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# Compact Kähler Manifolds of Positive Bisectional Curvature*

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In this paper we prove the following conjecture of Frankel [7].

**Main Theorem.** *Every compact Kähler manifold of positive bisectional curvature is biholomorphic to the complex projective space.*

The case of dimension two was proved by Andreotti-Frankel [7] and the case of dimension three by Mabuchi [18] using the result of Kobayashi-Ochiai [12].

Our method of proof uses harmonic maps and the characterization of the complex projective space obtained by Kobayashi-Ochiai [15]. According to the result of Kobayashi-Ochiai the complex projective space of dimension $n$ is characterized by the fact that its first Chern class equals $\lambda c_1(F)$ for some $\lambda \geqq n+1$ and some positive holomorphic line bundle $F$ over it. Since by the result of Bishop-Goldberg [2] the second Betti number of a compact Kähler manifold $M$ of positive bisectional curvature is 1, for the Main Theorem it suffices to show that $c_1(M)$ is $\lambda$ times a generator of $H^2(M, \mathbb{Z})$ for some $\lambda \geqq 1+\dim M$. This can be done by proving that a generator of the free part of $H_2(M, \mathbb{Z})$ can be represented by a rational curve, because the tangent bundle of $M$ restricted to the rational curve splits into a direct sum of holomorphic line bundles over the rational curve according to the result of Grothendieck [11]. The existence of the rational curve is obtained in the following way. According to the result of Sacks-Uhlenbeck [22] and its improved formulation by Meeks-Yau [19], the infimum of the energies of maps from $S^2$ to $M$ representing the generator of $\pi_2(M)$ can be achieved by a sum of stable harmonic maps $f_i$ from $S^2$ to $M$ $(1 \leqq i \leqq m)$. The key step in our proof is to show that each $f_i$ is either holomorphic or conjugate holomorphic. The known methods of proving the complex-analyticity of a harmonic map use the formula for the Laplacian of the energy function [23, 25, 26] or a variation of it [24]. Here we use instead the second variation formula of the energy function. In this second variation formula a 2-parameter variation has to be used to imitate the situation of holomorphic deformation. After this key step we use holomorphic deformations of rational curves in $M$ to show that

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$m=1$. This is done by proving that in case $m>1$ we can holomorphically deform the images of some holomorphic $f_i$ and some conjugate holomorphic $f_j$ so that they are tangential to each other at some point. By removing a disc centered at the point of contact from each and joining the two disc boundaries by a suitable surface, we obtain a map from $S^2$ to $M$ with energy smaller than the minimum energy. Thus $m=1$ and the image of $f_1$ is a rational curve representing a generator of the free part of $H_2(M, \mathbb{Z})$.

In our proof the existence of a rational curve plays a very important role. This important role of a rational curve has already been observed earlier by Hartshorne and Kobayashi-Ochiai [13].

We learn that very recently Mori [20] has also given a proof of Frankel's conjecture by using the method of algebraic geometry of characteristic $p>0$. Mori's result is stronger than ours. He needs only the assumption that the tangent bundle of the manifold is ample, whereas we have to assume that the manifold has positive holomorphic bisectional curvature. Though our result is weaker, our proof has the advantage that it uses only methods of Kähler geometry to answer a question in Kähler geometry. On the other hand, even in the case of complex manifolds Mori's proof involves the use of methods of algebraic geometry of characteristic $p>0$.

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## §1. Second Variation Formula

Suppose $M$, $N$ are compact Kähler manifolds whose Kähler metrics are respectively

$$ds_M^2 = 2\,\mathrm{Re} \sum_{\alpha,\beta=1}^{m} h_{\alpha\bar{\beta}}\, dz^\alpha\, d\overline{z^\beta}$$

and

$$ds_N^2 = 2\,\mathrm{Re} \sum_{i,j=1}^{n} g_{i\bar{j}}\, dw^i\, d\overline{w^j}.$$

Let $f(t)\colon N \to M$, $t \in \mathbb{C}$, $|t|<\varepsilon$, be a family of smooth maps parametrized by an open disc in $\mathbb{C}$. The pointwise $\bar{\partial}$-energy of $f$ is defined by

$$|\bar{\partial} f|^2 = g^{\bar{i}j} f_{\bar{i}}^{\alpha} \overline{f_{\bar{j}}^{\beta}} h_{\alpha\bar{\beta}},$$

where the summation convention is used and

$$f_{\bar{i}}^{\alpha} = \frac{\partial f^\alpha}{\partial \overline{w^i}}.$$

In order to compute $\dfrac{\partial^2}{\partial t\,\partial \bar{t}} |\bar{\partial} f|^2$ at a point $P$ of $N$, we choose local holomorphic coordinate systems at $P$ and $Q=f(P)$ such that

$$dg_{i\bar{j}}=0 \quad \text{at} \quad P$$

and

$$dh_{\alpha\bar{\beta}}=0$$
$$\partial_\gamma \partial_\delta h_{\alpha\bar{\beta}}=0 \quad \text{at} \quad Q.$$

This is possible because the metrics of $M$ and $N$ are both Kähler. Direct computation yields

$$\begin{aligned}\frac{\partial^2}{\partial t\,\partial \bar{t}}\,|\bar{\partial} f|^2 &= 2\operatorname{Re} g^{\bar{i}j}\left(\frac{\partial^2}{\partial t\,\partial \bar{t}}\, f^\alpha_{\bar{i}}\right)\overline{f^\beta_{\bar{j}}}\, h_{\alpha\bar{\beta}} + g^{\bar{i}j}\left(\frac{\partial}{\partial t} f^\alpha_{\bar{i}}\right)\overline{\left(\frac{\partial}{\partial t} f^\beta_{\bar{j}}\right)} h_{\alpha\bar{\beta}} \\ &\quad + g^{\bar{i}j}\left(\frac{\partial}{\partial \bar{t}} f^\alpha_{\bar{i}}\right)\overline{\left(\frac{\partial}{\partial \bar{t}} f^\beta_{\bar{j}}\right)} h_{\alpha\bar{\beta}} + g^{\bar{i}j} f^\alpha_{\bar{i}}\overline{f^\beta_{\bar{j}}}(\partial_\mu \partial_{\bar{\nu}} h_{\alpha\bar{\beta}})\,\frac{\partial f^\mu}{\partial \bar{t}}\,\overline{\frac{\partial f^\nu}{\partial \bar{t}}} \\ &\quad + g^{\bar{i}j} f^\alpha_{\bar{i}}\overline{f^\beta_{\bar{j}}}(\partial_\mu \partial_{\bar{\nu}} h_{\alpha\bar{\beta}})\,\frac{\partial f^\mu}{\partial t}\,\overline{\frac{\partial f^\nu}{\partial t}}.\end{aligned}$$

Consider the vector field $\xi$ on $N$ defined by

$$\xi^{\bar{i}} = g^{\bar{i}j}\left(\frac{D}{\partial t}\,\frac{\partial}{\partial \bar{t}}\, f^\alpha\right)\overline{f^\beta_{\bar{j}}}\, h_{\alpha\bar{\beta}},$$

where $\dfrac{D}{\partial t}$ is the covariant differentiation with respect to the connection of the tangent bundle of $M$, i.e.

$$\frac{D}{\partial t}\,\frac{\partial}{\partial \bar{t}}\, f^\alpha = \frac{\partial^2}{\partial t\,\partial \bar{t}}\, f^\alpha + {}^M\Gamma^\alpha_{\beta\gamma}\,\frac{\partial f^\beta}{\partial t}\,\frac{\partial f^\gamma}{\partial \bar{t}},$$

