0))$ such that*

$$\int_{p_k^{-1}(\gamma)} \widetilde{\omega} \neq 0.$$

where the cycle $p_k^{-1}(\gamma) \in H_n(X_0\backslash Sing(X_0), \mathbb{Q})$ is defined in Definition 6.

Lemma 27 *Let $\gamma \in H_{n-k}(Sing_k(X_0), \mathbb{Q})/\mathrm{Im}\, G_k$ and $\gamma \notin \mathrm{Im}\, G_k$, then γ_t is a non zero element in $H_n(X_t, \mathbb{Q})$.*

Proof of Lemma 25: We will prove part **i.** Suppose that

$$\gamma \in \mathrm{Im}\, G_k \subseteq H_{n-k}(\mathrm{Sing}_k(X_0), \mathbb{Q}).$$

From the Definition 13 of the Gysin map it follows that there exists a cycle

$$\Gamma \in \underset{i_1,..,i_k}{\oplus} H_{n-k+2}(C_{i_1} \cap .. \cap C_{i_k}, \mathbb{Q})$$

such that $\Gamma \cap \underset{i_0,..,i_k}{\oplus} [C_{i_0} \cap .. \cap C_{i_k}] = \gamma$. The definition 3 of the Clemens map it follows that the boundary of $\mathcal{C}_t^{-1}(\Gamma \backslash \gamma)$ is exactly $\mathcal{C}_t^{-1}(\gamma)$. Lemma 25 is proved. $\blacksquare$.

Proof of Lemma 26: The construction of the form ω is based on the following fact about the cohomology (homology) of $(X_0 \backslash \mathrm{Sing}(X_0))$, where X_0 is a Kähler variety and $\mathrm{Sing}(X_0)$ is a divisor with normal crossings in X_0.

Suppose that $\gamma \in H_{n-k}(\mathrm{Sing}_k(X_0), \mathbb{Q})$ and $\gamma \notin \mathrm{Im}\, G_k$. Let $\gamma \in H_{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$. In order to construct the form ω we will need to recall the how the dual $G_k^*$to the Gysin map $G_k : H_{k+2}(X, \mathbb{Z}) \to H_k(C, \mathbb{Z})$ is defined in Definition 15 as follows:

$$G_k^* : H^k(C, \mathbb{Z}) \to H^{k+2}(X, \mathbb{Z}), \text{ where } G_k^*(\alpha) = \alpha \wedge c_1[C].$$

$c_1[C]$ is the Chern class of the line bundle defined by the divisor with normal crossings C in X.

We will need the following Proposition:

Proposition 28 *Let $\gamma \in H_{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$ and $\gamma \notin \mathrm{Im}\, G_k$. Let $\omega_{n-k} \in H^{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$ and*

$$\int_\gamma \omega_{n-k} \neq 0$$

then ω_{n-k} can not be represent as follows $\omega_{n-k} = \left(c_1[C_{i_0} \cap ... \cap C_{i_k}]|_{C_{i_0} \cap ... \cap C_{i_k}} \right) \wedge \omega_1$ on $C_{i_0} \cap ... \cap C_{i_k}$, where $\omega_1 \in H^{n-k-2}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$.

Proof of Proposition 28: Suppose that $\omega_{n-k} \in H^{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$,

$$\int_\gamma \omega_{n-k} \neq 0.$$

and $\omega_{n-k} = \left(c_1[C_{i_0} \cap ... \cap C_{i_k}]|_{C_{i_0} \cap ... \cap C_{i_k}} \right) \wedge \omega_1$ *on* $C_{i_1} \cap ... \cap C_{i_k}$. Let η be anon zero section of the line bundle $\mathcal{O}([C_{i_0} \cap ... \cap C_{i_k}])$ on $C_{i_1} \cap ... \cap C_{i_k}$ such that the zero set of η is exactly $C_{i_0} \cap ... \cap C_{i_k}$. Let us consider the form

$$(d \log(\eta)) \wedge \left(c_1[C_{i_0} \cap ... \cap C_{i_k}]|_{C_{i_0} \cap ... \cap C_{i_k}} \right) \wedge \omega_1$$

on $C_{i_1} \cap ... \cap C_{i_k} \backslash C_{i_0} \cap ... \cap C_{i_k}$. Let us consider the cycle $p_1^{-1}(\gamma) \in H_{n-k+1}(C_{i_1} \cap ... \cap C_{i_k} \backslash C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$ as defined in Definition 6. Let us compute

$$\int_{p_1^{-1}(\gamma)} (d \log(\eta)) \wedge \left(c_1[C_{i_0} \cap ... \cap C_{i_k}]|_{C_{i_0} \cap ... \cap C_{i_k}} \right) \wedge \omega_1.$$

Since locally around a point $w \in C_{i_0} \cap ... \cap C_{i_k}$ the divisor $C_{i_0} \cap ... \cap C_{i_k}$ in $C_{i_1} \cap ... \cap C_{i_k}$ is given by $z = 0$, where $w \in U \subset C_{i_1} \cap ... \cap C_{i_k}$, we see that

$$d\log(\eta)|_U = \frac{dz}{z}.$$

From this local expression of $d\log(\eta)$ and the definition of $p_1^{-1}(\gamma)$ we deduce that

$$\int_{p_1^{-1}(\gamma)}(d\log(\eta)) \wedge (c_1[C_{i_0} \cap ... \cap C_{i_k}]) \wedge \widetilde{\omega}_1 = 2\pi \int_\gamma \omega_{n-k}$$

where $\omega_{n-k} = \left(c_1[C_{i_0} \cap ... \cap C_{i_k}]|_{C_{i_0}\cap ... \cap C_{i_k}}\right) \wedge \omega_1$ and $\widetilde{\omega}_1$ is a closed form in then tubular neighborhood of $C_{i_0} \cap ... \cap C_{i_k}$ in $C_{i_i} \cap ... \cap C_{i_k}$ such that the restriction of $\widetilde{\omega}_1$ on $C_{i_0} \cap ... \cap C_{i_k}$ is ω_1. On the other hand since the restriction of the line bundle $\mathcal{O}([C_{i_0} \cap ... \cap C_{i_k}])$ on $C_{i_1} \cap ... \cap C_{i_k} \backslash C_{i_0} \cap ... \cap C_{i_k}$ is the trivial line bundle, we deduce that the form $c_1[C_{i_0} \cap ... \cap C_{i_k}]$ will be an exact form on $C_{i_1} \cap ... \cap C_{i_k} \backslash C_{i_0} \cap ... \cap C_{i_k}$, i.e. $c_1[C_{i_0} \cap ... \cap C_{i_k}] = d\beta$. This implies that

$$(d\log(\eta)) \wedge (c_1[C_{i_0} \cap ... \cap C_{i_k}]) \wedge \widetilde{\omega}_1 = (d\log(\eta)) \wedge d\beta \wedge \widetilde{\omega}_1 = d(d\log(\eta)) \wedge \beta \wedge \widetilde{\omega}_1).$$

So Stoke's Theorem implies that

$$\int_{p_1^{-1}(\gamma)}(d\log(\eta)) \wedge (c_1[C_{i_0} \cap ... \cap C_{i_k}]) \wedge \widetilde{\omega}_1 = \int_{p_1^{-1}(\gamma)} d\left((d\log(\eta)) \wedge \beta \wedge \widetilde{\omega}_1\right) =$$
$$\int_{\partial\left(p_1^{-1}(\gamma)\right)}(d\log(\eta)) \wedge \beta \wedge \widetilde{\omega}_1 = 0$$

The last equality follows from the fact that $\partial\left(p_1^{-1}(\gamma)\right) = \emptyset$. So we can conclude that

$$\int_{p_1^{-1}(\gamma)}(d\log(\eta)) \wedge (c_1[C_{i_0} \cap ... \cap C_{i_k}]) \wedge \widetilde{\omega}_1 = 2\pi \int_\gamma \omega_{n-k} = 0.$$

On the other hand we know that

$$\int_\gamma \omega_{n-k} \neq 0.$$

So we got a contradiction. Proposition 28 is proved. $\blacksquare$.

In order to finish the proof of our Theorem we will need some facts from the Theory of Mixed Hodge Structures.

It is easy to see that $\gamma \notin \operatorname{Im} G_k$ implies that there exists $\omega_{n-k} \in H^{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$ such that

$$\int_\gamma \omega_{n-k} \neq 0.$$

We may assume that ω_{n-k} is the Poincare dual of γ. The condition $\gamma \notin \operatorname{Im} G_k$ implies that $\omega_{n-k} \notin \operatorname{Im} G_k^*$. So from here, Theorem 23 and Proposition 28 we conclude that we can find a form $\widetilde{\omega}$ on $X_0 \backslash \operatorname{Sing}(X_0)$ such that $R^{[k]}(\widetilde{\omega}) = \omega_{n-k}$. From the theory of Leray residues it follows that

$$\int_{p_k^{-1}(\gamma)} \widetilde{\omega} = \int_\gamma \omega_{n-k} \neq 0.$$

Lemma 26 is proved. ■.

