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Conformally flat manifolds, Kleinian groups and scalar curvature

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One of the natural generalizations of conformal structure on a two dimensional surface is a conformally flat structure on an n-manifold. In higher dimensions, not every manifold admits such a structure and it is a difficult problem to give a good classification of conformally flat manifolds. Recall that conformally flat Riemannian manifolds are manifolds whose metrics are locally conformally equivalent to the Euclidean metric. Kuiper [Ku1, Ku2] was the first to study the global properties of these manifolds. He classified those compact conformally flat manifolds with abelian fundamental group. For dimensions greater than two, the Liouville theorem tells us that the conformal transformations of S^n are determined locally and are given by Möbius transformations. Hence by a standard monodromy argument, a simply connected conformally flat manifold (with dimension $\geqq 3$) has a conformal immersion into S^n which is unique up to composition with a Möbius transformation of S^n. For a general locally conformally flat manifold, we can determine such a conformal immersion from its universal cover and this immersion is called the developing map. The fundamental group of the manifold acts on its universal cover and by the above uniqueness statement, is mapped into the Möbius group by a homomorphism called the holonomy representation.

The developing map and the homomorphism of π_1 into the Möbius group form the most important invariants for the study of conformally flat manifolds. An important class of locally conformally flat manifolds arises from the case in which the developing maps are injective and the manifolds are therefore quotients of open subsets of S^n by certain Kleinian groups. This class of conformally flat manifolds has been extensively studied by many mathematicians including Mostow, Thurston, Kulkarni, Goldman, Kamishima and others.

One of the main accomplishments of this paper is that we find an extensive class of locally conformally flat manifolds whose developing maps are injective. Such manifolds are, in particular, quotients of a simply connected domain in S^n by a Kleinian group. To explain this class of manifolds, we need to explain

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some new global conformal invariants. For a complete conformally flat manifold which can be immersed conformally into S^n, we can construct a Green's function for the conformally invariant Laplace operator. We are able to show that the "Sobolev constant" of such a manifold is the same as that of S^n. From this we prove that the minimal Green's function is integrable to the power $\frac{2n}{(n-2)}$ outside a neighborhood of its singularity. The minimum exponent for the integrability of the Green's function is an important invariant of the manifold. Multiplying such an exponent by $\frac{n-2}{2}$, we obtain a number $d(M)$ between zero and n. When M is an open subset of S^n, $d(M)$ is not less than the Hausdorff dimension of the complement of M. The invariant $d(M)$ measures "the conformal dimension" of M at infinity.

We demonstrate that when $d(M)$ is less than $\frac{(n-2)^2}{n}$ and when the scalar curvature is bounded, any nontrivial conformal map into S^n is injective and the complement of the image has zero Newtonian capacity. It is easy to see from here that such a manifold can be embedded as a simply connected domain in S^n.

When a conformally flat manifold M does not admit a nontrivial conformal map to S^n, we can consider its holonomy cover and define $d(M)$ in terms of this cover. When M is compact, $d(M)$ is an invariant depending only on the conformal structure. The previous theorem implies that if $d(M)<\frac{(n-2)^2}{n}$, M is the quotient of a simply connected domain in S^n by a Kleinian group.

There is a close relation between scalar curvature and the invariant $d(M)$. We prove that for a complete manifold with nonnegative scalar curvature, we have $0 \leqq d(M) \leqq \frac{n}{2}$. When the scalar curvature of M is greater than a positive constant or when the first eigenvalue of the Laplacian is positive, the stronger bound $0 \leqq d(M) \leqq \frac{n-2}{2}$ holds. Combining this with the previous statements, we see that when n is large enough, M is always quotient of a simply connected domain in S^n by a Kleinian group. By using the energy function which was introduced by the first author in [Sc1], we prove the same statement for arbitrary n.

We also get topological information for locally conformally flat manifolds. For any integer $k \geqq d$, we prove that when $d(M) < \min\left\{n-k-1, \frac{(n-2)^2}{n}\right\}$, then $\Pi_i(M)=0$ for $i=2, \ldots, k$. When M is complete and conformally flat with bounded curvature and with scalar curvature bounded from below by a positive constant, then $\Pi_i(M)=0$ for $i=2, \ldots, \left[\frac{n}{2}\right]$ where $\left[\frac{n}{2}\right]$ denotes the integer part of $\frac{n}{2}$.

If M is a compact locally conformally flat manifold which is the quotient of a simply connected domain by a Kleinian group Γ, then we prove that M admits a metric with nonnegative scalar curvature if and only if $d \leqq \frac{n-2}{2}$. Here d is the Hausdorff dimension of the limit set of Γ. Unless Γ is an elementary group, we prove that $d(M)=d$. A corollary of this theorem says that if the holomomy group of a compact conformally flat manifold M is amenable, then M is either flat or covered by $S^{n-1} \times S^1$. Results related to this last theorem were proven in the literature by several people including Goldman [G], Fried [F] and Kamishima [Ka]. Goldman has been able to extend these techniques to prove the same statement for amenable holonomy.

In the last section, we discuss the relation of conformally flat manifolds with weak solutions of a well-known nonlinear elliptic equation on S^n. We demonstrate examples of weak solutions whose singular sets are precisely given by limit sets of Kleinian groups including Cantor sets and various sets of fractional Hausdorff dimension.

1. Preliminaries on conformally flat manifolds

A manifold M of dimension n is said to be *locally conformally flat* if it admits a coordinate covering $\{U_\alpha, \phi_\alpha\}$, $\phi_\alpha: U_\alpha \to S^n$, such that whenever $U_\alpha \cap U_\beta \neq 0$, the change of coordinates map $\phi_\beta \circ \phi_\alpha^{-1}$ is a conformal diffeomorphism from $\phi_\alpha(U_\alpha \cap U_\beta)$ onto $\phi_\beta(U_\alpha \cap U_\beta)$. Note that if $n>2$ it follows from Liouville's theorem [Sp] that $\phi_\beta \circ \phi_\alpha^{-1}$ on any connected component of $\phi_\alpha(U_\alpha \cap U_\beta)$, is the restriction of a Möbius transformation of S^n. If M is locally conformally flat and g is a Riemannian metric on M, then we say that g is *compatible* with the conformally flat structure if for each α the map $\phi_\alpha: (U_\alpha, g) \to S^n$ is a conformal mapping. This is equivalent to saying that g has an expression of the form

$$g \equiv \lambda(x) \sum_{i=1}^{n} (dx^i)^2$$

for a function $\lambda > 0$ on U_α where $\phi_\alpha = (x^1, \ldots, x^n)$. It follows that the Weyl tensor of g vanishes. Conversely, if $n \geqq 4$ and (M, g) is a Riemannian manifold such that the Weyl tensor of g vanishes, then there is a unique locally conformally flat structure on M such that g is a compatible metric. A standard partition of unity argument shows that a paracompact locally conformally flat manifold admits a compatible metric g.

Suppose M is a smooth manifold and $\Phi: M \to S^n$ is an immersion, i.e., the differential of Φ is a linear isomorphism at each point. The immersion Φ then induces a locally conformally flat structure on M since for any point $p \in M$ there exists a neighborhood U_p of p such that $\Phi: U_p \to S^n$ is a diffeomorphism onto its image. The change of coordinates transformation is then the identity and hence M has a unique locally conformally flat structure with respect to which $\Phi: M \to S^n$ is a conformal map. If g_0 denotes the standard Riemannian metric on S^n, it then follows that $\Phi^* g_0$ is a compatible (incomplete) Riemannian metric on M. For a manifold M^n which is simply connected and of dimension n greater than or equal to three, every locally conformally flat structure on M is induced by an immersion $\Phi: M^n \to S^n$ called the *developing map.*

To see this, observe that if $p_0 \in M$ and (U_0, ϕ_0) is a chart with $p_0 \in U_0$, then we can define $\Phi \equiv \phi_0$ in a small neighborhood of p_0. We can then analytically continue Φ along any curve passing through p_0 in the following way: If Φ is defined on an open arc γ, and if $p \in M$ is an endpoint of γ, then we can choose a coordinate chart (U, ϕ) containing p and by Liouville's theorem we have a Möbius transformation $\psi: S^n \to S^n$ which agrees with $\Phi \circ \phi^{-1}$ in a neighborhood of $\phi(p)$. We can then extend Φ to a neighborhood of p by defining $\Phi = \psi \circ \phi$. Since M is simply connected we get a well-defined locally conformal map $\Phi: M \to S^n$.

For a general locally conformally flat manifold M we have the immersion $\Phi: \tilde{M} \to S^n$ where $\tilde{M}$ is the universal covering manifold of M. The uniqueness of Φ up to composition with a conformal transformation of S^n implies the existence of a homomorphism $\rho: \pi_1(M) \to C_n$, where C_n denotes the conformal group of S^n, satisfying $\Phi \circ \gamma = \rho(\gamma) \circ \Phi$ for $\gamma \in \pi_1(M)$ where we view $\pi_1(M)$ as a group of deck transformations on $\tilde{M}$. The homomorphism ρ is called the *holonomy representation* of the conformally flat structure. Conversely, conformally flat structures on M are determined by a homomorphism $\rho: \pi_1(M) \to C_n$ together with an equivariant immersion $\Phi: \tilde{M} \to S^n$.

In general, let ker (ρ) denote the kernel of the homomorphism ρ. Then ker (ρ) is a normal subgroup of $\pi_1(M)$, and the developing map is defined on the covering $\hat{M} = \tilde{M}/\ker(\rho)$. Thus we have $\Phi: \hat{M} \to S^n$ and the group $\hat{\Gamma} \equiv \pi_1(M)/\ker(\rho)$ acts by deck transformations on $\hat{M}$ with $M = \hat{M}/\hat{\Gamma}$. We then have $\hat{\rho}: \hat{\Gamma} \to C_n$ with $\hat{\rho}$ injective. It is often convenient to work with the covering $\hat{M}$ instead of the universal covering $\tilde{M}$. We refer to $\hat{M}$ as the *holonomy covering* of M.

Given a locally conformally flat manifold M together with a compatible metric g, the operator L is defined by

$$L\phi = \Delta\phi - \frac{n-2}{4(n-1)} R(g)\phi$$

where $R(g)$ is the scalar curvature of g. L is conformally invariant in the sense that if $\bar{g} = u^{\frac{4}{n-2}} g$, then we have for any function ϕ

$$L(u\phi) = u^{\frac{n+2}{n-2}} \bar{L}(\phi). \tag{1.1}$$

We will refer to the operator L as the *conformal Laplacian.* A basic fact about L is that for a metric $\bar{g} = u^{\frac{4}{n-2}} g$ the scalar curvature $R(\bar{g})$ is given by

$$R(\bar{g}) = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} Lu. \tag{1.2}$$

A consequence of (1.1) and (1.2) is the following well-known result.

