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Harmonic Mappings and Kähler Manifolds

Jürgen Jost[1] and Shing-Tung Yau[2]

1 Mathematisches Institut der Universität Bonn, Wegelerstrasse 10, D-5300 Bonn 1, Federal Republic of Germany
2 Institute for Advanced Study, Princeton, NJ 08540, USA

0. Introduction

In this paper, we develop an approach to the study of compact Kähler manifolds which admit mappings of everywhere maximal rank into quotients of polydiscs, e.g. into Riemann surfaces or products of them.

One main tool will be a detailed study of the harmonic maps in the corresponding homotopy classes (for definition and general properties of harmonic maps between Riemannian manifolds see [3]). Starting with a result of Siu, we prove in Sect. 2 that the local level sets of the components of these mappings are analytic subvarieties of the domain. This, together with a generalization of the similarity principle of Bers and Vekua which is proved in the appendix and a residue argument, enables us to give conditions involving the Chern and Kähler classes of the considered manifolds, under which this harmonic map is of maximal rank everywhere and, in case domain and image have the same dimension, in particular a local diffeomorphism (see Corollary 4.1, Theorem 5.1, and Corollary 5.1). In the latter case this condition can be formulated as follows:

$$c_1(M)\cup\Omega^{m-1}(M)[M]=f^*c_1(N)\cup\Omega^{m-1}(M)[M],$$

where $[M]$ denotes the fundamental homology class of the domain M, Ω its Kähler class, $c_1(M)$ its first Chern class and $c_1(N)$ the first Chern class of the image N. We always assume that N is a compact quotient of polydiscs with the usual induced metric and complex structure and that the functional determinant of $f:M\to N$ does not vanish identically.

Under the same condition a compact Kähler manifold which is homeomorphic to a quotient of polydiscs must necessarily be such a quotient also (Theorem 6.1). With the same arguments we can also study deformations of compact quotients of polydiscs (Theorem 6.2).

Another application of our methods is given to the Kodaira surfaces. These were first introduced by Kodaira [10] and independently by Atiyah [1] to provide examples of algebraic surfaces with positive index and of fibre spaces whose signature is different from the product of the signatures of the base space and the fibre. They can also be considered as locally nontrivial families of Riemann surfaces. They were also investigated by Hirzebruch [6] and Kas [7]. In Sect. 7 we prove that every deformation (in the sense of [11, p. 334f.]) of such a surface is again a Kodaira surface. Thus, their moduli space turns out to be the moduli space of a Riemann surface underlying the construction. This is a considerable extension of the corresponding local result of Kas, obtained by quite different, namely sheaf theoretic methods.

Our methods certainly have the potential to be applied to many more similar problems. In this paper, however, we restrict ourselves to the two applications mentioned above because they display already the essential features of our arguments.

We thank the referee for his comments on our first version of this paper. He pointed out to us that our original proof of Theorem 6.2 had to be modified and could be extended. Furthermore, he noted that for the purpose of the present paper, Proposition A.1 can be replaced by the following.

Lemma. *Suppose D is a Stein manifold, f a smooth function and ω a smooth $(0,1)$ form on D. Assume that the zero set of f is nowhere dense in D and $\bar\partial f = f\omega$. Then f can be represented as $f = e^h g$, where h is smooth and g is holomorphic on D.*

Proof. $\bar\partial\omega = 0$, since $\omega = \bar\partial \log f$. Hence $\omega = \bar\partial h$ for some smooth function h on D, and $g := fe^{-h}$ is holomorphic. q.e.d.

We think, however, that the higher dimensional generalization of the similarity principle of Bers and Vekua is of much more general interest than just for the application in the present paper, and therefore we still include the appendix.

1. Basic Formulae and Notations

Suppose M is a compact Kähler manifold of complex dimension m, and let N be a compact quotient of the n-fold product of the hyperbolic unit disc D by a discrete group of automorphisms Γ without singularities, i.e.

$$N = D \times \dots \times D/\Gamma .$$

Since the identity component of $\mathrm{Aut}(D^n)$ is $(\mathrm{Aut}(D))^n$, N is locally a product in the metric induced from the hyperbolic metric on D. If we restrict ourselves to local coordinates which respect this local product structure, the coordinate transformations will be of the form

$$z^1 = z^1(w^{\sigma(1)}), \dots, z^n = z^n(w^{\sigma(n)}), \tag{1.1}$$

where $(\sigma(1), \dots, \sigma(n))$ is a permutation of $(1, \dots, n)$, the metric $\gamma_{\alpha\bar\beta}$ and the curvature tensor are diagonal and the only nonvanishing Christoffel symbols will be

$$\Gamma^\alpha_{\alpha\alpha}, \Gamma^{\bar\alpha}_{\bar\alpha\bar\alpha}, \alpha \in \{1, \dots, n\} . \tag{1.2}$$

If $R^N(\cdot,\cdot,\cdot,\cdot)$ is the curvature tensor of N, we have

$$R^N(\xi, \eta, \zeta, \omega) = \sum_{\alpha=1}^{n} R_{\alpha\bar\alpha\alpha\bar\alpha}(\xi^\alpha \eta^{\bar\alpha} - \eta^\alpha \xi^{\bar\alpha}) \cdot \overline{(\zeta^\alpha \omega^{\bar\alpha} - \omega^\alpha \zeta^{\bar\alpha})}, \tag{1.3}$$

where $\xi = \sum_{\alpha=1}^{n} \left(\xi^\alpha \frac{\partial}{\partial w^\alpha} + \xi^{\bar\alpha} \frac{\partial}{\partial w^{\bar\alpha}} \right), \ldots$ and w is a local coordinate on N. Also, the sectional curvature in the direction given by the α^{th} local factor of N is

$$K^\alpha = -1. \tag{1.4}$$

In the following, if not otherwise indicated, a Latin capital used as summation index ranges from 1 to m, and a Greek one from 1 to n. This convention will be violated, however, in Sect. 7.

Denote the metric tensor of M by $(h_{i\bar j})$ and its inverse by $(h^{i\bar j})$, i.e.

$$\sum_{j=1}^{m} h^{i\bar j} \cdot h_{k\bar j} = \delta_{ik}. \tag{1.5}$$

Since the metric tensor is Hermitian, we have

$$h_{i\bar j} = h_{\bar j i} = \overline{h_{\bar i j}} = \overline{h_{j\bar i}}, \tag{1.6}$$

and since the metric is Kählerian,

$$\begin{aligned} \frac{\partial}{\partial z^i} h_{k\bar l} &= \frac{\partial}{\partial z^k} h_{i\bar l} \\ \frac{\partial}{\partial z^{\bar j}} h_{k\bar l} &= \frac{\partial}{\partial z^{\bar l}} h_{k\bar j}, \end{aligned} \tag{1.7}$$

where $z = (z^1, \ldots, z^m)$ is a local holomorphic coordinate on M and $z^{\bar i} = \overline{z^i}$. Moreover, the Riemannian curvature tensor on M is given by

$$R_{i\bar j k\bar l} = \frac{\partial}{\partial z^{\bar l}} \left(\frac{\partial}{\partial z^k} h_{i\bar j} \right) - \sum_{s,t} h^{s\bar t} \frac{\partial}{\partial z^k} h_{i\bar t} \frac{\partial}{\partial z^{\bar l}} h_{s\bar j}. \tag{1.8}$$

Finally, put

$$h = \det(h_{i\bar j}). \tag{1.9}$$

Now let $f = M \to N$ be a smooth map. We define the second covariant derivatives of f in a local coordinate representation by the formulae

$$\begin{aligned} D_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} f^\alpha &= \frac{\partial}{\partial z^A} \frac{\partial}{\partial z^B} f^\alpha + \sum_{\beta,\gamma} \Gamma^\alpha_{\beta\gamma} \frac{\partial f^\beta}{\partial z^A} \cdot \frac{\partial f^\gamma}{\partial z^B} \\ D_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} f^{\bar\alpha} &= \frac{\partial}{\partial z^A} \frac{\partial}{\partial z^B} f^{\bar\alpha} + \sum_{\beta,\gamma} \Gamma^{\bar\alpha}_{\bar\beta\bar\gamma} \frac{\partial f^{\bar\beta}}{\partial z^A} \cdot \frac{\partial f^{\bar\gamma}}{\partial z^B} \end{aligned} \tag{1.10}$$

whenever $A, B \in \{1, \ldots, m, \bar 1, \ldots, \bar m\}$. Here "covariant" means in the vector bundle over M obtained by the pull-back of the tangent bundle of N by f^*. Occasionally, we shall use the notation

$$\begin{aligned} e^\alpha(f) &= |\partial f^\alpha|^2 \\ e^{\bar\alpha}(f) &= |\bar\partial f^\alpha|^2, \end{aligned} \tag{1.11}$$

where

$$\partial f^\alpha = \sum_i \frac{\partial f^\alpha}{\partial z^i} dz^i$$

$$\bar\partial f^\alpha = \sum_i \frac{\partial f^\alpha}{\partial z^{\bar i}} dz^{\bar i}.$$

For notational convenience, we shall sometimes write

$$\partial_i f^\alpha := \frac{\partial f^\alpha}{\partial z^i}$$

$$\partial_{\bar i} f^\alpha := \frac{\partial f^\alpha}{\partial z^{\bar i}}$$

and the same for $f^{\bar\alpha}$, when the coordinate z is fixed.

