Proceedings of Symposia in Pure Mathematics
Volume 36, 1980

ESTIMATES OF EIGENVALUES OF A COMPACT RIEMANNIAN MANIFOLD

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Let M be a compact manifold with (possibly empty) boundary. Then in this paper, we study the eigenvalues of M with respect to various boundary conditions.

Since the Poincaré inequality plays a very important role in analysis and since a lower bound of the first eigenvalue gives an upper bound of the constant in the Poincaré inequality, it is very desirable to find a good lower estimate of the first eigenvalue. For domains in euclidean space, there are classical works of Faber-Krahn, Polyá-Szegö, Payne, Weinberger, etc. The works of these authors are not only beautiful and important, but also give a deep impact to estimate eigenvalues on curved spaces. For many geometric problems, we often need to estimate the Poincaré inequality for domains on a curved space. Thus in this paper, we concentrate our attention to this case. The first major result in this direction was due to Lichnerowicz [10] and Obata [11]. In their beautiful work, they assumed the Ricci curvature of the compact manifold (without boundary) is greater than a positive constant and they estimated the first eigenvalue from below in terms of this constant. It is remarkable that this constant is sharp. This estimate of Lichnerowicz-Obata was generalized later by Reilly [14] to manifolds with boundary where he treated

1980 Mathematics Subject Classification 53-XX.
*This research was supported in part by the NSF Grant MCS78-04872

the Dirichlet boundary valued problem. Besides the assumption of Lichnerowicz-Obata, he also assumed the boundary of the manifold has non-negative mean curvature.

After this work of Lichnerowicz-Obata, J. Cheeger [3] studied the first eigenvalue by estimating it from below by a constant which is involved in certain type of isoperimetric inequality. In [1], Aubin gave a lower estimate of the first eigenvalue in terms of a lower bound of the volume, an upper bound of the diameter, a lower bound of the sectional curvature, an upper bound of the Ricci curvature and a lower bound of the injectivity radius. For the purpose of applications, it is important to relax the dependency of the lower bound on the geometric quantities. For this purpose, the second author [15] showed that one can estimate the first eigenvalue from below by lower bound of the volume, an upper bound of the diamter and a lower bound of the Ricci curvature. Basing on an upper estimate of Cheng [5], the second author conjectured that one should be able to drop the dependency of the volume in the above estimate. Combining with Cheng's result, this would give the best possible estimate of the first eigenvalue for a general compact manifold.

It turns out that the first author [9] was able to demonstrate the above conjecture in all the major cases. His method depended on a gradient estimate of the first eigenfunction. In this paper, we indicate how to make a slight modification of this method to give a complete demonstration of the above conjecture. For the special case of compact manifolds with non-negative Ricci curvature, we show that $\lambda_1 \geq \frac{\pi^2}{4d^2}$ where d is the diameter of the manifold. On the other hand, Cheng's estimate gives $\lambda_1 \leq \frac{n\pi^2}{d^2}$ where n is the dimension of the manifold. (Cheeger [2] had a weaker estimate prior to Cheng's result.) We also extend these estimates to compact manifolds with boundary. For Dirichlet boundary valued problem, the estimate also depends on the lower bound of the mean curvature of the boundary. For Neumann boundary valued problem, we have to assume the second fundamental form of the boundary is positive semidefinite. In the Dirichlet boundary valued problem, the diameter d can be replaced by the radius of the largest geodesic ball that can be inscribed into the manifold.

In the second part of the paper, we demonstrate how to use the variational principle to obtain both upper and lower estimates of higher eigenvalues λ_m. Let V be the volume of the compact manifold. Then the famous estimate of H. Weyl shows that when m tends to infinity, $\lambda_m\left(\frac{V}{m}\right)^{\frac{2}{n}}$ approaches to a constant $C(n)$ depending only on the dimension of the manifold. We demonstrate how to find an upper and lower estimate of $\lambda_m\left(\frac{V}{m}\right)^{\frac{2}{n}}$. For example, when the Ricci curvature of M is non-negative, we show that $\frac{\lambda_m}{C(n)}\left(\frac{V}{m+1}\right)^{\frac{2}{n}}$ is bounded from above by an absolute constant. When the Ricci curvature is bounded from below by $(n-1)K$, then we prove that $\lambda_m \leq C_1\left(\frac{m+1}{V}\right)^{\frac{2}{n}} + C_2$ where C_1 depends on K, d and n; C_2 depends on K and n. Our method depends on Cheng's result [5].

The lower estimate of $\lambda_m\left(\frac{V}{m}\right)^{\frac{2}{n}}$ is more complicated. When the sectional curvature is non-negative, $\lambda_m\left(\frac{V}{m}\right)^{\frac{2}{n}}$ has a lower bound depending only on n when m is greater than a constant depending on the upper bound of sectional curvature and the radius of convexity. When the sectional curvature is allowed to be negative, then the lower bound of $\lambda_m\left(\frac{V}{m}\right)^{\frac{2}{n}}$ should be replaced by a positive constant depending on n, d and the lower bound of the sectional curvature.

Finally we should mention that Professor Gromov has pointed out that in a classical book of P. Levy [8] there was an indication of an estimate of the first eigenvalue. This estimate depends on a very non-trivial analysis of the regularity of certain hypersurface with constant mean curvature. Up to now, it is not known whether this analysis can be carried out when dimension $n \geqslant 7$. Furthermore, the constant involved in this analysis is not sharp whereas our estimate is sharp for manifolds whose Ricci curvature is non-negative and whose boundary has non-negative mean curvature. We should also mention that for convex domain in euclidean space, Payne-Weinberger [13] has already estimated λ_1 for Neumann problem. During the conference, Chavel-Feldman generalized the result of Payne-Weinberger to convex surfaces with non-negative curvature. Our theorem is more general while our method is very different. After we indicated our result to Professor Protter, he pointed out a paper of Payne and Stakgold [12] on the estimate of the first eigenvalue of a domain in euclidean space. It turns out that part of our estimates is similar to theirs.

1. GRADIENT ESTIMATE. In this section we consider the solution of the equation

$$\Delta u = F(u) \tag{1.1}$$

defined on a compact manifold M of dimension n. In the case if M is a manifold with boundary ∂M, we imposed one of the following boundary conditions:

$$u \equiv 0 \quad \text{on} \quad \partial M \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} \equiv 0 \quad \text{on} \quad \partial M \tag{1.3}$$

The first one is known to be the Dirichlet boundary condition and the latter is the Neumann boundary condition, where $\frac{\partial}{\partial \nu}$ denotes the outward normal to ∂M.

Suppose the Ricci curvature of M is bounded below by $(n - 1)K$. In the next three theorems we will derive an upper estimate for the gradient of the solution of (1.1)(with different boundary conditions) in terms of K, u, $F(u)$ and its derivative.

THEOREM 1. Let M be a compact manifold. If u is a solution of (1.1) and $\mu > 1$ is any constant, then

$$|\nabla u|^2 \leqslant \max\left\{4(n-1)\left[-(n-1)K - F_u + \left\|\frac{F(u)}{\mu \sup u - u}\right\|_\infty\right],\ \sqrt{8}\left\|\frac{F(u)}{\mu \sup u - u}\right\|_\infty\right\}(\mu \sup u - u)^2$$

where F_u denotes the derivative of F with respect to u.

