Math. Z. 179, 151–168 (1982)

Mathematische Zeitschrift

The Existence of Embedded Minimal Surfaces and the Problem of Uniqueness

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It was in the early nineteen thirties that Douglas and Radó solved the Plateau problem for a rectifiable Jordan curve in Euclidean space. The minimal surfaces that they obtained are realized by conformal harmonic maps from the unit disk. In 1948 Morrey generalized the theorem of Douglas and Radó to minimal surfaces in a homogeneously regular Riemannian manifold. In both cases the solutions are defective as they may possess branch points. It was not until 1968 that Osserman [24] was able to prove that interior branch points do not exist on the Douglas-Radó-Morrey solution. (Osserman's theorem in this full generality was proved by Alt [2, 3] and Gulliver [8].) If the Jordan curve is real analytic, Gulliver and Lesley [9] showed that the solution surface is free of branch points even on its boundary. Hence, at least in this case, we know that the Douglas-Radó-Morrey solution is an immersion and represents a classical surface.

However, the Douglas-Radó-Morrey solution may in general have selfintersections. It was then a classical problem to find conditions on the Jordan curve to guarantee that the solution is an embedded disk. In 1932 Radó was the first to produce a general condition of this type. He proved that if the curve can be projected onto a convex curve in the plane, then it bounds a unique embedded minimal disk which is a graph over this plane. It was not until the seventies that there was further progress on the problem of embeddedness. Osserman had conjectured that if a Jordan curve is extreme, that is, lies on the boundary of its convex hull, then the Douglas-Radó solution is embedded. Gulliver and Spruck [10] proved this conjecture under the additional assumption that the absolute total curvature of the Jordan curve is less than 4π. Right after this work, Tomi and Tromba [32] showed that an extreme curve must bound an embedded minimal disk which need not be stable. In 1976 Almgren and Simon [1] established the existence for such a curve of an embedded stable minimal disk (which need not be a Douglas-Radó solution). Finally in the same year, the authors [18] proved that any Douglas-Radó solution must be embedded.

* Supported in part by National Science Foundation grant MCS 79-12938.

0025-5874/82/0179/0151/$02.00

Both the arguments of Almgren-Simon and ours can be generalized to minimal surfaces of high topological type.

The method used by us in proving Osserman's conjecture is a topological one which goes back to Papakyriakopoulos, Whitehead, Shapiro, Stallings, and Epstein. It turns out that one can generalize Osserman's conjecture to minimal surfaces in a compact Riemannian manifold whose boundary has positive mean curvature with respect to the outward normal. This generalization enables us to strengthen the classical Dehn's lemma, loop theorem and sphere theorem to the equivariant setting and hence gives non-trivial applications to finite group actions on three-dimensional manifolds [16, 17].

In this paper we pursue the topological argument and show the relations between the problem of embeddedness and the problem of uniqueness. The last problem has a long history. Radó was also the first to find a condition on a Jordan curve to guarantee the uniqueness of the minimal disk. More recently Meeks [15] has generalized Radó's results to prove, for example, that a Jordan curve with a monotonic projection onto a convex plane curve is the boundary of a unique compact branched minimal surface and the interior of this surface is a graph over this plane. In 1973 Nitsche [22] was able to use a theorem of Barbosa and do Carmo [5] and the Morse-Tompkins-Shiffman [21, 29] result to demonstrate that a smooth Jordan curve with absolute total curvature less than 4π can bound only one disk. To this date it remains a conjecture that this curve cannot bound any minimal surface of high topological type. In the same year Tomi [33] used his work with Böhme [4] and the work of Osserman [24], Gulliver-Lesley [9] to prove a beautiful result that a real analytic regular Jordan curve can bound only a finite number of minimal disks of least area. Tomi's argument is very general and applies to surfaces of higher genus. If one could prove the absence of boundary branch points for a smooth regular curve, then one could generalize his theorem to these curves. Besides these results, there were results due to Böhme, Tomi, Tromba, and Morgan who proved that most curves bound only a finite number of solutions to the Plateau problem.

The purpose of this paper is to demonstrate the relation between the problem of embeddedness and the problem of uniqueness. The first part of the discussion can be considered as a preparation for the latter part of the paper. It generalizes the previously known theorems for the existence of minimal disks. We allow the boundary of the three-dimensional manifold to be non-smooth and to have non-negative mean curvature. As a consequence, we give a simple criterion for a smooth Jordan curve to bound an embedded stable minimal disk. We also apply the general existence theorem to prove that if a smooth Jordan curve is on the boundary of its convex hull, then either it bounds two distinct embedded stable disks or the only minimal surface (of any genus) that it can bound is unique and is an embedded disk. This was first observed by the first author and is proved here by a different method. As a corollary of this theorem and the theorem of Nitsche, one proves that any extremal curve with absolute total curvature less than 4π can bound only one minimal surface which is an embedded disk. In the last section we apply the similar argument in this paper to give a proof of the bridge theorem. This is a classical fact which was first established by Kruskal [12]. Kruskal's proof is not complete, however. We provide a proof for the theorem so that the heuristic construction due to Lévy [13] and Courant [6] can be carried out rigorously. On the basis of this construction, we obtain examples of rectifiable Jordan curves which can bound uncountably many stable minimal disks. The study of the bridge theorem was brought to our attention by Professor Nitsche in 1977. We completed the proof soon afterwards. We were also informed that Almgren and Solomon had independently found a version of the bridge theorem.

§ 1. General Boundary Conditions for the Existence of an Embedded Minimal Disk

In this section, we generalize the existence theorem in [18] to a more general class of manifolds. In [18], we assumed the boundary of the three-dimensional manifold to be convex in order to guarantee the existence of a minimal surface in the manifold. Here we shall only assume the boundary of the manifold to have non-negative mean curvature.

For the purpose of application, we allow the boundary ∂M of our manifold M to be non-smooth. More precisely, we assume that M is a compact subdomain of another smooth manifold $\bar{M}$ so that, after a suitable triangulation of $\bar{M}$, ∂M is a two-dimensional subcomplex of $\bar{M}$ consisting of smooth two-dimensional simplexes $\{H_1, \ldots, H_l\}$ which have the following properties:

(1) Each H_i is a C^2 surface in $\bar{M}$ whose mean curvature is non-negative with respect to the outward normal.

(2) Each surface H_i is a compact subset of some smooth surfaces $\bar{H}_i$ in $\bar{M}$ where $\bar{H}_i \cap M = H_i$ and $\partial \bar{H}_i \subset \partial \bar{M}$.

Theorem 1. *Let M be a three-dimensional compact manifold with piecewise smooth boundary which has the above properties* (1) *and* (2). *Let σ be a Jordan curve in ∂M. Then there exists a branched minimal immersion from the disk D into M with boundary σ which is smooth in the interior of D and has minimal area among all such maps. Furthermore, any branched minimal immersion of the above form must be an embedding.*

Proof. We will use the approximation method in [18]. It will depend on the following assertion.

Assertion 1. There exists a sequence of metrics ds_ε^2 which converges to the metric of $\bar{M}$ in a compact neighborhood of M. Furthermore, there exists a sequence of smooth domains M_ε and a sequence of Jordan curves σ_ε on ∂M_ε so that both σ_ε and ∂M_ε converge to σ and ∂M respectively in the uniform topology and M_ε has positive mean curvature with respect to the metric ds_ε^2.