Lemma 1.1. *If M is a compact locally conformally flat manifold, then M admits a compatible metric whose scalar curvature does not change sign. The sign is* *uniquely determined by the conformal structure, and so there are three mutually exclusive possibilities: M admits a compatible metric of*

(i) *positive,* (ii) *negative, or* (iii) *identically zero scalar curvature.*

Proof. The three possibilities are distinguished by the sign of $\lambda_0(-L)$, the lowest eigenvalue. If $u_0>0$ denotes a lowest eigenfunction, then the metric $u_0^{\frac{4}{n-2}} g$ has scalar curvature of one sign, and it is straightforward that (i), (ii), and (iii) are mutually exclusive and exhaustive possibilities.

We illustrate the analytic method by proving a result about the fundamental group of conformally flat manifolds of negative scalar curvature.

Proposition 1.2. *If (M, g) is locally conformally flat with strongly negative scalar curvature, then the universal cover $(\tilde{M}, \tilde{g})$ has positive lowest eigenvalue (for the Laplace operator), and in particular has exponential volume growth rate for geodesic balls. Therefore, if M is compact, locally conformally flat with negative scalar curvature, then $\pi_1(M)$ is a non-amenable group.*

Proof. Let $\Phi\colon \tilde{M}\to S^n$ be the developing map, and write $\Phi^*(g_0)=u^{\frac{4}{n-2}}\tilde{g}$ where g_0 is the standard metric on S^n. Thus there is a constant $c>0$ such that $\Delta u+cu<0$ on $\tilde{M}, u>0$. This implies that $\lambda_0(\tilde{M})\geqq c$. The exponential volume growth then follows by a standard variational argument. The final statement of the proposition follows from a theorem of R. Brooks [B] which shows that for a compact Riemannian manifold M the condition $\lambda_0(\tilde{M})>0$ is equivalent to the non-amenability of $\pi_1(M)$. This proves Proposition 1.2.

The following result derives from the same idea.

Corollary 1.3. *Suppose (M, g) is locally conformally flat. The conformal Laplacian L of $\hat{M}$ has a minimal positive Green's function on $\hat{M}$.*

Proof. Let $p_0\in\hat{M}$, and let $\Phi\colon \hat{M}\to S^n$ be the developing map, and let $q_o=\Phi(p_0)$. Let G_0 be the Green's function on (S^n, g_0) for the conformal Laplacian L_0 with pole at q_0 say $L_0 G_0=-\delta_{q_0}$. If we write $\Phi^*(g_0)=|\Phi'|^2 g$ on $\hat{M}$, then by (1.1) we have

$$LH=-\sum_{p\in\Phi^{-1}(q_0)}|\Phi'(p)|^{\frac{n+2}{2}}\delta_p$$

where $H=|\Phi'|^{\frac{n-2}{2}} G_0\circ\Phi$. The existence of a minimal positive solution G of $LG=-\delta_{p_0}$ then follows by standard arguments. Moreover, we have the bound

$$G\leqq|\Phi'(p_0)|^{-\frac{n+2}{2}} H.$$

2. Some global conformal invariants

Let M^n be a conformal Riemannian manifold (not necessarily locally conformally flat). The invariance properties of the operator L imply that the "Sobolev" constant $Q(M)$ defined below is a conformal invariant

$$Q(M)=\inf\left\{-\int_M \phi L\phi\,dv\colon \phi\in C_c^\infty(M),\ \int_M \phi^{\frac{2n}{n-2}}\,dv=1\right\}$$

where $C_c^\infty(M)$ denotes the space of smooth functions with compact support on M. If M is a compact manifold, it is an easy consequence of Lemma 1.1 that $Q(M)$ is positive (resp. zero, negative) if and only if M admits a conformally compatible metric with everywhere positive (resp. zero, negative) scalar curvature. A consequence of the work of [Au], [Sc1] on the Yamabe problem is that (for compact M) $Q(M)$ is strictly less than $Q(S^n)$ unless M is conformally equivalent to S^n with its standard conformal structure. The following lemma is well known.

Lemma 2.1. *If M is an open subset of S^n, then $Q(M)=Q(S^n)$.*

Proof. It is clear that $Q(\Omega_1)\leqq Q(\Omega_2)$ if $\Omega_1\supseteq\Omega_2$, and also it is clear that

$$\lim_{\varepsilon\to 0} Q(S^n-B_\varepsilon)=Q(S^n)$$

where B_ε denotes a ball of radius $\varepsilon>0$ in the standard metric. Now if $M=\Omega$ is a domain in the sphere, then there is a conformal transformation $\Phi\colon S^n\to S^n$ such that $\Phi(\Omega)\supseteq S^n-B_\varepsilon$ for any preassigned $\varepsilon>0$. Thus we have

$$Q(\Omega)=Q(\Phi(\Omega))\leqq Q(S^n-B_\varepsilon).$$

Since ε is arbitrary it follows that $Q(\Omega)\leqq Q(S^n)$, and the opposite inequality follows by inclusion.

We now generalize this result to manifolds M^n which can be mapped conformally into S^n. This includes those manifolds which are simply connected and locally conformally flat.

Proposition 2.2. *Assume there exists a conformal map $\Phi\colon M^n\to S^n$. We then have $Q(M)=Q(S^n)$.*

Proof. Since Φ is locally one to one, it follows from Lemma 2.1 that there exists an open subset $U\subset M$ with $Q(U)=Q(S^n)$. It then follows by inclusion that $Q(M)\leqq Q(S^n)$.

On the other hand, if we let $\{U_i\}$, $i=1,2,\ldots$ be an exhaustion of M by compact domains with smooth boundary, we then have $Q(M)=\lim\limits_{i\to\infty} Q(U_i)$. Thus, in order to show $Q(M)\geqq Q(S^n)$, it is enough to show that $Q(U)\geqq Q(S^n)$ for any domain $U\subset M$ with $\bar U$ compact and ∂U smooth. Suppose on the contrary we have $Q(U)<Q(S^n)$. The existence theory (see [Sc1]) then implies that $Q(U)$ can be realized by a smooth function $u>0$ on U satisfying $\int\limits_U u^{\frac{2n}{n-2}}\,dv=1$ as well as

$$Lu+Q(U)\,u^{\frac{n+2}{n-2}}=0 \quad \text{on } U$$
$$u=0 \quad \text{on } \partial U.$$

If we extend u by defining $u\equiv 0$ on $M-U$ we then have

$$Lu+Q(U)\,u^{\frac{n+2}{n-2}}\geqq 0 \quad \text{on } M \tag{2.1}$$
$$\int\limits_M u^{\frac{2n}{n-2}}\,dv=1$$

where the inequality in (2.1) is understood in the distributional sense. We now define a function v on S^n as follows: Set $v \equiv 0$ on $S^n - \Phi(\bar{U})$, and for $y \in \Phi(\bar{U})$, define

$$v(y) = \max\{|\Phi'(x)|^{-\frac{n-2}{2}} u(x) \colon x \in \Phi^{-1}(y) \cap \bar{U}\}.$$

Since Φ is an immersion, the set $\Phi^{-1}(y) \cap \bar{U}$ is finite, and for each $x \in \Phi^{-1}(y) \cap \bar{U}$ there is a neighborhood η_x of x such that Φ is a diffeomorphism of η_x onto $\Phi(\eta_x)$, a neighborhood of y. Let Φ_x^{-1} denote the inverse of this local diffeomorphism, and observe that the function v_x defined on $\Phi(\eta_x)$ by

$$v_x(y_1) = |(\Phi_x^{-1})'|^{\frac{n-2}{2}} u(\Phi_x^{-1}(y_1))$$

satisfies $L_0 v_x + Q(U) v_x^{\frac{n+2}{n-2}} \geqq 0$ on $\Phi(\eta_x)$. This follows from the conformal invariance properties of L. Here L_0 denotes the conformal Laplacian of S^n. Thus we see that v is a nonnegative Lipschitz function on S^n satisfying $L_0 v + Q(U) v^{\frac{n+2}{n-2}} \geqq 0$ on S^n. Again by conformal invariance we have

$$\int_{\Phi(\eta_x)} v_x^{\frac{2n}{n-2}} \, dv_0 = \int_{\eta_x} u^{\frac{2n}{n-2}} \, dv,$$

and hence we see that $\int_{S^n} v^{\frac{2n}{n-2}} \, dv_0 \leqq 1$. By integrating the differential inequality satisfied by v we have

$$E_0(v) \equiv -\int_{S^n} v L_0 v \leqq Q(U) \int_{S^n} v^{\frac{2n}{n-2}} \, dv_0.$$

Since $\int_{S^n} v^{\frac{2n}{n-2}} \, dv_0 \leqq 1$, this inequality implies that $Q(S^n) \leqq Q(U)$, a contradiction. This completes the proof of Proposition 2.2.

A consequence of the above result is the following corollary which limits the growth of the minimal Green's function for a manifold which can be mapped conformally into S^n.

Corollary 2.3. *Suppose there exists a conformal map $\Phi \colon M^n \to S^n$. Let g be any compatible metric on M, and let G_0 be the minimal Green's function for L with pole at $0 \in M$. (Note that G_0 exists by Corollary 1.3.) For any open neighborhood of 0 we have*

$$\int_{M \setminus \mathcal{O}} G_0^{\frac{2n}{n-2}} \, dv < \infty.$$

Proof. Let $U_1, U_2, \ldots$ be an exhaustion of M by smooth precompact domains with $\mathcal{O} \subset U_1$ and $\bar{U}_i \subset U_{i+1}$. Let $G_0^{(i)}$ denote the Green's function of U_i with pole at 0, and with $G_0^{(i)} \equiv 0$ on ∂U_i. By definition of G_0 we then have $\lim_{i \to \infty} G_0^{(i)} = G_0$. For an open set $U \subset M$, and $\phi \in C^\infty(U)$, we let

$$E_U(\phi) = \int_U (|\nabla \phi|^2 + c(n) R \phi^2) \, dv$$

denote the energy of ϕ taken on U. Let ζ be a smooth function with compact support in U_1 and with $\zeta \equiv 1$ on $\mathcal{O}$. Since $G_0^{(i)}$ minimizes energy for its boundary values on $U_i \backslash \mathcal{O}$, we have $E_{U_i \smallsetminus \mathcal{O}}(G_0^{(i)}) \leqq E_{U_i \smallsetminus \mathcal{O}}(\zeta G_0^{(i)})$. Therefore it follows that $E_{U_i \smallsetminus \mathcal{O}}(G_0^{(i)}) \leqq C$ for a constant C independent of i. It follows immediately that $E_{U_i}((1-\zeta) G_0^{(i)}) \leqq C'$ for a constant C' The Sobolev inequality of Proposition 2.2 now implies a bound on

$$\int_{U_i} [(1-\zeta) G_0^{(i)}]^{2n/(n-2)} \, dv.$$

This gives the conclusion of Corollary 2.3.