Proof of Lemma 27: We need to prove that if $\gamma \notin \operatorname{Im} G_k$, then $\gamma_t = \mathcal{C}_t^{-1}(\gamma)$ represent a non zero class of cohomology in $H_n(X_t, \mathbb{Q})$. Let ω_t=$\mathcal{C}_t^*(\widetilde{\omega})$. It is easy to see from the definition of the Clemenc map that ω_t is a well defined closed $n-$form on X_t. On the other hand since $\mathcal{C}_t$ is a diffeomorphism between $X_t \backslash \mathcal{C}_t^{-1}(\mathrm{Sing}(X_0))$ and $X_0 \backslash \mathrm{Sing}(X_0)$ we deduce that

$$\int_{p_k^{-1}(\gamma)} \widetilde{\omega} = \int_{\gamma_\tau} \omega_t = \int_\gamma \omega_{n-k} \neq 0.$$

The last inequality implies Lemma 27. Lemma 27 is proved. ■.

Proof of Theorem 24 i: Theorem 24 follows directly from Theorem 10 and Lemma 27. Theorem 24 part **i** is proved. ■.

Proof of Theorem 24 **ii:**Let $\alpha_1, .., \alpha_k$ be a basis of cycles of $H_n(\Pi(X_0), \mathbb{Q})$, where $\Pi(X_0)$ is defined in Definition 12. From the definition of $\Pi(X_0)$ we can assume that the cycle α_k consists of the n-dimensional simplices $S_{1_k}, ..., S_{N_k}$ such that the boundary of the cycle α_k is zero. Each n dimensional simplex S_i corresponds to a point $q_i = C_{j_{0,k}} \cap ... \cap C_{j_{n,k}}$ according to Definition 12. The fact that the n-dimensional simplexes $S_{1_k}, ..., S_{N_k}$ form a cycle means say that any singular points q_i and q_j can be joint by Riemann surface, which means that they lie on some $C_{j_{1,k}} \cap ... \cap C_{j_{n,k}}$. This follows directly from the fact that the boundary of the cycle formed from $S_{1_k}, ..., S_{N_k}$ is zero. So on each Riemann surface of the form $C_{j_{1,k}} \cap ... \cap C_{j_{n,k}}$ that contains the points q_i and q_j we can find a meromorphic form of the third kind ω_{ij} which has poles only at the points q_i and q_j with residues say $+1$ at q_i and -1 at q_j. From the spectral sequence defined in Definition 21, Theorem 22 and Theorem 23 we deduce that there is a holomorphic form ω_k in $X_0 \backslash \mathrm{Sing}(X_0)$ such that Poincare residue of this form on each Riemann surface of the form $C_{j_{1,k}} \cap ... \cap C_{j_{n,k}}$ that contains the points q_i and q_j is equal to ω_{ij}. Let the meromorphic form ω_k will be non-zero on the component $C_{j_{0,k}}$ of $\mathrm{Sing}(X_0)$, where $q_i = C_{j_{0,k}} \cap ... \cap C_{j_{n,k}}$ and q_i was defined as above. Suppose that the divisors $C_{j_{0,i}} \cap C_{j_{m,i}}$ are given by the equation $z_m = 0$ in $C_{j_{0,i}}$. Let us consider the cycle $p_n^{-1}(q_i)$ defined by $|z_1| = \varepsilon, ..., |z_n| = \varepsilon$ in $C_{j_0} \backslash (Sing(X_0) \cap C_{j_0})$. It is easy to see that the form ω_k locally around the point $q_i = C_{j_{0,k}} \cap ... \cap C_{j_{n,k}}$ will be given by

$$\omega_k|_U = \frac{dz_1 \wedge ... \wedge dz_n}{z_1 \times ... \times z_n}.$$

So we can conclude that

$$\int_{p_n^{-1}(q_i)} \omega_k = (2\pi \sqrt[2]{-1})^n \neq 0.$$

From Lemma 9 we deduce that the cycles $\mathcal{C}_t^{-1}(p_n^{-1}(q_i))$ and $\mathcal{C}_t^{-1}(q_i)$ are homological to each other in X_t. From the fact that all direct images $R^i \pi_* \omega_{\mathcal{X}/\mathcal{D}}(\log(X_0))$ of the sheaf $\omega_{\mathcal{X}/\mathcal{D}}(\log(X_0))$ are locally free sheaves on D, we deduce from the exact sequence

$$0 \to \omega_{\mathcal{X}/\mathcal{D}}(\log(X_0)) \stackrel{\otimes t}{\to} \omega_{\mathcal{X}/\mathcal{D}}(\log(X_0)) \to \omega_{\mathcal{X}/\mathcal{D}}(\log(X_0))|_{X_0} \to 0$$

that we have the following exact sequence:

$$0 \to H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathcal{D}}(\log(X_0)))$$
$$\stackrel{\otimes t}{\to} H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathcal{D}}(\log(X_0))) \to H^0(X_0, \omega_{\mathcal{X}/\mathcal{D}}(\log(X_0))|_{X_0}) \to 0$$

It is easy to see that the theory of mixed Hodge structures implies that as a free module over $H^0(\mathcal{D}, \mathcal{O}_{\mathcal{D}})$ the module $H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathcal{D}}(\log(X_0)))$ is of rank bigger or equal to $\dim_{\mathbb{Q}} H_n(\Pi(X_0), \mathbb{Q})$. So we can find $\omega \in H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathcal{D}}(\log(X_0)))$ such that

$$\omega|_{D_{j_0,k}} = \omega_k.$$

Let us define $\omega_t = \omega|_{X_t}$. So we get a holomorphic family of holomorphic n-forms $\omega_{t,k}$ on X_t such that

$$\lim_{t\to 0}\omega_t = \omega_0 \text{ and } \omega_0|_{D_{j_{0,k}}} = \omega_k.$$

From here we deduce that for small enouph t and Lemma 9 we have that

$$\int_{\mathcal{C}_t^{-1}(q_i)} \omega_t \neq 0.$$

From the last inequality we derive that the cycle $\mathcal{C}_t^{-1}(q_i)$ is a non zero element of $H_n(X_t, \mathbb{Q})$. Now Theorem 24 part **ii** follows directly from Corollary 11. Theorem 24 part **ii** is proved. ■.

Definition 29 *Let us define the geometric genus of X_t as follows:*

$$p_g(X_t) = \dim_{\mathbb{C}} H^0(X_t, \Omega_t^n) \text{ for } t \neq 0$$

Corollary 30 *We have the following formula for the geometric genus*:

$$p_g(X_t) =$$
$$\sum_{i=0}^m p_g(C_i) + \sum_{k=0}^{n-1} \sum_{|I|=k+1} \dim_{\mathbb{C}} H^0(C_I, \Omega^{n-k}) + \dim_{\mathbb{C}} H_n(\Pi(X_0), \mathbb{Q}).$$

5 Applications of Clemens' Theory to Complete Intersections

5.1 A Simple Criteria for the Existence of Jordan Block of Maximal Dimension

In this paragraph we will prove the following Theorem, which we will apply later to complete intersections in toric varieties:

Theorem 31 *Let $\pi : \mathcal{X} \to \mathcal{D}$ be a family of Kähler varieties over the disk such that:*

1. *For* $t \neq 0$, $\pi^{-1}(t) := X_t$ *is a non singular variety of complex dimension* $n \geq 1$.

2. $\dim_{\mathbb{C}} H^0(X_t, \Omega^n) \geq 1$ *for* $t \neq 0$.

3. $\pi^{-1}(0) = X_0 = \cup_{i=0}^{m} C_i$ *is a divisor of normal crossing and each* C_i is irreducible.

4. $C_0 \cap .. \cap C_n = q$ *is a point.*

5. *There exists a holomorphic section* $\omega \in H^0(\mathcal{X}, \Omega^n_{\mathcal{X}/\mathcal{D}} < \log X_0 >)$ *such that the restriction of* ω to C_0, $\omega_0 := \omega\,|_{C_0}$ *has the following expression in an open set* $\mathcal{U}$ *around the singular point* $q := C_0 \cap .. \cap C_n$ *:*

$$\omega_0\,|_{C_0 \cap \mathcal{U}} := \omega\,|_{C_0 \cap \mathcal{U}} = \frac{dz_1 \wedge ... \wedge dz_n}{z_1 \times ... \times z_n}$$

Then the monodromy operator of the family $\pi : \mathcal{X} \to \mathcal{D}$ *has a Jordan block of size* $n+1$.

Proof of Theorem 31: Let $z_0, ..., z_n$ be local coordinates near q such that $z_i = 0$ local equation of for C_i, for $i = 1, .., n$. Then $z_1, .., z_n$ can be regarded as coordinates on C_0 near q. Let T_0 be the n dimensional real torus in $C_0 \backslash (C_i \cap C_0)$ defined by $|z_i| = \varepsilon$ for $i = 1, .., n$.

