ON THE UPPER ESTIMATE OF THE HEAT KERNEL OF A COMPLETE RIEMANNIAN MANIFOLD

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Let M be a complete non-compact Riemannian manifold whose sectional curvature is bounded between two constants $-k$ and K. Then one expects that the heat diffusion in such a manifold behaves like the heat diffusion in Euclidean space. The purpose of this paper is to give a justification of such a statement.

In [5], J. Cheeger and the third author found a lower estimate of the heat kernel by "comparing" it with the heat kernel of the space form whose curvature is the lower bound of the curvature of the manifold. This lower estimate is sharp if we insist the dependence should be on the lower bound of the Ricci curvature alone. It remains to give an upper estimate of the heat kernel.

One does not, however, expect to have a comparison theorem for the upper bound because it is more sensitive to the geometry of the manifold. In fact, the heat kernel of the upper half space and the heat kernel of the complete hyperbolic manifold with finite volume have quite different behavior. This is reflected by the fact that the Laplacian has no discrete spectrum in the first case while infinite number of discrete eigenvalues may exist in the latter case.

What we will prove here is that in any case, the heat kernel has to decay in a manner similar to the Euclidean heat kernel. Thus we will prove that for any constant $C > 4$, there exists C_1 depending on C, T, the bound of the curvature of M and x so that for all $t \in [0, T]$

$$H(x, y, t) \leq C_1(C, T, x)t^{-n/2} \exp\left(-\frac{r^2}{Ct}\right)$$

where n is the dimension of M and r is the distance between x and y.

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This inequality shows that heat diffusion decays very fast when $t \neq 0$ and r tends to infinity. The constant C_1 accounts for the spectral difference between the upper half space and the hyperbolic space with finite volume. In our estimate, it arises from the fact that the injectivity radius decays exponentially for the latter manifold. In fact, we spend a substantial part of the paper proving that for complete manifolds with bounded curvature, injectivity radius decays at most exponentially. Then we derive suitable Sobolev inequality for this class of manifolds. We feel that this has interest in its own right. Based on the Sobolev inequality, we make use of a method of Aronson-Serrin [1], Moser [10] and Nash [11] to estimate the heat kernel. We also estimate the higher derivatives of the heat kernel in a similar manner.

We wish to express our gratitude to Professor A. Borel who brought the problem to us. He also mentioned that R. Beals has already proved the estimate holds for the heat kernel of G/Γ where G is semi-simple Lie group and Γ is a discrete subgroup. Finally, we point out as a corollary of our estimate here and the result of [4], we can derive that

$$\lim_{t \to 0} 4t \log H(x, y, t) = -r^2.$$

This was derived by Varadhan in [13] by different methods.

We also want to thank Professor J. Cheeger for discussions on the use of the triangle inequality in estimating the injectivity radius.

1. Injectivity radius estimate for non-compact manifolds. The injectivity radius $\delta(x)$ at a point $x \in M$ is defined to be the distance from x to its cut locus. In the event when we consider the injectivity radius of a point on a curve $\gamma(t)$ parametrized by t, we will write

$$\delta(t) = \delta(\gamma(t)). \tag{1.1}$$

The distance between any two points $x, y \in M$ will be denoted by $r(x, y)$. When ambiguity does not arise, the distance function from a fixed point p to x will be written as $r(x)$. It is known that $r(x)$ is a Lipschitz function defined on M.

In order to study the analytic aspect of a non-compact manifold, it is important to know how the geometry behaves at infinity. One of the most important quantities which reflects greatly on the analytic properties of M

is the concept of injectivity radius. This section is devoted to study the decay rate of $\delta(x)$ on a manifold with bounded curvature. We recall our notation that K and $-k$, with K, $k \geq 0$, are the upper and lower bounds of the sectional curvature of M.

THEOREM 1. *Let γ be a minimal geodesic joining two points, p and x, in M. Suppose γ is parametrized by arclength such that*

$$\gamma(0) = p \quad \textit{and} \quad \gamma(d) = x,$$

where $d = r(p, x)$. If $\delta(t) < \frac{1}{4}A = \pi/12\sqrt{K}$, for all $t \in [0, d]$, then for any $0 < T < d$,

$$\delta(d - T) \geq \delta(0) \exp\left[-\left(\sqrt{k} + \frac{C}{T}\right)(d - T)\right]$$

where $C > 0$ is some universal constant.

Proof. One verifies easily that the concept of injectivity radius $\delta(t)$ is continuously defined on $\gamma(t)$. Moreover by a standard argument involving the first variation formula for arclength one concludes that there exists a smooth geodesic loop $\eta_t(s)$ with vertex at $\gamma(t)$ which realizes

$$\begin{aligned} l(\eta_t) &= \text{length of } \eta_t \\ &= 2\delta(t). \end{aligned} \tag{1.2}$$

We now claim that there exists at most two values t_1, $t_2 \in [0, d]$ such that

$$\frac{d}{ds}\eta_t(0) = \pm\gamma'(t), \tag{1.3}$$

where $\eta_t(s)$ is assumed to be parametrized by arclength with

$$\eta_t(0) = \gamma(t) = \eta_t(2\delta(t)).$$

Indeed, if (1.3) holds, then by uniqueness of geodesic, $\eta_t(s)$ must be part of $\gamma(t)$. On the other hand, since $\eta_t(s)$ is a geodesic loop, and $\gamma(t)$ is minimal, this can only happen if the loop η_t is located at one of the ends of γ. However γ has only two ends; this establishes the claim.

We will proceed to estimate the left derivative of $\delta(t)$ at points not equal to t_1 or t_2. Let t_0 be such a point. We consider B, the geodesic ball centered at $\gamma(t_0)$ with radius $2A$. The exponential map

$$\pi : T_{\gamma(t_0)}M \cap B(2A) = TB \to B$$

is a covering map because $A = \pi/3\sqrt{K}$ implies B lies completely within the conjugate locus of $\gamma(t_0)$. The lifted geodesic segments $\tilde{\eta}_{t_0}$ of η_{t_0} is a straight line joining the origin $0 \in TB$ to a point

$$\bar{y} = \tilde{\eta}_{t_0}(2\delta(t)) \in \pi^{-1}(\eta_{t_0}(2\delta(t))).$$

We denote the geodesics γ_0 and γ_1 to be the lifts of γ in TB which contain 0 and $\bar{y}$ respectively. By the assumption that $t_0 \neq t_i$ for $i = 1, 2$, and the fact that γ_0 is just a straight line passing through 0, $\gamma_1 \neq \gamma_0$. We will parametrize γ_0 and γ_1 by t, the induced parameter from γ.

We claim that for all $t \in [t_0 - A/8, t_0]$, there exists a unique minimal geodesic $\tilde{\eta}_t(s)$ in TB joining $\gamma_0(t)$ to $\gamma_1(t)$. The proof of this claim will be postponed until after we complete this theorem and will be presented as Lemma 1. This asserts that in the interval $[t_0 - A/8, t_0]$, there exists a smooth 1-parameter family of geodesics $\tilde{\eta}_t$ joining $\gamma_0(t)$ to $\gamma_1(t)$. In fact, this family defines a Jacobi field $V(s)$ along $\tilde{\eta}_{t_0}(s)$.

Let us consider the first variation formula for arclength at t_0 with respect to this 1-parameter family $\tilde{\eta}_t$. Computation shows that

$$\frac{d}{dt} l(\tilde{\eta}_t)|_{t=t_0} = \left\langle V, \frac{d\tilde{\eta}_t}{ds} \right\rangle \Bigg|_{s=0}^{s=l(\tilde{\eta}_t)}. \tag{1.4}$$

However, since $\pi(\tilde{\eta}_t)$ are geodesic loops with vertices at $\gamma(t)$,

$$l(\tilde{\eta}_t) \geq 2\delta(t). \tag{1.5}$$

Therefore,

$$2\frac{\delta(t) - \delta(t_0)}{t - t_0} \geq \frac{l(\tilde{\eta}_t) - l(\tilde{\eta}_{t_0})}{t - t_0} \tag{1.6}$$

for $t_0 - \epsilon \leq t \leq t_0$. This implies that the left derivative of $\delta(t)$ at t_0 satisfies

$$\lim_{t \to t_0} \frac{\delta(t) - \delta(t_0)}{t - t_0} \geq \frac{1}{2} \left\langle V, \frac{d\eta_{t_0}}{ds} \right\rangle \Bigg|_{s=0}^{s=l(\tilde{\eta}_{t_0})} . \tag{1.7}$$

By definition of V, $|V(0)| = 1 = |V(l(\tilde{\eta}_{t_0}))|$, and also $|d\eta_{t_0}/ds| = 1$, therefore the right hand side of (1.7) gives

$$-\frac{1}{2}(\cos \alpha + \cos \beta) \tag{1.8}$$

where α and β are angles between

$$\frac{d}{ds}(\pi \circ \eta_{t_0})|_{s=0}, \qquad \gamma'(t_0) \quad \text{and} \quad -\frac{d}{ds}(\pi \circ \eta_{t_0})|_{s=l(\eta_{t_0})}, \qquad \gamma'(t_0)$$

respectively. To estimate (1.8), we apply Toponogov's comparison theorem.

Consider the geodesic triangle with sides η_{t_0}, $\gamma[t_0, t_0 + T]$ and $\gamma[t_0 + T, t_0]$ in M. This triangle has two identical sides with length T. Also, since $l(\eta_0) = 2\delta(t_0) < \pi/\sqrt{K}$, the comparison theorem applies. Let ABC be the geodesic triangle in $H^2(k)$ = the hyperbolic disk of constant curvature $-k$, such that $\overline{BC}$ and $\overline{AC}$ have length T and $\overline{AB}$ has length $2\delta(t_0)$. If $\bar{\alpha}$ and $\bar{\beta}$ are the angles BAC and ABC respectively, then Toponogov's comparison theorem (see [3]) asserts that

$$\alpha \geq \bar{\alpha} \quad \text{and} \quad \beta \geq \bar{\beta}.$$

Hence

$$-\cos \alpha \geq -\cos \bar{\alpha} \tag{1.9}$$

and

$$-\cos \beta \geq -\cos \bar{\beta}.$$

On the other hand, in $H^2(k)$, one has the formula

$$
\begin{aligned}
-\cos\bar{\alpha} &= -\cos\bar{\beta} \\
&= \frac{\cosh(\sqrt{k}T)}{\sinh(\sqrt{k}T)} \cdot \frac{(1 - \cosh(2\sqrt{k}\,\delta(t_0)))}{\sinh(\sqrt{k}\,\delta(t_0))} \\
&\geq -\frac{\cosh(\sqrt{k}T)}{\sinh(\sqrt{k}T)}\sqrt{k}\,\delta(t_0) \\
&\geq -\left(1 + \frac{C}{\sqrt{k}T}\right)\sqrt{k}\,\delta(t_0)
\end{aligned}
\tag{1.10}
$$

for some universal constant $C > 0$. Therefore we conclude that the left derivative of $\delta(t)$ at t_0 has a lower bound. In fact, for any $\epsilon > 0$, then for small enough Δt,

$$
\frac{\delta(t_0) - \delta(t_0 - \Delta t)}{\Delta t} \geq -\left(\sqrt{k} + \frac{C}{T}\right)\delta(t_0) - \epsilon \tag{1.11}
$$

for all $t_0 \in [0, d - T]$.