For a locally conformally flat Riemannian manifold M, let $\hat{M}$ denote the holonomy covering, and define a number $p(M)$ by

$$p(M) \equiv \inf \{p \colon \int_{\hat{M} \smallsetminus \mathcal{O}} G_0^p \, dv < \infty \text{ for any open } \mathcal{O} \ni 0\}.$$

We then define $d(M) = \left(\frac{n-2}{2}\right) p(M)$. The following proposition outlines some properties of $p(M)$ and $d(M)$.

Proposition 2.4. *The quantities $d(M)$, $p(M)$ satisfy:*

(i) *If M is compact, then $d(M)$ is independent of the metric g but depends only on the conformally flat structure of M.*

(ii) *For any M we have $d(M) \in [0, n]$, and if there is a constant $R_0 > 0$ with $R \geqq R_0$ on M, then we have $d(M) \in [0, (n-2)/2]$.*

(iii) *Suppose (M, g) satisfies $R \geqq 0$. Let $\lambda_0(\hat{M})$ denote the lowest eigenvalue for the Laplace operator on $\hat{M}$, i.e., the limit of the lowest Dirichlet eigenvalue. We distinguish two cases:*

(a) *If $\lambda_0(\hat{M}) > 0$, then $d(M) \in [0, (n-2)/2]$.*

(b) *If $\lambda_0(\hat{M}) = 0$, then $d(M) \in [0, n/2]$.*

Proof of Proposition 2.4. Part (i) follows because a new conformal metric $\bar{g}$ on M is of the form $\bar{g} = v^{\frac{4}{n-2}} g$, and hence the metric on $\hat{M}$ becomes $\bar{g} = u^{\frac{4}{n-2}} g$ where u is a positive bounded function. We then have for any $0 \in \hat{M}$, $\bar{G}_0 = u(0)^{\frac{n+2}{n-2}} u^{-1} G_0$ and hence it is clear that $\bar{G}_0$ is in L^p if and only if G_0 is L^p.

The first statement of (ii) follows from Corollary 2.3. If we assume $R \geqq R_0 > 0$ on M, then we will show that $\int_{\hat{M} \smallsetminus \mathcal{O}} G_0 \, dv < \infty$, and this implies $d(M) \leqq \frac{n-2}{2}$. Let $\{U_i\}$ be a smooth exhaustion of $\hat{M}$ by compact domains with $\bar{\mathcal{O}} \subset U_i \subset \bar{U}_i \subset U_{i+1}$ for $i \geqq 1$, and $\hat{M} = \bigcup_i U_i$. Let $G_0^{(i)}$ be the Dirichlet Green's function on U_i with pole at 0, and observe that we have

$$v_i(0) = \int_{U_i} G_0^{(i)}(x) \, dv_x$$

where v_i is the solution of the problem

$$L v_i = -1 \quad \text{on } U_i, \quad v_i \equiv 0 \quad \text{on } \partial U_i.$$

Since $R \geqq R_0 > 0$, the maximum principle immediately implies

$$\max_{U_i} v_i \leqq (c(n) R_0)^{-1}.$$

Thus $v_i(0)$ is bounded independent of i, and hence we have $\int_{\hat{M} \setminus \mathcal{O}} G_0 \, dv < \infty$ as required.

We now treat (iii), part (a). Let $\{U_i\}$ be an exhaustion of $\hat{M}$ as above, and observe that since $R > 0$ we have $\Delta G_0^{(i)} \geqq 0$ on $U_i \setminus \mathcal{O}$. Multiplying by $(G_0^{(i)})^\varepsilon$ for any $\varepsilon > 0$ and integrating by parts we conclude

$$\int_{U_i \setminus \mathcal{O}} |\nabla (G_0^{(i)})^{\frac{1+\varepsilon}{2}}|^2 \, dv \leqq c(\varepsilon). \tag{2.2}$$

Since $\lambda_0(\hat{M}) > 0$ we then conclude that

$$\int_{U_i \setminus \mathcal{O}} (G_0^{(i)})^{1+\varepsilon} \, dv \leqq c(\varepsilon).$$

This is derived in the same way as we derived Corollary 2.3. This establishes (a).

To prove (b) we use $\Delta G_0^{(i)} \geqq c(n) R G_0^{(i)}$, multiply by $(G_0^{(i)})^\varepsilon$ and integrate by parts as above to show

$$\int_{U_i \setminus \mathcal{O}} R (G_0^{(i)})^{1+\varepsilon} \, dv \leqq c.$$

Combining this with (2.2) and using $R \geqq 0$ we get

$$E_{U_i \setminus \mathcal{O}}((G_0^{(i)})^{\frac{1+\varepsilon}{2}}) \leqq c(\varepsilon).$$

As in the proof of Corollary 2.3 we may use the Sobolev inequality on $\hat{M}$ to derive

$$\int_{U_i \setminus \mathcal{O}} (G_0^{(i)})^{(1+\varepsilon)\frac{n}{n-2}} \, dv \leqq c(\varepsilon).$$

This implies $p(M) \leqq \dfrac{n}{n-2}$ and hence $d(M) \leqq \dfrac{n}{2}$ as required.

We now discuss the case of domains in S^n. If M is a domain in S^n, then it follows that the metric g may be written $g = u^{\frac{4}{n-2}} g_0$ where $u > 0$ on M and g_0 is the standard metric on S^n. If $S^n \setminus M$ has zero Newtonian capacity, it then follows that $G_0 = u(0)^{\frac{n+2}{n-2}} u^{-1} H_0$ where H_0 is the Green's function (on S^n) of the conformal Laplacian for g_0 with pole at $0 \in M$. (To see this, observe that $G_0 \leqq u(0)^{\frac{n+2}{n-2}} u^{-1} H_0$ because G_0 is minimal, and the function $u(0)^{\frac{n+2}{n-2}} u^{-1} H_0$ is a solution with the required singularity. Since $u G_0$ is a solution of $L_0(u G_0) = 0$ where L_0 is the conformal Laplacian of g_0, and $u G_0 \leqq u(0)^{-1} H_0$ and hence is bounded near ∂M we conclude that $u G_0$ extends across ∂M, a set of zero Newtonian capacity. Thus equality holds above.) Thus we see that the condition $\int_{M \setminus \mathcal{O}} G_0^p \, dv < \infty$ is equivalent to $\int_M u^{\frac{2n}{n-2} - p} \, dv_0 < \infty$ where dv_0 is the volume form of the standard metric g_0 on S^n. If we assume that the eigenvalues of g with respect to g_0 dominate δ^{-2} near ∂M where $\delta(x)=\operatorname{dist}(x, \partial M)$, then we see that the above condition implies

$$\int_M \delta(x)^{\frac{n-2}{2}p-n}\, dv_0(x) < \infty. \tag{2.3}$$

If ∂M consists of manifolds of dimension d, then we see easily that the infimum of p for which (2.2) holds is $\frac{2}{n-2}\, d$. Hence we see that $d(M) \geqq d$. (In fact, $d(M)=d$ if the eigenvalues of g are bounded in ratio with δ^{-2} near ∂M.) More generally we have the following result.

Proposition 2.5. *If M is a domain in S^n with ∂M of zero Newtonian capacity, and g is a complete conformal metric on M, then $d(M)$ is greater than or equal to the Hausdorff dimension of ∂M.*

Proof. If $p>0$ with $\int_{M \setminus \mathcal{O}} G_0^p\, dv < \infty$, then as we saw above

$$\int_M u^{\frac{2n}{n-2}-p}\, dv_0 < \infty$$

where $u>0$ is the function defining the metric g, i.e., $g=u^{\frac{4}{n-2}} g_0$. Let ϕ be a function of geodesic distance ρ (w.r.t. g) from $0 \in M$ satisfying $\phi(\rho)=1$ for $\rho \leqq a$, $\phi(\rho)=0$ for $\rho \geqq 2a$, and $|\nabla \phi| \leqq a^{-1}$. From the definition of u we have $|\partial \phi| = u^{\frac{2}{n-2}} |\nabla \phi|$ where $|\partial \phi|$ denotes the gradient with respect to g_0. Thus if we choose $q=n-\frac{n-2}{2}\, p$ we see from above

$$\int_M |\partial \phi|^q\, dv_0 \leqq a^{-q} \int_M u^{\frac{2n}{n-2}-p}\, dv_0 \leqq c\, a^{-q}.$$

Since g is complete, we may choose a arbitrarily large, and hence $\phi = \phi_a \to 1$ on compact subsets of M, each $\phi_a \equiv 0$ near ∂M, and $\int_M |\partial \phi|^q\, dv_0 \to 0$. This implies that ∂M has zero q-capacity, and a standard result [AM] then implies that the Hausdorff dimension of ∂M is less than or equal to $n-q=\frac{n-2}{2}\, p$. This completes the proof of Proposition 2.5.

Under very general conditions we can show that the eigenvalues of g dominate δ^{-2}, and hence relate $p(M)$ to inequality (2.3).

