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A New Conformal Invariant and Its Applications to the Willmore Conjecture and the First Eigenvalue of Compact Surfaces

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Introduction

Let M be a compact Riemannian manifold with a fixed conformal structure. Then we introduce the concept of conformal volume of M in the following manner. For each branched conformal immersion φ of M into the unit sphere S^n, we consider the set of all branched conformal immersions obtained by composition of φ with the conformal automorphisms of S^n. We let $V_c(n, \varphi)$ be the maximum volume of these branched immersions. The conformal volume of M is defined to be the infimum of $V_c(n, \varphi)$ where φ ranges over all branched conformal immersions of M into the unit sphere S^n.

In this paper, we study the case when M is a compact surface and we call the conformal volume of M to be the conformal area of M. We demonstrate that this conformal invariant is non-trivial. In fact, we prove that if there exists a minimal immersion of M into S^n where coordinate functions are first eigenfunctions, then the conformal area of M is given by the area of M with respect to the induced metric. This enables us to compute the conformal area for several surfaces. For example, the conformal area of $\boldsymbol{RP}^2$ is 6π and the conformal area of the square torus is $2\pi^2$. We believe that the computation of the conformal area for general surfaces will be very important in studying the geometry of compact surfaces. We demonstrate this claim by applying the concept of conformal area to two different branches of surface theory.

The first application is to study the total curvature of a compact surface in R^n. This problem has a long history. Fenchel and Fary [9] proved that for a closed curve σ in R^n, $\int_\sigma |k| \geqq 2\pi$ where k is its curvature. Then Milnor [12] proved that $\int_\sigma |k| \geqq 4\pi$ if σ is knotted in R^3. For an account of these, one can read the beautiful article of Chern [7]. In 1957, Chern and Lashof [8] studied the absolute total curvature of a compact manifold in R^n where k is replaced by the Gauss-Kronecker curvature and the lower bound can be expressed in terms of the sum of the Betti numbers of the manifold. Since then, topological

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methods have been introduced in the study of the total curvature, and the concept of tight manifolds has been studied extensively by Kuiper, Pohl, Banchoff, and others. In fact, a whole book is needed to give a full account of this theory. On the other hand, in 1965 Willmore [16] proposed to study a different total curvature for surfaces M embedded in R^3. Instead of the Gauss curvature, he proposed to estimate the quantity $\int_M H^2$, where H is the mean curvature of M. In [16] Willmore computed $\int H^2$ for all circular tori in R^3 and found that $\int H^2 \geqq 2\pi^2$ with equality only for the Clifford torus. Based on his computation, he conjectured that $\int H^2 \geqq 2\pi^2$ for all embedded surfaces in R^3 which has genus one. For a long time the only known lower estimate for $\int H^2$ came from the following argument. By using a simple arithmetic, one can estimate $\int H^2$ in terms of $\int K^+$ from below where K^+ is the positive part of the curvature. Since $\int K^+ \geqq 4\pi$, one obtains $\int H^2 \geqq 4\pi$. Although there are deeper and more intricate arguments due to Wingten [17] and Rodriguez and Guadalope [14] for knotted compact surfaces in R^4, their basic ideas were of similar structure and special properties of $\int H^2$ were not utilized. (For more arguments of this type, we refer the reader to articles of B.Y. Chen [4] and Proposition 2 of §5.)

In this paper we prove that for any compact surface in R^n, $\int_M H^2$ is not less than the conformal area of M and equality holds iff M is the sterographic projected image of a minimal surface in S^n whose coordinate functions are first eigenfunctions. As a corollary, we prove that $\int_{\boldsymbol{RP}^2} H^2 \geqq 6\pi^2$ for any branched immersion of $\boldsymbol{RP}^2$ into R^n and that equality holds only if it is a sterographic projection of a minimal surface in S^4 to R^4. We also prove that if M is conformally equivalent to the square torus, then for any branched immersion of M into R^n, $\int H^2 \geqq 2\pi^2$. In fact the same inequality holds if M is conformally equivalent to elements in an open region in the modular space of genus one. Equality holds only if M is the sterographic projected image of some embedded minimal torus of S^3 which is conformally equivalent to the square torus.

For the purpose of estimating $\int H^2$, one can redefine the concept of conformal area so that it depends also on the topology of the immersion. In particular, we can prove that for any branched minimal immersion of M into R^n, if the self-intersection of M is non-trivial then $\int_\Sigma H^2 \geqq 8\pi$.

The second application of the concept of conformal area is to give an upper estimate of the first eigenvalue of a compact surface M. We prove that the first eigenvalue is not greater than $\frac{2V_c(M)}{V(M)}$ where $V(M)$ is the area of M. Equality holds if and only if M is a minimal surface of the unit sphere whose coordinate functions are first eigenfunctions. This inequality is sharp if M is $\boldsymbol{RP}^2$ or if M is the square torus. Since we can estimate $V_c(M)$ in terms of the genus of M, we can estimate $\lambda_1 V(M)$ from above by a absolute constant depending only on the genus of M. This type of estimates originated from Szegö [15], who was the first one to estimate λ_1 for a simply connected domain in R^2. His method was then used by Hersch [10] to give an upper bound for the first eigenvalue of any metric on S^2. Then using a method of branched covering, Yang and Yau [18] gave an upper estimate of $\lambda_1 V(M)$ for any Riemann surface in terms of its genus. Our method here has the advantage of being applicable to non-orientable surfaces and the estimates are sharp in many cases. We would also like to point out that in an article [1] of Berger, he raised some interesting questions concerning the above problem.

Some other simple applications are that for a minimal surface M in the unit sphere S^n:

(i) if M is homeomorphic to $\boldsymbol{RP}^2$, then $V(M) \geqq 6\pi$.

(ii) if M is conformally equivalent to the square torus, then $V(M) \geqq 2\pi^2$.

(iii) if $V(M) < 8\pi$, then M must be embedded.

Finally, we give upper estimate for the first eigenvalue of a Kähler manifold in terms of its Kähler class and some information of a meromorphic function on the manifold.

We shall point out that J.P. Bourguignon has obtained upper estimate for the first eigenvalue of $\boldsymbol{RP}^2$ by utilizing some special properties of the projective group.

This work was done while both authors were visiting the University of California at San Diego in 1980. We would like to thank them for their hospitality.

§1. Non-degenerate Conformal Maps from a Manifold Into the Unit Sphere

Let M be a m-dimensional compact manifold which admits a conformal map ϕ into the n-dimensional unit sphere S^n. Let ds^2 be the metric on M and ds_0^2 be the standard metric on S^n. Then

$$\phi^* ds_0^2 = \alpha(x)\, ds^2 \tag{1.1}$$

where α is a non-degenerate function defined on M.

If G denotes the group of conformal diffeomorphisms of S^n, then we define the *n-conformal volume of* ϕ by

$$V_c(n, \phi) = \sup_{g\in G} \int_M dV_g = \sup_{g\in G} \int_M |\nabla(g\circ\varphi)|^2\, dV_M, \tag{1.2}$$

where dV_g is the volume element (possibly degenerate) associated to the tensor $\phi^* g^* ds_0^2$.

The *n-conformal volume of* M is then defined to be

$$V_c(n, M) = \inf_{\phi} V_c(n, \phi) \tag{1.3}$$

where ϕ runs over all non-degenerate conformal mappings of M into S^n.