We are now ready to estimate $\delta(d - T)$ in terms of $\delta(0)$. Let us first assume that $\delta(t)$ is a decreasing function of t. Since we do not know a priori that δ is Lipschitz, we have to mollify δ as follows: For any $0 < h < T$, we define the mollifier

$$
\rho_h(s) = \frac{\bar{\rho}_h(s)}{\int_0^d \bar{\rho}_h(s)\,ds} \tag{1.12}
$$

where $\bar{\rho}_h(s)$ is a smooth cut-off function satisfying

$$
\bar{\rho}_h(s) = \begin{cases} 1 & \text{if} \quad -\dfrac{h}{2} < s < \dfrac{h}{2} \\ 0 & \text{if} \quad |s| > h \end{cases} \tag{1.13}
$$

and $\bar{\rho}_h(s) \leq 1$ for all s. The mollified function $\delta_h(t)$ will be defined by

$$
\delta_h(t) = \int_0^d \rho_h(t - s)\,\delta(s)\,ds \tag{1.14}
$$

for $h \leq t \leq d - T$. Clearly $\delta_h(t)$ is a C^1 function. Its difference quotient

$$\begin{aligned}
&\frac{\delta_h(t) - \delta_h(t - \Delta t)}{\Delta t} \\
&\quad = \frac{1}{\Delta t}\left[\int_0^d \rho_h(t - s)\delta(s)ds - \int_0^d \rho_h(t - s - \Delta t)\delta(s)ds\right] \\
&\quad = \frac{1}{\Delta t}\left[\int_0^d \rho_h(t - s)\delta(s)ds - \int_{\Delta t}^{d+\Delta t} \rho_h(t - s)\delta(s - \Delta t)ds\right] \\
&\quad = \int_0^d \rho_h(t - s)\frac{\delta(s) - \delta(s - \Delta t)}{\Delta t}ds \qquad (1.15)
\end{aligned}$$

for small $\Delta t < h$. By the assumption that δ is decreasing and equation (1.12), we obtain, as $\Delta t \to 0$,

$$\begin{aligned}
\delta_h{}'(t) &\geq -\int_0^d \rho_h(t - s)\left[\left(\sqrt{k} + \frac{C}{T}\right)\delta(s) + \epsilon\right]ds \\
&= -\left(\sqrt{k} + \frac{C}{T}\right)\delta_h(t) - \epsilon \qquad (1.16)
\end{aligned}$$

for all $h < t < d - T$. Since ϵ is arbitrary, this yields

$$[\log \delta_h(t)]' \geq -\left(\sqrt{k} + \frac{C}{T}\right). \qquad (1.17)$$

Integrating both sides with respect to t from h to $d - T$, we conclude

$$\delta_h(d - T) \geq \delta_h(h) \exp\left[-\left(\sqrt{k} + \frac{C}{T}\right)(d - T - h)\right].$$

Letting $h \to 0$, δ_h becomes δ, and the above inequality gives

$$\delta(d - T) \geq \delta(0) \exp\left[-\left(\sqrt{k} + \frac{C}{T}\right)(d - T)\right]. \qquad (1.18)$$

When δ is not a decreasing function, we simply break the interval

$[0, d - T]$ into subintervals $[t_{2i-1}, t_{2i}]$, $1 \leq i \leq n$, such that δ is decreasing in each and non-decreasing outside. We may then apply (1.19) to each subinterval and obtain

$$\delta(t_{2i}) \geq \delta(t_{2i-1}) \exp\left[-\left(\sqrt{k} + \frac{C}{T}\right)(t_{2i} - t_{2i-1})\right].$$

However, by assumption δ is non-decreasing outside, this means

$$\delta(t_{2i-1}) \geq \delta(t_{2i-2}).$$

Putting all these inequalities together, we derive

$$\delta(d - T) \geq \delta(0) \exp\left[-\left(\sqrt{k} + \frac{C}{T}\right)(d - T)\right] \tag{1.19}$$

as to be shown.

LEMMA 1. *Let $x \in M$ such that $\delta(x) \leq A/4 = \pi/12\sqrt{K}$. Suppose the exponential map $\pi: TB \to B$ is defined as in Theorem 1, and γ_0 and γ_1 are geodesics in TB from lifting a minimal geodesic γ which passes through x with $\gamma(0) = x$. If $\tilde{\eta}_0$ is the lift of η, the geodesic loop at x with $l(\eta) = 2\delta(x)$, such that $0 = \tilde{\eta}_0(0) \in \gamma_0$ and $\tilde{\eta}_0(2\delta(x)) \in \gamma_1$, then for $t \in [-A/8, 0]$, there exists unique minimal geodesics joining $\gamma_0(t)$ to $\gamma_1(t)$.*

Proof. Let us first define the notion of ϵ-homotopy. A curve C_0 with endpoints at x_1 and x_2 is said to be ϵ-homotopic to another curve C_1 with the same endpoints if there exists a homotopy $C(t, s)$ with endpoints x_1 and x_2, such that $C(0, s) = C_0$, $C(1, s) = C_1$, and $l(C(t, s)) \leq \epsilon$ for any $t \in [0, 1]$. Of course C_0 is said to be ϵ-homotopically trivial if $x_1 = x_2$, and C_0 is ϵ-homotopic to the trivial curve $C_1 = x$ with length $l(c_1) = 0$.

Under this notation, we claim that if $p \in M$ is a point such that $\delta(p) \leq 3A/4$, then the minimal geodesic loop $\overline{\eta}$ with vertex at p which realizes

$$l(\overline{\eta}) = 2\delta(p).$$

is $2A$-homotopically non-trivial. If not, by lifting the homotopy to T_pB by π_p, this provides a homotopy which is completely contained in T_pB

and it deforms $\pi_p^{-1}(\bar{\eta})$ to a point. However $\pi_p^{-1}(\bar{\eta})$ is a straight line segment passing through the origin. This is a contradiction. Hence $\bar{\eta}$ is a $2A$-homotopically non-trivial loop.

To prove the lemma, we first observe that by the triangle inequality

$$\begin{aligned} r(\gamma_0(t), \gamma_1(t)) &\leq r(\gamma_0(0), \gamma_1(t)) + |t| \\ &\leq l(\bar{\eta}_0) + 2|t| \\ &\leq l(\bar{\eta}_0) + \frac{A}{4}. \end{aligned} \tag{1.20}$$

for all $t \in [-A/8, 0]$. If there exists more than one minimal geodesics joining $\gamma_0(t)$ to $\gamma_1(t)$, this implies that the injectivity radius $\delta(\gamma_0(t))$ at $\gamma_0(t)$ in TB is less than or equal to

$$2\delta(0) + \frac{A}{4} \leq \frac{3A}{4}.$$

By the above discussion, we conclude that the geodesic loop σ with vertex at $\gamma_0(t)$ which realizes

$$l(\sigma) = 2\delta(\gamma_0(t))$$

is $2A$-homotopically non-trivial.

We now claim that the curve formed by the geodesic segments $\gamma_0[0, t]$, σ, and $\gamma_0[t, 0]$ denoted by θ is $7A/4$-homotopically non-trivial. If not, let $\theta(u, s)$ be a homotopy such that $\theta(0, s) = \theta(s)$ and $\theta(1, s) = 0$ with $l(\theta(u, s)) \leq 7A/4$ for all $u \in [0, 1]$. However if we combine the homotopy with the geodesic segments $\gamma_0[t, 0]$ and $\gamma_0[0, t]$, we obtain a homotopy which deforms σ to the curve $\gamma_0[t, 0] + \gamma[0, t]$. The latter certainly is $2A$-homotopically trivial. Hence together with the fact that

$$2|t| = l(\gamma_0[t, 0]) + l(\gamma_0[0, t])$$

and $\theta(u, s)$ is a $7A/4$ homotopy, this provides a $2A$-homotopy from σ to 0, which is a contradiction.

We can now minimize among all curves with endpoints at $\gamma_0(0)$ and are $7A/4$-homotopic to θ. This way we obtain a non-trivial shortest geodesic loop with vertex at $\gamma_0(0)$ and its length is less than or equal to $7A/4$. This contradicts the fact that the injectivity radius at $\gamma_0(0)$ is at least $2A$, and the lemma follows.

COROLLARY 1. *If p is a fixed point in M, and if γ is a minimal geodesic joining p to another point $x \in M$ such that $\gamma(0) = p$ and $\gamma(d) = x$. Then for any $T \in (0, d)$,*

$$\delta(d - T) \geq \bar{\delta}(p) \exp\left[-\left(\sqrt{K} + \frac{C}{T}\right)(d - T)\right]$$

where $\bar{\delta}(p) = \min\{\pi/12\sqrt{K}, \delta(p)\}$. In particular, if γ is a geodesic ray emanating from p to infinity, then

$$\delta(t) \geq \bar{\delta}(p) \exp[-(\sqrt{k}t)]$$

for all $t \in [0, \infty]$.

The proof of this corollary follows directly from Theorem 1 and will be omitted. In case if $K = 0$, i.e., M is non-positively curved, one obtains a more direct estimate for $\delta(x)$.

COROLLARY 2. *Let M be a complete non-positively curved Riemannian manifold with finite volume. Then there exists a compact set $N \subset M$, such that for all $x \in M - N$*

$$\delta(x) \geq \delta(N) \exp[-\sqrt{k}\tilde{r}(x, N)]$$

where $\delta(N) = \inf_{p \in N} \delta(p)$, and $\tilde{r}(x, N)$ is defined to be the maximum distance from x to N.

Proof. By a theorem of Siu and Yau [12], such a manifold contains a compact set N with the property that for all $x \in M - N$, there exists a unique geodesic ray emanating from N passing through x to infinity. They have only discussed the case when M is strongly negatively curved, however the proof can be carried over to our situation. Clearly the corollary follows by applying Corollary 1.

2. Sobolev inequalities. The Sobolev inequality for compactly supported functions defined on a ball $B(p, R)$ centered at a fixed point p with radius R asserts that

$$\int_{B(p,R)} |\nabla f| \geq C_0\left(\int_{B(p,R)} |f|^{n/(n-1)}\right)^{(n-1)/n} \tag{2.1}$$

for all functions $f \in H_{1,1}(B(p, R))$ such that $f|_{\partial B(p,R)} \equiv 0$, where the Sobolev constant C_0 depends only on the geometry of $B(p, R)$.

It is known that the Sobolev inequality is equivalent to the isoperimetric inequality

$$A(N) \geq C_0(V(M_1))^{(n-1)/n}, \tag{2.2}$$

where N is any codimension -1 submanifold in $B(p, R)$, and M_1 is the part in $B(p, R) - N$ which does not contain $\partial B(p, R)$. In fact, it was demonstrated [2] that the Sobolev constant is equal to the isoperimetric constant.

In [6], it was shown that

$$C_0 \geq \alpha(n)\omega^{(n+1)/n}(R) \tag{2.3}$$

for some universal constant $\alpha(n)$ which depends only on n. From here on, $\alpha(n)$ will denote a constant which depends on n alone though its specific value may vary in other situations. $\omega(R)$ in (2.3) is defined to be

$$\omega(R) = \inf_{x \in B(p,R)} \omega(x, R), \tag{2.4}$$

where

$$\begin{aligned}\omega(x, R) &= \text{measure of } U(x, R)\\ &= \text{measure of } \{v \in T_xM,\ |v| = 1 \mid \text{the geodesic from } x \text{ emanating in the direction of } v \text{ is minimal up to } \partial B(p, R)\}.\end{aligned}$$

The measure is the canonical $(n-1)$-measure on the unit tangent sphere S^{n-1}. The next three lemmas give estimates of $\omega(R)$ from below, hence combining with (2.3) enable us to obtain lower bounds for the Sobolev constant.