Proposition 2.6. *Suppose M is a domain in S^n and g is a complete conformal metric on M with bounded curvature and that R_g has bounded gradient with respect to g. Then we have the inequality $u(x) \geqq c\,\delta(x)^{-\frac{n-2}{2}}$ for $x \in M$, c a positive constant.*

Proof. Let $0 \in M$ be a fixed point, and choose a Euclidean metric on $\mathbb{R}^n = S^n - \{0\}$. Then we write $g_{ij} = v^{\frac{4}{n-2}} \delta_{ij}$ in terms of Euclidean coordinates. The function

v then satisfies the equation $Lv^{-1}=0$ since $v^{-\frac{4}{n-2}}g$ is a flat metric. Since the Ricci curvature of g is bounded, the Harnack inequality [CY] implies $|\nabla \log v| \leqq c$ near ∂M. Translated to the Euclidean metric this says $|\partial v| \leqq c v^{\frac{n}{n-2}}$ where $|\partial v|$ is the length of the Euclidean gradient of v. Thus we have $|\partial v^{-\frac{2}{n-2}}| \leqq c'$. For a point $x \in \mathbb{R}^n \backslash \partial M$, let γ be a Euclidean line of length $\delta(x)$ from x to a point of ∂M. Integrating from x to any point y of γ we find $v(x)^{-\frac{2}{n-2}} \leqq c' \delta(x) + v(y)^{-\frac{2}{n-2}}$. Since g is complete, we have $\int_\gamma v^{\frac{2}{n-2}} d\zeta = \infty$, and hence there is a sequence $\{y_i\} \subset \gamma$ so that $v(y_i) \to \infty$. Thus we conclude $v(x) \geqq \delta(x)^{-\frac{n-2}{2}}$. Since the spherical metric is equivalent to the Euclidean metric near ∂M, we have established Proposition 2.6.

Theorem 2.7. *Suppose M is a domain in S^n which has a complete conformal metric g with $R(g) \geqq 0$. Then ∂M has zero Newtonian capacity and the Hausdorff dimension of ∂M is at most $\frac{n}{2}$. If $R(g)$ is bounded below by a positive constant then the Hausdorff dimension of ∂M is at most $\frac{n-2}{2}$.*

Proof. The fact that ∂M has zero Newtonian capacity is shown in the next section. The conclusions concerning the Hausdorff dimension of ∂M follow from Propositions 2.4 and 2.5.

It is not known whether smallness of the Hausdorff dimension of ∂M is sufficient for the existence of a complete conformal metric with nonnegative scalar curvature on M. The following example provides negative evidence.

Example 2.8. We claim there exists a convergent sequence of points $\{x_i\}$ with limit x in S^n such that if we let $M = S^n \backslash \{x_i, x\}$, then the infimum of p for which (2.3) holds is as close as we like to $\frac{2n}{n-2}$. It then follows from Proposition 2.6 that for any complete conformal metric g on M having bounded curvature we must have $d(M)$ nearly n. In particular M does not have a complete conformal metric g with $R(g) \geqq 0$ and with the curvature tensor of g bounded (even though ∂M is a countable set of points). To construct such a sequence $\{x_i\}$, let $\alpha > 0$, and choose for any $i \geqq 1$ a maximal set of points S_i in the sphere of radius $i^{-\alpha}$ centered at x whose distances apart are at least $i^{-1-\alpha}$. The number of points in S_i is then of the order i^{n-1}. We take our sequence to be the union of the S_i, and observe that for any $y \in S_i$ the nearest point of $\bigcup_i S_i$ to y has distance $c i^{-1-\alpha}$ from y. Therefore it follows that (2.3) holds if and only if the series

$$\sum_{i=1}^{\infty} (i^{-1-\alpha})^{\frac{n-2}{2}p} i^{n-1}$$

converges. This converges only if $p > (1+\alpha)^{-1} \frac{2n}{n-2}$.

3. An embedding theorem for locally conformally flat manifolds

In this section we show that a large class of locally conformally flat manifolds are conformally equivalent to domains in S^n. The hypothesis which is required to establish this is smallness of the invariant $d(M)$ introduced in the previous section.

Theorem 3.1. *Let (M, g) be a complete Riemannian manifold and $\Phi: M \to S^n$ a conformal map. Assume that the scalar curvature $R(g)$ is bounded below by a negative constant on M, and assume also that $d(M) < \frac{(n-2)^2}{n}$. For $n=3, 4$ we make the assumption that $|R(g)|$ is bounded. Then Φ is one-one and gives a conformal diffeomorphism of M onto $\Phi(M) \subset S^n$. Moreover the boundary of $\Phi(M)$ has zero Newtonian capacity.*

Proof. Let $0 \in M$ and let G_0 denote the minimal Green's function for L with pole at 0. We also let $\bar{G}_0$ denote the pull-back of the Green's function of S^n (with pole at $\Phi(0)$) under Φ normalized so that $\bar{G}_0 - G_0$ is bounded near 0. Thus $\bar{G}_0$ is a multiple of the function $|\Phi'|^{\frac{n-2}{2}} H \circ \Phi$ where H is the S^n Green's function, and $|\Phi'|$ denotes the linear stretch factor of the conformal map Φ. Note that the metric $\bar{g}$ on M defined by $\bar{g} = \bar{G}_0^{\frac{4}{n-2}} g$ is the pull-back of a Euclidean metric on $S^n - \{\Phi(0)\}$ and hence is a flat metric. Since the metric

$$G_0^{\frac{4}{n-2}} g = \left(\frac{G_0}{\bar{G}_0}\right)^{\frac{4}{n-2}} \bar{g}$$

has vanishing scalar curvature, it follows that the function $v = G_0(\bar{G}_0)^{-1}$ is a harmonic function with respect to $\bar{g}$. Since G_0 is the *minimal* Green's function we also have $0 < v \leqq 1$. Our starting point is the Bochner formula for v with respect to the (incomplete) flat metric $\bar{g}$. This is

$$\bar{\Delta} |\bar{\nabla} v|^2 = 2 |\bar{\nabla} \bar{\nabla} v|^2 \tag{3.1}$$

where the "bars" denote quantities taken with respect to $\bar{g}$. We need the following lemma.

Lemma 3.2. *Suppose (M, g) is complete and $\Phi: M \to S^n$ is a conformal map. In addition assume that $R(g) \geqq -c$ for some constant c. It then follows that*

$$\lim_{\sigma \to \infty} (\sup \{G_0(x) : \rho(x, 0) \geqq \sigma\}) = 0.$$

Proof. Since $\int_{M \setminus \mathcal{O}} G_0^{\frac{2n}{n-2}} dv < \infty$ for any open neighborhood $\mathcal{O}$ of 0 by Corollary 2.3, we have

$$\lim_{\sigma \to \infty} \int_{\{x: \rho(x, 0) \geqq \sigma\}} G_0^{\frac{2n}{n-2}} dv = 0.$$

Our proof will be complete if we can establish the estimate

$$G_0(x) \leqq c\left(\int_{B_1(x)} G_0^{\frac{2n}{n-2}}\,dv\right)^{\frac{n-2}{2n}} \tag{3.2}$$

for any $x \in M \backslash B_2(0)$. First observe that the Sobolev inequality of Proposition 2.2 implies, for any ϕ of compact support on M

$$\left(\int_M \phi^{\frac{2n}{n-2}}\,dv\right)^{\frac{n-2}{n}} \leqq c \int_M (|\nabla\phi|^2 + c(n)\,R_+\,\phi^2)\,dv$$

where R_+ denotes the positive part of R. The equation for G_0 combined with the lower bound on R imply

$$\Delta G_0 - c(n)\,R_+\,G_0 \geqq -c_1\,G_0$$

for a constant c_1. Multiplying by $G_0^{p-1}\,\phi^2$ and integrating by parts gives

$$\begin{aligned}(p-1)\int_M \phi^2\,G_0^{p-2}\,|\nabla G_0|^2\,dv + c(n)\int_M \phi^2\,G_0^p\,R_+\,dv \\ \leqq 2\int_M \phi G_0^{p-1}\,|\nabla\phi|\,|\nabla G_0| + c_1\int_M G_0^p\,\phi^2.\end{aligned}$$

Assuming $p \geqq \dfrac{2n}{n-2}$ we get

$$\int_M (|\nabla\phi\,G_0^{p/2}|^2 + c(n)\,R_+\,\phi^2\,G_0^p)\,dv \leqq cp\int_M G_0^p(|\nabla\phi|^2 + \phi^2)\,dv.$$

Combining this with the above Sobolev inequality

$$\left(\int_M (\phi G_0^{p/2})^{\frac{2n}{n-2}}\,dv\right)^{\frac{n-2}{n}} \leqq cp\int_M G_0^p(|\nabla\phi|^2 + \phi^2)\,dv.$$

The standard iteration argument [M] starting with $p_0 = \dfrac{2n}{n-2}$ then implies (3.2). This completes the proof of Lemma 3.2.

Using the observation of [CY] we see that $|\bar\nabla\bar\nabla v|^2 \geqq \dfrac{n}{n-1}\,|\bar\nabla|\bar\nabla v||^2$, and hence from (3.1) we find

$$\bar\Delta|\bar\nabla v|^\alpha \geqq c(\alpha)\,|\bar\nabla v|^{\alpha-2}\,|\bar\nabla|\bar\nabla v||^2 \tag{3.3}$$

for $\alpha > \dfrac{n-2}{n-1}$. Multiplying (3.3) by a smooth function ϕ^2 with compact support away from the poles of $\bar G$ we easily get

$$\int_M \phi^2\,|\bar\nabla v|^{\alpha-2}\,|\bar\nabla|\bar\nabla v||^2\,\overline{dv} \leqq c\int_M |\bar\nabla\phi|^2\,|\bar\nabla v|^\alpha\,\overline{dv}.$$

We will choose $\alpha = \dfrac{2(n-2)}{n}$ and note that for $n \geqq 3$ we have $\alpha > \dfrac{n-2}{n-1}$. We express the right hand side of the previous inequality in terms of g

$$\text{l.h.s.} \leqq c\int_M |\nabla\phi|^2\,\bar G_0^\alpha\,|\nabla v|^\alpha\,dv \tag{3.4}$$

where we have used the choice of α. Next observe that near a pole of $\bar{G}_0$ we have

$$|\nabla v| = 0(|x-0|^{n-3}) \quad \text{and} \quad \bar{G}_0 = 0(|x-0|^{2-n}).$$

Thus if we choose ψ to be a function vanishing near 0 with $\psi \leqq 1$, and replace ϕ by $\psi\phi$ where ϕ is not necessarily zero near x_0 we have from (3.4)

$$\text{l.h.s.} \leqq c \int_M |\nabla\phi|^2 \bar{G}_0^\alpha |\nabla v|^\alpha \, dv + c \int_M \phi^2 |\nabla\psi|^2 |x-x_0|^{-\alpha} \, dv.$$

Therefore, if we choose $\psi = 0$ in $B_r(x_0)$, $\psi \equiv 1$ on $M \setminus B_{2r}(x_0)$ with $|\nabla\psi| \leqq r^{-1}$ we see that the last term above is bounded by $cr^{n-2-\alpha}$. Since $n-2-\alpha > 0$ we may let $r \to 0$ and establish (3.4) for any smooth ϕ with compact support on M.