Fact 1. *If M admits a degree d conformal map onto another manifold N, then*

$$V_c(n, M) \leqq |d|\, V_c(n, N). \tag{1.4}$$

Since any compact surface can be conformally branched over S^2 or $\boldsymbol{RP}^2$, we can apply the above inequality to give an upper estimate of the conformal area of any compact surface in terms of the degree of the conformal map and the conformal area of S^2 or $\boldsymbol{RP}^2$. The conformal area of S^2 is easily seen to be 4π (see Fact 2). In §3, we shall prove that $V_c(n, \boldsymbol{RP}^2)=6\pi$ for $n \geqq 4$. If Σ is an orientable surface without boundary, then by the theorem of Riemann-Roch, we can choose a conformal map onto S^2 with degree $\leqq(g+1)$ where g is the genus of Σ. If Σ is a non-orientable surface, then a double cover $\tilde{\Sigma}$ of Σ is orientable and we can choose a conformal map from $\tilde{\Sigma}$ onto S^2 which commutes with the deck transformation of $\tilde{\Sigma}$ and the antipodal map of S^2. For example, let s be a meromorphic one form on $\tilde{\Sigma}$ and σ be the deck transformation. Then $\overline{s(\sigma)}$ is also a meromorphic one form and $\sqrt{-1}\, s\, \overline{s(\sigma)}^{-1}$ defines a meromorphic function from $\tilde{\Sigma}$ onto S^2. Since antipodal map of S^2 is given by $z \to -\bar{z}^{-1}$, we obtain a conformal map onto $\boldsymbol{RP}^2$. The degree of this map is $2(g(\tilde{\Sigma})+1)$. In conclusion, if Σ is orientable, $V_c(n, \Sigma) \leqq 4(g(\Sigma)+1)\pi$ for $n \geqq 2$. If Σ is non-orientable, $V_c(n, \Sigma) \leqq 12(g(\Sigma)+1)\pi$ for $n \geqq 4$. Note that we do not compute $V_c(3, \boldsymbol{RP}^2)$. However, the arguments in §4 show that $V_c(3, \boldsymbol{RP}^2) \geqq 8\pi$.

Fact 2. *If M is of dimension m, then*

$$V_c(n, M) \geqq V_c(n, S^m) = V(S^m). \tag{1.5}$$

To see this, let θ be a point on S^n, and $g_\theta(t)$ be the one parameter conformal subgroup of G generated by the gradient of the linear function of $\boldsymbol{R}^{n+1}$ in the direction θ. Then, for all t, $g_\theta(t)$ fixes the points θ and $-\theta$. Moreover

$$\lim_{t \to \infty} g_\theta(t)(x) = \theta \tag{1.6}$$

for all $x \in S^n - \{-\theta\}$. Notice that g_θ corresponds to the homotheties of $\boldsymbol{R}^n$ by stereographic projection with poles at θ. Therefore, if ϕ: $M \to S^n$ is a conformal map whose differential has rank m at x, the volume of $g_{-\phi(x)}(t) \circ \phi(M)$ will tend to some non-trivial integral multiple of the volume of S^m as $t \to \infty$ because $g_{-\phi(x)}(t)$ has the same effect as blowing up the picture on the tangent space of S^n at $\phi(x)$. From now on, we will denote this procedure as *blowing up at the point* $\phi(x)$. The identity $V_c(n, S^m) = V(S^m)$ for all n will be discuss in §3.

Fact 3. *If M is of dimension m, and φ: $M \to S^n$ is a conformal map with the property that there exists exactly k distinct points on M $\{x_i\}_{i=1}^k$ such that $\varphi(x_i) = p \in S^n$ for all $1 \leqq i \leqq k$, then $V_c(n, \varphi) \geqq k V(S^m)$.*

This follows from the same arguments as in Fact 2. One blows up at the point p and obtains k totally geodesic spheres through p in the limit, hence provide the lower bound. In general, we will denote such a point p to be a k-point indicating its preimage set consists of k points.

Fact 4. *For an integer n such that a conformal map ϕ: $M \to S^n$ exists, we have obviously the inequality*

$$V_c(n, M) \geqq V_c(n+1, M). \tag{1.7}$$

In view of this, we define the conformal volume of M to be

$$V_c(M) = \lim_{n\to\infty} V_c(n, M). \tag{1.8}$$

Question 1. For any compact manifold M, can one find a conformal mapping ϕ of M into some S^n such that $V_c(n,\phi) = V_c(n,M)$. If M is of dimension two, this is closely related to the question of whether the area of a minimally embedded surface in S^n is the same as its n-conformal area. We shall prove that for compact minimal surfaces (without boundary) of S^n, and for minimal submanifolds of dimension $m \geqq 3$ which are obtained as orbits in S^n under the action of some subgroup of $O(n+1)$,

$$V_c(n, M) \leqq V(M). \tag{1.9}$$

In fact, for a special class of minimal surfaces in S^n, we shall also prove that

$$V_c(n, M) \geqq V(M). \tag{1.10}$$

Question 2. If the n-conformal volume of a m-dimensional manifold is equal to the volume of S^m, is the manifold necessarily conformal to S^m?

When the manifold has non-trivial boundary, we define $V_c(n,M)$ in the same way as above. Then clearly we have the following

Fact 5. *If M is a subdomain of another manifold N, then for all n, $V_c(n,M) \leqq V_c(n,N)$. Hence to obtain an upper estimate of the conformal area for any compact surface Σ with boundary, we can embed Σ conformally into another compact surface $\hat{\Sigma}$ without boundary so that each component of $\partial\Sigma$ bounds a disk in $\hat{\Sigma}$. The conformal area of $\hat{\Sigma}$ has already been estimated in Fact 1. Therefore we have an upper estimate of the conformal area of any compact surface depending only on the topology of the surface.*

Finally, we remark that we can refine the concept of conformal volume so that it is sensitive to the topology of the map $\phi\colon M \to S^n$. This is very useful for the study of the Willmore conjecture in §4. This concept can be described as follows.

Let $\phi\colon M \to S^n$ be a fixed non-degenerate conformal map. Then we define $V_{c,\phi}(n,M)$ to be the infimum of all $V_c(n,\psi)$ where ψ is a conformal map so that $\psi = F \circ \phi$ for some diffeomorphism of F of S^n. We see from Fact 3 that if ϕ is an immersion, $V_{c,\phi}(n,M)$ is not less than the product of the volume of S^m with the number of times of its multiplicity at one point.

§2. Relationship Between the First Eigenvalue and the Conformal Area for Surfaces

Theorem 1. *Let M be a compact surface (possibly with boundary). Let $\lambda_1 > 0$ be its first non-zero eigenvalue for the Laplacian (with Neumann boundary condition if $\partial M \neq 0$). Then*

$$\lambda_1 \cdot V(M) \leqq 2 V_c(n, M) \tag{2.1}$$

for all n where $V_c(n, M)$ is defined (i.e., there exists a conformal mapping ϕ: $M \to S^n$). Equality implies M must be a minimal surface of S^n, moreover the immersion is given by a subspace of the first eigenspace. Hence by a result of Cheng [5], we can assume that n is less than a constant depending only on the genus of M.

Proof. We shall utilize the variational characteristic of λ_1 given by

$$\lambda_1 = \inf \frac{\int_M |\nabla f|^2}{\int_M f^2}, \tag{2.2}$$

where inf is taken over all Lipschitz functions with $\int_M f = 0$.

Let ϕ be a conformal map of M into S^n so that

$$V_c(n, \phi) \leqq V_c(n, M) + \varepsilon. \tag{2.3}$$

Suppose X_i's are the coordinate functions of $\mathbf{R}^{n+1}$, we claim that there exists an element $g \in G$, the conformal group of S^n, such that

$$\int_M X_i \circ g \circ \phi = 0, \qquad \text{for all } 1 \leqq i \leqq n+1. \tag{2.4}$$

The action of G can be extended to the unit ball B^{n+1} bounded by S^n in $\mathbf{R}^{n+1}$. The isotropic subgroup at the origin is simply $O(n+1)$. For each point $A \in B^{n+1}$ with $A \neq 0$, one obtains a conformal vector field V_A on S^n by projecting $A/\|A\|$ on the tangent space of each point of S^n. The conformal vector field extends to be a conformal vector field of B^{n+1}. It generates a one-parameter family of conformal automorphism $g(t)$ of B^{n+1} onto itself. In the group $g(t)$, there exists a unique conformal automorphism g_A which maps the origin to A. In this way, one obtains an embedding of B^{n+1} into G. Call this embedding F.

