# Arithmetic Properties of Mirror Map and Quantum Coupling ${ }^{\diamond}$ 

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#### Abstract

We study some arithmetic properties of the mirror maps and the quantum Yukawa coupling for some 1-parameter deformations of Calabi-Yau manifolds. First we use the Schwarzian differential equation, which we derived previously, to characterize the mirror map in each case. For algebraic K3 surfaces, we solve the equation in terms of the $J$-function. By deriving explicit modular relations we prove that some K3 mirror maps are algebraic over the genus zero function field $\mathbf{Q}(J)$. This leads to a uniform proof that those mirror maps have integral Fourier coefficients. Regarding the maps as Riemann mappings, we prove that they are genus zero functions. By virtue of the Conway-Norton conjecture (proved by Borcherds using Frenkel-Lepowsky-Meurman's Moonshine module), we find that these maps are actually the reciprocals of the Thompson series for certain conjugacy classes in the Griess-Fischer group. This also gives, as an immediate consequence, a second proof that those mirror maps are integral. We thus conjecture a surprising connection between K3 mirror maps and the Thompson series. For threefolds, we construct a formal nonlinear ODE for the quantum coupling reduced $\bmod p$. Under the mirror hypothesis and an integrality assumption, we derive mod $p$ congruences for the Fourier coefficients. For the quintics, we deduce (at least for $5 \nmid d$ ) that the degree $d$ instanton numbers $n_{d}$ are divisible by $5^{3}$ - a fact first conjectured by Clemens.


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## 1. Introduction

For background on Mirror Symmetry, the readers are referred to the reference [1] [2] (see especially the articles therein by Greene-Plesser, Candelas-de la Ossa-Green-Parkes, Katz, Morrison, Vafa and Witten.)

It is known that the so-called mirror map and the quantum coupling have many interesting number theoretic properties based on numerical experiments - as previously observed by many [1] [3] [4] [5]. For example the Fourier coefficients of the mirror map appears to be integral in all known cases. In some cases, the coefficients even appear to be alternating. The instanton numbers $n_{d}$ in the quantum coupling on the other hand, apparently have some striking divisibility property. Clemens conjectured that for the quintics in $\mathbf{C P}^{4}, 5^{3} \mid n_{d}$ for all $d[6]^{*}$. This was supported by Katz' proof that $5 \mid n_{d}$, along with similar divisibility properties for other manifolds [7]. Our main motivation here is to develop some techniques, along with mirror symmetry, to understand some of these remarkable "arithmetic" properties.

The technique for studying the mirror map is based on the following simple idea: fix a known integral series $f(q)$. Study when is $z(q)$ commensurable with $f(q)$ (ie. when do $z(q), f(q)$ satisfy a polynomial relation)? Hopefully when enough is known about $f(q)$, then given a polynomial $p(X, Y)$ we can understand some of the arithmetic properties of the root $z(q)$ to $p(f(q), Y)=0$. Note that this is still a difficult Diophantine type problem in general, involving infinitely many variables consisting of the Fourier coefficients of $z(q)$.

However, a lot is known about the j-function both number theoretically and geometrically. Thus it is natural to try to find commensurability relations (also known as modular relations) between $j(q), z(q)$. We will see that this idea works well in the case of elliptic curves and K3 surfaces.

The technique for studying the instanton numbers is based on the fact that there is a canonical polynomial ODE for the quantum coupling, which is defined over $\mathbf{Q}$. This raises the possibility of deriving similar equations, but reduced $\bmod p$. The $\bmod p$ arithmetic properties of the quantum couplings should then be reflected in these reduced equations. In [8] [9], in collaboration with A. Klemm and S.S. Roan we have constructed an ODE over Q. However, its mod $p$ counterpart derived here appears much simpler and more manageable. We now summarize our discussion.

[^1]First we discuss, along the lines of [4] [5], the construction of deformation coordinates based on which the mirror map is defined. Then We review the polynomial differential equations for the mirror maps, studied in [8] [9]. First we classify the analytic solutions, on a disk, to the $n^{t h}(\mathrm{n}=2,3$ or 4$)$ Schwarzian equations associated to certain $n^{\text {th }}$ order linear ODEs of Fuchsian type. As a consequence, the so-called mirror map $z$ can be given a simple characterization. Corresponding to $\mathrm{n}=2,3$ or 4 , there is a universal family of polynomials whose evaluation at certain integral points recovers the Fourier coefficients of a mirror map.

We consider some examples in which the linear ODEs are the Picard-Fuchs equations of several distinguished families of smooth Calabi-Yau varieties in weighted projective spaces. We revisit the case of elliptic curves ( $n=2$ ).

In the case of $\mathrm{n}=3$, we consider some distinguished families of K3 surfaces in weighted projective spaces. We prove that the the mirror map $z$ for each of those families is algebraic over the function field $\mathbf{Q}(J)$ generated by the Dedekind-Klein J-function - a rather surprising connection between modular functions and K3 surfaces. Our result on the $\mathrm{n}=3$ Schwarzian equation is an important tool in this connection. We then use our explicit modular relations to give a uniform proof that the Fourier coefficients of $z$ are integral - a fact which has been previously observed experimentally. This is also the first confirmation of the integrality property in the case of K3 surfaces.

We then discuss a mysterious connection between our mirror maps and the Thompson series. We offer some speculation as to why the two might be related. We conjecture that any 1-parameter deformation of algebraic K3, determined by an orbifold construction, gives rise to a Thompson series.

In the last section under an integrality assumption and the Mirror Hypothesis, we study our differential equations reduced $\bmod p$. We derived some general $\bmod p$ congruences and then specialize to $p=2,3,5,7$. For the quintic hypersurface, we deduce Clemens' conjecture that the degree $d$ instanton numbers $n_{d}$ are divisible by $5^{3}$ (at least for $5 \times d$ ).

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## 2. Deformation coordinates

The purpose here is to review the orbifold construction of the mirror map and to give an explicit description of the deformation coordinates for complex structures. The orbifold
construction in weighted projective spaces has now been superceded by toric geometry construction [10] [4]. But in the former, the description of the deformation coordinates can be made rather explicit. Here we will adopt the multi-indexed convention for monomials $y^{I}=y_{1}^{i_{1}} \cdots y_{m}^{i_{m}}$ whenever the meaning of the variables $y$ is clear.

Let $X$ be a $l:=(n-k-1)$-dimensional Calabi-Yau variety defined as the zero locus of $k$ homogeneous polynomials $p_{1}, . ., p_{k}$ whose degrees are $d_{1}, . ., d_{k}$, in the weighted projective space $\mathbf{P}^{n-1}[w]$. Here $w=\left(w_{1}, . ., w_{n}\right)$ are the weights consisting of coprime positive integers. Suppose that $X$ has a mirror $X^{*}$ given by an orbifold construction [11] |12] [1]. (Thus $X^{*}$ is assumed to have the usual mirror Hodge diamond.) The space $X^{*}$ is a resolution $\widehat{X / G}$ of the singular quotient $X / G$, where $G$ is a finite abelian group acting on the homogeneous coordinates $x_{i}$ of the ambient space $\mathbf{P}^{n-1}[w]$ by characters $\chi_{i}: G \rightarrow S^{1}$. They are assumed to satisfy $\chi_{1} \cdots \chi_{n}=1$, which ensures that the holomorphic top form on $X$ is $G$-invariant, and hence induces a form on $X^{*}$. We now want to describe some complex structure deformations of $X^{*}$. We do so by describing a family of $G$-invariant holomorphic top forms $\Omega(z)$ on $X$. Thus ultimately, our deformation space $M$ will be parameterizing a family of $G$-invariant deformations of the $p_{j}$.

Fix $m_{j} G$-invariants monomials which we denote by $x^{I_{i}}, i \in N_{j}:=\left\{\left(m_{1}+. .+m_{j-1}\right)+\right.$ $\left.1, . .,\left(m_{1}+. .+m_{j-1}\right)+m_{j}\right\}$. Let $m:=m_{1}+\cdots+m_{k}$. Let $F$ be the following holomorphic family, fibered over $\left(\mathbf{C}^{\times}\right)^{m}$, of varieties. Its fiber at $a=\left(a_{1}, . ., a_{m}\right)$ is defined as the zero locus of

$$
\begin{equation*}
\tilde{p}_{j}(a ; x)=\sum_{i \in N_{j}} a_{i} x^{I_{i}} . \tag{2.1}
\end{equation*}
$$

We assume that $X$ is the fiber at some limiting value of $a$ and that the generic fibers are homeomorphic to $X$. Now two fibers can be biholomorphic simply by coordinate transformation. Specifically for any $\alpha=\left(\alpha_{1}, . ., \alpha_{n+k}\right) \in\left(\mathbf{C}^{\times}\right)^{n+k}$, the transformation $x_{i} \mapsto x_{i} / \alpha_{i}, a_{i} \mapsto a_{i} \alpha_{1}^{i_{1}} \cdots \alpha_{n}^{i_{n}} \alpha_{n+j} \equiv a_{i} \alpha^{J_{i}}$ where $\left(i_{1}, . ., i_{n}\right)=I_{i}$, transforms $\tilde{p}_{j} \mapsto \alpha_{n+j} \tilde{p}_{j}$. Call this transformation group $H$, and consider the quotient family $F / H \rightarrow\left(\mathbf{C}^{\times}\right)^{m} / H=$ $M$. The base $M$ will be our deformation space. The isotropy group of the $H$-action on $\left(\mathbf{C}^{\times}\right)^{m}$ is $H^{\prime}=\left\{\alpha \mid \alpha^{J_{i}}=1\right.$, all $\left.i\right\}$. The conditions $\alpha^{J_{i}}=1$ mean that the dot products $J_{i} \cdot\left(\log \alpha_{1}, . ., \log \alpha_{n+k}\right)=0$. Or if $K$ is the $m \times(n+k)$ matrix of exponents whose rows are the $J_{i}$, then $K \cdot(\log \alpha)^{t}=0$. Thus $r k K=\operatorname{dim} H-\operatorname{dim} H^{\prime}$, implying that $\operatorname{dim} M=m-r k K$.

We wish to construct some coordinates on $M$. Now $H$ acts on the coordinate ring $\mathbf{C}\left[a_{1}^{ \pm 1}, . ., a_{m}^{ \pm 1}\right]$ of $\left(\mathbf{C}^{\times}\right)^{m}$ by $a_{i} \mapsto a_{i} \alpha^{J_{i}}$. A simple way to coordinatize $M=\left(\mathbf{C}^{\times}\right)^{m} / H$
would be to find enough suitable $H$-invariant functions $f:\left(\mathbf{C}^{\times}\right)^{m} \rightarrow \mathbf{C}^{\times}$. An $H$-invariant Laurent monomial $a^{\mu}$ is one with $\mu_{1} J_{1}+\cdots+\mu_{m} J_{m}=0$. The set of such $\mu$ is the lattice $L=\left(\right.$ ker $\left.K^{t}\right) \cap \mathbf{Z}^{m}$ which has rank $\operatorname{dim}\left(\right.$ ker $\left.K^{t}\right)=m-r k K^{t}=\operatorname{dim} M$. The subalgebra of $H$-invariants in $\mathbf{C}\left[a_{1}^{ \pm 1}, \ldots, a_{m}^{ \pm 1}\right]$ is the group algebra $\mathbf{C}[L]$ which is canonically generated by $a^{ \pm B}$, for a given basis $\{B\}$ of $L$. Thus for every basis $\{B\}$, we get a canonical set of functions $z=\left(a^{B}\right)$ globally defined on $M$. The $z$ will be our deformation coordinates.

Since any two bases $\{B\},\left\{B^{\prime}\right\}$ are related by $B=\sum m_{B B^{\prime}} B^{\prime}$ where $\left(m_{B B^{\prime}}\right)$ is an integral matrix, $z=\left(a^{B}\right) \mapsto z^{\prime}=\left(a^{B^{\prime}}\right)$ is a birational change. Note that if the $z$ take the value of integral $q$-series, then so do the image $z^{\prime}$ under the above change. Thus when the $z_{i}$ takes the value of the mirror map which has a q -series expansion, the question of integrality of the Fourier coefficients is independent of the choice of basis of $L$. In case $\operatorname{dim} M=1$, which is all we are going to deal with here, the coordinate $z$ above is unique up to $z \mapsto 1 / z$. But demanding that $z=0$ is the point with maximum unipotent monodromy [13] for the Picard-Fuchs equation, fixes the choice completely.

### 2.1. Some examples

Let's first consider the simplest example: cubics in $\mathbf{P}^{2}$. Let $G$ be a cyclic group of order 3 and act on the homogeneous coordinates by $x_{i} \mapsto \xi^{i} x_{i}$ where $\xi=e^{2 \pi i / 3}$. Then the $G$-invariant cubic monomials are $x_{1}^{3}, x_{2}^{3}, x_{3}^{3}, x_{1} x_{2} x_{3}$. We consider the invariant family $F$ which is the zero locus of the polynomial

$$
\begin{equation*}
\tilde{p}(a ; x)=a_{1} x_{1}^{3}+a_{2} x_{2}^{3}+a_{3} x_{3}^{3}+a_{4} x_{1} x_{2} x_{3} . \tag{2.2}
\end{equation*}
$$

The group $H=\left(\mathbf{C}^{\times}\right)^{3+1}$ acts on $F$ as described above. On the base space $\left(\mathbf{C}^{\times}\right)^{4}$, it acts by $\alpha: a \mapsto\left(\alpha_{1}^{3} a_{1}, \alpha_{2}^{3} a_{2}, \alpha_{3}^{3} a_{3}, \alpha_{1} \alpha_{2} \alpha_{3} a_{4}\right) \alpha_{4}$. We see that the $H$-invariant functions are generated by $z=a_{1} a_{2} a_{3} a_{4}^{-3}$. It's easy to check that this function defines an isomorphism $\left(\mathbf{C}^{\times}\right)^{4} / H \rightarrow \mathbf{C}^{\times}$.