In the notations of Definition 5, T_0 is just $p_n(\gamma)^{-1}(q)$, where γ is the homology class of the point q. From the properties of the Clemens map $\mathcal{C}_t$ it follows Lemma 9 that the real n cycle $\mathcal{C}_t^{-1}(T_0) = T_t$ is homological to $\gamma_t := \mathcal{C}_t^{-1}(\gamma)$. Let $\omega_t = \omega|_{X_t}$. The idea of the proof is to show that for t small enough, T_t is not homological to zero, by showing that

$$\int_{T_t} \omega_t \neq 0.$$

This fact together with Corollary 11 implies Theorem 31. So we need to prove the following Proposition:

Proposition 32 *The real cycle* T_t *is not homological to zero on* X_t*.*

Proof of Proposition 32: Condition **5** implies that $\lim\limits_{t \to 0} \omega_t = \omega_0$ exists and locally around the singular point $q \in C_0$ of X_0, ω_0 is given by the following expression:

$$\omega_0\,|_{C_0 \cap \mathcal{U}} = \frac{dz_1 \wedge ... \wedge dz_n}{z_1 \times ... \times z_n}.$$

From this expression we obtain that

$$\int_{T_0} \omega_0 = \int_{T_0} \frac{dz_1 \wedge ... \wedge dz_n}{z_1 \times ... \times z_n} = \left(2\pi\varepsilon\sqrt{-1}\right)^n \neq 0.$$

From the fact that

$$\lim_{t \to 0} \int_{T_t} \omega_t = \int_{T_0} \omega_0 \neq 0$$

we obtain that small t we have:

$$\int_{T_t} \omega_t \neq 0$$

This proves Proposition 32. ■. Theorem 31 is proved. ■.

5.2 Families of Complete Intersection CY Manifolds in Projective Space with Maximal Unipotent Monodromy

Definition 33 *We will consider the following one parameter family of complete intersections of CY manifolds in* $\mathbb{P}^{n+k}$ *for* $n \geq 4$ *and* $k \geq 1$ *defined by the following equations*:

$$G_{1,t} = tF_1 - \prod_{i=0}^{n_1} x_i = 0, .., G_{k,t} = tF_k - \prod_{j=n_1+..+n_{k-1}}^{n_k} x_j = 0,$$

where the system $F_1 = .. = F_k = 0$ *defines a non singular CY manifolds,*

$$n_i = \deg F_i \geq 2 \text{ and } \sum n_i = n+k+1$$

and x_i *are the standard homogeneous coordinates in* $\mathbb{P}^{n+k}$.

The condition $\sum n_i = n+k+1$ implies that the fibers $\pi^{-1}(t) = X_t$ for $t \neq 0$ are CY manifolds of complex dimension n.

Let us denote the family defined in Definition 33 by

$$\mathcal{X} \to \mathcal{D}, \text{ where } \mathcal{X} \subset \mathbb{P}^N \times \mathcal{D}.$$

Theorem 34 *The family* $\pi : \mathcal{X} \to \mathcal{D}$ *has a monodromy* T *of maximal unipotent index, i.e.* $(T^M - id)^{n+1} = 0$ *and* $(T^M - id)^n \neq 0$.

Proof of Theorem 34: Our proof is based on checking the conditions of Theorem 31 for this particular family. First we will check that the singular fiber $X_0 := \pi^{-1}(0)$ consists of linear subspaces C_i isomorphic to $\mathbb{P}^n$. We will prove that there exists a component say C_0 such that another n linear spaces C_i intersect C_0 transversely along $\mathbb{P}^{n-1}$. First we will prove the following Proposition:

Proposition 35 *The singular fiber of the family* $\mathcal{X} \to \mathcal{D}$ *consists of exactly* $\prod_{i=1}^{k}(n_i - 1)$ n *dimensional linear subspaces isomorphic to* $\mathbb{P}^n$.

Proof of Proposition 35: From the Definition 33 it follows that $X_0 := \pi^{-1}(0)$ is defined by the equations:

$$\prod_{i=0}^{n_1} x_i = 0, .., \prod_{j=n_1+..+n_{k-1}}^{n_k} x_j = 0$$

and thus X_0 is a union of linear subspaces of dimension n defined by the equations:

$$x_{j_1} = 0, .., x_{j_k} = 0,$$

where $0 \leq x_{j_1} \leq n_1, .., n_1 + .. + n_{k-1} < x_{j_k} \leq N$. From these equations Proposition 35 follows directly. $\blacksquare$.

Proposition 36 *There exists a component* C_0 *of the singular fiber* X_0 *of the family* $\pi : \mathcal{X} \to \mathcal{D}$ *such that* C_0 *is isomorphic to* $\mathbb{P}^n$. *Let* $(z_0 : .. : z_n)$ *be the homogeneous coordinates in* $\mathbb{P}^n$, *then there exists* n *components of the singular fiber* X_0 *say* $C_1, .., C_n$ *such that* $C_i \cap C_0$ *is defined in* $D_0 \approxeq \mathbb{P}^n$ *by the equation* $z_i = 0$.

Proof of Proposition 36: Let us denote by

$$d_1 = n_1,\ d_2 := n_1 + n_2, .., d_j := n_1 + ... + n_j, .., d_k = n_1 + ... + n_k.$$

We already proved that each component of X_0 is a linear subspace isomorphic to $\mathbb{P}^n$. Let D_0 be the subspace defined by the equations:

$$x_0 = x_{d_1} = x_{d_2} = ... = x_{d_{k-1}} = 0.$$

The coordinates $(x_{i_1} : x_{i_2} : ... : x_{n+k})$ define a homogeneous coordinates on C_0 if all x_{i_m} are different from the x_{d_j} that define C_0. Let us define C_{i_m} for $1 \leq m \leq n$ by the following system of equations: $x_{j_1} = .. = x_{j_k} = 0$, where all the indexes $(j_1, .., j_k)$ with an exception of one is equal to indexes $(0, d_1, .., d_{k-1})$. These equations define a component of the singular fiber. Indeed the necessary and sufficient condition for a system of linea equations $x_{j_1} = .. = x_{j_k} = 0$ to define a component of the singular fiber X_0 of the family $\pi : \mathcal{X} \to \mathcal{D}$ is $0 \leq j_1 \leq d_1,\ d_1 < j_2 \leq d_3, .., d_{k-1} < j_k \leq d_k$. Here we are using the conditions that $n_i \geq 2$ and $n_1 + ... + n_k = n + k$. Clearly $C_{i_m} \cap C_0$ will be a $n-1$ dimensional linear subspace in $C_0 \approxeq \mathbb{P}_{n-1}$. From here it follows that we constructed C_i for $i = 1, .., n$ such that $C_0 \cap ... \cap C_n = (0 : ... : 1) = q$. So we proved Proposition 36. $\blacksquare$.

Proposition 37 *There exists a section* $\omega \in H^0(\mathcal{X}, \Omega^n_{\mathcal{X}/\mathcal{D}} \log < X_0 >)$ *such that if* $\omega|_{X_t} = \omega_t$ *for* $t \neq 0$ *then:*

$$\lim_{t \to 0} \omega_t = \omega_0$$

where when we restrict ω_0 *on* $C_0 \approxeq \mathbb{P}^n$*, then* $\omega_0 |_{U_i}$ *is given by the formula:*

$$\omega_0|_{U_i} = \tfrac{dz^1 \wedge .. \wedge dz^n}{z^1 ... z^n}.$$

where U_i *is the standard covering of* $\mathbb{P}^n$.

Proof of Proposition 37: Since we choose $p = (0, 0, .., 1)$ we will work in the open set $U_{n+k} := \{(x_0 : .. : x_{n+k}) | x_{n+k} \neq 0\}$. Let us consider the meromorphic form

$$\Omega_t := \tfrac{dz^1 \wedge .. \wedge dz^{n+k}}{\tilde{G}_{t,1} .. \tilde{G}_{t,k}}, \text{ where } \tilde{G}_{t,i} := G_{t,i}(\tfrac{x_0}{x_{n+k}}, .., \tfrac{x_{n+k-1}}{x_{n+k}}, 1) \text{ and } z^i = \tfrac{x_i}{x_{n+k}}.$$

on $U_{n+k} \times \mathcal{D}$. By taking k times the Leray residue of Ω_t we define the holomorphic n-form ω_t on X_t. Suppose that C_0 in $U_{n+k} \times \mathcal{D}$ is given by the equations (by suitably reordering the coordinates):

$$z^{n+1} = .. = z^{n+k} = 0 \text{ and } t = 0.$$

(See [9].) Let $z^1, .., z^n$ be the rest of the coordinates in U_{n+k}. Let us choose an open set $\mathcal{W}$ in $\mathcal{X} \subset \mathbb{P}^N \times D^*$ where $z^1, .., z^n, t$ are local coordinates of $\mathcal{W}$, then from the definition of the Leray residue it follows that on $\mathcal{W}$ we have:

$$\omega_t|_{\mathcal{W}} := \frac{dz^1\wedge..\wedge dz^n}{\partial_{n+1}...\partial_{n+k}(\tilde{G}_{t,1}\times..\times\tilde{G}_{t,N-n})}, \text{ where } \partial_i = \frac{\partial}{\partial z^i}.$$

From the last formulas we get that

$$\lim_{t\to 0}\partial_{n+1}...\partial_{n+k}(\tilde{G}_{t,1}\times..\times\tilde{G}_{t,k}) = z^1\times..\times z^n.$$

So we deduce that

$$\lim_{t\to 0}\omega_t|_{\mathcal{W}} = \omega_0 = \frac{dz^1..\wedge dz^n}{z^1\times..\times z^n}.$$

Proposition 37 is proved. ■.