Proof of Assertion 1. We assert that there is a smooth function φ defined in a neighborhood of ∂M so that $d\varphi(n_i) > 0$ at points of ∂M where n_i is the outward normal of the smooth piece H_i of M. This can be done by piecing functions together in the following way.

For each point $p \in \partial M$, we have a neighborhood N_p so that up to a diffeomorphism $N_p \cap M$ is convex. We may assume that N_p is homeomorphic to the set $(0,1) \times \tilde{N}_p$ where $\tilde{N}_p$ is an open set in ∂M. It is clear that we may choose a function φ_p in N_p so that φ_p has support in $(0,1) \times \bar{N}_p$ where $\bar{N}_p$ is a compact neighborhood of p in $\tilde{N}_p$ and $d\varphi_p(n_i) > 0$ at p where n_i are the outward normals of H_i. By a compactness argument, we can produce our desired function φ by a suitable sum of these φ_p's.

Let ds^2 be the metric of M. Then we deform ds^2 to the metric $\exp(\varepsilon\varphi)\, ds^2$ where ε is a number tending to zero. By direct computation, one checks that with respect to the metric $\exp(\varepsilon\varphi)\, ds^2$, the smooth pieces H_i have positive mean curvature with respect to the outward normal.

By slight perturbation of the surfaces H_i, we may assume that the H_i's are surfaces with positive mean curvature which intersect each other transversally and satisfy property (2) mentioned before the statement of Theorem 1.

We are going to smooth out the corners of ∂M by induction on the number of edges of ∂M. By using a diffeomorphism of M, we may assume that $H_1 \cap H_2$ is a line segment of the x_2-axis passing through the origin, part of H_1 is given by $x_3 = x_1$ for $\varepsilon > x_1 > 0$ and part of H_2 is given by $x_3 = -x_1$ for $-\varepsilon \leqq x_1 \leqq 0$ where ε is a fixed positive number.

If f is any function of x_1 and g_{ij} is the metric tensor of M, then the mean curvature of the surface defined by f is positive with respect to the downward direction if and only if

$$(g_{22} g_{33} - g_{32}^2) f'' + p_1 (f')^2 + p_2 f' + p_3 > 0 \tag{1.1}$$

where p_1, p_2 and p_3 are polynomials in g_{ij} and its first derivatives.

Hence for $\varepsilon \geqq x_1 \geqq 0$ or $0 \geqq x_1 \geqq -\varepsilon$, the above expression is positive if f is replaced by x_1 and $-x$, respectively. Let $C_1 > 0$ be the minimum of such expressions for $|x_1| \leqq \varepsilon$ and $0 \leqq x_3 \leqq \varepsilon$. Then we will construct an *even* convex function f of x_1 so that

$$f(t) = |t| \quad \text{for } |t| > \frac{\varepsilon}{2} \tag{1.2}$$

and for $t \geqq 0$.

$$C_3 f'' - C_2 |f' - 1| + C_1 > 0$$

where

$$C_3 = \min_{0 \leqq x_3 \leqq \varepsilon} (g_{22} g_{33} - g_{32}^2) > 0$$

and

$$C_2 = \max_{0 \leqq x_3 \leqq \varepsilon} (2|p_1| + |p_2|).$$

Since $f'(0) = 0$ and $f'' \geqq 0$, we have $|f'| \leqq 1$ and $\varepsilon \geqq f(t) \geqq |t|$. The function f will then satisfy the inequality (1.1) and give a smooth surface with positive mean curvature with respect to the outward normal. This function f can be constructed in the following way. By suitably choosing a constant $a > 0$, we can define $f'(t) = \left(1 + \frac{a C_2}{C_1}\right)(1 - e^{-C_1 t})$ for $\frac{\varepsilon}{2} \geqq t \geqq 0$ and $f'(t) = 1$ for $t \geqq \frac{\varepsilon}{2}$. The function f'' is piecewise smooth. However, we can smooth out f'' and integrate to obtain the required f.

After smoothing the edge created by H_1 and H_2, we can continue the process to the other edge and obtain a smooth ∂M with positive mean curvature with respect to the outward normal. Since it is clear that one can produce a Jordan curve σ_ε on ∂M which is close to σ in uniform topology, Assertion 1 is proved.

Assertion 2. Let M be a compact three-dimensional manifold whose boundary ∂M has positive mean curvature with respect to the outward normal. Let σ be a Jordan curve on ∂M. Then there exists a smooth embedded minimal disk in M whose boundary is σ.

This assertion is rather well-known and can be proved as follows (see also [18] Theorem 1). We embed M as a smooth compact domain of a large manifold $\bar{M}$. Let f be a smooth function defined in a neighborhood so that $f(x)\leqq 0$ in M. If $d(x)$ is the distance function to ∂M, we assume that for $x\in M$, $f(x)=-d(x)$ and for $x\notin M$, $f(x)=d(x)$ when $d(x)\leqq\varepsilon$. The positivity of the mean curvature of ∂M implies that $\Delta f>0$ in a neighborhood of M.

Let φ be a smooth function defined on the interval $(-\infty,\varepsilon)$ so that $\varphi(x)=1$ for $x\leqq 0$ and $\varphi(x)=\dfrac{1}{(\varepsilon-x)^2}$ for x near ε. Then for ε small enough, $\varphi(f)\,ds^2$ is a complete metric defined in a neighborhood of M which is homogeneously regular in the sense of Morrey [16].

By the theorem of Morrey [20], we can find a branched conformal immersion ψ from the unit disk into the manifold $\{x\,|\,f(x)<\varepsilon\}$ which has boundary σ and which has minimal area with respect to the metric $\varphi(f)\,ds^2$. We claim that the image of D under ψ lies in the interior of M.

In fact, if this were not true, the function $f\circ\psi$ will achieve its maximum at an interior point x_0 of the disk so that $f(\psi(x_0))\geqq 0$. Let e_1, e_2, e_3 be an orthonormal frame with respect to the metric $\varphi(f)\,ds^2$ so that e_3 is proportional to ∇f. Then computing with respect to this frame and the metric $\varphi(f)\,ds^2$, one sees that

$$|\nabla f(\psi)|=|\nabla f|\,|\nabla\psi^3| \tag{1.4}$$

$$\Delta f(\psi)=\sum_{i,j,k} f_{ij}\,\psi^i_k\,\psi^j_k, \tag{1.5}$$

where we have used the fact that ψ^i is harmonic.

The conformality of ψ shows that

$$(\psi^1_1)^2-(\psi^1_2)^2=-(\psi^2_1)^2+(\psi^2_2)^2+(\psi^3_2)^2-(\psi^3_1)^2 \tag{1.6}$$

$$\psi^1_1\,\psi^1_2=-\psi^2_1\,\psi^2_2-\psi^3_1\,\psi^3_2. \tag{1.7}$$

Squaring (1.6), (1.7) and adding a suitable multiple together, one obtains

$$|\nabla\psi^2|^2+O(|\nabla\psi^3|^2)\geqq|\nabla\psi^1|^2\geqq|\nabla\psi^2|^2-O(|\nabla\psi^3|^2). \tag{1.8}$$

We may have chosen our frame so that $f_{12}=0$. Then (1.4), (1.5) and (1.8) show that at points where $|\nabla f|$ is bounded from below by a positive constant,

$$\Delta f(\psi)=(f_{11}+f_{22})|\nabla\psi^1|^2+O(|\nabla f(\psi)|^2). \tag{1.9}$$

Direct computation shows that $f_{11}+f_{22}>0$ at points not in M. The strong maximum principle then shows that $f\circ\psi$ cannot have an interior maximum at points where $\psi(x)$ does not belong to the interior of M. Hence the image of the open unit disk under ψ lies in the interior of M. The proof that ψ is an embedding is the same as in [18].