Consider now the sextics in $\mathbf{P}^{3}[1,1,2,2]$. Let $G$ be a finite abelian group of type $(6,3,3)$, which acts on the homogeneous coordinates by

$$
\begin{equation*}
x_{1} \mapsto \xi_{1} \xi_{2} \xi_{3} x_{1}, x_{2} \mapsto \xi_{1}^{-1} x_{2}, x_{3} \mapsto \xi_{2}^{-1} x_{3}, x_{4} \mapsto \xi_{3}^{-1} x_{4} \tag{2.3}
\end{equation*}
$$

where $\xi_{1}, \xi_{2}, \xi_{3}$ are respectively arbitrary 6 th, 3 rd, 3 rd roots of 1 . The $G$-invariants monomials are $x_{1}^{6}, x_{2}^{6}, x_{3}^{3}, x_{4}^{3}, x_{1} x_{2} x_{3} x_{4}, x_{1}^{3} x_{2}^{3}$. As before we can write down a linear sum of them with coefficients $a_{1}, . ., a_{6}$. The matrix of exponents $K$ defined above is the $6 \times 5$ matrix with
rows: $(6,0,0,0,1),(0,6,0,0,1),(0,0,3,0,1),(0,0,0,3,1),(1,1,1,1,1),(3,3,0,0,1)$. The lattice $L$ therefore has a base $(1,1,0,0,0,-2),(0,0,1,1,-3,1)$. This gives us the deformation coordinates $z=\left(z_{1}, z_{2}\right)$ with $z_{1}=a_{1} a_{2} a_{6}^{-2}, z_{2}=a_{3} a_{4} a_{5}^{-3} a_{6}$.

As a third example, we consider the sextics in $\mathbf{P}^{3}[1,1,1,3]$. Let $G$ be a finite abelian group of type $(6,6,2)$, which acts on the homogeneous coordinates by

$$
\begin{equation*}
x_{1} \mapsto \xi_{1} \xi_{2} \xi_{3} x_{1}, x_{2} \mapsto \xi_{1}^{-1} x_{2}, x_{3} \mapsto \xi_{2}^{-1} x_{3}, x_{4} \mapsto \xi_{3}^{-1} x_{4} \tag{2.4}
\end{equation*}
$$

where $\xi_{1}, \xi_{2}, \xi_{3}$ are respectively arbitrary 6 th, 6 th, 2 nd roots of 1 . The $G$-invariants monomials are $x_{1}^{6}, x_{2}^{6}, x_{3}^{6}, x_{4}^{2}, x_{1} x_{2} x_{3} x_{4}, x_{1}^{2} x_{2}^{2} x_{3}^{2}$. As before we can write down a general linear sum of them to define our family of sextics. But note that any sum of the form $a x_{4}^{2}+b x_{1} x_{2} x_{3} x_{4}+c x_{1}^{2} x_{2}^{2} x_{3}^{2}$ can be written as $a x_{4}^{2}+b^{\prime} x_{1} x_{2} x_{3} x_{4}$, by a suitable redefinition $x_{1} \mapsto x_{1}, x_{2} \mapsto x_{2}, x_{3} \mapsto x_{3}, x_{4} \mapsto x_{4}+\lambda x_{1} x_{2} x_{3}$. Thus we consider the family:

$$
\begin{equation*}
p(a ; x)=a_{1} x_{1}^{6}+a_{2} x_{2}^{6}+a_{3} x_{3}^{6}+a_{4} x_{4}^{2}+a_{5} x_{1} x_{2} x_{3} x_{4} . \tag{2.5}
\end{equation*}
$$

The matrix of exponents $K$ is the $5 \times 5$ matrix with rows: $(6,0,0,0,1),(0,6,0,0,1)$, $(0,0,3,0,1),(0,0,0,3,1),(1,1,1,1,1)$. The lattice $L$ therefore has a base $(1,1,1,3,-6)$. This gives us the deformation coordinate $z=a_{1} a_{2} a_{3} a_{4}^{3} a_{5}^{-6}$.

### 2.2. Definition of the mirror map

Following [1] [13] [4], we now define the mirror map in terms of the coordinates $z$ by studying variation of the periods for $X^{*}$. By construction, the holomorphic top form $\Omega(z)$ on $X$ induces a form on $X^{*}$. Integrating the form against the $G$-invariant cycles, we obtain periods for $X^{*}$. By the Dwork-Griffith-Katz reduction method, we get a system of PicardFuchs differential equation(s) for those periods. If $z=0$ is a maximal unipotent point, then the system admits a unique solution $\omega_{0}(z)$ which is holomorphic near $z=0$ with $\omega_{0}(0)=1$, and $s:=\operatorname{dim} M$ independent solutions of the form $\omega_{i}(z)=\omega_{0}(z) \log z_{i}+g_{i}(z)$ where the $g_{i}$ are holomorphic near $z=0$ with $g_{i}(0)=0$. We call the mapping defined by $q_{j}:=e^{2 \pi i t_{j}}=z_{j} e^{\frac{g_{j}(z)}{\omega_{0}(z)}}$ the mirror map. For convenience, we also refer to the inverse $z_{j}(q)$ as the mirror map. Thus by the construction above, the mirror map for an $s$-parameter family of Calab! i-Yau mirror pairs can be regarde

## 3. Construction of the Schwarzian equations

We now discuss the construction of the differential equation which governs the mirror map $z(t)$. We begin with an $n^{t h}$ order ODE of Fuchsian type:

$$
\begin{equation*}
L f:=\left(\frac{d^{n}}{d z^{n}}+\sum_{i=0}^{n-1} q_{i}(z) \frac{d^{i}}{d z^{i}}\right) f=0 \tag{3.1}
\end{equation*}
$$

( $n$ will specialize to 2,3 and 4 later.) In particular, the $q_{i}(z)$ are rational functions of $z$. Let $f_{1}, f_{2}$ be two linearly independent solutions of this equation and consider the ratio $t:=f_{2}(z) / f_{1}(z)$. Inverting this relation (at least locally), we obtain $z$ as a function of $t$. Our goal is to derive a polynomial ODE, in a canonical way, for $z(t)$.

We first perform a change of coordinates $z \rightarrow t$ on (3.1) and obtain:

$$
\begin{equation*}
\sum_{i=0}^{n} b_{i}(t) \frac{d^{i}}{d t^{i}} f(z(t))=0 \tag{3.2}
\end{equation*}
$$

where the $b_{i}(t)$ are rational expressions of the derivatives $z^{(k)}$ (including $z(t)$ ). For example we have $b_{n}(t)=a_{n}(z(t)) z^{\prime}(t)^{-n}$. It is convenient to put the equation in reduced form. We do a change of variable $f=A g$, where $A=\exp \left(-\int \frac{b_{n-1}(t)}{n b_{n}(t)}\right)$, and multiply (3.2) by $\frac{1}{A b_{n}}$ so that it becomes

$$
\begin{equation*}
\tilde{L} g:=\left(\frac{d^{n}}{d t^{n}}+\sum_{i=0}^{n-2} c_{i}(t) \frac{d^{i}}{d t^{i}}\right) g(z(t))=0 \tag{3.3}
\end{equation*}
$$

where $c_{i}$ is now a rational expression of $z(t), z^{\prime}(t), . ., z^{(n-i+1)}$ for $i=0, . ., n-2$. Now $g_{1}:=f_{1} / A$ and $g_{2}:=f_{2} / A=t g_{1}$ are both solutions to the equation (3.3). In particular we have

$$
\begin{align*}
& P:=\tilde{L} g_{1}=\left(\frac{d^{n}}{d t^{n}}+\sum_{i=0}^{n-2} c_{i}(t) \frac{d^{i}}{d t^{i}}\right) g_{1}=0 \\
& Q:=\tilde{L}\left(t g_{1}\right)-t \tilde{L} g_{1}=\left(n \frac{d^{n-1}}{d t^{n-1}}+\sum_{i=0}^{n-3}(i+1) c_{i+1}(t) \frac{d^{i}}{d t^{i}}\right) g_{1}=0 . \tag{3.4}
\end{align*}
$$

Note that since $c_{i}$ is a rational expression of $z(t), z^{\prime}(t), . ., z^{(n-i+1)}(t)$, it follows that $P$ involves $z(t), . ., z^{(n+1)}(t)$ while $Q$ involves only $z(t), . ., z^{(n)}(t)$. Eqns (3.4) may be viewed as a coupled system of differential equations for $g_{1}(t), z(t)$. Our goal is to eliminate $g_{1}(t)$ so that we obtain an equation for just $z(t)$. One way to construct this is as follows. By (3.4), we have

$$
\begin{align*}
& \frac{d^{i}}{d t^{i}} P=0, \quad i=0,1, . ., n-2  \tag{3.5}\\
& \frac{d^{j}}{d t^{j}} Q=0, \quad j=0,1, . ., n-1
\end{align*}
$$

We now view (3.5) as a homogeneous linear system of equations:

$$
\begin{equation*}
\sum_{l=0}^{2 n-2} M_{k l}\left(z(t), . ., z^{(2 n-1)}(t)\right) \frac{d^{l}}{d t^{l}} g_{1}=0, \quad k=0, . ., 2 n-2 \tag{3.6}
\end{equation*}
$$

where $\left(M_{k l}\right)$ is the following $(2 n-1) \times(2 n-1)$ matrix:

$$
\left(\begin{array}{cccccccc}
c_{0} & c_{1} & . . & c_{n-2} & 0 & 1 & 0 & . .  \tag{3.7}\\
c_{0}^{\prime} & c_{0}+c_{1}^{\prime} & . . & c_{n-3}+c_{n-2}^{\prime} & . & 0 & 1 & 0 \\
0 \\
c_{0}^{(n-2)} & (n-2) c_{0}^{(n-3)}+c_{1}^{(n-2)} & . . & . . & \ldots & & & . \\
c_{1} & 2 c_{2} & . . & (n-2) c_{n-2} & . & . & . & 0 \\
1 \\
c_{1}^{\prime} & c_{1}+2 c_{2}^{\prime} & . . & (n-3) c_{n-3}+(n-2) c_{n-2}^{\prime} & 0 & 0 & 0 & . . \\
& \ldots & \ldots & 0 & . . & 0 \\
c_{1}^{(n-1)} & (n-1) c_{1}^{(n-2)}+2 c_{2}^{(n-1)} & . . & \ldots & \ldots & & . . & \\
c_{1}^{\prime} & \ldots & . & . & 0 & n
\end{array}\right)
$$

More precisely if we define the $1^{\text {st }}$ and $n^{t h}$ ( $n$ fixed) row vectors to be $\left(M_{1 l}\right)=$ $\left(c_{0}, c_{1}, . ., c_{n-2}, 0,1,0, . ., 0\right)$ and $\left(M_{n l}\right)=\left(c_{1}, 2 c_{2}, . .,(n-2) c_{n-2}, 0, n, 0, . ., 0\right)$ respectively, then the matrix $\left(M_{k l}\right)$ is given by the recursion relation:

$$
\begin{equation*}
M_{k+1, l}=M_{k, l-1}+M_{k, l}^{\prime}, \quad l=1, . ., 2 n-1 ; \quad k=1, . ., n-2, n, . ., 2 n-2 \tag{3.8}
\end{equation*}
$$

Thus the $\left(M_{k l}\right)$ depends rationally on $z(t), . ., z^{(2 n-1)}(t)$. Since $g_{1}$ is nonzero, it follows that

$$
\begin{equation*}
\operatorname{det}\left(M_{k l}\left(z(t), . ., z^{(2 n-1)}(t)\right)\right)=0 \tag{3.9}
\end{equation*}
$$

We call this the $n^{t h}$ Schwarzian equation associated with (3.1). Note that by suitably clearing denominators, the equation becomes a $(2 n-1)^{s t}$ order polynomial ODE for $z(t)$ with constant coefficients. It is clear that this equation depends only on the data $q_{i}(z)$ we began with.

## 4. The analytic solutions to the Schwarzian equation

It was already known to H.A. Schwarz that all the solutions to his equation ( $\mathrm{n}=2$ ) can be constructed from the original 2 nd order ODE. In this section, we wish to generalizes his result to the other higher Schwarzian equations. We will focus on those which arise only from cases in which the data $q_{i}(z)$ come from the Picard-Fuchs equation of a smooth Calabi-Yau hypersurface (or complete intersections) in a weighted projective space. We will show that there is exactly 1-parameter family of single-valued analytic solutions, on a
disk, to the Schwarzian equation. It is instructive to first go back to the $n=2$ case (elliptic curves). While Schwarz's treatment in this case focus on the solution ratios (triangle functions), it is less direct for studying their inverses. Instead we will study the inverses directly. This will have the advantages of a) exhausting all the analytic solutions directly; b) seeing manifestly that the solutions have polynomial dependence on the parameters; and c) generalizing immediately to higher Schwarzian equations.

## 4.1. $n=2$ Schwarzian equation

We begin with a Fuchsian equation of the general form:

$$
\begin{equation*}
\left(\Theta^{2}-\delta z(\Theta+a)(\Theta+b)\right) f(z)=0 \tag{4.1}
\end{equation*}
$$

where the $\delta, a, b$ are constants with $\delta \neq 0$ (later we will restrict to rational numbers with $0<a, b<1$ ), and $\Theta=z \frac{d}{d z}$. The Picard-Fuchs equations for elliptic curves in a weighted projective space (For the general form of the Picard-Fuchs equation see [7] [5].) is of this form where $z$ is a suitable coordinate for the complex structure moduli space of the curves. The periods of the curves are given by the solutions to (4.1). This equation is projectively equivalent to a hypergeometric equation:

$$
\begin{equation*}
z(1-z) y^{\prime \prime}+(1-(a+b+1) z) y^{\prime}-a b y=0 \tag{4.2}
\end{equation*}
$$

ie. there exists a function $A$ and a Mobius transformation $z \rightarrow \frac{\alpha z+\beta}{\gamma z+\delta}$ such that $f=A y$ transforms (4.1) into (4.2). (Take $z \rightarrow z / \delta$.) The Schwarzian equation associated to (4.1) is

$$
\begin{equation*}
2 Q(z(t)) z^{\prime 2}+\{z, t\}=0 \tag{4.3}
\end{equation*}
$$

where $\{z, t\}=\frac{z^{\prime \prime \prime}}{z^{\prime}}-\frac{3}{2}\left(\frac{z^{\prime \prime}}{z^{\prime}}\right)^{2}$ and

$$
\begin{equation*}
Q(z)=\frac{1+2(-1+a+b-2 a b) \delta z+(-1+a-b)(-1-a+b) \delta^{2} z^{2}}{4 z^{2}(1-\delta z)^{2}} \tag{4.4}
\end{equation*}
$$

The Schwarzian equation associated with (4.2) is (4.3) with $\delta=1$. For convenience, we let $\delta=1$ for now. To restore the generality, one simply replace $z$ by $\delta z$ below. It was known to H.A. Schwarz that the every solution $z(t)$ to (4.3) is the inverse of a ratio (known as a Schwarz triangle function of type $s(0, b-a, 1-a-b ; z)$ ) of two hypergeometric functions which solve (4.2). In this note, we will only be interested in solutions $z(q)$ to (4.3) which are analytic in some disk $|q|<R$ where $q=e^{2 \pi i t}$. It turns out that every
analytic solution $z$ must have either $z(q=0)=0$ or 1 . We will only consider those with $z(0)=0$. They can be obtained as follows. Eqn (4.1)has a unique power series solution $f_{1}(z)$ with $f_{1}(0)=1$, namely $f_{1}(z)=\sum_{n \geq 0} c_{n} z^{n}$ where $c_{n}=\frac{\Gamma(n+a) \Gamma(n+b)}{\Gamma(a) \Gamma(b) \Gamma(n+1)^{2}}$. There is also a unique solution of the form $f_{2}(z)=\log (z) f_{1}(z)+\sum_{n \geq 1} d_{n} z^{n}$. Note that both the $c_{n}, d_{n}$ depends polynomially on the parameters $a, b$. Let $t=\frac{f_{2}(z)}{2 \pi i f_{1}(z)}$. Then

$$
\begin{equation*}
q=z \exp \left(\sum d_{n} z^{n} / \sum c_{n} z^{n}\right) \tag{4.5}
\end{equation*}
$$

This defines an invertible analytic map from a disk $|z|<S$ to some $|q|<R$ sending 0 to 0 . The inverse map $z_{f}(q)$ therefore gives a particular analytic solution to (4.3), which we will call the fundamental solution. Note also the the Fourier coefficients of $z_{f}(q)$ also depend polynomially on $a, b$. (Thus the above construction makes sense with no restriction at all on the values of $a, b$.) It is convenient to transform (4.3) by $q=e^{2 \pi i t}$ so that the equation now has $q$ as an independent variable and $z(q)$ dependent variable.

