Proc. Natl. Acad. Sci. USA
Vol. 87, pp. 80–82, January 1990
Mathematics

Holomorphic curves in surfaces of general type

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Communicated by Shiing-Shen Chern, September 11, 1989

ABSTRACT This note answers some questions on holomorphic curves and their distribution in an algebraic surface of positive index. More specifically, we exploit the existence of natural negatively curved "pseudo-Finsler" metrics on a surface S of general type whose Chern numbers satisfy $c_1^2 > 2c_2$ to show that a holomorphic map of a Riemann surface to S whose image is not in any rational or elliptic curve must satisfy a distance decreasing property with respect to these metrics. We show as a consequence that such a map extends over isolated punctures. So assuming that the Riemann surface is obtained from a compact one of genus q by removing a finite number of points, then the map is actually algebraic and defines a compact holomorphic curve in S. Furthermore, the degree of the curve with respect to a fixed polarization is shown to be bounded above by a multiple of $q - 1$ irrespective of the map.

Section 1. Introduction

It has been a well-known question that a holomorphic map from the complex line into an algebraic manifold of general type be algebraically degenerate. Classically, this type of question has been studied by E. Borel, H. Cartan, S. Bloch, H. Weyl, L. Ahlfors, S. S. Chern, and others. One of the fundamental tools is the lemma of Ahlfors on the distance decreasing property of holomorphic maps for manifolds with negative curvature. It was Chern who generalized and pointed out the importance of such a lemma to higher dimensional manifolds. He also coined the concept of hyperbolic analysis. Under the leadership of Chern, the geometers at Berkeley developed an extensive research on this type of hyperbolic analysis. Outstanding results were obtained by Griffiths, who applied such analysis to the period map. However, several important questions remained unanswered. As was pointed out by Lang (1), some of these questions may be related to higher dimensional generalizations of the Mordell conjecture.

A compact complex manifold is called hyperbolic if there exists no nontrivial holomorphic map from the complex line $\mathbb{C}$ into the manifold (cf. ref. 2). More generally, we will call a complex variety M Chern hyperbolic (or C-hyperbolic) if there exists a proper subvariety V of M so that the image of every nonconstant holomorphic map from the complex line $\mathbb{C}$ into M must be in V. M will be called strongly C-hyperbolic if every holomorphic map of the punctured disk to M not into V extends over the puncture and weakly C-hyperbolic if the image of a smaller punctured subdisk under each such map lies in a compact subvariety with boundary and of positive codimension. A fundamental problem in the subject of hyperbolic analysis asks which variety is C-hyperbolic. A general feeling is that varieties of general type are C-hyperbolic. Green and Griffiths (3) have made some groundbreaking contributions on this problem. Some ideas of Bogomolov (4) played an important role in their analysis. Those ideas were later refined by Miyaoka (5) and led to a stronger result on surfaces of positive index, claiming that their cotangent bundles are "almost" ample [cf. also Schneider and Tancredi (6)]. In this note, we also use the same ideas to demonstrate that if the minimal model of a two-dimensional projective variety S is of positive index,† then S is C-hyperbolic‡ and in fact strongly C-hyperbolic. In a future paper, we shall weaken the hypothesis of positive index and study the case when S is quasiprojective.

Section 2. Some Preliminaries

This section will serve mainly to establish the notation to be used. Unless otherwise specified, objects such as maps, bundles, and their sections are assumed to be holomorphic. No distinction will be made between bundles and their sheaves of sections.

Let M be a complex analytic variety. Given a complex manifold N and $f: N \mapsto M$ a holomorphic map, the pullback action of f on bundles and their sections over M will be denoted by f^{-1}. If M is nonsingular, a section ω of the canonical bundle K_M of M can also be thought of as a holomorphic top form. To clarify matters, we differentiate between the pullback on sections f^{-1} and that on forms f^*, so that for example if N has the same dimension as M, then $f^*\omega = (\det df)f^{-1}\omega$ where we view $\det df$ as a section of $K_N \otimes f^{-1}K_M^*$.