With Assertion 1 and Assertion 2, we can finish the proof of Theorem 1 in the following way. For each σ_ε, we can find a smoothly embedded disk ψ_ε in M_ε which minimizes area with respect to the metric ds_ε^2. Since ds_ε^2 converges in a fixed neighborhood of M to ds^2, standard arguments (see [18]) show that ψ_ε converges to a smoothly embedded disk ψ in M which minimizes area with respect to the metric ds^2.

Once we have proved the existence of such a minimal embedded disk the arguments in [18] show that every disk which minimizes area must be embedded. This finishes the proof of Theorem 1.

Remark. One can allow a more general boundary condition for the existence of minimal disks in the following way. Let f be a Lipschitz subharmonic function defined on a manifold M so that the length of the gradient of f is almost everywhere equal to one. Suppose the set $\{f\leqq 0\}$ is compact and σ is a contractable Jordan curve in this set. Then σ bounds a stable minimal disk in this set.

For later purposes, we record the following theorem. A slightly special case of it was known to Nitsche [23].

Theorem 2. *Let M be a three-dimensional compact manifold with piecewise smooth boundary which satisfies properties* (1) *and* (2) *in Theorem* 1. *Let f: $\Sigma\to M$ be any branched conformal minimal immersion of a compact smooth surface into M so that $f(\partial\Sigma)\subset\partial M$. Then either $f(\Sigma)\subset\partial M$ or $f(\Sigma)\cap\partial M=f(\partial\Sigma)$. Furthermore, the differential of f is not zero along $\partial\Sigma$, i.e., $f(\Sigma)$ has no boundary branch point.*

Proof. Suppose $f(\Sigma)$ is not a subset of ∂M. Then if the first assertion is not true, there is a point x_0 in the interior of Σ so that $f(x_0)\in\partial M$. We claim that f maps a neighborhood of x_0 into ∂M.

There are several cases to be considered. First assume that $f(x_0)$ is in the interior of H_i. Let d be the distance function from H_i. Then as the mean curvature of H_i is non-negative with respect to the outward normal, Eq. (1.9) shows that in a neighborhood of x_0,

$$\Delta(-d(f))\geqq -C_1|\nabla d(f)|^2-C_2 d(f) \tag{1.10}$$

where C_1 and C_2 are positive constants. Since $-d(f)$ achieves its maximum at x_0, the Hopf maximum principle shows that $d(f)\equiv 0$ in a neighborhood of x_0.

In the second case, we assume $f(x_0)\in H_i\cap H_j$ for $i\neq j$. If x_0 is a branched point of f, the local representation (see [8]) of $f(\Sigma)$ near x_0 shows that locally,

$f(\Sigma)$ cannot be on one side of $\bar{H}_i$, the smooth surface extending H_i. Hence we may assume that f is an immersion at x_0. In this case, we may represent a neighborhood of $f(\Sigma)$ at x_0 to be a graph over the tangent plane at x_0. For simplicity we choose a coordinate system so that the tangent plane at x_0 is the $x-y$ plane and the graph is given by a smooth function h. We may also assume that $\bar{H}_i$ is locally defined by a smooth function k on the half space $x \geqq 0$. If we assume the outward normal of ∂M is pointing downward at the origin, then $h \geqq k$ for $x \geqq 0$. If $f(x_0)$ is not a vertex of ∂M, then we argue as follows. The function h satisfies the mean curvature equation and k satisfies the inequality that the mean curvature is non-negative with respect to the outward normal for $x \geqq 0$. Hence $h-k$ satisfies an inequality of the form $L(h-k) \leqq 0$ where L is a second order homogeneous elliptic operator. Since $h-k \geqq 0$ and $h-k=0$ at the origin, the Hopf maximum principle shows that $\frac{\partial(h-k)}{\partial x} \neq 0$ at the origin. This is impossible because $f(\Sigma)$ cannot lie locally on two sides of $\bar{H}_i$.

If $f(x_0)$ is a vertex of the simplicial decomposition of M, we argue as follows. (At this point, the simplices cannot intersect each other transversally.) As before, we choose a coordinate system so that the tangent plane at $f(x_0)$ is the $x-y$ plane. Each of the surfaces $\bar{H}_i$ must be tangential to the $x-y$ plane at the origin. Hence locally the $x-y$ plane is divided by sectors with vertex at the origin and each H_i is defined by a function k_i on a sector. We claim that the Lipschitz function k defined by combining all the k_i's together satisfies the property that the mean curvature of k (in the sense of distribution) is non-negative with respect to the downward normal. Since the mean curvature operator is in divergence form, it suffices to find functions $\tilde{k}$ so that $\tilde{k}$ approximates k in L_1^2 sense and the mean curvature of $\tilde{k}$ is bounded from below by small negative constant. By the proof of Theorem 1, we can deform the metric conformally to obtain such a function $\tilde{k}$ where the mean curvature of $\tilde{k}$ with respect to the new metric is positive. Computing the mean curvature of $\tilde{k}$ with respect to the original metric, we can check that all the required properties are verified. Hence the function $h-k$ satisfies an inequality $L(h-k) \leqq 0$ weakly where L is a second ordder homogeneous elliptic operator. Since $h-k>0$ except at the origin, the generalized minimum principle shows that this is impossible. Hence we have proved that f maps a neighborhood of x_0 in ∂M. As Σ is connected, $f(\Sigma) \subseteq \partial M$.

It remains to prove the last statement of the theorem. If $x_0 \in \partial\Sigma$ and $f(x_0)$ is a point in the interior of H_i, then we let d be the distance function from H_i and prove as above that near x_0, $-d \circ f$ satisfies an equality $L(-d \circ f) \geqq 0$ where L is a second order homogeneous elliptic operator. As $-d \circ f \leqq 0$ and $-d \circ f(x_0) =0$, the maximum principle shows that $\frac{\partial}{\partial n}(d \circ f)(x_0) \neq 0$ which implies that the differential of f is not zero at x_0. If $f(x_0)$ is at the vertex or a point along an edge, we approximate M by M_i so that with respect to a deformed metric, ∂M_i has positive mean curvature with respect to the deformed metric. Let d_i be the distance function to ∂M_i. Then from the way that we construct ∂M_i, we can prove that $\lim_{i \to \infty}(-d_i \circ f)$ satisfies a differential inequality weakly as above.

(Note that for an embedded hypersurface H in a Riemanian manifold, the distance function d from H need not be smooth. However if we restrict it on one side of H and compute the Hessian of $-d$ in the sense of distribution, we can always obtain an upper bound as if d is smooth. Hence we can conclude that the differential of f is not zero at x_0.)