2. Consequences of Siu's Analysis

Let $f: M \to N$ be a harmonic map, i.e.

$$\sum_{i,j}\left(h^{i\bar j}\partial_{\bar j}\partial_i f^\alpha + h^{i\bar j}\sum_{\beta,\gamma}\Gamma^\alpha_{\beta\gamma}\partial_i f^\beta \partial_{\bar j} f^\gamma\right) = 0. \tag{2.1}$$

(The fundamental theorem of Eells and Sampson implies existence and C^∞-regularity of a harmonic map in our setting, cf. [3].)

Since we assumed $K^\alpha < 0$ for $\alpha = 1, \ldots, m$, the curvature of N is strongly seminegative in the sense of [16, p. 77]. Therefore, it follows from [16, pp. 81 and 82] that

$$D_{\frac{\partial}{\partial z^{\bar j}}} \frac{\partial f^\alpha}{\partial z^i} = D_{\frac{\partial}{\partial z^{\bar j}}} \frac{\partial f^{\bar\alpha}}{\partial z^i} = 0, \quad \text{for} \quad \begin{matrix} i,j \in \{1, \ldots, m\} \\ \alpha \in \{1, \ldots, n\} \end{matrix} \tag{2.2}$$

and, since $K^\alpha < 0$, also that

$$\partial_{\bar i} f^\alpha \overline{\partial_j f^\alpha} - \partial_{\bar j} f^\alpha \overline{\partial_i f^\alpha} = 0 \tag{2.3}$$

for $i,j \in \{1, \ldots, m\}$, $\alpha \in \{1, \ldots, n\}$.

If $e^\alpha(f) \neq 0$, it follows that

$$\partial_{\bar i} f^\alpha(z) = \overline{\partial_i f^\alpha(z)} \cdot \lambda^\alpha(z), \tag{2.4}$$

where $\lambda^\alpha(z)$ is independent of the index $i \in \{1, \ldots, m\}$.

Suppose again, that $e^\alpha(f) \neq 0$ at a given point $z_0 \in M$. In a suitable chart, we have therefore $\frac{\partial f^\alpha}{\partial z^1} \neq 0$. We set

$$\frac{\partial f^\alpha}{\partial z^j} = k^\alpha_j(z) \cdot \frac{\partial f^\alpha}{\partial z^1}, \quad \text{for} \quad j \in \{2, \ldots, m\}. \tag{2.5}$$

Then $k_j^\alpha(z)$ is a smooth function in a neighborhood of z_0 where $e^\alpha(f)\neq 0$. From (2.2), we have for $i\in\{1,\dots,m\}$,

$$\begin{aligned}0&=\frac{\partial}{\partial z^{\bar{i}}}\frac{\partial}{\partial z^j}f^\alpha+\sum_\alpha \Gamma^\alpha_{\alpha\alpha}\frac{\partial f^\alpha}{\partial z^{\bar{i}}}\frac{\partial f^\alpha}{\partial z^j}\\&=\frac{\partial}{\partial z^{\bar{i}}}k_j^\alpha\cdot\frac{\partial f^\alpha}{\partial z^1}+k_j^\alpha\left(\frac{\partial}{\partial z^{\bar{i}}}\frac{\partial}{\partial z^1}f^\alpha+\sum_\alpha \Gamma^\alpha_{\alpha\alpha}\frac{\partial f^\alpha}{\partial z^{\bar{i}}}\frac{\partial f^\alpha}{\partial z^1}\right),\end{aligned}$$

using (1.2).

Now, again using (2.2) and $e^\alpha(f)\neq 0$, it follows that

$$\frac{\partial}{\partial z^{\bar{i}}}k_j^\alpha(z)=0.$$

Since $k_j^\alpha(z)$ is smooth and holomorphic in each variable z^i separately, $k_j^\alpha(z)$ is a holomorphic function of z by a well-known theorem of Hartogs.

Assuming again $e^\alpha(f)\neq 0$ locally for a fixed α, we want to perform a holomorphic change of coordinates $(z^1,\dots,z^m)\to(\zeta^1,\dots,\zeta^m)$ with the property that

$$\frac{\partial f^\alpha}{\partial \zeta^j}=0\quad\text{for}\quad j=2,\dots,m.$$

This condition is equivalent to

$$\sum_i\frac{\partial f^\alpha}{\partial z^i}\cdot\frac{\partial z^i}{\partial \zeta^j}=0, \tag{2.6}$$

since the coordinate change will be holomorphic.

Since $e^\alpha(f)\neq 0$ locally, we can assume again $\frac{\partial f^\alpha}{\partial z^i}\neq 0$, and applying (2.5), we get

$$\frac{\partial z^1}{\partial \zeta^j}+\sum_{i=2}^m k_i^\alpha(z)\frac{\partial z^i}{\partial \zeta^j}=0,\quad j=2,\dots,m \tag{2.7}$$

as equations for the tangent space of the local analytic subsets $\zeta^1=\text{const}$ [the distribution given by the vectors

$$(k_2^\alpha(z),-1,0,\dots,0),(k_3^\alpha(z),0,-1,0,\dots,0),(k_m^\alpha(z),0,\dots,0,-1),$$

clearly satisfies the integrability condition by (2.5)].

As ζ^1-curves, i.e. lines $\zeta^2=\text{const},\dots,\zeta^m=\text{const}$, we can choose any family of local analytic curves which intersect the analytic sets $\zeta^1=\text{const}$ transversally.

For later purposes, however, we note that we can obtain the coordinate ζ^1 more explicitly in the following way: If we replace ζ^1 by ζ^j in the previous considerations, we get for each $j\in\{2,\dots,m\}$ an equation of the same type as (2.7). ζ^1 can then be defined as the common solution of these $m-1$ equations.

By (2.4) it follows that for $j=2,\dots,m$

$$\begin{aligned}\frac{\partial f^\alpha}{\partial \zeta^{\bar{j}}}&=\sum_i\frac{\partial f^\alpha}{\partial z^{\bar{i}}}\cdot\frac{\partial z^{\bar{i}}}{\partial \zeta^{\bar{j}}}=\sum_i\lambda^\alpha(z)\overline{\frac{\partial f^\alpha}{\partial z^i}}\cdot\overline{\frac{\partial z^i}{\partial \zeta^j}}\\&=0\ \text{by (2.6)},\end{aligned}$$

and consequently f^α is independent of the coordinates $\zeta^2, \ldots, \zeta^m$ in our special coordinate system, which means that the analytic sets $\zeta^1 = \text{const}$ describe local level sets.

[More precisely, at a point where $e^\alpha(f) \neq 0$, the local level sets of f^α are unions of analytic sets. Points where $e^\alpha(f) = 0$ are treated in Sect. 4.]

In the case where f^α is a map with $e^\alpha(f) \neq 0$ everywhere to a closed Riemann surface, we can piece the local ζ^1 coordinates together to give a holomorphic map from M to a closed Riemann surface by using a result of Kaup as follows:

Since the level sets of f^α coincide locally with the level sets of ζ^1, they give rise to an analytic and open equivalence relation R on M. By [8], it follows that the quotient space M/R is an analytic space and the projection $\pi: M \to M/R$ is holomorphic. Since in our case obviously M/R is a closed Riemann surface of the same genus as the image of f^α, this proves the above claim.

3. Important Local Formulae

We assume here and in the following that $n \leqq m$.