PROOF. For $\mu > 1$, we define the function

$$f = \frac{|\nabla u|^2}{(\mu \sup u - u)^2} \tag{1.4}$$

By the compactness of M, there is a point $x_o \in M$ such that f achieves its supremum. Hence at x_o

$$\nabla f = 0$$

(1.5) and

$$\Delta f \leqslant 0$$

This gives

$$\sum_j u_j u_{ji} + \frac{|\nabla u|^2 u_i}{\mu \sup u - u} = 0 \tag{1.6}$$

and

$$0 \geqslant \frac{\sum_{i,j} u_{ij}^2 + \sum_{i,j} u_j u_{jii}}{(\mu \sup u - u)^2} + \frac{4 \sum_{i,j} u_i u_{ij} u_j}{(\mu \sup u - u)^3} + \frac{|\nabla u|^2 \sum_i u_{ii}}{(\mu \sup u - u)^3} + \frac{3|\nabla u|^4}{(\mu \sup u - u)^4} \tag{1.7}$$

By choosing suitable orthonormal frame at x_o, we may assume $u_\alpha = 0$ for $\alpha > 1$. Substituting (1.6) into (1.7) and using the Ricci formula, we have

$$\sum_{\alpha>1} u_{\alpha\alpha}^2 + \sum_i u_i (\Delta u)_i + \sum_{i,j} R_{ij} u_i u_j + \frac{|\nabla u|^2 \Delta u}{\mu \sup u - u} \leqslant 0 \tag{1.8}$$

Clearly

$$\begin{aligned} \sum_{\alpha>1} u_{\alpha\alpha}^2 &\geqslant \frac{1}{n-1}\left(\sum_{\alpha>1} u_{\alpha\alpha}\right)^2 \\ &= \frac{1}{n-1}(\Delta u - u_{11})^2 \\ &\geqslant \frac{u_{11}^2}{2(n-1)} - \frac{(\Delta u)^2}{n-1} \end{aligned} \tag{1.9}$$

By (1.6)

$$\sum_{\alpha>1} u_{\alpha\alpha}^2 \geqslant \frac{1}{2(n-1)} \frac{|\nabla u|^4}{(\mu \sup u - u)^2} - \frac{(\Delta u)^2}{n-1} \tag{1.10}$$

Therefore

$$\begin{aligned} &\frac{1}{2(n-1)} \frac{|\nabla u|^4}{(\mu \sup u - u)^2} - \frac{F^2(u)}{n-1} \\ &+ F_u |\nabla u|^2 + (n-1)K|\nabla u|^2 \\ &+ \frac{|\nabla u|^2 F(u)}{\mu \sup u - u} \leqslant 0 \end{aligned} \tag{1.11}$$

Hence

$$\frac{1}{2(n-1)} f^2(x_o) + \left(F_u + (n-1)K + \frac{F(u)}{\mu \sup u - u}\right) f(x_o) \leq \frac{1}{n-1}\left(\frac{F(u)}{\mu \sup u-u}\right)^2 \tag{1.12}$$

and

$$f(x_o) \leq \max\left\{4(n-1)\left[-(n-1)K - F_u + \left\|\frac{F(u)}{\mu \sup u-u}\right\|_\infty\right], \sqrt{8}\left\|\frac{F(u)}{\mu \sup u-u}\right\|_\infty\right\}$$

This proves theorem 1.

THEOREM 2. Let M be a compact manifold with boundary. Suppose H denotes the mean curvature of ∂M, and if u is a non-negative solution of (1.1) and (1.2). Then for $\mu > 1$,

either

$$|\nabla u| \leq \max\left\{-2(n-1)H, 2\left\|\frac{F(u)}{\mu \sup u-u}\right\|_{\partial M}^{1/2}\right\} \times (\mu \sup u - u)$$

or

$$|\nabla u|^2 \leq \max\left\{4(n-1)\left[-(n-1)K - F_u + \left\|\frac{F(u)}{\mu \sup u-u}\right\|_\infty\right], \sqrt{8}\left\|\frac{F(u)}{\mu \sup u-u}\right\|_\infty\right\}(\mu \sup u-u)^2$$

where $\| \ \|_{\partial M}$ denotes the supremum norm on ∂M.

PROOF. Again, we consider the function defined by (1.4). If $x_o \in \partial M$ is a point where f attains its supremum. By the strong maximal principle

$$\frac{\partial f}{\partial \nu}(x_o) > 0 \tag{1.14}$$

Hence

$$\sum_i u_i u_{i\nu} + \frac{|\nabla u|^2 u_\nu}{\mu \sup u-u} > 0 \tag{1.15}$$

By a suitable choice of orthonormal frame $e_1, \ldots, e_n$ such that $e_n = \frac{\partial}{\partial \nu}$ and e_α are tangential to ∂M for $\alpha < n$, since $u|_{\partial M} \equiv 0$

$$u_\alpha = 0 \quad \text{for } \alpha < n \tag{1.16}$$

Direct computation then gives

$$\begin{aligned}(n-1)Hu_\nu &= \sum_{\alpha<n} u_{\alpha\alpha} \\ &= \Delta u - u_{\nu\nu}\end{aligned} \tag{1.17}$$

Therefore (1.15) becomes

$$u_\nu \Delta u - (n-1)Hu_\nu^2 + \frac{u^3}{\mu \sup u - u} > 0 \tag{1.18}$$

Since $u_\nu < 0$,

$$f^{3/2}(x_o) < -(n-1)Hf(x_o) - \frac{\Delta u}{\mu \sup u - u} f^{1/2}(x_o) \tag{1.19}$$

Hence

$$f^{1/2}(x_o) < \max\left\{-2(n-1)H,\ 2\left\|\frac{F(u)}{\mu \sup u - u}\right\|_{\partial M}^{1/2}\right\} \tag{1.20}$$

This gives the first part of the theorem. On the other hand, if x_o is in the interior of M, theorem 1 gives the second half of the estimate.

THEOREM 3. Let M be a compact manifold with boundary. Suppose the principal curvatures on ∂M are non-negative (i.e. ∂M is convex). If u is a solution of (1.1) and (1.3), then for $\mu > 1$.

$$|\nabla u|^2 \leqslant \max\left\{4(n-1)\left[-(n-1)K - F_u + \left\|\frac{F(u)}{\mu \sup u - u}\right\|_\infty\right],\right.$$
$$\left.\sqrt{8}\left\|\frac{F(u)}{\mu \sup u - u}\right\|_\infty\right\}(\mu \sup u - u)^2$$

PROOF. In view of theorem 1, it suffices to show that the function

$$f = \frac{|\nabla u|^2}{(\mu \sup u - u)^2}$$

attains its supremum in the interior of M.

Assuming the contrary if $x_o \in \partial M$ is the maximum point of f. Then

$$\sum_i u_i u_{i\nu} + \frac{|\nabla u|^2 u_\nu}{\mu \text{ sip } u - u} > 0 \tag{1.21}$$

If $h_{\alpha\beta}$ are the second fundamental form elements of ∂M, then by direct computation one shows that

$$u_{\alpha\nu} = -h_{\alpha\beta}u_\beta \qquad 1 \leqslant \alpha,\beta < n \tag{1.22}$$

where we used the fact that $u_\nu \equiv 0$ on ∂M. Together with (1.21), we have

$$-\sum_{\alpha,\beta} h_{\alpha\beta}u_\alpha u_\beta > 0$$

which is a contradiction to the convexity of ∂M. The theorem follows.

2. ANOTHER GRADIENT ESTIMATE. In this section, another form of gradient estimate for the solution of (1.1) is obtained. In the case when the Ricci curvature of M is non-negative, the estimates in the following theorems are much sharper than those of the previous section. However for general manifolds theorem 1-3 are more effective as will be demonstrated in the next section.

THEOREM 4. Let M be a compact manifold. Suppose u is a solution of (1.1). If α is any constant and $\beta^2 \geq \sup(\alpha + u)^2$, then

$$|\nabla u|^2 \leq \sup_{x \in M} \left[\frac{-(F_u + (n-1)K)(\beta^2 - (\alpha + u)^2) - (\alpha + u)F(u)}{\beta^2} \right] \times (\beta^2 - (\alpha + u)^2)$$

PROOF. Consider the function

$$g = \frac{|\nabla u|^2}{\beta^2 - (\alpha + u)^2} \tag{2.1}$$

where α = constant and $\beta^2 \geq \sup(\alpha + u)^2$. If $x_o \in M$ is a point where g achieves its supremum, then at x_o, g satisfies

$$0 = \frac{g_i}{2} = \frac{u_j u_{ji}}{\beta^2 - (\alpha + u)^2} + \frac{|\nabla u|^2 (\alpha + u) u_i}{(\beta^2 - (\alpha + u)^2)^2} \tag{2.2}$$

and

$$\begin{aligned} 0 \geq \frac{\Delta g}{2} = {} & \frac{u_{ji}^2 + u_j u_{jii}}{\beta^2 - (\alpha + u)^2} + \frac{4 u_j u_{ji} u_i (\alpha + u)}{(\beta^2 - (\alpha + u)^2)^2} \\ & + \frac{|\nabla u|^4}{(\beta^2 - (\alpha + u)^2)^2} + \frac{|\nabla u|^2 (\alpha + u) F(u)}{(\beta^2 - (\alpha + u)^2)^2} \\ & + \frac{4 |\nabla u|^4 (\alpha + u)^2}{(\beta^2 - (\alpha + u)^2)^3} \end{aligned} \tag{2.3}$$

Substituting (2.2) and using the Ricci identity, we have

$$\begin{aligned} 0 \geq {} & \frac{u_{ji}^2}{\beta^2 - (\alpha + u)^2} + \frac{F_u |\nabla u|^2}{\beta^2 - (\alpha + u)^2} + \frac{(n-1)K|\nabla u|^2}{\beta^2 - (\alpha + u)^2} \\ & + \frac{|\nabla u|^4}{(\beta^2 - (\alpha + u)^2)^2} + \frac{|\nabla u|^2 (\alpha + u) F(u)}{(\beta^2 - (\alpha + u)^2)^2} \end{aligned} \tag{2.4}$$

By picking suitable orthonormal frame and using (2.2) again

$$u_{ji}^2 \geq u_{11}^2 \geq \frac{|\nabla u|^4 (\alpha + u)^2}{(\beta^2 - (\alpha + u)^2)^2} \tag{2.5}$$

Hence

$$\begin{aligned} 0 \geq {} & g^2 \frac{(\alpha + u)^2}{\beta^2 - (\alpha + u)^2} + (F_u + (n-1)K) g \\ & + g^2 + \frac{(\alpha + u) F(u)}{\beta^2 - (\alpha + u)^2} g \end{aligned} \tag{2.6}$$

Therefore

$$-(F_u + (n-1)K)(\beta^2 - (\alpha+u)^2) - (\alpha+u)F(u) \geqslant \beta^2 g \tag{2.7}$$

and the theorem follows.