We defined a map from $F(B^{n+1})$ to B^{n+1} in the following manner. For $g \in F(B^{n+1})$, we associate the point

$$H(g) = \frac{1}{V(M)} \int_M X \circ g \circ \phi \in B^{n+1}. \tag{2.5}$$

$H \circ F$ maps B^{n+1} into itself, and when prolonged by continuity to $\partial B^{n+1} = S^n$ is the identity map. By a standard topological argument, $H \circ F$ must be surjective. Hence, in particular, there exists $g \in F(B^{n+1})$ such that $H(g) = 0$. This establishes our claim (2.4).

Now we may use $X_i \circ g \circ \phi$ as trial functions in (2.2). One observes that

$$\int_g |\nabla X_i \circ g \circ \phi|^2 = \int_M (g \circ \phi)^* [|\nabla X_i|^2 \, dV] \tag{2.6}$$

for each $1 \leqq i \leqq n+1$. Therefore

$$\begin{aligned}\sum_{i=1}^{n+1} \int_M |\nabla X_i \circ g \circ \phi|^2 &= \int_M (g \circ \phi)^* [\sum_i |\nabla X_i|^2 \, dV] \\ &= 2 \int_M (g \circ \phi)^* \, dV \\ &\leqq 2 V_c(n, \phi) \\ &\leqq 2(V_c(n, M) + \varepsilon). \end{aligned} \tag{2.7}$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{n+1} \int_M (X_i \circ g \circ \phi)^2 &= \int_M \sum_i (X_i \circ g \circ \phi)^2 \\ &= V(M) \end{aligned} \tag{2.8}$$

since $\sum_i X_i^2 = 1$. By (2.2), we find that

$$\lambda_1 \sum_i \int_M (X_i \circ g \circ \phi)^2 \leqq \sum_i \int_M |\nabla (X_i \circ g \circ \phi)|^2. \tag{2.9}$$

This implies

$$\lambda_1 V(M) \leqq 2(V_c(n, M) + \varepsilon). \tag{2.10}$$

Inequality (2.1) follows by letting $\varepsilon \to 0$.

We now assume that (2.1) holds with equality. By scaling, we may assume

$$\lambda_1 = 2, \tag{2.11}$$

hence

$$V(M) = V_c(n, M). \tag{2.12}$$

Suppose ϕ_k: $M \to S^n$ is a sequence of conformal mappings such that

$$\lim_{k \to \infty} V_c(n, \phi_k) = V_c(n, M) \tag{2.13}$$

and

$$\int_M X_i \circ \phi_k = 0 \tag{2.14}$$

for all i and k. Then by choosing coordinates appropriately, we may assume that

$$\lim_{k \to \infty} \int_M (X_i \circ \phi_k)^2 > 0 \tag{2.15}$$

for $1 \leqq i \leqq N$, and

$$\lim_{k \to \infty} \int_M (X_i \circ \phi_k)^2 = 0 \tag{2.16}$$

for $N+1 \leqq i \leqq n+1$, with some N. Equations (2.2), (2.7), and (2.11) imply

$$\begin{aligned} 2 V_c(n, \phi_k) &\geqq \sum_{i=1}^{n+1} \int_M |\nabla X_i \circ \phi_k|^2 \\ &\geqq 2 \sum_{i=1}^{n+1} \int_M (X_i \circ \phi_k)^2. \end{aligned} \tag{2.17}$$

Taking the limit as $k\to\infty$, this yields

$$V_c(n,M)\geqq \lim_{k\to\infty}\sum_{i=1}^{n+1}\int_M (X_i\circ\phi_k)^2. \tag{2.18}$$

On the other hand, the fact that

$$\sum_{i=1}^{n+1}\int_M (X_i\circ\phi_k)^2=V(M) \tag{2.19}$$

implies

$$\lim_{k\to\infty}\sum_{i=1}^{n+1}\int_M (X_i\circ\phi_k)^2=V(M). \tag{2.20}$$

This together with (2.15) and (2.16) means

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_M (X_i\circ\phi_k)^2=V(M). \tag{2.21}$$

The assumption (2.12) therefore implies that the inequalities in (2.17) become equalities when passed to the limit as $k\to\infty$. We can therefore assume that for each $1\leqq i\leqq n+1$, the functions $X_i\circ\phi_k$ converges weakly in $H_{1,2}(M)$ and strongly in $L_2(M)$ to some function ψ_i. Clearly

$$\sum_{i=1}^{n+1}\psi_i^2=1 \tag{2.22}$$

almost everywhere, and

$$\psi_i=0 \qquad \text{for } N+1\leqq i\leqq n+1. \tag{2.23}$$

Moreover,

$$\lim_{k\to\infty}\int_M |\nabla X_i\circ\phi_k|^2=\lambda_1\int_M \psi_i^2$$

for $1\leqq i\leqq N$. Therefore the sequence $\{X_i\circ\phi_k\}$ in fact converge to $\{\psi_i\}$ strongly in $H_{1,2}(M)$ and ψ_i are first eigenfunctions of M. In particular, the map $(\psi_1,\ldots,\psi_N)$ defines a smooth conformal mapping of M in S^{N-1}.

Taking the Laplacian of (2.22), we obtain

$$\sum_{i=1}^{N}|\nabla\psi_i|^2=2\sum_{i=1}^{N}\psi_i^2=2=\lambda_1. \tag{2.24}$$

Together with the fact that $(\psi_1,\ldots,\psi_N)$ is conformal, we conclude that it must be an isometry. Hence M is therefore a minimal submanifold of S^{N-1} given by $(\psi_1,\ldots,\psi_N)$.

A theorem of Yang and the second author can be obtained as a corollary of Theorem 1. They proved that if M is a surface of genus g, then

$$\lambda_1 V(M)\leqq 8\pi(1+g). \tag{2.25}$$

According to Theorem 1, in order to prove this, we only need to estimate the conformal volume of M from above. By Riemann-Roch, we know there exist a map ϕ: $M \to S^2$ which is a conformal branch covering of M over S^2 with $|\deg(\phi)| \leqq (1+g)$. Hence, by Fact 1

$$2V_c(2,M) \leqq 2(1+g)\,V_c(2,S^2) \\ = 8\pi(1+g),$$

as to be proved. The theorem in [18] is however stronger than ours, because they have proved

$$\sum_{i=1}^{3} \frac{1}{\lambda_i} \geqq \frac{3V(M)}{8\pi(1+g)}$$

which implied (2.25).

Corollary 1. *Let M be a compact surface without boundary of genus 1. If M is conformally equivalent to a flat torus with lattice generated by $\{(1,0),(x,y)\}$, where $0 \leqq x \leqq \frac{1}{2}$ and $\sqrt{1-x^2} \leqq y \leqq 1$, then*

$$2\pi^2 \leqq V_c(M).$$

Proof. In view of Theorem 1, we only need to show that

$$4\pi^2 \leqq \lambda_1(T^2)\,V(T^2) \tag{2.26}$$

where $\lambda_1(T^2)$ and $V(T^2)$ are computed with respect to the flat metric. If $(1,0)$ and (x,y) are the generators of the lattice corresponding to T^2, then the volume

$$V(T^2) = y. \tag{2.27}$$

Moreover, by the computation in [1],

$$\lambda_1(T^2) = \frac{4\pi^2}{y^2} \tag{2.28}$$

when $\sqrt{1-x^2} \leqq y \leqq 1$. Therefore

$$\lambda_1(T^2)\,V(T^2) = \frac{4\pi^2}{y} \geqq 4\pi^2$$

which was to be proved.