${}^M\Gamma^\alpha_{\beta\gamma}$ being the Christoffel symbol of $M$. The divergence of $\xi$ at $P$ is

$$\begin{aligned}\nabla_{\bar{i}}\, \xi^{\bar{i}} &= g^{\bar{i}j}\left(\frac{\partial^2}{\partial t\,\partial \bar{t}}\, f^\alpha_{\bar{i}}\right)\overline{f^\beta_{\bar{j}}}\, h_{\alpha\bar{\beta}} + g^{\bar{i}j}\left(\frac{\partial^2}{\partial t\,\partial \bar{t}}\, f^\alpha\right)\overline{\partial_i f^\beta_{\bar{j}}}\, h_{\alpha\bar{\beta}} \\ &\quad + g^{\bar{i}j}(\partial_\mu \partial_{\bar{\nu}} h_{\alpha\bar{\beta}})\,\overline{f^\nu_{\bar{i}}}\,\overline{f^\beta_{\bar{j}}}\,\frac{\partial f^\mu}{\partial t}\,\frac{\partial f^\alpha}{\partial \bar{t}}.\end{aligned}$$

Now assume that $f$ is harmonic at 0. Then at $P$ and $t=0$

$$g^{\bar{i}j}\,\overline{\partial_i f^\beta_{\bar{j}}}\, h_{\alpha\bar{\beta}}=0$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t\,\partial \bar{t}}\,|\bar{\partial} f|^2 - 2\operatorname{Re}\operatorname{div}\xi &= g^{\bar{i}j}\left(\frac{D}{\partial t} f^\alpha_{\bar{i}}\right)\overline{\left(\frac{D}{\partial t} f^\beta_{\bar{j}}\right)} h_{\alpha\bar{\beta}} \\ &\quad + g^{\bar{i}j}\left(\frac{D}{\partial \bar{t}} f^\alpha_{\bar{i}}\right)\overline{\left(\frac{D}{\partial \bar{t}} f^\beta_{\bar{j}}\right)} h_{\alpha\bar{\beta}} + g^{\bar{i}j} f^\alpha_{\bar{i}}\,\overline{f^\beta_{\bar{j}}}\, R_{\mu\bar{\nu}\alpha\bar{\beta}}\,\frac{\partial f^\mu}{\partial \bar{t}}\,\overline{\frac{\partial f^\nu}{\partial \bar{t}}} \\ &\quad + g^{\bar{i}j} f^\alpha_{\bar{i}}\overline{f^\beta_{\bar{j}}}\, R_{\mu\bar{\nu}\alpha\bar{\beta}}\,\frac{\partial f^\mu}{\partial t}\,\overline{\frac{\partial f^\nu}{\partial t}} \quad - 2\operatorname{Re} g^{\bar{i}j} R_{\mu\bar{\nu}\alpha\bar{\beta}}\,\overline{f^\nu_{\bar{i}}}\,\overline{f^\beta_{\bar{j}}}\,\frac{\partial f^\mu}{\partial \bar{t}}\,\frac{\partial f^\alpha}{\partial t}.\end{aligned}$$

Hence at $t=0$

$$\begin{aligned}\frac{\partial^2}{\partial t\,\partial \bar t}\int_N |\bar\partial f|^2 =& \int_N g^{\bar\imath j}\left(\frac{D}{\partial t} f_{\bar\imath}^{\alpha}\right)\overline{\left(\frac{D}{\partial t} f_{\bar\jmath}^{\beta}\right)} h_{\alpha\bar\beta} + \int_N g^{\bar\imath j}\left(\frac{D}{\partial \bar t} f_{\bar\imath}^{\alpha}\right)\overline{\left(\frac{D}{\partial \bar t} f_{\bar\jmath}^{\beta}\right)} h_{\alpha\bar\beta}\\ &+\int_N g^{\bar\imath j} f_{\bar\imath}^{\alpha}\overline{f_{\bar\jmath}^{\beta}} R_{\mu\bar\nu\alpha\bar\beta}\frac{\partial f^{\mu}}{\partial \bar t}\overline{\frac{\partial f^{\nu}}{\partial \bar t}}\\ &+\int_N g^{\bar\imath j} f_{\bar\imath}^{\alpha}\overline{f_{\bar\jmath}^{\beta}} R_{\mu\bar\nu\alpha\bar\beta}\frac{\partial f^{\mu}}{\partial t}\overline{\frac{\partial f^{\nu}}{\partial t}}\\ &-2\,\mathrm{Re}\int_N g^{\bar\imath j} R_{\mu\bar\nu\alpha\bar\beta}\overline{f_{i}^{\nu}}\,\overline{f_{\bar\jmath}^{\beta}}\frac{\partial f^{\mu}}{\partial \bar t}\frac{\partial f^{\alpha}}{\partial t}\end{aligned}$$

where

$$R_{\mu\bar\nu\alpha\bar\beta}=\partial_\mu\partial_{\bar\nu}h_{\alpha\bar\beta}-h^{\bar\gamma\delta}\partial_\mu h_{\alpha\bar\gamma}\overline{\partial_\nu h_{\beta\bar\delta}}$$

is the curvature tensor of $M$.

## § 2. Relation of Energy and $\bar\partial$-Energy

Suppose $M$, $N$ are as in § 1. Let $f\colon N\to M$ be a smooth map. Similar to $|\bar\partial f|^2$, the pointwise $\partial$-energy $|\partial f|^2$ of $f$ is defined by

$$|\partial f|^2 = g^{\bar\imath j} f_j^{\alpha}\overline{f_i^{\beta}}\, h_{\alpha\bar\beta}$$

where

$$f_j^{\alpha}=\frac{\partial f^{\alpha}}{\partial w^j}.$$

The pointwise energy $e(f)$ of $f$, which is defined as the trace of $f^*(ds_M^2)$ with respect to $ds_N^2$, is therefore equal to $|\bar\partial f|^2+|\partial f|^2$. Now assume $\dim_{\mathbb C} N=1$. The pullback of the Kähler form of $M$ under $f$ is

$$\sqrt{-1}\, h_{\alpha\bar\beta}\, df^{\alpha}\wedge \overline{df^{\beta}} = \sqrt{-1}\, h_{\alpha\bar\beta}\left(\frac{\partial f^{\alpha}}{\partial w}\overline{\frac{\partial f^{\beta}}{\partial w}}-\frac{\partial f^{\alpha}}{\partial \bar w}\overline{\frac{\partial f^{\beta}}{\partial \bar w}}\right) dw\wedge d\bar w.$$

Hence

$$\int_N |\partial f|^2-\int_N |\bar\partial f|^2=\int_N \sqrt{-1}\, h_{\alpha\bar\beta}\, df^{\alpha}\wedge\overline{df^{\beta}}$$

which is equal to the Kähler class $\omega(M)$ of $M$ evaluated at the homology class $[f(N)]$ defined by $f\colon N\to M$. It follows that

$$\int_N |\partial f|^2=\tfrac12\int_N e(f)+\tfrac12\omega(M)\,[f(N)],$$

$$\int_N |\bar\partial f|^2=\tfrac12\int_N e(f)-\tfrac12\omega(M)\,[f(N)].$$

As a consequence, the energy-minimizing maps from $N$ to $M$ are precisely the same as the $\bar\partial$-energy-minimizing maps.

*Remark.* This last fact was first observed by Lichnerowicz [16].