Corollary 1. *Let M be a three-dimensional compact manifold with piecewise smooth boundary with positive mean curvature which satisfies property* (2) *in Theorem* 1. *Let σ be any $C^{4,\alpha}$ Jordan curve on ∂M. Then given any constant k, σ can bound only a finite number of stable minimally immersed disks with areas less than k.*

Proof. Note that an estimate of Schoen [26] shows that a stable minimal disk which is the limit of stable minimal immersions has no interior branch point. Hence Tomi's argument in [30] shows that either the conclusion of the corollary is correct or there is an immersion $F: S^1 \times D \to M$ so that each $f(\{x\} \times D)$ is a minimal disk and F can be extended to $S^1 \times \bar{D}$ as a map with rank 1 on the boundary. The last statement is impossible topologically.

§2. Necessary Condition for a Jordan Curve to Bound an Embedded Stable Surface

In Section 1, we found a sufficient condition for a Jordan curve to bound an embedded stable minimal surface. In this section, we demonstrate that this condition is also necessary.

Theorem 3. *Let $f: \Sigma \to M$ be a strictly stable minimal immersion of a compact two-dimensional smooth surface into a three-dimensional manifold so that $f|\partial\Sigma$ is $C^{2,\alpha}$. Then there exists a three-dimensional manifold N whose boundary ∂N has positive mean curvature with respect to the outward normal. Furthermore, $\Sigma \subset N$, $\Sigma \cap \partial N = \partial\Sigma$ and the map f can be extended to be an isometric immersion of N into M. If $f: \Sigma \to M$ is an embedding, we can also assume the extension is an embedding.*

Proof. Let n be the normal vector field defined by the immersion f. Then if φ is any smooth function defined on Σ with small C^1-norm, $f+\varphi n$ defines an immersed surface.

Let $C^{k,\alpha}(\Sigma)$ and $C^{k,\alpha}(\partial\Sigma)$ be the space of functions on Σ and $\partial\Sigma$ respectively whose k-th derivatives are Hölder continuous with Hölder exponent $0<\alpha<1$. They form a Banach space by the standard norm. We claim that if $\varphi|\partial\Sigma$ is small enough in $C^{2,\alpha}$-norm, then there is an extension φ of $\varphi|\partial\Sigma$ to Σ so that $f+\varphi n$ defines a surface with (small) constant positive mean curvature with respect to the upward normal and φ has small $C^{2,\alpha}$-norm. Furthermore if $\varphi|\partial\Sigma$ is a small positive constant, φ will be positive.

We prove this claim by the implicit function theorem. Define a map from

$$C^{2,\alpha}(\Sigma) \times C^{2,\alpha}(\partial\Sigma) \quad \text{to} \quad C^{0,\alpha}(\Sigma) \times C^{2,\alpha}(\partial\Sigma)$$

by assigning (φ_1, φ_2) to the pair $(H(f+\varphi_1 n), \varphi_1|\partial\Sigma - \varphi_2)$ where $H(f+\varphi_1 n)$ is the mean curvature of the surface $f+\varphi_1 n$ and $\varphi_1|\partial\Sigma$ is the restriction map.

The linearized operator of H at $(\varphi_1, \varphi_2)=(0,0)$ has the form $\Delta - 2K$ where Δ is the Laplacian of the induced metric of f and K is the Gaussian curvature of Σ. By the standard Schauder theory, this operator is invertible if there exists no non-trivial function φ so that $\Delta\varphi - 2K\varphi = 0$ with $\varphi = 0$ on $\partial\Sigma$. However strict stability of Σ guarantees the last condition and hence our nonlinear map is an open map in a neighborhood of $(0,0)$. Hence when $\varphi|\partial\Sigma$ is small in $C^{2,\alpha}$-norm, we can extend it in the required manner.

It remains to prove that the extended solution is positive if $\varphi|\partial\Sigma$ is positive and sufficiently small. Otherwise φ will be non-positive somewhere. As two minimal surfaces cannot touch without crossing each other, φ is in fact negative somewhere. Hence the set $\{x|\varphi(x) \leqq 0\}$ is a compact subdomain of Σ which does not intersect $\partial\Sigma_\varepsilon$. The first eigenvalue of the operator $\Delta - 2K$ with the Dirichlet condition on the domain $\{x|\varphi(x) \leqq 0\}$ is greater than the corresponding first eigenvalue on Σ which is positive (by the strict stability of Σ). On the other hand, the equation for φ can be written in the form $\Delta\varphi - 2K\varphi = O(\|\varphi\|_2^2)$. Hence multiplying this equation by φ and integrating over the domain $\{x|\varphi(x) \leqq 0\}$, one finds an upper bound of the first eigenvalue of Σ in terms of the C^2-norm of φ which is assumed to be small. (The min-max principle is used here with φ as a trial function.) This is a contradiction and we have proved our claim.

We can now prove Theorem 2 in the following way. In the above construction, we can let σ be the curve on Σ which has distance ε (measured in M) from $\partial\Sigma$. Let T_ε be the boundary of the tube with radius ε around σ in M. Then $\partial\Sigma \subset T_\varepsilon$ and when ε is small, T_ε has positive mean curvature with respect to the outward normal.
For each small constant $\alpha > 0$, consider the surface Σ_α^+ and Σ_α^- whose boundaries are given by $f|\partial\Sigma + \alpha n$ and $f|\partial\Sigma - \alpha n$ respectively and whose mean curvature is $+\alpha$ and $-\alpha$ respectively. Then for small α, Σ_α^+ intersects T_ε transversally along two collections of Jordan curves. Denote the collection of Jordan curves closer to $f|\partial\Sigma + \alpha n$ by $\partial\Sigma_\alpha^+$. They bound a subdomain $\tilde{\Sigma}_\alpha^+$ of Σ_α^+. Similarly we define a subdomain $\tilde{\Sigma}_\alpha^-$ in Σ_α^-.

The surface $\tilde{\Sigma}_\alpha^+$, $\tilde{\Sigma}_\alpha^-$ and the part of T_ε bounded by the curves $\partial\Sigma_\alpha^+$ and $\partial\Sigma_\alpha^-$ form a closed surface in a normal bundle of Σ which enclose a domain N'. This three-dimensional manifold N' satisfies properties (1) and (2) of Theorem 1. The proof of Theorem 1 shows that we can smooth out the corner of N' to form the required manifold N.

Corollary 1. *Let $f\colon \Sigma \to M$ be a stable minimal immersion of a compact two-dimensional smooth surface with boundary into a three-dimensional manifold so that $f|\partial\Sigma$ is $C^{2,\alpha}$. Then for any sequence of smooth proper subdomains Σ_i of Σ, we can find a sequence of three-dimensional manifolds N_i so that $\Sigma_i \subset N_i$, $\Sigma_i \cap \partial N_i = \partial\Sigma_i$ and the map f can be extended to be an immersion of N_i into M in such a way that ∂N_i has positive mean curvature with respect to the metric pulled back by f. The areas of ∂N_i can be assumed to be uniformly bounded and the extension of f can be assumed to be an embedding if f is an embedding.*

§3. Nonuniqueness, Embeddedness and the Genus of the Solutions to the Plateau Problem

In this section, we demonstrate that if an extremal Jordan curve bounds a non-embedded stable minimal disk or a minimal surface of genus $\geqq 1$, then it also bounds two distinct embedded minimal disks.