Corollary 2. *Let M be a compact surface. Let $0 < \lambda_1 \leqq \ldots \leqq \lambda_n \ldots$ be the eigenvalues of M. Then for any $n \geqq k$,*

$$\sum_{i=1}^{k} \frac{1}{\lambda_i} \geqq \frac{k}{n} V(M)\,V_c(n-1,M)^{-1}. \tag{2.29}$$

Proof. As in Theorem 1, we can find a conformal map X: $M \to S^{n-1}$ so that $\int_M X_i = 0$ for $1 \leqq i \leqq n$. By making an orthonormal change of basis in R^n, we may assume that X_i is orthogonal to X_j for $i \neq j$. Therefore

$$\sum_{i=1}^{k} \frac{1}{\lambda_i} \geqq \sum_{i_1 < i_2 < \ldots < i_k} \left(\int_M X_{i_j}^2\right)\left(\int_{X(M)} |\nabla X_{i_j}|^2\right)^{-1} \tag{2.30}$$

where $i_1 < i_2 < \ldots < i_k$ is an arbitrary chosen subsequence of $\{1, \ldots, n\}$.

Since $\sum_{i=1}^{n} X_i^2 = 1$, we can choose $i_1 < \ldots < i_k$ so that

$$\sum_{i_1 < i_2 < \ldots < i_k} \int_M X_{i_j}^2 = \frac{k}{n} V(M). \tag{2.31}$$

The corollary follows from the fact that $|\nabla X_{i_j}| \leqq 1$.

Corollary 3. *Let M be an m-dimensional compact manifold. Then*

$$\lambda_1(M) \leqq m(n+1)\, V_c(n, M)^{\frac{2}{m}} (V(M))^{\frac{-2}{m}} \tag{2.32}$$

Proof. Since

$$\int_M |\nabla f|^2 \leqq \left(\int_M |\nabla f|^m\right)^{\frac{2}{m}} V(M)^{\frac{m-2}{m}} \tag{2.33}$$

and $\int_M |\nabla f|^m$ is a conformal invariant, all the previous arguments apply. (The number $n+1$ occurs because among the functions $X_1, \ldots, X_{n+1}$, we can choose X_i so that

$$\int_M X_i^2 \geqq \frac{1}{n+1} V(M). \tag{2.34}$$

Remark. In both Corollary 2 and Corollary 3, the constants are not sharp. The reason is that we do not have good control on the quantities $\int |\nabla X_i|^m$ in these situations. Therefore we propose in these cases, we should study the *conformal moment* of an m dimensional manifold M which is defined as follows.

For each non-degenerate conformal map $X: M \to S^n$, we define

$$\mathcal{M}_c(n, X) = \sup_{g \in G} \sup_{A \in S^n} \int_{g(X(M))} |\nabla(g \circ X, A)|^m. \tag{2.35}$$

The *n conformal moment of M* is defined to be

$$\mathcal{M}_c(n, M) = \inf_X \mathcal{M}_c(n, X).$$

Then in both Corollaries 2 and 3, the number $V_c(n, M)$ can be replaced by $\mathcal{M}_c(n, M)$. However, at this moment, we do not have good computation of the conformal moment yet.

§3. Minimal Surfaces and Equivariant Minimal Submanifolds of the Unit Spheres

In this section, we shall compute the conformal volume for minimal surfaces of S^n, and for minimal submanifolds (of dimension $m \geqq 3$) in S^n which are given as orbits of some subgroups of the isometry group $O(n+1)$ of S^n.

Proposition 1. *Let M^2 be a compact minimal surface of S^n given by the isometric immersion $\phi: M^2 \to S^n$. Then*

$$V_c(n, \phi) = V(M).$$

Proof. Let $\pi: S^n \to \mathbf{R}^n$ denote a stereographic projection. The composition $\pi \circ \phi$ is a conformal mapping of M into $\mathbf{R}^n$. For each normal vector ν^α of M in $\mathbf{R}^n$, let $\{\mu_i^\alpha\}$ be the principal curvatures associated to ν^α. Then it is well-known that the quantity

$$\int_{\pi \circ \phi(M)} \sum_\alpha (\mu_1^\alpha - \mu_2^\alpha)^2$$

is invariant under any conformal change of metric on $\mathbf{R}^n$. Hence

$$\int_{\pi \circ \phi(M)} \sum_\alpha (\mu_1^\alpha - \mu_2^\alpha)^2 = \int_{g \circ \phi(M)} \sum_\alpha (\bar{\mu}_1^\alpha - \bar{\mu}_2^\alpha)^2 \tag{3.1}$$

where the $\bar{\mu}_i^\alpha$'s are the corresponding principal curvatures of $g \circ \phi(M)$ in S^n, with $g \in G$.

The Gauss curvature equation, on the other hand, enables us to write (3.1) in the form

$$4 \int_{\pi \circ \phi(M)} (|H|^2 - K) = 4 \int_{g \circ \phi(M)} (|\bar{H}|^2 - \bar{K}) + 4V(g \circ \phi(M)) \tag{3.2}$$

for all $g \in G$. Gauss-Bonnet then gives

$$\int_{\pi \circ \phi(M)} |H|^2 = \int_{g \circ \phi(M)} |\bar{H}|^2 + V(g \circ \phi(M)). \tag{3.3}$$

However the left hand side is independent of g, therefore by assumption that $\phi(M)$ is minimal,

$$\begin{aligned} V(\phi(M)) &= \int_{\pi \circ \phi(M)} |H|^2 \\ &= \int_{g \circ \phi(M)} |\bar{H}|^2 + V(g \circ \phi(M)) \\ &\geqq V(g \circ \phi(M)). \end{aligned} \tag{3.4}$$

This shows that

$$V_c(n, \phi) \leqq V(M)$$

as claimed.

The volume function corresponding to an immersion $\phi: M \to S^n$ is defined on G given by

$$V(n, \phi, g) = \int_M dV_g, \tag{3.5}$$

where dV_g is the volume element associated to the tensor $\phi^* g^* ds_0^2$. Clearly

$$V(n, \phi, g) = V(n, \phi, hg),$$

if $h \in O(n+1) \subseteq G$. Hence, the volume function can be viewed as a function defined on the cosets space $G/O(n+1)$. For simplicity, we will denote $gO(n+1)$ by g. Then

$$V_c(n,\phi) = \sup_{g \in G/O(n+1)} V(n,\phi,g).$$

Theorem 2. *Let M be a homogeneous Riemannian manifold of dimension m. Suppose $\phi: M \to S^n$ is an immersion of M into S^n which satisfies the properties:*

(i) ϕ is an isometric minimal immersion

(ii) The transitive subgroup H of the isometry group of M is induced by a subgroup, also denoted by H, of $O(n+1)$ (i.e., ϕ is equivariant).

(iii) $\phi(M)$ does not lie on any hyperplane of $\mathbf{R}^{n+1}$, i.e., ϕ is a "full" immersion. Then

$$V_c(n,\phi) = V(M).$$

In fact, the identity element $0 \in G/O(n+1)$ is the only local maximum for the volume function defined on $G/O(n+1)$.