THEOREM 5. Let M be a compact manifold with boundary. Suppose u is a positive solution of (1.1) and (1.2). If $H \leqslant 0$ is the lower bound of the mean curvature of ∂M with respect to the outward normal $\frac{\partial}{\partial\nu}$, then either

$$|\nabla u|^2 \leqslant \sup_{x\in M}\left[\frac{-(F_u + (n-1)K)(\beta^2 - (\alpha+u)^2) - (\alpha+u)F(u)}{\beta^2}\right] \times (\beta^2 - (\alpha+u)^2)$$

or

$$|\nabla u|^2 \leqslant (n-1)^2 H^2\left(\frac{\beta^2 - \alpha^2}{\alpha^2}\right)(\beta^2 - (\alpha+u)^2)$$

for any $\alpha \geqslant 0$ and $\beta^2 \geqslant \sup(\alpha+u)^2$.

PROOF. Again we consider the function g defined by (2.1). If the supremum point x_o of g is in the interior of M, the estimates of theorem 4 can be applied. If $x_o \in \partial M$, then

$$0 < \frac{1}{2} g_\nu (x_o) \tag{2.8}$$

Hence at x_o

$$0 < u_{\nu\nu}u_\nu + \frac{|\nabla u|^2 \alpha u_\nu}{\beta^2 - \alpha^2} \tag{2.9}$$

By (1.17)

$$0 < -(n-1)Hu_\nu^2 + \frac{|\nabla u|^2 \alpha u_\nu}{\beta^2 - \alpha^2} \tag{2.10}$$

Since u is a positive solution, this implies

$$-(n-1)H > \frac{\alpha|\nabla u|}{\beta^2 - \alpha^2} \tag{2.11}$$

Hence

$$(n-1)^2 H^2 > \frac{\alpha^2 |\nabla u|^2}{(\beta^2 - \alpha^2)^2} \tag{2.12}$$

Therefore

$$(n-1)^2 H^2 \left(\frac{\beta^2 - \alpha^2}{\alpha^2}\right) > \frac{|\nabla u|^2 (x_o)}{\beta^2 - \alpha^2} \geqslant g \tag{2.13}$$

which proves the theorem.

THEOREM 6. Let M be a compact manifold with boundary. Suppose u is a solution of (1.1) and (1.3). If ∂M is convex with respect to the outward normal, then

$$|\nabla u|^2 \leqslant \sup_{x\in M} \left[\frac{-(F_u + (n-1)K)(\beta^2 - (\alpha+u)^2) - (\alpha+u)F(u)}{\beta^2}\right] \times (\beta^2 - (\alpha+u)^2)$$

for any constant α and $\beta^2 \geqslant \sup(\alpha+u)^2$.

PROOF. Similar to theorem 3, we only need to show that the supremum of the function g occurs in the interior of M.

If not, say $x_0 \in \partial M$ is the supremum point of g. Then

$$0 < \frac{u_j u_{j\nu}}{\beta^2 - (\alpha+u)^2} + \frac{|\nabla u|^2(\alpha+u)u_\nu}{(\beta^2-(\alpha+u)^2)^2} \tag{2.14}$$

Since $u_\nu \equiv 0$ on ∂M, this gives

$$\begin{aligned} 0 &< \sum_j u_j u_{j\nu} \\ &= -h_{\alpha\beta}u_\alpha u_\beta \qquad 1 \leqslant \alpha,\beta < n \end{aligned} \tag{2.15}$$

which is a contradiction to the convexity assumption of ∂M. Hence the estimate of theorem 4 can be applied.

3. APPLICATIONS AND EIGENVALUE ESTIMATES. We will give applications of the previous theorems to obtain lower bounds for the first non-zero eigenvalues of the Laplacian.

For manifold without boundary, we consider the eigenvalues of the equation

$$\Delta u = -\lambda u \tag{3.1}$$

In the case if M has boundary ∂M, again we impose either the Dirichlet or Neumann condition:

$$u \equiv 0 \quad \text{on} \quad \partial M \tag{3.2}$$

$$\frac{\partial u}{\partial \nu} \equiv 0 \quad \text{on} \quad \partial M . \tag{3.3}$$

THEOREM 7. Let M be a compact manifold. Suppose d denotes the diameter of M. If λ_1 is the first non-zero eigenvalue of (3.1), then

$$\lambda_1 \geqslant \frac{\exp - [1 + (1 - 4(n-1)^2 d^2 K)^{1/2}]}{2(n-1)d^2} .$$

PROOF. If u is the first eigenfunction, by theorem 1

$$|\nabla u|^2 \leqslant \max\left\{4(n-1)\left[-(n-1)K + \lambda_1 + \frac{\lambda_1}{\mu-1}\right], \sqrt{8}\,\frac{\lambda_1}{\mu-1}\right\}(\mu \sup u - u)^2 \tag{3.4}$$

for any $\mu > 1$. Therefore

$$\frac{|\nabla u|}{\mu \sup u - u} \leq \left[4(n-1)\left(\frac{\mu\lambda_1}{\mu - 1} - (n-1)K\right)\right]^{1/2} \tag{3.5}$$

However since u satisfied

$$\int_M u = 0 \tag{3.6}$$

This implies that the nodal set N of u divides M into two parts. If $x \in M$ is the point where u achieves its supremum and γ be the shortest geodesic joining x and N, then γ has length at most d. Integrating (3.5) along γ, we have

$$\begin{aligned} \log \frac{\mu}{\mu - 1} &\leq \int_\gamma \frac{|\nabla u|}{\mu \sup u - u} \\ &\leq \left[4(n-1)\left(\frac{\mu\lambda_1}{\mu - 1} - (n-1)K\right)\right]^{1/2} d \end{aligned} \tag{3.7}$$

Hence

$$\frac{\mu - 1}{\mu}\left[\frac{1}{4(n-1)d^2}\left(\log \frac{\mu}{\mu - 1}\right)^2 + (n-1)K\right] \leq \lambda_1 \tag{3.8}$$

Clearly the left hand side can be made to be positive by choosing μ closed enough to 1. The theorem is then proved by maximizing (3.8) with

$$\frac{\mu}{\mu - 1} = \exp[1 + (1 - 4(n-1)^2 d^2 K)^{1/2}] .$$

THEOREM 8. Let M be a compact manifold with boundary. Suppose i denotes the inscribed radius of M, i.e. the radius of the biggest geodesic ball than can be fitted into M. If μ_1 is the first eigenvalue of (3.1) and (3.2), then

$$\mu_1 \geq \frac{1}{\gamma}\left[\frac{1}{4(n-1)i^2}(\log \gamma)^2 + (n-1)K\right]$$

where

$$\gamma = \max\left\{\exp[1 + (1 - 4(n-1)^2 i^2 K)^{1/2}],\ \exp[-2(n-1) \times Hi]\right\}$$

PROOF. It is well known that the first eigenfunction u of the Dirichlet boundary problem does not change sign. We may then assume u is non-negative and apply theorem 2. We get

$$|\nabla u| < -2(n-1)H(\mu \sup u - u) \tag{3.9}$$

or

$$|\nabla u|^2 \leq 4(n-1)\left(\frac{\mu\lambda}{\mu - 1} - (n-1)K\right)(\mu \sup u - u)^2 \tag{3.10}$$

If (3.9) holds, we have

$$\frac{|\nabla u|}{\mu \sup u - u} < -2(n-1)H \tag{3.11}$$

By integrating along the shortest geodesic γ joining $\sup u$ to ∂M, we have

$$\log \frac{\mu}{\mu - 1} < -2(n-1)Hi \tag{3.12}$$

Hence if we take μ such that

$$\frac{\mu}{\mu - 1} \geqslant e^{-2(n-1)Hi} \tag{3.13}$$

we get a contradiction. Hence (3.10) holds, and the estimate of theorem 7 follows for μ satisfying (3.13).