Let L be a line bundle over M with metric g. We view g as a smooth section of $L^*\overline{L}^*$ so that $g|v|^2 = gv\overline{v}$ is real and positive over the domain of any local nowhere-vanishing section v of L. Note that $dd^c\log g|v|^2$ is independent of the local section v (since $\partial\overline{\partial}\log|h|^2 = \partial\overline{\partial}\log h + \partial\overline{\partial}\log\overline{h} = 0$ for a holomorphic function h without zero where $2\pi dd^c = \sqrt{-1}\partial\overline{\partial}$) and thus defines locally a smooth global (1, 1)-form $c_1(g)$ called the Chern (or Ricci) form of g. We note that the natural metric on $\mathbb{C}^{n+1}$ defines a canonical metric on $\mathcal{O}(-1)$ over $\mathbb{P}^n$. Its Chern form is positive definite and gives rise to the well-known Fubini study metric on $\mathbb{P}^n$. As another example, if M is one-dimensional and $L = TM$ then (M, g) defines a Riemann surface with (Gaussian) curvature $\kappa = -c_1(g)g^{-1}$.

We will need to consider a slight generalization of a metric on a line bundle: g will be allowed to degenerate, but locally it will differ from a metric only by a factor of $|h|^\nu$, where $h \not\equiv 0$ is a holomorphic function and $\nu > 0$. This means that g is degenerate only along proper subvarieties and that $c_1(g)$ defines a current which is smooth aside from delta functions supported along these subvarieties. In this case, g will be called a pseudometric. Note the identity $c_1(f^{-1}g) = f^*c_1(g)$.

We now make some observations about metrics on $\mathbb{D}$ and $\mathbb{D}^*$. The Poincaré (punctured) disk of radius c is defined to be

†Implicit is that the minimal model of S is unique. So S cannot be $\mathbb{CP}^2$ and, by Kodaira classification of surfaces with positive index, must in fact be of general type.
‡In principle we can find the corresponding algebraic subvariety explicitly.

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$\mathbb{D}_c$ (respectively, $\mathbb{D}_c^*$), the (punctured) disk of radius c, equipped with the Poincaré metric:

$$\rho_c(z) = \frac{2c^4}{(c^2 - |z|^2)^2} \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}$$

$$\left(\text{respectively, } \rho_c^*(z) = \frac{2c^2}{|z|^2(\log|z/c|^2)^2} \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}\right).$$

One can verify directly that both the Poincaré disk and the Poincaré punctured disks of radius c are complete and have constant curvature $\kappa = -1/c^2$ and that the latter has a finite area in a neighborhood of the puncture. The following is self-evident and will be applied to show the Schwarz lemma in *Section 3*.

LEMMA 1. *Let* g *be a complete Hermitian metric on* $\mathbb{D}$. *Define* $m_a : \mathbb{D} \mapsto \mathbb{D}$ *by* $m_a(z) = az$, $0 < a < 1$. *If* $Q_a = g^{-1}m_a^*g$, *then* $\lim_{a\to 1}Q_a(z) = 1$ *for all* z *in* $\mathbb{D}$ *and* $\lim_{|z|\mapsto 1}Q_a(z) = 0$.

Section 3. The Main Theorem

Let S *be a surface of general type with cotangent bundle denoted by* Ω_S. *Let* P *be the projectivized tangent bundle of* S *and* $L = \mathcal{O}_{\mathbb{P}(\Omega_S)}(-1)$ *be the tautological line bundle defined over* P. In what follows, we may assume that S is minimal without affecting any conclusions.

3.1: The Case $c_1^2 > c_2$. LEMMA 2. *Let* H *be a line bundle over* P. *If* $c_1^2(S) > c_2(S)$, *then there are positive constants* a *and* m_0 *such that* $h^0(P, L^{*m}H^{-1}) > am^3$ *for* $m > m_0$.