We will argue that $\left\{z_{f}(k q)\right\}_{k \in \mathbf{C}^{\times}}$exhausts all analytic solutions $z(q)$ with $z(0)=0$ and $\frac{d z(0)}{d q} \neq 0$, ie. every such solution can be obtained from the fundamental one by scaling $q$. First note that scaling $q$ by $k$ corresponds to a translation $t \rightarrow t+\beta$ for some $\beta$. But we know that (4.3) is invariant under Mobius transformations on $t$. This shows that each $z_{f}(k q)$ is an analytic solution to (4.3). It remains to show that if $z(q)=\sum_{n \geq 1} a_{n} q^{n}$ is an analytic solution to (4.3) with $a_{1} \neq 0$, then the $a_{n}$ are determined by $a_{1}$. Since (4.3) is 3rd order and is linear in $\frac{d^{3} z}{d q^{3}}, a_{n}$ for $n=3,4, \ldots$ are determined by $a_{1}$ and $a_{2}$. But a simple computation using (4.3)gives

$$
\begin{equation*}
a_{2}=(2 a b-a-b) a_{1} . \tag{4.6}
\end{equation*}
$$

As argued, the Fourier coefficients of $z_{f}(k q)$ are polynomials in $a, b, k$. The first few terms of $z_{f}(q)$ are

$$
\begin{align*}
z_{f}(q) & =q+(-a-b+2 a b) q^{2} \\
& -\left(a-5 a^{2}+b-11 a b+19 a^{2} b-5 b^{2}+19 a b^{2}-19 a^{2} b^{2}\right) q^{3} / 4 \\
& -\left(6 a-45 a^{2}+93 a^{3}+6 b-94 a b+408 a^{2} b-515 a^{3} b-45 b^{2}\right. \\
& \left.+408 a b^{2}-1116 a^{2} b^{2}+987 a^{3} b^{2}+93 b^{3}-515 a b^{3}+987 a^{2} b^{3}-658 a^{3} b^{3}\right) q^{4} / 54+\ldots \tag{4.7}
\end{align*}
$$

This is obtained by either inverting (4.5) or by directly solving (4.3).

## 4.2. $n=3$ Schwarzian equation

It is now clear how to generalize the above. Consider the following 3rd order ODE of Fuchsian type:

$$
\begin{equation*}
\left(\Theta^{3}-z\left(\sum_{i=0}^{3} r_{i} \Theta^{i}\right)\right) f(z)=0 \tag{4.8}
\end{equation*}
$$

where the $r_{i}$ are constants with $r_{3} \neq 0$. The Picard-Fuchs equations for (a 1-parameter deformations of) smooth K3 hypersurfaces or complete intersections in weighted projective space is of this form (see (4] 5]) where the $r_{i}$ are integer valued and satisfy some restriction given as follows. There is a "uniformizing" coordinate $\tau$ in which (4.8) becomes $\frac{d^{3} \tilde{f}}{d \tau^{3}}=0$. This implies that the following three possibilities:

$$
\begin{align*}
& \text { a) } r_{0}=r_{1}=r_{2}=0 \\
& \text { b) } 3 r_{0}=r_{1}=r_{2}=3 r_{3}  \tag{4.9}\\
& \text { c) } r_{0}=\frac{2 r_{1}-r_{3}}{4}, \quad r_{2}=\frac{3 r_{3}}{2}
\end{align*}
$$

Cases a) and b) turns out to be projectively equivalent, ie. they both result in the same reduced form for (4.8):

$$
\begin{equation*}
\left(\frac{d^{3}}{d z^{3}}+\frac{1}{4 z^{2}} \frac{d}{d z}-\frac{1}{4 z^{3}}\right) \tilde{f}(z)=0 . \tag{4.10}
\end{equation*}
$$

This case will not arise in our discussion below. Thus from now on we impose c) and hence (4.8) becomes, after simplication:

$$
\begin{equation*}
\left(\Theta^{3}-r_{3} z\left(\Theta+\frac{1}{2}\right)\left(\Theta^{2}+\Theta+\frac{r_{1}}{r_{3}}-\frac{1}{2}\right)\right) f(z)=0 \tag{4.11}
\end{equation*}
$$

According to our general construction. The Schwarzian equation associated with (4.11) is then (see eqn (3.9))

$$
\begin{equation*}
\operatorname{det}\left(M_{k l}\right):=27 c_{0}^{2}+4 c_{1}^{3}-18 c_{1} c_{0}^{\prime}-3{c_{1}^{\prime}}^{2}+6 c_{1} c_{1}^{\prime \prime}=0 \tag{4.12}
\end{equation*}
$$

where

$$
\begin{align*}
c_{1}(t):= & q_{1}(z) z^{\prime 2}+2\{z, t\} \\
c_{0}(t):= & q_{0}(z) z^{\prime 3}+q_{1}(z) z^{\prime} z^{\prime \prime}+\frac{3 z^{\prime \prime}}{z^{3}} \\
& -\frac{4 z^{\prime \prime} z^{(3)}}{z^{\prime 2}}+\frac{z^{(4)}}{z^{\prime}}  \tag{4.13}\\
q_{1}(z):= & \frac{4-4 r_{1} z-2 r_{3} z+4 r_{1} r_{3} z^{2}+r_{3}^{2} z^{2}}{4 z^{2}\left(-1+r_{3} z\right)^{2}} \\
q_{0}(z):= & \frac{1}{2} \frac{d q_{1}(z)}{d z} .
\end{align*}
$$

It is easy to check that $c_{0}(t)=\frac{1}{2} c_{1}^{\prime}(t)$. This simplifies (4.12) to

$$
\begin{equation*}
15 c_{1}^{2}+16 c_{1}^{3}-12 c_{1} c_{1}^{\prime \prime}=0 \tag{4.14}
\end{equation*}
$$

Note that this simplification is a direct consequence of imposing c) above. This $\mathrm{n}=3$ Schwarzian equation will be useful for understanding the mirror map for K3 surfaces (see below).

We now construct all analytic solutions $z(q)$ to (4.14) with $z(0)=0$ and $\frac{d z(0)}{d q} \neq 0$. The situation here is completely analogous to the $\mathrm{n}=2$ case. Eqn (4.11) has a unique power series solution $f_{1}(z)=\sum_{n>0} c_{n} z^{n}$ with $f_{1}(0)=1$. There is also a unique solution of the form $f_{2}(z)=\log (z) f_{1}(z)+\sum_{n \geq 1} d_{n} z^{n}$. The $c_{n}, d_{n}$ are polynomials of the parameter $r_{1}, r_{3}$. If we let $t=\frac{f_{2}(z)}{2 \pi i f_{1}(z)}$, then

$$
\begin{equation*}
q:=e^{2 \pi i t}=z \exp \left(\sum d_{n} z^{n} / \sum c_{n} z^{n}\right) \tag{4.15}
\end{equation*}
$$

defines an invertible analytic map from a disk $|z|<S$ to some $|q|<R$. The inverse map $z_{f}(q)$ therefore gives a particular analytic solution to (4.14). By an argument similar to the $\mathrm{n}=2$ case, we conclude that $\left\{z_{f}(k q)\right\}_{k \in \mathbf{C}^{\times}}$exhausts all such analytic solutions. The Fourier coefficients of the fundamental solution $z_{f}(q)$ can be computed using (4.14):

$$
\begin{align*}
z_{f}(q)= & q+\left(2 r_{1}-3 r_{3}\right) q^{2} / 4+\left(76 r_{1}^{2}-212 r_{1} r_{3}+135 r_{3}^{2}\right) q^{3} / 256 \\
& +\left(2632 r_{1}^{3}-10468 r_{1}^{2} r_{3}+12862 r_{1} r_{3}^{2}-5007 r_{3}^{3}\right) q^{4} / 13824  \tag{4.16}\\
& +\left(1806544 r_{1}^{4}-9219424 r_{1}^{3} r_{3}\right. \\
& \left.+16526488 r_{1}^{2} r_{3}^{2}-12589560 r_{1} r_{3}^{3}+3479157 r_{3}^{4}\right) q^{5} / 14155776+\ldots
\end{align*}
$$

The above result has an interesting consequence: every analytic solution to (4.14) is a solution to:

$$
\begin{equation*}
c_{1}:=q_{1}(z) z^{\prime 2}+2\{z, t\}=0 \tag{4.17}
\end{equation*}
$$

To prove this, note that (4.17) is an $n=2$ Schwarzian equation (4.3) with $Q(z)=\frac{1}{4} q_{1}(z)$. Associated to it are (in general) four 2nd order Fuchsian equation of the type (4.1) with parameters $\delta=r_{3}$ and $a, b$ satisfying

$$
\begin{align*}
2 a b-a-b & =\frac{r_{1}}{2 r_{3}}-\frac{3}{4}  \tag{4.18}\\
(a-b)^{2} & =-\frac{r_{1}}{r_{3}}+\frac{3}{4} .
\end{align*}
$$

(Note that if $(a, b)$ solve (4.18), so do $(b, a)$ and $(1-a, 1-b)$.) By our previous result on $\mathrm{n}=2$ Schwarzian equation, (4.17) has a family of solutions $\left\{z_{f}(k q)\right\}_{k \in \mathbf{C}^{\times}}$. Obviously they are solutions to (4.14) as well. Thus by our result on $n=3$ Schwarzian equation, this family exhaust the solutions to (4.14). Thus we have effectively reduced the $n=3$ Schwarzian equation to the $n=2$ equation.

## 4.3. $n=4$ Schwarzian equation

For completeness, we briefly discuss the $\mathrm{n}=4$ case even though we will not be applying the result later. The situation here is quite similar to the $n=3$ case, except the last part above. Consider the following 4rd order ODE:

$$
\begin{equation*}
\left(\Theta^{4}-z\left(r_{4} \Theta^{4}+2 r_{4} \Theta^{3}+r_{2} \Theta^{2}+\left(r_{2}-r_{4}\right) \Theta+r_{0}\right)\right) f(z)=0 \tag{4.19}
\end{equation*}
$$

where the $r_{i}$ are constants with $r_{4} \neq 0$. The Picard-Fuchs equations for certain 1-parameter family of Calabi-Yau threefolds in weighted projective space is of this form where the $r_{i}$ are integers. According to our general construction, the Schwarzian equation associated with (4.11) is then (see eqn (3.9))

$$
\begin{align*}
\operatorname{det}\left(M_{k l}\right): & =16 c_{2}^{4} c_{0}-128 c_{2}^{2} c_{0}^{2}+256 c_{0}^{3}+4 c_{2}^{3} c_{2}^{\prime} \\
& +240 c_{2} c_{0}{c_{2}^{\prime 2}}^{2}-15{c_{2}^{\prime}}^{4}-144 c_{2}^{2} c_{2}^{\prime} c_{0}^{\prime}-448 c_{0} c_{2}^{\prime} c_{0}^{\prime} \\
& +256 c_{2} c_{0}^{\prime 2}-8 c_{2}^{4} c_{2}^{\prime \prime}+128 c_{0}^{2} c_{2}^{\prime \prime}-48 c_{2} c_{2}^{2} c_{2}^{\prime \prime} \\
& +48 c_{2}^{\prime} c_{0}^{\prime} c_{2}^{\prime \prime}+12 c_{2}^{2} c_{2}^{\prime \prime 2}-48 c_{0} c_{2}^{\prime \prime}+32 c_{2}^{3} c_{0}^{\prime \prime}  \tag{4.20}\\
& -128 c_{2} c_{0} c_{0}^{\prime \prime}+48 c_{2}^{\prime 2} c_{0}^{\prime \prime}+32 c_{2}^{2} c_{2}^{\prime} c_{2}^{(3)}+64 c_{0} c_{2}^{\prime} c_{2}^{(3)} \\
& -96 c_{2} c_{0}^{\prime} c_{2}^{(3)}+8 c_{2} c_{2}^{(3)^{2}}-8 c_{2}^{3} c_{2}^{(4)}+32 c_{2} c_{0} c_{2}^{(4)}-12 c_{2}^{\prime 2} c_{2}^{(4)}=0
\end{align*}
$$

where

$$
\begin{align*}
c_{2}(t):= & q_{2}(z) z^{\prime 2}+5\{z(t), t\}_{2} \\
c_{0}(t):= & q_{0}(z) z^{4}+\frac{3}{2} \frac{d q_{2}(z)}{d z} z^{\prime 2} z^{\prime \prime}-\frac{3}{4} q_{2}(z) z^{\prime \prime 2}-\frac{135 z^{\prime \prime 4}}{16 z^{\prime 4}} \\
+ & \frac{3}{2} q_{2}(z) z^{\prime} z^{(3)}+\frac{75 z^{\prime \prime 2} z^{(3)}}{4 z^{\prime 3}}-\frac{15 z^{(3)^{2}}}{4{z^{\prime}}^{2}}-\frac{15 z^{\prime \prime} z^{(4)}}{2 z^{\prime 2}}+\frac{3 z^{(5)}}{2 z^{\prime}} \\
q_{2}(z):= & \frac{5-2 z r_{2}-4 z r_{4}+2 z^{2} r_{2} r_{4}+2 z^{2} r_{4}^{2}}{2 z^{2}\left(-1+z r_{4}\right)^{2}} \\
q_{0}(z):= & \left(81-16 z r_{0}-12 z r_{2}-280 z r_{4}+48 z^{2} r_{0} r_{4}+44 z^{2} r_{2} r_{4}+340 z^{2} r_{4}^{2}-48 z^{3} r_{0} r_{4}^{2}\right. \\
& \left.-64 z^{3} r_{2} r_{4}^{2}-128 z^{3} r_{4}^{3}+16 z^{4} r_{0} r_{4}^{3}+32 z^{4} r_{2} r_{4}^{3}+32 z^{4} r_{4}^{4}\right) /\left(16 z^{4}\left(-1+z r_{4}\right)^{4}\right) \tag{4.21}
\end{align*}
$$