## § 3. Complex-Analyticity of Energy-Minimizing Maps

Suppose $M$ is a compact Kähler manifold with positive holomorphic bisectional curvature. Let $f_0\colon \mathbb{P}_1 \to M$ be an energy-minimizing map.

**Proposition 1.** *If $f_0^* c_1(M)$ evaluated at $\mathbb{P}_1$ is nonnegative (respectively nonpositive), then $f_0$ is holomorphic (respectively conjugate holomorphic).*

*Proof.* Since the proof of the other case is similar, we prove here only the holomorphic case.

Let $T_M$ be the holomorphic tangent bundle of $M$, let $w$ be a local coordinate of $\mathbb{P}_1$, and let $\frac{D}{\partial \bar{w}}$ be the covariant differentiation, in the antiholomorphic direction, of local cross sections of $f_0^* T_M$ with respect to the connection of $T_M$. Let $\mathscr{F}$ be the sheaf of germs of local cross sections $s$ of $f_0^* T_M$ with $\frac{D}{\partial \bar{w}} s = 0$. Clearly $\mathscr{F}$ is an analytic sheaf over $\mathbb{P}_1$. We claim that $\mathscr{F}$ is locally free and the holomorphic vector bundle associated to $\mathscr{F}$ is topologically isomorphic to $f_0^* T_M$.

It suffices to show that for an arbitrary point $P$ of $\mathbb{P}_1$ there exist local cross sections $s_1, \ldots, s_m$ at $P$ such that $\frac{D}{\partial \bar{w}} s_i \equiv 0$, $1 \leqq i \leqq m$, and $s_1(P), \ldots, s_m(P)$ form a basis of the fiber $f_0^* T_M$ at $P$, where $m = \dim_{\mathbb{C}} M$.

These local sections $s_i$ can be constructed as follows. Choose the local coordinate $(z^\alpha)$ of $M$ at $Q = f_0(P)$ such that the metric tensor $h_{\alpha\bar{\beta}}$ of $M$ satisfies $d h_{\alpha\bar{\beta}} = 0$ at $Q$. Every local cross section $s$ of $f_0^* T_M$ at $P$ can be written as

$$s = \sum_\alpha s^\alpha \frac{\partial}{\partial z^\alpha}.$$

Then at $P$

$$\frac{D}{\partial \bar{w}} s = \sum_\alpha \frac{\partial s^\alpha}{\partial \bar{w}} \frac{\partial}{\partial z^\alpha}.$$

Choose smooth local cross sections

$$t_i = \sum_\alpha t_i^\alpha \frac{\partial}{\partial z^\alpha}$$

of $f_0^* T_M$ at $P$, $1 \leqq i \leqq m$, such that the matrix

$$(t_i^\alpha)_{1 \leqq \alpha, i \leqq m}$$

is nonsingular at $P$ and each $t_i^\alpha$ is holomorphic at $P$ as a function of $w$. Then $\frac{D}{\partial \bar{w}} t_i$ vanishes at $P$. Solve the equations

$$\frac{D}{\partial \bar{w}} u_i = \frac{1}{w - w(P)} \frac{D}{\partial \bar{w}} t_i \qquad (*)$$

for $u_i$ locally at $P$ by using the Cauchy kernel

$$\frac{1}{2\pi\sqrt{-1}}\,\frac{dw\wedge d\bar{w}}{w-\zeta}$$

and the standard classical iteration process of Korn-Lichtenstein (see e.g. p. 394 of [21]; cf. pp. 254–267 of [1]). Since the right-hand side of (*) is bounded, it follows that $u_i$ is $\varepsilon$-Hölder continuous for any $0<\varepsilon<1$. Define $s_i$ by

$$s_i=t_i-(w-w(P))\,u_i.$$

Then $s_i$ satisfies the requirements.

Now we can regard $f_0^* T_M$ as a holomorphic vector bundle over $\mathbb{P}_1$ (after its identification with the holomorphic vector bundle associated to $\mathscr{F}$). By the theorem of Grothendieck [11], $f_0^* T_M$ is a direct sum of holomorphic line bundles $L_1, \ldots, L_m$ over $\mathbb{P}_1$. Since the first Chern class of $f_0^* T_M$ evaluated at $\mathbb{P}_1$ is nonnegative, it follows that for some $i$, $c_1(L_i)$ evaluated at $\mathbb{P}_1$ is nonnegative. By the theorem of Riemann-Roch, we can find a nontrivial global holomorphic section

$$s=\sum s^\alpha \frac{\partial}{\partial z^\alpha}$$

of $L_i$ (and hence of $f_0^* T_M$) over $\mathbb{P}_1$.

Construct a smooth family of smooth maps $f(t)\colon \mathbb{P}_1\to M$, $t\in\mathbb{C}$, $|t|<\varepsilon$, such that $f(0)=f_0$ and at $t=0$

$$\frac{\partial}{\partial \bar{t}} f^\alpha(t)=0$$

and

$$\frac{\partial}{\partial t} f^\alpha(t)=s^\alpha.$$

Since $s$ is a holomorphic cross section of $f_0^* T_M$, it follows that at $t=0$

$$\frac{D}{\partial t}\frac{\partial}{\partial \bar{w}} f^\alpha=\frac{D}{\partial \bar{w}}\frac{\partial f^\alpha}{\partial t}=\frac{D}{\partial \bar{w}} s^\alpha=0.$$

Moreover,

$$\frac{D}{\partial \bar{t}}\frac{\partial}{\partial \bar{w}} f^\alpha=\frac{D}{\partial \bar{w}}\frac{\partial f^\alpha}{\partial \bar{t}}=0$$

at $t=0$.

From the second variation formula derived in § 1, it follows that at $t=0$

$$\frac{\partial^2}{\partial t\,\partial \bar{t}}\int_{\mathbb{P}_1} |\bar{\partial} f|^2=\int_{\mathbb{P}_1} \frac{\partial f^\alpha}{\partial \bar{w}}\,\overline{\frac{\partial f^\beta}{\partial \bar{w}}}\,R_{\mu\bar{\nu}\alpha\bar{\beta}}\,\frac{\partial f^\mu}{\partial t}\,\overline{\frac{\partial f^\nu}{\partial t}}\,(\sqrt{-1}\,dw\wedge d\bar{w}), \qquad (\dagger)$$

where $R_{\mu\bar{\nu}\alpha\bar{\beta}}$ is the curvature tensor of $M$. By § 2, $f_0$ is $\bar{\partial}$-energy minimizing. Hence at $t=0$ we have

$$\frac{\partial^2}{\partial t\,\partial \bar{t}}\int_{\mathbb{P}_1} |\bar{\partial} f|^2 \geqq 0.$$

Since the holomorphic bisectional curvature of $M$ is positive, i.e.

$$R_{\mu\bar{\nu}\alpha\bar{\beta}}\xi^{\mu}\overline{\xi^{\nu}}\eta^{\alpha}\overline{\eta^{\beta}}<0$$

for $(\xi^{\mu})$, $(\eta^{\alpha})\in\mathbb{C}^m-0$, it follows from (†) that

$$\frac{\partial f_0^{\alpha}}{\partial \bar{w}}=0$$

on $\mathbb{P}_1-Z$ for all $\alpha$, where $Z$ is the zero-set of $s$. The map $f_0$ is holomorphic, because $Z$ is a finite subset of $\mathbb{P}_1$. Q.E.D.