Propositions 36 and 37 imply that the conditions of Theorem 31 are satisfied by the family $\pi : \mathcal{X} \to \mathcal{D}$. So Theorem 31 implies Theorem 34. Theorem 34 is proved. ■.

Remark 38 *It is not at all difficult to generalized the contsruction of the one parameter family of complete intersections of CY manifold in* $\mathbb{CP}^N$ *with maximal unipotent element in the monodromy group to the case of of complete intersections of general type. More precisely it is very easy to prove that there exists a family* $\pi : \mathcal{Y} \to \mathcal{D}$ *of complete intersections in* $\mathbb{CP}^N$ *of general type such that if the dimension of the fiber is* n, *then the monodromy operator* T *contains exactly* $p_g(Y_t)$ *Jordan blocks of dimension* $n+1$, *where* $p_g = \dim_{\mathbb{C}} H^0(Y_t, \Omega^n_{Y_t})$.

6 Construction of a Family of CY Manifolds in Toric Varieties with Maximal Unipotent Monodromy

6.1 Introduction to Toric Varieties

Let $T = (\mathbb{C}^*)^N$. Let $\mathbb{N}$ be a rank N lattice, and Σ be a complete fan relative to $\mathbb{N}$. (See [14].)

Notations:

$\mathbb{P}_\Sigma = V$: toric variety defined by Σ.

$\mathbb{N}^\vee$: lattice dual to $\mathbb{N}$.

$\Sigma(k)$: the set of k-cones in Σ.

$\tau^\vee$: the dual of a cone τ.

D_ρ : the toric divisor corresponding to $\rho \in \Sigma(1)$.

$\mathcal{O}(D) = \mathcal{O}_V(D)$: the line bundle (invertible sheaf) associated to the divisor D.

We follow the usual combinatorial description of a cone τ in Σ, and often use the set of edges of τ or its primitive generators to denote the cone. Let O_ρ, $\rho \in \Sigma(1)$, be the codimension 1 T-orbit and D_ρ be their respective closures. Thus they are irreducible T-invariant Weil divisors in V. Let R be the polynomial ring $\mathbb{C}[x_\rho,\ \rho \in \Sigma(1)]$. If we declare that $\deg x_\rho = [D_\rho] \in A_{N-1}(V)$, then R becomes a $A_{N-1}(V)$-graded ring, where $A_{N-1}(V)$ is the Picard group of V. We denote the degree $[D]$ subspace in R by $R_{[D]}$. It is known that (see [1][5] [13]):

$$R_{[D]} \approxeq H^0(V, \mathcal{O}_V(D)).$$

The graded ring R is called the homogeneous coordinate ring of V. The isomorphism above can be described as follows. The dual lattice $\mathbb{N}^\vee$ can be viewed as the lattice of characters of the group T. We denote a character multiplicatively corresponding to $\nu \in \mathbb{N}^\vee$ by χ^ν. Then

$$H^0(V, \mathcal{O}_V(D)) = \oplus_{\nu \in P_D \cap \mathbb{N}^\vee} \mathbb{C} \cdot \chi^\nu$$

where

$$P_D = \{\nu \in \mathbb{N}^\vee \otimes_{\mathbb{Z}} \mathbb{R} |\ \langle \nu, \rho \rangle \geq -a_\rho\ \forall \rho\}.$$

Here ρ denotes the primitive generators of the 1-cones, and $D = \sum a_\rho D_\rho$ is the Weil divisor defining the line bundle $\mathcal{O}_V(D)$. The isomorphism ϕ_D above is given by $\phi_D : \chi^\nu \to \prod_\rho x_\rho^{\langle \nu, \rho \rangle + 1}$. The assertion above gives a description of the space of sections of all equivariant line bundles over V.

Let us consider the section x_ρ. Note that $\nu = 0$ is in P_{D_ρ}, hence by the isomorphism above $\phi_{D_\rho}(\chi^0) = x_\rho$ is a section of $\mathcal{O}_V(D)$. In fact the zero set of $x_\rho = 0$ is exactly the divisor D_ρ.

For mirror symmetry, the most interesting case is when the convex hull of the 1-cone generators $\rho \in \mathbb{N}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ form a reflexive polytope Δ, and that D is the anticanonical divisor. (See [2] and [3].) In this case P_D is the polar dual of Δ. Here are some examples:

1. $V = \mathbb{P}^2$, $D = D_1 + D_2 + D_3$, D$_i$ being the $i^{th}-$hyperplane. The homogeneous coordinate ring of V is the usual $\mathbb{C}[x_1, x_2, x_3]$. If we identify $A_1(V)$ with $\mathbb{Z}$ such that $[H] \to 1$, then x_i has degree 1. The polytope Δ is generated by the 1-cone generators ρ_i is the triangle with vertices (1,0), (0,1), (-1,-1). Its dual ∇ is the triangle with vertices (2,-1), (-1,2), (-1,-1). It is easy to check that ∇ has exactly 10 lattice points m. They are in 1-1 correspondence with the degree 3 monomials in x_i.

2. Let ξ be a primitive third root of unity. Let $\mathbb{Z}_3$ acts on $\mathbb{P}^2$ by $[x_1, x_2, x_3] \mapsto [\xi x_1, \xi^{-1} x_2, x_3]$. Resolve all the singularities in $\mathbb{P}^2/\mathbb{Z}_3$ "minimally". The result is a toric variety V with 9 hyperplanes D_i all together (with two linear relations). The anticanonical divisor in V is $D = D_1 + ... + D_9$. The polytope Δ generated by the 1-cone generators n_ρ is now the triangle with vertices (2,-1), (-1,2), (-1,-1). Its dual ∇ is the triangle with vertices (1,0), (0,1), (-1,-1). Since ∇ has exactly 4 lattice points, K_V^{-1} has 4 sections – one of them being $x_1 \cdots x_9$.

6.2 Construction of One Parameter Family of CY Complete Intersection in a Toric Variety with Maximal Jordan Block in the Monodromy Operator

Let $\pi_1, ..., \pi_k$ be a partition of the set $\Sigma(1)$. We assume that $F_i \in H^0(V, \mathcal{O}_V(\sum_{\rho\in\pi_i} D_\rho))$ together define a nonsingular subvariety X of codimension k:

$$F_1 = ... = F_k = 0.$$

Put $n = N - k$. Since the anticanonical class is

$$[K_V^{-1}] = \sum_\rho D_\rho,$$

by adjunction X is a Calabi-Yau manifold. We consider the following 1-parameter family of complete intersections X_t defined by

$$G_{i,t} = tF_i - \prod_{\rho\in\pi_i} x_\rho = 0,\ i = 1, ..., k.$$

Let us denote this family when $|t| < 1$ by $\pi : \mathcal{X} \to \mathcal{D}$. The fiber X_0 is the union of toric subvarieties

$$C_\sigma := \cap_{\rho\in\sigma} D_\rho$$

where σ is any subset of $\Sigma(1)$ such that $|\sigma \cap \pi_i| = 1$ for all i. It turns out that (see Appendix) C_σ is nonempty iff the 1-cones in σ generate a cone in Σ. In this case, C_σ is a nonsingular irreducible toric subvariety in V of codimension k corresponding to the k-cone $\sigma \in \Sigma(k)$. Thus such a nonempty C_σ is a component in X_0.