Proof: From $(L^*)^2 + \pi^*c_1(S)\cdot(L^*) + \pi^*c_2(S) = 0$, we see $(L^*)^3 = c_1^2(S) - c_2(S) > 0$ so that $\chi(L^{*m})$ dominates a positive multiple of m^3 for $m >> 0$ by the Riemann–Roch formula. Now $h^2(L^{*m}) = h^2(S, S^m\Omega_S) = h^0(S, S^m\Omega_S \otimes K_S^{-(m-1)})$ by Serre duality. K_S^{m-1} being effective thus gives $h^2(L^{*m}) \le h^0(X, S^m\Omega_S) = h^0(L^{*m})$ for $m >> 0$. So, $2h^0(L^{*m}) \ge h^0(L^{*m}) + h^2(L^{*m}) \ge \chi(L^{*m})$ for $m >> 0$. As $h^0(L^{*m}|_{(H)}) = O(m^2)$, the exact sequence $0 \to L^{*m}H^{-1} \to L^{*m} \to L^{*m}|_{(H)} \to 0$ now shows $h^0(L^{*m}H^{-1}) > am^3$ for some $a > 0$ and $m >> 0$, as desired. □

Now take $H = \pi^*H_0$, where H_0 is a very ample line bundle on S. Let $t_1, \ldots, t_N$ form a basis of sections of H_0 giving a projective embedding for S and $s_i = \pi^*t_i$. Let σ be a section of $L^{*m}H^{-1}$ and Z^σ be its set of zeros in P. Then $\sigma s_1, \ldots, \sigma s_N$ are sections of L^{*m}, and $g' = (\Sigma_{i=1}^n |\sigma s_i|^2)^{1/m}$ defines a pseudometric on L. Consider a nontrivial holomorphic map f_0 from a complete Riemann surface Σ to S. It naturally induces a map f from Σ to P. We may view df_0 as a section τ of $K_\Sigma \otimes f^{-1}L$. Then $u_i = \tau^m f^{-1}(\sigma s_i)$ is a section of K_Σ^m, $i = 1, \ldots, N$. We assume $f(\Sigma) \not\subset Z^\sigma$ so that $(u_1, \ldots, u_N)$ has only isolated zeros on Σ. We define a pseudometric g_f on $K_\Sigma^{-1} = T\Sigma$ via $g_f(v)^m = \Sigma_i|u(v)|^2$, $v \in T\Sigma$. Note that $g_f = (f^{-1}g')|\tau|^2$. Let g be the metric defining the Riemann surface Σ. $\phi = g^{-1}g_f$ is then a well-defined function on Σ with isolated zeros. We thus obtain, as a current, that $dd^c\log(\phi^m) = m[\kappa g + f^*c_1(g') + Z_{(\tau)}] = m\kappa g + f_0^*\omega + Z_{(\sigma\circ f)} + mZ_{(\tau)}$, where κ is the curvature of g and ω is the pullback of the Fubini study metric under the embedding of S.[§] Integrating this equation, we obtain $H_0 \cdot f_0(\Sigma) \le -m\chi(\Sigma)$ when Σ is compact where $\chi(\Sigma)$ is the Euler characteristic of Σ.

When Σ is not compact, one can still deduce a pointwise estimate (after a simple rescaling of g')[¶] that $\sup_\Sigma\phi \le -\inf_\Sigma\kappa$ from either of the following arguments:

(*i*) The metric ω on S endows naturally a metric g_0 on L, which by compactness of P, dominates a constant multiple of the pseudometric mg' on L. The constant multiple can be absorbed into the definition of g' without changing its Chern form, and we assume this has been done. Observe then that $f_0^*\omega = (f^{-1}g_0)|\tau|^2 > m(f^{-1}g')|\tau|^2 = mg_f$. We thus see that $\phi^{-1}\Delta\phi \ge \Delta\log(\phi) = \kappa + (mg)^{-1}f_0^*\omega > \kappa + \phi$. The method of section 2 of ref. 7 now carries over verbatim to give the desired estimate (cf. *Appendix* for detail).