Once again, the set of analytic solutions to (4.20) with $z(q=0)=0$ and $\frac{d z(0)}{d q} \neq 0$ is of the form $\left\{z_{f}(k q)\right\}_{k \in \mathbf{C}^{\times}}$, where $z_{f}(q)$ is the fundamental solution whose Fourier coefficients are polynomials in the data $\left(r_{4}, r_{2}, r_{0}\right)$. The construction is almost identical to that for $\mathrm{n}=2,3$. The first few coefficients $z_{f}(q)$ are given by:

$$
\begin{align*}
z_{f}(q) & =q+q^{2}\left(4 r_{0}-r_{2}+r_{4}\right) \\
& +q^{3}\left(163 r_{0}^{2}-84 r_{0} r_{2}+11 r_{2}^{2}+90 r_{0} r_{4}-24 r_{2} r_{4}+13 r_{4}^{2}\right) / 8 \\
& +q^{4}\left(112798 r_{0}^{3}-88840 r_{0}^{2} r_{2}+23610 r_{0} r_{2}^{2}-2115 r_{2}^{3}\right. \\
& +98969 r_{0}{ }^{2} r_{4}-53280 r_{0} r_{2} r_{4}+7245 r_{2}^{2} r_{4}+29958 r_{0} r_{4}^{2} \\
& \left.-8253 r_{2} r_{4}^{2}+3123 r_{4}^{3}\right) / 972 \\
& +q^{5}\left(702344153 r_{0}^{4}-747165436 r_{0}^{3} r_{2}+300965204 r_{0}^{2} r_{2}^{2}-54365280 r_{0} r_{2}^{3}\right. \\
& +3713328 r_{2}^{4}+853805796 r_{0}^{3} r_{4}-694587576 r_{0}^{2} r_{2} r_{4}+189938400 r_{0} r_{2}^{2} r_{4} \\
& -17448192 r_{2}^{3} r_{4}+400090708 r_{0}^{2} r_{4}^{2}-220959840 r_{0} r_{2} r_{4}^{2}+30726432 r_{2}^{2} r_{4}^{2} \\
& \left.+85542240 r_{0} r_{4}^{3}-24023808 r_{2} r_{4}^{3}+7032240 r_{4}^{4}\right) / 995328 \\
& +q^{6}\left(2322744173252 r_{0}^{5}-3118439852795 r_{0}^{4} r_{2}+1688073976516 r_{0}^{3} r_{2}^{2}\right. \\
& -460304004892 r_{0}^{2} r_{2}^{3}+63195785760 r_{0} r_{2}^{4}-3493177200 r_{2}^{5} \\
& +3628517760463 r_{0}^{4} r_{4}-3959225703216 r_{0}^{3} r_{2} r_{4}+1631456931956 r_{0}^{2} r_{2}^{2} r_{4} \\
& -300755456640 r_{0} r_{2}^{3} r_{4}+20919798000 r_{2}^{4} r_{4}+2319777533708 r_{0}^{3} r_{4}^{2} \\
& -1926736953572 r_{0}^{2} r_{2} r_{4}^{2}+536717533440 r_{0} r_{2}^{2} r_{4}^{2}-50124477600 r_{2}^{3} r_{4}^{2} \\
& +757958047180 r_{0}^{2} r_{4}^{3}-425557676160 r_{0} r_{2} r_{4}^{3}+60050628000 r_{2}^{2} r_{4}^{3} \\
& \left.+126449580000 r_{0} r_{4}^{4}-35961865200 r_{2} r_{4}^{4}+8609094000 r_{4}^{5}\right) / 518400000+\ldots \tag{4.22}
\end{align*}
$$

It is interesting to note that this series appears to be integral when evaluated at certain integral points. For example, if the data $\left(r_{4}, r_{2}, r_{0}\right)=(3125,4375,120)$ corresponding to the quintic threefold in $\mathbf{P}^{4}$, then (4.22) becomes:

$$
\begin{equation*}
z(q)=q-770 q^{2}+171525 q^{3}-81623000 q^{4}-35423171250 q^{5}-54572818340154 q^{6}-\ldots \tag{4.23}
\end{equation*}
$$

This integrality phenomenon apparently continues to hold in other examples as has been previously observed.

At this point, one might wonder if $z_{f}$ would satisfy an $\mathrm{n}=2$ Schwarzian equation (4.3) as in the previous case, ie. is there a rational function $Q(z)$ similar to (4.4) such that (4.22) solves the equation (4.3)? The answer is no in general. In fact, we have checked that in many cases of a 1-parameter family of Calabi-Yau threefolds, such a $Q(z)$ doesn't
exist.(cf. [1] section 2.) However the mirror map $z_{f}$ of such a threefold does satisfy an equation similar to (4.3), but with some "quantum" corrections, namely:

$$
\begin{equation*}
2 Q(z){z^{\prime}}^{2}+\{z, t\}=\frac{2}{5} y^{\prime \prime}-\frac{1}{10} y^{\prime 2} \tag{4.24}
\end{equation*}
$$

where $Q(z)=\frac{1}{10} q_{2}(z)$ (eqn (4.21)) and $y=\log K(t), K$ being the quantum Yukawa coupling (also written as $\left.\partial^{3} F\right)$. Thus the right hand side of $(4.24)$ should be thought of as a "quantum" correction to the classical equation (4.3).

### 4.4. Remarks

1. In the last section, we have seen that the Schwarzian equations can essentially be solved in terms of certain hypergeometric functions. Moreover the analytic solution is essentially unique. In fact, it follows easily from the previous discussion that in each case $\mathrm{n}=2,3$ or 4 , the fundamental solution to the $n^{\text {th }}$ Schwarzian equation is the unique analytic solution (single-valued) $z_{f}(q)$ with $z_{f}(0)=0, \frac{d z_{f}(0)}{d q}=1$. This is exactly the socalled mirror map. Thus the mirror map can be characterized by means of an ODE and analyticity. We also find that there is a similar characterization for the quantum coupling $K(t)$ in a number of examples.
2. Even though we have obtained only the singled-valued analytic solutions to the Schwarzian equations in the disk $|q|<R$, the results actually allows us to classify certain multi-valued solutions as well. For simplicity, let's focus on the $n=2$ case. What we say here will hold true for $\mathrm{n}=3,4$ as well. For example, given any positive constant $\alpha$, one can easily classify the analytic solutions $z(q)$ to (4.3) such that $z(q) \sim k q^{\alpha}$ as $q \rightarrow 0$ for some constant $k$. First note that there is a 1-parameter family of such solutions, namely $z_{f}\left(k q^{\alpha}\right)$ where $z_{f}(q)$ is our fundamental solution. This follows immediately from Mobius invariance of (4.3). This family also exhausts all such solutions.
3. We studied the Schwarzian equations associated with Fuchsian equations of special type which arises as the Picard-Fuchs equations for certain Calabi-Yau manifolds. It is clear that one can study this Schwarzian equations without reference to special geometry, in which case, some of our assumption can be weaken. For example, we have checked that the uniqueness of analytic solutions to these equations continues to hold true under much weaker assumptions on the form of the associated Fuchsian equations. (For instance, take any Fuchsian equation whose indicial equation is maximally degenerate, ie. whenever the differential operator takes the form $\Theta^{n}-p(z, \Theta)$ where $p$ is a polynomial of degree at most $n$ in $\Theta$, with $p(0, \Theta)=0$.)
4. We have shown that the mirror map in the case of K 3 surfaces satisfies an $\mathrm{n}=2$ Schwarzian equation, as in the case of elliptic curves. This is the first hint that there might be some relationship between K3 surfaces and elliptic curves. It also says that the mirror map is nothing but the inverse of a Schwarz triangle function. An even more interesting hint comes from the following numerical experiment. If we consider a 1-parameter family of sextic hypersurfaces in the weighted projective space $\mathbf{P}^{3}[1,1,1,3]$ and compute the Fourier series of the mirror map, we get

$$
\begin{equation*}
z(q)=q-744 q^{2}+356652 q^{3}-140361152 q^{4}+49336682190 q^{5}-16114625669088 q^{6}+\ldots \tag{4.25}
\end{equation*}
$$

The observant reader would have realized that the coefficient 744 is the constant term in the J-function times 1728 . In fact if we compute $1 / z(q)$, we get exactly the first 6 coefficients of the 1728 J . We will prove that $1728 J=1 / z(q)$. This will be another confirmation that the mirror map for a Calabi-Yau variety is integral.

## 5. Transcendence of the mirror map over $\mathbf{Q}(J)$

In this section analytic functions are assumed to be defined on some disk $|q|<R$. Let $\mathbf{Q}(f)$ be the field generated by the analytic functions $f(q)$ over the rationals $\mathbf{Q}$. Two analytic functions $f(q), g(q)$ are called commensurable if there exists a nontrivial polynomial relation (with coefficients in $\mathbf{Q}$ ) $P\left(f(q), g\left(k q^{\alpha}\right)\right)=0$ for some rational numbers $k, \alpha$. The statement that $f(q)$ and $g(q)$ are commensurable is equivalent to that $g\left(k q^{\alpha}\right)$ is algebraic over the field $\mathbf{Q}(f)$. Note also that commensurability is an equivalence relation.

We will be mainly concerned with the question of commensurability of an analytic function $f(q)$ with the J-function. Perhaps the simplest example of a function commensurable to $J$ is the so-called elliptic modular function (see below) $\lambda$. It bears the following well-known modular relation to $J$ :

$$
\begin{equation*}
J=\frac{4}{27} \frac{\left(1-\lambda+\lambda^{2}\right)^{3}}{\lambda^{2}(1-\lambda)^{2}} \tag{5.1}
\end{equation*}
$$

The function $\lambda$ will show up again in the next section.
In this section, we will prove that the mirror map $z(q)$ for K3 surfaces modelled in various weighted projective spaces is commensurable with $J$. Before discussing K3 surfaces, it is useful to recall some known examples of mirror maps which are commensurable with $J$ [14] [15] (see also [8] section 3.1). For both the elliptic curves and the K3 surfaces, we will use the commensurability relation to prove that the mirror maps in those cases are integral.

### 5.1. Elliptic curves

In the following table, we have four realizations of a 1-parameter family of elliptic curves as hypersurfaces (or complete intersections) in weighted projective spaces $\mathbf{P}^{2}(1,1,1), \mathbf{P}^{2}(1,1,2), \mathbf{P}^{2}(1,2,3)$ and $\mathbf{P}^{3}(1,1,1,1)$.

|  | deformations | diff. operator | $1728 J(z)$ |
| :--- | :--- | :--- | :--- |
| 1. | $x_{1}^{3}+x_{2}^{3}+x_{3}^{3}+z^{-1 / 3} x_{1} x_{2} x_{3}=0$ | $\theta^{2}-3 z(3 \theta+2)(3 \theta+1)$ | $\frac{(1+216 z)^{3}}{z(1-27 z)^{3}}$ |
| 2. | $x_{1}^{4}+x_{2}^{4}+x_{3}^{2}+z^{-1 / 4} x_{1} x_{2} x_{3}=0$ | $\theta^{2}-4 z(4 \theta+3)(4 \theta+1)$ | $\frac{(1+192 z)^{3}}{z(1-64 z)^{2}}$ |
| 3. | $x_{1}^{6}+x_{2}^{3}+x_{3}^{2}+z^{-1 / 6} x_{1} x_{2} x_{3}=0$ | $\theta^{2}-12 z(6 \theta+5)(6 \theta+1)$ | $\frac{1}{z(1-432 z)}$ |
| $\left.4.2256 z^{2}\right)^{3}$ |  |  |  |
| $x_{1}^{2}+x_{2}^{2}+z^{-1 / 4} x_{3} x_{4}=0$ | $\theta^{2}-4 z(2 \theta+1)^{2}$ | $\frac{(1+224 z+25 z)^{4}}{z(-1+16 z)^{4}}$ |  |
| $x_{3}^{2}+x_{4}^{2}+z^{-1 / 4} x_{1} x_{2}=0$ |  |  |  |

Thus in those four examples, the mirror maps $z$ are algebraic over the field $\mathbf{Q}(J)$. The relations between $z$ and $J$ in the some of these examples are derived in [8] by using the Weierstrass model for the elliptic curves. We will instead illustrate the proof using a slightly different approach which will prove useful in the case of K3 surfaces. Namely, we will effectively use the results on the uniqueness of analytic solutions to the associated $\mathrm{n}=2$ Schwarzian equations. We will consider example 4 because it has some interesting connection with elliptic functions and it hasn't been treated in [8]. Our goal is to prove the relation between $z$ and $J$ in example 4 above. (We also have similar proofs for all the other cases above.)

Recall that $J(t):=\frac{g_{2}^{3}(t)}{\Delta(t)}$ where $\Delta(t)$ is the discriminant of the Weierstrass elliptic curve $W$ with period ratio $t$ :

$$
\begin{equation*}
W: y^{2}=4 x^{3}-g_{2}(t) x-g_{3}(t) . \tag{5.2}
\end{equation*}
$$

The periods $\Omega$ of $W$ are solutions to the Picard-Fuchs equation:

$$
\begin{equation*}
\frac{d^{2} \Omega}{d J^{2}}+\frac{1}{J} \frac{d \Omega}{d J}+\frac{31 J-4}{144 J^{2}(1-J)^{2}} \Omega=0 \tag{5.3}
\end{equation*}
$$

Thus $J$ is a solution to the associated Schwarzian equation (4.3) with:

$$
\begin{equation*}
Q(z):=\frac{32-41 z+36 z^{2}}{144 z^{2}(-1+z)^{2}} \tag{5.4}
\end{equation*}
$$

The function $J(q)\left(q:=e^{2 \pi i t}\right)$ can be characterized as the unique meromorphic solution to the Schwarzian equation on a disk $|q|<R$ with the leading behavior $J(q) \sim \frac{1}{1728 q}$ near 0 . It is more convenient to work with $z_{W}(q):=\frac{1}{1728 J(q)}$ which satisfies (4.3) with:

$$
\begin{equation*}
Q(z)=Q_{W}(z):=\frac{1-1968 z+2654208 z^{2}}{4 z^{2}(-1+1728 z)^{2}} \tag{5.5}
\end{equation*}
$$

which is in the standard form (4.4). Since $z_{W}$ has the leading behavior $z_{W}(q) \sim q$, it follows that (section 4.1) $z_{W}$ is the fundamental solution to (4.3) with (5.5).