LEMMA 2. *Let M be a complete Riemannian manifold with sectional* *curvature bounded from above by K. If $R \le \bar\delta(p)/8$, where $\bar\delta(p) = \min\{\delta(p), \pi/12\sqrt{K}\}$, then*

$$\omega(R) \ge \alpha(n)$$

for some universal constant $\alpha(n)$.

Proof. By the definition of $\omega(R)$, it is clear that if $B(p, R)$ lies inside the cut locus of any point $x \in B(p, R)$, then $\omega(x, R) = A(S^{n-1}) = \omega(R)$. Hence it suffices to show that for all $x \in B(p, R)$, $\delta(x) \ge 2R$. However this follows directly from the proof of Lemma 1.

LEMMA 3. *Let M be a complete Riemannian manifold with bounded curvature. If $\omega(R) < \alpha(n)$, then*

$$\omega(R) \ge \alpha(n)k^{n/2}[\sqrt{k}R \sinh^{(n-1)}(4\sqrt{k}R]^{-1} \times \bar\delta^n(p)\exp[-2n\sqrt{k}R].$$

Here $-k$ is a lower bound for the sectional curvature.

Proof. Let q be an arbitrary point in M such that $r(p, q) = 2R$. We consider the ball $B(q, R)$ of radius R around q. For any $x \in B(p, R)$, we define the set

$$W = \{v \in T_xM,\ |v| = 1 \mid \text{there exists a point } y \in B(q, R) \text{ such that } v \text{ is the tangent vector at } x \text{ to the minimal geodesic joining } x \text{ and } y\}.$$

Clearly $B(p, R)$ and $B(q, R)$ are disjoint, and any geodesic with tangent vector in W must intersect $\partial B(p, R)$ before it connects with $y \in B(q, R)$, hence $W \subseteq U(x, R)$. The cone $C(W, 4R)$ is defined to be the set of points in M which are contained in a geodesic segment of length $4R$ from x with tangent vector in W. One verifies that $B(q, R) \subseteq C(W, 4R)$, therefore

$$V(B(q, R)) \le V(C(W, 4R)). \tag{2.5}$$

On the other hand

$$V(C(W, 4R)) = \int_0^{4R}\int_W \sqrt{g_x}(\theta, t)t^{n-1}d\theta\, dt$$

$$\leq \int_0^{4R} \int_W (\sqrt{k^{-1}} \sinh \sqrt{k}t)^{n-1} d\theta\, dt$$

$$\leq m(W)\alpha(n)Rk^{-((n-1)/2)} \sinh^{(n-1)}(4\sqrt{k}R), \qquad (2.6)$$

where $m(W)$ stands for the measure of W in ζ^m, the unit sphere in T_xM. Since $W \subseteq U(x, R)$, this implies,

$$\omega(x, R) \geq m(W)$$
$$\geq V(B(q, R))\alpha(n)k^{n/2}[\sqrt{k}R \sinh^{(n-1)}(4\sqrt{k}R)]^{-1}. \qquad (2.7)$$

It remains to estimate $V(B(q, R))$ from below. Since we assume $w(R) < \alpha(n)$, in view of Lemma 2, this implies $R > \bar{\delta}(p)/8$. If we define $R(q) = \min\{\delta(q), R\}$, obviously

$$V(B(q, R)) \geq \int_0^{R(q)} \int_{S^{n-1}} (\sqrt{K^{-1}} \sin(\sqrt{K}t))^{n-1} d\theta\, dt$$
$$= \alpha(n)K^{-n/2} \int_0^{\sqrt{K}R(q)} \sin^{(n-1)} t\, dt \qquad (2.8)$$
$$\geq \alpha(n)R^n(q).$$

To complete the proof of this lemma, we will need to estimate $R(q)$. Let γ be a minimal geodesic ray emanating from p to infinity. By completeness of M, such γ always exists. Since the choice of q is arbitrary on $\partial B(p, 2R)$, we may pick q to lie on γ. By Corollary 1, the injectivity radius of q satisfies

$$\delta(q) \geq \bar{\delta}(p) \exp[-2\sqrt{k}R]. \qquad (2.9)$$

Obviously, the fact that $R > \bar{\delta}(p)/8$ and (2.7), (2.8) and (2.9) imply the lemma.

LEMMA 4. *Let $p, z \in M$ such that $d = r(p, z)$. Suppose the hypothesis of Lemma 3 is satisfied with $\omega(R) < \alpha(n)$, then:*

(i) *When $0 < R \leq d/2$,*

$$\omega(R) \geq \alpha(n) \min\{1, k^{n/2}\delta^n(z)\} \exp[-3(n-1)\sqrt{k}d]$$

(ii) *When* $R > d/2$,

$$\omega(R) \geq \alpha(n)k^{n/2}[\sqrt{k}R \sinh^{(n-1)}(4\sqrt{k}R]^{-1}\bar{\delta}^n(z) \times \exp(-n\sqrt{k}(d + 2R)).$$

Proof. The proof of Lemma 3 can be carried through up to equation (2.7). We will now give an alternate way of estimating $V(B(q, R))$.

(i) When $0 < R \leq d/2$. By monotonicity of ω, it suffices to estimate $\omega(d/2)$. In this case we pick $q = z$. If $d/2 \leq \delta(q) = \delta(z)$ then by computation similar to (2.8)

$$V\left(B\left(z, \frac{d}{2}\right)\right) \geq \alpha(n)R^n. \tag{2.10}$$

Hence combining with (2.7), we get

$$\begin{aligned}\omega\left(\frac{d}{2}\right) &\geq \alpha(n)\left[\frac{\sqrt{k}d}{\sinh(2\sqrt{k}d)}\right]^{n-1} \\ &\geq \alpha(n)\exp(-2(n-1)\sqrt{k}d).\end{aligned} \tag{2.11}$$

On the other hand, if $d/2 > \delta(z)$, then by (2.8), we have

$$V\left(B\left(z, \frac{d}{2}\right)\right) \geq \alpha(n)\delta^n(z),$$

and (2.7) implies

$$\begin{aligned}\omega\left(\frac{d}{2}\right) &\geq \alpha(n)k^{n/2}[\sqrt{k}d \sinh^{(n-1)}(2\sqrt{k}d)]^{-1}\delta^n(z). \\ &\geq \alpha(n)k^{n/2}\delta^n(z)\exp(-3(n-1)\sqrt{k}d).\end{aligned} \tag{2.12}$$

(ii) When $R > d/2$. Since $r(p, q) = d$, $z \in B(p, 2R)$. Let γ be a geodesic ray emanating from z to infinity. We now pick q to be a point in $\gamma \cap \partial B(p, 2R)$. By Corollary 1, and the fact that

$$\delta(q) \geq \bar{\delta}(z)\exp[-\sqrt{k}(d + 2R)], \tag{2.13}$$

together with (2.7) and (2.8) yields

$$\omega(R) \geq \alpha(n)k^{n/2}[\sqrt{k}R \sinh^{(n-1)}(4\sqrt{k}R)]^{-1} \times \bar{\delta}^n(z) \exp[-n\sqrt{k}(d+2R)],$$

and the lemma is proved.

For all practical purposes, we will consider the following two weaker versions of the Sobolev inequality. Suppose f is any function with $f|_{\partial B(p,R)} \equiv 0$ and also $f \in H_{1,2}(B(p, R))$. Then

$$\int_{B(p,R)} |\nabla f|^2 \geq C(p, R)\left(\int_{B(p,R)} |f|^{2n/(n-2)}\right)^{(n-2)/n} \tag{2.14}$$

and

$$\int_{B(p,R)} |\nabla f|^2 \geq C(p, R)\left(\int_{B(p,R)} f^2\right)^{(n+2)/n}\left(\int_{B(p,R)} |f|\right)^{-4/n} \tag{2.15}$$

are valid. It was demonstrated in [5] and [7] that both Sobolev constants can be estimated below by $\alpha(n)C_0{}^2$, hence are denoted by $C(p, R)$. Applying (2.3) and Lemmas 2, 3, and 4, we have proved:

THEOREM 2. *Let M be a complete Riemannian manifold with bounded curvature. There exists a universal constant $\alpha(n)$, such that if $C(p, R) < \alpha(n)$, then*

(i) $C(p, R) \geq \alpha(n)k^{n+1}\bar{\delta}^{2(n+1)}(p)(\sqrt{k}R)^{-2(n+1)/n} \exp[-\alpha(n)\sqrt{k}R]$. *In case if $z \in M$ such that $d = r(p, z)$, then*

(ii) *for* $0 < R \leq d/2$,

$$C(p, R) \geq \alpha(n) \min\{1, k^{n+1}\delta^{2(n+1)}(z)\} \exp[-\alpha(n)\sqrt{k}d].$$

(iii) *for* $R > d/2$

$$C(p, R) \geq \alpha(n)k^{n+1}\bar{\delta}^{2(n+1)}(z)(\sqrt{k}R)^{-2(n+1)/n} \times \exp[-\alpha(n)\sqrt{k}(d+2R)].$$

Remark. The estimates in Lemmas 2 and 3 are all local, except in the proof of Lemma 3 where we pick a geodesic γ to be a ray emanating from p to infinity. One can modify this and apply Corollary 1 to the minimal geodesic segment of length $4R$ emanating from p. We then obtain the estimate

$$\delta(q) \geq \bar{\delta}(p) \exp\left[-\sqrt{k} + \frac{C}{2R}\right] 2R = \bar{\delta}(p) \exp[-(2\sqrt{k}R + C)]. \quad (2.16)$$

Now all the argument that we employed in Section 1 and Lemmas 2 and 3 can be restricted to the ball of radius $4R$ around p. Hence we obtain the local estimate:

COROLLARY 3. *Let M be a complete Riemannian manifold. Suppose $K(4R)$ and $-k(4R)$ are upper and lower bounds of the sectional curvature of $B(p, 4R)$. Then if $C(p, R) < \alpha(n)$,*

$$C(p, R) \geq \alpha(n) k^{n+1}(4R) \bar{\delta}^{2(n+1)}(p) (\sqrt{k(4R)}R)^{-2(n+1)/n} \times \exp[-\alpha(n)\sqrt{k(4R)}R].$$

In view of Theorem 2 in [5], we obtain estimates on the i^{th} eigenvalue for the Laplace operator defined on $B(p, R)$ with Dirichlet boundary condition.