## § 4. Existence of Energy-Minimizing Maps

Let $M$ be a compact Riemannian manifold and $S^2$ be the 2-sphere. For each $C^1$ map $f\colon S^2\to M$, we let $E(f)$ be the energy of $f$ and let $E([f])$ be the infimum of the sum of the energies of maps whose sum is homotopic to $f$. The following proposition is proved by using the method of Sacks-Uhlenbeck [22] (see also [19] for results related to the present situation).

**Proposition 2.** *For every $C^1$ map $f\colon S^2\to M$ there exist energy-minimizing maps $f_i\colon S^2\to M$, $1\leqq i\leqq m$, such that the sum of $f_i$ is homotopic to $f$ and $E([f])=\sum_{i=1}^{m}E(f_i)$.*

*Proof.* It is proved in [22] that any nonconstant harmonic map or any homotopically nontrivial $C^1$ map from $S^2$ into $M$ has energy $>c$ for some fixed positive constant $c$. Let $k$ be the smallest nonnegative integer such that

$$E(f)\leqq\frac{kc}{2}.$$

We are going to prove the proposition by induction on $k$. The case $k=0$ is true. We want to prove the case $k=n+1$ under the assumption that the case $k=n$ is true.

By [22] for $\alpha>1$ we can find smooth maps $f_\alpha\colon S^2\to M$ which minimize the functional

$$E_\alpha(g)=\int_{S^2}(1+|dg|^2)^{\alpha}$$

over the space of all $C^1$ maps homotopic to $f$. Let $x_\alpha$ be a point of $S^2$ so that

$$|df_\alpha|^2(x_\alpha)=\sup_{S^2}|df_\alpha|^2.$$

If $\sup_{S^2}|df_\alpha|^2$ is uniformly bounded in $\alpha$, the $f_\alpha$ converges to an energy-minimizing map and the assertion is proved. If $\sup_{S^2}|df_\alpha|^2$ is unbounded in $\alpha$, then for some disc $D_\alpha$ with center at $x_\alpha$ the map $f_\alpha|D_\alpha$, after identifying $D_\alpha$ with an open disc in $\mathbb{R}^2$ by a suitable homothetic map, converges on compact subsets to a harmonic map $g: \mathbb{R}^2 \to M$. It is proved in [22] that $g$ can be extended to a harmonic map $\tilde{g}: S^2 \to M$ and we can assume that $f_\alpha | \partial D_\alpha$ has arbitrarily small length.

As in [19], we can construct a map $f_\alpha^1: S^2 \to M$ so that $E(f_\alpha^1)$ is arbitrarily close to $E(f_\alpha) - E(\tilde{g})$ and $f$ is homotopic to the sum of $f_\alpha^1$ and $\tilde{g}$. For later purpose, we outline the argument as follows. Fix a disc $D$ so that $f_\alpha | D$ converges smoothly on $D$ to $\tilde{g} | D$ and the energy of $\tilde{g}$ over $D$ is close to the energy of $\tilde{g}$. We define maps $\tilde{f}_\alpha$ from $S^2$ into $M$ by extending $f_\alpha | D$ to $S^2$ in the following way. Since $f_\alpha | \partial D$ is close to $\tilde{g} | \partial D$, we can join each point $f_\alpha(x)$ to $\tilde{g}(x)$ (for $x \in \partial D$) by a short geodesic and obtain a map $h_\alpha$ from an annulus into $M$ whose image has small area. Then we obtain a map $\tilde{f}_\alpha$ from $S^2$ into $M$ by putting together the maps $f_\alpha | S^2 - D$, $h_\alpha$ and $\tilde{g} | S^2 - D$.

By approximation, we may assume that the map $\tilde{f}_\alpha$ is a smooth immersion from $S^2$ into $M$ and the area of the image of $\tilde{f}_\alpha$ is close to the sum of the areas of the images of $f_\alpha | S^2 - D$, $h_\alpha$, and $\tilde{g} | S^2 - D$. The map $\tilde{f}_\alpha$ pulls back the metric from $M$ to a metric on $S^2$ which defines a conformal structure on $S^2$. Since there is only one conformal structure on $S^2$, there is an orientation preserving diffeomorphism from $S^2$ into $S^2$ which pulls this conformal structure back to the standard conformal structure on $S^2$. We define $f_\alpha^1$ to be $\tilde{f}_\alpha$ composed with this diffeomorphism so that $f_\alpha^1$ is conformal. Observe that the area of the image of a map from a real surface to a Riemannian manifold is the same as the energy when the map is conformal. Since $\tilde{g}$ is conformal (see for example [19]) and $E(\tilde{g} | S^2 - D)$ is small, the area of the image of the map $\tilde{g} | S^2 - D$ is small. This together with the smallness of the area of the image of $h_\alpha$ implies that $E(f_\alpha^1)$ is close to $E(f_\alpha) - E(\tilde{g})$.

Since $\tilde{g}$ is harmonic, $E(\tilde{g}) > c$ and $E(f_\alpha^1) \leqq \frac{nc}{2}$ for $\alpha$ sufficiently close to 1. By induction hypothesis, for $\alpha$ sufficiently close to 1, we can find energy-minimizing maps $f_i: S^2 \to M$, $1 \leqq i \leqq m-1$, such that the sum of $f_i$ $(1 \leqq i \leqq m-1)$ is homotopic to $f_\alpha^1$ and $E([f_\alpha^1]) = \sum_{i=1}^{m-1} E(f_i)$.

Set $f_m = \tilde{g}$, then

$$\begin{aligned} E([f]) &= \lim_{\alpha \to 1} E(f_\alpha) = \lim_{\alpha \to 1} E(f_\alpha^1) + E(\tilde{g}) \\ &\geqq E([f_\alpha^1]) + E(\tilde{g}) \\ &= \sum_{i=1}^{m} E(f_i). \end{aligned}$$

On the other hand, from the definition of $E([f])$, it is clear that

$$E([f]) \leqq \sum_{i=1}^{m} E(f_i).$$

Hence $E([f]) = \sum_{i=1}^{m} E(f_i)$. It follows from the definition of $E([f])$ that $f_m$ is energy-minimizing. Q.E.D.

## § 5. Holomorphic Deformation of Rational Curves

Let $M$ be a compact complex manifold of complex dimension $m$ whose tangent bundle $T_M$ is positive in the sense of Griffiths [10]. Let $C_0$ be a rational curve in $M$, possibly with singularities, that is, $C_0$ is the image of a holomorphic map $f_0\colon \mathbb{P}_1 \to M$, which is the normalization of $C_0$.

Let $V \subset \mathbb{P}_1 \times M$ be the graph of $f_0$. Let

$$\pi\colon \mathbb{P}_1 \times M \to \mathbb{P}_1,$$

$$\sigma\colon \mathbb{P}_1 \times M \to M$$

be the natural projections. Since $T_{\mathbb{P}_1 \times M}$ is isomorphic to $\pi^* T_{\mathbb{P}_1} \oplus \sigma^* T_M$ and $T_V$ is isomorphic to $\pi^* T_{\mathbb{P}_1}$, it follows that $\sigma^* T_M | V$ is isomorphic to the normal bundle $N_V$ of $V$ in $\mathbb{P}_1 \times M$, that is, $f_0^* T_M$ is isomorphic to $N_V$.

Let $\tilde{D}$ be the moduli space when $V$ is deformed as a subspace of $\mathbb{P}_1 \times M$ (see [3]). Let $D$ be the irreducible component of $\tilde{D}$ which contains the point $x_0$ of $\tilde{D}$ corresponding to $V$. The infinitesimal deformation of $V$ is given by $\Gamma(V, N_V)$. We claim that $H^1(V, N_V)$ vanishes so that all infinitesimal deformations are realized as actual deformations and $x_0$ is a regular point of $\tilde{D}$ of dimension equal to the dimension of $\Gamma(V, N_V)$.