Theorem 4. *Let M be a compact subdomain in a three-dimensional real analytic manifold such that ∂M has non-negative mean curvature with respect to the outward normal. Let $f: \Sigma \to M$ be a branched conformal minimal immersion of a compact surface Σ (with possibly higher genus) into M such that $f|\partial\Sigma$ is an embedding which separates ∂M into components Σ_i. Suppose that $f(\partial\Sigma)$ is smooth and is homotopically trivial in the component of $M \setminus f(\Sigma)$ which contains Σ_i. Then in each such component, $f(\partial\Sigma)$ bounds an embedded stable minimal disk which is disjoint from $f(\Sigma)$ unless $f(\Sigma)$ is an embedded stable disk.*

Proof. Let M_i be the component of $M \setminus f(\Sigma)$ which contains Σ_i. Then we claim that ∂M_i has non-negative mean curvature in the sense of Section 1.

Indeed, as the metric is real analytic, Morrey's theorem [20] shows that the map f is real analytic in the interior of Σ. On the other hand, as $f(\Sigma) \subset M$ and ∂M has non-negative mean curvature with respect to the outward normal, f has no boundary branch point (see Theorem 2) and f is an embedding in a neighborhood of ∂M. Hence by a suitable triangulation of Σ and M, we can assume that f is a simplicial map so that f is an embedding on each triangle of Σ and each branch point of f is mapped to a vertex of the simplex of M.

From the explicit local representation of the branch point given by Osserman [24] and Gulliver [8], we can give a finer triangulation of Σ and M so that the component of $M \setminus f(\Sigma)$ which contains Σ_i satisfies all the hypothesis of Theorem 1. Therefore Theorem 1 guarantees the existence of a stable embedded minimal disk in this component and it cannot touch $f(\Sigma)$ unless it intersects with $f(\Sigma)$ in a nontrivial open set. The well-known unique continuation property of minimal surface (see e.g. [18] Lemma 1) then shows that $f(\Sigma)$ is equal to the stable embedded minimal disk.

Corollary 1. *Let M be a compact subdomain of a real analytic three-dimensional manifold such that ∂M has non-negative mean curvature with respect to the outward normal. Let σ be a smooth embedded Jordan curve in a component C of ∂M which is homeomorphic to S^2. Then either γ bounds at least two distinct stable embedded minimal disks in M or σ has the following strong uniqueness property: The only compact branched minimally immersed surfaces, including surfaces of higher genus, that σ can bound is a unique stable embedded minimal disk.*

Remark. If σ has the strong uniqueness property, then the only compact minimal surface that it bounds is an embedded stable disk given by Douglas-Radó-Morrey.

Proof. The Jordan curve σ divides C into two components C_1 and C_2. If there is a branched minimal immersion $f: \Sigma \to M$ which is not a stable embedded minimal disk, then we can apply Theorem 1 to create two distinct stable embedded minimal disks. These two disks obtained in Theorem 1 locally lie on different sides of $f(\Sigma)$ near γ and hence are distinct which proves the corollary.

Corollary 2. *Let σ be a smooth extremal regular Jordan curve in R^3. Then either σ bounds two distinct stable embedded minimal disks or σ has the strong uniqueness property. In particular, if σ has total curvature not greater than 4π, then the only minimal surface that σ can bound is a unique embedded area minimizing disk.*

Proof. By definition, σ lies on the boundary of some convex surface in R^3. If this convex surface is not smooth, we can approximate it by a smooth one (see e.g. [18]). Hence we can apply Corollary 1. The last part of the corollary follows because of a theorem of Nitsche [22] (see also [10]) states that σ cannot bound more than one minimal disk.

Remark. For a different proof of the above corollary, we refer the reader to [15].

§4. The Existence of Embedded Minimal Surface of Higher Connectivity

In this section, we generalize the theorems in the previous sections to surfaces of higher genus.

Theorem 5. *Let M be a compact subdomain of a three-dimensional real analytic manifold whose boundary has non-negative mean curvature (in the sense of Section 1). Let Σ be a compact subdomain of ∂M such that each homotopically nontrivial closed curve in Σ is also homotopically nontrivial in M. Then there is a stable minimal embedding $f: \Sigma \to M$ so that $f(\partial\Sigma) = \partial\Sigma$. Furthermore if g: $\Sigma' \to M$ is any branched minimal immersion of a compact surface so that $g(\partial\Sigma') = \partial\Sigma$, then we can assume $f(\Sigma) \subset M \setminus g(\Sigma')$.*

Proof. The existence of f was already proved in [28] and [34]. We have to prove that we can choose f to be an embedding.

Suppose $f_1: \Sigma \to M$ is the minimal immersion guaranteed by the existence theorem ([34]). Let C be the component of $M \setminus f_1(\Sigma)$ which contains Σ. Let $\bar{C}$ be the (abstract) metric completion of C. (This means that we cut along the part of $f(\Sigma)$ which is in the interior of the closure of C.)

By the existence theorem mentioned in [34] and the proof of Theorem 1, we can produce a stable branched minimal immersion $g: \Sigma \to C$ whose boundary is mapped homeomorphically onto $\partial\Sigma$ and whose area is not greater than the area of Σ (as a subset of ∂M). For each such g, we define $I(g)$ to be the integral along $\partial\Sigma$ of the dot product of the inward normal of ∂M with the normal of $g(\Sigma)$ which is pointing inward C. (Since $g(\Sigma) \subset C$, it is clear that the dot product is non-negative along $\partial\Sigma$.) Since the area of $g(\Sigma)$ has a fixed upper bound, Morrey and Hildebrandt's proof of the differentiability of g shows that the minimum of I is achieved by some branched minimal immersion which we still denote by g.

By projecting g from C into M, we obtain a branched minimal immersion of Σ into M. We claim that g is an embedding. Indeed, if g were not an embedding into C, we could do the same procedure as above to the component of $C \setminus g(\Sigma)$ which contains Σ and obtain another branched minimal immersion $\bar{g}$: $\Sigma \to M$. Since along $\partial \Sigma$, the dot product of the inward normal of ∂M with the normal of $\bar{g}(\Sigma)$ which points inward C is not greater than the corresponding dot product for $g(\Sigma)$, the minimality of $I(g)$ shows that they are in fact equal everywhere along $\partial \Sigma$. Hence we have two branched minimal immersions g and $\bar{g}$ whose tangent planes are equal along $\partial \Sigma$. The strong maximum principle and the unique continuation then shows that $g = \bar{g}$ (see [18]). If g is not an embedding, g will intersect itself transversally almost everywhere and the construction of $\bar{g}$ shows that $g \neq \bar{g}$ in this case. This contradiction shows that g is an embedding into C. Similar reasoning shows that g is an embedding into M. This finishes the proof of Theorem 5.

The following theorem can be proved in the same way as Theorem 5.

Theorem 6. *Let M be a compact subdomain of a three-dimensional real analytic manifold. Let f: $\Sigma \to M$ be a branched minimal immersion of some compact Riemann surface such that f is an embedding of $\partial \Sigma$ into ∂M. Let Δ be a compact subset of ∂M whose boundary is the same as $\partial \Sigma$.*

If Δ is a subset of some component of $M \setminus f(\Sigma)$ so that the inclusion map of Δ into this component induces an injective map on the fundamental group, then there is a stable minimal embedding of Δ into this component whose boundary is the same as $\partial \Sigma$.