COROLLARY 4. *Let M be a complete Riemannian manifold. Suppose $K(4R)$ and $-k(4R)$ are the upper and lower bounds of the sectional curvature of $B(p, 4R)$. If $\mu_i(R)$ denotes the i^{th} eigenvalue for the Laplace operator on $B(p, R)$ with Dirichlet boundary condition, then*

$$\mu_i(R) \geq \left[\frac{i}{V(B(p, R))}\right] \cdot \alpha(n) \min\{1, k^{n+1}(4R) \times \bar{\delta}^{2(n+1)}(p) \exp[-\alpha(n)\sqrt{k(4R)}R] \times (\sqrt{k(4R)}R)^{-2(n+1)/n}\}.$$

Remark. In [14], the third author gave estimates on the L^1-type Poincaré inequality for functions which are not compactly supported. Later it was indicated that this implied the standard L^2-type Poincaré inequality for functions which are not compactly supported (see [8]). It is then rather standard to see that this together with the estimate we obtain in the section imply an estimate for the Sobolev constant $\bar{C}(p, R)$, where

$$\int_{B(p,2R)} |\nabla f|^2 \geq \bar{C}(p, R)\left(\int_{B(p,R)} |f|^{2n/(n-2)}\right)^{(n-2)/n}$$

for all functions $f \in H_{1,2}(B(p, 2R))$ which satisfy $\int_{B(p,2R)} f = 0$.

3. Heat kernel estimates. In this section, we will demonstrate some applications of Theorem 2 to obtain estimates on the fundamental solution of the heat equation (heat kernel),

$$\square F(x, t) \equiv \left(\Delta - \frac{\partial}{\partial t}\right)F(x, t) = 0. \tag{3.1}$$

Standard properties of the heat kernel will be assumed.

THEOREM 3. *Let M be a complete Riemannian manifold. Suppose $H(x, y, t)$ is the heat kernel defined on $M \times M \times [0, \infty)$. Then*

$$\int_{M-B(p,R)} H^2(p, y, t)dy \leq \tilde{C}(p, \beta, T)t^{-n/2} \exp\left[\frac{-R^2}{2\beta t}\right]$$

for all $\beta > 1$, where

$$\tilde{C}(p, \beta, t) \leq \exp\left(\frac{1}{8(\beta - 1)}\right)\alpha(n)C^{-\theta(n)}\left(p, \sqrt{\frac{t}{4}}\right)$$

for some $\theta(n) \geq n/2$, and $C(p, \sqrt{t/4})$ is the Sobolev constant.

Proof. Let us define the function

$$F(y, s) = \int_{M-B(p,R)} H(p, \xi, t)H(y, \xi, s)d\xi. \tag{3.2}$$

By the fact that the heat kernel is a delta function at $t = 0$,

$$F(y, 0) = \begin{cases} 0 & \text{if } y \in B(p, R) \\ H(p, y, t) & \text{if } y \notin B(p, R). \end{cases} \tag{3.3}$$

We define the Lipschitz function

$$g(y, s) = \frac{-r^2(p, y)}{2(\beta t - s)}, \quad \text{for} \quad s < t. \tag{3.4}$$

for any fixed $\beta > 1$ and $t > 0$. One verifies easily that

$$\frac{1}{2}|\nabla g|^2 + g_s = 0 \tag{3.5}$$

almost everywhere. Consider the cut-off function

$$\varphi_\nu(y) = \begin{cases} 0 & \text{outside } B(p, \nu) \\ 1 & \text{on } B(p, \nu - 1) \end{cases} \tag{3.6}$$

where $0 \leq \varphi_\nu(y) \leq 1$ for all $y \in M$ and $|\nabla \varphi_\nu| \leq 2$, with $\nu \geq 2$.

Since $F(y, s)$ satisfies (3.1), one derives that

$$\begin{aligned} 0 &= \int_0^T \int_M \varphi_\nu{}^2 e^g F \square F \, dy ds \\ &= -\int_0^T \int_M \varphi_\nu{}^2 e^g |\nabla F|^2 - 2\int_0^T \int_M \varphi_\nu e^g F(\nabla \varphi_\nu, \nabla F) \\ &\quad -\int_0^T \int_M \varphi_\nu{}^2 e^g F(\nabla g, \nabla F) - \frac{1}{2}\int_0^T \int_M \varphi_\nu{}^2 e^g (F^2)_s. \end{aligned} \tag{3.7}$$

On the other hand,

$$\left|\int_0^T \int_M \varphi_\nu{}^2 e^g F(\nabla g, \nabla F)\right| \leq \int_0^T \int_M \varphi_\nu{}^2 e^g |\nabla F|^2 + \frac{1}{4}\int_0^T \int_M \varphi_\nu{}^2 e^g F^2 |\nabla g|^2,$$

and also integration by parts gives

$$\int_0^T \int_M \varphi_\nu{}^2 e^g (F^2)_s = \int_M \varphi_\nu{}^2 e^g F^2 |_{s=T} - \int_M \varphi_\nu{}^2 e^g F^2 |_{s=0} - \int_0^T \int_M \varphi_\nu{}^2 e^g g_s F^2.$$

Substituting these into (3.7) yield

$$\frac{1}{2}\int_M \varphi_\nu{}^2 e^g F^2|_{s=\tau} \le \frac{1}{2}\int_M \varphi_\nu{}^2 e^g F^2|_{s=0}$$

$$+\frac{1}{2}\int_0^\tau\int_M \varphi_\nu{}^2 e^g F^2\left(\frac{1}{2}|\nabla g|^2 + g_s\right)$$

$$-2\int_0^\tau\int_M \varphi_\nu e^g F(\nabla\varphi_\nu, \nabla F).$$

The second term on the right-hand side vanishes because (3.5), and the last term approaches to zero as $\nu \to \infty$, therefore

$$\int_{B(p,\sqrt{t/4})} e^{g(y,\tau)}F^2(y, \tau) \le \int_M e^{g(y,0)}F^2(y, 0) \tag{3.8}$$

for all $0 \le \tau \le t$. However when $y \in B(p, \sqrt{t/4})$ and $\tau \in [0, t]$, $g(y, \tau) \ge -1/8(\beta - 1)$, also when $y \notin B(p, R)$, $g(y, 0) \le -R^2/2\beta t$. Since $F(y, 0)$ satisfies (3.3), we have

$$\exp\left(-\frac{R^2}{2\beta t}\right)\|F(y, 0)\|_2{}^2$$

$$\ge \max_{(0,t)}\int_{B(p,\sqrt{t/4})} e^{g(y,\tau)}F^2(y, \tau)$$

$$\ge \exp\left(\frac{-1}{8(\beta-1)}\right)\max_{(0,t)}\int_{B(p,\sqrt{t/4})} F^2(y, \tau). \tag{3.9}$$

An iteration argument of Moser [10] (also see [1]) asserts that

$$|F(p, t)| \le \tilde{C}^{1/2}(p, t)t^{-(n+2)/4}\left(\int_0^t\int_{B(p,\sqrt{t/4})} F^2(y, s)\right)^{1/2}, \tag{3.10}$$

where $\tilde{C}(p, t) \le \alpha(n)C^{-\theta(n)}(p, \sqrt{t/4})$ with $C(p, \sqrt{t/4})$ being the Sobolev constant described in (2.14), and $\theta(n) \ge n/2$ is some universal constant. This together with (3.9) shows that

$$F^2(p, t) \le \tilde{C}(p, \beta, t)t^{n/2}\exp\left[\frac{-R^2}{2\beta t}\right]\|F(y, 0)\|_2{}^2. \tag{3.11}$$

But (3.2) and (3.3) imply

$$F(p, t) = \int_{M-B(p,R)} H^2(p, \xi, t)\,d\xi$$

$$= \|F(y, 0)\|_2^2,$$

which proves the theorem.

COROLLARY 5. *Let M be a complete Riemannian manifold (not necessarily with bounded curvature). Then*

$$\lim_{t\to 0} [-4t \log H(p, x, t)] = r^2(p, x)$$

for all $p, x \in M$.

Proof. By Theorem 3,

$$\int_{M-B(p,R)} H^2(p, y, t) \le \tilde{C}_{(p,\beta,t)} t^{-n/2} \exp\left[\frac{-R^2}{2\beta t}\right]. \tag{3.12}$$

To prove the corollary, clearly one only need to consider small enough t. In particular we may assume $t \le \min\{\bar{\delta}(p)/8, \bar{\delta}(x)/8\}, \epsilon^2$ where ϵ is to be chosen later. If $p = x$, the well known asymptotic formula for the heat kernel gives the corollary trivially. If $r(p, x) \ne 0$, we pick ϵ to be any small fixed constant between 0 and $r(p, x)$. Let $R = r(p, x) - \epsilon$, then by Lemma 2, (3.12), and the assumption on t

$$\int_{M-B(p,R)} H^2(p, y, t) \le \alpha(n) \exp\left(\frac{1}{8(\beta - 1)}\right) t^{-n/2} \tag{3.13}$$

$$\times \exp\left[\frac{-R^2}{2\beta t}\right].$$

On the other hand,

$$\int_{M-B(p,R)} H^2(p, y, t) \ge \int_{B(x,\sqrt{t}/4)} H^2(p, y, t) \tag{3.14}$$

because $t \leq \epsilon^2$. Applying Moser's iteration argument and Lemma 2 again, we have

$$\alpha(n)t^{-(n+2)/2}\int_0^t\int_{B(x,\sqrt{t}/4)} H^2(p, y, s) \geq H^2(p, x, t). \tag{3.15}$$

Combining (3.13) and (3.15), we obtain

$$H^2(p, x, t) \leq \alpha(n)t^{-(n+2)/2}\int_0^t s^{-n/2}\exp\left(\frac{-1}{8(\beta-1)}\right) \times \exp\left[\frac{-R^2}{2\beta s}\right] ds. \tag{3.16}$$

One can easily check that for any $\theta < 1$

$$\int_0^t s^{-n/2}\exp\left[\frac{-R^2}{2\beta s}\right] \leq C(\beta, \theta, R)\exp\left[\frac{-\theta R^2}{2\beta t}\right],$$

where the constant $C(\beta, \theta, R)$ only depends on β, θ, and R. Clearly

$$\begin{aligned}\lim_{t\to 0}\,[-4t\log H(p, x, t)] &= \lim_{t\to 0}\,[-2t\log H^2(p, x, t)] \\ &\geq \lim_{t\to 0} - 2t\Big[\log\alpha(n) + \log C(\beta, \theta, R) \\ &\qquad - \frac{n+2}{2}\log t - \frac{\theta R^2}{2\beta t}\Big] \\ &= \frac{\theta R^2}{\beta} = \frac{\theta(r(p, x) - \epsilon)^2}{\beta}. \end{aligned} \tag{3.17}$$

However, since β, ϵ, and θ are arbitrary constant with $\beta > 1$, $\epsilon > 0$ and $\theta < 1$, taking limits as $\beta \to 1$, $\epsilon \to 0$, and $\theta \to 1$, we obtain

$$\lim_{t\to 0}\,[-4t\log H(p, x, t)] \geq r^2(p, x). \tag{3.18}$$

In order to get an upper bound, we apply a theorem of Cheeger and Yau [4]. Let $d = 2r(p, x)$, we consider the heat kernel with Dirichlet boundary condition on $B(p, d)$, denoted by $H_d(y, z, t)$. Let $(n - 1)K$ be the lower bound of the Ricci curvature of $B(p, d)$, then according to the comparison theorem in [4],

$$H_d(p, y, t) \geq \bar{H}_{k,d}(r(p, y), t) \tag{3.19}$$

where $\bar{H}_{k,d}(r(p, y), t)$ is the heat kernel on a ball of radius d in the space form of constant curvature k, which satisfies the Dirichlet boundary condition. On the other hand, it is known that

$$H(p, y, t) \geq H_d(p, y, t), \tag{3.20}$$

therefore

$$\lim_{t\to 0} [-4t \log H(p, y, t)] \leq \lim_{t\to 0} [-4t \log \bar{H}_{k,d}(r(p, y), t)].$$

However, one readily checks that the upper bound is true for geodesic balls in constant space forms (see Appendix). This gives the desired upper bound and the proof of the corollary is completed.