Let $\sigma \subset \{1, \ldots, n\}$, $|\sigma| = k$, $\sigma = \{\alpha_1, \ldots, \alpha_k\}$. We define

$$a_\sigma := \sum_{I_k = \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}} \det(\gamma_{\alpha\bar{\beta}})_{\alpha, \beta \in \sigma} \cdot \det(h^{i\bar{j}})_{i, j \in I_k} \left| \det\left(\frac{\partial f^\alpha}{\partial z^i}\right)_{\alpha \in \sigma, i \in I_k} \right|^2$$

(it is crucial that we take only derivatives with respect to z^i, but not with respect to $z^{\bar{i}}$).

a_σ is invariant under local coordinate transformations, but in general not necessarily globally defined. It is globally defined, however, if $n = m$ and $\sigma = \{1, \ldots, n\}$, or if $N = D^n/\Gamma_0$, where Γ_0 belongs to the identity component $(\mathrm{Aut}(D))^n$ of $\mathrm{Aut}(D^n)$.

We note that

$$\det(h^{r\bar{s}})_{r, s \in I_k} = \det(h^{i\bar{j}})_{i, j \in \{1, \ldots, m\}} \cdot \det H_{I_k}, \tag{3.1}$$

where H_{I_k} is the minor of $(h_{i\bar{j}})_{i, j = 1, \ldots, m}$ obtained by deleting the rows and columns $i_1, \ldots, i_k$. (If $I_k = \{1, \ldots, m\}$, we put $\det H_{I_k} = 1$.)

We suppose now $a_\sigma \neq 0$ at a given point z_0 in the domain, and we want to calculate $\Delta \log a_\sigma$ in a neighborhood of z_0. By the results of Sect. 2, we can introduce local coordinates around z_0 with the property that

$$\frac{\partial f^{\alpha_1}}{\partial z^j} = 0 \quad \text{for} \quad j = 2, \ldots, m,$$

since $a_\sigma = 0$ certainly implies $e^{\alpha_1}(f) \neq 0$, and by induction we can also achieve

$$\frac{\partial f^{\alpha_l}}{\partial z^j} = 0 \quad \text{for} \quad j = \alpha_l + 1, \ldots, m, \qquad l = 1, \ldots, k. \tag{3.2}$$

Using (3.1) and (3.2), we obtain that in such a coordinate system

$$\begin{aligned} \Delta \log a_\sigma &= \Delta \log \det(h^{i\bar{j}})_{i, j = 1, \ldots, m} \\ &\quad + \Delta \log \det H_\sigma \\ &\quad + \sum_{j=1}^{k} \Delta \log \langle \partial_j f^j, \partial_{\bar{j}} f^{\bar{j}} \rangle \end{aligned} \tag{3.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the metric of the corresponding local factor of N.

Now we need some calculations.

First of all, we have

$$\Delta \log\det(h^{i\bar{j}}) = -\Delta \log\det(h_{i\bar{j}}) = R, \tag{3.4}$$

where R is the scalar curvature of M.

Moreover, for a positive smooth function u on M, we have

$$\begin{aligned}\Delta \log u &= \sum_{i,j=1}^{m} h^{i\bar{j}}\partial_{\bar{j}}\partial_i \log u \\ &= \frac{1}{u}\Delta u - \frac{1}{u^2}\sum_{i,j=1}^{m} h^{i\bar{j}}\partial_{\bar{j}}u\partial_i u.\end{aligned} \tag{3.5}$$

Furthermore

$$\begin{aligned}\Delta\langle\partial_k f^k, \partial_{\bar{k}} f^{\bar{k}}\rangle &= \sum_{i,j=1}^{m} h^{i\bar{j}}\partial_{\bar{j}}\left\langle D_{\frac{\partial}{\partial z^i}}\partial_k f^k, \partial_{\bar{k}} f^{\bar{k}}\right\rangle \\ &= \sum_{i,j=1}^{m} h^{i\bar{j}}\left\langle D_{\frac{\partial}{\partial \bar{z}^j}} D_{\frac{\partial}{\partial z^i}}\partial_k f^k, \partial_{\bar{k}} f^{\bar{k}}\right\rangle \\ &\quad + \sum_{i,j=1}^{m} h^{i\bar{j}}\left\langle D_{\frac{\partial}{\partial z^i}}\partial_k f^k, D_{\frac{\partial}{\partial \bar{z}^j}}\partial_{\bar{k}} f^{\bar{k}}\right\rangle \\ &= \sum_{i,j=1}^{m} h^{i\bar{j}} R^N\left(f_*\left(\frac{\partial}{\partial \bar{z}^j}\right), f_*\left(\frac{\partial}{\partial z^i}\right), \frac{\partial f^k}{\partial z^k}, \frac{\partial f^{\bar{k}}}{\partial \bar{z}^k}\right) \\ &\quad + \sum_{i,j=1}^{m} h^{i\bar{j}}\left\langle D_{\frac{\partial}{\partial z^i}}\partial_k f^k, D_{\frac{\partial}{\partial \bar{z}^j}}\partial_{\bar{k}} f^{\bar{k}}\right\rangle \\ &= -K^k|\partial_k f^k|^2(e^k(f) - e^{\bar{k}}(f)) \\ &\quad + \sum_{i,j=1}^{m} h^{i\bar{j}}\left\langle D_{\frac{\partial}{\partial z^i}}\partial_k f^k, D_{\frac{\partial}{\partial \bar{z}^j}}\partial_{\bar{k}} f^{\bar{k}}\right\rangle\end{aligned} \tag{3.6}$$

by (1.3), for each $k \in \{1, \ldots, m\}$, and

$$\begin{aligned}&\sum_{i,j=1}^{m} h^{i\bar{j}}\partial_{\bar{j}}\langle\partial_k f^k, \partial_{\bar{k}} f^{\bar{k}}\rangle\partial_i\langle\partial_k f^k, \partial_{\bar{k}} f^{\bar{k}}\rangle \\ &= \sum_{i,j=1}^{m} h^{i\bar{j}}\left\langle\partial_k f^k, D_{\frac{\partial}{\partial \bar{z}^j}}\partial_{\bar{k}} f^{\bar{k}}\right\rangle\cdot\left\langle D_{\frac{\partial}{\partial z^i}}\partial_k f^k, \partial_{\bar{k}} f^{\bar{k}}\right\rangle \text{ by (2.2)} \\ &= \sum_{i,j=1}^{m} h^{i\bar{j}}\langle\partial_k f^k, \partial_{\bar{k}} f^{\bar{k}}\rangle\cdot\left\langle D_{\frac{\partial}{\partial z^i}}\partial_k f^k, D_{\frac{\partial}{\partial \bar{z}^j}}\partial_{\bar{k}} f^{\bar{k}}\right\rangle\end{aligned} \tag{3.7}$$

since the local factors of N are complex 1-dimensional.

From (3.3)–(3.7) we obtain

$$\begin{aligned}\Delta \log a_\sigma &= R + \Delta \log H_{\{1,\ldots,k\}} \\ &\quad - \sum_{\alpha\in\sigma} K^\alpha(e^\alpha(f) - e^{\bar{\alpha}}(f)).\end{aligned} \tag{3.8}$$

By virtue of (1.1) we still get an (at least locally) invariantly defined expression if we replace one or several of the f^α by $f^{\bar\alpha}$ in the definition of a_σ. This means that σ can be a subset of $\{1, \ldots, n, \bar{1}, \ldots, n\}$ which does not contain both α and $\bar\alpha$ for any $\alpha \in \{1, \ldots, n\}$. Since complex conjugation on the image can be considered as a change of orientation, using (3.8) we find that at points where $a_\sigma \neq 0$

$$\begin{aligned}\Delta \log a_\sigma = R + \Delta \log H_{\{1, \ldots, k\}} \\ - \sum_{A \in \sigma} K^A(e^A(f) - e^{\bar{A}}(f)),\end{aligned} \tag{3.9}$$

where we now define $\bar{\bar\alpha} = \alpha$.

In order to keep track of the signs, note that

$$e^A(f) - e^{\bar A}(f) = -(e^\alpha(f) - e^{\bar\alpha}(f))$$

in case $A = \bar\alpha$, i.e. $A \in \{\bar 1, \ldots, \bar n\}$.

We shall be mostly interested in the following two special cases of the preceding formula.

The first one is $m = n = k$, and $\sigma = \{1, \ldots, n\}$. We derive from (3.9)

$$\Delta \log a_{1, \ldots, n} = R - \sum_{\alpha=1}^{n} K^\alpha(e^\alpha(f) - e^{\bar\alpha}(f)). \tag{3.10}$$

If we replace one of the indices in σ by its conjugate, this changes again the sign of the corresponding term in the sum.