Corollary 1. *Let f: $\Sigma \to M$ be a strictly stable minimal immersion of a compact smooth surface with boundary so that f: $\partial \Sigma \to M$ is $C^{4,\alpha}$. Then for some $\varepsilon > 0$, there is no branched minimal immersion g: $\Sigma \to M$ such that for some homeomorphism σ: $\Sigma \to \Sigma$, $g|\partial \Sigma = f \circ \sigma | \partial \Sigma$ and $\|g - f \circ \sigma\|_0 < \varepsilon$.*

Proof of Corollary 1. Let Σ be a compact subdomain of another compact smooth surface Σ' of the same topological type. By continuity, we can extend f to be an immersion of Σ' into M and by using the exponential map in M, we can extend f to be an immersion of a neighborhood of Σ' in its normal bundle into M. As in the proof of Theorem 3, we can find a family of neighborhoods N_t of Σ in the normal bundle of Σ' with the following properties: (1) $\partial \Sigma \subset \partial N_0$, (2) Σ is in the interior of N_t for $t > 0$ and (3) ∂N_t has positive mean curvature with respect to the pulled back metric $f^* ds^2$.

We claim that for ε small enough and for any map g satisfying the hypothesis of the theorem, we can find a branched immersion g': $\Sigma \to N_1$ so that $g'(\partial \Sigma) = \partial \Sigma$ and $g = f \cdot g'$. In fact, since f is an immersion we can find $\delta > 0$ so that for every ball B_δ with center on $f(\Sigma)$ and radius δ, $f^{-1}(B_\delta) \cap N_1 = \bigcup_i B_i$ where the B_i's are disjoint and $f | B_i$ is a diffeomorphism onto B_δ when B_i is a subset of some fixed neighborhood of Σ. When ε is small, we can choose (small) constant α so that the ε neighborhood of the image (under f) of the disk of radius α is a subset of B_δ. We shall define g' in each such neighborhood so that the image of g' is in an ε-neighborhood of Σ. To define g', we start from the boundary where we simply take $g' = \sigma$. A simple monodromy argument then shows that we can define g' globally.

Clearly g' is a branched minimal immersion with respect to $f^* ds^2$. Since each N_t has positive mean curvature with respect to $f^* ds^2$, Theorem 2 shows that $g'(\Sigma) \subset N_t$ for all $t \geqq 0$ and $g'(\Sigma) \cap \partial N = \partial \Sigma$. We have therefore proved that each branched minimal immersion close to Σ in C^0 sense is a subset of N.

Suppose the corollary is wrong, then we have a sequence g_i': $\Sigma \to N$ so that the C^0 norm between $g_i'(\Sigma)$ and Σ is tending to zero. If $g_i'(\Sigma)$ is not a stable minimal embedding in N, we can apply Theorem 6 to produce two distinct stable minimal embeddings in N. Therefore one of them is distinct from Σ. Call it $\tilde{g}_i$. By choosing N smaller successively, we can then produce a sequence of distinct stable minimal embeddings $\tilde{g}_i$: $\Sigma \to N$. From the way that we produce these g_i's, it is clear that the area of $\tilde{g}_i$ is not greater than the area of ∂N. Hence $\tilde{g}_i(\Sigma)$ has uniformly bounded area.

Since every minimal surface in N with boundary $\partial \Sigma$ has no boundary branch point and cannot touch ∂N in its interior, the arguments of Tomi [30] show that one can produce a circle family of stable minimal embeddings of Σ into N with a fixed boundary $\partial \Sigma$. This contradicts the strict stability of Σ.

§5. The Bridge Principle

The bridge principle is related to a physical property of soap films. This principle can be illustrated by the following experiment. Suppose two soap film surfaces are bounded by two bent steel wires. We can change this wire configuration by joining these wires by parallel wire segments which are close to each other. The experiment shows that usually one can form a soap film surface bounded by this new configuration and the new surface is close to the old surfaces joined together with a soap film bridge joining the old surfaces.

Since soap films correspond to strictly stable minimal surfaces, the bridge principle can be reformulated using the concept of stable minimal surfaces. However, the formulation has to be careful because the bridge principle is not true in its full generality. This can be sen by considering two disks S_1 and S_2 with radius equal to one and two respectively with S_1 contained in the interior of S^2. Suppose both of them are subsets of $x-y$ plane and suppose they are joined by two parallel line segments close to each other on the $x-y$ plane, then the unique compact minimal surface with the new boundary does not contain a bridge.

This example shows that we must not allow the bridge to be arbitrary. We shall show that by perturbing the bridge slightly, the bridge principle is valid. We shall call two disjoint line segments which are close to each other a bridge pair.

Theorem 7 (Bridge Principle). *Let Σ_1 and Σ_2 be two stable orientable compact minimally immersed surfaces in R^n and γ be a Jordan curve joining $\partial \Sigma_1$ and $\partial \Sigma_2$. Then for any tubular neighborhood of γ, we can find a bridge pair joining $\partial \Sigma_1$ and $\partial \Sigma_2$ such that the new configuration is the boundary of a compact minimal surface which is close to the union of Σ_1 and Σ_2 joint by a strip in the tubular neighborhood of γ.*

Proof. Since the proof is similar, we shall assume $n=3$. We first consider the case when Σ_1 and Σ_2 are embedded disjoint surfaces and the curve γ: $[0,1]\to R^3$ is an embedding that intersects $\Sigma_1\cup\Sigma_2$ only in its endpoints.

Suppose that $p=\gamma(0)\in\partial\Sigma_1$ and $q=\gamma(1)\in\partial\Sigma_2$ are endpoints of γ. Then after a small continuous change of γ near p and q, we may assume that $\langle\gamma'(0), n(p)\rangle>0$ and $\langle\gamma'(1), n(q)\rangle>0$ where $n(p)$ is the outward normal of $\partial\Sigma_1$ at p and $n(q)$ is the inward normal of $\partial\Sigma_2$ at q. Furthermore, we can assume that γ is a straight line in a neighborhood of p and q.

By pushing $\partial\Sigma_i$ inside a little in a neighborhood of p and q, we can assume that Σ_i are strictly stable minimal surfaces. Then Theorem 3 shows that we can find neighborhoods N_i of Σ_i with positive mean curvature such that in a neighborhood of $\partial\Sigma_i$, ∂N_i is part of the boundary of a tube $T_i(\varepsilon)$ containing $\partial\Sigma_i$. We claim that when the tube $T_i(\varepsilon)$ has small radius, we can construct a small regular neighborhood R of γ such that the union of R, N_1 and N_2 is a smooth three-dimensional manifold with positive mean curvature.

First of all, we assert that in a neighborhood of $p\in\partial N_1$, we may push N_1 in the outward normal direction to obtain a new neighborhood N_1' of Σ_1 so that $\partial N_1'$ has positive mean curvature and is strictly convex in a neighborhood of p.