The estimate in the proof of Corollary 5 is not quite sharp when t is large. In order to establish a good upper bound for $H(x, y, t)$ for all time t, we will employ another method. First we prove the following:

LEMMA 5. *Let M be a complete Riemannian manifold. For any point $p \in M$, and $t \in [0, T]$,*

$$H(p, p, t) \leq \alpha(n) C^{-\theta(n)}\left(p, \sqrt{\frac{T}{8}}\right) t^{-n/2}$$

where $\theta(n) \geq n/2$ is a constant depending only on n.

Proof. By the semi-group property of the heat kernel

$$\begin{aligned} H(p, p, t) &= \int_M H^2\left(p, y, \frac{t}{2}\right) dy \\ &= \int_{B(p,R)} H^2\left(p, y, \frac{t}{2}\right) dy + \int_{M-B(p,R)} H^2\left(p, y, \frac{t}{2}\right) dy \end{aligned}$$

$$\leq \int_{B(p,R)} H^2\left(p, y, \frac{t}{2}\right) + \tilde{C}\left(p, \frac{t}{2}\right) t^{-n/2} \exp\left[\frac{-R^2}{2t}\right] \quad (3.21)$$

where the last inequality follows from Theorem 3 by setting $\beta = 2$.

Let $\eta(y)$ be a cut-off function with the properties that

$$\eta(y) = \begin{cases} 1 & \text{on } B(p, R) \\ 0 & \text{outside } B(p, 2R), \end{cases}$$

and $0 \leq \eta(y) \leq 1$ for all $y \in M$. Also $|\nabla \eta|^2 \leq 2/R^2$. Consider

$$\frac{\partial}{\partial t} \int_{B(p,2R)} \eta^2(y) H^2\left(p, y, \frac{t}{2}\right)$$

$$= \int_{B(p,2R)} \eta^2(y) H\left(p, y, \frac{t}{2}\right) \Delta_y H\left(p, y, \frac{t}{2}\right). \quad (3.22)$$

This is due to the fact that $H(p, y, t/2)$ satisfies (3.1). Integration by parts a few times, we conclude that

$$\frac{\partial}{\partial t} \int_{B(p,2R)} \eta^2(y) H^2\left(p, y, \frac{t}{2}\right) = -\int_{B(p,2R)} \left| \nabla \left(\eta(y) H\left(p, y, \frac{t}{2}\right)\right)\right|^2 + \int_{B(p,2R)} H^2\left(p, y, \frac{t}{2}\right) |\nabla \eta(y)|^2. \quad (3.23)$$

Since $|\nabla \eta| \equiv 0$ on $B(p, R)$, and $|\nabla \eta|^2 \leq 2/R^2$ on $M - B(p, R)$, the second term on the right-hand side of (3.23) is dominated by

$$\frac{2}{R^2} \int_{B(p,2R)} H^2\left(p, y, \frac{t}{2}\right) \leq \frac{2}{R^2} \tilde{C}\left(p, \frac{t}{2}\right) t^{-n/2} \exp\left[\frac{-R^2}{2t}\right], \quad (3.24)$$

where we apply Theorem 3 again. By (2.15), and the fact that

$$\int_{B(p,2R)} \left| \eta(y) H\left(p, y, \frac{t}{2}\right)\right| \leq \int_M H\left(p, y, \frac{t}{2}\right) = 1,$$

we conclude that

$$\int_{B(p,2R)} \left| \nabla\left(\eta(y)H\left(p, y, \frac{t}{2}\right)\right)\right|^2$$

$$\geq C(p, 2R)\left(\int_{B(p,2R)} \eta^2(y)H^2\left(p, y, \frac{t}{2}\right)\right)^{(n+2)/n}. \quad (3.25)$$

Substituting (3.24), (3.25) into (3.23) gives

$$\frac{\partial}{\partial t}\int_{B(p,2R)} \eta^2(y)H^2\left(p, y, \frac{t}{2}\right)$$

$$\leq -C(p, 2R)\left(\int_{B(p,2R)} \eta^2(y)H^2\left(p, y, \frac{t}{2}\right)\right)^{(n+2)/n} \quad (3.26)$$

$$+\frac{2}{R^2}\tilde{C}\left(p, \frac{t}{2}\right)t^{-n/2}\exp\left[\frac{-R^2}{2t}\right].$$

By the monotonicity of the Sobolev constant and the fact that $x\exp(-1/x)$ is an increasing function, $h(t) = \int_{B(p,2R)} \eta^2(y)H^2(p, y, t/2)$ therefore satisfies the differential inequality

$$h'(t) \leq -C(p, 2R)h^{(n+2)/n}(t)$$

$$+\tilde{C}\left(p, \frac{T}{2}\right)\frac{2T}{R^2}\exp\left[\frac{-R^2}{2T}\right]t^{-(n+2)/n}. \quad (3.27)$$

Setting $2R = \sqrt{T/8}$, this becomes

$$h'(t) \leq -C\left(p, \sqrt{\frac{T}{8}}\right)h^{(n+2)/n}(t) + \tilde{C}\left(p, \frac{T}{2}\right)t^{-(n+2)/2}. \quad (3.28)$$

Now we claim that there exists a constant C_1 which satisfies

$$C_1 \leq \alpha(n)C^{-\theta(n)}\left(p, \sqrt{\frac{T}{8}}\right) \quad (3.29)$$

for some $\alpha(n)$ and $\theta(n) \geq n/2$, such that

$$h(t) \leq C_1 t^{-n/2} \tag{3.30}$$

for all $t \in [0, T]$. To see this, we consider

$$\frac{d}{dt}[C_1 t^{-n/2}] + c\left(p, \sqrt{\frac{T}{8}}\right)[C_1 t^{-n/2}]^{(n+2)/n} - \tilde{C}\left(p, \frac{T}{2}\right) t^{-(n+2)/2}$$

$$= \left[-\frac{n}{2}C_1 + c\left(p, \sqrt{\frac{T}{8}}\right) C_1^{(n+2)/n} - \tilde{C}\left(p, \frac{T}{2}\right)\right] t^{-(n+2)/2}. \tag{3.31}$$

By definition of $\tilde{C}(p, T/2)$, clearly C_1 can be chosen to satisfy (3.29) such that the right-hand side of (3.31) becomes positive. Now consider the function

$$f(t) = h(t) - C_1 t^{-n/2}. \tag{3.32}$$

By definition of

$$h(t) = \int_{B(p, 2R)} \eta^2(y) H^2\left(p, y, \frac{t}{2}\right) \leq \int_M H^2\left(p, y, \frac{t}{2}\right) = H(p, p, t),$$

and the asymptotic formula for $H(p, p, t)$ as $t \to 0$, C_1 can be chosen so $f(t)|_{t=0} < 0$. On the other hand,

$$f'(t) = h'(t) - [C_1 t^{-n/2}]'$$

$$< -c\left(p, \sqrt{\frac{T}{8}}\right)[h^{(n+2)/n}(t) - (C_1 t^{-n/2})^{(n+2)/n}] \tag{3.33}$$

because of (3.28) and (3.31). If inequality (3.30) is violated, let t_0 be the first $t > 0$ such that

$$h(t_0) = C_1 t_0^{-n/2},$$

then at t_0, (3.33) gives

$$f'(t_0) < 0.$$

However, this is impossible because $f(t_0) = 0$ and $f(t) < 0$ for all $0 \le t < t_0$. This proves (3.30). Putting (3.21), (3.29) and (3.30) together with the definition of $h(t)$, they imply the lemma.

The following theorem gives a global estimate for $H(p, x, t)$ for all time t.

THEOREM 4. *Let M be a complete Riemannian manifold with bounded curvature. Suppose $H(p, x, t)$ is the heat kernel for the heat equation* (3.1). *Then there exists a constant $C(n, k, T)$ depending only on n, k, and T, such that for all $t \in [0, T]$*

$$H(p, x, t) \le C(n, k, T)\tilde{\delta}^{-\alpha(n)}(p)t^{-n/2}\exp\left(-\frac{d^2}{16t}\right)$$

for some universal constant $\alpha(n) > 0$, where

$$\tilde{\delta}(p) = \min\left\{\frac{\pi}{12\sqrt{K}}\quad 1, \delta(p)\right\}, \textit{ and } d = r(p, x).$$

Proof. We will consider the following cases:

(i) When $d^2/8 \le t \le T$. We write

$$\begin{aligned} H(p, x, t) &= \int_M H\left(p, y, \frac{t}{2}\right)H\left(y, x, \frac{t}{2}\right) \\ &\le \left(\int_M H^2\left(p, y, \frac{t}{2}\right)\right)^{1/2}\left(\int_M H^2\left(x, y, \frac{t}{2}\right)\right)^{1/2}. \qquad (3.34) \end{aligned}$$

Now applying Lemma 5, we conclude

$$\begin{aligned} H(p, x, t) &\le H^{1/2}(p, p, t)H^{1/2}(x, x, t) \\ &\le \alpha(n)C^{-\theta(n)/2}\left(p, \sqrt{\frac{T}{8}}\right)C^{-\theta(n)/2}\left(x, \sqrt{\frac{T}{8}}\right)t^{-n/2} \\ &\le \alpha(n)C^{-\theta(n)/2}\left(p, \sqrt{\frac{T}{8}}\right)C^{-\theta(n)/2}\left(x, \sqrt{\frac{T}{8}}\right)t^{-n/2}\exp\left[\frac{-d^2}{16t}\right]. \end{aligned}$$

(3.35)

By Theorem 2,

$$C\left(p, \sqrt{\frac{T}{8}}\right) \geq C(n, k, T)\bar{\delta}^{2(n+1)}(p) \tag{3.36}$$

and

$$C\left(x, \sqrt{\frac{T}{8}}\right) \geq C(n, k, T)\bar{\delta}^{-2(n+1)}(p), \tag{3.37}$$

which completes the proof for this case.