Proof. We begin by observing that O is a local maximum for the volume function on $G/O(n+1)$. Indeed, the minimality condition (i) implies that O (i.e., $\phi(M)$) is a critical point. If $\{e_1, \ldots, e_{n+1}\}$ form an orthonormal basis for $\mathbf{R}^{n+1}$, then $g_{e_i}(t)$ form a one-parameter subgroup of G for each i. The second variational formula (see [11]) for the volume in the direction e_i is given by

$$\frac{d^2}{dt^2} V(n,\phi,g_{e_i}(t)|_{t=0} = m[m \int_M X_i^2 - \int_M |\bar{\nabla} X_i|^2], \tag{3.6}$$

where X_i is the coordinate function of $\mathbf{R}^{n+1}$ in the direction of e_i, and $\bar{\nabla}$ denotes the gradient computed on S^n. However (i) implies

$$\Delta X_i = -mX_i \quad \text{for all } 1 \leqq i \leqq n+1.$$

Therefore

$$m \int_M X_i^2 = \int_M |\nabla X_i|^2 < \int_M |\bar{\nabla} X_i|^2 \tag{3.7}$$

for all $1 \leqq i \leqq n+1$, where the strict inequality follows from assumption (iii). Substituting (3.7) into (3.6), we conclude that

$$\frac{d^2}{dt^2} V(n,\phi,g_{e_i}(t)|_{t=0} < 0 \tag{3.8}$$

for all $1 \leqq i \leqq n+1$. Together with the fact that $\{e_1, \ldots, e_{n+1}\}$ spans the tangent space at O of $G/O(n+1)$, we deduce that O is a local maximum.

Suppose O is not the only local maximum, then by the minimax method we conclude that there exists another critical point $g_0 \in G/O(n+1)$ which is a saddle point. The existence of g_0 follows from a rather standard construction argument, and we will only give its outline as follows:

We consider the number

$$\sup_{\gamma\in p}\inf_{g\in\gamma} V(n,\phi,g),$$

where inf is taken over all g on the path γ joining the two local maximum in $G/O(n+1)$, and sup is taken over all such paths. The point g_0 on some $\gamma\in p$ which achieves this number is then the saddle point which we seek for.

Let A be the level set of the volume function which contains g_0. We claim that locally through g_0, A contains a m-dimensional submanifold. To see this, we write

$$g_0=g_\theta(t_0) \tag{3.9}$$

for some unit vector θ in $\mathbf{R}^{n+1}$, the tangent space of $G/O(n+1)$ at O. The equivariant property of ϕ, (ii), implies that

$$V(n,\phi,g_\theta(t_0))=V(n,\phi,g_{h(\theta)}(t_0)) \tag{3.10}$$

for all $h\in H\subseteq O(n+1)$, where $h(\theta)$, of course, stands for the image of θ under the action of h. On the other hand, if we identify unit vectors in $\mathbf{R}^{n+1}$ with points in S^n, then property (iii) implies that θ can be written as

$$\theta=a\theta_0+b\theta_1 \tag{3.11}$$

where a, b are constants, $\theta_0\in\phi(M)\subseteq S^n$ and $\theta_1\perp\theta_0$. Since H acts transitively on M, around θ_0, $H(\theta_0)=\{h(\theta_0)|h\in H\}$ can be parametrized by a piece of M and clearly so is $H(\theta)$ by (3.11). This together with Eq. (3.10) confirms our claim.

We shall now proceed to compute the second variational formula for the volume at g_0. We observe that the coordinate functions X_i satisfy

$$\begin{aligned}|\bar{V}X_i|^2&=\sup X_i^2-X_i^2\\&=1-X_i^2\end{aligned} \tag{3.12}$$

on S^n, for all $1\leqq i\leqq n+1$. Therefore substituting into (3.6), we obtain

$$\frac{d^2}{dt^2}V(n,g_0\circ\phi,g_{e_i}(t)|_{t=0}=m\Big[(m+1)\int_{g_0\circ\phi(M)}X_i^2-V(n,g_0\circ\phi,O)\Big]. \tag{3.13}$$

Summing this over all $1\leqq i\leqq n+1$, we have

$$\begin{aligned}&\sum_{i=1}^{n+1}\frac{d^2}{dt^2}V(n,g_0\circ\phi,g_{e_i}(t)|_{t=0}\\&=m\left[(m+1)\int_{g_0\circ\phi(M)}\sum_{i=1}^{n+1}X_i^2-(n+1)V(n,g_0\circ\phi,O)\right]\\&=m(m-n)V(n,g_0\circ\phi,O).\end{aligned} \tag{3.14}$$

On the other hand, since A contains a m-dimensional manifold through g_0 which is a critical point, there are at least m directions such that

$$\frac{d^2}{dt^2} V(n, g_0 \circ \phi, g_{e_i}(t))|_{t=0} = 0, \tag{3.15}$$

say $1 \leqq i \leqq m$. Moreover, the fact that g_0 is a saddle point implies that there is at least one more direction, say $i = m+1$, such that the second variation is non-negative. Summing over $1 \leqq i \leqq n+1$, we now have

$$\begin{aligned}
&\sum_{i=1}^{n+1} \frac{d^2}{dt^2} V(n, g_0 \circ \phi, g_{e_i}(t))|_{t=0} \\
&\quad \geqq \sum_{i=m+2}^{n+1} \frac{d^2}{dt^2} V(n, g_0 \circ \phi, g_{e_i}(t))|_{t=0} \\
&\quad = m\left[(m+1) \sum_{i=m+2}^{n+1} \int_{g_0 \circ \phi(M)} X_i^2 - (n-m) \times V(n, g_0 \circ \phi, O)\right].
\end{aligned} \tag{3.16}$$

Combining with (3.14), this implies

$$0 \geqq \sum_{i=m+2}^{n+1} \int_{g_0 \circ \phi(M)} X_i^2. \tag{3.17}$$

However this is impossible by virtue of (iii). This proves our claim that O is the only local maximum.

To conclude the theorem, we first observe that any minimal submanifold of dimension m in S^n must have the property that

$$V(M) > V(S^m). \tag{3.18}$$

In fact, it was proved in [6] that there exists an $\varepsilon > 0$ which only depends on n and m, such that,

$$V(M) \geqq V(S^m) + \varepsilon. \tag{3.19}$$

The fact that O is the absolute maximum for the volume function defined on $G/O(n+1)$ follows from Fact 2 in § 1.

Corollary 4. *Let M be a compact minimal surface immersed in S^n, then for any metric ds^2 which is conformally equivalent to the original induced metric*

$$\bar{\lambda}_1 \bar{V}(M) \leqq 2V(M) = 2V_c(n, \phi), \tag{3.20}$$

where $\bar{\lambda}_1$ and $\bar{V}(M)$ stand for the first eigenvalue and the volume computed with respect to ds^2. Moreover, if the minimal immersion ϕ is given by a subspace of the first eigenspace, i.e., $\lambda_1 = 2$, then

$$V_c(M) = V_c(n, M) = V_c(n, \phi) = V(M). \tag{3.21}$$

Proof. This first part of the corollary simply follows from Theorem 1 and Proposition 1. If $\lambda_1 = 2$, then

$$2V(M) \leqq 2V_c(M) \leqq 2V_c(n, M)$$

by Theorem 1. However, by definition

$$\begin{aligned} V_c(n, M) &\leqq V_c(n, \phi) \\ &= V(M), \end{aligned}$$

hence they must all be equal.

When M is a two dimensional sphere, S^2, because it has unique conformal structure the corollary implies that for any metric ds^2 on S^2

$$\bar{\lambda}_1 \bar{V}(S^2) \leqq 8\pi = 2V_c(S^2).$$

This was a theorem of Hersch [10].

Corollary 5. *For any metric ds^2 on $\boldsymbol{RP}^2$,*

$$\bar{\lambda}_1 \bar{V}(\boldsymbol{RP}^2) \leqq 2V_c(\boldsymbol{RP}^2) = 12\pi.$$

Equality implies there exists a subspace of the first eigenspace of ds^2 which gives an isometric minimal immersion of $\boldsymbol{RP}^2$ into S^4.