If the tube $T_1(\varepsilon)$ is part of a straight cylinder C of radius one and $\partial\Sigma_1\cap T_1(\varepsilon)$ is a part of the generator of the cylinder, then the above assertion is clear. For the general case, we proceed as follows: Let X_C be such a surface with mean curvature $\geqq\frac{1}{2}$. Suppose the radius of the tube $T_1(\varepsilon)$ is ε and p is the origin of R^3. Consider the new tube $\frac{1}{\varepsilon}T_1(\varepsilon)$ obtained by homothetic change. Then as $\varepsilon\to 0$, the $\frac{1}{4}$-neighborhood of p in $\frac{1}{\varepsilon}T_1(\varepsilon)$ converges to part of a cylinder C with radius one smoothly and $\frac{1}{2}\partial\Sigma$ converges to a generator of C. Consider a diffeomorphism g: $R^3\to R^3$ with $g(0)=0$ which takes the $\frac{1}{4}$ neighborhood of $\frac{1}{\varepsilon}T_1(\varepsilon)$ to C which is close enough to the identity so that g^{-1} takes surfaces with mean curvature $\geqq\frac{1}{2}$ to surfaces with positive mean curvature. Now replace the $\frac{1}{4}$ neighborhood of $\frac{1}{\varepsilon}T_1(\varepsilon)$ by $g^{-1}(X_C)$. This new surface has the required convexity property.

The second step is to put the neighborhood of p in R^3 in such a way that p is the origin, γ is on the positive z-axis and N' is in the lower half space. Let C' be the catenoid which is a minimal surface of revolution around the z-axis. It is symmetric with respect to the $x-y$ plane. Multiplying C' by $\frac{1}{r}$ where r is a big constant. As $\partial N_1'$ is strictly convex in a neighborhood of p, $\frac{1}{r}C'$ will be transversal to $\partial N_1'$ and intersect $\partial N_1'$ in two embedded circles near p. (One can write the lower half of the catenoid as the graph of a function defined outside a circle. Instead of scaling the catenoid towards the origin, one can blow up the graph of the function defining $\partial N_1'$. Using the fact that the catenoid grows slowly at infinity, one can estimate the sign of the difference of the above two functions to achieve our conclusion.) Now change N_1' to N_1'' by replacing part of N_1' by the region of R^3 which is "inside" $\frac{1}{r}C'$ and which has non-positive z-coordinate. Since the lower half of the catenoid is strictly stable, we may assume that each piece of N_1'' has positive mean curvature with respect to the outward normal. We can apply a similar construction to N_2 and obtain N_2''. We can also assume the inner radius of the catenoid are the same for both N_1'' and N_2''.

Let R be a tubular neighborhood of γ with radius corresponding to the radius of the circle on the $x-y$ plane defined by the catenoid. Then when this radius is small enough, ∂R has positive mean curvature in the radial direction. Hence the manifold $M = N_1'' \cup R \cup N_2''$ is a piecewise smooth compact three-dimensional manifold satisfying properties (1) and (2) of Section 1. As in Theorem 1, we can perturb M to be a smooth manifold.

Assertion. Let δ_1, δ_2 be a disjoint pair of Jordan arcs in $\partial M \cap \partial R$ such that $\delta_i \cap \Sigma_1 = \delta_i(0)$ and $\delta_i \cap \Sigma_2 = \delta_i(1)$. Suppose that δ_i intersects each two-dimensional plane perpendicular to γ in exactly one point. Then δ_1, δ_2 can serve as a bridge pair.

Proof of the assertion. Let $\{\alpha_1, \ldots, \alpha_{k+1}\}$ be the boundary curves of Σ_1 and $\{\beta_1, \ldots, \beta_{n+1}\}$ be the boundary curves of Σ_2. Let $\alpha_{k+1} \# \beta_{n+1}$ be the connected sum of α_{k+1} with β_{n+1} along the arcs δ_1 and δ_2. Since N_1'' and N_2'' are regular neighborhoods of Σ_1 and Σ_2 respectively, it is straightforward to verify that the system of curves $\Gamma = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_n, \alpha_{k+1} \# \beta_{n+1}\}$ disconnects ∂M into two connected surfaces which are diffeomorphic to $\Sigma_1 \# \Sigma_2$, the connected sum of Σ_1 with Σ_2 along the bridge. Furthermore the inclusion map of each such surface in M is injective on the fundamental group. Hence Theorem 5 shows that there is a stable minimal embedding of $\Sigma_1 \# \Sigma_2$ into M with boundary given by Γ. Call this embedding f. The existence of such f provides a solution to Theorem 7.

Let us now give a geometric analysis of f. First note that $M \setminus (N_1'' \cup N_2'')$ has a foliation by plane disks and the boundary of each disk intersects, by hypothesis, the curve δ_i in one point. We claim that each of these disks intersects $f(\Sigma_1 \# \Sigma_2)$ in a single arc. Otherwise there is a simple closed Jordan curve in the intersection and this Jordan curve bounds a disk on $f(\Sigma_1 \# \Sigma_2)$ as f is injective in fundamental group. However a Jordan curve in a plane is the boundary of a unique minimal disk contained in the plane. Thus $f(\Sigma_1 \# \Sigma_2) \cap R$ has the topological type of a bridge or strip.

We claim that the area of $f(\Sigma_1 \# \Sigma_2) \cap R$ is not greater than the area of ∂R which tends to zero when the radius of R tends to zero. This follows easily from the minimizing property of $f(\Sigma_1 \# \Sigma_2)$. In fact, in the proof of Theorem 5, we know that the area of $f(\Sigma_1 \# \Sigma_2)$ is not greater than the area of any surface which has the same boundary and topology as $\Sigma_1 \# \Sigma_2$ and which lies in a component of $M \setminus f(\Sigma_1 \# \Sigma_2)$. Since the planes perpendicular to γ intersect $f(\Sigma_1 \# \Sigma_2)$ along Jordan arcs, it is easy to deform the part of $f(\Sigma_1 \# \Sigma_2)$ in R to ∂R and form a new surface with the same boundary and topology as $\Sigma_1 \# \Sigma_2$. Comparing the area of this surface with $f(\Sigma_1 \# \Sigma_2)$, we conclude that $\mathrm{Area}(f(\Sigma_1 \# \Sigma_2) \cap R) \leq \frac{1}{2}\mathrm{Area}(\partial R)$.

In a similar manner, one can prove that $\mathrm{Area}(f(\Sigma_1 \# \Sigma_2) \cap N_1'') \leq \frac{1}{2}\mathrm{Area}(\partial N_1'')$ and $\mathrm{Area}(f(\Sigma_1 \# \Sigma_2) \cap N_2'') \leq \frac{1}{2}\mathrm{Area}(\partial N_2'')$. By the construction of N_i'', we can clearly assume that $\frac{1}{2}\mathrm{Area}(\partial N_i'')$ is close to $\mathrm{Area}(\Sigma_i)$. On the other hand, as Corollary 1 of Theorem 6 shows, we can also choose N_i'' so that Σ_i is area minimizing in N_i''. Hence $\mathrm{Area}(\Sigma_i)$ is not greater than $\mathrm{Area}(f(\Sigma_1 \# \Sigma_2) \cap N_i'')$ plus the area of the disk perpendicular to $\gamma(0)$ or $\gamma(1)$. In conclusion we have proved $\mathrm{Area}(f(\Sigma_1 \# \Sigma_2) \cap N_i'') \to \mathrm{Area}(\Sigma_i)$ as M tends to $\Sigma_1 \cup \Sigma_2 \cup \gamma$. (Note that one can give an explicit estimate for the above convergence.)