We now estimate the right hand side of (3.4) by noting

$$\bar{G}_0^\alpha |\nabla v|^\alpha = \bar{G}_0^\alpha |\bar{G}_0^{-1} \nabla G_0 - G_0 \bar{G}_0^{-2} \nabla \bar{G}_0|^\alpha$$

and hence

$$\bar{G}_0^\alpha |\nabla v|^\alpha \leqq c |\nabla G_0|^\alpha + c G_0^\alpha |\nabla \log \bar{G}_0|^\alpha.$$

For a large radius σ we choose $\phi \equiv 1$ on $B_{\sigma/2}(0)$, $\phi \equiv 0$ on $M \setminus B_\sigma(0)$, and $|\nabla\phi| \leqq 2\sigma^{-1}$. Thus (3.4) implies

$$\text{l.h.s.} \leqq c\sigma^{-2} \int_{B_\sigma(0) \setminus B_{\sigma/2}(0)} |\nabla G_0|^\alpha \, dv + c\sigma^{-2} \int_{B_\sigma(0) \setminus B_{\sigma/2}(0)} |\nabla \log \bar{G}_0|^\alpha \, dv. \tag{3.5}$$

Since $\alpha < 2$ we have $|\nabla G_0|^\alpha \leqq c(G_0^\alpha + G_0^\alpha |\nabla \log G_0|^2)$. Away from the poles of $\bar{G}_0$ we have

$$\Delta \log \bar{G}_0 = c(n) R - |\nabla \log \bar{G}_0|^2 = G_0^{-1} \Delta G_0 - |\nabla \log \bar{G}_0|^2.$$

We multiply by a smooth function ψ^2 with compact support and integrate by parts to get

$$\int_M \psi^2 |\nabla \log \bar{G}_0|^2 \, dv \leqq \int_M \psi^2 |\nabla \log G_0|^2 \, dv + 2 \int_M \psi \nabla\psi \cdot (\nabla \log \bar{G}_0 - \nabla \log G_0) \, dv.$$

This implies

$$\int_M \psi^2 |\nabla \log \bar{G}_0|^2 \, dv \leqq c \int_M \psi^2 |\nabla \log G_0|^2 \, dv + c \int_M |\nabla\psi|^2 \, dv$$

for any ψ with compact support away from the poles of $\bar{G}_0$. It is easy to remove the restriction that ψ vanish near the poles of $\bar{G}_0$, and hence to derive the same inequality with ψ of compact support on M. We now replace ψ by $G_0^{\alpha/2}\psi$ and require ψ to vanish near 0. This implies

$$\int_M \psi^2 G_0^\alpha |\nabla \log \bar{G}_0|^2 \, dv \leqq c \int_M \psi^2 G_0^\alpha |\nabla \log G_0|^2 \, dv + c \int_M G_0^\alpha |\nabla\psi|^2 \, dv.$$

We choose $\psi \equiv 0$ on $B_{\sigma/4}(0)$ and on $M \backslash B_{2\sigma}(0)$ and $\psi \equiv 1$ on $B_\sigma(0) \backslash B_{\sigma/2}(0)$ with $|\nabla \psi| \leqq 1$. Then from (3.5) we have

$$\text{l.h.s.} \leqq c\sigma^{-2} \int_{M \smallsetminus B_1(0)} G_0^\alpha \, dv + c\sigma^{-2} \int_{B_{2\sigma}(0) \smallsetminus B_{\sigma/4}(0)} G_0^\alpha |\nabla \log G_0|^2 \, dv. \tag{3.6}$$

Note that for $n \geqq 5$ we have $\alpha > 1$ and we can estimate the second term on the right as follows: We have $\Delta G_0 \geqq -cG_0$ since R has a lower bound. Multiply by $\phi^2 G_0^{\alpha-1}$ and integrate by parts and absorb to get

$$\int_M G_0^{\alpha-2} |\nabla G_0|^2 \phi^2 \, dv \leqq c \int_M G_0^\alpha (|\nabla \phi|^2 + \phi^2) \, dv. \tag{3.7}$$

Thus we may use this in (3.6) to finally derive

$$\text{l.h.s.} \leqq c\sigma^{-2} \int_{M \smallsetminus B_1(0)} G_0^\alpha \, dv. \tag{3.8}$$

To derive (3.8) for $n=3,4$ we need to use a slightly different argument. For $n=3$ we have $\alpha = \frac{2}{3}$, and we use the assumed upper bound on $R(g)$ to get $\Delta G_0 \leqq cG_0$. We multiply by $G_0^{\alpha-1} \phi^2$ and integrate by parts to get

$$(1-\alpha) \int_M \phi^2 G_0^{\alpha-2} |\nabla G_0|^2 \, dv \leqq 2 \int_M \phi G_0^{\alpha-1} |\nabla \phi| |\nabla G_0| \, dv + c \int_M G_0^\alpha \phi^2 \, dv.$$

This implies as before the inequality (3.7) and hence (3.8). For $n=4$ we have $\alpha = 1$, so we choose a positive number $\alpha_1 < \alpha = 1$. From Lemma 3.2 we know that G_0 is bounded on $M \smallsetminus B_1(0)$, and hence (3.6) holds with α replaced by α_1. Since $\alpha_1 < 1$ the argument used for $n=3$ implies (3.7) with α replaced by α_1 and hence

$$\text{l.h.s.} \leqq c\sigma^{-2} \int_{M \smallsetminus B_1(0)} G_0^{\alpha_1} \, dv \tag{3.8$'$}$$

for $n=4$ and $\alpha_1 \in (0,1)$ arbitrary. Since $d(M) < \frac{(n-2)^2}{n}$ we have $p(M) < \frac{2(n-2)}{n} = \alpha$. By Lemma 3.2 we see that for *any* $p > p(M)$ $\int_{M \smallsetminus B_1(0)} G_0^p \, dv < \infty$ since by definition there is a sequence $p_i \downarrow p(M)$) with $\int_{M \smallsetminus B_1(0)} G_0^{p_i} \, dv < \infty$ and hence for i large we have $p_i < \alpha$ and by Lemma 3.2

$$\int_{M \smallsetminus B_1(0)} G_0^\alpha \, dv \leqq c \int_{M \smallsetminus B_1(0)} G_0^{p_i} \, dv < \infty.$$

Therefore the integral appearing on the right of (3.8) and (3.8)$'$ can be taken finite and letting $\sigma \to \infty$ we conclude that $|\nabla v| \equiv$ constant, and since $|\nabla v|(0) = 0$ we have $v = \bar{G}_0^{-1} G_0$ is a constant. Since $v(0) = 1$ we have $v \equiv 1$ on M and that implies $G_0 \equiv \bar{G}_0$. Since $\bar{G}_0$ has a pole at every point of $\Phi^{-1}(\Phi(0))$ it follows that $\Phi^{-1}(\Phi(0)) = \{0\}$. Since $0 \in M$ was arbitrary we have shown that Φ is injective.

Let $\Omega = \Phi(M)$. To show that $\partial\Omega$ has zero Newtonian capacity, choose a point $0 \in \Omega$ and consider a Euclidean metric on $S^n - \{0\} = \mathbb{R}^n$. The minimal

Green's function G_0 is then gotten by minimizing the Euclidean Dirichlet integral for functions ϕ vanishing near $\partial\Omega$ with $\phi\to 1$ near 0. The statement $G_0=\bar{G}_0$ says precisely that this minimizing function is identically one and this is the statement that $\partial\Omega$ has zero Newtonian capacity. This completes the proof of Theorem 3.1.

Proposition 3.3. *Let (M, g) be a complete Riemannian manifold with $R(g)\geqq 0$ and let $\Phi\colon M\to S^n$ be a conformal map. Under any of the hypotheses* (i), (ii), *or* (iii) *below we can conclude that Φ is injective and $\partial\Phi(M)$ has zero Newtonian capacity.*

(i) *Suppose $n\geqq 4$, $R(g)\geqq R_0$ for some number $R_0>0$, and if $n=4$ assume the Ricci curvature of g is bounded.*

(ii) *Suppose $n\geqq 5$ and $\lambda_0(M)>0$ where λ_0 denotes the limit of the lowest Dirichlet eigenvalue for compact domains in M.*

(iii) *Suppose $n\geqq 7$.*

Proof. We use the results of Proposition 2.4 which show that under hypotheses (i) or (ii) we have $d(M)\leqq\frac{n-2}{2}$. Since $\frac{n-2}{2}<\frac{(n-2)^2}{n}$ for $n\geqq 5$ we may apply Theorem 3.1 to establish our conclusion under hypothesis (ii) or (i) with $n\geqq 5$. For $n=4$ we have $\frac{n-2}{2}=\frac{(n-2)^2}{n}=1$. We show that under the hypothesis of bounded Ricci curvature we actually have $d(M)<\frac{n-2}{2}$. To see this, let $G_0^{(i)}$ be the Dirichlet Green's function for U_i, and observe that

$$\Delta G_0^{(i)}\geqq c(n)\,R_0\,G_0^{(i)} \quad \text{on } U_i.$$

Multiply both sides by $e^{\delta\rho}$ where $\delta>0$ and ρ is the geodesic distance from 0 and integrate by parts on $U_i\setminus B_1(0)$. Since the normal derivative of $G_0^{(i)}$ is negative on ∂U_i we have

$$c(n)\,R_0\int_{U_i\setminus B_1(0)} e^{\delta\rho}\,G_0^{(i)}\,dv\leqq c+\int_{U_i\setminus B_1(0)} G_0^{(i)}\,\Delta e^{\delta\rho}\,dv.$$

Since the Ricci curvature of M is bounded we know that $\Delta e^{\delta\rho}\leqq c\,\delta\,e^{\delta\rho}$ distributionally (see [CY]). Therefore we have

$$\int_{U_i\setminus B_1(0)} e^{\delta\rho}\,G_0^{(i)}\,dv\leqq c+\delta c\int_{U_i\setminus B_1(0)} e^{\delta\rho}\,G_0^{(i)}\,dv.$$

Choosing δ small then enables us to prove

$$\int_{M\setminus B_1(0)} e^{\delta\rho}\,G_0\,dv<\infty.$$

Since the Ricci curvature of M is bounded, the Harnack inequality [CY] implies that $G_0\geqq e^{-c\rho}$ for some constant c. Therefore we have

$$\int_{M\setminus B_1(0)} G_0^{1-\delta c^{-1}}\,dv<\infty$$

and $p(M)<1$ as required. This proves the theorem under hypotheses (i) or (ii).