The second case is $k = 1$, and at points where $e^\alpha(f) \neq 0$, introducing coordinates with $\frac{\partial f^\alpha}{\partial z^j} = 0$ for $j = 2, \ldots, m$, from (3.9) we find that

$$\begin{aligned}\Delta \log e^\alpha(f) = R + \Delta \log H_{1\bar1} \\ - K^\alpha(e^\alpha(f) - e^{\bar\alpha}(f)).\end{aligned} \tag{3.11}$$

4. Global Considerations

Proposition 4.1. *We can represent each $a_\sigma(f)$ in the form $a_\sigma = \sum_l \zeta^l \cdot |k^l|^2$, where ζ^l is a nonvanishing C^∞ function, and k^l is holomorphic.*

Proposition 4.1 will be proved in the appendix.

Let us assume now that a_σ is globally defined and does not vanish identically on M. Then the zero set M_σ of a_σ consists of a finite number of analytic subvarieties of M. The standard residue argument then yields from (3.8).

Proposition 4.2. *If $a_\sigma(f) \not\equiv 0$ on M, then*

$$\begin{aligned}\mathcal{J}_\sigma(f) := \int_M R\,dM + \sum_{A \in \sigma} \int_M (e^A(f) - e^{\bar A}(f))\,dM + \int_M \Delta \log H_{\{1, \ldots, n\}}\,dM \\ = -2\pi \sum_{\lambda=1}^{l} \mu_\lambda \mathcal{H}^{2m-2}(M_\sigma^\lambda),\end{aligned}$$

where M_σ^λ, $\lambda = 1, \ldots, l$, are the irreducible components of M_σ, on which k vanishes to order μ_λ resp., and where $\mathscr{H}^{2m-2}$ denotes $(2m-2)$ dimensional Hausdorff measure.

(Note: $K^\alpha = -1$.)

In order to understand the meaning of this formula, we give the following definitions and observations:

For a smooth real-valued function g on M,

$$\Delta g dM = \partial\bar{\partial} g \cup \Omega^{m-1}, \tag{4.1}$$

where Ω is the Kähler form of M.

$$\int R dM = c_1(M) \cup \Omega^{m-1}(M)[M] \quad [\text{cf. } (3.4)], \tag{4.2}$$

where $c_1(M)$ is the first Chern class of M.

Suppose that $N = D^n/\Gamma_0$ with $\Gamma_0 \subset (\mathrm{Aut}(D))^n$. Then the local factors f^α are globally defined, and we denote the first Chern class of the line bundle over N determined by the α^{th} local factor by $c_1(N^\alpha)$ and put $c_1(N^{\bar{\alpha}}) = -c_1(N^\alpha)$. Then

$$\int (e^A(f) - e^{\bar{A}}(f)) dM = -f^* c_1(N^A) \cup \Omega^{m-1}(M)[M] \tag{4.3}$$

[cf. (3.5)–(3.7)].

By the results of Sect. 2, the intersections of the level surfaces of the $f^A, A \in \sigma$, consist of analytic subvarieties, and $\frac{-1}{2\pi i} \partial\bar{\partial} \log H_\sigma$ is the first Chern class of the line bundle $K^\sigma(f)$ canonically attached to these. Therefore we conclude

$$\begin{aligned} \frac{1}{2\pi} \mathscr{I}_\sigma(f) = {} & c_1(M) \cup \Omega^{m-1}[M] \\ & - c_1(K^\sigma(f)) \cup \Omega^{m-1}[M] \\ & - \sum_{A \in \sigma} f^* c_1(N^A) \cup \Omega^{m-1}[M]. \end{aligned} \tag{4.4}$$

In case $n = m$ and $\sigma = \{1, \ldots, n\}$, $a_\sigma(f)$ is globally defined for a general $N = D^n/\Gamma$, where Γ is not necessarily contained in the identity component of $\mathrm{Aut}(D^n)$, and we have in this case

$$\frac{1}{2\pi} \mathscr{I}_\sigma(f) = c_1(M) \cup \Omega^{m-1}[M] - f^* c_1(N) \cup \Omega^{m-1}[M].$$

As a corollary of Proposition 4.2, we obtain

Corollary 4.1. *If $\mathscr{I}_\sigma(f) = 0$, then $a_\sigma(f)$ vanishes either identically or at most on a set with complex codimension 2. If $\mathscr{I}_\sigma(f) > 0$, then $a_\sigma(f)$ vanishes identically on M.*

We now want to investigate what $a_\sigma \equiv 0$ for some σ means:

Lemma 4.1. *If $a_\sigma \equiv 0$, then $e^A(f) \equiv 0$, i.e. f^α is (anti)holomorphic if $A = \alpha$ ($A = \bar{\alpha}$ resp.) for some $A \in \sigma$, or f is degenerate in the sense that it has nowhere (maximal) rank n.*

Proof. If some f^α is (anti)holomorphic on a nonvoid open subset of M, then it is (anti)holomorphic on the whole of M by an easy application of the unique continuation theorem of Aronszajn (cf. Proposition 4 of [16]). At a point where $a_\sigma=0$ some of the vectors $\left(\frac{\partial f^A}{\partial z^i}\right)_{i=1,\ldots,m}$ $(A\in\sigma)$ must be linearly dependent. If none of the $e^A(f)$ vanishes there, then by the considerations of Sect. 2 the complex dimension of the intersection of the tangent spaces to the level hypersurfaces of the f^A, $A\in\sigma$, must exceed $n-k$. This implies that f does not have maximal rank at such a point. The set of points where this happens is closed and in the case that none of the $e^A(f)$, $A\in\sigma$, vanishes identically, it must contain a dense subset of M by the above, since we assumed $a_\sigma\equiv 0$, and must therefore coincide with M, which proves the lemma.

Proposition 4.3. *If $\mathscr{I}_\sigma(f)>0$ or $\mathscr{I}_\sigma(f)=0$ and $a_\sigma(f)\equiv 0$, then f is degenerate, i.e. has nowhere maximal rank n.*

Proof. By Corollary 4.1 we have in both cases $a_\sigma(f)\equiv 0$. By Lemma 4.1 it follows that

(1) f is degenerate, or

(2A) $e^A(f)\equiv 0$ for at least one $A\in\sigma$.

In case (2A), we have

(2Ā) $e^{\bar{A}}(f)\equiv 0$,

which, together with (2A), implies that $f^A\equiv\text{const}$ and consequently f is degenerate, or

(3A) $\mathscr{I}_{\sigma_A}>0$, where $\sigma_A=(\sigma\backslash\{A\})\cup\{\bar{A}\}$, i.e. A is replaced by $\bar{A}$ in σ.

In case (3A) Corollary 4.1 implies that

$$a_{\sigma_A}\equiv 0$$

and Lemma 4.1 implies then again that (1) or (2Ā) or (2B) holds for some $B\in\sigma$, $B\neq A$.

Continuing in this way, we find that (1) or (2A) and (2Ā) for some $A\in\sigma$ or (2B) for all $B\in\sigma$ must hold. In the latter case, if none of the f^A is constant

$$\mathscr{I}_{\bar{\sigma}}>0, \text{ where } \bar{\sigma} \text{ is obtained from } \sigma \text{ by replacing every element by its conjugate.}$$

This implies, as before,

$$a_{\bar{\sigma}}\equiv 0$$

and therefore that (1) or (2Ā) for some $A\in\sigma$ holds. In the latter case, f^A must be constant, and in either case f is degenerate. q.e.d.

For the proof of Corollary 5.1 it is important to note that the arguments of the proofs of Lemma 4.1 and Proposition 4.3 are still valid even if the components f^α are not globally defined. Only $a_\sigma(f)$ has to be globally defined.

5. Mappings of Maximal Rank

In this section we generalize the methods of [15] (cf. also [14]). We assume $n\leqq m$, and for a moment also that N is again of the special form D^n/Γ_0, where $\Gamma_0\subset(\text{Aut}(D))^n$, in order to have the factors f^α globally defined.

$$\text{Suppose } a_\sigma(f)\neq 0 \text{ in } M \text{ for some } \sigma\subset\{1,\ldots,n,\bar{1},\ldots,\bar{n}\}, \tag{5.1}$$

i.e. $a_\sigma(f)$ is *nowhere* vanishing. We note already here, that in case $m=n$ (5.1) is implied by the condition

$$c_1(M)\cup\Omega^{m-1}[M]=f^*c_1(N)\cup\Omega^{m-1}[M],$$

if f is nondegenerate, by Corollary 4.1 and Proposition 4.1. This will be exploited in Sect. 6.