Now consider the mirror map $z_{X}$ for example 4 above. This is the fundamental solution to the Schwarzian equation (4.3) with:

$$
\begin{equation*}
Q(z)=Q_{X}(z):=\frac{1-16 z+256 z^{2}}{4 z^{2}(-1+16 z)^{2}} \tag{5.6}
\end{equation*}
$$

We want to prove that

$$
\begin{equation*}
z_{W}=\frac{z_{X}\left(-1+16 z_{X}\right)^{4}}{\left(1+224 z_{X}+256 z_{X}^{2}\right)^{3}} \tag{5.7}
\end{equation*}
$$

Note that the right hand side, which we will call $\zeta$, has the correct leading behavior. By subsituting $\zeta$ into (4.3) with (5.5) and use the fact the $z_{X}$ satisfies (4.3) with (5.6), we find that indeed (4.3) holds. By uniqueness of analytic solution to the Schwarzian equation we conclude that $\zeta=z_{W}$. We remark that one can also prove (5.7) simply by applying the variable change $z \rightarrow \frac{z(-1+16 z)^{4}}{\left(1+224 z+256 z^{2}\right)^{3}}$ to the (reduced) Fuchsian equation $f^{\prime \prime}(z)+Q_{W}(z) f(z)=0$ and see that you get the new reduced Fuchsian equation $\tilde{f}^{\prime \prime}(z)+$ $Q_{X}(z) \tilde{f}(z)=0$. It follows that their respective associated Schwarzian equations must transform from one to another under the same variable change.

### 5.2. Integrality of $z_{X}$ and the elliptic modular function $\lambda$

There is an amusing connection which we should point out between $z_{X}$ and $\lambda$. The latter is defined to be the inverse function of the Schwarz triangle function [16]:

$$
\begin{equation*}
t=S(0,0,0 ; z):=\frac{\sqrt{-1} F\left(\frac{1}{2}, \frac{1}{2}, 1 ; 1-z\right)}{F\left(\frac{1}{2}, \frac{1}{2}, 1 ; z\right)} \tag{5.8}
\end{equation*}
$$

where the $F$ are hypergeometric functions with $(a, b, c)=\left(\frac{1}{2}, \frac{1}{2}, 1\right)$. As a result we can also characterize $\lambda$ as the unique analytic solution, with the leading behavior $\lambda(q) \sim 16 q^{\frac{1}{2}}$, to the Schwarzian equation (4.3) (associated to (4.2)) with:

$$
\begin{equation*}
Q(z)=Q_{E}(z):=\frac{1-z+z^{2}}{4 z^{2}(-1+z)^{2}} \tag{5.9}
\end{equation*}
$$

The function $\lambda$ has a q-series expansion:

$$
\begin{equation*}
\lambda(q)=16 q^{1 / 2}-128 q+704 q^{3 / 2}-3072 q^{2}+11488 q^{5 / 2}-38400 q^{3}+117632 q^{7 / 2}-\ldots \tag{5.10}
\end{equation*}
$$

From its expression in terms of theta functions, it is known that $\lambda$ has integral Fourier coefficients divisible by 16 . It is also well-known that $\lambda$ defines a modular function of level 2 on the upper half-plane and it is related to $J$ by:

$$
\begin{equation*}
J=\frac{4}{27} \frac{\left(1-\lambda+\lambda^{2}\right)^{3}}{\lambda^{2}(1-\lambda)^{2}} \tag{5.11}
\end{equation*}
$$

(This relation can also be proved using the approach we used to prove (5.7).) Since commensurability is an equivalence relation, it follows from (5.7) and (5.11) that $z_{X}(q)$, $\lambda\left(q^{2}\right)$ are commensurable also. In fact, using the method above, one can easily prove that

$$
\begin{equation*}
\lambda\left(q^{2}\right)=16 z_{X}(q) \tag{5.12}
\end{equation*}
$$

Since the Fourier coefficients on the left hand side are integers divisible by 16, it follows that $z_{X}$ also has integral Fourier coefficients. We should also point out that combining the relations (5.7), (5.11) and (5.12), we get a rather perculiar identity for $\lambda$ :

$$
\begin{equation*}
\frac{16 \lambda\left(q^{2}\right)\left(-1+\lambda\left(q^{2}\right)\right)^{4}}{\left(1+14 \lambda\left(q^{2}\right)+\lambda\left(q^{2}\right)^{2}\right)^{3}}=\frac{\lambda(q)^{2}(1-\lambda(q))^{2}}{\left(1-\lambda(q)+\lambda(q)^{2}\right)^{3}} . \tag{5.13}
\end{equation*}
$$

It also says that you can write $J$ in terms of $\lambda$ in two different ways.

### 5.3. Remarks

1. The above approach for studying the commensurability of mirror maps also applies easily to examples 1-3 in the previous section.
2. In the last example we studied, it is less clear how to deduce, from the relation (5.7), the integrality of $z_{X}$ from the known integrality property of $z_{W}$. But upon relating $z_{X}$ to $\lambda$ via (5.12), this property becomes immediately clear. The upshot of this is that using the arithmetic properties of any particular one mirror map, such as $z_{W}$, alone may only give partial information about mirror maps commensurable to it. One should instead use other series, such as $\lambda$, in the same commensurability class to help obtain further information about other members in the same class. The lesson is that the larger the commensurability class, the more arithmetical information we can get about its members because every pair of members are related by some modular relations.
3. The key steps we used repeatedly above to prove commensurability of two fundamental solutions are:
a) identify the appropriate modular relation;
b) relate their corresponding Schwarzian equations (or equivalently their reduced Fuchsian equations) by a change of variables using the modular relation;
c) use analyticity (asymptotic behavior) and uniqueness of solutions to the Schwarzian equation;

This idea will continue to work well, as we will see, in the case of fundamental solutions arising from the Picard-Fuchs equations for K3 surfaces.
4. Finally, we note that in e.g. 3 of elliptic curves above, we have the relation, for $\operatorname{Im} t \gg 0$

$$
\begin{equation*}
w(t)=\frac{j(t)+(j(t)(j(t)-1728))^{1 / 2}}{2} \tag{5.14}
\end{equation*}
$$

where $w=1 / z, j=1728 J$. We claim that $w(t)$ admits a double-valued analytic continuation in the upper half plane $\operatorname{Im} t>0$. This implies in particular that the Fourier series of the mirror map $z=1 / w$ has radius of convergence strictly less than 1 - a fact that is not obvious from the construction of $z$. If we denote by $z(t), \tilde{z}(t)$ the two branches of $z$, we see that $z(t)+\tilde{z}(t)=\frac{1}{432}$. Thus the two branches differ by an affine transformation $x \rightarrow-x+\frac{1}{432}$.

To prove our claim, let's recall some properties of the modular function $j(t)$. We know that $j(t)$ is a single-valued function in $\operatorname{Im} t>0$. Thus (5.14) implies that $w(t)$ admits an analytic continuation which is at most double-valued. Also for $\rho:=\frac{1}{2}+\frac{i \sqrt{3}}{2}$ we have $j(\rho)=0$. In a small punctured disk centered at $\rho,(j(t)-1728)^{1 / 2}$ is single-valued. If we move around a small loop enclosing $\rho$ in that disk, $j(t)$ will move around a small loop (3 times) enclosing 0 in the j-plane. It follows that $(j(t)(j(t)-1728))^{1 / 2}$ is necessarily double-valued in that disk, implying that $w(t)$ is at least double-valued.
5.4. K3 surfaces: $X_{z}: x_{1}^{6}+x_{2}^{6}+x_{3}^{6}+x_{4}^{2}+z^{-1 / 6} x_{1} x_{2} x_{3} x_{4}=0$

We now apply what we learned in section 4.2 to this family of K3 surfaces. The Picard-Fuchs equation for the above 1-parameter family of K 3 hypersurfaces in $\mathbf{P}^{3}[1,1,1,3]$ is given by:

$$
\begin{equation*}
\left(\Theta^{3}-8 z(6 \Theta+1)(6 \Theta+3)(6 \Theta+5)\right) f(z)=0 \tag{5.15}
\end{equation*}
$$

which is of the form (4.11) with $\left(r_{3}, r_{1}\right)=(1728,1104)$. By the results in section 4.2, the analytic solutions of the associated $n=3$ Schwarzian equation (4.14) can be obtained from the $\mathrm{n}=2$ Schwarzian equation (4.3) with:

$$
\begin{equation*}
Q(z)=Q_{X}(z):=\frac{1-1968 z+2654208 z^{2}}{4 z^{2}(-1+1728 z)^{2}} \tag{5.16}
\end{equation*}
$$

In particular the mirror map $z_{X}$ for our K3 surfaces $X$ which is the fundamental solution to (4.14), is now the fundamental solution to (4.3) with (5.16). But observe that $Q_{X}(z)$ is identical to $Q_{W}(z)$ given in (5.5). It follows that our mirror map is given by:

$$
\begin{equation*}
z_{X}=z_{W}:=\frac{1}{1728 J} \tag{5.17}
\end{equation*}
$$

This also proves, in particular, that $z_{X}$ also has an integral Fourier coefficients, and that the mirror map $z_{X}$ is commensurable to $J$.

### 5.5. Other deformations of K3 surfaces

We now consider three other families of K3 surfaces: quartic hypersurfaces $X_{(4)}$ in $\mathbf{P}^{3}$, the complete intersections of quadrics and cubics $X_{(2,3)}$ in $\mathbf{P}^{4}$ and the complete intersections of 3 quadrics $X_{(2,2,2)}$ in $\mathbf{P}^{5}$. The following table lists the types of abelian group $G$ for the orbifold constructions, the 1-parameter deformations, the data (4.4) for the corresponding $\mathrm{n}=2$ Schwarzian equation (4.3), (4.17). We also include the previous example in $\mathbf{P}^{3}[1,1,1,3]$, which we denote $X_{(6)}$.

| $X, G$ | deformations | $Q(z)$ in (4.3) |
| ---: | :--- | :--- |
| $X_{(6)},(6,6,2)$ | $x_{1}^{6}+x_{2}^{6}+x_{3}^{6}+x_{4}^{2}+z^{-1 / 6} x_{1} x_{2} x_{3} x_{4}=0$ | $\frac{1-1968 z+2654208 z^{2}}{4 z^{2}(-1+1728 z)^{2}}$ |
| $X_{(4)},(4,4)$ | $x_{1}^{4}+x_{2}^{4}+x_{3}^{4}+x_{4}^{4}+z^{-1 / 4} x_{1} x_{2} x_{3} x_{4}=0$ | $\frac{1-304 z+61440 z^{2}}{4 z^{2}(-1+256 z)^{2}}$ |
| $X_{(2,3)},(2,2,3)$ | $x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+z^{-1 / 5} x_{4} x_{5}=0$ | $\frac{1-132 z+11340 z^{2}}{4 z^{2}(-1+108 z)^{2}}$ |
|  | $x_{4}^{3}+x_{5}^{3}+z^{-1 / 5} x_{1} x_{2} x_{3}=0$ |  |
| $X_{(2,2,2)},(2,2,2)$ | $x_{1}^{2}+x_{2}^{2}+z^{-1 / 6} x_{3} x_{4}=0$ | $\frac{1-80 z+4096 z^{2}}{4 z^{2}(-1+64 z)^{2}}$ |
|  | $x_{3}^{2}+x_{4}^{2}+z^{-1 / 6} x_{5} x_{6}=0$ |  |
|  | $x_{5}^{2}+x_{6}^{2}+z^{-1 / 6} x_{1} x_{2}=0$ |  |

The respective values of $\left(r_{3}, r_{1}\right)$ in eqn. (4.11) are $(1728,1104),(256,176),(108,78)$, $(64,48)$. The corresponding fundamental solutions in the 4 cases have Fourier series:

$$
\begin{align*}
z_{X_{(6)}}(q) & =q-744 q^{2}+356652 q^{3}-140361152 q^{4}+49336682190 q^{5} \\
& -16114625669088 q^{6}+\ldots \\
z_{X_{(4)}}(q) & =q-104 q^{2}+6444 q^{3}-311744 q^{4}+13018830 q^{5}-493025760 q^{6}+\ldots  \tag{5.18}\\
z_{X_{(2,3)}}(q) & =q-42 q^{2}+981 q^{3}-16988 q^{4}+244230 q^{5}-3089394 q^{6}+\ldots \\
z_{X_{(2,2,2)}}(q) & =q-24 q^{2}+300 q^{3}-2624 q^{4}+18126 q^{5}-105504 q^{6}+\ldots
\end{align*}
$$

We claim that, as for $z_{X_{(6)}}$, all the other mirror maps are commensurable with the J-function (or equivalently with $z_{W}$ ). By numerical experiment, we identified the following modular relations:

$$
\begin{align*}
& P_{X_{(4)}}\left(z_{W}, z_{X_{(4)}}\right):=-z_{W}^{2}+z_{W} z_{X_{(4)}}-432 z_{W}^{2} z_{X_{(4)}}-207 z_{W} z_{X_{(4)}}^{2}-62208 z_{W}^{2} z_{X_{(4)}}^{2} \\
&-z_{X_{(4)}}^{3}+3456 z_{W} z_{X_{(4)}}^{3}-2985984 z_{W}^{2} z_{X_{(4)}}^{3}=0 \\
& P_{X_{(2,3)}}\left(z_{W}, z_{X_{(2,3)}}\right):=z_{W}^{2}-z_{W} z_{X_{(2,3)}}+576 z_{W}^{2} z_{X_{(2,3)}}+126 z_{W} z_{X_{(2,3)}}^{2} \\
&+110592 z_{W}^{2} z_{X_{(2,3)}}^{2}-2944 z_{W} z_{X_{(2,3)}}^{3}+7077888 z_{W}^{2} z_{X_{(2,3)}}^{3}+z_{X_{(2,3)}}^{4}=0 \\
& P_{X_{(2,2,2)}}\left(z_{W}, z_{X_{(2,2,2)}}\right):=z_{W}^{2}-z_{W} z_{X_{(2,2,2)}}+624 z_{W}^{2} z_{X_{(2,2,2)}}+96 z_{W} z_{X_{(2,2,2)}}^{2} \\
&+129840 z_{W}^{2} z_{X_{(2,2,2)}}-2352 z_{W} z_{X_{(2,2,2)}}^{3}+9018880 z_{W}^{2} z_{X_{(2,2,2)}} \\
&+10495 z_{W} z_{X_{(2,2,2)}}^{4}+2077440 z_{W}^{2} z_{X_{(2,2,2)}^{4}}^{4}+z_{X_{(2,2,2)}^{5}}^{5} \\
&-1488 z_{W} z_{X_{(2,2,2)}^{5}}+159744 z_{W}^{2} z_{X_{(2,2,2)}^{5}}^{5}+4096 z_{W}^{2} z_{X_{(2,2,2)}}^{6}=0 . \tag{5.19}
\end{align*}
$$

We now proceed to prove the first of the relations (5.19). The other cases are similar. By solving $P_{X_{(4)}}\left(w, z_{X_{(4)}}(q)\right)=0$ for $w$, we see that there is a branch of $w$ which admits a qseries expansion near $q=0$ and which has the leading behavior $w(q) \sim q$. Now it is enough to show that every branch of $w$ solves the Schwarzian equation (4.3) with (5.5). For then $w(q)$ above must coincide with the fundamental solution, by uniqueness. Now applying the relation $P_{X_{(4)}}\left(w(q), z_{X_{(4)}}(q)\right)=0$, we can compute $w^{\prime}(q), w^{\prime \prime}(q), w^{\prime \prime \prime}(q)$ in terms of $w(q), z_{X_{(4)}}(q)$ and its derivatives. Substituting these expressions into (4.3) and using the fact that $z_{X_{(4)}}(q)$ satisfies its Schwarzian equation, we find that (4.3) with (5.5) holds identically.