By applying theorem 3 instead, one can easily obtain lower estimate for λ_1 of the Neumann problem.

THEOREM 9. Let M be a compact manifold with convex boundary. (i.e. the principal curvatures of ∂M are non-negative). If λ_1 is the first non-zero eigenvalue of (3.1) and (3.3), then

$$\lambda_1 \geqslant \frac{\exp - [1 + (1 - 4(n-1)^2 d^2 K)^{1/2}]}{2(n-1)d^2}$$

PROOF. Similar to theorem 7.

For compact manifolds with non-negative Ricci curvature we apply the gradient estimates of section 2 to obtain the following sharp estimates for the first non-zero eigenvalues of (3.1), (3.2) and (3.3).

THEOREM 10. Let M be a compact manifold. Suppose λ_1 is the first non-zero eigenvalue of (3.1). Then

$$\lambda_1 + \max\{-(n-1)K, 0\} \geqslant \frac{\pi^2}{4d^2}.$$

In particular, if the Ricci curvature of M is non-negative

$$\lambda_1 \geqslant \frac{\pi^2}{4d^2}$$

PROOF. Applying theorem 4, we have

$$\begin{aligned} |\nabla u|^2 &\leqslant \sup \left[\frac{(\lambda_1 - (n-1)K)(\beta^2 - (\alpha + u)^2)}{\beta^2} \right. \\ &\quad \left. + \frac{\lambda_1 u(\alpha + u)}{\beta^2} \right] (\beta^2 - (\alpha + u)^2) \\ &= \sup \left[\frac{\lambda_1(\beta^2 - \alpha(\alpha + u)) - (n-1)K(\beta^2 - (\alpha + u)^2)}{\beta^2} \right] \\ &\quad \times (\beta^2 - (\alpha + u)^2) \end{aligned} \tag{3.14}$$

By setting $\alpha = 0$ and $\beta^2 = (\sup u)^2$ (we may assume that $\sup u > |\inf u|$) gives

$$|\nabla u|^2 \leq [\lambda_1 + \max\{-(n-1)K, 0\}] \times ((\sup u)^2 - u^2) \tag{3.15}$$

Hence

$$\frac{|\nabla u|}{\sqrt{(\sup u)^2 - u^2}} \leq [\lambda_1 + \max\{-(n-1)K, 0\}]^{1/2} \tag{3.16}$$

Integrating along the shortest geodesic joining $\sup u$ and the nodal set of u yields

$$\frac{\pi}{2} \leq [\lambda_1 + \max\{-(n-1)K, 0\}]^{1/2} d \tag{3.17}$$

This proves the theorem.

THEOREM 11. Let M be a manifold with boundary. Suppose μ_1 is the first eigenvalue of (3.1) and (3.2), then

$$\mu_1 + \max\{-(n-1)K, 0\} \geq \frac{1}{i^2}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{-(n-1)H}{(\mu_1 + \max\{-(n-1)K, 0\} + (n-1)^2H^2)^{1/2}}\right)\right]^2$$

where i = inscribe radius and $H \leq 0$ is the lower bound of the mean curvature of ∂M.

In particular, if M has non-negative Ricci curvature then

$$\mu_1 \geq \frac{1}{i^2}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{-(n-1)H}{\mu_1 + (n-1)^2H^2}\right)\right]^2$$

PROOF. By theorem 5, if we set $\alpha = \gamma \sup u$ and $\beta = (\gamma + 1)\sup u$, then either

$$g = \frac{|\nabla u|^2}{\beta^2 - (\alpha + u)^2} \leq (\mu_1 + \max\{-(n-1)K, 0\}) \times \frac{2\gamma + 1}{(\gamma + 1)^2} \tag{3.18}$$

or

$$g \leq (n-1)^2H^2\left(\frac{2\gamma + 1}{\gamma^2}\right) \tag{3.19}$$

where u is the non-negative first eigenfunction. We claim that for appropriate choice of γ

$$(3.20)\qquad g \leq \mu_1 + \max\{-(n-1)K, 0\}\ .$$

If not, since $\dfrac{2\gamma+1}{(\gamma+1)^2} \leq 1$, g must satisfied (3.19). Therefore, we have

$$(3.21)\qquad (n-1)^2H^2\left(\frac{2\gamma+1}{\gamma^2}\right) \geq g > \mu_1 + \max\{-(n-1)K, 0\}$$

However if $\gamma = \left[\dfrac{(n-1)^2H^2}{\mu_1+\max\{-(n-1)K,0\}}\left(\dfrac{(n-1)^2H^2}{\mu_1+\max\{-(n-1)K,0\}}+1\right)\right]^{1/2}$

$+\dfrac{(n-1)^2H^2}{\mu_1+\max\{-(n-1)K,0\}}$, it can be easily be checked that

$$(n-1)^2H^2(2\gamma+1) = \gamma^2(\mu_1 + \max\{-(n-1)K,0\})$$

which is a contradiction to (3.21). Hence for the above choice of γ, we have

$$(3.22)\qquad \frac{|\nabla u|}{\sqrt{(\gamma+1)^2(\sup u)^2 - (\gamma \sup u - u)^2}} \leq (\mu_1 + \max\{-(n-1)K,0\})^{1/2}$$

Integrating from $\sup u$ to ∂M yields the theorem.

REMARK. If M is a compact domain in $\mathbb{R}^n$ with non-negative mean curvature, theorem 11 generalizes the result obtained by Hersch [7].

THEOREM 12. Let M be a manifold with convex boundary. Suppose λ_1 is the first non-zero eigenvalue of (3.1) and (3.3), then

$$\lambda_1 + \max\{-(n-1)K,0\} \geq \frac{\pi^2}{4d^2}$$

PROOF. Follows from theorem 6.

The gradient estimate of theorem 4 is in some sense best possible for compact manifold with non-negative Ricci curvature, as the next theorem will show.

THEOREM 13. Let M be a compact manifold with non-negative Ricci curvature. If u is an eigenfunction on M then

$$\|u\|_\infty^2 \geqslant \frac{2\|u\|_2^2}{V}$$

where V = volume of M. Equality holds iff the universal covering $\tilde{M}$ isometrically splits into $\mathbb{R}^k \times N$ and u is an eigenfunction on $\mathbb{R}^k$ alone.

PROOF. By setting $\alpha = 0$, $\beta = \|u\|_\infty$ and applying theorem 4, we have

$$|\nabla u|^2 \leqslant \lambda \left[\|u\|_\infty^2 - u^2\right] \tag{3.23}$$

Integrating both sides yields

$$2\lambda \|u\|_2^2 \leqslant \|u\|_\infty^2 V\lambda \tag{3.24}$$

Hence equality holds iff

$$|\nabla u|^2 = \lambda(\|u\|_\infty^2 - u^2) \tag{3.25}$$

Differentiating equation (3.25) gives

$$u_j u_{ji} = -\lambda u u_i \tag{3.26}$$

If one chooses suitable orthonormal frame, this yields

$$\begin{aligned} u_{11} &= -\lambda u \\ &= \Delta u \end{aligned} \tag{3.27}$$

and

$$u_1 = \nabla u$$

Covariant differentiating (3.26) in the ith direction and sum

$$\begin{aligned}
(3.28) \qquad 0 &= u_{ji}^2 + u_j u_{jii} + \lambda u_i^2 + \lambda u u_{ii} \\
&\geqslant u_{11}^2 + (n-1)K|\nabla u|^2 - \lambda^2 u^2 \\
&= (n-1)K|\nabla u|^2 \qquad \text{(by 3.27)} \\
&\geqslant 0
\end{aligned}$$

This implies that equality holds on each step of (3.28). Hence the Ricci curvature vanishes along ∇u and $u_{ij} = 0$ unless $i = j = 1$.

We define a 1-form ω to be the dual of $(\operatorname{sgn} u)\frac{\nabla u}{|\nabla u|}$. We claim that ω can be defined smoothly on M. In fact, since $u_{11} \equiv \Delta u$, this means $e_1 = (\operatorname{sgn} u)\frac{\nabla u}{|\nabla u|}$ is well-defined up to a sign even at the critical points of u. Hence ω is a smooth 1-form.