(*ii*) As Σ is one dimensional, one can emulate the usual proof of Ahlfor's lemma: Observe that if ϕ attains its maximum, then $0 \ge dd^c\log\phi$ at the maximum so that $-\kappa g = c_1(g) \ge dd^c\log\phi + c_1(g) = c_1(g_f) = f^*c_1(g') = m^{-1}f_0^*\omega > g_f = (\max_\Sigma\phi)g$ there by the preceding observation. In particular, the estimate holds if Σ is compact. Pulling back the metric, we may replace Σ by its universal cover $\mathbb{D}$, $\mathbb{C}$, or $\mathbb{P}^1$. We first consider the case $\Sigma = \mathbb{D}$. Replacing f by $f \circ m_d$ for $0 < d < 1$ where $m_d(x) = dx$, we see with the help of *Lemma 1* that ϕ approaches zero at the boundary of $\mathbb{D}$ and so must attain its maximum ϕ_d in $\mathbb{D}$. Hence, $\phi_d < -\kappa$ for all $d < 1$. The result follows in this case since $\lim_{d\to 1}\phi_d = \sup_\Sigma\phi$. The remaining cases are now easily excluded from our consideration as $\mathbb{C}$ with its flat metric can be exhausted by Poincaré disks of ever-increasing radius whose curvatures go to zero forcing $\phi \equiv 0$, contradicting our assumption on f (see also ref. 8). □

Remark: We need this pointwise estimate only for the Poincaré disk.

Armed with this, a big Picard type theorem lends itself in a standard fashion: Making the same assumptions, we take Σ to be the Poincaré punctured disk $\mathbb{D}_c^*$ with $c > 1$. We will show that $f_0^*\omega$ has finite mass in a neighborhood of the puncture, namely in $\mathbb{D}^*$, so that f_0 extends over the puncture by a well-known theorem of Bishop (9). Since $-\kappa g = c^{-2}g$ has bounded mass near the puncture as the g area is bounded there, we need only to bound the integral of $\mu = dd^c\log(\phi)$ there as a current. But as such, $\int_{\mathbb{D}^*}\mu = \Phi'(1) - \lim_{\varepsilon\to 0}\varepsilon\Phi'(\varepsilon)$, where $\Phi(r) = \int_0^{2\pi}\log\phi(re^{i\theta})$. Now over the interval (0, 1), Φ is bounded from above as we know ϕ is and is smooth except for harmless isolated discontinuities contributed from the "positive" delta functions. So one can produce a sequence $r_i \to 0$ such that $r_i\Phi'(r_i) \ge -\varepsilon$, for otherwise we may assume $\Phi'(r) < -\varepsilon/r$ for $r < \delta$, which would contradict the boundedness of Φ as r approach 0. Hence we are done.

To recapitulate, assuming $f(\Sigma)$ is not a subset of Z^σ, we have (*i*) if Σ is compact, then the degree of $f_0(\Sigma)$ is bounded in terms of its genus[‖]; (*ii*) compact or not, Σ cannot have nonnegative curvature; and (*iii*) the big Picard theorem holds if $\Sigma = \mathbb{D}^*$. To deal with the case when $f(\Sigma) \subset Z^\sigma$, we need the following:

3.2: The Case of Positive Index [$=(c_1^2 - 2c_2)/3$]. LEMMA 3. *Let* Y *be any nonvertical component of* Z^σ, $L_Y = L|_Y$, *and* H′ *be a line bundle over* Y. *If* $c_1^2(S) > 2c_2(S)$, *then we have for* $m >> 0$ *that* $h^0(Y, L_Y^{*m} \otimes H'^{-1}) > bm^2$, *where* $b > 0$.

Proof: As a divisor up to linear equivalence, $Y = n(L^*) + \pi^*(F)$ for some line bundle F over S and $n > 0$. We obtain

$$\begin{aligned}(L_Y^*)^2 &= (L^*)^2\cdot Y \\ &= (L^*)^2\cdot(n(L^*) + \pi^*(F)) \\ &= n(c_1^2(S) - c_2(S)) + c_1(S)\cdot(F) \\ (L_Y^*)\cdot(\pi^*K_S) &= n(K_S)^2 + (K_S)\cdot(F).\end{aligned}$$

As $[Y] = L^{*n} \otimes \pi^*F$ is effective, $\{0\} \neq H^0(P, [Y]) = H^0(S, S^n\Omega_S \otimes F) = \mathrm{Hom}(F^{-1}, S^n\Omega_S)$. Therefore, F^{-1} has a

§We use Z_D to denote the current associated to the divisor D.
¶Independent of f_0 or Σ.