## 6. Uniform proof of integrality

Applying the modular relations derived above between the mirror maps and the j function, we will now prove that the Fourier coefficients of the mirror maps are integral. We begin with the following lemma. Let $z_{0}(q)$ be an integral $q$-series and

$$
\begin{equation*}
P(x, y)=\sum a_{i, j} x^{i} y^{j} \tag{6.1}
\end{equation*}
$$

be a nonzero polynomial with integral coefficients $a_{i, j}$. Let $z(q)=\sum_{n \geq 1} c_{n} q^{n}$ with $c_{1}=1$. Let $l:=\min \left\{i+j \mid a_{i, j} \neq 0\right\}$. Now suppose that

$$
\begin{equation*}
m:=\sum_{i \geq 0} a_{l-i, i} i \neq 0 \tag{6.2}
\end{equation*}
$$

We claim that the relation $P\left(z_{0}(q), z(q)\right)=0$ determines $z(q)$ uniquely and that the coefficients $c_{n}$ are in $\mathbf{Z}\left[\frac{1}{m}\right]$.

Consider the coefficient $K_{N}$ of $q^{N}$, for $N>l$, in the series $P\left(z_{0}(q), z(q)\right)$. Since $z_{0}(q)^{j}=q^{j}+O\left(q^{j+1}\right)$ for $j \geq 0$, the contribution to $K_{N}$ from the term $a_{p-i, i} z_{0}(q)^{p-i} z(q)^{i}$ ( $p \geq l$ ) must be an integral linear combination of monomials of the form $c_{n_{1}} \cdots c_{n_{i}}$, with $n_{1}+\cdots+n_{i} \leq N-(p-i)$. From this inequality, we find that $n_{k} \leq N-p+i-\left(n_{1}+\right.$ $\left.\cdots n_{i}-n_{k}\right) \leq N-p+1$ for $k=1, . ., i$. Thus the contribution to $K_{N}$ from the term $a_{p-i, i} z_{0}(q)^{p-i} z(q)^{i}$ lives in $\mathbf{Z}\left[c_{1}, . ., c_{N-p+1}\right]$. In particular, for $p>l$ this contribution is in $\mathbf{Z}\left[c_{1}, . ., c_{N-l}\right]$.

Consider the case $p=l$. It is easy to see that the order $q^{N}$ term in $a_{l-i, i} z_{0}(q)^{l-i} z(q)^{i}$ takes the form

$$
\begin{equation*}
a_{l-i, i} q^{l-i} i q^{i-1} c_{N-l+1} q^{N-l+1}+g q^{N} \tag{6.3}
\end{equation*}
$$

where $g \in \mathbf{Z}\left[c_{1}, . ., c_{N-l}\right]$. Setting $c_{1}=1$, we see that

$$
\begin{equation*}
K_{N}=m c_{N-l+1}+h \tag{6.4}
\end{equation*}
$$

for some $h \in \mathbf{Z}\left[c_{1}, . ., c_{N-l}\right]$. By induction, the relations $K_{N}=0$ imply that $c_{N-l+1} \in \mathbf{Z}\left[\frac{1}{m}\right]$.
It follows immediately from the above lemma that if $m= \pm 1$, then $z(q)$ is an integral q-series.

Now consider the polynomial relation between $z_{W}(q)$ and the mirror map $z(q)$ for each case of elliptic curves (see table in section 5.1). For each of the polynomials, we have $l=1, m=1$. It follows that $z(q)$ is integral in each case. Now for the four cases of K3
surfaces, we found the relations (5.19). For these polynomials, we have $l=2, m=1$. It follows again that $z(q)$ is integral in each case.

We should remark that the above lemma gives us an effective way to construct many integral q-series $f(q)$ which are commensurable with $1 / 1728 J(q)$. For example let $r(x, y)$ be any integral polynomial with no constant or linear terms. Then our lemma implies that the equation

$$
\begin{equation*}
\frac{1}{1728 J(q)}-z(q)+r\left(\frac{1}{1728 J(q)}, z(q)\right)=0 \tag{6.5}
\end{equation*}
$$

determines a unique integral q-series $z(q)$. This also implies that $z(q)$ admits an analytic continuation (with at most finitely many sheets) to the upper half plane $\operatorname{Im} t>0$. Moreover, since $J(q)$ satisfy the Schwarzian equation (4.3) with rational $Q(z), z(q)$ also satisfies a similar equation with algebraic $Q(z)$.

### 6.1. Relations with genus zero functions and the Thompson series

In this section, we point out some tantalizing evidence that the mirror maps for K3 surfaces are possibly related to the theory of the Griess-Fischer group. We will in fact give an alternative proof of integrality for the mirror maps. This proof, though more technical than the previous one, suggests some deeper connection with the theory of modular functions and the Thompson series.

Let's us recall a few facts about genus zero functions and the Thompson series (see [17].) Let $H$ be the upper half plane, $G$ be a discrete subgroup of $\operatorname{PSL}(2, \mathbf{R})$ which acts on $H$ by linear fractional transformations. We consider meromorphic functions $f$ on $H$ which has a Fourier expansion $\sum_{n>m} a_{n} q^{n}$ for $\operatorname{Im} t \gg 0\left(q=e^{2 \pi i t}\right)$.

If $f$ is invariant under $G$, then $f$ defines a function on the quotient $H / G$. If $H / G$ is a genus zero Riemann surface with finitely many punctures, we call $G$ a genus zero group and $f$ a genus zero function. In this case, the field of genus zero functions for $G$ has a canonical generator $h$ known as the normalized hauptmodul for $G$. It has a Fourier expansion of the form $h(t)=\frac{1}{q}+O(q)$. For example $\Gamma_{0}(1):=P S L(2, \mathbf{Z})$ is a genus zero group and the Dedekind-Klein j-function $j(t)-744$ is the hauptmodul for this group. More generally, a list of 174 hauptmoduls corresponding to certain congruent subgroups is known [18]. Let $\Gamma_{0}(N)$ be the group consisting of integral transformations $t \rightarrow(a t+b) /(c t+d)$ with $N \mid c$. Then the 174 genus zero groups are subgroups of $\operatorname{PSL}(2, \mathbf{R})$ generated by $\Gamma_{0}(N)$ together with a set of Atkin-Lehner involutions (see [19).

It turns out that there is a deep connection between the theory of modular functions and the Griess-Fischer group $M$ also known as the Monster. It is the largest sporadic simple finite group, whose order is:

$$
\begin{equation*}
2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29.31 \cdot 41 \cdot 59.71 \tag{6.6}
\end{equation*}
$$

For a long and glory history of this group, see the paper of Conway-Norton [17]. Based on some remarkable observations of MacKay and Thompson, Conway-Norton conjectured that there exist a natural Z-graded representation $V$ of $M$ with the following property: for every $g \in M$, the q-series, called the Thompson series of $g$,

$$
\begin{equation*}
T_{g}(q):=\sum \operatorname{tr}\left(\left.g\right|_{V_{n}}\right) q^{n-1} \tag{6.7}
\end{equation*}
$$

is a normalized hauptmodul in the list of 174 mentioned above. It turns out that all but 3 of those 174 hauptmoduls correspond to Thompson series. At the time the conjecture was made, neither $M$ nor $V$ had been known to exist, though the evidence for them was overwhelming. On some hypotheses, the first few coefficients of the Thompson series were computed in [17] and were seen to coincide with the coefficients of the appropriate hauptmoduls. (see table 4 in 17 ). Later, Griess proved that $M$ exists by constructing a 196883 -dimensional algebra $B$ in which $M$ acts by automorphisms. Subsequently, Frenkel-Lepowsky-Meurman [20] constructed a Z-graded representation $V^{\#}$ of $M$ with the property that $V_{2}^{\#}$ is the direct sum of $B$ with the 1-dimensional representation of $M$ and that $\sum \operatorname{tr}\left(\left.1\right|_{V_{n}^{\#}}\right) q^{n-1}=j(q)-744$. Borcherds [21] finally completed the proof of the Conway-Norton conjecture by showing that $V^{\#}$ does indeed have the property that the corresponding Thompson series coincide with the hauptmoduls suggested in [17].

Our brief account of the Monster by no means does justice to the many of the remarkable developments surrounding the theory of the Monster. The above paragraph is meant to provide just enough background to enter a discussion on the relation, which we are about to show, between the Thompson series and our mirror maps. Based on numerical experiments, we have

Observation: The reciprocal of the four mirror maps (5.18) agree respectively, up to an additive constant, with the Thompson series $T_{1 A}, T_{2 A}, T_{3 A}, T_{4 A}$ (see table 4 of [1才]).

Before we give a proof of this assertion, we argue that together with the ConwayNorton conjecture, the above assertion implies the integrality of mirror maps (5.18). The character values of a finite group are algebraic integers. But since $V^{\#}$ is defined over the
rationals, the coefficients of the Thompson series which are character values of $M$, must be rational integers. Note also that this property can also be deduced from the explicit representation of the hauptmoduls in terms of the Dedekind eta function.

To prove our assertion above, we will use Schwarz' theorem on the Riemann mapping and the fact that the Thompson series above are hauptmoduls for the following genus zero groups $\Gamma_{0}(1), \Gamma_{0}(2)+, \Gamma_{0}(3)+, \Gamma_{0}(4)+$ (see [17] on notations). If $G$ is a genus zero group and $h(t)$ its normalized hauptmodul, then it is easy to check that the expression $\{h(t), t\} /\left(2 h^{\prime}(t)^{2}\right)$ is a meromorphic function which is invariant under $G$. Since $h(t)$ is a generator of the function field for $G$, it follows that $\{h(t), t\} /\left(2 h^{\prime}(t)^{2}\right)$ is a rational expression $Q$ in $h(t)$. This shows that a hauptmodul satisfies a Schwarzian equation of the type (4.3). Our problem is to construct $Q$ in each of the cases of interest. Once we determine the Schwarzian equations governing the hauptmoduls $T_{1 A}, T_{2 A}, T_{3 A}, T_{3 A}$ in each case, it is enough to see that they coincide with the corresponding Schwarzian equation which we already know governing $\frac{1}{z}+c$ where $z$ is one of the $z_{X_{(6)}}, z_{X_{(4)}}, z_{X_{(2,3)}}, z_{X_{(2,2,2)}}$. For if two $q$-series satisfy the same Schwarzian equation and has the same asymptotic behavior $1 / q$, then they must coincide by uniqueness. The constant $c$ above will turn out to be the coefficient of $q^{2}$ in each case (so that $\frac{1}{z}+c$ has no constant term.)

We now proceed to determine $Q$ for each of the four hauptmoduls $h(t)$ above. Since $h(t)$ is an isomorphism from $H / G$ onto a punctured Riemann sphere, it defines a univalent mapping from a simply connected region $R$ (half a fundamental region of $G$ ) onto the upper half $h$-plane. If $R$ is a polygon with circular edges, then Schwarz' Theorem tells us that $h(t)$ satisfies the eqn. (4.3) with $Q$ having the form

$$
\begin{equation*}
Q(z)=\sum_{i=1}^{n}\left(\frac{A_{i}}{\left(z-a_{i}\right)^{2}}+\frac{B_{i}}{z-a_{i}}\right) \tag{6.8}
\end{equation*}
$$

where $a_{1}, \ldots, a_{n}$ are images of the vertices of $R$ and the $A_{i}, B_{i}$ are constants. To determine $Q$ completely it is enough to know the number $n$ of vertices (actually a reasonable bound would suffice). For then we can use the first few Fourier coefficients of $h(t)$ to determine the $A_{i}, B_{i}$. To find the number of vertices, we need to determine a domain $R$ and see that it is a circular polygon in each case. But since we know $G$ explicitly as a congruent subgroup in each case, constructing $R$ is easy. We find that the $R$ corresponding to the hauptmodul $T_{k A}(k=1,2,3,4)$ is $R=\left\{t \in H\left|0<\operatorname{Im} t<\frac{1}{2},|t|>\frac{1}{\sqrt{k}}\right\}\right.$. Each of these domains is a circular polygon with 3 vertices. Using the first 6 Fourier coefficients (actually

3 is enough because the $A_{i}, B_{i}$ satisfies 3 relations.) of the $T_{k A}$, we find that the respective $Q$ for their Schwarzian equations are:

$$
\begin{align*}
& T_{1 A}: Q(z)=\frac{19896824+53527 z+36 z^{2}}{144(743+z)^{2}(744+z)^{2}} \\
& T_{2 A}: Q(z)=\frac{40640-96 z+z^{2}}{4(-152+z)^{2}(104+z)^{2}}  \tag{6.9}\\
& T_{3 A}: Q(z)=\frac{7560-48 z+z^{2}}{4(-66+z)^{2}(42+z)^{2}} \\
& T_{4 A}: Q(z)=\frac{2752-32 z+z^{2}}{4(-40+z)^{2}(24+z)^{2}}
\end{align*}
$$

Now consider the four Fourier series $\frac{1}{z}+c$ mentioned above. We know that each one satisfies a Schwarzian equation which can be easily obtained from that of $z$. We find that the $Q$ indeed agrees with (6.9) in each case. This completes the proof of our assertion.