(5.1) implies in particular

$$e^A(f)>0 \quad \text{for} \quad A\in\sigma. \tag{5.2}$$

Using the arguments of Sect. 2, we conclude that we can introduce local coordinates around each point of M with the property that

$$a_\sigma=\det(h^{i\bar{j}})\prod_{\alpha\in\sigma_1}\langle\partial_\alpha f^\alpha,\partial_{\bar{\alpha}}f^{\bar{\alpha}}\rangle \cdot\prod_{\beta\in\sigma_2}\langle\partial_{\bar{\beta}}f^\beta,\partial_\beta f^{\bar{\beta}}\rangle, \tag{5.3}$$

where $\sigma_1=\sigma\cap\{1,\ldots,n\}$ and $\sigma_2=\sigma\cap\{\bar{1},\ldots,\bar{n}\}$. A corresponding formula is valid for a_{σ_A}, $A\in\sigma$ $(\sigma_A=(\sigma\setminus\{A\})\cup\{\bar{A}\})$.

We can now prove

Proposition 5.1. *Suppose $a_\sigma(f)\neq 0$ in M for some σ. Then $e^A(f)\geqq e^{\bar{A}}(f)$ for $A\in\sigma$.*

Proof. Suppose that

$$B=\{p\in M:(e^A(f)-e^{\bar{A}}(f))(p)<0\}$$

is not empty. From (5.2) it follows that

$$e^{\bar{A}}(f)>0 \text{ in } B,$$

and therefore

$$a_{\sigma_A}>0 \text{ in } B,$$

and from this

$$\Delta\log\frac{e^A(f)}{e^{\bar{A}}(f)}=\Delta\log a_\sigma-\Delta\log a_{\sigma_A} =2(e^A(f)-e^{\bar{A}}(f))<0 \text{ in } B, \tag{5.4}$$

by (3.9), since $K^A=-1$.

Therefore, $\log\frac{e^A(f)}{e^{\bar{A}}(f)}$ is a superharmonic function on B, which is negative in the interior and vanishes on the boundary of B. This, however, contradicts the minimum principle, from which we conclude $B=\emptyset$, which completes the proof.

Theorem 5.1. *Assume again $n\leqq m$ and $N=D^n/\Gamma_0$, $\Gamma_0\subset(\mathrm{Aut}(D))^n$. If $a_\sigma(f)\neq 0$ in M for some $\sigma\subset\{1,\ldots,n,\bar{1},\ldots,\bar{n}\}$ with $|\sigma|=n$ and f is not degenerate, then f has maximal rank n everywhere.*

Proof. Since $a_\sigma(f) \neq 0$ in M, we can introduce coordinates around each point of M satisfying

$$\frac{\partial f^\alpha}{\partial z^k} = \frac{\partial f^\alpha}{\partial \bar{z}^k} = 0 \quad \text{if} \quad \alpha \neq k,$$

using the results of Sect. 2. We define the σ-Jacobian of f by

$$I_\sigma(f) := \prod_{A \in \sigma} (e^A(f) - e^{\bar{A}}(f)). \tag{5.5}$$

By Proposition 5.1 it follows that

$$I_\sigma(f) \geqq 0. \tag{5.6}$$

The claim of the theorem is now equivalent to the fact that $I_\sigma(f)$ vanishes either identically or nowhere. To prove this, we shall show in the following that we can have $I_\sigma(f) = 0$ only if $e^A(f) = e^{\bar{A}}(f) = 0$ for some $A \in \sigma$ which would, however, contradict the nonvanishing of a_σ, or if $I_\sigma(f) \equiv 0$.

So assume to the contrary that $I_\sigma(f) = 0$ at a point p of M and

$$e^A(f) = e^{\bar{A}}(f) > 0 \quad \text{for some} \quad A \in \sigma. \tag{5.7}$$

As in the proof of Proposition 5.1 we conclude

$$\Delta \log \frac{e^A(f)}{e^{\bar{A}}(f)} = 2(e^A(f) - e^{\bar{A}}(f)) \quad \text{[cf. (5.4)]}.$$

Since this expression is nonnegative by Proposition 5.1, we can find positive constants c_1, c_2 such that in a neighborhood of p we have, using (5.7),

$$\Delta \log \frac{e^A(f)}{e^{\bar{A}}(f)} \leqq c_1 \left(\frac{e^A(f)}{e^{\bar{A}}(f)} - 1 \right)$$
$$\leqq c_2 \log \frac{e^A(f)}{e^{\bar{A}}(f)}.$$

Therefore, the function $h = \log \frac{e^A(f)}{e^{\bar{A}}(f)}$ satisfies the inequality

$$\Delta h \leqq c_2 \cdot h$$

locally. Since Lemma 6′ of [5] obviously generalizes to arbitrary dimension, we get

$$\int_{|z| \leqq R} h(z) \leqq c_3 h(0),$$

where z is a coordinate system centered at p and R is sufficiently small.

Since $h \geqq 0$ and $h(0) = 0$, we conclude that h vanishes identically for $|z| \leqq R$. Therefore, the set

$$\{p \in M : (e^A(f) - e^{\bar{A}}(f))(p) = 0\}$$

is open and closed and by assumption nonempty. It thus coincides with M, which proves the theorem.

We apply Theorem 5.1 to obtain

Corollary 5.1. *Assume $m=n$, i.e. that domain and image have the same dimension, and $N=D^n/\Gamma$, $\Gamma \subset \mathrm{Aut}(D^n)$, but not necessarily $\Gamma \subset (\mathrm{Aut}(D))^n$. Suppose that $a_\sigma(f)$ is globally defined for some σ with $|\sigma|=n$ (this is always the case, if, e.g. $\sigma=\{1,\dots,n\}$), and $\mathscr{I}_\sigma(f)\geqq 0$. If f is not degenerate, then it is a local diffeomorphism, and we have in fact $\mathscr{I}_\sigma(f)=0$.*

Proof. By Proposition 4.3 and Corollary 4.1 $\mathscr{I}_\sigma(f)=0$ and $a_\sigma(f)$ can vanish at most on a set with complex codimension 2. Since we assume $m=n$, near a vanishing point of $a_\sigma(f)$, we have the representation

$$a_\sigma(f)=\zeta\cdot|k|^2$$

by Proposition 4.1, where ζ is a nonvanishing C^∞ function and k is holomorphic. This, however, implies, by the Weierstrass Preparation Theorem, that the zero set of $a_\sigma(f)$ is of codimension 1 in M. Consequently, we have $a_\sigma(f)\neq 0$ in M.

We now look at the following exact sequence of groups

$$1\to(\mathrm{Aut}(D))^n\to\mathrm{Aut}(D^n)\xrightarrow{h}S(n)\to 1\,,$$

where $S(n)$ is the permutation group of n elements. Taking $\Gamma_0:=\Gamma\cap\ker(h)$, we see that D^n/Γ_0 is a finite cover of D^n/Γ, and D^n/Γ_0 satisfies the assumptions of Theorem 5.1. In order to apply this theorem, let $\Sigma:=\pi_1(M)$, and $\Sigma_0:=\ker(hf')$, where $f':\Sigma\to\Gamma$ is the map induced by f, and $h:\Gamma\to\Gamma/\Gamma_0$ comes from the exact sequence above. Then f lifts to a map $\bar{f}:\tilde{M}/\Sigma_0\to D^n/\Gamma_0$ ($\tilde{M}$ is the universal cover of M), and since $a_\sigma(f)\neq 0$, also $a_\sigma(\bar{f})\neq 0$ everywhere. Theorem 5.1 now implies that $\bar{f}$ and consequently also f is of maximal rank everywhere, and the proof is complete.

6. Quotients of Polydiscs

In this section we consider the case $m=n$ and apply Corollary 5.1.

Theorem 6.1. *Assume $m=n$ and that there is a homotopy class of nondegenerate mappings $g:M\to N$* which satisfy

$$c_1(M)\cup\Omega^{m-1}[M]\geqq g^*c_1(N)\cup\Omega^{m-1}[M]\,. \tag{6.1}$$

Then the universal cover of M is biholomorphically equivalent to D^n, the n-fold product of the unit disc. Furthermore, we have equality in (6.1).

Proof. Let f again be the harmonic mapping in the given homotopy class, and denote the lift to the universal covers by

$$\tilde{f}:\tilde{M}\to D^n\,.$$

Corollary 5.1 implies that $\tilde{f}$ is a diffeomorphism and that equality holds in (6.1). Looking at the intersections of the level sets of the components $\tilde{f}^\alpha$, $\alpha=1,\dots,n$, and using the results of Sect. 2, we infer that $\tilde{M}$ must be biholomorphically equivalent to a product of unit discs and complex planes. Since M is diffeomorphic to a quotient of D^n by a discrete group of automorphisms, it follows that $\tilde{M}=D^n$.