Since $T_M$ is positive, the $\mu^{\text{th}}$ symmetric tensor product $T_M^{(\mu)}$ of $T_M$ is ample in the sense of Griffiths [10] for $\mu \geqq$ some positive integer $\mu_0$. Take $v \in V$ such that at $y = \pi(v)$ the map $f_0$ is an immersion. Then for $\mu \geqq \mu_0$, we have an exact sequence

$$0 \to F \to \Gamma(V, N_V^{(\mu)}) \to (N_V^{(\mu)})_v \to 0$$

so that the natural map $F \to (N_V^{(\mu)})_v \otimes (T_V^*)_v$ is surjective. By a theorem of Grothendieck [11], $N_V$ splits into a sum of holomorphic line bundles $L_1, \ldots, L_m$ over $V$. Then for $\mu \geqq \mu_0$ and $1 \leqq i \leqq m$ we have an exact sequence

$$0 \to F_i \to \Gamma(V, L_i^\mu) \to (L_i^\mu)_v \to 0$$

so that the natural map $F_i \to (L_i^\mu)_v \otimes (T_V^*)_v$ is surjective. Hence each $L_i$ is a positive holomorphic line bundle over $V$. By the theorem of Riemann-Roch, $H^1(V, L_i) = 0$ for each $i$. It follows that $H^1(V, N_V) = 0$.

We have a complex subspace $\mathscr{C}$ of $D \times \mathbb{P}_1 \times M$ with the following property. Let

$$\alpha\colon \mathscr{C} \to D,$$

$$\beta\colon \mathscr{C} \to \mathbb{P}_1 \times M$$

be the natural projections. Then $\mathscr{C}$ is $\alpha$-flat and for every $x \in D$, $\beta$ maps $\alpha^{-1}(x)$ biholomorphically onto the complex subspace of $\mathbb{P}_1 \times M$ corresponding to the point $x$. In particular, $\beta$ maps $\alpha^{-1}(x_0)$ biholomorphically onto $V$.

Let $\mathscr{C}'$ be the subset of $\mathscr{C}$ consisting of all $w \in \mathscr{C}$ such that

i) the structure sheaf of $\mathscr{C}$ is reduced at $w$,
ii) $D$ is regular at $\alpha(w)$, and
iii) the map $\alpha$ is a submersion at $w$.

Clearly $\mathscr{C}-\mathscr{C}'$ is a subvariety of $\mathscr{C}$. The fiber $\alpha^{-1}(x_0)$ is contained in $\mathscr{C}'$. Let $D_1=\alpha(\mathscr{C}-\mathscr{C}')$. Then $D_1$ is a proper subvariety of $D$.

Let $\Omega_D^1$ (respectively $\Omega_{\mathscr{C}}^1$) be the sheaf of germs of holomorphic 1-forms on $D$ (respectively $\mathscr{C}$) in the sense that it is locally the sheaf of germs of holomorphic 1-forms on an ambient manifold modulo the defining functions and their differentials. Let $\mathscr{L}$ be the linear space over $\mathscr{C}$ associated to the coherent analytic sheaf $\Omega_{\mathscr{C}}^1/\alpha^*\Omega_D^1$ on $\mathscr{C}$, i.e., the sheaf of germs of holomorphic functions on $\mathscr{L}$ which are linear forms along the fibers of $\mathscr{L}$ is isomorphic to $\Omega_{\mathscr{C}}^1/\alpha^*\Omega_D^1$ (see [6]). When restricted to $\mathscr{C}'$, $\mathscr{L}$ is isomorphic to the vector bundle of tangent vectors along the fibers of $\mathscr{C}$. Let $\mathbb{P}(\mathscr{L})$ be obtained by replacing each fiber of $\mathscr{L}$ by the projective space of all complex lines, i.e. $\mathbb{P}(\mathscr{L})$ is the orbit space of $\mathscr{L}$ under the $\mathbb{C}^*$ action on the fibers. The projection $\sigma\beta\colon \mathscr{C}\to M$ induces a holomorphic map

$$(\sigma\beta)_*\colon \mathbb{P}(\mathscr{L})\to\mathbb{P}(T_M),$$

where $\mathbb{P}(T_M)$ is the projectivization of the tangent bundle $T_M$. We claim that $(\sigma\beta)_*$ is surjective.

Since $D$ is compact (see [8, 17]), it follows that $\mathscr{C}$ is compact and $\mathbb{P}(\mathscr{L})$ is compact. Hence the image of $(\sigma\beta)_*$ is a subvariety of $\mathbb{P}(T_M)$. To prove the surjectivity of $(\sigma\beta)_*$, it suffices to show that $(\sigma\beta)_*$ is open at some point. Take a regular point $y_0$ of $C_0$. Then it corresponds to a point $w_0$ in $\mathscr{C}$, i.e. $\alpha(w_0)=x_0$ and $(\sigma\beta)(w_0)=y_0$. Let $\tilde{y}_0\in\mathbb{P}_1$ be the point such that $f_0(\tilde{y}_0)=y_0$. Let $\zeta$ be a local coordinate of $\mathbb{P}_1$ at $\tilde{y}_0$ vanishing at $y_0$. Then $w_0$ and $\left(\frac{\partial}{\partial\zeta}\right)_{\tilde{y}_0}$ determine a point $\theta_0$ in $\mathbb{P}(\mathscr{L})$. We are going to show that $(\sigma\beta)_*$ is open at $\theta_0$.

Let $E$ be the divisor of the differential $df_0$ of $f_0$ and let $[E]$ be the line bundle over $\mathbb{P}_1$ associated to the divisor $E$. Then $T_{\mathbb{P}_1}\otimes[E]$ is a line subbundle of $f_0^*T_M$. Let $Q$ be the quotient bundle $f_0^*T_M/T_{\mathbb{P}_1}\otimes[E]$. By the theorem of Grothendieck [11], $Q$ splits into a direct sum of holomorphic line bundles $Q_2,\ldots,Q_m$ over $\mathbb{P}_1$. Since $f_0^*T_M$ is positive, each $Q_\nu$, $2\leqq\nu\leqq m$, is a positive line bundle over $\mathbb{P}_1$. Let $[\tilde{y}_0]$ be the line bundle over $\mathbb{P}_1$ associated to the divisor $\tilde{y}_0$. For $2\leqq\nu\leqq m$ there exists a holomorphic cross section $s_\nu$ of $Q_\nu\otimes[\tilde{y}_0]^{-1}$ over $\mathbb{P}_1$ which does not vanish at $\tilde{y}_0$. We can regard $s_\nu$ as a holomorphic cross section of $Q\otimes[\tilde{y}_0]^{-1}$. Consider the following exact sequence

$$\Gamma(\mathbb{P}_1,(f_0^*T_M)\otimes[\tilde{y}_0]^{-1})\to\Gamma(\mathbb{P}_1,Q\otimes[\tilde{y}_0]^{-1})\to H^1(\mathbb{P}_1,T_{\mathbb{P}_1}\otimes[E]\otimes[\tilde{y}_0]^{-1})$$

coming from the exact sequence

$$0\to T_{\mathbb{P}_1}\otimes[E]\otimes[\tilde{y}_0]^{-1}\to(f_0^*T_M)\otimes[\tilde{y}_0]^{-1}\to Q\otimes[\tilde{y}_0]^{-1}\to 0.$$

Since

$$H^1(\mathbb{P}_1,T_{\mathbb{P}_1}\otimes[E]\otimes[\tilde{y}_0]^{-1})=0$$

by the vanishing theorem of Kodaira, the cross sections $s_\nu$, $2\leqq\nu\leqq m$, can be lifted to holomorphic cross sections $\tilde{s}_\nu$ of $f_0^*T_M\otimes[\tilde{y}_0]^{-1}$ over $\mathbb{P}_1$.