Our observation above raises various tantalizing questions. Why do the mirror maps of our K3 surfaces correspond to the Thompson series of the Monster group? What happens to the other realizations of K3 surfaces in weighted projective spaces? Do they realize other Thompson series? On the last two questions our joint work with A. Klemm in this direction is now underway. Early indications have shown that the correspondence continue to hold. On the first question, we offer the following speculative remarks. It is known that there are K3 surfaces on which a subgroup of the Mathieu group $M i_{24}$ acts by automorphisms [22]. It happens that $M i_{24}$ is also a subgroup of the Monster. Thus it makes sense to speak of the Thompson series for those elements in $M i_{24}$. It is possible that the appearance of Thompson series as mirror maps is related to some appropriate action of a subgroup of $M i_{24}$. Precisely how this happens is unclear at this point. However, enough evidence has convinced us the following:

Conjecture: If $z(q)$ is the mirror map for a 1-parameter deformations of an algebraic K3 surface from an orbifold construction, then for some $c \in \mathbf{Z}, \frac{1}{z(q)}+c$ is a Thompson series $T_{g}(q)$ for some $g \in M$.

It seems that the above connection between the mirror map and Thompson series is peculiar to K3 surfaces. In the case of elliptic curves for instance, unlike the K3 case, the correspondence there is only partial. Applying the same technique as above, we find that all but the 3rd family of elliptic curves we studied in section 5.1, correspond to Thompson series. They are $T_{2 B}, T_{3 B}$, and $T_{4 C}$ respectively. The mirror map for the 3rd family is in fact a double-valued function on the upper half plane, hence cannot be a modular function of any type.

## 7. Congruences of the Quantum Coupling

We now move on to Calabi-Yau threefolds. We consider a 1-parameter family of hypersurfaces or complete intersections $X_{z}$ in $\mathbf{P}^{n}[w]$, where $z$ is the complex structure deformation coordinate as defined in section 2 in terms of a finite orbifold group $G$. In this case, the period vector $\omega=\left(\omega_{0}, \omega_{1}, \omega_{2}, \omega_{3}\right)$ of the holomorphic 3 -form for the mirror manifold $X^{*}$ satisfy a 4th order Picard-Fuchs equation which has the form (4.19). The prepotential is defined as

$$
\begin{equation*}
F:=\frac{1}{\omega_{0}^{2}}\left(\omega_{3} \omega_{0}+\omega_{1} \omega_{2}\right) \tag{7.1}
\end{equation*}
$$

which is a holomorphic section of a line bundle over the moduli space $M$. It is also related to the period vector by [1] [23] 24] $\omega=\omega_{0}\left(1, t, \partial_{t} F, 2 F-t \partial_{t} F\right)$ where $t$ is a special coordinate defined by $t=\omega_{1}(z) / \omega_{0}(z)$. Let

$$
\begin{equation*}
K=\partial_{t}^{3} F \tag{7.2}
\end{equation*}
$$

The mirror hypothesis [1] identifies $t$ with the flat coordinate on the Kahler cone of $X$ and asserts that $K$ is the quantum coupling for $X$, which has the form [1] [25] [26]

$$
\begin{equation*}
K=\int_{X} J \wedge J \wedge J+\sum_{d \geq 1} \frac{d^{3} n_{d} e^{2 \pi i t d}}{1-e^{2 \pi i t d}} \tag{7.3}
\end{equation*}
$$

Here $n_{d}$ is the "number" of rational curves of degree $d$ in a generic deformation of $X$, and $J$ is the Kahler class on $X$.

In this section, we would like to study some arithmetic properties of $K$ as a $q$-series ( $q=e^{2 \pi i t}$ ) under the assumption of the mirror hypothesis with the $n_{d}$ integers, and that the mirror map $z(q)$ has integral Fourier coefficients. Specifically, we will study congruences $\bmod p$ of the instanton numbers under the above hypothesis, using a formal nonlinear ODE. We should point out that the integrality of $z(q)$ has been checked numerically in many examples of threefolds up to at least order $q^{30}$, and has been proved for K3 surfaces and elliptic curves in numerous cases.

### 7.1. Differential equations mod $p$

In [8] [9], we derived certain polynomial diferential equations for $z(q), K(q)$. An important feature of these equations is that they are defined over the rationals. This is so because the Picard-Fuchs equation, from which our ODE are constructed, are defined over the rationals. The main idea here is to derive the mod $p$ version of our nonlinear ODE
from the Picard-Fuchs equation. We then use some data - basically an integer $N_{e}$ - which is extracted from the Picard-Fuchs equation in each case to derive congruences modulo certain prime powers. We will illustrate this in several well-known examples.

We begin with the following form of the Picard-Fuchs equation (4.19):

$$
\begin{equation*}
\left(\frac{d^{4}}{d z^{4}}+q_{2}(z) \frac{d^{2}}{d z^{2}}+q_{2}^{\prime}(z) \frac{d}{d z}+q_{0}(z)\right) f(z)=0 \tag{7.4}
\end{equation*}
$$

where $q_{2}(z), q_{0}(z)$ are defined in (4.21). By a change of coordinate $z \mapsto t$ (see [9] for details) and using the list of solutions $f_{0}(z)\left(1, t, \partial_{t} F, 2 F-t \partial_{t} F\right)$, we show that $z, K$ as functions of $t$, satisfy a pair of coupled nonlinear ODEs: The simpler of the two is given by

$$
\begin{equation*}
\lambda_{z} K^{4}=\rho_{K} \quad(*) \tag{7.5}
\end{equation*}
$$

where

$$
\begin{align*}
\rho_{K}:= & 5^{2} \cdot 7 K^{\prime 4}-2^{3} \cdot 5 \cdot 7 K{K^{\prime}}^{2} K^{\prime \prime}+7^{2} K^{2} K^{\prime \prime 2}+2 \cdot 5 \cdot 7 K^{2} K^{\prime} K^{(3)}-2 \cdot 5 K^{3} K^{(4)} \\
\lambda_{z}:= & \frac{e(z) z^{\prime 4}}{\Delta(z)^{4}} \\
e(z):= & 10\left(-10 r_{2}+3 r_{2}-4 r_{4}\right) z+\left(-9 r_{2}^{2}+300 r_{0} r_{4}-46 r_{2} r_{4}+64 r_{4}^{2}\right) z^{2} \\
& +2 r_{4}\left(9 r_{2}^{2}-150 r_{0} r_{4}+7 r_{2} r_{4}-22 r_{4}^{2}\right) z^{3}+r_{4}^{2}\left(-9 r_{2}^{2}+100 r_{0} r_{4}+2 r_{2} r_{4}+11 r_{4}^{2}\right) z^{4} . \tag{7.6}
\end{align*}
$$

Here $\Delta(z)=z\left(1-r_{4} z\right)$ is the discriminant of the equation (4.19). (Actually, the following discussion requires only that $e(z), \Delta(z) \in \mathbf{Z}[z]$, with $\Delta(z)$ having leading coefficient 1 . No specific forms need to be assumed.) We define $N_{e}$ to be the l.c.m. of the coefficients in $e(z)$. Then it is clear that $n \mid N_{e}$ iff $n$ divides all Fourier coefficients of $e(z(q))$. All the arithmetic properties we derive of $K(q)$ will be entirely controlled by this one integer $N_{e}$ and the classical intersection number $K_{c l}:=\int_{X} J \wedge J \wedge J$.

Notice that (a) eqn. $\left(^{*}\right)$ is defined over $\mathbf{Z}$; (b) eqn. $\left({ }^{*}\right)$ is invariant under an overall constant scaling of $K$; (c) the primes $2,5,7$ feature prominantly. It turns out that, though less obvious, the eqn. also simplifies considerably modulo 3 . The main purpose of having eqn. $\left(^{*}\right)$ is as follows. Suppose a 1-parameter family of mirror Calabi-Yau varieties is given such that $p \mid N_{e}$ for some prime $p$. Then the left hand side of $\left(^{*}\right)$ becomes zero over the field of $q$-series $\mathbf{Z} / p \mathbf{Z}((q))$. We can then study the $q$-series solution $K(7.3)$ of this simplified equation modulo $p$ or its powers. We should point out that in [8] [9], we derived a similar equation but over $\mathbf{Q}$. The result there was much more complicated.

Notations: In the following discussion, $K$ will be the quantum coupling (7.3). The ' in the equation $\left({ }^{*}\right)$ means $q \frac{d}{d q}$. Thus $\left(q^{n}\right)^{\prime} \equiv 0 \bmod p$ whenever $p \mid n$. More generally when $p$ is prime and $h=\sum a_{n} q^{n}$ is any integral q-series, the $p t h$ derivative $h^{(p)} \equiv h^{\prime}$ because $n^{p} \equiv n$ for all $n$. In what follows, we will use these few tricks repeatedly, sometimes without mention, to simplify our computations. We write $n \mid h$ if $n \mid a_{n}$ for all $n$. We also write $\rho_{h}$ to denote the right hand side of (7.5) but with $K$ replaced with $h$ :

$$
\begin{equation*}
\rho_{h}:=5^{2} \cdot 7{h^{\prime}}^{4}-2^{3} \cdot 5 \cdot 7 h h^{\prime 2} h^{\prime \prime}+7^{2} h^{2} h^{\prime \prime 2}+2 \cdot 5 \cdot 7 h^{2} h^{\prime} h^{(3)}-2 \cdot 5 h^{3} h^{(4)} . \tag{7.7}
\end{equation*}
$$

For each prime $p$, we introduce the notation:

$$
\begin{equation*}
h_{p}:=\sum_{p \backslash n} a_{n} q^{n} . \tag{7.8}
\end{equation*}
$$

Thus we have $h^{\prime} \equiv h_{p}^{\prime} \bmod p$. Finally, given $K(\overline{7.3})$, we denote the largest integer divisor of $K$ by $m(K)$.

## 7.2. $\bmod p$

Claim 1: Let $h:=\frac{K}{m(K)}$. For any integer $n, n \mid h^{\prime}$ implies that $n \mid \rho_{h}$, and $n \mid \rho_{h}$ implies that $n \mid N_{e}$. Conversely, if $n \mid N_{e}$ then $n \mid \rho_{h}$.

Pf: If $n \mid h^{\prime}$ then from (7.7), it follows that $n \mid \rho_{h}$. Assuming $n \mid \rho_{h}$, eqn. $\left(^{*}\right.$ ) becomes $e(z) z^{\prime 4} h^{4} \equiv 0 \bmod n$. Since neither $h$ nor $z^{\prime}$ has any overall factor, we have $e(z) \equiv 0$. This implies that $N_{e} \equiv 0$. The converse follows immediately from eqn. (*).

This says that the integer $N_{e}$ tells us much about congruences of $\frac{K}{m(K)}$.
Claim 2: Fix a prime $p$ and suppose $p^{r} \mid K_{p}$. Then $p^{r} \mid n_{d}$ for all $d \not \equiv 0 \bmod p$, and $p^{\min (3, r)} \mid\left(K-K_{c l}\right)$.

Pf: From (7.3), the mth $(m>0)$ Fourier coefficient of $K$ is

$$
\begin{equation*}
k_{m}=\sum_{d \mid m} d^{3} n_{d} \tag{7.9}
\end{equation*}
$$

By supposition, $p^{r} \mid k_{m}$ for all $m \not \equiv 0 \bmod p$. Let $d \not \equiv 0 \bmod p$ be the smallest such integer for which $p^{r} \quad \nmid n_{d}$. Now $p^{r} \mid k_{d}$ means that $p^{r} \mid\left(d^{3} n_{d}+\sum_{d^{\prime} \mid d, d^{\prime}<d} d^{\prime 3} n_{d^{\prime}}\right)$. Since $p \nmid d$, that $d^{\prime} \mid d$ implies that $p \nmid d^{\prime}$, hence $p^{r} \mid n_{d^{\prime}}$ by the minimality of $d$. So $p^{r} \mid\left(d^{3} n_{d}+\sum_{d^{\prime} \mid d, d^{\prime}<d} d^{\prime 3} n_{d^{\prime}}\right)$ implies that $p^{r} \mid n_{d}$ which is a contradiction.

In particular, we have $p^{r} \mid d^{3} n_{d}$ for $p \Lambda d$. Trivially $p^{3} \mid d^{3} n_{d}$ for $p \mid d$, and so we have $p^{\min (3, r)} \mid d^{3} n_{d}$ for all $d$. Now it follows immediately from (7.3) that $p^{\min (3, r)} \mid\left(K-K_{c l}\right)$.

Now let $h$ be an integral q-series. As pointed out above, $h^{\prime} \equiv h_{p}^{\prime} \bmod p$. It follows that $p \mid h_{p}$ implies $p \mid \rho_{h}$. On the other hand, we have the following:

Claim 3: Assume prime $p \neq 2,5$. Suppose for any integral $q$-series $h$ with $p \nmid h, p \mid \rho_{h}$ implies $p \mid h_{p}$. Then for all $r>0, p^{r} \mid \rho_{h}$ implies that $p^{r} \mid h_{p}$ and $p^{r} \mid \rho_{h-h_{p}}$.

Pf: If $p \mid \rho_{h}$ then $p \mid h_{p}$, and so we can write $h_{p}=p f$ for some integral series $f$. Write also $g=h-h_{p}$, which has only q-powers $q^{n}$ with $p \mid n$. Thus $p \mid g^{\prime}$, hence $p \mid \rho_{g}$. Thus our statement holds for $r=1$.

Suppose it holds up to some $r>0$. Assume $p^{r+1} \mid \rho_{h}$. By inductive hypothesis we have at least $p^{r} \mid h_{p}$, and so we can write $h_{p}=p^{r} f, g=h-h_{p}$ as before. Because $h=p^{r} f+g$, and $p \nmid h$ by assumption, we have $p \backslash g$. The equation $\rho_{h} \equiv 0 \bmod p^{r+1}$ becomes, after some simplifications using $p^{i} \mid g^{(i)}$,

$$
\begin{equation*}
\rho_{g}-10 p^{r} g^{3} f^{\prime \prime \prime \prime} \equiv 0 \bmod p^{r+1} \tag{7.10}
\end{equation*}
$$

But $g$ only has q-powers $q^{n}$ with $p \mid n$, and hence so does $\rho_{g}$. On the other hand $f$ (hence $\left.f^{\prime \prime \prime \prime}\right)$ has only $q^{n}$ with $p \nmid n$. Thus ( $\overline{7.10)}$ implies that $p^{r+1} \mid \rho_{g}$ and $p^{r+1} \mid\left(10 p^{r} g^{3} f^{\prime \prime \prime \prime}\right)$. From the fact that $p \neq 2,5$ and that $p \nmid g$, it follows that $p \mid f^{\prime \prime \prime \prime}$ and hence $p \mid f$. Thus $h_{p}=p^{r} f$ implies that $p^{r+1} \mid h_{p}$.