Under hypothesis (iii) we use the weak statement of Proposition 2.4 that $d(M) \leqq \frac{n}{2}$ and observe that $\frac{n}{2} < \frac{(n-2)^2}{n}$ when $n \geqq 7$.

Remark 3.4. In the next section we will show that the hypotheses (i), (ii), and (iii) can be removed by a different argument so that Proposition 3.3 is true for all n with no additional hypotheses.

4. Fundamental group and positive energy theorems

We first consider the simplest examples of locally conformally flat manifolds, those arising from Kleinian groups. Suppose Γ is a discrete subgroup of the conformal group $C_n = 0(n+1, 1)$ of S^n. The limit set Λ of Γ is the minimal closed invariant subset of S^n which may be characterized as the set of accumulation points of the orbit of any point. The complement Ω of Λ on S^n is then an open set on which Γ acts properly discontinuously and is called the domain of discontinuity of Γ. If Γ has no fixed points in Ω, then $M = \Omega/\Gamma$ is a locally conformally flat manifold (possibly disconnected).The following result realizes a large class of locally conformally flat manifolds as Ω/Γ for a Kleinian group Γ.

Theorem 4.1. *Suppose (M, g) is a complete locally conformally flat manifold with $R(g)$ bounded below. If $n=3, 4$ assume $|R(g)|$ is bounded. If $d(M) < \frac{(n-2)^2}{n}$ then we have $\hat{M} = \tilde{M}$, the developing map $\Phi\colon \tilde{M} \to S^n$ is injective, the holonomy representation $\rho\colon \pi_1(M) \to C_n$ is $1-1$, and $\Gamma = \rho(\pi_1(M))$ is a discrete subgroup of C_n. If M is compact, then $\Omega = \Phi(\hat{M})$ is the domain of discontinuity of Γ and $M = \Omega/\Gamma$ arises as above.*

Proof. We may apply Theorem 3.1 to $\tilde{M}$ to deduce that $\Phi\colon \tilde{M} \to S^n$ is injective. This implies that the holonomy covering $\hat{M}$ is the universal cover $\tilde{M}$, and hence ρ is injective. Let $\Omega = \Phi(\tilde{M})$, and observe that $\Gamma = \rho(\pi_1(M))$ acts properly discontinuously on Ω since the action of Γ is conjugate under Φ to the action of $\pi_1(M)$ on $\tilde{M}$. Therefore Γ is a discrete subgroup of C_n. Since $\partial\Omega (= S^n \backslash \Omega)$ is a closed invariant subset under Γ we have $\Lambda(\Gamma) \subset \partial\Omega$. If M is compact and $x \in \partial\Omega$, then any neighborhood of x must contain infinitely many translates of a compact fundamental domain $F \subset \Omega$ and hence $\Omega = \Lambda$. This completes the proof of Theorem 4.1.

We now discuss the case $R(g) \geqq 0$ and relate it to the positive energy theorem. Suppose (M, g) is complete, and suppose for every point $p_0 \in M$ the minimal Green's function G_0 for L with pole at p_0 exists. It is natural to think of $\bar{g} = G_0^{\frac{4}{n-2}} g$ as a metric on $M - \{p_0\}$. If we choose a conformally flat chart near p_0, say $\Phi_0\colon \mathcal{O} \to S^n$, then $\bar{g}$ has a nice form if we require $\Phi_0(p_0) = \infty$ and think of S^n as $\mathbb{R}^n \cup \{\infty\}$. Let g_0 denote a Euclidean metric on $\mathbb{R}^n$ with Euclidean coordinates $x^1, \ldots, x^n$. Expressing

$$\bar{g} = h^{\frac{4}{n-2}} g_0 = h(x)^{\frac{4}{n-2}} \sum_{i=1}^{n} (dx^i)^2$$

we see that $h(x)$ is an asymptotically constant harmonic function defined outside a compact set of $\mathbb{R}^n$. It follows that $h(x)$ has the following expansion for $|x|$ large,

$$h(x)=a+b|x|^{2-n}+O(|x|^{1-n})$$

where $a, b\in\mathbb{R}$ with $a>0$. If we had chosen another chart $\hat{\Phi}_0$ with $\hat{\Phi}_0(p_0)=\infty$ and denote Euclidean coordinates on the image of $\hat{\Phi}_0$ by $y^1, \ldots, y^n$ then we see that $x=\Phi_0\circ\hat{\Phi}_0^{-1}(y)$ is a conformal transformation fixing ∞, and hence $x=\mu By+x_0$ where $\mu>0$, $B\in O(n)$ is an $n\times n$ orthogonal matrices, and $x_0\in\mathbb{R}^n$. Therefore we have

$$\bar{g}=h(x)^{\frac{4}{n-2}}\sum_{i=1}^{n}(dx^i)^2=h(\mu By+x_0)^{\frac{4}{n-2}}\mu^2\sum_{i=1}^{n}(dy^i)^2$$

so that $\hat{h}(y)=\mu^{\frac{n-2}{2}}h(\mu By+x_0)$ which has the expansion

$$\hat{h}(y)=\hat{a}+\hat{b}|y|^{2-n}+O(|y|^{1-n})$$

where $\hat{a}=\mu^{\frac{n-2}{2}}a$ and $\hat{b}=\mu^{\frac{2-n}{2}}b$. Thus it follows that $\hat{a}\hat{b}=ab$ so we define $E(p_0)=ab$ and observe that $E(p_0)$ depends only on p_0 and the metric g. If we replace g by $\hat{g}=u^{\frac{4}{n-2}}g$, then we have $\hat{G}_0=u(p_0)^{\frac{n+2}{n-2}}u^{-1}G_0$ and hence $\hat{E}(p_0)=u(p_0)^{\frac{n+2}{n-2}}E(p_0)$. Note that if we choose Φ_0 so that $a=1$, we have $\hat{g}=h^{\frac{4}{n-2}}g_0$ where

$$h(x)=1+E|x|^{2-n}+O(|x|^{1-n})$$

for $|x|$ large. The quantity $E(p_0)$ generalizes the ADM energy of asymptotically flat spacetimes in general relativity (see [SY1]). The positive energy theorems imply a statement to the effect that E will be positive if the function $R(g)$ is positive. The next result relates the positivity of energy to the embedding theorems of the previous section.

Proposition 4.2. *Suppose M is a locally conformally flat manifold and $\Phi: M\to S^n$ is a conformal map. The function E is nonnegative at each point of M if and only if Φ is one to one and the image $\Omega=\Phi(M)$ has boundary of zero Newtonian capacity.*

Proof. We have seen that Φ is one-one onto a domain with boundary of zero capacity if and only if for every $p_0\in M$ the minimal Green's function G_0 for L with pole at p_0 is a multiple of the pullback under Φ of the Green's function on S^n with pole at $q_0=\Phi(p_0)$. Following the notation used in the proof of Theorem 3.1, we let $\bar{G}_0$ denote the multiple of the pullback Green's function with pole of the same strength as G_0 at p_0. We study the energy terms of G_0 and $\bar{G}_0$. Let $\Phi_0: O\to S^n$ be as above with $\Phi_0(p_0)=\infty$, and let h, $\bar{h}$ be the corresponding harmonic functions, say Φ_0 is chosen so that $h\to 1$ at infinity. Since the metric $\bar{h}^{\frac{4}{n-2}}\sum_{i=1}^{n}(dx^i)^2$ is a Euclidean metric we clearly have

$$\bar{h}(x)=1,$$

i.e., $\bar{E}=0$. Note that $\bar{h}\to 1$ because $\bar{G}_0$ has pole of the same strength as G_0 at p_0 and $h\to 1$. We also have $h\leqq\bar{h}$ near infinity. Thus by the maximum principle we either have $h\equiv\bar{h}$ or $h<\bar{h}$ near infinity. We write the expansion for h

$$h(x)=1+E|x|^{2-n}+O(|x|^{1-n}).$$

If $h<\bar{h}$, then $\bar{h}-h$ is a positive harmonic function on $\mathbb{R}^n\backslash B_\sigma(0)$ for some $\sigma>0$. If we choose a number δ with $0<\delta<\min\{\bar{h}(x)-h(x)\colon x\in\partial B_\sigma(0)\}$, then we must have $\bar{h}(x)-h(x)>\delta|x|^{2-n}$ for $|x|\geqq\sigma$ since $\delta|x|^{2-n}$ is a minimal solution. Therefore we have $E\leqq-\delta$. Thus we have shown that either $E(p_0)<0$ or $G_0\equiv\bar{G}_0$ on M. This completes the proof of Proposition 4.2.

In order to study non-simply connected manifolds we prove the following result on the behavior of the energy for coverings..

Proposition 4.3. *Suppose M_1 is a nontrivial covering of M, M is locally conformally flat, and for each $y_0\in M$ suppose the minimal Green's function with pole at y_0 exists. If $\pi\colon M_1\to M$ denotes a covering map, then the energy functions E, E_1 for M, M_1 satisfy the inequality $E_1(x)<E(\pi(x))$ for every $x\in M_1$.*

Proof. Let $x_0\in M_1$, and let $\bar{G}_0$ denote the pullback under π of the minimal Green's function of M with pole at $y_0=\pi(x_0)$. The existence of $\bar{G}_0$ then implies the existence of the minimal Green's function G_0 of M_1 with pole at x_0. Since π is nontrivial we have $G_0<\bar{G}_0$ on $M_1\backslash\{x_0\}$, and hence the argument used in the proof of Proposition 4.2 implies the inequality $E_1(x_0)<E(y_0)$ as required.

We now state a result which derives a positive energy theorem from the results of Section 3.

Proposition 4.4. *Let (M, g) be a complete locally conformally flat manifold with $R(g)\geqq 0$. Assume that for every point $x_0\in M$ the minimal Green's function with pole at x_0 exists. (This is automatic if $R(g)\geqq R_0>0$ or if M is compact and $R(g)$ is not identically zero.) Assume one of hypotheses* (i), (ii) *or* (iii) *of Proposition* 3.3 *holds for the holonomy covering $\hat{M}$. Then the energy function E on M is nonnegative at every point. If M is not simply connected then E is strictly positive at each point, while if $\pi_1(M)=0$ then E is identically zero.*

Proof. This is an immediate consequence of Theorem 3.3, Proposition 4.2, and Proposition 4.3.