Proof. First we observe that $\boldsymbol{RP}^2$ has unique conformal structure. Also the fact that the first eigenspace of $\boldsymbol{RP}^2$ with the standard metric gives, up to a constant factor, an isometric minimal embedding of $\boldsymbol{RP}^2$ into S^4 enable us to apply Theorem 1 and Proposition 1. This minimal embedding is known as the Veronese surface, which has volume 6π. The corollary follows.

Using the fact that the flat square torus can be isometrically minimally immersed into S^3 via its first eigenspace, we derive the following:

Corollary 6. *Let M be a compact surface without boundary of genus 1. Suppose the Riemannian metric ds^2 on M is conformally equivalent to the square torus with lattice generated by $(1,0)$ and $(0,1)$. Then $\bar{\lambda}_1 \bar{V}(M) \leqq 4\pi^2$. Equality implies M can be isometrically minimally immersed via its first eigenspace into S^3.*

In dimension greater than 2, Corollary 3 and Theorem 2 imply the following:

Corollary 7. *Let M_0 be a fully equivariantly immersed minimal submanifold in S^n of dimension m (i.e., conditions (ii) and (iii) of Theorem 2 are satisfied). If M is another manifold which is conformally equivalent to M_0, then*

$$\lambda_1(M)^{m/2} V(M) \leqq (m(n+1))^{m/2} V(M_0).$$

In particular, when M is conformally equivalent to an irreducible homogeneous manifold M_0, a theorem of Takahashi (see [11]) says that one can minimally immerse M_0 isometrically into $S^n \subseteq \mathbf{R}^{n+1}$ by its first eigenspace in an equivariant manner. Hence if $n+1$ is the dimension of the first eigenspace of M_0, then

$$\lambda_1(M)^{m/2} V(M) \leqq (m(n+1))^{m/2} V(M_0).$$

For example, when $M_0 = S^m$, we recover a theorem of Berger [1].

§ 4. The First Eigenvalue of a Compact Kähler Manifold

Let M be a m-dimensional compact Kähler manifold which admits a meromorphic map onto $\boldsymbol{CP}^1$. Let Ω be the Kähler form of M. Then we define $V_\Omega(M)$ to be $\inf_f\{\int_M \Omega^{m-1}\wedge f^*\omega$: ω is the standard Kähler form of $\boldsymbol{CP}^1$ and f is a meromorphic map from M onto $\boldsymbol{CP}^1\}$.

Theorem 3. *Let M be a m-dimensional compact Kähler manifold with Kähler form Ω. Then $\lambda_1(M)\leqq 2V_\Omega(M)\operatorname{Vol}(M)^{-1}$.*

Proof. Let f be a meromorphic map from M onto $\boldsymbol{CP}^1$ so that $\int_M \Omega^{m-1}\wedge f^*\omega\leqq V_\Omega(M)+\varepsilon$ where ε is an arbitrary preassigned positive number. By resolving the singularity of f, we may assume that f is holomorphic. (By the theorem of Hironaka, we may find $\hat{M}\to M$ so that the lifting of f to $\hat{M}$ is holomorphic. Then we lift Ω to $\hat{M}$ and perturb Ω to be a Kähler form.)

Let X_1, X_2, and X_3 be the first eigenfunctions of $\boldsymbol{CP}^1$. Then by composing f with an automorphism of $\boldsymbol{CP}^1$, we may assume that

$$\int_M X_i\cdot f=0 \tag{4.1}$$

for $i=1, 2, 3$. (See the argument of Theorem 1.)

As in Theorem 1, we need only to prove

$$\sum_{i=1}^{3}\int_M |\nabla X_i\circ f|^2\leqq 2(V_\Omega(M)+\varepsilon). \tag{4.2}$$

This follows by the observation that

$$\begin{aligned}4\sum_i\int_M \partial(X_i\circ f)\wedge\bar{\partial}(X_i\circ f)\wedge\Omega^{m-1}&=4\sum_i\int_M f^*(\partial X_i\wedge\bar{\partial}X_i)\wedge\Omega^{m-1}\\&=2\int_M f^*(\omega)\wedge\Omega^{m-1}\\&\leqq 2(V_\Omega(M)+\varepsilon).\end{aligned} \tag{4.3}$$

This completes the proof of Theorem 3.

§ 5. Willmore Conjecture and the Conformal Area

Let M be a compact surface without boundary in $\mathbf{R}^n$. Suppose H denotes its mean curvature vector and K its Gaussian curvature with respect to the induced metric. An interesting question is whether one can obtain a lower estimate of the quantity $\int_M |H|^2$. In [16] Willmore conjectured that

$$\int_{T^2}|H|^2\geqq 2\pi^2$$

for any immersed torus, T^2, in $\mathbf{R}^3$.

When the torus, T^2, is the boundary of a circular tubular neighborhood of the unit circle in $\mathbf{R}^3$, the conjecture was verified in [16].

In this section, we shall use the concept of conformal area to study the Willmore conjecture. We allow the surface to be immersed in any higher dimensional Euclidean space $\mathbf{R}^n$. The basis observation is the following:

Lemma 1. *Let M be a compact surface without boundary in $\mathbf{R}^n$. Then*

$$\int_M |H|^2 \geqq V_c(n, M).$$

Furthermore, equality implies M is the image of some minimal surface in S^n under some stereographic projection.

Proof. By using the inverse of stereographic projection, we form a conformal immersion ϕ of M into S^n. Compositing with a Mobius transformation, we may assume that the area of $\phi(M)$ is equal to the n-conformal area of ϕ, $V_c(n, \phi)$.

Using the same argument as in Proposition 1, we conclude that

$$\int_M |H|^2 = \int_{\phi(M)} |\bar{H}|^2 + V(\phi(M)), \tag{5.1}$$

where $\bar{H}$ is the mean curvature vector of $\phi(M)$ in S^n. Since $V(\phi(M)) = V_c(n, \phi)$, we arrive with the inequality

$$\int_M |H|^2 \geqq V_c(n, \phi) \tag{5.2}$$

as to be shown.

An immediate consequence of Lemma 1 and Theorem 1 is:

Lemma 2. *Let M be a compact surface in $\mathbf{R}^n$. Then*

$$\int_M |H|^2 \geqq \tfrac{1}{2} \sup \{\lambda_1 \cdot V(M)\}$$

where sup is taken over any metric which is conformally equivalent to the induced metric from $\mathbf{R}^n$.

The following theorems are direct consequences of Corollaries 1 and 5.

Theorem 4. *Let M be a compact surface in $\mathbf{R}^n$ homeomorphic to $\boldsymbol{RP}^2$. Then*

$$\int_M |H|^2 \geqq 6\pi.$$

Equality implies M is the image of a stereographic projection of some minimal surface in S^4 with $\lambda_1 = 2$.

Theorem 5. *Let M be a surface of genus 1 in $\mathbf{R}^n$. Suppose M is conformally equivalent to one of the flat tori described in Corollary 1. Then*

$$\int_M |H|^2 \geqq 2\pi^2.$$

Equality implies M must be conformally equivalent to the square torus and is the image of a stereographic projection of a minimal torus in S^3.

Proof. The first part of the theorem is a direct consequence of Corollary 1 and Lemma 1. When equality holds, the proof of Corollary 1 implies M is conformally equivalent to a flat torus with lattice generated by $(1,0)$ and $(x,1)$. However by Theorem 1, we conclude that x must be 0 because otherwise one checks easily that the first eigenspace do not give an isometric minimal immersion into S^3. The rest of the conclusion then follows from the last part of Lemma 1.