Consider the equation

$$\begin{aligned}
(3.29) \qquad (\omega,\Delta\omega) &= -\frac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 \quad (\text{since } R_{11} \equiv 0) \\
&= |\nabla\omega|^2 \qquad (|\omega|^2 \equiv 1)
\end{aligned}$$

Clearly $u_{ij} = 0$ when $i \neq 1$ or $j \neq 1$ implies

$$(3.30) \qquad |\nabla\omega|^2 = 0$$

Hence ω is a parallel harmonic 1-form on M. By the splitting theorem of Cheeger and Gromoll [4], $\tilde{M}$ splits into $\mathbb{R}^k \times N$. The rest of the theorem follows trivially.

4. UPPER BOUNDS FOR λ_m

Given any compact manifold of dimension n, the well-known Weyl formula gives the asymptotic relationship between the number of eigenvalues less than any given number T and the number itself. If we take the increasing ordering of eigenvalues $\{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ then the Weyl formula can be written as

$$\lambda_m \sim \left(\frac{m}{V}\right)^{\frac{2}{n}} \times 4\pi\Gamma\left(\frac{n}{2}+1\right)^{\frac{2}{n}}$$

The purpose of this section is to give upper bounds of λ_m with the right order in $\frac{m}{V}$ which corresponds to the Weyl formula. For general Riemannian manifold, we obtain

$$\lambda_m \leq C_1 + C_2\left(\frac{m+1}{V}\right)^{\frac{2}{n}}$$

where C_1 and C_2 are constants depending only on n, d = diameter of M and K = lower bound of Ricci curvature of M. When M is a manifold of non-negative Ricci curvature, we have

$$\lambda_m \leq \overline{C}\left(\frac{m+1}{V}\right)^{\frac{2}{n}} \times 4\pi\Gamma\left(\frac{n}{2}+1\right)^{\frac{2}{n}}$$

where $\overline{C}$ = constant depending on n along. Moreover $\overline{C}(n)$ is bounded above for all $n \in \mathbb{Z}^+$.

COVERING LEMMA. Let M be a compact manifold. For each $0 < r \leq d$, there exists a collection of geodesic balls $\{B_i(r)\}_{i=1}^{k(r)}$ such that they satisfied the following properties

(i) $\bigcup_i B_i(r)$ covers M

(ii) $B_j(\frac{r}{2}) \cap B_i(\frac{r}{2})$ is a set of measure zero for $i \neq j$

(iii) for $y \in M$, there are at most

$$P = 3\left(2\,\frac{\sinh\sqrt{-K}\,r}{\sqrt{-K}\,r}\right)^n$$

balls which contained y, where $\kappa \leq 0$ is the lower bound of the sectional curvature of M.

REMARK. If M is non-negatively curved, $P \leq 3 \cdot 2^n$.

PROOF. For any $0 < r \leq d$, there exists a maximal set of points $\{x_i\}_{i=1}^{k(r)}$ such that they are mutually of distance at least r apart. Property (i) follows trivially from the maximality of $\{x_i\}_{i=1}^{k(r)}$ if we take $\{B(x_i),r\}_{i=1}^{k(r)}$ to be our collection. The fact that all the x_i's are of at least distance r apart gives property (ii). For $y \in M$, the number of balls which contained y is the same as the number of x_i's which are in $B(y,r)$. By the comparison theorem and lifting $B(y,r)$ to the tangent space at y, it reduces to counting the maximal number of points with euclidean distance

$$\frac{r\sqrt{-K}}{\sinh r\sqrt{-K}}$$

apart that can be fitted into the euclidean ball of radius 1. A simple counting argument gives (iii).

REMARK. By (i) and (ii) one obtains the following estimates of $k(r)$

$$(4.1) \qquad \begin{aligned} V(M) &\leq \sum_{i=1}^{k(r)} V(B_i(r)) \\ &\leq k(r) \max\{V(B_i(r))\} \\ &\leq k(r)\alpha(n-1)\int_0^r [(-K)^{-1/2}\sinh\sqrt{-K}\, t]^{n-1}dt \end{aligned}$$

and

$$(4.2) \qquad \begin{aligned} V(M) &\geq \sum_{i=1}^{k(r)} V(B_i(\tfrac{r}{2})) \\ &\geq k(r) \min\{B_i(\tfrac{r}{2})\} \\ &\geq k(r)\alpha(n-1)(\sqrt{\tau})^{1-n}\int_0^{r/2} (\sin(\sqrt{\tau}\, t))^{n-1}dt \end{aligned}$$

where $\alpha(n-1)$ denotes the volume of the unit $n-1$ sphere in $\mathbb{R}^n$, and τ = upper bound of the sectional curvature of M.

Clearly $k(r)$ is a non-increasing function of r. In general, $k(r)$ is not a surjection onto $\mathbb{Z}^+$ = positive integers. However the following lemmas show that it is not far from being surjective.

LEMMA 14. For $\varepsilon > 0$, and $m \in \mathbb{Z}^+$, there exists a collection of geodesic balls $\{B_i(r_i)\}_{i=1}^m$ which satisfied:

(i) $\bigcup B_i(r_i) - M$ is a set of measure less than ε.

(ii) $B_j(\frac{r_j}{2}) \cap B_i(r_i/2)$ is a set of measure zero for $i \neq j$

(iii) there exists an r, such that $r \leqslant r_i \leqslant r + \varepsilon$ for all $1 \leqslant i \leqslant m$.

PROOF. In view of lemma 14, we know that for δ small enough, there exists $r > 0$ such that $k(r) = m_1 < m < m_2 = k(r - \delta)$. Consider $\{B_i(r)\}_{i=1}^m$ the collection which satisfied property (i-iii) of the covering lemma. We may assume that $B_1(r-\delta) \bigcup_{i=2}^m B_i(r)$ does not cover M. Let

$x \in M - \{B_1(r-\delta) \bigcup_{i=2}^m B_i(r) \subset B_1(r) - B_1(r-\delta)$.

Clearly the collection $\{B(x, r-\delta), B_1(r-\delta)\} \cup \{B_i(r)\}_{i=2}^m$ satisfies the required properties. Inductively by shrinking the balls $B_i(r)$ by δ and adding balls of radius $r - \delta$, we can obtain m balls which satisfied the conclusion of the lemma.

LEMMA 15. For $\varepsilon > 0$, and $m \in \mathbb{Z}^+$, there exists a collection of geodesic balls $\{B_i(r_i)\}_{i=1}^m$ of which satisfied

(i) $\bigcup B_i(r_i)$ covers M

(ii) $B_j(\frac{r_j}{2}) \cap B_i(r_i/2)$ is a set of measure less than ε for $i \neq j$

(iii) there exists an r, such that $r \leqslant r_i \leqslant r + \varepsilon$ for all $1 \leqslant i \leqslant m$

(iv) for $y \in M$, the number of $B_i(r_i)$'s which contained y is at most

$$P = 3\left(2\,\frac{\sinh((r+\varepsilon)\sqrt{-K})}{(r+\varepsilon)\sqrt{-K}}\right)^n$$

PROOF. We proceed the same way as in lemma 14. Instead we take $\{B(x,r)\} \cup \{B_i(r)\}_{i=1}^m$ to be our new collection. It is easy to check that properties (i-iv) are satisfied. Again by adding more balls inductively, the lemma follows.

We are now ready to give some estimates on the eigenvalues.

THEOREM 16. Let M be a compact manifold. If λ_m is the mth non-zero eigenvalue of the Laplacian on M, then

$$\lambda_m \leq \begin{cases} \dfrac{(2\beta+1)^2}{4}(-K) + 4(1+2^\beta)^2\pi^2\left(\dfrac{\sinh\sqrt{-K}\,d}{\sqrt{-K}\,d}\right)^{\frac{2n-2}{n}} \\ \qquad \times\left[(m+1)\,\dfrac{\alpha(n-1)}{n}\,\dfrac{1}{V(M)}\right]^{\frac{2}{n}} \\ \text{when } n = 2(\beta+1),\ \beta = 0,1,2,\ldots \\ \dfrac{(2\beta+2)^2}{4}(-K) + 4(1+\pi^2)(1+2^{2\beta})^2\left(\dfrac{\sinh\sqrt{-K}\,d}{\sqrt{-K}\,d}\right)^{\frac{2n-2}{n}} \\ \qquad \times\left[(m+1)\,\dfrac{\alpha(n-1)}{n}\,\dfrac{1}{V(M)}\right]^{\frac{2}{n}} \\ \text{when } n = 2\beta+3,\ \beta = 0,1,2,\ldots \end{cases}$$

where $K \leq 0$ is the lower bound of the Ricci curvature of M.