Remark 39 *The assumption that the F_i defines a codimension k subvariety is important. This put a strong constraint on the kind of partitions $\pi_1, ..., \pi_k$ that are allowed. A general partition of $\Sigma(1)$ will fail to satisfy this condition. Here is an example. Take the 4-dimensional weight projective space $\mathbb{P}[9, 6, 1, 1, 1]$. After a minimal desingularization, the resulting toric variety has 9 1-cones $\rho_1, ..., \rho_9$ in its fan Σ. After suitable ordering, it is easy to find a partition of the form $\pi_1 = \{\rho_1, \rho_2\}$, $\pi_2 = \{\rho_3, \rho_4\}$, $\pi_3 = \{\rho_5, ..., \rho_9\}$ such that $\{\rho_1, \rho_3\}$, $\{\rho_1, \rho_4\}$, $\{\rho_2, \rho_3\}$, $\{\rho_2, \rho_4\}$, are all primitive collections. In this case, X_0 would be empty because the $C_\sigma = D_{\rho_i} \cap D_{\rho_j} \cap D_{\rho_k}$, for $\rho_i \in \pi_1$, $\rho_j \in \pi_2$, $\rho_k \in \pi_3$, will all be empty.*

Throughout this section, we shall make the following additional assumption: that the convex hull of the set of primitive generators ρ of the 1-cones in Σ is a reflexive polytope, which we shall denote by Δ.

Theorem 40 *The monodromy operator of the family $\mathcal{X} \to \mathcal{D}$ has one Jordan block of dimension $n + 1$.*

6.3 Proof of Theorem 40

Lemma 41 *Let C_{σ_0} be a fixed component in X_0. Then there exists n other components $C_{\sigma_1}, ..., C_{\sigma_n}$ with the property that $C_{\sigma_0} \cap C_{\sigma_i}$ is a codimension $k+1$ toric subvariety, and that*

$$C_{\sigma_0} \cap ... \cap C_{\sigma_n}$$

is a toric fixed point.

Proof: Since the fan Σ is complete, we can find an $N-$ cone, say τ, contaning σ_0 as a k-face. Write $\sigma_0 = \langle \rho_1, ..., \rho_k \rangle$, where the $\rho_i \in \pi_i$ are the canonical generators of σ_0. Similarly, write τ in terms of its 1-cone generators:

$$\tau = \left\langle \rho_1, ..., \rho_k, \rho_1^{'}, ..., \rho_n^{'} \right\rangle .$$

Then each $\rho_j^{'}$ lies in a unique say π_{i_j}. Since V is non-singular, τ is a simplicial cone, so that any k of its generators generate a k-cone in Σ. In particular we have the k-cones

$$\sigma_j := \left\langle \rho_1, ..., \hat{\rho}_{i_j}, ..., \rho_k, \rho_j^{'} \right\rangle , \; j = 1, ..., n.$$

That is, σ_j is obtained from σ_0 by replacing the generator ρ_{i_j} of σ_0 by $\rho_j^{'}$. Since both ρ_{i_j} and $\rho_j^{'}$ live in the same π_{i_j}, it follows that C_{σ_j}, as defined above, is a component in X_0. By construction,

$$C_{\sigma_0} \cap C_{\sigma_j} = \cap_{\rho \in \sigma_0 \cup \sigma_j} D_\rho.$$

But $\sigma_0 \cup \sigma_j$ is the list σ_0 adjoint with $\rho_j^{'}$, hence gives a $(k+1)$-cone in Σ. So $C_{\sigma_0} \cap C_{\sigma_j}$ is a codimension $(k+1)$ toric subvariety in V. Moreover,

$$C_\tau = \cap_{\rho \in \sigma_0 \cup ... \cup \sigma_n} D_\rho = \cap_{\rho \in \tau} D_\rho$$

which is codimension $k + n = N$ toric subvariety of V. Hence it is a fixed point.$\blacksquare$.

Note that the nonsingular toric subvariety C_{σ_0} comes with a standard affine coordinates on the affine patch containing fixed point above. Namely, they are obtained from restricting the standard affine coordinates on the patch

$$U_\tau = Hom_{sg}(\tau^\vee, \mathbb{C}^*) \approxeq (\mathbb{C}^*)^N.$$

Here the isomorphism is determined by the choice of ordering of the set of primitive generators of the cone $\tau^\vee$.

Lemma 42 *There exists a meromorphic form on $V = \mathbb{P}_\Sigma$ with simple poles along each hypersurface $F_i = 0$.*

Proof: First let X be a complex N-fold, and K_X its canonical bundle. Let $\{U_\sigma\}$ be a covering of charts on X, whose coordinates $U_\sigma \to \mathbb{C}^N$ we denote $z^\sigma = (z_1^\sigma, ..., z_N^\sigma)$. From this data, we get a frame $dz^\sigma = dz_1^\sigma \wedge ... \wedge z_N^\sigma$ on each U_σ for the bundle K_X, hence a dual frame for the dual bundle K_X^{-1}. Suppose f is a nonzero global section of K_X^{-1}. Then relative to a dual frame above, f is represented as a holomorphic function $f_\sigma(z^\sigma)$. Note that on any overlap $U_\sigma \cap U_\tau \neq \emptyset$, the functions f_σ, f_τ transform by the same transition function as the frame dz^σ and dz^τ. It follows that the local expressions

$$\frac{dz^\sigma}{f_\sigma(z^\sigma)}$$

together define a global meromorphic N-form on X with poles along the zero locus $f = 0$. We now assume that X is an algebraic variety and the U_σ are affine patches. We now apply this to an N-dimensional complete toric variety $V = \mathbb{P}_\Sigma$, as before.

Recall that $\mathbb{P}_\Sigma$ is covered by the affine subvarieties $U_\sigma \approxeq \mathbb{C}^N$, labelled by N-cones σ in Σ. Here the isomorphism is determined by an ordering of the set of N integral generators of the cone σ. We denote by $z^\sigma = (z_1^\sigma, ..., z_N^\sigma)$ the coordinates of this isomorphism. Let

$$\nabla := \{\nu \in \mathbb{N}^\vee \otimes \mathbb{R} | \, \langle \nu, x\rangle \geq -1, \, \forall x \in \Delta\}.$$

This is the dual of the reflexive polytope Δ.

The global sections of the anticanonical bundle K_V^{-1} on $V = \mathbb{P}_\Sigma$ has the following description. There is a basis of $H^0(V, K_V^{-1})$ which corresponds 1-1 with the set $\nabla \cap \mathbb{N}^\vee$. Let's fix an ordering of the set $\Sigma(1)$ and denote the generators by $\rho_1, ..., \rho_S \in \mathbb{N}$. Let L be the kernel of the natural map

$$\mathbb{Z}^S \to \mathbb{N}, \, m \to \sum m_i \rho_i.$$

Let $L^\perp$ be the orthogonal complement of L in $\mathbb{Z}^S$ with respect to the standard inner product $l \cdot l'$. Then $\mathbb{N} \approxeq \mathbb{Z}^S/L$, and the natural pairing $L^\perp \times \mathbb{Z}^S/L \to \mathbb{Z}$, $(l, l' + L) \mapsto l \cdot l'$, defines a canonical isomorphism $L^\perp \approxeq \mathbb{N}^\vee$. Then a general section of K_V^{-1} is of the form

$$f = x_1 ... x_S \sum_{\nu \in \nabla \cap \mathbb{N}^\vee} c_\nu x^\nu.$$

(See [1], [5] and [13].) Here the c_ν are arbitrary complex numbers, x_i are the homogeneous coordinates of V, viewed as a section of $\mathcal{O}(D_{\rho_i})$, and x^ν is the monomial in the x_i with exponents $\nu \in L^\perp \subset \mathbb{Z}^S$.

As an example, when $V = \mathbb{P}^N$, we have $K_V^{-1} = \mathcal{O}_{\mathbb{P}^N}(N+1)$, and x_i are the usual homogeneous coordinates. In this case a general section of K_V^{-1} above is exactly a degree $N+1$ homogeneous polynomial in the x_i.

We now return to the general case. Let $e_1, ..., e_N$ denotes the standard basis of $\mathbb{Z}^N$. Let σ be any N-cone in the fan Σ, and fix an ordering $\nu_1, ..., \nu_N$ for the primitive generators of the dual cone $\sigma^\vee$.

Proposition 43 *Relative to the frame dual to dz^σ above, sections of K_V^{-1} are represented by polynomial functions of the form*

$$f_\sigma(z) = z_1...z_N \sum_{v\in\nabla_\sigma} c_v z^v.$$

Here $z_i = z_i^\sigma$, c_v are arbitrary numbers, $\nabla_\sigma \subset \mathbb{Z}^N$ is the image of $\nabla \cap \mathbb{N}^\vee$ under the isomorphism $\mathbb{N}^\vee \to \mathbb{Z}^N$ determined by $\nu_i \to e_i$.