We have now transform the question of extimating the quantity $\int_M |H|^2$ from below to the problem of estimating the n-conformal area of M. Based on the indication of Theorem 2, we conjecture that if M can be conformally embedded as a minimal surface in S^3, then $\int_M |H|^2$ is not less than the area of this minimal surface.

While we cannot prove this conjecture, we shall give a lower bound of $\int_M |H|^2$ depending on the topological configuration of the surface M. The following theorem is a direct consequence of Fact 3 and (5.2).

Theorem 6. *Let $\psi\colon M\to\mathbf{R}^n$ be an immersion of a compact surface. Suppose there is a point $p\in\mathbf{R}^n$ such that $\psi^{-1}(p)=\{x_1,\dots,x_k\}$ where x_i's are all distinct points in M. Then $\int_M |H|^2\geqq 4k\pi$.*

In particular, this proves that if an immersion $\psi\colon M\to\mathbf{R}^n$ has the property that $\int_M |H|^2<8\pi$, then ψ must be an embedding.

The following corollaries are trivial consequences of Theorems 4–6, together with (5.1).

Corollary 8. *Let M is a compact surface homeomorphic to $\boldsymbol{RP}^2$. If M is a minimal surface in some unit sphere S^n, then*

$$V(M)\geqq 6\pi=\text{volume of the Veronese surface.}$$

Corollary 9. *Let M be a compact surface of genus 1. Suppose M is conformally equivalent to one of the tori described in Corollary 1. If M is a minimal surface in some unit sphere S^n, then*

$$V(M)\geqq 2\pi^2.$$

Equality implies M is conformally equivalent to the flat torus.

Corollary 10. *Let $\phi\colon M\to S^n$ be a minimal immersion of a compact surface M into some unit sphere S^n. If there exists a point $p\in S^n$ such that its preimage set $\phi^{-1}(p)$ consists of k distinct points in M, Then*

$$V(M)\geqq 4k\pi.$$

In particular, if $V(M)<8\pi$, then ϕ must be a minimal embedding.

Remark. Let $\varphi: M \to S^n$ be an isometric immersion of an m-manifold M into $S^n \subseteq \mathbf{R}^{n+1}$. If $\{X_i\}_{i=1}^{n+1}$ are the coordinate functions of $\mathbf{R}^{n+1}$, then a standard elementary computation shows that

$$m^2 \int_M |H|^2 = \sum_{i=1}^{n+1} \int_M (\Delta X_i)^2. \tag{5.3}$$

By translation, one can assume that the center of gravity of M is at the origin of $\mathbf{R}^{n+1}$, i.e.,

$$\int_M X_i = 0 \qquad \text{for all } 1 \leqq i \leqq n+1. \tag{5.4}$$

Expanding the X_i's in terms of the eigenfunctions $\{\psi_\alpha\}$ of M, say

$$X_i = \sum_{\alpha=1}^{\infty} A_{i\alpha} \psi_\alpha, \tag{5.5}$$

the condition (5.4) simply means $A_{i0} = 0$ for all i. However

$$\begin{aligned} \sum_{i=1}^{n+1} \int (\Delta X_i)^2 &= \sum_{i=1}^{n+1} \int \left(\Delta \sum_{\alpha=1}^{\infty} A_{i\alpha} \psi_\alpha \right)^2 \\ &= \sum_{i=1}^{n+1} \int \left(\sum_{\alpha=1}^{\infty} A_{i\alpha} \lambda_\alpha \psi_\alpha \right)^2 \\ &= \sum_{i,\alpha} A_{i\alpha}^2 \lambda_\alpha^2, \end{aligned} \tag{5.6}$$

since $\int \psi_\alpha \psi_\beta = \delta_{\alpha\beta}$. Combining with (5.3), we obtain

$$\begin{aligned} \int_M |H|^2 \geqq \frac{\lambda_1}{m^2} \left(\sum_{i,\alpha} A_{i\alpha}^2 \lambda_\alpha \right) &= \frac{-\lambda_1}{m^2} \sum_{i=1}^{n+1} \int X_i \Delta X_i \\ &= \frac{\lambda_1}{m^2} \sum_{i=1}^{n+1} \int |\nabla X_i|^2 \\ &= \frac{\lambda_1}{m} V(M), \end{aligned} \tag{5.7}$$

where the last equality follows from the fact that φ is an isometric immersion, hence $\sum_{i=1}^{n+1} |\nabla X_i|^2 = m$. Using Hölder inequality, one derives theorems of Bleecker-Weiner [3], Reilly [13], and Chen [4].

The above observation together with some elementary algebraic manipulation, we see that the Willmore conjecture for isometrically immersed flat tori follows directly.

Proposition 2. Let M be a two dimensional flat torus in $\mathbf{R}^n$. Up to a homothety M must be $\mathbf{R}^2$ divided out by a lattice generated by two vectors of the form $\{(1, 0), (x, y)\}$ where $0 \leqq x \leqq \frac{1}{2}$ and $\sqrt{1-x^2} \leqq y$. Then

$$\int_M |H|^2 \geqq \pi^2 \left(y + \frac{1}{y}\right).$$

Proof. In view of the above remark, it suffices to estimate the quantity in (5.3) under the assumption (5.4). On the other hand, it is known that [2] the eigenfunctions on M when lifted to $\mathbf{R}^2$ are functions of the form

$$\cos(2\pi\langle v, w\rangle) \quad \text{and} \quad \sin(2\pi\langle v, w\rangle) \tag{5.8}$$

where v is an element in the dual lattice Γ^* and $w \in \mathbf{R}^2$. Moreover, the eigenvalues are $\{4\pi^2|v|^2\}$, for $v \in \Gamma^*$, where each has multiplicity 2 unless $v = 0 \in \mathbf{R}^2$. The dual lattice Γ^* clearly is generated by the pair of vectors $\{(0, 1/y), (1, -x/y)\}$, hence, all elements in Γ^* are of the form $\left\{\left(q, \frac{p-qx}{y}\right) \,\middle|\, p, q \in \mathbb{Z}\right\}$.

Following the same argument as above, if we express X_i in terms of the non-constant eigenfunctions

$$\begin{aligned} X_i = &\sum_{p,q} A_{ipq} \cos\left(2\pi\left\langle\left(q, \frac{p-qx}{y}\right), \cdot\right\rangle\right) \\ &+ \sum_{p,q} B_{ipq} \sin\left(2\pi\left\langle\left(q, \frac{p-qx}{y}\right), \cdot\right\rangle\right), \end{aligned} \tag{5.9}$$

then

$$\begin{aligned} \sum_{i=1}^{n+1} \int (\Delta X_i)^2 \geqq 16\pi^4 &\Bigg[\sum_{\substack{i,p,q \\ (p,q)\neq(0,0)}} A_{ipq}^2 \left(q^2 + \left(\frac{p-qx}{y}\right)^2\right)^2 \\ &+ \sum_{\substack{i,p,q \\ (p,q)\neq(0,0)}} B_{ipq}^2 \left(q^2 + \left(\frac{p-qx}{y}\right)^2\right)^2 \Bigg] \\ \geqq 16\pi^4 &\Bigg[\sum_{\substack{i,p,q \\ (p,q)\neq(0,0)}} (A_{ipq}^2 + B_{ipq}^2) \left(q^2 + \left(\frac{p-qx}{y}\right)^2 \frac{1}{y^2}\right)\Bigg]. \end{aligned} \tag{5.10}$$

Here we have used the algebraic inequalities

$$\left[q^2 + \left(\frac{p-qx}{y}\right)^2\right]^2 \geqq q^2 + \left[2q^2 + \left(\frac{p-qx}{y}\right)^2\right] \left(\frac{p-qx}{y}\right)^2 \geqq q^2 + \frac{1}{y^2}\left(\frac{p-qx}{y}\right)^2$$

for $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ where $(p, q) \neq (0, 0)$, $0 \leqq x \leqq \frac{1}{2}$, and $1 - x^2 \leqq y^2$.