(ii) When $0 < t < d^2/8$ and $t \leq T$. We follow the proof of Corollary 5, but here we have to keep track of the constants. Similar as before, by Theorem 3, we have

$$\int_{M-B(p,4d/5)} H^2(p, y, t)dy \leq \tilde{C}(p, \beta, t)t^{-n/2} \times \exp\left[\frac{-16d^2}{50\beta t}\right] \tag{3.38}$$

for all $\beta > 1$. Pick $\beta = 32/25$, (3.38) becomes

$$\int_{M-B(p,4d/5)} H^2(p, y, t) \leq \tilde{C}(p, t)t^{-n/2} \exp\left[\frac{-d^2}{4t}\right]. \tag{3.39}$$

However, applying Moser's iteration argument again, gives

$$\int_0^t \int_{B(x,\sqrt{t}/4)} H^2(p, y, t)dy \geq \tilde{C}^{-1}(x, t)t^{(n+2)/2}H^2(p, x, t). \tag{3.40}$$

Combining (3.39) and (3.40), we obtain

$$\begin{aligned} H^2(p, x, t) &\leq \tilde{C}(x, t)t^{-((n+2)/2)} \int_0^t \tilde{C}(p, s)s^{-n/2} \exp\left(\frac{-d^2}{4s}\right) ds \\ &\leq \tilde{C}(x, T)\tilde{C}(p, T)t^{-((n+2)/2)} \int_0^t s^{-n/2} \exp\left(\frac{-d^2}{4s}\right) ds \end{aligned} \tag{3.41}$$

where the last inequality follows from the mononicity of the Sobolev constant. One verifies directly that

$$\int_0^t s^{-n/2} \exp\left(\frac{-d^2}{4s}\right) ds \le \alpha(n)\left(\frac{d^2}{4}\right)^{-((n+2)/2)} \exp\left(\frac{-d^2}{4t}\right)$$

$$\le \alpha(n) t^{-((n+2)/2)} \exp\left(\frac{-d^2}{4t}\right), \tag{3.42}$$

since $t < d^2/8$. Applying Theorem 2, (3.41) and (3.42) imply that

$$H(p, x, t) \le C(n, k, T)\bar{\delta}^{-\alpha(n)}(p) t^{-n/2}$$

$$\times \exp\left[-\frac{d^2}{8t} + \alpha(n)\sqrt{k}d\right]. \tag{3.43}$$

However, we observe that

$$\exp\left[-\frac{d^2}{8t} + \alpha(n)\sqrt{k}d\right] \le \exp\left[-\frac{d^2}{8t} + \frac{\alpha(n)\sqrt{k}T^{1/2}d}{t^{1/2}}\right]. \tag{3.44}$$

Clearly, if

$$\frac{\alpha(n)\sqrt{k}T^{1/2}d}{t^{1/2}} > \frac{d^2}{16t},$$

then

$$16\alpha(n)\sqrt{k}T^{1/2} \ge \frac{d}{t^{1/2}},$$

hence

$$\exp\left[\frac{-d^2}{8t} + \frac{\alpha(n)\sqrt{k}T^{1/2}d}{t^{1/2}}\right] \le C(n, k, T) \exp\left[\frac{-d^2}{16t}\right]. \tag{3.45}$$

On the other hand, if

$$\frac{\alpha(n)\sqrt{k}T^{1/2}d}{t^{1/2}} \le \frac{d^2}{16t}$$

then

$$\exp\left[-\frac{d^2}{8t}+\frac{\alpha(n)\sqrt{k}T^{1/2}d}{t^{1/2}}\right] \le \exp\left[\frac{-d^2}{16t}\right]. \tag{3.46}$$

Therefore the theorem follows for both cases.

If we fixed another point $0 \in M$, we can apply part (ii) and (iii) of Theorem 2 to estimate $C(p, \sqrt{T/4})$ and $C(x, \sqrt{T/4})$ in terms of information at 0. In that case, the proof of Theorem 4 implies:

THEOREM 5. *Let M be a complete Riemannian manifold with bounded curvature. Let d_1, d_2, and d denote the distances $r(0, p)$, $r(0, x)$, and $r(p, x)$ respectively. Then*

$$H(p, x, t) \le C(n, k, T)\tilde{\delta}^{-\alpha(n)}(0)t^{-n/2}\exp\left(-\frac{d^2}{8t}+\alpha(n)\sqrt{k}(d_1+d_2)\right)$$

for all $t \in [0, T]$.

COROLLARY 6. *Let M be a complete Riemannian manifold with bounded curvature. Suppose $\varphi \in L^2(M)$ is an eigenfunction satisfying the equation*

$$\Delta\varphi = -\lambda\varphi$$

for some $\lambda > 0$. Then φ is of at most exponential growth.

More precisely, if $0 \in M$ is a fixed point and $d = r(0, x)$, then

$$|\varphi(x)| \le C(n, k, \tilde{\delta}(0), \lambda)\exp(\alpha(n)\sqrt{k}d)\|\varphi\|_2$$

where $C(n, k, \tilde{\delta}(0), \lambda)$ is a constant depending on the quantities prescribed. Moreover if $x \notin B(0, \sqrt{2/\lambda})$. The estimate takes the form

$$|\varphi(x)| \le C(n, k, \tilde{\delta}(0))\exp(\alpha(n)\sqrt{k}d)\lambda^{n/4}\|\varphi\|_2$$

where the constant $C(n, k, \tilde{\delta}(0))$ depends only on n, k, and $\tilde{\delta}(0)$ alone.

Proof. Since φ is an eigenfunction and $\varphi \in L^2(M)$, one verifies that

$$\square_x \int_M H(x, y, t)\varphi(y) = 0 \tag{3.47}$$

and also

$$\square_x e^{-\lambda t}\varphi(x) = 0. \tag{3.48}$$

By the fact that $H(x, y, t)$ is a delta function when $t \to 0$,

$$\lim_{t\to 0}\int_M H(x, y, t)\varphi(y) = \varphi(x) = \lim_{t\to 0} e^{-\lambda t}\varphi(x).$$

Hence by uniqueness of parabolic equation

$$\int_M H(x, y, t)\varphi(y) = e^{-\lambda t}\varphi(x). \tag{3.49}$$

However, the left-hand side of (3.49) satisfies

$$\left|\int H(x, y, t)\varphi(y)\right| \le \left(\int H^2(x, y, t)\right)^{1/2}\left(\int \varphi^2(y)\right)^{1/2}$$

$$= H^{1/2}(x, x, 2t)\|\varphi\|_2. \tag{3.50}$$

Now applying Lemma 5, we have

$$|\varphi(x)| \le \alpha(n)C^{-\theta(n)/2}\left(x, \sqrt{\frac{t}{4}}\right)t^{-n/4}e^{\lambda t}\|\varphi\|_2. \tag{3.51}$$

Clearly, the corollary follows if we set $t = 2/\lambda$ and apply Theorem 2. When $x \notin B(0, \sqrt{2/\lambda})$, using Theorem 2(ii) we obtain the estimate

$$|\varphi(x)| \le C(n, k, \bar{\delta}(0)) \exp(\alpha(n)\sqrt{k}d)\lambda^{n/4}\|\varphi\|_2 \tag{3.52}$$

as claimed.

Remark. Estimates similar to that of Theorem 4 can be obtained for compact manifolds with or without boundary. In that case Theorem 3 still holds, where we take r to be the length of the shortest curve contained in M joining two points. In fact the introduction of the cut off function φ_ν presented in the proof of Theorem 3 can be omitted. To finish the proof of Theorem 4 for the compact case, instead of using

Lemma 5, one simply applies a similar estimate obtained for compact manifolds in [5]. Since the estimates of Cheeger and Yau holds for compact manifolds without boundary. Corollary 5 is hence valid for the case also.

4. Higher order estimates. For general applications, sometimes it is essential to derive estimates for the derivatives of the heat kernel. In view of the upper bounds obtained for compact manifolds from the parametrix method, one would expect the derivatives of $H(x, y, t)$ to satisfy similar inequalities to that of $H(x, y, t)$, except now the magnitudes in time are different and the constants will depend on the curvature of M and its covariant derivatives. In fact, the purpose of this section is to demonstrate that all higher order estimates follow from the upper bound for $H(x, y, t)$ itself.

Since it is more convenient to set up the problem in an intrinsic manner, instead of deriving estimates for each partial derivative of $H(x, y, t)$, we will consider the totality of its derivatives, i.e., $|\nabla H|$, $|\text{hess}\, H|$, etc. The estimate for $|\nabla H|(x, y, t)$ can be obtained as follows:

LEMMA 6. *Suppose κ denotes the lower bound of the Ricci curvature of M, then the function $|\nabla_x H|(x, y, t)$ satisfies the differential inequality*

$$\Box_x |\nabla_x H|(x, y, t) - \kappa |\nabla_x H|(x, y, t) \geq 0$$

for all $x, y \in M$ and $t \in (0, \infty)$.

Proof. By the Böchner's formula

$$\begin{aligned} \Delta|\nabla H|^2 &= 2|\text{hess}\, H|^2 + 2(\nabla H, \nabla \Delta H) + 2\,\text{Ric}(\nabla H, \nabla H) \\ &\geq 2|\text{hess}\, H|^2 + (|\nabla H|^2)_t + 2\kappa|\nabla H|^2 \end{aligned} \tag{4.1}$$

where we have used that fact that H satisfies (3.1). On the other hand,

$$\Delta|\nabla H|^2 = 2|\nabla H|\Delta|\nabla H| + 2|\nabla|\nabla H||^2. \tag{4.2}$$

One checks easily that

$$|\text{hess}\, H|^2 \geq |\nabla|\nabla H||^2,$$

hence (4.1) and (4.2) imply

$$2|\nabla H|\Delta|\nabla H| \geq 2|\nabla H|(|\nabla H|)_t + 2\kappa|\nabla H|^2. \tag{4.3}$$

Therefore

$$\Delta|\nabla H| - |\nabla H|_t - \kappa|\nabla H| \geq 0$$

as asserted.

LEMMA 7. *Let M be a complete Riemannian manifold (not necessarily with bounded curvature). Then*

$$\left|\int_M H(x, y, t)\Delta_y{}^{\beta}H(x, y, t)dy\right| \leq t^{-\beta}C(\beta)\int H^2\left(x, y, \frac{t}{2}\right)dy$$

for any $\beta \in Z^+$, and any $x \in M$.

Proof. Let $\{\Omega_i\}_{i=1}^{\infty}$ be a compact exhaustion of M. In particular, say $\Omega_i = B(x, R_i)$ where $R_i \to \infty$ as $i \to \infty$. We denote $H_i(x, y, t)$ to be the heat kernel on $B(x, R_i)$ which satisfies the Dirichlet boundary condition. We claim that $H_i(x, y, t)$ converge to $H(x, y, t)$ uniformly on compact sets of M. In fact, it is known that $H_i(x, y, t) \leq H_j(x, y, t)$ for $i \leq j$, hence $H_i(x, y, t)$ is a monotone increasing sequence of functions. Also, since $H_i(x, y, t) \leq H(x, y, t)$ and Theorem 4, we conclude that

$$H_i(x, y, t) \leq C(n, k(2d), \tilde{\delta}(x), T)t^{-n/2}\exp\left(\frac{-d^2}{16t}\right) \tag{4.4}$$

for all $t \in [0, T]$, where $d = r(x, y)$ and $k(2d) =$ lower bound of the sectional curvature on $B(x, 2d)$. Therefore the monotone sequence $H_i(x, y, t)$ which is bounded must converge uniformly on compact sets. Moreover, the limit must be $H(x, y, t)$ by uniqueness of solution for (3.1) with initial condition

$$\lim_{t\to 0} H(x, y, t) = \delta_x.$$

To prove the lemma, we first show that it is true for $H_i(x, y, t)$ using its eigenfunction expansion

$$H_i(x, y, t) = \sum_{\alpha=1}^{\infty} e^{-\lambda_\alpha{}^i t}\varphi_\alpha{}^i(x)\varphi_\alpha{}^i(y). \tag{4.5}$$