Corollary 6.1. *If, under the assumptions of Theorem 6.1, the image N is a product of Riemann surfaces, then M is also a product of Riemann surfaces.*

Corollary 6.2. *If D^n/Γ does not have a closed complex 1-dimensional global factor, then under the hypotheses of Theorem 6.1, M and $N = D^n/\Gamma$ are locally biholomorphically equivalent. Also f is holomorphic or antiholomorphic in this case.*

This follows from Mostow's rigidity theorem (cf. [13]).

Theorem 6.2. *Every deformation of a compact quotient of polydiscs has again the complex structure of such a quotient, where we mean by a deformation a deformation in the sense of Kodaira and Spencer (cf. [11, p. 334f.]).*

Proof. Denote the given complex manifold by M_0. It suffices to consider a deformation over the open unit 1-dimensional disc D. Let M_z be the fiber over z and M the total fiber space with projection $p: M \to D$. Let L be the line bundle over M whose restriction to M_z is the canonical line bundle of M_z. Since the restriction of L to $M_0 = D^n/\Gamma$ is positive, $\Gamma(M_0, L^k)$, i.e. the sections of L^k, embeds M_0 into some projective space $\mathbb{P}^N$ for sufficiently large k. Since D is Stein, sections can be extended to M, and it is then easily seen (cf. [4, p. 180]) that the set of $z \in D$ for which $\Gamma(M, L^k)$ does not embed M_z into $\mathbb{P}^N$ is an analytic subvariety of D and therefore a discrete set Z. Thus each $z_0 \in D/Z$ can be joined in D/Z to 0 by a smooth deformation path $\lambda \to z(\lambda)$. On $M_{z(\lambda)}$, we can choose a Kähler-Hodge metric depending smoothly on λ. By uniqueness and a-priori estimates for the harmonic map $f_\lambda: M_{z(\lambda)} \to M_0$ in our given homotopy class, we can easily infer with the usual Arzela-Ascoli argument that f_λ depends continuously on the given metric. In particular, $\mathscr{I}(f_\lambda)$ (cf. Proposition 4.2) varies continuously with λ. Since the metric on $M_{z(\lambda)}$ is chosen in such a way that the volume of an analytic subvariety of codimension one can only assume values 2π times an integer, i.e. discrete values, it follows that $\mathscr{I}(f_\lambda) \equiv 0$ through the deformation. Since f_0 is the identity, Theorem 6.1 implies that each M_z, $z \in D \backslash Z$, is a quotient of polydiscs.

Let T be the product of the Teichmüller spaces corresponding to the compact Riemann surfaces which are global factors of M_0 (there might be none of them). Let B be the total fiber space over T whose fibers are products of compact Riemann surfaces. Since topologically M is globally trivial, we get a holomorphic map $g: D\backslash Z \to T$ and a holomorphic map $h: M \backslash p^{-1}(Z) \to B$ which covers g. Since both T and B are hyperbolic, we can extend g and h respectively to holomorphic maps $\hat{g}: D \to T$ and $\hat{h}: M \to B$, and M is biholomorphic to the pullback of B by $\hat{h}$.

This completes the proof.

Since every automorphism of D^n can be composed of automorphisms of the separate factors and permutations of these factors, Corollary 6.2 and Theorem 6.2 imply

Corollary 6.3. *The moduli space for deformations of a given compact quotient of polydiscs is nothing but the product of the moduli spaces of the closed Riemann surfaces which are global factors of that quotient of polydiscs, divided by the group of permutations of topologically equal factors.*

7. Deformations of Kodaira Surfaces

A Kodaira surface is constructed in the following way:

Let R_0 be a compact Riemann surface of genus $g_0 \geqq 2$, and let R be a two-sheeted unramified covering surface of R_0. R has genus $g = 2g_0 - 1$. Let $\mathbb{Z}_m$ be the cyclic group of order m. $H_1(R, \mathbb{Z}_m) = (\mathbb{Z}_m)^{2g}$ is in a canonical way a homomorphic image of $\pi_1(R)$. We obtain now an m^{2g}-sheeted unramified covering surface S of R by the requirement that its fundamental group is mapped onto the kernel of this homomorphism by the covering map v:

$$0 \to v(\pi_1(S)) \to \pi_1(R) \to (\mathbb{Z}_m)^{2g} \to 0.$$

The genus of S is $h = m^{2g}(g-1)+1$.

We consider the graphs Γ_v and $\Gamma_{\tau v}$ of v and $\tau \circ v$, where τ is the fixed point free automorphism of R corresponding to the covering map $R \to R_0$. By construction, the integral homology class of the divisor $D = \Gamma_v - \Gamma_{\tau v}$ is divisible by m, and therefore (cf. [6]) there exists an algebraic surface M which is a ramified covering of order m of $R \times S$ with branch set D. This is our Kodaira surface.

We have the diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle\varphi} & & \\ S & \xleftarrow{p_1} & R \times S & \xrightarrow{p_2} & R, \end{array}$$

where the p_i are projections.

$\pi_1 := p_1 \circ \varphi$ and $\pi_2 := p_2 \circ \varphi$ have both maximal rank everywhere. Their level surfaces become tangent to each other along the branch locus of φ. Each level surface of π_1 is a branched covering of R with two branch points, namely its intersection points with the branch locus. Therefore these level surfaces $\pi_1^{-1}(u)$, $u \in S$, have varying complex structure (in fact, the infinitesimal deformation

$$\frac{\partial \pi_1^{-1}(u)}{\partial u}$$

does not vanish at any point $u \in S$, as is shown in [10]). Thus, M is a fibre space which is a fibre bundle in the differentiable sense, but not in the complex analytic sense.

In this section we want to study the global moduli space of a Kodaira surface and prove

Theorem 7.1. *Every deformation of a Kodaira surface is again a Kodaira surface. The deformation of the complex structure arises from a deformation of the complex structure of R_0. Therefore the moduli space of a Kodaira surface is the moduli space of the corresponding Riemann surface R_0.*

By a deformation of complex analytic structures we mean a deformation in the sense of [11, p. 334f.].

For the proof of Theorem 7.1 we will show the following lemma first:

Lemma 7.1. *Suppose the compact complex surface M is an analytic fibre space over a closed Riemann surface Σ. Assume that base and fibre have genus at least 2. Then* *each deformation of M is again such a fibre space, and the deformation of M gives rise to a deformation of Σ.*

Proof. Let M_λ, $\lambda \in [0,1]$, be a smooth path of complex structures on the underlying differentiable manifold with $M_0 = M$. By Kodaira's classification of compact complex surfaces, each M_λ is algebraic and hence Kähler (cf. e.g. [9]). For each M_λ, we consider the harmonic map $f_\lambda : M_\lambda \to \Sigma$ which is homotopic to the projection $\pi : M \to \Sigma$. Because these maps are unique and satisfy a priori estimates, they depend continuously on λ by a well-known argument.

We want to show first that each f_λ has maximal rank everywhere. Certainly this is true for $\lambda = 0$, since f_0 coincides with π. By Corollary 4.1, $e(f_\lambda)$ could vanish at most at isolated points. But in that case, the foliation of M_λ given by the level surfaces of f_λ (compare Sect. 2) would have a singular fibre through such a point. This fibre would be of the form $C = \sum_{j=1}^{k} C_j$, where the C_j are its irreducible components. We note that each C_j has multiplicity 1, since in case of higher multiplicity $e(f_\lambda)$ would have to vanish along such a C_j. Therefore C is homologous to a regular fibre. Since C, however, is assumed to be singular, it must have a strictly larger Euler number than a regular fibre (cf. [4, p. 508]). But this cannot happen because it would result in a change of the second Chern class of M_λ.

Therefore, each f_λ has maximal rank everywhere, applying Theorem 5.1.

Therefore, using the result at the end of Sect. 2, we obtain for each λ a holomorphic mapping $\pi_\lambda : M_\lambda \to \Sigma_\lambda$, where Σ_λ is a Riemann surface of the same topological type as Σ.

We want to show now that the complex structure of Σ varies continuously with λ. For this we consider (2.7) and the corresponding equations which determined the coordinate ζ^1 in Sect. 2 as real differential equations on the common underlying differentiable manifold. Because f_λ depends smoothly on λ, so does (2.7) and therefore the coordinate ζ_λ^2 and finally also ζ_λ^1. ζ_λ^1 gives local complex coordinates on the Riemann surface Σ_λ obtained as quotient of M via the harmonic map $f_\lambda : M_\lambda \to \Sigma$. Since Σ_0 coincides with Σ, we see therefore that Σ_λ varies continuously with λ. [In fact, using $s : U \cap \Sigma_\lambda \to M_\lambda$ with $s(\zeta^1) \mapsto (\zeta^1, \zeta^2 = \text{const})$ as local sections, we obtain a harmonic map from Σ_λ to Σ, if we take also (2.2) into account. On the underlying differentiable manifold, we get thus a smooth deformation of the identity.]