Let us now sketch a proof that when M tends to $\Sigma_1 \cup \Sigma_2 \cup \gamma$, $f(\Sigma_1 \# \Sigma_2)$ is a graph over $\Sigma_1 \# \Sigma_2$ which is the connected sum of Σ_1 with Σ_2 along the bridge in R.

By choosing $\alpha_{k+1} \# \beta_{n+1}$ and M suitably, we may assume that in a neighborhood of p, $\alpha_{k+1} \# \beta_{n+1}$ is a graph over a two-dimensional plane A such that locally M lies on one side of the surface generated by e, the normal of A, along $\alpha_{k+1} \# \beta_{n+1}$. If v is the normal of $f(\Sigma_1 \# \Sigma_2)$, then the boundary maximum principle (Theorem 2) shows that $\langle v, e\rangle \neq 0$ along $\alpha_{k+1} \# \beta_{n+1}$.

We show that when M is close enough to $\Sigma_1 \cup \Sigma_2 \cup \gamma$, the surface $f(\Sigma_1 \# \Sigma_2)$ in this neighborhood is a graph over A. First of all, note that if Σ is an immersed simply connected surface bounded by two parallel lines in R^3, then Σ is linear. In fact, by the Radó argument, one can prove that Σ is a graph over the linear space containing $\partial\Sigma$. By reflecting Σ along its boundary, one can extend Σ to be a minimal graph defined over R^2. Bernstein's theorem then shows that Σ is linear. (By Theorem 4, we do not have to assume Σ is simply connected.)

Since we assume that the bridge pair (δ_1, δ_2) is a pair of parallel line segments in the neighborhood of p, we can blow up the minimal surface $f(\Sigma_1 \# \Sigma_2)$ bounded by (δ_1, δ_2) and apply the above assertion to prove the following statement. If δ_1 and δ_2 is close enough, we can join two points on δ_1 and δ_2 by a Jordan curve in R which lies on $f(\Sigma_1 \# \Sigma_2)$ and projects one to one into the plane A.

On the other hand, the classical interior estimate and the boundary estimate for the Plateau problem [11] shows that outside a fixed neighborhood of p, $f(\Sigma_1 \# \Sigma_2) \cap N_1''$ is close to Σ_1 in smooth norm. Hence we can prove that in a suitable neighborhood of p, $f(\Sigma_1 \# \Sigma_2)$ is bounded by a Jordan curve along which $\langle v, e\rangle \neq 0$. For convenience we assume $\langle v, e\rangle > 0$.

However, it is well-known that over $f(\Sigma_1 \# \Sigma_2)$, $\Delta\langle v, e\rangle = -(\sum_{i,j} h_{ij}^2)\langle v, e\rangle$ where h_{ij} is the second fundamental form of $f(\Sigma_1 \# \Sigma_2)$. If $\langle v, e\rangle$ were not positive in the domain bounded by the above Jordan curve in $f(\Sigma_1 \# \Sigma_2)$, then the maximum principle shows that the set $\langle v, e\rangle < 0$ is a non-empty proper subdomain. (It is proper because $\langle v, e\rangle > 0$ on the boundary of the domain.) This proper subdomain must be strictly stable. However, one can deform this subdomain in the direction $\langle v, e\rangle v$ and obtain a contradiction. Hence we conclude that $\langle v, e\rangle > 0$ in the domain bounded by the above mentioned Jordan curve and so this domain is a graph over A.

The part of $f(\Sigma_1 \# \Sigma_2)$ in the middle part of R can be treated in a similar way as above and we have therefore proved that $f(\Sigma_1 \# \Sigma_2)$ is a graph over $\Sigma_1 \# \Sigma_2$ when M tends to $\Sigma_1 \cup \Sigma_2 \cup \gamma$. This finishes our claim when Σ_1 and Σ_2

are embedded. The proof for the general case is done by looking at the normal bundles of these surfaces abstractly.

Classically the bridge principle is used to construct a rectifiable curve can bound an infinite number of stable minimal disks. To achieve this, we have to modify Theorem 7 to the following theorem.

Theorem 8. *Let Σ_1^i be a finite set of stable minimally immersed surfaces with the same boundary $\partial\Sigma_1$ in R^3. Suppose that for each i, there is a compact three-dimensional manifold N_1^i such that ∂N_1^i has positive mean curvature with respect to the outward normal, $\partial\Sigma_1 \subset \partial N_1^i$ and N_1^i is a regular neighborhood of Σ_1^i. Let Σ_2^i be another finite set of stable minimally immersed surfaces with the similar properties. Let γ be a Jordan arc joining a point p on $\partial\Sigma_1$ to a point q on $\partial\Sigma_2$ so that $\gamma'(0)$ is transversal to Σ_1^i at p for all i and $\gamma'(1)$ is transversal to Σ_2^i at q for all i. Then we can find a bridge pair (δ_1, δ_2) close to γ joining $\partial\Sigma_1$ to $\partial\Sigma_2$ so that the new boundary configuration bounds distinct stable minimal surfaces $\Sigma_1^i \# \Sigma_2^j$ which are close to $\Sigma_1^i \cup \Sigma_2^j \cup \gamma$ respectively. Furthermore we can assume that these stable minimal surfaces have regular neighborhoods with properties similar to N_1^i.*

Remark. In order to construct minimal surfaces satisfying the hypothesis of Theorem 8, it suffices to assume that the Σ_1^i's are strictly stable (by Theorem 3). However there is a criterion due to do Carmo-Barbosa which guarantees that a minimal surface is strictly stable if its total absolute total curvature is less than 2π. One can easily find a smooth Jordan curve which bounds two distinct minimal disks with absolute total curvature less than 2π. Theorem 8 then provides a rigorous proof of the classical example due to Levy and Courant (see [6]) of a Jordan curve bounding uncountably many stable minimal disks.

Proof. We shall show how to modify the proof of Theorem 5 to obtain a bridge pair which works for all Σ_1^i's. For each i, we can construct $N_1^{i\prime\prime}$ as in Theorem 7. Clearly part of the line segment of γ near p belongs to $\bigcap_i N_1^{i\prime\prime}$. Hence we can choose δ_1 and δ_2 close to γ so that for all (i,j) $\alpha_{k+1} \# \beta_{n+1}$ is a subset of $N_1^{i\prime\prime} \cup R_{ij} \cup N_2^{j\prime\prime}$ where R_{ij} and $N_2^{j\prime\prime}$ are constructed as in Theorem 5. Since $N_1^{i\prime\prime} \cup R_{ij} \cup N_2^{j\prime\prime}$ has positive mean curvature with respect to the outward normal and since $\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_n, \alpha_{k+1} \# \beta_{n+1}\}$ bounds an incompressible surface homeomorphic to $\Sigma_1^i \# \Sigma_2^j$, we can apply the existence theorems in [25], [28], and [34] to prove the existence of a stable incompressible minimal surface in $N_1^{i\prime\prime} \cup R_{ij} \cup N_2^{j\prime\prime}$ homeomorphic to $\Sigma_1^i \# \Sigma_2^j$.

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Received June 23, 1980; received in final form March 26, 1981