PROOF. Let $\{B_i(r_i)\}_{i=1}^{m+1}$ be the collection of balls which satisfied lemma 14. Consider φ_i, the first eigenfunctions of the Dirichlet boundary problem on the $B_i(r_i)$. If u_α, $0 \leq \alpha \leq m-1$, are the first m eigenfunctions on M (including the constant function), then by the variational principle

$$\lambda_m = \inf_{f \perp u_\alpha} \frac{\int |\nabla f|^2}{\int f^2} \tag{4.3}$$

where inf is taken over all f orthogonal to u_α, $0 \leq \alpha \leq m-1$.

By the essential disjointness of $B_i(\frac{r_i}{2})$, the set of functions φ_i

on M are linearly independent. Hence there exists $0 \neq (a_i)_{i=1}^{m+1} \in \mathbb{R}^{m+1}$, such that

$$\varphi = \Sigma\, a_i \varphi_i \tag{4.4}$$

is orthogonal to u_α, for all α.
Therefore

$$\lambda_m \leqslant \frac{\int |\nabla\varphi|^2}{\int \varphi^2} \tag{4.5}$$

But

$$\begin{aligned} \int |\nabla\varphi|^2 &= \Sigma \int a_i^2 |\nabla\varphi_i|^2 \\ &= \Sigma\, \mu_1(B_i)\, a_i \int \varphi_i^2 \\ &\leqslant \max \mu_1(B_i) \int \varphi^2 \end{aligned}$$

where $\mu_1(B_i)$ denotes the first eigenvalue of the Dirichlet problem on $B_i(\frac{r_i}{2})$. This gives

$$\lambda_m \leqslant \max \mu_1(B_i) \tag{4.6}$$

However by the monotonuity of μ_1 and a theorem of Cheng [5]

$$\lambda_m \leqslant \mu_1(B(K, \tfrac{r}{2})) \tag{4.7}$$

$$\leqslant \begin{cases} \dfrac{(2\beta + 1)^2}{4}(-K) + \dfrac{4(1 + 2^{\beta})^2 \pi^2}{r^2} \\ \text{when } n = 2(\beta + 1), \quad \beta = 0,1,2,\ldots \\ \dfrac{(2\beta + 2)^2}{4}(-K) + \dfrac{4(1 + \pi^2)(1 + 2^{2\beta})^2}{r^2} \\ \text{when } n = 2\beta + 3, \quad \beta = 0,1,2,\ldots \end{cases}$$

On the other hand, by (4.1)

$$V(M) \leq (m+1)\alpha(n-1)(-K)^{\frac{1-n}{2}} \int_0^{r+\varepsilon} (\sinh\sqrt{-K}\, t)^{n-1} dt \tag{4.8}$$

$$\leq (m+1)\alpha(n-1)(-K)^{\frac{-n}{2}} \int_0^{\sqrt{K}\,(r+\varepsilon)} \left(\frac{\sinh\sqrt{-K}\,(r+\varepsilon)}{\sqrt{-K}\,(r+\varepsilon)}\, t\right)^{n-1} dt$$

$$= (m+1)\alpha(n-1)(-K)^{\frac{-2n+1}{2}} \left(\frac{\sinh\sqrt{-K}\,(r+\varepsilon)}{r+\varepsilon}\right)^{n-1} \times \frac{[\sqrt{-K}\,(r+\varepsilon)]^n}{n}$$

$$= (m+1)\frac{\alpha(n-1)}{n}(r+\varepsilon)(\sinh\sqrt{-K}\,(r+\varepsilon))^{n-1}(-K)^{\frac{-n+1}{2}}$$

$$\leq (m+1)\frac{\alpha(n-1)}{n}(r+\varepsilon)^n \left(\frac{\sinh\sqrt{-K}\, d}{\sqrt{-K}\, d}\right)^{n-1}$$

Hence

$$\frac{1}{r^2} \leq \frac{(r+\varepsilon)^2}{r^2}\left[(m+1)\frac{\alpha(n-1)}{n}\frac{1}{V(M)}\right]^{\frac{2}{n}} \left(\frac{\sinh\sqrt{-K}\, d}{\sqrt{-K}\, d}\right)^{\frac{2n-2}{n}} \tag{4.9}$$

Together with (4.6)

$$\lambda_m \leq \begin{cases} \leq \dfrac{(2\beta+1)^2}{4}(-K) + 4(1+2^{\beta})^2\pi^2 \dfrac{(r+\varepsilon)^2}{r^2}\left(\dfrac{\sinh\sqrt{-K}\, d}{\sqrt{-K}\, d}\right)^{\frac{2n-2}{n}} \\ \qquad \times \left[(m+1)\dfrac{\alpha(n+1)}{n}\dfrac{1}{V(M)}\right]^{\frac{2}{n}} \\ \text{when } n = 2(\beta+1),\ \beta = 0,1,2,\ldots \\ \text{and} \\ \dfrac{(2\beta+2)^2}{4}(-K) + \dfrac{4(1+\pi^2)(1+2^{2\beta})^2(r+\varepsilon)^2}{r^2} \\ \qquad \times \left(\dfrac{\sinh\sqrt{-K}\, d}{\sqrt{-K}\, d}\right)^{\frac{2n-2}{n}} \left[(m+1)\dfrac{\alpha(n-1)}{n}\dfrac{1}{V(M)}\right]^{\frac{2}{n}}. \\ \text{when } n = 2\beta+3,\ \beta = 0,1,2,\ldots \end{cases}$$

Letting $\varepsilon \to 0$, we obtain the desired result.

When $K = 0$, we can obtain a sharper estimate on λ_m.

THEOREM 17. Let M be a compact manifold of non-negative Ricci curvature. Then

$$\lambda_m \leq (n + 4)n^{1-\frac{2}{n}} \left[\frac{m + 1}{V} \alpha(n - 1)\right]^{\frac{2}{n}}$$

PROOF. As in theorem 16, we have

$$\lambda_m \leq \mu_1(B(0,\tfrac{r}{2}))$$

However, it is known [5] that

$$\mu_1(B(0,\tfrac{r}{2})) \leq \frac{n(n + 4)}{r^2}$$

Similarly to (4.7)

$$V(M) \leq (m + 1)\max\{V(B(x_i,r + \varepsilon)\}$$

$$\leq \frac{(m + 1)\alpha(n - 1)}{n} (r+\varepsilon)^n$$

Hence

$$\lambda_m \leq n(n + 4) \left[\frac{(m + 1)\alpha(n - 1)}{nV}\right]^{\frac{2}{n}}$$

5. LOWER BOUNDS FOR λ_m

In this section we will obtain lower bounds for λ_m with the right order in $\frac{m}{V}$. The first part is devoted to estimate higher eigenvalues. For m big enough, we have

$$\lambda_m \geq C_3 \left(\frac{m}{V}\right)^{\frac{2}{n}}$$

where C_3 depends only on K, d, upper bound of δ = radius of convexity, and κ = lower bound of the sectional curvature of M. When M has non-negative sectional curvature we show that

$$\lambda_m \geq \bar{\bar{C}}\left(\frac{m}{V}\right)^{\frac{2}{n}}$$

where $\bar{\bar{C}}$ is a constant depending on n alone.

For general $m \geq 2$, we employ a method of the second author [15] to obtain lower bounds for λ_m, namely

$$\lambda_m \geq C_4\left(\frac{m}{V}\right)^{\frac{2}{n}}$$

However C_4 now depends on n, δ, d, k and V.

THEOREM 18. Let M be a compact manifold of dimension n. Suppose λ_m is the mth non-zero eigenvalue of M. If

$$m \geq \frac{V(\sqrt{\tau})^{n-1}}{\alpha(n-1)\int_0^{\delta/2}(\sin(\tau t))^{n-1}dt}$$

where τ = upper bound of the sectional curvature of M and δ = radius of convexity. Then

$$\lambda_m \geq \frac{\exp -[1+(1-16(n-1)^2d^2K)^{1/2}]}{32(n-1)p}\left(\frac{\alpha(n-1)}{n}\right)^{\frac{2}{n}}\left(\frac{m}{V}\right)^{\frac{2}{n}}$$

where

$$P = 3\left(2\,\frac{\sinh \delta\sqrt{-\kappa}}{\delta\sqrt{-\kappa}}\right)^n$$

with $0 \geq \kappa$ = lower bound of the sectional curvature of M.