This finishes the proof of the lemma.

Proof of Theorem 7.1. Let M_λ, $\lambda \in [0,1]$ now be a deformation of our given Kodaira surface M, with $M_0 = M$. Again, all M_λ are Kähler manifolds. We apply Lemma 7.1 to $\pi_1 : M \to S$ and $\pi_2 : M \to R$. Thus, for each λ, we get a holomorphic map $\varphi_\lambda : M_\lambda \to R_\lambda \times S_\lambda$. The area of its branch locus, counted with multiplicity, is determined by Proposition 4.2 and hence is independent of λ. We want to prove now that the branch set does not split into more than two components during the deformation, i.e. that for each λ, all the m sheets come together at the branch locus.

The branch locus is the zero set of the Jacobian of φ_λ, i.e. the zero set of the section of $\varphi_\lambda^*(K_\lambda)$ which is given by the Jacobian, where K_λ is the canonical bundle of $R_\lambda \times S_\lambda$.

During the deformation, $\varphi_\lambda^*(K_\lambda)$ and the Jacobian change continuously, and therefore the zero set remains homologous to the branch locus of $\varphi = \varphi_0$. Consequently, also their images, i.e. the branch sets, are homologous, and in particular,

$$(m-1)C_0 = \sum_i (m_i - 1)C_{\lambda,i},$$

where $C_0 = \Gamma_v$ is one component of the branch set of φ and where $C_{\lambda,i}$ are the components of the branch set of φ_λ emerging from C_0. $(m-1)$ and $(m_i - 1)$ denote the resp. vanishing orders of the Jacobians on these sets.

Since the degree of φ_λ remains m, we have

$$(m-1) = \sum_i (m_i - 1). \tag{7.1}$$

We cannot have $C_{\lambda,i} \cdot R = 0$ (the product is the intersection number of homology classes), because the $C_{\lambda,i}$ cannot become vertical, since they are pointwise close to C_0. Since on the other hand $C_0 \cdot R = 1$, we conclude from (7.1)

$$C_{\lambda,i} \cdot R = 1 \quad \text{for all } i.$$

Since each $C_{\lambda,i}$ is analytic, it is therefore the graph of a holomorphic mapping from S to R. The corresponding mappings are homotopic, where the homotopy is given from the deformation. By the uniqueness of holomorphic mappings in a given homotopy class (the image has genus at least 3), C_λ can consist only of one component which proves that the branch set cannot split. Since, therefore, for each λ the branch set consists of two holomorphic curves each of which has intersection number 1 with R_λ, we get two holomorphic maps v_λ, w_λ from S_λ to R_λ. These are clearly unramified coverings. They give rise to a $\mathbb{Z}_2$ action on R in the following manner:

For each $r \in R_\lambda$, we take $v_\lambda^{-1}(r) \subset S$. $v_\lambda^{-1}(r) = \{s_1, \ldots, s_q\}$, where $q = \deg v_\lambda = \deg v$. We define $\tau(r) = w_\lambda(s_1)$. This is well-defined, i.e. $\tau(r) = w_\lambda(s_i)$ for all $i = 1, \ldots, q$, for the following reason. Since the branch set depends continuously on λ, so do v_λ and w_λ. As unramified coverings they give rise to covering transformations, and if the images of the s_i would become different at some point, this would give rise to homotopic but different covering transformations, which is certainly impossible, since they are holomorphic and S_λ has genus larger than 1.

If we divide R_λ by this $\mathbb{Z}_2$ action, we get a Riemann surface $R_{0,\lambda}$, also continuously depending on λ.

Thus, we have proved that each deformation of M is again a Kodaira surface, and that this deformation is induced by a deformation of the Riemann surface R_0. This gives a continuous and surjective mapping from the moduli space of R_0 onto the moduli space of M.

It remains to prove that this mapping is also injective. For this we proceed by contradiction and assume that we have two different holomorphic mappings $\varphi: M \to S \times R$ and $\varphi': M \to S' \times R'$ of the type which gives rise to a Kodaira surface and is described at the beginning of this paragraph. Because φ and φ' are related via a deformation of R_0 into R'_0, they are homotopic (regarded as mappings in the common underlying differentiable manifold). Since we assume that R_0 and R'_0

have different complex structures, the structures of S and S' are also different. Now let Δ be one of the components of the branch locus of φ. π_1 restricted to Δ is holomorphic and one-to-one since Δ is transversal to the level sets of π_1. Thus, Δ and S are biholomorphically the same. On the other hand, also π'_1 restricted to Δ is holomorphic and cannot be constant since Δ has the wrong genus for being a level set of π'_1. This gives a nonconstant holomorphic and hence nonsingular holomorphic mapping from S to S' which contradicts the assumption that R_0 and R'_0 and hence S and S' have different complex structures.

This finishes the proof of Theorem 7.1.

Remark. The local version of the preceding theorem was proved by Kas [7]. He investigated also a slightly more general class of surfaces. All our arguments, however, apply also to this class without any difficulties.

A. Proof of Proposition 4.1

The tool to prove Proposition 4.1 will be the similarity principle of Bers and Vekua. References for the following are [2, 5].

We shall prove first

Proposition A.1. *Suppose that $g \in C^m(B, \mathbb{C})$, where B is a bounded domain in $\mathbb{C}^m$. Assume that for every $\tau = \{\tau_1, \ldots, \tau_l\} \subset \{1, \ldots, m\}$ there exists a constant c_τ such that*

$$|g_{\bar{z}^{\tau_1}, \ldots, \bar{z}^{\tau_l}}| \leqq c_\tau |g| \quad \text{in } B. \tag{A.1}$$

Then

$$g(z) = e^{s(z)} h(z), \tag{A.2}$$

where $s(z)$ is Hölder continuous and $h(z)$ is holomorphic.

In the proof of Proposition A.1 we shall make use of three lemmata.

Lemma A.2. *Let $D_R(w_0) = \{w \in \mathbb{C} : |w - w_0| < R\}$, and suppose $f \in C^1(D_R)$. Furthermore, suppose that f and $k = f_{\bar{w}}$ are continuous in $\overline{D_R(w_0)}$. Then we have for $w \in D_R(w_0)$ the representation*

$$f(w) = \frac{1}{2\pi i} \oint_{|\zeta - w_0| = R} \frac{f(\zeta)}{\zeta - w_0} d\zeta - \frac{1}{\pi} \iint_{D_R(w_0)} \frac{k(\zeta)}{\zeta - w} d\xi d\eta \quad (\zeta = \xi + i\eta).$$

Lemma A.3. *Let $D \subset D_R(w_0)$ be an open set, and let $k(w)$ be continuous in D and $|k(w)| \leqq a$. Then*

$$s(w) = -\frac{1}{\pi} \iint_D \frac{k(\xi)}{\zeta - w} d\xi d\eta$$

satisfies

$$|s(w)| \leqq 4aR \quad \text{for} \quad w \in D$$

$$|s(w_2) - s(w_1)| \leqq c(R, a, \beta) |w_2 - w_1|^\beta$$

for $w_1, w_2 \in D$, $0 < \beta < 1$, where $c(R, a, \beta) < \infty$.

Lemma A.4. *Suppose that, in addition to the assumptions of Lemma A.2,*

$$k(w) \in C^{0,\alpha}(D).$$

Then $s(w)$ satisfies the differential equality

$$s_{\bar{w}} = k \ \text{in} \ D.$$

These lemmata are taken over from [5]. Furthermore, we shall use the similarity principle of Bers and Vekua (cf. [2] or [5]):

Lemma A.5. *Suppose that $|f_{\bar{w}}| \leqq M|f|$ in a domain $D \subset D_R(0) \subset \mathbb{C}$. Then we have the representation*

$$f(w) = e^{s_0(w)}\varphi(w),$$

where $\varphi(w)$ is holomorphic in $D_R(0)$, and $s_0(w)$ satisfies

$$|s_0(w)| \leqq 4MR \ \text{in} \ D$$
$$|s_0(w_2) - s_0(w_1)| \leqq c(R, M, \beta)|w_2 - w_1|^\beta,$$

for $w_1, w_2 \in D$, $0 < \beta < 1$, where $c(R, M, \beta)$ is given from Lemma A.3.