Let $t$ be the holomorphic cross section of $[\tilde{y}_0]$ over $\mathbb{P}_1$ whose divisor is $\tilde{y}_0$. Let $u_\nu=\tilde{s}_\nu t$. Then $u_\nu$ is a holomorphic cross section of $f_0^*T_M$ over $\mathbb{P}_1$, $2\leqq\nu\leqq m$, and these sections have the property that $\frac{1}{\zeta} u_\nu$, $2 \leqq \nu \leqq m$, form a basis of $(f_0^* T_M / T_M)_{\tilde{y}_0}$.

For $1 \leqq \nu \leqq m-1$ there exists a holomorphic cross section $s_{m+\nu}$ of $Q_{\nu+1}$ over $\mathbb{P}_1$ which does not vanish at $\tilde{y}_0$. We can regard $s_{m+\nu}$, $1 \leqq \nu \leqq m-1$, as a holomorphic cross section of $Q$. Consider the following exact sequence

$$\Gamma(\mathbb{P}_1, f_0^* T_M) \to \Gamma(\mathbb{P}_1, Q) \to H^1(\mathbb{P}_1, T_{\mathbb{P}_1} \otimes [E])$$

coming from the exact sequence

$$0 \to T_{\mathbb{P}_1} \otimes [E] \to f_0^* T_M \to Q \to 0.$$

Since

$$H^1(\mathbb{P}_1, T_{\mathbb{P}_1} \otimes [E]) = 0$$

by the vanishing theorem of Kodaira, the sections $s_{m+\nu}$, $1 \leqq \nu \leqq m-1$, can be lifted up to a holomorphic cross section $u_{m+\nu}$, $1 \leqq \nu \leqq m-1$, of $f_0^* T_M$ over $\mathbb{P}_1$.

Each $u_\nu$, $2 \leqq \nu \leqq 2m-1$, defines a tangent vector $v_\nu$ of $D$ at $x_0$ and defines $\xi_\nu \in \Gamma(\alpha^{-1}(x_0), T_{\mathscr{C}})$ such that

i) $(d\alpha)(\xi_\nu) \equiv v_\nu$ at every point of $\alpha^{-1}(x_0)$, and
ii) $(d(\sigma\beta))(\xi_\nu)$ at $w \in \alpha^{-1}(x_0)$ equals $u_\nu$ at $\pi\beta(w)$.

Choose a local submanifold $R$ of $D$ at $x_0$ such that the tangent space of $R$ at $x_0$ is spanned by $v_2, \ldots, v_{2m-1}$. At $w_0$, we choose a local coordinate system $t_1, \ldots, t_{2n-1}$ of $\alpha^{-1}(R)$ such that

i) $t_\nu$ is of the form $t'_\nu \circ \alpha$ for $2 \leqq \nu \leqq 2m-1$,
ii) $t_1(w_0) = \ldots = t_{2m-1}(w_0) = 0$,
iii) $\frac{\partial}{\partial t_\nu} = \xi_\nu$ $(2 \leqq \nu \leqq 2m-1)$ at $\alpha^{-1}(x_0)$, and
iv) $t_1 = \zeta \circ f_0^{-1} \circ (\sigma\beta)$ on $\alpha^{-1}(x_0)$.

Choose a local coordinate system $z_1, \ldots, z_m$ of $M$ at $y_0$ such that

i) $C_0$ is defined by $z_2 = \ldots = z_m = 0$, and
ii) $z_1 = \zeta \circ f_0^{-1}$ on $C_0$.

Let

$$u_\nu = \sum_{\mu=1}^{m} u_{\nu\mu} \frac{\partial}{\partial z_\mu} \qquad (2 \leqq \nu \leqq m-1)$$

where $u_{\nu\mu}$ is a holomorphic function on $\mathbb{P}_1$ near $\tilde{y}_0$. Since $\frac{1}{\zeta} u_\nu$, $2 \leqq \nu \leqq m$, form a basis of $(f_0^* T_M / T_M)_{\tilde{y}_0}$, it follows that the matrix

$$\left(\frac{1}{\zeta} u_{\nu\mu}\right)_{2 \leqq \nu, \mu \leqq m}$$

is nonsingular at $\tilde{y}_0$. Let

$$a_{\nu\mu} = \left(\frac{1}{\zeta} u_{\nu\mu}\right) \circ f_0^{-1}$$

on $C_0$ near $y_0$. Then on $C_0$ near $y_0$ we have

$$(d(\sigma\beta))\frac{\partial}{\partial t_1}=\frac{\partial}{\partial z_1},$$

$$(d(\sigma\beta))\frac{\partial}{\partial t_\nu}=z_1\sum_{\mu=1}^{m}a_{\nu\mu}\frac{\partial}{\partial z_\mu}\qquad(2\leqq\nu\leqq m).$$

Let

$$(d(\sigma\beta))\frac{\partial}{\partial t_\nu}=\sum_{\mu=1}^{m}b_{\nu\mu}\frac{\partial}{\partial z_\mu}\qquad(m<\nu<2m).$$

Since $u_\nu$, $m<\nu<2m$, form a basis of $(f_0^* T_M/T_M)_{\tilde{y}_0}$, it follows that the matrix

$$(b_{\nu\mu})_{m<\nu<2m,\,2\leqq\mu\leqq m}$$

is nonsingular at $y_0$.

Now we calculate the Jacobian matrix of the map $(\sigma\beta)_*$, restricted to $\mathbb{P}(\mathscr{L})|R$, with respect to local coordinate systems we are going to describe and verify that the Jacobian matrix has rank $2m-1$ over $\mathbb{C}$ at $\theta_0$.

Since $\alpha^{-1}(x)$ is of complex dimension 1 for every $x\in D$, for $w\in\mathscr{C}'$ the fiber of $\mathbb{P}(\mathscr{L})$ at $w$ consists only of a single point. Hence we can identify $\mathbb{P}(\mathscr{L})|\mathscr{C}'$ with $\mathscr{C}'$ and use $t_1,\ldots,t_{2m-1}$ as local coordinates at $w_0$ for $\mathbb{P}(\mathscr{L})$ (after identification of $\mathbb{P}(\mathscr{L})|\mathscr{C}'$ with $\mathscr{C}'$).