Conversely, one can use the existing positive energy theorems to strengthen the embedding theorem for manifolds with $R(g)\geqq 0$. For this application it is necessary to extend the positive energy theorems (see [SY1], [SY2]) to the case of complete manifolds; that is, assuming M has an asymptotically flat end and other ends which are merely complete. This extension will be carried out in a future work.

Proposition 4.4′. *Proposition* 4.4 *holds without the assumption of* (i), (ii), *or* (iii).

Proof. It is sufficient to prove Proposition 3.3 without hypotheses (i), (ii), or (iii). Thus we may assume $\Phi\colon \mathrm{M}\to S^n$ is a conformal map. By Proposition 4.2 it is enough to show $E(x_0)\geqq 0$ for each $x_0\in M$. Let G_0 be the minimal Green's function with pole at x_0, and observe that for any $\delta>0$ the metric $\bar{g}=(G_0+\delta)^{\frac{4}{n-2}}g$ is complete on $M\backslash\{x_0\}$ and has an asymptotically flat end at x_0.

Moreover, since $R(g) \geqq 0$ we have

$$L(G_0 + \delta) = -c(n)\, \delta R(g) \leqq 0,$$

and hence $\bar{g}$ has nonnegative scalar curvature. The positive energy theorem then shows that the energy of $\bar{g}$ is positive, hence $E(x_0) + \delta > 0$ for any $\delta > 0$. This implies $E(x_0) \geqq 0$ and completes the proof of Theorem 4.4.

Theorem 4.5. *Suppose $M(g)$ is complete, locally conformally flat with $R(g) \geqq 0$. Then we have $\hat{M} = \tilde{M}$, the developing map $\Phi\colon \tilde{M} \to C_n$ is $1-1$, and $\Gamma = \rho(\pi_1 M)$ is a discrete subgroup of C_n. If M is compact, then $\Omega = \Phi(\tilde{M})$ is the domain of discontinuity of Γ and $M = \Omega/\Gamma$.*

The previous theorem follows from Theorem 4.4 in the same way as Theorem 4.1 follows from Theorem 3.1. Recall that the invariant $d(M)$ dominates the Hausdorff dimension of $\partial(\Phi(\tilde{M}))$. We can use this bound to derive some topological information concerning locally conformally flat manifolds.

Theorem 4.6. (i) *Suppose $k \geqq 2$ is an integer, and (M, g) is a locally conformally flat manifold with $d(M) < \min\left\{n-k-1, \dfrac{(n-2)^2}{n}\right\}$. Then $\pi_i(M) = 0$ for $i = 2, \ldots, k$.*

(ii) *Suppose (M, g) is complete locally conformally flat with $R(g) \geqq R_0 > 0$. Then $\pi_i(M) = 0$ for $i = 2, \ldots, \left[\left[\dfrac{n}{2}\right]\right]$ where $[[\cdot]]$ denotes the integer part.*

Proof. By Theorem 3.1 and Proposition 2.5, the universal covering of M is a domain Ω with $\partial\Omega$ having Hausdorff dimension at most $n-k-1$. Given a map $f\colon S^i \to \Omega$ we can extend f to $\bar{f}\colon B^{i+1} \to S^n$, and since $i \leqq k$ we can perturb $\bar{f}$ so that $\bar{f}$ misses $\partial\Omega$ and hence $\pi_i(\Omega) = 0$, $i = 2, \ldots, k$. This establishes (i), and Part (ii) follows because of Theorem 4.4 and the bound $d(M) < \dfrac{n-2}{2} \leqq n - \left[\left[\dfrac{n}{2}\right]\right] - 1$. This proves Theorem 4.6.

We now focus on the case of compact M, and we show that in this case $d(M)$ is precisely the Hausdorff dimension of $\partial\Omega$.

Theorem 4.7. *Suppose Γ is a Kleinian group with $M = \Omega/\Gamma$ compact, and let Λ be the limit set of Γ and let $d(\Lambda)$ denote the Hausdorff dimension of Λ. M has a compatible metric g with $R(g) \geqq 0$ if and only if $d(\Lambda) \leqq \dfrac{n-2}{2}$. In any case we have $d(M) = d(\Lambda)$ unless Γ is an elementary group, i.e., unless Λ is either one point or two points.*

Proof. We first show $d(M) = d(\Lambda)$. Suppose $p > 0$ with $\int_{M \smallsetminus \mathcal{O}} G_0^p\, dv < \infty$. Let $F \subset M$ be a compact fundamental domain for the action of Γ with $0 \in F$ and hence

$$\sum_{\substack{\gamma \in \Gamma \\ \gamma \neq 1}} \int_F G_0^p(\gamma x)\, dv(x) < \infty.$$

Since γ acts on $\tilde{M}$ by isometries we have $G_0(\gamma x) = G_{\gamma^{-1}0}(x)$. Since F is compact, the Harnack inequality implies that $\max\limits_F G_{\gamma^{-1}0} \leqq c \min\limits_F G_{\gamma^{-1}0}$, and hence for each x we have convergence of the series

$$\sum_{\substack{\gamma\in\Gamma \\ \gamma\neq 1}} G^p_{\gamma^{-1}0}(x) < \infty.$$

Since $G(x, y) = u^{-1}(x)\,u^{-1}(y)\,\bar{G}(x, y)$ where $\bar{G}$ is the Green's function for L_0 on S^n, we see that convergence of the previous series is equivalent to

$$\sum_{\gamma\in P} u^{-p}(\gamma 0) < \infty.$$

The transformation law for u is $|\gamma'|^{\frac{n-2}{2}}\, u\circ\gamma = u$ for $\gamma\in\Gamma$, and hence we get convergence of the Poincaré series $\sum_{\gamma\in\Gamma} |\gamma'|^{\frac{n-2}{2}p}$. Thus we see that $G_0\in L^p(\tilde{M})$ if and only if the Poincaré series with exponent $\frac{n-2}{2}\,p$ converges. A theorem of Patterson [P] and Sullivan [Su] shows that the minimal exponent for which the Poincaré series converges is $d(\Lambda)$ provided Γ is not elementary. Therefore $d(M) = d(\Lambda)$ proving the last assertion of the theorem.

First suppose $R(g) \geqq 0$. It then follows from Proposition 2.4 that $d(M) \leqq \frac{n-2}{2}$ unless $\lambda_0(\tilde{M}) = 0$. A theorem of Brooks [B] shows that this occurs only if $\pi_1(M)$ is an amenable group. Since any non-elementary Kleinian group contains a non-abelian free subgroup, it then follows that $\Gamma = \pi_1(M)$ is elementary; that is, the limit set consists of either one or two points. If Λ is one point, then M is a flat manifold since each element of Λ is parabolic with the same fixed point. If Λ consists of two points, then M must be covered by $S^1 \times S^{n-1}$ and it is easy to see that G_0 decays exponentially, on $\tilde{M}$ so that $d(M) = 0$. Since $d(\Lambda) \leqq d(M)$ we have shown that unless M is a flat manifold, we have $d(\Lambda) \leqq \frac{n-2}{2}$.

Conversely, suppose $d(\Lambda) \leqq \frac{n-2}{2}$. Since M is compact we either have a conformal metric g with $R(g) \geqq 0$ on M or we have a g with $R(g) < 0$ on M. Suppose we have g with $R(g) < 0$. Lifting g to Ω we get a complete metric g with $R(g) \leqq -R_0$, $R_0 > 0$. Let G_0 be the minimal Green's function with pole at $0\in\Omega$. Since $G_0 = u^{-1}(0)\,u^{-1}\,\bar{G}_0$ where $g = u^{\frac{4}{n-2}} g_0$, we have that $G_0 \to 0$ at $\partial\Omega$. Let $\mathcal{O} = \{G_0 > 1\}$ and observe that $\mathcal{O}$ is a compact neighborhood of 0. Now the equation of G_0 implies $\Delta G_0 \leq -R_0\, G_0$. Let $\delta > 0$ and compute

$$\Delta G_0^{1+\delta} \leqq -(1+\delta)\, R_0\, G_0^{1+\delta} + \delta(1+\delta)\, G_0^{-1+\delta}\,|\nabla G_0|^2.$$

The Harnack inequality [CY] implies $|\nabla G_0|^2 \leqq c\, G_0^2$ on $M\setminus\mathcal{O}$, and hence if δ is small we have

$$\Delta G_0^{1+\delta} \leqq -c\, G_0^{1+\delta}$$

on $M\setminus\mathcal{O}$ for a positive constant c. If it were true that $\int_{M\setminus\mathcal{O}} G_0^{1+\delta}\, dv < \infty$, then we could choose a sequence of radii $\sigma_i \to \infty$ so that

$$\lim_{i\to\infty} \int_{\partial B_{\sigma_i}} G_0^{1+\delta}\, d\Sigma = 0$$

where $d\Sigma$ is the surface measure (w.r.t. g) of ∂B_{σ_i}. Integrating the above inequality on $B_{\sigma_i}\setminus\mathcal{O}$ we get

$$\int_{\partial B_{\sigma_i}} \frac{\partial}{\partial v} G_0^{1+\delta} \leqq -c \int_{B_{\sigma_i}\setminus\mathcal{O}} G_0^{1+\delta}$$

after applying Stokes theorem and the fact that the normal derivative of G_0 is nonnegative on $\partial\mathcal{O}$. Again by Harnack we have $\left|\frac{\partial}{\partial v} G_0^{1+\delta}\right| \leqq c G_0^{1+\delta}$ and hence letting i tend to infinity we get a contradiction. Therefore we must have $\int_{M\setminus\mathcal{O}} G_0^{1+\delta}\, dv = \infty$ and hence $d(M) > \frac{n-2}{2}$. Since $d(M) = d(\Lambda)$ we have a contradiction. This completes the proof of Theorem 4.7.

5. PDE aspects of the theory

In this section we describe the relationship of the geometric problem of construction of complete conformal metrics of constant scalar curvature on domains $\Omega \subset S^n$ to the study of weak (distributional) solutions of the equation

$$L_0 u + u^{\frac{n+2}{n-2}} = 0. \tag{5.1}$$

where L_0 is the conformal Laplacian for the standard metric g_0 on S^n. There is a substantial interest in the study of weak solutions of (5.1) with u positive and $u \in L_{\text{loc}}^{\frac{n+2}{n-2}}$. In particular, progress has been made on the case of isolated singularities by Serrin, Caffarelli, Nirenberg, Spruck, Polking, etc. We will also consider the differential inequality

$$L_0 u + u^{\frac{n+2}{n-2}} \leqq 0. \tag{5.2}$$

The following theorem relates the study of weak solutions of (5.1), (5.2) to the complete metrics on domains $\Omega \subset S^n$.