Furthermore, s_0 can be represented as

$$s_0(w) = -\frac{1}{\pi}\iint_{\mathring{D}} \frac{f_{\bar{\zeta}}}{f(\zeta)} \frac{d\xi d\eta}{\zeta - w}, \tag{A.4}$$

where $\mathring{D} = \{w \in D : f(w) \neq 0\}$, and we have

$$s_{0_{\bar{w}}} = \frac{f_{\bar{w}}}{f}. \tag{A.5}$$

Now we can prove Proposition A.1: We shall give details for $m = 2$. The general case follows easily by induction.

By Lemma A.5 we have for each z^2 the representation

$$g(\cdot, z^2) = e^{s^1(\cdot, z^2)}\varphi^1(\cdot, z^2), \tag{A.6}$$

exploiting (A.1), where $s^1(\cdot, z^2)$ depends Hölder continuously on z^1, and $\varphi^1(\cdot, z^2)$ is an analytic function of z^1 for every z^2.

Also,

$$g(z^1, \cdot) = e^{s^2(z^1, \cdot)}\varphi^2(z^1, \cdot), \tag{A.7}$$

where s^2 and φ^2 display now the resp. properties as functions of z^2 for every z^1. This implies that, for every z^2, $g(\cdot, z^2)$ either vanishes identically or has at least isolated zeros, and the same holds for $g(z^1, \cdot)$, z^1 fixed.

We now assume w.l.o.g. that $g \not\equiv 0$. Let $z_0 \in B$. Performing a transformation and a unitary rotation of $\mathbb{C}^2$, we can assume that $z_0 = 0$ and that $g(z^1, 0)$ has only isolated zeros as a function of z^1, and that $g(0, z^2)$ has only isolated zeros as a function of z^2. Consequently, the same holds for $g(z^1, z^2)$, if $|z^1| \leqq \varepsilon$, $|z^2| \leqq \varepsilon$, viewed as a function of one coordinate ($\varepsilon>0$ sufficiently small). Therefore, we obtain from (A.4)

$$s^1(z) = -\frac{1}{\pi}\iint_{D^1}\frac{g_{\bar\zeta_1}(\zeta_1, z_2)}{g(\zeta_1, z_2)}\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}, \tag{A.8}$$

if $|z_2|\leqq\varepsilon$, where $D^1=\{z_1:|z_1|\leqq\varepsilon\}$, $\zeta_1=\xi_1+i\eta_1$, and

$$\varphi^1(z) = g(z)e^{-s^1(z)} \tag{A.9}$$

with

$$\varphi^1_{\bar z^1}(z)=0 \text{ for every } z^2. \tag{A.10}$$

By assumption and Lemma A.3, $s^1(z)$ is a continuous function of z. Furthermore, by (A.1) the $\bar z^1$-derivative of the integrand of (A.8) is bounded independently of z^2 by a function which is by Lemma A.3 integrable with respect to ζ^1, and we can therefore differentiate under the integral sign to get

$$s^1_{\bar z^2}(z) = -\frac{1}{\pi}\iint_{D^1}\left(\frac{g_{\bar\zeta^1}}{g}\right)_{\bar z^2}\frac{d\xi^1 d\eta^1}{\zeta^1 - z^1} \tag{A.11}$$

(if $|z^2|\leqq\varepsilon$) as a continuous function of z. Therefore, we obtain from (A.9)

$$\varphi^1_{\bar z^2} = \left(\frac{g_{\bar z^2}}{g} - s^1_{\bar z^2}\right)\cdot\varphi^1, \tag{A.12}$$

and in particular

$$|\varphi^1_{\bar z^2}|\leqq c_3|\varphi^1|.$$

Now applying Lemma A.5 to φ^1, we get

$$\varphi^1(z^1, z^2) = \psi(z^1, z^2)e^{\sigma(z^1, z^2)} \tag{A.13}$$

where ψ and σ have the desired properties as functions of z^2. In particular, we have from (A.4)

$$\sigma(z) = -\frac{1}{\pi}\iint_{D^2}\frac{\varphi^1_{\bar\zeta^2}(z^1, \zeta^2)}{\varphi^1(z^1, \zeta^2)}\frac{d\xi^2 d\eta^2}{\zeta^2 - z^2}, \tag{A.14}$$

($D^2=\{z^2:|z^2|\leqq\varepsilon\}$, $\zeta^2=\xi^2+i\eta^2$), where we can integrate over D^2 since $g(z^1,\cdot)$ and hence $\varphi^1(z^1,\cdot)$ by (A.9) has only isolated zeros as a function of z^2 in D^2 (by the choice of our coordinates). And we have

$$\psi_{\bar z^2}=0.$$

For each $z^2\in D^2$, let $(p_n(\cdot, z^2))$ be a sequence of polynomials in z^1, which converge uniformly on D^1 towards

$$\left(\frac{g_{\bar z^2}}{g}\right)_{\bar z^1}.$$

We put

$$\varrho(z) = -\frac{1}{\pi}\iint_{D^1}\left(\frac{g_{\bar z^2}}{g}\right)_{\bar\zeta^1}\frac{d\xi^1 d\zeta^1}{\zeta^1 - z^1}$$

$$\varrho_n(z) = -\frac{1}{\pi}\iint_{D^1} p_n(z)\frac{d\xi^1 d\eta^1}{\zeta^1 - z^1}.$$

By Lemma A.3, $\varrho_n(z)\to\varrho(z)$ uniformly in $z^1\in D^1$, and by Lemma A.4 $(\varrho_n)_{\bar z^1} = p_n$. Consequently

$$\left(\varrho_n - \frac{g_{\bar z^2}}{g}\right)_{\bar z^1}\to 0 \text{ uniformly for } z^1\in D^1.$$

From Lemma A.2 it follows that $\varrho - \frac{g_{\bar z^2}}{g}$ is a holomorphic function of z^1, which implies that ϱ is a C^1-function of z^1 and that

$$\varrho_{\bar z^1} = \left(\frac{g_{\bar z^2}}{g}\right)_{\bar z^1}.$$

Since by (A.11) and assumption

$$s^1_{\bar z^2}(z) = \varrho(z),$$

we obtain

$$s^1_{\bar z^2\bar z^1} = \left(\frac{g_{\bar z^2}}{g}\right)_{\bar z^1} = \left(\frac{g_{\bar z^1}}{g}\right)_{\bar z^2} = s^1_{\bar z^1\bar z^2} \tag{A.15}$$

(cf. [5, p. 211] for the last equality as well as for the technique used above).

Consequently, by (A.9)

$$\begin{aligned}\varphi^1_{\bar z^2\bar z^1} &= (g_{\bar z^2\bar z^1} - g_{\bar z^1}s^1_{\bar z^2} - g_{\bar z^2}s^1_{\bar z^1} - g s^1_{\bar z^2\bar z^1} + g s^1_{\bar z^2}s^1_{\bar z^1})e^{-s^1}\\ &= \varphi^1_{\bar z^1\bar z^2} = 0.\end{aligned} \tag{A.16}$$

Then we obtain from (A.14)

$$\sigma_{\bar z^1} = 0.$$

Since we also have $\varphi^1_{\bar z^1} = 0$, it follows from (A.13)

$$\psi_{\bar z^1} = 0. \tag{A.17}$$

Since g, s^1, φ^1, σ, and consequently ψ are continuous functions of z, and since $\psi_{\bar z^1} = \psi_{\bar z^2} = 0$, we conclude from a well-known theorem that ψ is a holomorphic function of z. Therefore

$$g = \psi e^{\sigma + s^1}$$

gives the desired representation, since σ and s^1 are Hölder continuous by Lemma A.3, and Proposition A.1 is proved.

Now we can easily prove Proposition 4.1:

We put

$$g = \det\left(\frac{\partial f^A}{\partial z^i}\right)_{A\in\sigma,\, i\in I_k}.$$

Using (2.2) we get

$$g_{\bar{z}^j} = g \cdot \sum_{A\in\sigma} (-\Gamma^A_{AA} \partial_{\bar{j}} f^A),$$

i.e. we have

$$g_{\bar{z}^j} = c^j(z) \cdot g,$$

where $c^j(z)$ is a C^∞-function.

Therefore, we can apply Proposition A.1, and we conclude furthermore from (A.11), (A.12), and (A.14) that $s(z) = s^1(z) + \sigma(z)$ in the representation of g is a C^∞-function. Since the metrics of M and N are also assumed to be C^∞, and their determinants do not vanish, Proposition 4.1 follows from this.

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