Proof:A priori**,** $f_\sigma(z)$ given above is a Laurent poly nomial in z. We will first show that it is in fact a polynomial. Recall that every element $\nu \in \nabla$ satisfies $\langle \nu, x\rangle \geq -1$ for all $x \in \Delta$. Let v be the image of ν under the isomorphism $\mathbb{N}^\vee \to \mathbb{Z}^N$ determined by $\nu_i \to e_i$, and let a be the preimage of $x \in \Delta$ under the dual map $\mathbb{Z}^N \to \mathbb{N}$. Then $v \cdot a = \langle \nu, x\rangle \geq -1$. In particular if x is then preimage generator of the cone σ, then $\langle \nu_k, x\rangle = 1$ for one k and zero otherwise. This means that the preimage a of x is such that $e_i \cdot a = 1$ if $i = k$ and zero otherwise, i.e. $a = e_k$. Clearly, every e_k can be realized as the preimage of some primitive generator $x \in \Delta$ of σ. This shows that for any $v \in \nabla_\sigma$, we have $v \cdot e_k = \langle \nu, x\rangle \geq -1$ for all k. It follows that every Laurent monomial $z_1...z_N \cdot z^v$ appearing in $f_\sigma(z)$ has a nonnegative exponent, i.e. $f_\sigma(z)$ is a polynomial function on $\mathbb{C}^N$.

Let τ be any other N-cone in the fan Σ, and fix an ordering $\mu_1, ..., \mu_N$ for the primitive generators of $\tau^\vee$. Clearly the respective generators ν_i, μ_j for $\sigma^\vee, \tau^\vee$, determine an $A \in (a_{ij}) \in \mathbb{GL}(N, \mathbb{Z})$ such that

$$\nu_i = \sum_j a_{ij}\mu_j.$$

For simplicity, we write $z = z^\sigma$, $w = z^\tau$. The coordinates transition function is then given by

$$z_i = w^{A_i},\ A_i = (a_{i1}, ..., a_{iN}).$$

We consider the region of overlap of $U_\sigma \cap U_\tau$ where $z_i \neq 0$ and $w_j \neq 0$ for all i, j. First we show that the frames dz, dw of N-forms are related by the transformation law

$$dz = \det(A)\tfrac{z_1...z_N}{w_1...w_n}dw.$$

We have

$$\tfrac{dz}{z_1...z_N} = d\log z_1 \wedge ...d\log z_N.$$

Now $\log z_i = \sum_j a_{ij} \log w_j$ on the overlap. So the transformation law above follows immediately.

It remains to show that there is a polynomial function $f_\tau(w)$ such that

$$f_\sigma(z) = \det(A)\tfrac{z_1...z_N}{w_1...w_N} f_\tau(w).$$

The matrix A defines an isomorphism $B : \mathbb{R}^N \to \mathbb{R}^N$, $Bv = \sum_j a_{ij}v_i$. Then

$$z^v = w^{Bv} \ .$$

Moreover the image of $\nabla_\sigma \subset \mathbb{R}^N$ under B is ∇_τ. It follows that

$$f_\sigma(z) = z_1...z_N \cdot \sum_{v\in\nabla_\sigma} c_v z^v = z_1...z_N \sum_{u\in\nabla_\tau} c_{B^{-1}u} w^u.$$

So if we set

$$f_\tau(w) := w_1...w_N \cdot \sum_{u\in\nabla_\tau} \det(A)^{-1} c_{B^{-1}u} w^u,$$

then we obtain the desired function. ■.

Proposition 43 implies Lemma 42. ■.

As an example, when $V = \mathbb{P}^N$, we can choose the fan Σ in $\mathbb{Z}^N$, so that the 1-cone primitive generators are $e_0 = -e_1 - \cdots - e_N, e_1, .., e_N$. In this case, we have $N+1$ standard affine patches U_{σ_i}, where σ_i is the N-cone generated by $e_0, .., \hat{e}_i, ..., e_N$. The affine coordinate are $z^{\sigma_i} = (\frac{x_0}{x_i}, ..., 1_i, ... \frac{x_N}{x_i})$. For simplicity, we label each cone σ_i simply by the integer i. Then the coordinate transition functions are given by

$$z_i^\sigma = z_j^\tau \tfrac{x_\tau}{x_\sigma},\ i \neq \sigma, \tau;\ z_\tau^\sigma = \tfrac{1}{z_\sigma^\tau}.$$

The transition functions for coordinate N-forms are given by

$$dz^\sigma = (-1)^{\sigma+\tau} \tfrac{x_\tau}{x_\sigma} dz^\tau.$$

We now prove the general toric version of Proposition 37.

Proposition 44 *There exists a family of holomorphic n-forms ω_t on X_t for $t \neq 0$ such that*

$$\lim_{t\to 0} \omega_t = \omega_0$$

where ω_0 is a section on X_0 with the following properties: restricted to the component C_{σ_0}, ω_0 has the form:

$$\omega_0|_{U_\varepsilon} = \tfrac{dz_1 \wedge ... dz_n}{z_1 ... z_n}$$

where $(z_1, ..., z_n)$ are coordinates on the affine patch $U_\varepsilon \subset C_{\sigma_0}$.

Proof: Let $U_\tau \approxeq \mathbb{C}^N$ be the affine patch in $V = \mathbb{P}_\Sigma$ as given in the proof of Proposition 43. Let $z^\tau = z = (z_1, ..., z_N)$ be the affine coordinates. Here we choose the coordinates so that the affine subvariety $U_\varepsilon = U_\tau \cap C_{\sigma_0}$ is defined by

$$z_{n+1} = ... = z_N = 0$$

and the $z_1, ..., z_n$ are affine coordinates of U_ε. Choose a small analytic neighborhood $W \subset \mathcal{X} \cap U_\tau \times \mathcal{D}$ so that $z_1, ..., z_n, t$ are coordinates on $X_t \cap W$ for small t.

Clearly

$$f := G_{t,1}...G_{t,k} \in H^0(V, \mathcal{O}(\sum_{\rho\in\Sigma(1)} D_\rho)) = H^0(V, K_V^{-1}).$$

By Lemma 42, we have a meromorphic N-form given by

$$\Omega_t = \tfrac{dz}{g_t(z)}$$

where

$$g_t(z) = z_1...z_N(1 - tf_1(z))...(1 - tf_k(z)).$$

Here $f_i(z)$ is a Laurent polynomial representing the section F_i of $\mathcal{O}(\sum_{\rho\in\pi_i} D_\rho)$ on the algebraic torus $z_1...z_N \neq 0$. Note that for all t, g_t is polynomial in the z by construction. Integrating the form Ω_t via Leray residue k times, the result is a section ω_t of the sheaf of n-forms, whose restriction on $X_t \cap W$ is given by

$$\omega_t = \tfrac{dz_1\wedge...\wedge dz_n}{\frac{\partial}{\partial z_{n+1}}\cdots\frac{\partial}{\partial z_N} g_t(z)} \ .$$

It is clear that the dominator of ω_t goes to $z_1...z_n$ as $t \to 0$. This proved our assertion. ■.

Theorem 40 now follows directly from Theorem 31. ■.

Remark 45 *One can generalize the same constructions to the case of compete intersections of general type in toric varieties for which the canonical class is very ample. Namely one can prove that there exists one parameter family of complete intersections of general type $\mathcal{Y} \to \mathcal{D}$ for which the generic fibers is of dimension n and the canonical class is very ample, such that the monodromy operator contains exactly $p_g(Y_t)$ Jordan blocks of dimension $n + 1$.*

7 Comparisons

7.1 Relationship with Hypergeometric Functions

In [10], the construction of the point of maximal unipotent monodromy has been done for the family of CY hypersurfaces consisting of all anticanonical hypersurfaces modulo equivalence under T in a fixed toric variety. The construction there uses an entirely different approach. Namely, one exploits the fact that the periods of the family are solutions to a system of linear partial differential equations, known as a GKZ hypergeometric system and its generalization. (See [10] and the references there.) The construction is then done by carefully analyzing the structure of certain singularities of the PDE system. Explicit computation of certain period is also necessary in that approach.

We now briefly compare the approach of [10] with the present approach. For simplicity, we consider the case of hypersurfaces in an N dimensional toric variety $V = \mathbb{P}_\Sigma$. Recall that in order to apply Theorem 31, we must construct a meromorphic N-form Ω_t on V with a simple pole along the CY hypersurface X_t :

$$G_t := tF - \prod_\rho x_\rho = 0.$$

Here $F \in H^0(V, K_V^{-1})$ is a fixed section. In the affine coordinates $z^\sigma = (z_1, ..., z_N)$ on an affine patch $U_\sigma \approxeq \mathbb{C}^N$, Ω_t takes the form:

$$\Omega_t = \frac{dz_1 \wedge ... \wedge dz_N}{z_1 ... z_N(1 - tf(z))}$$

where $f(z)$ is a Laurent polynomial representing the section F. The Leray residue of Ω_t gives a holomorphic n-form on X_t. We then integrate ω_t over a real n-torus, giving a holomorphic function of t bounded near $t = 0$. This function is a period of the CY manifold X_t near $t = 0$. It is easy to see that this period can also be obtained by integrating Ω_t over a real N-torus α_0, $|z_1| = ... = |z_N| = \varepsilon$, in T :

$$\int_{\alpha_0} \Omega_t.$$

But this is precisely a special value of the holomorphic solution of GKZ system given in [10]. In fact, in [10], G_t is replaced by a general section, and the special Laurent polynomial $1 - tf(z)$ is replaced by its general counterpart. The result is a multi-variable holomorphic function (the variables being the coefficients of the general Laurent polynomial). A power series expansion of this function can be easily computed. (See [10].) The period $\int_{\alpha_0} \Omega_t$ above can be obtained by specializing the multi-variable function to 1-variable function by restricting it to the one-parameter family $G_t = 0$.