‖In fact a simple rigidity argument bounds the number of such curves, and indeed the Mordell conjecture over complex function field follows in this case (cf. ref. 8).

nontrivial map into $S^n\Omega_S$. By the semistability of Ω_S (and therefore of $S^n\Omega_S$) with respect to K_S (cf. refs. 10 and 11), we have

$$-(F)\cdot(K_S) \leq \frac{(\det(S^n\Omega_S))\cdot(K_S)}{\operatorname{rank}(S^n\Omega_S)} = \frac{n}{2}(K_S)^2.$$

Therefore,

$$\begin{aligned}(L_Y^*)\cdot\pi^*(K_S) &= nc_1^2(S) + (K_S)\cdot(F)\\ &\geq nc_1^2(S) - \frac{n}{2}(K_S)^2\\ &= \frac{n}{2}c_1^2(S) > 0.\\ (L_Y^*)^2 &= n(c_1^2(S) - c_2(S)) - (K_S)\cdot(F)\\ &\geq \frac{n}{2}(c_1^2(S) - 2c_2(S))\\ &> 0.\end{aligned}$$

Serre duality says $h^2(L_Y^{*m}) = h^0(L_Y^m K_Y)$, which vanishes for $m >> 0$ by virtue of $\pi^*(K_S)\cdot(-m(L_Y^*) + (K_Y)) < 0$ and K_S being numerically effective. Therefore, $h^0(L_Y^{*m}) \geq \chi(L_Y^{*m}) > bm^2$ for some $b > 0$ and $m >> 0$. So $h^0(L_Y^{*m}H'^{-l}) > bm^2$ follows exactly as before. □

So assume $f(\Sigma)$ lies in a nonvertical component Y of Z^σ and let $H' = \pi^*H_{0|Y}$. The same argument as before shows that if $0 \neq \sigma' \in H^0(L_Y^{*m}H'^{-1})$,[††] then either $f_0(\Sigma) \subset \pi(Z^{\sigma'})$ or $H_0\cdot f_0(\Sigma)$ is bounded by $-m\chi(\Sigma)$ and Σ is complete negatively curved. In this latter case, the image of f_0 lies in a finite number of fixed algebraic curves, necessarily rational or elliptic. Also if Σ is the punctured disk, then f extends as a holomorphic map to the whole disk unless $f(\Sigma)$ lies in those fixed algebraic curves. Collecting these facts gives the main theorem, which we state as follows.

THEOREM 1. *Let* S *be a variety that dominates a minimal surface of general type and positive index.*[‡‡] *Then* S *is weakly C-hyperbolic. Further, if* S *is two-dimensional and, in particular, if* S *has a minimal model of positive index, then there is only a finite set of rational or elliptic curves in* S. *Let* E *be the union of these curves. We conclude that holomorphic images of complete nonnegatively curved Riemann surfaces must lie in* E. *Also, a holomorphic map of* $\mathbb{D}^*$ *not into* E *must extend to* $\mathbb{D}$. *In particular,* S *is strongly C-hyperbolic. Furthermore, a compact curve* Σ *in* S *must either sit in* E *or have degree bounded by* $-\alpha\chi(\Sigma)$, *where* $\alpha > 0$ *depends only on* S *and its polarization.*

Appendix

Here, we will give a proof following ref. 7 of the pointwise estimate of section 3.1 in a more general setting, namely, of the following proposition.

PROPOSITION A.1. *Let* M *be a complete Riemannian manifold with Ricci curvature bounded from below. If a nonnegative* C^0 *function* $\phi \not\equiv 0$ *is* C^2 *and satisfies* $\Delta\phi \geq \kappa\phi + \phi^2$ *away from its zero set, then* $sup_M\phi \leq -inf_M\kappa$.

As in section 2 of ref. 7, the proof is based on the following theorem from ref. 12.