Every element $\eta$ of $T_M$ can be written as

$$\sum_{\mu=1}^{m}\eta_\nu\frac{\partial}{\partial z_\nu}$$

and we use as local coordinates of $\mathbb{P}(T_M)$ near $\left(y_0,\frac{\partial}{\partial z_1}\right)$ the functions

$$z_1,\ldots,z_m,\frac{\eta_2}{\eta_1},\ldots,\frac{\eta_m}{\eta_1}.$$

The map $\sigma\beta$ makes $z_1,\ldots,z_m$ functions of $t_1,\ldots,t_{2m-1}$. The Jacobian matrix of $(\sigma\beta)_*$, restricted to $\mathbb{P}(\mathscr{L})|R$, with respect to the coordinate systems

$$t_1,\ldots,t_{2m-1},$$

and

$$z_1,\ldots,z_m,\frac{\eta_2}{\eta_1},\ldots,\frac{\eta_m}{\eta_1},$$

equals to the $(2m-1)\times(2m-1)$ matrix $(A,B)$, where

$$A=\left(\frac{\partial z_\mu}{\partial t_\nu}\right)_{1\leqq\nu\leqq 2m-1,\,1\leqq\mu\leqq m},$$

$$B=\left(\frac{\partial}{\partial t_\nu}\left(\frac{\frac{\partial z_\mu}{\partial t_1}}{\frac{\partial z_1}{\partial t_1}}\right)\right)_{1\leqq\nu\leqq 2m-1,\,2\leqq\mu\leqq m}$$

We have

$$\frac{\partial z_\mu}{\partial t_1}=\delta_{\mu 1} \qquad \text{(the Kronecker delta)} \quad \text{for } 1 \leqq \mu \leqq m,$$

$$\frac{\partial z_\mu}{\partial t_\nu}=t_1 a_{\nu\mu} \qquad \text{for } 2 \leqq \nu \leqq m,\ 1 \leqq \mu \leqq m,$$

$$\frac{\partial z_\mu}{\partial t_\nu}=b_{\nu\mu} \qquad \text{for } m<\nu<2m,\ 1 \leqq \mu \leqq m$$

on $\alpha^{-1}(x_0)$ near $w_0$. Now

$$\frac{\partial}{\partial t_\nu}\left(\frac{\dfrac{\partial z_\mu}{\partial t_1}}{\dfrac{\partial z_1}{\partial t_1}}\right)=\frac{\dfrac{\partial z_\mu}{\partial t_1}\dfrac{\partial z_\mu}{\partial t_\nu}}{\dfrac{\partial z_1}{\partial t_1}}-\frac{\dfrac{\partial}{\partial t_1}\dfrac{\partial z_1}{\partial t_\nu}}{\dfrac{\partial z_1}{\partial t_1}}\frac{\partial z_\mu}{\partial t_1}=a_{\nu\mu}$$

at $w_0$ for $2 \leqq \nu \leqq m$, $2 \leqq \mu \leqq m$. Since the matrices

$$(a_{\nu\mu})_{2 \leqq \nu, \mu \leqq m}$$

and

$$(b_{\nu\mu})_{m<\nu<2m,\, 2 \leqq \mu \leqq m}$$

are nonsingular at $y_0$, it follows that $(A, B)$ is nonsingular at $w_0$. This concludes the proof that $(\sigma\beta)_*$ is open at $\theta_0$.

Let $G$ be the subset of $\mathbb{P}(T_M)$ consisting of all points $\hat{y}$ of $\mathbb{P}(T_M)$ such that $(\sigma\beta)_*^{-1}(\hat{y})$ is entirely contained in the restriction of $\mathbb{P}(\mathscr{L})$ to $\mathscr{C}'-\alpha^{-1}(D_1)$. Let $Z=\mathbb{P}(T_M)-G$. Then $Z$ is a proper subvariety of $\mathbb{P}(T_M)$. We thus have proved the following.

**Proposition 3.** *Let $M$ be a compact complex manifold whose tangent bundle $T_M$ is positive. Let $C_0$ be a rational curve in $M$ and $f_0$: $\mathbb{P}_1 \to C_0$ be its normalization. Then there exists a proper subvariety $Z$ of $\mathbb{P}(T_M)$ with the following property. If $y \in M$ and $\xi \in (T_M)_y-0$ define an element of $\mathbb{P}(T_M)-Z$, then there exists a holomorphic map $f$: $\mathbb{P} \to M$ homotopic to $f_0$ (when $f_0$ is regarded as a map from $\mathbb{P}_1$ to $M$) such that $y$ is a regular point of $f(\mathbb{P}_1)$ and the tangent vector of $f(\mathbb{P}_1)$ at $y$ is a nonzero multiple of $\xi$.*

## § 6. Proof of the Main Theorem

Suppose $M$ is an $m$-dimensional compact Kähler manifold of positive bisectional curvature. Since the Ricci curvature of $M$ is positive, it follows from the theorem of Bonnet-Myers that the universal covering of $M$ is compact. Since $\mathbb{P}_m$ has no fixed-point-free automorphism, in order to prove the Main Theorem, by replacing $M$ by its universal covering, we can assume without loss of generality that $M$ is simply connected. It follows that $\pi_2(M)$ is isomorphic to $H_2(M, \mathbb{Z})$.

Since the holomorphic bisectional curvature of $M$ is positive, by the result of Bishop-Goldberg [2] (see also [9]) the second Betti number of $M$ is 1. From the universal coefficient theorem it follows that $H^2(M, \mathbb{Z}) \approx \mathbb{Z}$. There exists a positive holomorphic line bundle $F$ over $M$ whose first Chern class $c_1(F)$ is a generator of $H^2(M, \mathbb{Z})$. Let $g$ be a generator of the free part of $H_2(M, \mathbb{Z})$ such that the value of $c_1(F)$ at $g$ is 1. Let $f\colon \mathbb{P}_1 \to M$ be a smooth map so that the element in $\pi_2(M)$ defined by $f$ corresponds to $g$ in the isomorphism between $\pi_2(M)$ and $H_2(M, \mathbb{Z})$.

By Proposition 2, there exist energy-minimizing maps $f_i\colon \mathbb{P}_1 \to M$, $0 \leqq i \leqq k$, such that the sum of $f_i$ $(0 \leqq i \leqq k)$ is homotopic to $f$ and $E([f]) = \sum_{i=0}^{k} E(f_i)$.

By Proposition 1, each $f_i$ $(0 \leqq i \leqq k)$ is either holomorphic or conjugate holomorphic. So each $f_i(\mathbb{P}_1)$ $(0 \leqq i \leqq k)$ is a rational curve. Since $c_1(T_M)$ is a positive integral multiple of $c_1(F)$ and the value of $c_1(F)$ at $g$ is 1, it follows that at least one $f_i$ is holomorphic. If $k > 0$, then at least one $f_j$ is conjugate holomorphic. We distinguish now between two cases.

*Case 1.* $k = 0$.

We use the notations of § 5. The line bundle $T_M \otimes [E]$ is a subbundle of $f_0^* T_M$ and the quotient bundle $(f_0^* T_M)/(T_M \otimes [E])$ splits into a direct sum of line bundles $Q_2, \ldots, Q_m$. Each $Q_i$ $(2 \leqq i \leqq m)$ is a positive line bundle. It follows that

$$c_1(f_0^* T_M) = c_1(T_M) + c_1([E]) + \sum_{i=2}^{m} c_1(Q_i).$$

Hence $c_1(T_M)$ evaluated at $g$ is $\geqq n+1$. That is, $c_1(T_M) = \lambda c_1(F)$ for some integer $\lambda \geqq n+1$. By the result of Kobayashi-Ochiai [14], $M$ is biholomorphic to $\mathbb{P}_m$.

*Case 2.* $k > 0$.