7.2 Comparison of Maximal Unipotent Monodromy and the SYZ Conjecture

Clemens' theorem 10 and our construction of the maximal unipotent monodromy operator T for complete intersection CY manifolds in a toric variety show that there exist special cycles, namely the invariant cycle γ and the cycles $\alpha_1, ..., \alpha_n$, such that

$$T(\alpha_n) = \gamma + \alpha_1 + ... + \alpha_n.$$

From Clemens' theory, we know that the cycle γ can be realized as a n dimensional torus. The cycle α_n can probably be realized as an n dimensional sphere. A theorem of Clemens says that γ and α_n have intersection number 1.

On the other hand, the SYZ conjecture asserts that a CY n-fold should admit the structure of a fibration over an n-sphere with generic fibers given by special Lagrangian n-tori. Moreover this fibration admits a canonical section. Thus the section is an n-cycle having intersection number 1 with the fiber cycle. It is quite clear that the cycle γ and α_n above should be realized precisely as the n-sphere section and the n-torus fiber respectively. This general relation deserves further investigation. For families of polarized K3 surfaces having maximal unipotent monodromy, this can be verified by using isometric deformation (relative to the Calabi-Yau metric) of the surface, as we will discuss next.

Theorem 46 *Let $\pi : \mathcal{X} \to \mathcal{D}$ be a family of polarized K3 surfaces with polarization class $e \in H^2(X,\mathbb{Z})$ such that $< e, e >= 2n$ for a fixed positive integer n. Suppose that $\pi^{-1}(0) = X_0$ is a singular surface. Fix $t \neq 0$ and suppose the monodromy operator T acting on the second homology group of $X = \pi^{-1}(t)$ is such that $(T - id)^3 = 0$ and $(T - id)^2 \neq 0$. Let g_t be the Calabi-Yau metrics on $X = X_t$ such that $[Im\ g_t] = e$. Let γ, α_1 and α_2 be cycles such that*

$$T(\gamma) = \gamma, \quad T(\alpha_1) = \gamma + \alpha_1, \quad T(\alpha_2) = \gamma + \alpha_1 + \alpha_2,$$

$$< e, \gamma >=< \gamma, \gamma >=< \alpha_2, \alpha_2 >= 0, \quad < \gamma, \alpha_2 >= 1.$$

Then there exists a torus fibration $\psi : X \to S^2$ such that the generic fiber $\psi^{-1}(s)$ is a Lagrangian 2-torus with respect to $Re\ g_t$ representing the homology class γ. Moreover, there is a section $\sigma : S^2 \to X$ such that $\sigma(S^2) \subset X$ is a Lagrangian submanifold representing the homology class $\alpha_2 - \gamma$.

Before we prove the theorem, let's construct a family of polarized K3 surface that fulfills the conditions stated in Theorem 46. Let $\mathcal{E}_t : y^2 = x(x-1)(x-t)$ be a family of elliptic curves when $t \in \mathcal{D}$, where $\mathcal{D} := \{t \in \mathbb{C} |\ |t| < 1\}$ and $\mathcal{E}_t \subset \mathbb{CP}^2 \times \mathcal{D}$. Then $\mathcal{E}_t \times_{\mathcal{D}} \mathcal{E}_t \to \mathcal{D}$ is a family of abelian surfaces over the unit disc. Let $\pi : \mathcal{K}_t \to \mathcal{D}$ be the family of Kummer surfaces associated with this family of abelian surfaces. In [7] it was proved that the family $\pi : \mathcal{K}_t \to \mathcal{D}$ satisfies the conditions of Theorem 46. See page 252-253 in 46. On page 252 an explicit construction of the cycles γ and α_2 is given and moreover from Clemens' theory it follows that

$$< \gamma, \gamma >=< \alpha_2, \alpha_2 >= 0 \text{ and } < \gamma, \alpha_2 >= 1.$$

In particular, γ and α_2 span a hyperbolic lattice

$$\mathbb{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is a well known fact that for a K3 surface X $H_2(X,\mathbb{Z})$ is a lattice isomorphic to $\Lambda_{K3} := \mathbb{H}^3 \oplus (-E_8)^2$. Let us define (transcendental lattice)

$$T_e := \{v \in \Lambda_{K3} | < e, v >= 0\}.$$

We know also that $T_e \approxeq \mathbb{Z}e^* \oplus \mathbb{H}^2 \oplus (-E_8)^2$, where $< e^*, e^* >= - < e, e >$. So we may suppose that $T_e = \mathbb{Z}e^* \oplus \{\gamma, \alpha_2\} \oplus \mathbb{H} \oplus (-E_8)^2$. See [7]. In [7] it is shown that the cycles γ and $\alpha_2 \in T_e$.

PROOF of Theorem 46: We will use isometric deformation with respect to the CY metric g_t on a K3 surface whose class of cohomology $[Img_t] = e$ and the epimorphism of the period map for K3 surfaces to prove that K3 surafce can be realized as a Lagrangian fibration with respect to each CY metric g_t. The Lagrangian fibration that we will construct will have the properties that the homology class of the generic fibre will be γ and the homology class of the base will be $\delta = \alpha_2 - \gamma$. For the description and applications of isometric deformations see [15]. Theorem 46 will follow directly from the following Lemma:

Lemma 47 ***i.*** *Let Y be a K3 surface whose Neron-Severi group $NS(Y)$ spanned by a cycle γ and such that $<\gamma,\gamma>=0$, then γ can be realized as a non singular elliptic curve C on Y and the linear system $|C|$ defines an elliptic fibration $|C| : Y \to \mathbb{CP}^1$.* ***ii*** *Let Y be a K3 surface whose Neron Severi group $NS(Y)$ spanned by a cycle δ and such that $<\delta,\delta>=-2$, then either δ or $-\delta$ can be realized as an embedded complex line $\mathbb{CP}^1 \subset Y$.*

PROOF: First we will prove part **i** of Lemma 47. From the condition that Neron Severi group $NS(Y)$ is spanned by cycles γ such that $<\gamma,\gamma>=0$, we deduce that there exists a line bundle $\mathcal{L}_\gamma$, whose Chern class $c_1(\mathcal{L}_\gamma)=\gamma$. From Riemann-Roch theorem we deduce that

$$\dim H^0(Y,\mathcal{L}_\gamma)+\dim H^2(Y,\mathcal{L}_\gamma)\geq 2.$$

Serre's duality implies that $\dim H^2(Y,\mathcal{L}_\gamma)=\dim H^2(Y,\mathcal{L}_\gamma^*)$, where $\mathcal{L}_\gamma^*$ is the dual line bundle of $\mathcal{L}_\gamma$. From here we obtain that either $\dim H^0(Y,\mathcal{L}_\gamma)>1$ or $\dim H^0(Y,\mathcal{L}_\gamma^*)>1$. So either γ or $-\gamma$ can be realized as an effective divisor on Y. Recall

Theorem 48 *([16] p559.) If an effective divisor $D>0$ on a K3 surface Y satisfies the conditions $<D,D>=0$ and $<D,E>\geq 0$ for any effective divisor $E>0$, then the linear system $|D|$ contains a divisor of the form mC, where $m>0$ and C is an elliptic curve.*

Since $NS(Y)$ is spanned by a cycle γ such that $<\gamma,\gamma>=0$, it follows that that γ or $-\gamma$ can be realized a non sigular elliptic curve C. From the standard exact sequence on C:

$$0\to\mathcal{O}_Y(-C)\to\mathcal{O}_Y\to\mathcal{O}_C\to 0,$$

we have, in the associated long exact sequence,

$$H^0(\mathcal{O}_Y)\to H^0(\mathcal{O}_C)\to H^1(\mathcal{O}_Y(-C))\to H^1(\mathcal{O}_Y)=0.$$

Since the restriction map $H^0(\mathcal{O}_Y)=\mathbb{C}\to H^0(\mathcal{O}_C)=\mathbb{C}$ is an isomorphism, we have $H^1(\mathcal{O}_Y(-C))=0$. From Serre's duality applied to K3 surafces we get $H^1(\mathcal{O}_Y(-C))=H^1(\mathcal{O}_Y(C))=0$ and $H^2(\mathcal{O}_Y(C))=0$. It is a standard fact that the linear system $|C|$ defines a map $\phi : Y\to\mathbb{CP}^1$. For details see page 560 of [16]. Part **i** of Lemma 47 is proved.