PROOF. By (4.2) and lemma 15, it is clear that if

$$m \geqslant \frac{V}{\alpha(n-1)(\sqrt{\tau})^{1-n}\int_0^{\delta/2}(\sin(\sqrt{\tau}\, t))^{n-1}\,dt} \tag{5.1}$$

and for any $\varepsilon > 0$, there exists a collection of geodesic balls $\{B_i(r_i)\}_{i=1}^m$ which satisfied properties (i-iv) of lemma 15. Moreover
$r_i \leqslant r + \varepsilon \leqslant \delta$ for all $i = 1,\dots, m$.

The maximum principle of λ_m gives

$$\lambda_m = \max_{\{\varphi_i\}_{i=1}^m} \min_{f \perp \varphi_i} \frac{\int|\nabla f|^2}{\int f^2} \tag{5.2}$$

where we maximize over any set of m functions on M and minimize over all function f which are perpendicular to φ_i, $1 \leqslant i \leqslant m$. In particular, if we take φ_i to be the charateristic functions on $B_i(r_i)$, then the conditions $f \perp \varphi_i$ means

$$\int_{B_i(r_i)} f = 0 \tag{5.3}$$

Hence

$$\lambda_m \geqslant \min_{f \perp \varphi_i} \frac{\int|\nabla f|^2}{\int f^2} \tag{5.4}$$

However if $\lambda_1(B_i)$ denotes the first non-zero eigenvalue of the Neumann problem on $B_i(r_i)$ then (5.3) implies

$$\int_{B_i(r_i)} |\nabla f|^2 \geqslant \lambda_1(B_i) \int_{B_i(r_i)} f^2 \tag{5.5}$$

Summing up both sides of (5.5) over i, we have

$$(5.6)\qquad \sum_{i=1}^{m} \int_{B_i(r_i)} |\nabla f|^2 \geq \min \{\lambda_1(B_i)\} \sum_{i=1}^{m} \int_{B_i(r_i)} f^2$$

$$\geq \min\{\lambda_1(B_i)\} \int_M f^2$$

by property (i) of lemma 15. On the other hand, property (iv) gives

$$(5.7)\qquad P\int_M |\nabla f|^2 \geq \sum_{i=1}^{m} \int_{B_i(r_i)} |\nabla f|^2$$

Therefore

$$(5.8)\qquad \lambda_m \geq \frac{\min\{\lambda_1(B_i)\}}{P}$$

Applying theorem 9, we have

$$(5.9)\qquad \lambda_1(B_i) \geq \frac{\exp - [1 + (1 - 16(n - 1)^2 r_i^2 K)^{1/2}]}{8(n - 1) r_i^2}$$

$$\geq \frac{\exp - [1 + (1 - 16(n - 1)^2 (r + \varepsilon)^2 K)^{1/2}]}{8(n - 1)(r + \varepsilon)^2}$$

by property (iii). However (4.2) and property (ii) gives

$$(5.10)\qquad V \geq m\alpha(n - 1) \frac{1}{(\sqrt{\tau})^{n-1}} \int_0^{\frac{r}{2}} (\sin(\sqrt{\tau}\, t))^{n-1}\, dt - m\varepsilon$$

$$\geq m \left[\alpha(n - 1) \frac{1}{(\sqrt{\tau})^{n-1}} \int_0^{\frac{r}{2}} \left(\frac{2 \sin\left(\frac{r\sqrt{\tau}}{2}\right)}{t}\, t \right)^{n-1} dt - \varepsilon \right]$$

since $r \leqslant \delta$ = radius of convexity

$$= m\left[\alpha(n-1)\left(\frac{2\sin\left(\frac{r\sqrt{\tau}}{2}\right)}{r\sqrt{\tau}}\right)^{n-1}\left(\frac{r}{2}\right)^n \frac{1}{n} - \varepsilon\right]$$

$$\geqslant m\left[\frac{\alpha(n-1)}{n2^n} r^n - \varepsilon\right]$$

Combining (5.8), (5.9) and (5.10) yields

$$\lambda_m \geqslant \frac{\exp - [1 + (1 - 16(n-1)^2(r+\varepsilon)^2 K)^{1/2}]}{8(n-1)p} \tag{5.11}$$

$$\times \left(\frac{r}{r+\varepsilon}\right)^2 \frac{1}{4}\left(\frac{m\alpha(n-1)}{n}\right)^{\frac{2}{n}} \frac{1}{(V + m\varepsilon)^{\frac{2}{n}}}$$

Theorem 18 is proved by letting $\varepsilon \to 0$.

THEOREM 19. Let M be a compact manifold of non-negative Ricci curvature. If

$$m \geqslant \frac{V(\sqrt{\tau}\,)^{n-1}}{\alpha(n-1)\int_0^{\delta/2}(\sin(\sqrt{\tau}\, t))^{n-1}\, dt}$$

then

$$\lambda_m \geqslant \frac{\pi^2}{3 \cdot 2^{n+6}}\left(\frac{\alpha(n-1)}{n}\right)^{\frac{2}{n}}\left(\frac{m}{V}\right)^{\frac{2}{n}}$$

PROOF. We follow the idea of the proof of theorem 18. The non-negative Ricci curvature assumption allows us to apply theorem 12, hence we can replace (5.9) by

$$\lambda_1(B_i) \geqslant \left(\frac{\pi}{4(r+\varepsilon)}\right)^2 \tag{5.12}$$

and the theorem follows.

COROLLARY 20. Let M be a compact manifold of non-negative sectional curvature. If

$$m \geq \frac{V(\sqrt{\tau})^{n-1}}{\alpha(n-1)\int_0^{\delta/2}(\sin(\tau t))^{n-1}dt}$$

then

$$\lambda_m \geq \frac{\pi^2}{64 \cdot 3 \cdot 2^n}\left(\frac{\alpha(n-1)}{n}\right)^{\frac{2}{n}}\left(\frac{m}{V}\right)^{\frac{2}{n}}$$

For lower eigenvalues λ_m, the task of getting a lower bound becomes more difficult since theorem 9 and theorem 12 applied only to manifolds with convex boundaries. However utilizing the method of Yau [15], we can avoid the difficulty of obtaining a lower bound for the first eigenvalue of the Neumann problem.

Let M be a compact manifold, consider the ball of radius $r \leq d$, $B_p(r)$, centered at a given point $p \in M$. We define the isoperimetric constant $\Pi(B_p(r))$ by

$$\Pi(B_p(r) = \inf\left[\frac{A(\partial M_1 \cap \partial M_2 \cap B_p(r))}{\min\{V(M_1 \cap B_p(\frac{r}{2})), V(M_2 \cap B_p(\frac{r}{2}))\}}\right]$$

where inf is taken over all decompositions $B_p(\frac{r}{2}) = M_1 \cup M_2$ with $V(M_1 \cap M_2) = 0$. This constant $\Pi(B_p(r))$ is similar to the constant $I(M)$ defined by Yau [15, p. 499]. Moreover, theorem 4 and corollary 1 in [15] can also be generalized to $\Pi(B_p(r))$ with the appropriate modification in the proofs and the statements. We will give their analogue for $\Pi(B_p(r))$ in the next two propositions, however since their proofs follow directly as in [15] and will be omitted.

PROPOSITION 21. $\Pi(B_p(r)) = \inf\left[\frac{\int_{B_p(r)}|\nabla f|}{\inf_{\beta\in\mathbb{R}}\int_{B_p(\frac{r}{2})}|f-\beta|}\right]$

where inf is taken over all functions $f \in H_{1,1}(B_p(r))$

PROPOSITION 22. For any $f \in H_{1,2}(B_p(r))$, we have

$$\int_{B_p(r)} |\nabla f|^2 \geq \frac{\Pi(B_p(r))^2}{4} \int_{B_p(\frac{r}{2})} (f - k)^2$$

for any $k \in \mathbb{R}$ satisfying

$$V(B_p(\tfrac{r}{2}) \cap \{x \mid f(x) \geq k\}) \geq \tfrac{1}{2} V(B_p(\tfrac{r}{2}))$$

and

$$V(B_p(\tfrac{r}{2}) \cap \{x \mid f(x) \leq k\}) \geq \tfrac{1}{2} V(B_p(\tfrac{r}{2}))$$

In particular, if $\int_{B_p(\frac{r}{2})} f = 0$ then

$$\int_{B_p(r)} |\nabla f|^2 \geq \frac{\Pi(B_p(r))^2}{4} \int_{B_p(\frac{r}{2})} f^2$$

The following proposition is essentially theorem 8 of [15] together with proposition 21.