Without loss of generality we can assume that $f_0$ is holomorphic and $f_1$ is conjugate holomorphic. From Proposition 3 we obtain a proper subvariety $Z$ of $\mathbb{P}(T_M)$ for the rational curve $f_0(\mathbb{P}_1)$ of $M$. Similarly we obtain a proper subvariety $Z'$ of $\mathbb{P}(T_M)$ for the rational curve $f_1(\mathbb{P}_1)$ of $M$. Take a point $y \in M$ and a nonzero tangent vector $\xi$ of $M$ at $y$ so that the point of $\mathbb{P}(T_M)$ defined by $y$ and $\xi$ does not belong to $Z \cup Z'$. The map $f_0$ (respectively $f_1$) is homotopic to a holomorphic map $f_0'$ (respectively conjugate holomorphic map $f_1'$) from $\mathbb{P}_1$ to $M$ such that $y$ is a regular point of $f_0'(\mathbb{P}_1)$ (respectively $f_1'(\mathbb{P}_1)$) whose tangent vector at $y$ is a multiple of $\xi$. Since for $v = 0, 1$ both $E(f_v)$ and $E(f_v')$ are equal to the absolute value of the Kähler class of $M$ at $[f_v(\mathbb{P}_1)]$, by replacing $f_v$ by $f_v'$ $(v = 0, 1)$ we can assume without loss of generality $f_v = f_v'$ $(v = 0, 1)$.

Choose a local coordinate system $z_1, \ldots, z_n$ at $y$ with $y$ as the origin such that $\xi = \frac{\partial}{\partial z_1}$. Choose a local coordinate system $\zeta$ of $\mathbb{P}_1$ so that both $f_0^{-1}(y)$ and $f_1^{-1}(y)$ correspond to $\zeta = 0$ and $f_v$ $(v = 0, 1)$ is of the form

$$z_1 = \zeta$$

$$z_\mu = f_{v\mu}(\zeta) \quad (2 \leqq \mu \leqq m)$$

near $\zeta = 0$.

For $\delta > 0$ let $\Delta_\delta$ be the closed disc in $\mathbb{C}$ of radius $\delta$ centered at 0. For $\delta$ sufficiently small, we remove $f_\nu(\Delta_\delta)$ from $f_\nu(\mathbb{P}_1)$ $(\nu = 1, 2)$ and replace these two discs by the surface $S_\delta$ defined by

$$z_1 = \delta e^{i\theta}$$

$$z_\mu = t f_0(\delta e^{i\theta}) + (1-t) f_1(\delta e^{i\theta})$$

for $0 \leqq \theta \leqq 2\pi$ and $0 \leqq t \leqq 1$. The surface

$$S_\delta \cup \Big( \bigcup_{\nu=1,2} f_\nu(\mathbb{P}_1 - \Delta_\delta) \Big)$$

oriented in the obvious way is the image of a map $\tilde{f}$ from $S^2$ to $M$ which is homotopic to the sum of $f_0$ and $f_1$. Moreover, for $\delta$ sufficiently small the area of this surface is strictly less than the combined area of $f_0(\mathbb{P}_1)$ and $f_1(\mathbb{P}_1)$. By smoothing out $\tilde{f}$ we obtain a smooth map $\hat{f}$ from $S^2$ to $M$ such that the area of $\hat{f}(S^2)$ is strictly less than the combined area of $f_0(\mathbb{P}_1)$ and $f_1(\mathbb{P}_1)$. We make $\hat{f}$ conformal by replacing it by its composite with a suitable diffeomorphism of $S^2$. Then $E(\hat{f}) < E(f_0) + E(f_1)$. The sum of $\hat{f}$ and $f_2, \ldots, f_k$ is homotopic to $f$ and yet

$$E(\hat{f}) + \sum_{i=2}^{k} E(f_i) < \sum_{i=0}^{k} E(f_i) = E([f]),$$

which is a contradiction. Hence $k=0$ and $M$ is biholomorphic to $\mathbb{P}_m$.

## References


1. Bers, L., Fohn, F., Schechter, M.: Partial Differential Equations. New York: Interscience Publishers 1964
2. Bishop, R.L., Goldberg, S.I.: On the other second cohomology group of a Kähler manifold of positive curvature. Proc. Amer. Math. Soc. **16**, 119–122 (1965)
3. Douady, A.: Le problème de modules pour les sous-espaces analytiques compacts d'un espace analytique donné. Ann. Inst. Fourier (Grenoble) **16**, 1–95 (1966)
4. Eells, J., Lemaire, L.: A report on harmonic maps. Bull. London Math. Soc. **10**, 1–68 (1978)
5. Eells, J., Sampson, J.H.: Harmonic mappings of Riemannian manifolds. Amer. J. Math. **86**, 109–160 (1964)
6. Fischer, G.: Lineare Faserräume und kohärente Modulgarben über komplexen Räumen. Archiv der Math. **18**, 609–617 (1967)
7. Frankel, T.: Manifolds with positive curvature. Pacific J. Math. **11**, 165–174 (1961)
8. Fujiki, A.: Closedeness of the Douady spaces of compact Kähler manifolds. Publ. Math. R.I.M.S. Kyoto Univ. **14**, 1–52 (1978)
9. Goldberg, S.I., Kobayashi, S.: Holomorphic bisectional curvature. J. Diff. Geom. **1**, 225–234 (1967)
10. Griffiths, P.: Hermitian differential geometry, Chern classes, and positive vector bundles. In: Global Analysis, (D.C. Spencer and S. Iyanaga, eds.) pp. 185–251. Princeton Univ. Press 1969
11. Grothendieck, A.: Sur la classification des fibrés holomorphes sur la sphère de Riemann. Amer. J. Math. **79**, 121–138 (1957)
12. Kobayashi, S., Ochiai, T.: On complex manifolds with positive tangent bundles. J. Math. Soc. Japan **22**, 499–525 (1970)
13. Kobayashi, S., Ochiai, T.: Compact homogeneous complex manifolds with positive tangent bundle. In: Differential Geometry, in honor of K. Yano, pp. 233–242, (S. Kobayashi, M. Obata, and T. Takahashi, eds.) pp. 233–242. Tokyo: Kinokuniya 1972
14. Kobayashi, S., Ochiai, T.: Three-dimensional compact Kähler manifolds with positive holomorphic bisectional curvature. J. Math. Soc. Japan **24**, 465–480 (1972)

15. Kobayashi, S., Ochiai, T.: Characterizations of complex projective spaces and hyperquadrics. J. Math. Kyoto Univ. **13**, 31–47 (1973)
16. Lichnerowicz, A.: Applications harmoniques et variétés kählériennes, Symp. Math. **3**, 341–402 (1970)
17. Lieberman, D.: Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds. Séminaire F. Norguet pp. 140–186, 1976
18. Mabuchi, T.: $\mathbb{C}$-actions and algebraic threefolds with ample tangent bundle. Nagoya Math. J. **69**, 33–64 (1978)
19. Meeks, W., Yau, S.-T.: Topology of three dimensional manifolds and the embedding problems in minimal surface theory
20. Mori, S.: Projective manifolds with ample tangent bundles. Ann. of Math. **110**, 593–606 (1979)
21. Newlander, A., Nirenberg, L.: Complex-analytic coordinates in almost complex manifolds. Ann. of Math. **65**, 391–404 (1957)
22. Sacks, J., Uhlenbeck, K.: The existence of minimal immersion of 2-spheres
23. Schoen, R., Yau, S.-T.: On univalent harmonic maps between surfaces. Invent. Math. **44**, 265–278 (1978)
24. Siu, Y.-T.: The complex-analyticity of harmonicc maps and the strong rigidity of compact Kähler manifolds. Ann. of Math. in press (1980)
25. Toledo, D.: On the Schwarz lemma for harmonic maps and characteristic numbers of flat bundles
26. Wood, J.C.: Holomorphicity of certain harmonic maps from a surface to complex projective $n$-space

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