Part **ii** of Lemma 47 follows directly from Corollary 3 on page 560 of [16]. Lemma 47 is proved. $\blacksquare$.

One of the main consequences of the existence of a Calabi Yau metric on a K3 surface X with a holomorphic form ω_X is the fact that the covariant derivative of the holomorphic form ω_X is zero, i.e. $\nabla\omega_X=0$. This implies that the following three forms $\{Re\omega_X, Im\omega_X$ and $Im(g)\}$ are parallel, closed and non degenerate forms on X. It is easy to see that:

$$\langle Re\omega_X, Im\omega_X\rangle=\langle Re\omega_X, Img\rangle=\langle Img, Im\omega_X\rangle=0.$$

Here $Im(g)$ is the imaginary part of the Calabi Yau metric on X. If $\|Re\omega_X\|^2 = \|Im\omega_X\|^2 = \|Im(g)\|^2 = 1$, then these three forms define three integrable complex structures I, J and K on X such that $I^2 = J^2 = K^2 = -id$ and $IJ + JI = IK + KI = JK + KJ = 0$. For all these facts see [15]. The isometric deformation is defined as a new integrable complex structure $aI + bJ + cK$ on X, where $a^2 + b^2 + c^2 = 1$. We'll denote this new complex structure on X by Y. In [15] it was proved that $\mathrm{Re}\omega_Y = A(Re\omega_X)$, $Im\omega_Y = A(Im\omega_X)$, where $A \in SO(3)$. The class $A(Img)$ will be a class of cohomology of type $(1,1)$ on Y and will define a new Calabi Yau metric on Y which is isometric to the Calabi Yau metric on X as a Riemannian metrics. In [15] we proved that there exists $A \in SO(3)$ such that

$$\int_\gamma A(Re\omega_X) = \int_\gamma A(Im\omega_X) = 0.$$

Let Y and C be defined as in Lemma 47. Notice that when we do an isometric deformation, we are not changing either the C^∞ structure or the Riemannian structure on Y. So we may consider that C realizes the cycle γ as an embedded two dimensional torus in X.

Lemma 49 *C is a Lagrangian cycle on X*

PROOF: We need to prove the following two facts: **1.** The volume form of the restriction of the CY metric g_X on C is given by the following expression:

$$Vol(g_X|_C) = aRe(\omega_X) + bIm(\omega_X) \tag{1}$$

and **2.** that

$$Im(g_X|_C) = 0 \ . \tag{2}$$

Since C is an elliptic curve in Y, it follows that $\omega_Y|_C = 0$, so that $Re\omega_Y|_C = Im\omega_Y|_C = 0$. The volume form of the restriction of the Ricci flat Riemannian metric on C is equal to $Im(g_Y)|_C$, ie.

$$Vol(g_X|_C) = Im(g_Y)|_C. \tag{3}$$

So from (3) and the general facts about isometric deformation we deduce that

$$Vol(g_X|_C) = aRe(\omega_X)|_C + bIm(\omega_X)|_C + cIm(g_X)|_C \ . \tag{4}$$

where a, b, c are some real numbers given by

$$a = \int_C Re(\omega_X), \quad b = \int_C Im(\omega_X), \quad c = \int_C Im(g_X) \ . \tag{5}$$

Since $\gamma \in T_e$ and the cohomology class $[Im\ g_X] = e$ we can conclude that:

$$0 = \int_\gamma e = \int_C Im(g_X) = c. \tag{6}$$

This proves fact **1.** We now prove fact **2,** i.e. that the restriction of the form $Im(g_X)$ on C is identically zero. This follows directly from the following formula, proved in [15]:

$$Im(g_X)|_C = cIm(g_X)|_C \text{ and } c = \int_C Img_X \ . \tag{7}$$

So Lemma 49 is proved. ■.

Remark 50 *Exactly in the same way we can prove that the cycle $\delta = \alpha_2 - \gamma$ can be realized as a Lagrangian sphere. Indeed we can deform isometrically the complex structure on X to Z with respect to the CY metric g_X so that δ will be realized as a complex projective line $\mathbb{P}^1$ embedded in Z. Repeating the arguments from Lemma 49 we deduce that $\mathbb{P}^1$ is a Lagrangian sphere embedded in X.*

The End of the Proof of Theorem 46: Indeed we realized the cycles γ and $\alpha_2 - \gamma$ as complex analytic curves on Y and on Z. Then we know from Theorem 48 that Y is an elliptic fibration whose "generic" fibre is an elliptic curve. Lemma 49 implies that the fibres of this fibration are Lagrangian submanifolds. Since we know that $\langle \gamma, \delta \rangle = 1$, we can conclude that the basis of the Lagrangian fibration is the Lagrangian sphere that realizes the cycle δ. ■.

8 Appendix: Clemens' Cell Complex for Toric Hypersurfaces

As before let Δ be the convex hull of generators ρ of the 1-cones in Σ. This is an N dimensional polytope. But it has more. It comes equipped with a canonical simplicial decomposition induced by Σ. In particular the boundary $\partial\Delta$ is topologically an N-sphere which comes equipped with a simplicial decomposition.

Lemma 51 *Let τ be a set of primitive generators of 1-cones in Σ. Put $k = |\tau|$. The following are equivalent:*

i. *$\cap_{\rho\in\tau} D_\rho$ is non empty.*
ii. *the cone is generated by τ is a k-cone in Σ.*
iii. *the convex hull of τ is a $(k-1)$-cell in $\partial\Delta$.*

Proof:(**ii**)⇔(**iii**) is obvious. We will show that (**i**) and (**ii**) are equivalent. Recall that there is an order reversing 1-1 correspondence between T-orbits O_σ and cones σ in Σ, and that the closure of V(σ) of O_σ is given by

$$\mathrm{V}(\sigma) = \coprod_{\gamma\supset\sigma} O_\gamma.$$

Note that $D_\rho = V(\rho)$ for $\rho \in \Sigma(1)$. Thus

$$\cap_{\rho\in\tau} D_\rho = \coprod_{\gamma\supset\tau} O_\gamma.$$

If this is nonempty, then we have some cone $\gamma \in \Sigma$ containing τ. Since the toric variety V is nonsingular, γ is a simplicial cone. this implies that any collection of k edges of γ generates a k-face of γ. So τ generates a k-face, which we also call τ, of γ. In particular $\tau \in \Sigma$. Conversely, if τ generates a cone in Σ, obviously every D_ρ, $\rho \in \tau$, contains O_ρ. In this case, $\cap_{\rho\in\tau} D_\rho$ is nonempty. $\blacksquare$.

Note that if the intersection $\cap_{\rho\in\tau} D_\rho$ is nonempty, then it has dimension $N - |\tau|$. Moreover, the correspondence above between nonempty intersections of the D_ρ's and the cells of $\partial\Delta$ is order reversing. Namely, if two sets τ, τ' of primitive generators yield nonempty intersections, then $\cap_{\rho\in\tau} D_\rho \subset \cap_{\rho\in\tau'} D_\rho$ iff the $conv(\tau) \supset conv(\tau')$.

We now specialize our family $\mathcal{X} \to \mathcal{D}$ to the case of hypersurfaces

$$tF - \prod_\rho x_\rho = 0.$$

Thus $N = n + 1$.

Theorem 52 *The Clemens polytope* $\Pi(X_0)$ *is a simplicial complex which is naturally isomorphic to* $\partial\Delta$.

Proof: Recall that $X_0 = \cup_\rho D_\rho$. By definition of $\Pi(X_0)$, each D_ρ corresponds to a vertex in $\Pi(X_0)$. For $\rho \neq \rho'$, $D_\rho \cap D_{\rho'}$ corresponds to 1-cell if $D_\rho \cap D_{\rho'}$ has dimension $n - 1 = N - 2$, and so on. Thus given a set τ of 1-cones with $|\tau| = k, \cap_{\rho\in\tau} D_\rho$ corresponds to $(k-1)$-cell in $\Pi(X_0)$ iff $\cap_{\rho\in\tau} D_\rho$ has dimension $n-k+1$. This is an order reversing correspondence between $n-k+1$ dimensional intersections $\cap_{\rho\in\tau} D_\rho$ of D_ρ's and $(k-1)$-cells of the simplicial complex $\partial\Delta$ as simplicial complex. $\blacksquare$.

Corollary 53 $H_n(\Pi(X_0), \mathbb{Q}) \approxeq \mathbb{Q}$.

In the case when $\mathcal{X}$ is a family of complete intersections in $\mathbb{P}_\Sigma$, one can still define a cell complex $\Pi(X_0)$ similar to the definition of Clemens' complex. It is expected that the cell complex $\Pi(X_0)$ will again be isomorphic to n-sphere. We have verified this for many examples. It turns out that the complex is typically non-simplicial, unlike in the case of hypersurfaces. It is an interesting combinatorial problem to describe the complex $\Pi(X_0)$ in simple terms.

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