The cases $p=2,5$ are dealt with separately. We will also discuss the cases $p=3,7$ in more details.

## 7.3. $\bmod 2$

Claim 4: Let h be integral $q$-series with $2 \not \backslash h$. Then $2 \mid \rho_{h}$ implies that $2 \mid h_{2}$ or $2 \mid\left(h-h_{2}\right)$. If moreover $2 \mid h_{2}$ then (a) $2^{r} \mid \rho_{h}$ implies that $2^{r} \mid \rho_{h-h_{2}}$, and (b) $2^{r+1} \mid \rho_{h}$ implies that $2^{r} \mid h_{2}$, for all $r>0$.

Pf: Suppose $2 \mid \rho_{h}$. Then we get

$$
\begin{equation*}
h^{\prime 4}+h^{2} h^{\prime 2} \equiv 0 \bmod 2 \tag{7.11}
\end{equation*}
$$

This says that either $2 \mid h^{\prime}$ or $2 \mid\left(h+h^{\prime}\right)$. But $h^{\prime} \equiv h_{2}^{\prime} \equiv h_{2} \bmod 2$, implying that either $2 \mid h_{2}$ or $2 \mid\left(h-h_{2}\right)$. This proves the first half.

For part (a), the proof of Claim 3 applies to $p=2$ without change.
For part (b), we modify the inductive argument there as follows. Suppose $2^{3} \mid \rho_{h}$ and write $h_{2}=2 f, g=h-h_{2}$ as before. The equation $\rho_{h} \equiv 0 \bmod 2^{3}$ becomes

$$
\begin{equation*}
\rho_{g}+49.2^{2} g^{2} f^{\prime \prime 2}-10.2 g^{3} f^{\prime \prime \prime \prime} \equiv 0 \bmod 2^{3} . \tag{7.12}
\end{equation*}
$$

Part (a) implies that $2^{3} \mid \rho_{g}$. Also $g^{2} f^{\prime \prime 2}$ only has q-powers $q^{n}$ with $2 \mid n$ while $g^{3} f^{\prime \prime \prime \prime}$ has only those with $2 \nmid n$. Thus (7.12)implies that all three terms vanish separately. It follows that $2 \mid f^{\prime \prime \prime \prime}$ and hence $2 \mid f$. Thus $h_{2}=2 f$ implies that $2^{2} \mid h_{2}$. Suppose (b) holds up to some $r>1$. Assume $2^{r+2} \mid \rho_{h}$. By inductive hypothesis we have at least $2^{r} \mid h_{2}$, and so we can write $h_{2}=2^{r} f, g=h-h_{2}$ as before. The equation $\rho_{h} \equiv 0 \bmod 2^{r+2}$ becomes

$$
\begin{equation*}
\rho_{g}-10.2^{r} g^{3} f^{\prime \prime \prime \prime} \equiv 0 \bmod 2^{r+2} . \tag{7.13}
\end{equation*}
$$

Part (a) implies that $2^{r+2} \mid \rho_{g}$. By assumptions, $2 \not \backslash h$ and $2 \mid h_{2}$ hence $2 \not \chi g$. It follows that $2 \mid f^{\prime \prime \prime \prime}$, hence $2 \mid f$. Thus $h_{2}=2^{r} f$ implies that $2^{r+1} \mid h_{2}$. This completes our proof of part (b).

Claim 5: If $2 \mid\left(K+K^{\prime}\right)$ then $2 \mid n_{d}$ for all odd $d$, and $2 \mid K$.
Pf: Let $l$ be the smallest odd integer such that $2 \nmid n_{l}$. Now $2 \mid\left(K+K^{\prime}\right)$ means that $2 \mid k_{m}$ for all even $m$, where $k_{m}$ is the $m$ th Fourier coefficients of $K$. In particular $2 \mid k_{2 l}$. But

$$
\begin{equation*}
k_{2 l}=\sum_{d \mid 2 l} d^{3} n_{d}=\sum_{d \mid 2 l, d \text { odd }} d^{3} n_{d}+\sum_{d \mid 2 l, d \text { even }} d^{3} n_{d} . \tag{7.14}
\end{equation*}
$$

The even part obviously divides 2 , while the odd part is

$$
\begin{equation*}
\sum_{d \mid 2 l, d \text { odd }} d^{3} n_{d}=l^{3} n_{l}+\sum_{d \mid l, d \text { odd, } l>d} d^{3} n_{d} \tag{7.15}
\end{equation*}
$$

By the minimality of $l$, the second sum divides 2 , implying that $2 \mid l^{3} n_{l}$. But $l$ is odd, hence $2 \mid n_{l}$ which is a contradiction. Thus $2 \mid n_{d}$ for odd $d$.

Now consider (7.3). If $2 \mid n_{d}$ for odd $d$, then obviously $2 \mid d^{3} n_{d}$ for all $d$. Also $2 \mid\left(K+K^{\prime}\right)$ implies that $2 \mid K_{c l}$. So we have $2 \mid K$.

Remarks: Let $h:=\frac{K}{m(K)}$ and suppose $2^{r} \mid N_{e}, r>0$. By Claim 1, we have $2^{r} \mid \rho_{h}$. By Claim 4, we have either $2 \mid h_{2}$ or $2 \mid\left(h-h_{2}\right)$. Suppose first $2 \mid h_{2}$. Then $2^{r-1} \mid h_{2}$ by Claim $4 b$, hence $\left(m(K) 2^{r-1}\right) \mid K_{2}$. Thus by Claim 2, we see that $2^{s+r-1} \mid n_{d}$ for all $d$ odd, where $s$ is the largest integer such that $2^{s} \mid m(K)$. Moreover, $2^{\min (3, s+r-1)} \mid\left(K-K_{c l}\right)$. To summarize: if $2^{r} \mid N_{e}$ and $2 \left\lvert\, \frac{K_{2}}{m(K)}\right.$, then (a) $2^{s+r-1} \mid n_{d}$ for all $d$ odd, where $s$ is the largest integer such that $2^{s} \mid m(K)$, and (b) $2^{\min (3, s+r-1)} \mid\left(K-K_{c l}\right)$.

Suppose now $2 \mid\left(h-h_{2}\right)$ instead. This means in particular that $2 \mid\left(K+K^{\prime}\right)$, and so by Claim 5 we have $2 \mid n_{d}$ for all odd $d$, and $2 \mid K$. Thus $m(K) \geq 2$. So if $2 \mid N_{e}$ and $2 \left\lvert\,\left(\frac{K-K_{2}}{m(K)}\right)\right.$, then $2 \mid n_{d}$ for all odd $d$ and $2 \mid K$.

## 7.4. $\bmod 3$

Suppose $h$ is any integral series with $3 \mid \rho_{h}$ and $3 \not \backslash h$. We get by differentiating $\rho_{h} \equiv$ $0 \bmod 3$ once,

$$
\begin{equation*}
h h^{\prime}\left(2 h^{2}+h^{\prime 2}+h h^{\prime \prime}\right) \equiv 0 \bmod 3 . \tag{7.16}
\end{equation*}
$$

If $3 \backslash h^{\prime}$ then $3 \mid\left(2 h^{2}+h^{\prime 2}+h h^{\prime \prime}\right)$. Differentiating $2 h^{2}+h^{\prime 2}+h h^{\prime \prime}$ and simplifying it by $h^{\prime \prime \prime} \equiv h^{\prime} \bmod 3$, we get $3 \mid\left(2 h h^{\prime}\right)$ implying that $3 \mid h^{\prime}$, hence $3 \mid h_{3}$. Thus we have shown that $3 \mid \rho_{h}$ implies $3 \mid h^{\prime}$ and $3 \mid h_{3}$.

Remark: Let $h:=\frac{K}{m(K)}$ and suppose $3^{r} \mid N_{e}, r>0$. Then by Claim 1, we have $3^{r} \mid \rho_{h}$. We have just shown that $3 \mid \rho_{h}$ implies $3 \mid h_{3}$. Then $3^{r} \mid h_{3}$ by Claim 3, hence $\left(m(K) 3^{r}\right) \mid K_{3}$. Thus by Claim 2, we see that $3^{s+r} \mid n_{d}$ for all $d \not \equiv 0 \bmod 3$, where $s$ is the largest integer such that $2^{s} \mid m(K)$. Moreover, $3^{\min (3, s+r)} \mid\left(K-K_{c l}\right)$. To summarize: if $3^{r} \mid N_{e}$, then (a) $3^{s+r} \mid n_{d}$ for all $d \not \equiv 0 \bmod 3$, where $s$ is the largest integer such that $3^{s} \mid m(K)$, and (b) $3^{\min (3, s+r)} \mid\left(K-K_{c l}\right)$.

## 7.5. $\bmod 5$

Claim 6: Let $h$ be integral $q$-series with 5 久h. Then $5 \mid \rho_{h}$ implies that $5 \mid h_{5}$. Moreover we have (a) $5^{r} \mid \rho_{h}$ implies that $5^{r} \mid \rho_{h-h_{5}}$, and (b) $5^{r+1} \mid \rho_{h}$ implies that $5^{r} \mid h_{5}$, for all $r>0$.

Pf: Suppose $5 \mid \rho_{h}$. Then the eqn. $\rho_{h} \equiv 0 \bmod 5$ reads $4 h h^{\prime \prime} \equiv 0 \bmod 5$. Hence $5 \mid h^{\prime \prime}$ which implies $5 \mid h_{5}$. This proves the first statement.

For part (a), the proof of Claim 3 applies to $p=5$ without change.
For part (b), the proof of Claim $4 b$ applies here with only minor changes.
Remarks: Let $h:=\frac{K}{m(K)}$ and suppose $5^{r} \mid N_{e}, r>0$. By Claim 1, we have $5^{r} \mid \rho_{h}$. Then $5^{r-1} \mid h_{5}$ by Claim 6b, hence $\left(m(K) 5^{r-1}\right) \mid K_{5}$. Thus by Claim 2, we see that $5^{s+r-1} \mid n_{d}$ for all $d \not \equiv 0 \bmod 5$, where $s$ is the largest integer such that $5^{s} \mid m(K)$. Moreover, $5^{\min (3, s+r-1)} \mid\left(K-K_{c l}\right)$. To summarize: if $5^{r} \mid N_{e}$, then (a) $5^{s+r-1} \mid n_{d}$ for all $d \not \equiv 0 \bmod 5$, where $s$ is the largest integer such that $5^{s} \mid m(K)$, and (b) $5^{\min (3, s+r-1)} \mid\left(K-K_{c l}\right)$.

## 7.6. mod 7

We summarize the result here: if $7^{r} \mid N_{e}$, then (a) $7^{s+r} \mid n_{d}$ for all $d \not \equiv 0 \bmod 7$, where $s$ is the largest integer such that $7^{s} \mid m(K)$, and (b) $7^{\min (3, s+r)} \mid\left(K-K_{c l}\right)$. The argument here is quite similar to the case of $\bmod 3$.

### 7.7. Examples

Under the assumption of the integrality of the mirror map $z(q)$ and the mirror hypothesis' assertion that quantum coupling $K(q)$ is given by (7.3) with integers $n_{d}$, we consider the following examples applying the above results:

1. Let $X$ be the Fermat quintics in $\mathbf{P}^{4}$. The orbifold group here is an abelian group of type $(5,5,5)\left[12\right.$ [1]. In this case, we have $N_{e}=-5^{3}$ and $K_{c l}=5$. It follows from our $\bmod 5$ analysis that at least $5^{2} \mid\left(K-K_{c l}\right)$. But since $K_{c l}=5$, we have $5 \mid K$ and $m(K)=5$. Thus $5^{3} \mid n_{d}$ for all $d \neq 0 \bmod 5$. Because $n_{1}=23.5^{3}, 5^{3}$ is the upper bound on the prime power dividing all $n_{d}$. Experimentally however, it seems that $5^{3}$ divides all $n_{d}$, as previously observed by others (see table 4 of [1]). Note that since $N_{e}$ has no prime divisor other than 5 , there is no other prime $p$ dividing all $n_{d}$, by Claim 1 .
2. Consider $X$ the complete intersection of a quartic $x_{1}^{4}+x_{2}^{4}+x_{3}^{4}=0$ and a sextic $x_{4}^{3}+x_{5}^{3}+x_{6}^{2}=0$ in $\mathbf{P}^{5}[1,1,1,2,2,3]$. The orbifold group is of type (4, 4, 3). In this case, we have $N_{e}=-2^{6} .3^{5}$ and $K_{c l}=2$. It follows from our mod 3 analysis that $3^{5} \mid n_{d}$ for all $d \xlongequal{=} 0 \bmod 3$. Once again, this lower bound on the powers of 3 is sharp because already $n_{1}=2^{6} .3^{5}$. For mod 2 , our analysis shows that either $2 \mid h_{2}$ our $2 \mid\left(h-h_{2}\right)$, where $h=\frac{K}{m(K)}$. Suppose $2 \mid\left(h-h_{2}\right)$. Then our analysis above implies that $2 \mid K$ and $m(K) \geq 2$. But $K_{c l}=2 \geq m(K)$, implying that $m(K)=2$. Thus $\left(h-h_{2}\right)$ has constant term $\frac{K_{c l}}{m(K)}=1$, contradicting $2 \mid\left(h-h_{2}\right)$. It follows that we must have $2 \mid h_{2}$. By the mod 2 analysis above, we conclude that at least $2^{6} \mid n_{d}$ for all $d$ odd. Again this lower bound on the power of 2 is also sharp because already $n_{1}=2^{6} .3^{5}$. Note that since $N_{e}$ has no prime divisor other than 2,3 , there is no other prime $p$ dividing all $n_{d}$, by Claim 1 .
3. Finally consider $X$ the degree 10 hypersurface of Fermat type in $\mathbf{P}^{4}[1,1,1,2,5]$. The orbifold group is of type $(10,5,2)$. In this case, we have $N_{e}=-2^{6} .5^{3}$ and $K_{c l}=1$. Thus $m(K)=1$. It follows from our $\bmod 2$ and $\bmod 5$ analyses that at least $2^{5} \mid n_{d}$ for $d$ odd; and $5^{2} \mid n_{d}$ for $d \equiv 0 \bmod 5$. These bounds on the prime power are also sharp because already $n_{1}=2^{5} .5^{2} .17$. Note that since $N_{e}$ has no prime divisor other than 2,5 , there is no other prime $p$ dividing all $n_{d}$, by Claim 1 .

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