PROPOSITION 23. Let M be a compact manifold. Consider the ball $B_p(r) \subset M$ with $r \leq d$. Define $G = \sup_{\sigma} \exp\left[\int_0^{\ell} \left(- \frac{1}{t^2} \int_0^t K(\sigma^1(s)) s^2 ds\right) dt\right]$ where σ ranges over all minimal geodesic segments with length $\ell \leq r$ and $\sigma(0) \in B_p(\frac{r}{2})$. Then for all $f \in H_{1,1}(B_p(r))$,

$$\Pi(B_p(r)) \geq \frac{nV(B_p(\frac{r}{2}))}{2r^{n+1}\alpha(n-1)G}$$

Combining the above propositions, we can show that for any

$f \in H_{1,2}(B_p(r))$ satisfying $\int_{B_p(\frac{r}{2})} f = 0$ then

$$(5.13) \qquad \int_{B_p(r)} |\nabla f|^2 \geq \left(\frac{nV(B_p(\frac{r}{2}))}{4r^{n+1}\alpha(n-1)G} \right)^2 \int_{B_p(\frac{r}{2})} f^2 .$$

The next lemma enables us to estimate the volume of $B_p(\frac{r}{2})$ from below.

LEMMA 24. Let M be a compact manifold. The ball $B_p(\frac{r}{2})$ has volume bounded below by either

$$V(B_p(\tfrac{r}{2})) \geq \frac{V(M)}{2}$$

or

$$V(B_p(\tfrac{r}{2})) \geq \frac{r^n C_o}{(2n)^n}$$

where C_o is the isoperimetric constant defined by

$$C_o = \inf \frac{A(N)^n}{\min\{V(M_1), V(M_2)\}^{n-1}}$$

with inf taken over all codimension-1 submanifold N which divides M into M_1 and M_2.

PROOF. Suppose $V(B_p(\frac{r}{2})) \leq \frac{V(M)}{2}$. In particular $V(B_p(t)) \leq \frac{V(M)}{2}$ for all $t \leq \frac{r}{2}$. Hence by the definition of C_o

$$(5.14) \qquad \frac{A(\partial B_p(t))}{V(B_p(t))^{\frac{n-1}{n}}} \geq C_o^{\frac{1}{n}}$$

However it is well known that $A(\partial B_p(t)) = \frac{d}{dt}V(B_p(t))$. Therefore integrating (5.14) gives

$$(5.15) \qquad nV(B_p(\tfrac{r}{2}))^{\frac{1}{n}} = \int_o^{\frac{r}{2}} \frac{\frac{d}{dt}V(B_p(t))}{V(B_p(t))^{\frac{n-1}{n}}}\, dt \geq \frac{rC_o^{\frac{1}{n}}}{2}$$

The lemma follows.

THEOREM 25. Let M be a compact manifold. Suppose λ_m is the mth non-zero eigenvalue of M with $m \geq 2$. Then

$$\lambda_m \geq C_5 \left(\frac{m}{V}\right)^{\frac{2}{n}}$$

where C_5 is a constant bounded below by

$$C_5 \geq \frac{\min\{\frac{2\delta}{d}, 1\}^{2n+2}\, \alpha(n-1)^{\frac{2}{n}} (d\sqrt{-k})^n}{3 \cdot 2^{6n+10} (\sinh(d\sqrt{-k}))^n n^{\frac{2}{n}} G^2}$$

or

$$C_5 \geq \frac{\min\{\frac{2\delta}{d}, 1\}^{2} (d\sqrt{-k})^n C_o^2}{3 \cdot 2^{4n+8} (\sinh(d\sqrt{-k}))^n n^{2n-2+\frac{2}{n}} \alpha(n-1)^{2-\frac{2}{n}} G^2}$$

where C_o is the isoperimetric constant defined in lemma 24 and

$$G = \sup_{\sigma} \exp\left[\int_o^d \left(-\frac{1}{t^2}\int_o^t K(\sigma'(s))s^2 ds\right) dt\right]$$

PROOF. For $m \in \mathbb{Z}^+$, let $\{B_i(r_i)\}_{i=1}^m$ be a collection of geodesic balls which satisfied properties (i-iv) of lemma 15. In view of the proof of theorem 18 and 19 where we obtain the estimates for λ_m with $r \leq r_i \leq r + \varepsilon$ and by letting $\varepsilon \to 0$ in the final process, we may hence assume $r_i = r$ for $1 \leq i \leq m$.

If we take φ_i to be charateristic functions on $B_i(r)$, equation (5.2) gives

$$(5.16) \qquad \lambda_m \geq \min_{f \perp \varphi_i} \frac{\int |\nabla f|^2}{\int f^2}$$

However proposition 23 and lemma 24 imply either

$$\int_{B_i(2r)} |\nabla f|^2 \geq \left(\frac{nV}{8(2r)^{n+1}\alpha(n-1)G}\right)^2 \int_{B_i(r)} f^2 \tag{5.17}$$

or

$$\int_{B_i(2r)} |\nabla f|^2 \geq \left(\frac{C_o}{2^{n+3}n^{n-1}\alpha(n-1)rG}\right)^2 \int_{B_i(r)} f^2 \tag{5.18}$$

Summing over i on both sides gives

$$\bar{p}\int_M |\nabla f|^2 \geq \xi \int_M f^2 \tag{5.19}$$

where

$$\xi = \min\left\{\left(\frac{nV}{2^{n+4}r^{n+1}\,\alpha(n-1)G}\right)^2, \left(\frac{C_o}{2^{n+3}n^{n-1}\,\alpha(n-1)rG}\right)^2\right\}$$

and

$$\bar{p} = 3\left(4\frac{\sinh d\sqrt{-K}}{d\sqrt{-K}}\right)^n$$

$\geq$ maximal number of $B_i(2r)$ that contained any given point in M

On the other hand by (4.2) and property (ii) of lemma 15

$$V \geq m\alpha(n-1)\frac{1}{(\sqrt{T})^{n-1}}\int_o^{\frac{r}{2}} (\sin(\sqrt{T}\,t))^{n-1}\,dt \tag{5.20}$$

Since $r \leq d$, in the case when $2\delta \geq d$ we have $\delta \geq \frac{r}{2}$. If $2\delta < d$, then $\frac{r}{2} \geq \frac{r\delta}{d}$.

Therefore

$$(5.21)\qquad V \geqslant m\alpha(n-1)\frac{1}{(\sqrt{\tau})^{n-1}}\int_0^{\frac{r}{2}\times\min\{2\frac{\delta}{d},1\}}(\sin(\sqrt{\tau}))^{n-1}\,dt$$

$$\geqslant m\alpha(n-1)\frac{(\min\{\frac{2\delta}{d},1\})^n}{n2^n}r^n$$

$$\left(\text{since } \frac{r}{2}\times\min\{\frac{2\delta}{d},1\}\leqslant\delta\right)$$

Combining with (5.16) and (5.19), we have

$$\lambda_m \geqslant \frac{\xi}{\bar{p}}$$

where $\bar{p} = 3\left(4\frac{\sinh d\sqrt{-\kappa}}{d\sqrt{-\kappa}}\right)^n$

and

$$\xi \geqslant \min\left\{\frac{m^{2+\frac{2}{n}}\min\{\frac{2\delta}{d},1\}^{2n+2}\,\alpha(n-1)^{\frac{2}{n}}}{V^{\frac{2}{n}}n^{\frac{2}{n}}\,2^{4n+10}G^2},\ \frac{M^{\frac{2}{n}}\min\{\frac{2\delta}{d},1\}^2C_o^2}{V^{\frac{2}{n}}n^{2n-2+\frac{2}{n}}\,2^{2n+8}\,\alpha(n-1)^{2-\frac{2}{n}}G^2}\right\}$$

The theorem follows

COROLLARY 26. Let M be a compact manifold. Then

$$\lambda_m \geqslant C_4\left(\frac{m}{V}\right)^{\frac{2}{n}}$$

where C_4 = constant depending on n, δ, d, k, and V.

PROOF. This follows from theorem 25, plus the fact that $K \geqslant \kappa$, and a result of Croke [6] which gives a lower bound for C_o, namely

$$C_o \geqslant \frac{V^{n+1}}{4\alpha(n)^{n-1}\,\alpha(n-1)\left(\int_o^d (\sqrt{-K}^{\,-1} \sinh\sqrt{-K}\,t)^{n-1}\,dt\right)^{n+1}}$$

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