PROPOSITION A.2. *Let* M *be as above, and let* f *be a continuous function bounded from below. If for some* $C > inf_M f$, f *is* C^2 *over the set defined by* $f < C$, *then for all* $\varepsilon > 0$, *there exist* $p \in M$ *such that at* p: $|\nabla f| < \varepsilon$, $\Delta f > -\varepsilon$ *and* $f(p) < inf_M f + \varepsilon$.

We base the proof here on the following lemma,[§§] of independent interest:

LEMMA A.1. *Let* M *be as above, and* $p \in M$. *There is a smooth (i.e.,* C^2*) function* h *on* M *such that*

$$|\nabla h| < C_0, \qquad h > C_1\rho, \qquad |\Delta h| < C_2$$

for positive constants C_0, C_1, C_2, *and* $\rho(x) = dist(p, x)$.

Proof of Proposition A.2. If h is as in *Lemma A.1*, the set $F_i = \{h \leq i\}$ is contained in the ball of radius i and so is compact. Let $h_i = \max(1 - h/i, 0)$; then h_i is supported on F_i and $\lim_{i\to\infty}h_i(x) = 1$ for all $x \in M$. We may assume without loss of generality that $\inf_M f = 0$. If $f(p) = 0$ for some $p \in M$ then we have proved the proposition. Hence, we may assume $f > 0$ so that $1/f$ is also C^2 and positive. Now h_i/f is supported on F_i and so achieves its maximum H_i there, say at x_i. Given $\varepsilon > 0$, there exists an x in M such that $f(x) < \varepsilon/2$ and an integer n such that $h_i(x) > 1/2$ for all $i > n$. So with this choice of x and i, we see $f(x_i) < 1/H_i < f(x)/h_i(x) < \varepsilon$. As $R_i = \log(h_i/f)$ also attains its maximum at x_i, $\nabla R_i = 0$ and $\Delta R_i \leq 0$ at x_i. So we obtain, respectively, that at x_i

$$\frac{\nabla f}{f} = \frac{\nabla h_i}{h_i} \text{ and } \frac{\Delta f}{f} \geq \frac{\Delta h_i}{h_i}.$$

These give $|\nabla f| = H_i^{-1}|\nabla h_i| < \varepsilon C_0/i$ and $\Delta f \geq H_i^{-1}\Delta h_i > -\varepsilon C_2/i$ at x_i for $i > n$.[¶¶] □

Proof of Proposition A.1. Let $f = (\phi + c)^{-1/2}$; then by direct computation

$$\nabla f = \frac{\nabla\phi}{2(\phi + c)^{3/2}}, \qquad f\Delta f = \frac{-\Delta\phi}{2(\phi + c)^2} + \frac{3|\nabla\phi|^2}{(\phi + c)^3}.$$

Using $\Delta\phi \geq \kappa\phi + \phi^2$ at a neighborhood of the supremum of ϕ, we obtain

$$\frac{-\kappa\phi - \phi^2}{2(\phi + c)^2} \geq f\Delta f - 12|\Delta f|^2 > -(\inf_M f + \varepsilon)\varepsilon - 12\varepsilon = \delta,$$

where the last inequality holds at some point where $f < \inf_M f + \varepsilon$ by *Proposition A.2*. Hence, there are points where $-\kappa\phi \geq \phi^2 - \delta(\phi + c)^2 \geq (1 + \delta)\phi^2$ and where ϕ approaches its supremum as ε (and therefore δ) approaches zero. Q.E.D.

Note Added in Proof. Lang explicitly made the conjecture (cf. ref. 1) that varieties of general type are C-hyperbolic (or more properly pseudohyperbolic in the terminology of ref. 1). We have learned from a letter from him that he and Paul Vojta have independently arrived at the conjecture from two different angles.

[††]We take m here to be a multiple of that in *Lemma 2*.

[‡‡]Equivalently, the function field of S contains the function field of a surface of positive index other than that of $\mathbb{P}^2$. The conclusion of the theorem can also be stated in terms of homomorphism of the function field of S into the field of meromorphic functions on $\mathbb{C}$.

[§§]Theorem 1.4.2 on p. 30 of section 1.4 of ref. 13.

[¶¶]Note that we have used only the facts that h is proper and that ∇h, Δh are bounded.

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