Theorem 5.1. *Suppose Ω is a domain in S^n, and $g = u^{\frac{4}{n-2}} g_0$ is a complete metric on Ω with scalar curvature $R(g)$ satisfying $R(g) \geqq 1$ on Ω. Then it follows that $\partial\Omega = S^n \setminus \Omega$ has Hausdorff dimension at most $\frac{n-2}{2}$, $u \in L^{\frac{n+2}{n-2}}(S^n)$, and a multiple of u is a weak solution of* (5.2) *on S^n. If g has bounded curvature and $R(g) \equiv 1$, then a multiple of u is a weak solution of* (5.1) *on S^n.*

Proof. We assume $R(g) \geqq 1$ and establish (5.2) distributionally. First observe that a multiple of u, which we call u, satisfies (5.2) pointwise on Ω. The condition that $\partial\Omega$ has Hausdorff dimension at most $\frac{n-2}{2}$ follows from Proposition 2.4 and Proposition 2.5 (with the Newtonian capacity condition on $\partial\Omega$ implied by Theorem 4.4 and Proposition 4.2). In fact, from the discussion preceding Proposition 2.5 we have $u \in L^{\frac{n+2}{n-2}}(S^n)$. To establish (5.2) weakly we must show that

$$\int_{S^n} (uL_0\zeta + u^{\frac{n+2}{n-2}}\zeta)\, dv_0 \leqq 0 \tag{5.3}$$

for every $\zeta\in C^\infty(S^n)$ with $\zeta\geqq 0$. Let $\chi_a(t)$ be a smooth concave function with $\chi_a(t)=t$ for $t\leqq a$ and $\chi_a(t)\equiv\frac{3a}{2}$ for $t\geqq 2a$, and observe that

$$L_0(\chi_a(u)) \leqq -\frac{n(n-2)}{4}\,\chi_a(u)+\chi_a'(u)\,\Delta_0 u$$

since χ_a is concave and $L_0=\Delta_0-\frac{n(n-2)}{4}$. Thus on Ω we have

$$L_0(\chi_a(u)) \leqq \frac{n(n-2)}{4}\,(\chi_a'(u)\,u-\chi_a(u))-\chi_a'(u)\,u^{\frac{n+2}{n-2}}. \tag{5.4}$$

It follows from Lemma 3.2 that G_0 tends to zero at $\partial\Omega$ uniformly. Since $G_0 = u^{-1}\,\bar{G}_0$ where $\bar{G}_0$ is the Green's function for L_0, it follows that u tends uniformly to ∞ near $\partial\Omega$. Therefore $\chi_a(u)$ extends to a smooth function on S^n and (5.4) holds on all of S^n. Therefore we have for $\zeta\in C^\infty(S^n)$, $\zeta\geqq 0$

$$\int_{S^n} (\chi_a(u)(L_0\zeta)+\chi_a'(u)\,u^{\frac{n+2}{n-2}}\zeta)\,dv_0 \leqq \frac{n(n-2)}{4}\int_{S^n}\zeta(\chi_a'(u)\,u-\chi_a(u))\,d\sigma_0.$$

Since $u\in L^{\frac{n+2}{n-2}}(S^n)$ we may let $a\to\infty$ and use the dominated convergence theorem to get (5.3) as required.

Now we assume that $R(g)\equiv 1$ on Ω and hence a multiple of u satisfies (5.1) pointwise on Ω. We modify the previous argument to establish (5.1) weakly on S^n. As above we get $\chi_a(u)$ is a smooth global solution of

$$L_0(\chi_a(u))=\chi_a''(u)\,|\partial u|^2-\chi_a'(u)\,u^{\frac{n+2}{n-2}}+\frac{n(n-2)}{4}\,(\chi_a'(u)\,u-\chi_a(u)).$$

We can argue as above, letting $a\to\infty$ to get (5.1) satisfied weakly on S^n provided we can show

$$\lim_{a\to\infty}\int_{S^n}\zeta\chi_a''(u)\,|\partial u|^2\,dv_0=0 \tag{5.5}$$

for any $\zeta\in C^\infty(S^n)$. We may choose $\chi_a(t)$ so that $|\chi_a''(t)|\leqq c a^{-1}$. Therefore we have

$$\begin{aligned}\left|\int_{S^n}\zeta\chi_a''(u)\,|\partial u|^2\,dv_0\right| &\leqq c a^{-1}\int_{\{x:\,a\leqq u(x)\leqq 2a\}}|\partial u|^2\,dv_0\\ &\leqq 2c\int_{\{x:\,a\leqq u(x)\}}u^{-1}\,|\partial u|^2\,dv_0.\end{aligned}$$

Thus in order to establish (5.5) it is enough to show

$$\int_{\Omega} u^{-1}\,|\partial u|^2\,dv_0<\infty.$$

Since we are assuming g has bounded curvature, the Harnack inequality [CY] implies $|\nabla G_0| \leqq c G_0$ on $\Omega \backslash b_1(0)$. Since $G_0 = u^{-1} \bar{G}_0$, this implies $|\nabla u^{-1}| \leqq c u^{-1} + c|\nabla \bar{G}_0^{-1}| u^{-1}$. Writing this in terms of g_0 and using the fact that $u \geqq c > 0$ we see $|\partial u| \leqq c u^{\frac{n}{n-2}}$. Therefore we have

$$\int_\Omega u^{-1} |\partial u|^2 \, dv_0 \leqq c \int_\Omega u^{\frac{n+2}{n-2}},$$

and hence we have established (5.5). This completes the proof of Theorem 5.1.

As an application of Theorem 5.1 we can find a rich class of examples of global weak solutions of (5.1). We consider those domains Ω which are universal coverings of compact locally conformally flat manifolds with positive scalar curvature. The solution of the Yamabe problem [Sc1] on the quotient then lifts to a complete metric g on Ω with $R(g) \equiv 1$ and with bounded curvature. The first such example arises from the Kleinian group generated by a single hyperbolic element $\gamma \in C_n$. We then have $\Omega = S^n \backslash \{p, q\}$ where p, q are the fixed points of γ and we get a weak solution of (5.1) singular at the points p, q. Of course such solutions can also be constructed by putting (without loss of generality) $q = -p$ and looking for radial O.D.E. solutions of (5.1). The next simplest example arising from Kleinian groups is the group generated by two sufficiently strong hyperbolic elements γ_1, γ_2 with distinct fixed points. It is well-known that such a group is free on two generators and $\Omega = S^n \backslash \Lambda$ where Λ is a Cantor set whose Hausdorff dimension can be made arbitrarily small. This then generates weak solutions of (5.1) which are singular on a set of fractional Hausdorff dimension. This behavior is typical for the singular sets arising from domains which are invariant under a Kleinian group. Other singular sets which occur are round spheres (e.g. from products of spheres with compact hyperbolic manifolds), and certain quasiconformal deformations of round spheres. Again these tend to have fractional Hausdorff dimension.

This discussion leads naturally to the question of whether weak solutions of (5.1) exist which are singular on smooth submanifolds. This question for constant negative scalar curvature was solved by Loëwner-Nirenberg [LN]. The first author has shown [Sc2] that singular solutions exist which blow up at a prescribed set of at least two points. We close with a conjecture about weak solutions of (5.1).

Conjecture 5.2. Any $u \in L^{\frac{n+2}{n-2}}(S^n)$ with $u > 0$ which is a weak solution of (5.1) is regular on a domain Ω with $\partial\Omega$ of Hausdorff dimension at most $\frac{n-2}{2}$ and the metric $g = u^{\frac{4}{n-2}} g_0$ is a complete metric (of constant positive scalar curvature on Ω.)

References

[AM] Adams, D.R., Meyers, N.G.: Bessel potentials, inclusion relations among classes of exceptional sets. Indiana Univ. Math. J. **22**, 873–905 (1973)

[Au] Aubin, T.: Equations différentielles nonlinéaires et Problem de Yamabe concernant la courbure scalaire. J. Math. Pure Appl. **55**, 269–296 (1976)

[B] Brooks, R.: The fundamental group and the spectrum of the Laplacian. Comm. Math. Helv. **56**, 581–598 (1981)

[CG] Cheeger, J., Gromoll, D.: The splitting theorem for manifolds of nonnegative Ricci curvature. J. Differ. Geom. **6**, 119–128 (1971)

[CY] Cheng, S.Y., Yau, S.-T.: Differential equation on Riemannian manifolds and their geometric application. Comm. Pure Appl. Math. **28**, 333–354 (1975)

[F] Fried, D.: Closed similarity manifolds. Comm. Math. Helv. **55**, 576–582 (1980)

[G] Goldman, W.M.: Conformally flat manifolds with nilpotent holonomy and the uniformization problem for 3-manifolds.Trans. Am. Math. Soc. **278**, 573–583 (1983)

[Ka] Kamishima, Y.: Conformally flat manifolds whose development maps are not surjective I (Preprint)

[Ku1] Kuiper, N.: On conformally flat manifolds in the large. Ann. Math. **50**, 916–924 (1949)

[Ku2] Kuiper, N.: On compact conformally euclidean spaces of dimension >2. Ann. Math. **52**, 478–490 (1950)

[LN] Loewner, C., Nirenberg, L.: Partial differential equations invariant under conformal or projective transformations, Contribution to Analysis, 245–272. Academic Press, New York 1974

[M] Morrey, C.B.: Multiple Integrals in the Calculus of Variations, Springer, New York, 1966

[P] Patterson, S.J.: The limit set of a Fuchsian group. Acta Math. **136**, 241–273 (1976)

[Sc1] Schoen, R.: Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differ. Geom. **20**, 479–495 (1984)

[Sc2] Schoen, R.: The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation (to appear in Comm. Pure Appl. Math., 1988)

[Sp] Spivak, M.: A Comprehensive Introduction to Differential Geometry, Volume Four, Publish or Perish, Inc., Berkeley, 1979

[SY1] Schoen, R., Yau, S.-T.: On the proof of the positive mass conjecture in general relativity. Comm. Math. Phys. **65**, 45–76 (1979)

[SY2] Schoen, R., Yau, S.-T.: Proof of the positive mass theorem II. Comm. Math. Phys. **79**, 231–260 (1981)

[Su] Sullivan, D.: The density at infinity of a discrete group of hyperbolic motions. IHES Publications Mathematiques No. 50, 171–202 (1979)

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