Then

$$\Delta^\beta H(x, y, t) = \sum_{\alpha=1}^{\infty} (-\lambda_\alpha)^\beta e^{-\lambda_\alpha t} \varphi_\alpha(x) \varphi_\alpha(y) \tag{4.6}$$

where the subscript i is being suppressed. Therefore

$$\begin{aligned} \left| \int H(x, y, t) \Delta^\beta H(x, y, t) \right| &= \sum_{\alpha=1}^{\infty} \lambda_\alpha{}^\beta e^{-2\lambda_\alpha t} \varphi_\alpha{}^2(x) \\ &= t^{-\beta} \sum_{\alpha=1}^{\infty} (\lambda_\alpha t)^\beta e^{-2\lambda_\alpha t} \varphi_\alpha{}^2(x) \\ &\leq t^{-\beta} C(\beta, \epsilon) \sum_{\alpha=1}^{\infty} e^{-2(1-\epsilon)\lambda_\alpha t} \varphi_\alpha{}^2(x), \end{aligned} \tag{4.7}$$

where the last inequality follows from the fact that

$$x^\beta e^{-2x} \leq C(\beta, \epsilon) e^{-2(1-\epsilon)x}$$

for all $0 < \epsilon < 1$. However, since

$$\int H^2(x, y, t) dy = \sum_{\alpha=1}^{\infty} e^{-2\lambda_\alpha t} \varphi_\alpha{}^2(x), \tag{4.8}$$

$$\left| \int H(x, y, t) \Delta^\beta H(x, y, t) dy \right| \leq t^{-\beta} C(\beta, \epsilon) \int H^2(x, y, (1-\epsilon)t) dy. \tag{4.9}$$

Since this estimate is independent of the compact domain, by setting $\epsilon = 1/2$, we have

$$\begin{aligned} &\left| \int_{B(x, R_i)} H_i(x, y, t) \Delta^\beta H_i(x, y, t) dy \right| \\ &\qquad \leq t^{-\beta} C(\beta) \int_{B(x, R_i)} H_i^2\left(x, y, \frac{t}{2}\right) dy \\ &\qquad \leq t^{-\beta} C(\beta) \int_M H^2\left(x, y, \frac{t}{2}\right) dy, \end{aligned} \tag{4.10}$$

where the last inequality follows from the fact

$$H_i(x, y, t) \leq H(x, y, t).$$

On the other hand, integration by parts yields

$$\left| \int_{B(x,R_i)} H_i(x, y, t)\Delta^{\beta} H_i(x, y, t)dy \right|$$

$$= \begin{cases} \int_{B(x,R_i)} (\Delta^{\tau} H_i(x, y, t))^2 dy & \text{if} \quad \beta = 2\tau \\ \int_{B(x,R_i)} |\nabla\Delta^{\tau} H_i(x, y, t)|^2 dy & \text{if} \quad \beta = 2\tau + 1. \end{cases} \tag{4.11}$$

Now the integrands on the right-hand side are positive, hence for a fixed $j \in Z^+$ and for $i \geq j$

$$\int_{B(x,R_i)} (\Delta^{\tau} H_i(x, y, t))^2 \geq \int_{B(x,R_j)} (\Delta^{\tau} H_i(x, y, t))^2 \tag{4.12}$$

if $\beta = 2\tau$, and

$$\int_{B(x,R_i)} |\nabla\Delta^{\tau} H_i(x, y, t)|^2 \geq \int_{B(x,R_j)} |\nabla\Delta^{\tau} H_i(x, y, t)|^2 \tag{4.13}$$

when $\beta = 2\tau + 1$. Either case, since $H_i(x, y, t)$ converges to $H(x, y, t)$ on compact set, in particular on $B(x, R_j)$, hence (4.10), (4.11), (4.12) and (4.13) imply

$$t^{-\beta} C(\beta) \int_M H^2\left(x, y, \frac{t}{2}\right)$$

$$\geq \begin{cases} \int_{B(x,R_j)} (\Delta^{\tau} H(x, y, t))^2 & \text{if} \quad \beta = 2\tau \\ \int_{B(x,R_j)} |\nabla\Delta^{\tau} H(x, y, t)|^2 & \text{if} \quad \beta = 2\tau + 1. \end{cases} \tag{4.14}$$

However, because j is arbitrary, (4.14) gives

$$t^{-\beta}C(\beta)\int_M H^2\left(x, y, \frac{t}{2}\right) \geq \begin{cases} \int_M (\Delta^\tau H(x, y, t))^2 & \text{if } \beta = 2\tau \\ \int_M |\nabla\Delta^\tau H(x, y, t)|^2 & \text{if } \beta = 2\tau + 1. \end{cases} \tag{4.15}$$

Integration by parts again will imply the lemma.

THEOREM 6. *Let M be a complete Riemannian manifold with bounded curvature. Then there exists a constant $C(n, k, T)$, depending only on n, k and T, such that for all p, $x \in M$ and $t \in [0, T]$,*

$$|\nabla H|(p, x, t) \leq C(n, k, T)\tilde{\delta}^{-\alpha(n)}(p)t^{-((n+1)/2)}\exp\left[\frac{-\alpha(n)d^2}{t}\right]$$

where $d = r(p, x)$.

Proof. Let $\eta(y)$ be the cut off function defined by

$$\eta(y) = \begin{cases} 1 & \text{on } B(p, \nu) - B\left(p, \frac{3d}{4}\right) \\ 0 & \text{outside } B(p, 2\nu) \\ 0 & \text{on } B\left(p, \frac{d}{2}\right) \end{cases}$$

where ν is any constant strictly greater than d. We also assume that $0 \leq \eta(y) \leq 1$, and $|\nabla\eta|(y) \leq 4/d$ for all $y \in M$. Using integration by parts,

$$\int_M \eta^2|\nabla H|^2 = -2\int_M (H\nabla\eta, \eta\nabla H) - \int_M \eta^2 H\Delta H$$

$$\leq 2\int |\nabla\eta|^2 H^2 + \frac{1}{2}\int_M \eta^2|\nabla H|^2 - \int_M \eta^2 H\Delta H. \tag{4.16}$$

Therefore

$$\int_M \eta^2 |\nabla H|^2 \le 4\int_M |\nabla\eta|^2 H^2 - 2\int_M \eta^2 H\Delta H$$

$$\le \frac{64}{d^2}\int_{M-B(p,d/2)} H^2 + 2\int_{M-B(p,d/2)} |H\Delta H|. \quad (4.17)$$

Letting $\nu \to \infty$, we have

$$\int_{M-B(p,3d/4)} |\nabla H|^2 \le \frac{64}{d^2}\int_{M-B(p,d/2)} H^2 + 2\int_{M-B(p,d/2)} |H\Delta H|$$

$$\le \left(\int_{M-B(p,d/2)} H^2(p,y,t)\right)^{1/2}$$

$$\cdot\left[\frac{64}{d^2}\left(\int_M H^2\right)^{1/2} + 2\left(\int_M (\Delta H)^2\right)^{1/2}\right]. \quad (4.18)$$

Applying Theorem 3, Lemma 5, and Lemma 7, we obtain,

$$\int_{M-B(p,3d/4)} |\nabla H|^2 \le \tilde{C}^{1/2}(p,T)t^{-n/4}\exp\left[\frac{-\alpha(n)d^2}{t}\right]$$

$$\times\left[\frac{\alpha(n)}{d^2}C^{-\theta/2}\left(p,\sqrt{\frac{T}{4}}\right)t^{-n/4}\right.$$

$$\left. + \alpha(n)C^{-\theta/2}\left(p,\sqrt{\frac{T}{8}}\right)t^{-((n+4)/4)}\right]. \quad (4.19)$$

In the case when $4t \le d^2$, (4.19) becomes

$$\int_{M-B(p,3d/4)} |\nabla H|^2 \le C(n,k,T)\bar{\delta}^{-\alpha(n)}(p)$$

$$\times t^{-((n+2)/2)}\exp\left[\frac{-\alpha(n)d^2}{t}\right]. \quad (4.20)$$

However, by Lemma 6, $|\nabla H|$ satisfies

$$\square|\nabla H| - \kappa|\nabla H| \geq 0, \tag{4.21}$$

one verifies that the iteration method of Moser applies to subsolution of the form (4.21) (see [1]), hence

$$|\nabla H|(p, x, t) \leq \tilde{C}^{1/2}(x, t)t^{-((n+2)/4)}\left(\int_0^t \int_{B(x, \sqrt{t/4})} |\nabla H|^2\right)^{1/2}$$
$$\leq \tilde{C}^{1/2}(x, t)t^{-((n+2)/4)}\left(\int_0^t \int_{M-B(p, 3d/4)} |\nabla H|^2\right)^{1/2}$$

Clearly applying Theorem 2 and (4.20) the theorem follows for the case $4t \leq d^2$, where we have used the fact that $-(n-1)k \leq \kappa$.

When $4t > d^2$, we observe that by Lemma 5 and 7

$$\int_M |\nabla H|^2 = \left|\int_M H\Delta H\right| \leq t^{-1}\alpha(n)\int_M H^2\left(p, y, \frac{t}{2}\right)$$
$$\leq \alpha(n)C^{-\theta(n)}\left(p, \sqrt{\frac{T}{8}}\right)t^{-((n+2)/2)}$$
$$\leq \alpha(n)C^{-\theta(n)}\left(p, \sqrt{\frac{T}{8}}\right)t^{-((n+2)/2)}\exp\left[\frac{-d^2}{4t}\right]. \tag{4.22}$$

Applying the iteration method again, we obtain the desired estimate.

For the sake of simplicity, we will only outline the proof for the upper bound of $|\text{hess } H|$. Similar methods will yield higher order estimates, which will be pointed out as we proceed. These estimates are based on the same idea as in the case for $|\nabla H|$. It can be divided into three steps. First, we utilize the commutation formula to show that $|\text{hess } H| = (\Sigma_{i,j} h_{ij}^2)^{1/2}$ is a subsolution of some parabolic equation. Secondly, we use Lemma 7 to obtain an L^2 estimate of $|\text{hess } H|$. Finally, we invoke the iteration method in [1] to obtain the pointwise estimate.

LEMMA 8. *Let M be a complete Riemannian manifold with bound curvature tensor and its covariant derivatives. Then*

$$\square|\text{hess } H| - A_0|\text{hess } H| - A_1|\nabla H| \geq 0.$$

where A_0 is a bound for the curvature tensor and A_1 is a bound for its covariant derivatives.

Proof. Consider

$$\begin{aligned}\frac{1}{2}\Delta|\text{hess } H|^2 = \frac{1}{2}\Delta(H_{i,j}^2) &= H_{ijkk}H_{ij} + H_{ijk}^2 \\ &= R_{ijkl,k}H_l H_{ij} + R_{kikl,j}H_l H_{ij} \\ &\quad + R_{ijkl}H_{lk}H_{ij} + R_{kikl}H_{lj}H_{ij} - R_{likj}H_{lk}H_{ij} \\ &\quad - R_{lkkj}H_{il}H_{ij} + (\Delta H)_{ij}H_{ij} + H_{ijk}^2 \\ &\geq A_0|\text{hess } H|^2 + A_1|\text{hess } H||\nabla H| \\ &\quad + \frac{1}{2}(H_{ij}{}^2)_t + H_{ijk}^2. \end{aligned} \tag{4.23}$$

On the other hand,

$$\frac{1}{2}\Delta|\text{hess } H|^2 = |\text{hess } H|\Delta|\text{hess } H| + |\nabla|\text{hess } H||^2 \tag{4.24}$$

where

$$\begin{aligned}|\nabla|\text{hess } H||^2 &= \sum_k \left[\left(\sum_{i,j} H_{ij}{}^2\right)^{1/2}\right]_k^2 \\ &= \sum_k \left[\sum_{i,j} \frac{H_{ij}H_{ijk}}{|\text{hess } H|}\right]^2 \leq \sum_{i,j,k} H_{ijk}^2 .\end{aligned}$$

Therefore (4.23) and (4.24) imply

$$\Delta|\text{hess } H| \geq |\text{hess } H|_t + A_0|\text{hess } H| + A_1|\nabla H| \tag{4.25}$$

which was to be proved.