On the other hand, the assumption that the flat metric is the induced metric means

$$\begin{aligned} \sum_{i=1}^{n+1} \int_M |\nabla X_i|^2 &= 2 \sum_{i=1}^{n+1} \int_M (e_1 X_i)^2 \\ &= 2 \sum_{i=1}^{n+1} \int_M (e_2 X_i)^2 \\ &= 2V(M) \end{aligned} \tag{5.11}$$

for any orthonormal frame $\{e_1, e_2\}$ on M. In particular, if we pick e_1 and e_2 to be the canonical directions corresponding to the coordinates in $\mathbf{R}^2$, then

$$\sum_{i=1}^{n+1} \int_M (e_1 X_i)^2 = 4\pi^2 \sum_{\substack{i,p,q\\(p,q)\neq 0}} (A_{ipq}^2 + B_{ipq}^2) q^2$$

and

$$\sum_{i=1}^{n+1} \int_M (e_2 X_i)^2 = 4\pi^2 \sum_{\substack{i,p,q\\(p,q)\neq 0}} (A_{ipq}^2 + B_{ipq}^2) \left(\frac{p-qx}{y}\right)^2. \tag{5.12}$$

Therefore, (5.10) becomes

$$\sum_{i=1}^{n+1} \int (\Delta X_i)^2 \geqq 4\pi^2 V(M) \left(1 + \frac{1}{y^2}\right) = 4\pi^2 \left(y + \frac{1}{y}\right),$$

which proves the assertion.

Remark. The inequality $\int_M |H|^2 \geqq \pi^2 \left(y + \frac{1}{y}\right)$ is sharp, and is achieved by the isometric embedding of the torus with lattice generated by $\{(1,0), (0,y)\}$ via the eigenfunctions $\left\{\cos(2\pi\tilde{x}),\ \sin(2\pi\tilde{x}),\ \cos\left(\frac{2\pi}{y}\tilde{y}\right),\ \sin\left(\frac{2\pi}{y}\tilde{y}\right)\right\}$ into $\mathbf{R}^4$ with $\tilde{x}$ and $\tilde{y}$ the coordinates in $\mathbf{R}^2$. We shall point out that in Chen's most recent article [4, V], he proved the inequality $\int |H|^2 \geqq 2\pi^2$ under the same assumption in our Proposition 2. Clearly, since $\frac{1}{y} + y \geqq 2$, one derives Chen's theorem as a corollary of the above.

In dimensions greater than 2, since all compact flat manifolds are covered by some flat torus, and the fact that all eigenvalues and eigenfunctions of flat tori are explicitly computed as above, it is obvious that a similar argument will yield lower estimates for $\int_M |H|^2$ when M is isometrically embedded in $\mathbf{R}^n$.

Appendix

The first eigenvalue of a compact surface with boundary and a generalization of a result of Szegö [15].

If we apply Theorem 1 to a compact surface diffeomorphic to the disk, we find that the first eigenvalue (with Neumann condition) is bounded by $8\pi \times V(M)^{-1}$. However, Szegö's result says that for a compact simply connected domain in R^2, the above value can be improved to $p^2\pi V(M)^{-1}$, where $p \sim 1.8412$ is the first positive zero of the Bessel function $J_1'(r)$. Both these results are sharp. In the first case, we can illustrate the sharpness by taking the domain to be the sphere minus a small cap. In the second case, the value is achieved by the circular disk. We offer an explanation by generalizing Szegö's theorem in a more intrinsic manner.

Theorem. *Let M be a compact simply connected surface with non-positive curvature. Then the first eigenvalue of M (with the Neumann condition) is not greater than $p^2\pi V(M)^{-1}$.*

Proof. We follow the argument of Szegö's. By the uniformization theorem, we can assume that our surface M is parametrized by the unit disk with metric tensor $e^{2\rho}|dz|^2$. By the argument of Szegö, we may also assume that $\int e^{2\rho} u_1 r\,dr\,d\theta = \int e^{2\rho} u_2 r\,dr\,d\theta = 0$ where u_1 and u_2 form an orthonormal basis for the first eigenspace of the disk.

By the mini-max principle, we have therefore

$$\lambda_1(M) \leqq \left[\int_M ((\nabla u_1)^2 + |\nabla u_2|^2)\right] \left[\int_M (u_1^2 + u_2^2)\right]^{-1} \tag{A.1}$$

As the Dirichlet integral is invariant under conformal change,

$$\int_M (|\nabla u_1|^2+|\nabla u_2|^2)=\int_D (|\nabla u_1|^2+|\nabla u_2|^2)$$
$$=p^2\int_D (u_1^2+u_2^2). \tag{A.2}$$

On the other hand, as $u_1^2+u_2^2$ depends only on r,

$$\begin{aligned}\int_M (u_1^2+u_2^2)&=\int_0^1\int_0^{2\pi}(u_1^2+u_2^2)e^{2\rho}r\,d\theta\,dr\\&=\int_0^1(u_1^2+u_2^2)\left(\int_0^{2\pi}e^{2\rho}r\,d\theta\right)dr\\&=(u_1^2+u_2^2)(1)\int_0^1\int_0^{2\pi}e^{2\rho}r\,d\theta\,dr-\int_0^1\frac{\partial}{\partial r}(u_1^2+u_2^2)\left(\int_0^r\int_0^{2\pi}e^{2\rho}t\,d\theta\,dt\right)dr.\end{aligned} \tag{A.3}$$

We claim that

$$\int_0^r\int_0^{2\pi}e^{2\rho}t\,d\theta\,dt\leqq r^2\int_0^1\int_0^{2\pi}e^{2\rho}t\,d\theta\,dt.$$

In fact,

$$\begin{aligned}\frac{d}{dr}\left[\left(\int_0^r\int_0^{2\pi}e^{2\rho}t\,d\theta\,dt\right)r^{-2}\right]&=r^{-1}\int_0^{2\pi}e^{2\rho}\cdot d\theta-2r^{-3}\int_0^r\int_0^{2\pi}e^{2\rho}t\,d\theta\,dt\\&=r^{-3}\int_0^r t^2\left(\frac{d}{dt}\int_0^{2\pi}e^{2\rho}d\theta\right)dt.\end{aligned}$$

Our claim will be a consequence of

$$\frac{d}{dr}\int_0^{2\pi}d\theta\geqq 0. \tag{A.4}$$

This follows from the fact that M has non-positive curvature which implies $\frac{d^2}{dr^2}\int_0^{2\pi}e^{2\rho}d\theta\geqq 0$. $\left(\text{Note that by regularity, } \frac{d}{dr}\int_0^{2\pi}e^{2\rho}d\theta\Big|_{r=0}=0.\right)$ Hence we have proved our claim and inequality (A.3) implies that

$$\begin{aligned}\int_M (u_1^2+u_2^2)&\geqq\left(\int_0^1\int_0^{2\pi}e^{2\rho}r\,d\theta\,dr\right)\left[(u_1^2+u_2^2)(1)-\int_0^1\frac{\partial}{\partial r}(u_1^2+u_2^2)r^2\,dr\right]\\&=\pi^{-1}\left(\int_0^1\int_0^{2\pi}e^{2\rho}r\,d\theta\,dr\right)\int_D(u_1^2+u_2^2).\end{aligned}$$

Putting this inequality into (A.1) and (A.2) we conclude that $\lambda_1(M)\leqq p^2\pi V(M)^{-1}$.

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