Clearly, by the same token, one can obtain differential inequalities of a similar form for higher order derivatives of H. However one would require the equation to depend on the higher order covariant derivatives of the curvature tensor.

THEOREM 7. *Let M be a complete Riemannian manifold. Suppose $C(n, A_0, A_1, T)$ is a constant depending on n, A_0, A_1, and T, then for all p, $x \in M$ and $t \in [0, T]$,*

$$|\operatorname{hess} H|(p, x, t) \leq C(n, A_0, A_1, T)\tilde{\delta}^{-\alpha(n)}(p)t^{-((n+2)/2)} \exp\left[\frac{-\alpha(n)d^2}{t}\right]$$

where $d = r(p, x)$.

Proof. The only ingredient that we need is the L^2 estimates of $|\operatorname{hess} H|$ on $M - B(p, 3d/4)$. Let η be the cut off function defined in Theorem 6. Then

$$\begin{aligned}\int \eta^2 H_{ij}{}^2 &= \int (\eta^2)_j H_{ij} H_i + \int \eta^2 H_{ijj} H_i \\ &= 2\int \eta\eta_j H_{ij} H_i + \int \eta^2 R_{ij} H_i H_j + \int \eta^2 (\Delta H)_i H_i \\ &\leq \frac{1}{2}\int \eta^2 H_{ij}{}^2 + 2\int |\nabla\eta|^2 |\nabla H|^2 + (n-1)K\int \eta^2 |\nabla H|^2 \\ &\quad + \int 2\eta\eta_i (\Delta H)_i H + \int \eta^2 (\Delta^2 H) H, \end{aligned} \tag{4.26}$$

and

$$\begin{aligned}&\int_{M-B(p,3d/4)} H_{ij}{}^2 \\ &\quad \leq \frac{64}{R^2}\int_{M-B(p,d/2)} |\nabla H|^2 + (n-1)K\int_{M-B(p,d/2)} |\nabla H|^2 \\ &\quad\quad + \left(\int_{M-B(p,d/2)} H^2\right)^{1/2}\left[\left(\frac{64}{R^2}\int_M |\nabla\Delta H|^2\right)^{1/2} + \left(\int_M (\Delta^2 H)^2\right)^{1/2}\right] \\ &\quad = \left[\frac{64}{R^2} + (n-1)K\right]\int_{M-B(p,d/2)} |\nabla H|^2 + \left(\int_{M-B(p,d/2)} H^2\right)^{1/2} \\ &\quad\quad \times \left[\frac{8}{R}\left(\int_M H\Delta^2 H\right)^{1/2} + \left(\int_M H\Delta^4 H\right)^{1/2}\right]. \end{aligned} \tag{4.27}$$

Applying the estimates for $\int_{M-B(p,d/2)} |\nabla H|^2$ and Lemma 7, we obtain an upper bound for $\int_{M-B(p,3d/4)} H_{ij}^2$. The theorem then follows from the iteration method.

COROLLARY 8. *Let M be a complete manifold. Suppose $C(n, A_0, A_1, \ldots, A_{l-1}, T)$ is a constant depending on the quantities described, where the A_i's are bounds on the i^{th} covariant derivatives of the curvature tensor. Then if $|D_l H|$ denotes the norm of the l^{th} covariant derivative of H,*

$$|D_l H|(p, x, t) \leq C(n, A_0, \ldots, A_{l-1}, T) t^{-((n+l)/2)} \bar{\delta}^{-\alpha(n,l)}(p)$$

$$\exp\left[\frac{-\alpha(n) d^2}{t}\right]$$

where $d = r(p, x)$ and $t \in [0, T]$.

APPENDIX

PROPOSITION A. *Let $B(2R)$ be a geodesic ball of radius $2R$ in the hyperbolic space form of constant curvature -1. Suppose $H(r, t)$ denotes the heat kernel on $B(2R)$ with Dirichlet boundary condition such that $H(r, 0)$ is the delta function at the origin $0 \in B(2R)$. Then*

$$\lim_{t \to 0} [-4t \log H(R, t)] \leq R^2.$$

Proof. Consider the heat kernel $\bar{H}(r, t)$ for the simply connected hyperbolic space form of constant curvature -1 with the pole at 0. By Duhamel's principle

$$H(R, t) - \bar{H}(R, t) = \int_0^t \frac{\partial}{\partial s} \int_{B(2R)} H(0, \xi, s) \bar{H}(x, \xi, t - s) \qquad \text{(A.1)}$$

where $r(0, x) = R$. On the other hand since $H(0, \xi, s)$ and $\bar{H}(x, \xi, t - s)$ both satisfy the heat equation, (A.1) becomes

$$H(R, t) - \bar{H}(R, t) = \int_0^t \int_{B(2R)} \Delta_\xi H(0, \xi, s) \bar{H}(x, \xi, t - s)$$

$$- \int_0^t \int_{B(2R)} H(0, \xi, s)\Delta \bar{H}(x, \xi, t-s)$$

$$= \int_0^t \int_{\partial B(2R)} H_\nu(0, \xi, s)\bar{H}(x\xi, t-s). \tag{A.2}$$

The last equality follows from Green's 2nd identity and the fact that $H(0, \xi, s)$ satisfies Dirichlet boundary condition. $H_\nu(0, \xi, s)$ denotes the directional derivation of H with respect to the outward normal vector in the 2nd variable. Since $H(0, \xi, s)$ is rotationally symmetric

$$H_\nu(0, \xi, s) = H_\nu(2R, s) = \frac{\int_{\partial B(2R)} H_\nu(2R, s)}{A(2R)} \tag{A.3}$$

for $\xi \in \partial B(2R)$, where $A(2R) =$ area of $\partial B(2R)$. However,

$$\int_{\partial B(2R)} H_\nu(2R, s) = \int_{B(2R)} \Delta H(0, \xi, s) = \int_{B(2R)} \frac{\partial}{\partial s} H(0, \xi, s), \tag{A.4}$$

hence

$$H(R, t) - \bar{H}(R, t) = \int_0^t \left(\frac{\frac{\partial}{\partial s} \int_{B(2R)} H(0, \xi, s)}{A(2R)} \right) \cdot \left(\int_{\partial B(2R)} \bar{H}(x, \xi, t-s) \right). \tag{A.5}$$

Since $r(0, x) = R$, this implies that the ball $B(x, R)$ around x with radius R is contained in $B(2R)$, a result of Cheeger and Yau [4], asserts that $(\partial/\partial r)\bar{H}(x, \xi, t-s) \leq 0$. Therefore

$$\bar{H}(x, \xi, t-s) \leq \bar{H}(R, t-s)$$

for all $\xi \in \partial B(2R)$, where we have used the fact that $\bar{H}(R, t-s) = \bar{H}(x, y, t-s)$ for all $y \in \partial B(x, R)$. By (A.4) and the fact that $H_\nu(2R, s) \leq 0$, (A.5) becomes

$$H(R, t) - \bar{H}(R, t) \geq \int_0^t \frac{1}{A(2R)}\left(\frac{\partial}{\partial s}\int_{B(2R)} H(0, \xi, s)\right)A(2R)\bar{H}(R, t-s)$$

$$= \int_0^t \bar{H}(R, t-s)\frac{\partial}{\partial s}\int_{B(2R)} H(0, \xi, s). \tag{A.6}$$

A probabilistic argument shows that for small enough t, $(\partial/\partial s)\bar{H}(R, t-s) < 0$ for all $s \in [0, t]$. In fact, in this case one checks that this is so from the explicit formula for $\bar{H}(R, t-s)$ (see [9], also private communication with J. Cheeger, who has obtained explicit formula for all n). Hence for small t,

$$H(R, t) - \bar{H}(R, t) \geq \bar{H}(R, t)\int_0^t \frac{\partial}{\partial s}\int_{B(2R)} H(0, \xi, s)$$

$$= \bar{H}(R, t)\left[\int_{B(2R)} H(0, \xi, t) - 1\right]. \tag{A.7}$$

This implies that for t small enough such that

$$\int_{B(2R)} H(0, \xi, t) \geq \frac{1}{2},$$

then

$$H(R, t) \geq \frac{1}{2}\bar{H}(R, t).$$

On the other hand, from the formula for $\bar{H}(R, t)$, we see that

$$\lim_{t\to 0} [-4t \log \bar{H}(R, t)] = R^2.$$

Therefore

$$\lim_{t\to 0} [-4t \log H(R, t)] \leq R^2$$

as asserted.

Clearly, by scaling, the conclusion of Proposition A remains true for hyperbolic space form of constant curvature $-k$, for any $k > 0$.

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REFERENCES

[1] D. G. Aronson and J. Serrin, "Local behavior of solutions of quasilinear parabolic equations," *Arch. Rat. Mech. Anal.* **25** (1967), pp. 81-122.

[2] E. Bombieri, Theory of Minimal Surfaces and a Counter-Example to the Bernstein Conjecture in High Dimension, Lecture Notes, Courant Institute, 1970.

[3] J. Cheeger and D. G. Ebin, Comparison Theorems in Riemannian Geometry, North-Holland Mathematics Library, 1975.

[4] J. Cheeger and S. T. Yau, "A lower bound for the heat kernel," (preprint).

[5] S. Y. Cheng and P. Li, "Heat kernel estimates and lower bound of eigenvalues," (preprint).

[6] C. Croke, "Some isoperimetric inequalities and eigenvalue estimates," (preprint).

[7] P. Li, "On the Sobolev constant and the p-spectrum of a compact Riemannian manifold," to appear in *Ann. Sci. Ec. Norm. Sup.*

[8] P. Li and S. T. Yau, "Estimates of eigenvalues of a compact Riemannian manifold," *Proc., Symp. Pure Math.*, **36** (1980), pp. 205-240.

[9] H. P. McKean, "An upper bound to the spectrum of Δ on a manifold of negative curvature," *J. Diff. Geom.*, **4** (1970), pp. 359-366.

[10] J. Moser, "A Harnack inequality for parabolic differential equations," *Comm. Pure Applied Math.*, **17** (1964), pp. 101-134.

[11] J. Nash, "Continuity of solutions of parabolic and elliptic equations," *Am. J. Math.*, **80** (1958), pp. 931-954.

[12] Y. T. Siu and S. T. Yau, "Compactification of negatively curved complete Kähler manifolds of finite volume," (preprint).

[13] S. R. S. Varadhan, "On the behavior of the fundamental solution of the heat equation with variable coefficients," *Comm. Pure Applied Math.*, **20** (1967), pp. 431-455.

[14] S. T. Yau, "Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold," *Ann. Sci. Ec. Norm. Sup.*, 4e série, t. 8 (1975), pp. 487-507.