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REMARKS ON THE GROUP OF ISOMETRIES OF A RIEMANNIAN MANIFOLD

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IT IS well-known that if a compact group acts differentiably on a differentiable manifold, then this group must preserve some Riemannian metric on this manifold. From this point of view, we shall discuss certain facts about group actions on manifolds.

In §1, we study the group of isometries of a non-compact manifold. We discover that if the group is non-compact, the manifold has to split as a product of the Euclidean space and another manifold. This gives some information on problem 33 of [5]. It also gives some information on the group of biholomorphic transformations of a complex manifold whose Bergman metric is not trivial. Then we find a topological obstruction for a non-compact manifold to admit an infinite group to act freely and properly discontinuously. Namely, we prove that the natural map from the de Rham cohomology group with compact support to the de Rham cohomology group without compact support is trivial when the first group has finite dimension.

In §2, we obtain some topological obstructions for group actions by looking at the complex of invariant differential forms. We prove, for example, that if a compact group acts on a compact manifold with non-zero Euler number, then $\omega_1 \wedge \cdots \wedge \omega_{k+1} = 0$ for all closed invariant 1-forms $\omega_1, \ldots, \omega_{k+1}$ with $k \geq$ the codimension of the principle orbit. (It may be interesting to note that the vanishing is on the form level so that any secondary obstruction should also vanish.) We also prove a topological version of this theorem for circle actions. An interesting corollary is that if a manifold is the connected sum of a torus and a compact manifold with Euler number $\neq 2$, then it does not admit any circle actions. Another interesting corollary is that the only compact connected group acting effectively and differentiably on a compact complex submanifold of a complex torus must be a torus and the action is locally free.

Finally, we prove a fixed point theorem which may be of interest. If the first Betti number of a compact manifold M is zero and if there are rational cohomology classes $\Omega_1, \ldots, \Omega_k$ of dimension 2 with $2k = n$ and $\Omega_1 \cup \cdots \cup \Omega_k \neq 0$, then any circle group acting on M must have a fixed point and the fixed pint set is disconnected.

In §3, we point out that the theorem of Gromoll–Meyer on non-compact positively curved manifolds is also true equivariantly.

Recently, J. P. Bourguignon was also able to obtain Corollary 1 of Theorem 3. We thank him for several interesting comments on the original manuscript. Both he and Professor S. Kobayashi point out to us the paper "Dynamische Systeme and Topologische Aktionen" by S. Strantzalos (*Manuscripta Mathematics* **13**, (1974) 207–211) where the corollary of Theorem 1 was proved. R. S. Kulkarni also obtained Corollary 2 of Theorem 2 recently. Most of the results here were obtained when I was in Stony Brook.

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§1. GROUP OF ISOMETRIES OF A NON-COMPACT MANIFOLD

It is well-known that the group of isometries of a compact Riemannian manifold is compact. Conversely, any compact Lie group acting differentiably on a manifold must preserve some Riemannian metric. This fact makes the study of compact group actions a lot easier than that of non-compact group actions.

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In this section, we shall study the group of isometries of a non-compact Riemannian manifold. Of course, it is not true that every Lie group acting differentiably can preserve some metric. There are many restrictions for a group to preserve some metric. For example, the isotropy groups must be compact and hence the group action must be very regular. To study this in more detail, we begin with the following.

PROPOSITION 1. *Let G be a closed subgroup of the group of isometries of a Riemannian manifold M. Then every orbit of G is closed.*

Proof. Let $x \in M$ be an arbitrary point. Let y be a point in the closure of the orbit $G(x)$. Then we claim $y \in G(x)$.

In fact, let $\{g_1, g_2, \dots\} \subset G$ be such that $\lim_{i\to\infty} g_i(x) = y$. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame of M at the point x. Then by passing to a subsequence, if necessary, we may assume that $\lim_{i\to\infty} (g_i)_*(e_j)$ exists for all $j = 1, \dots, n$. (The limit is taken on the tangent bundle of M.)

Let $\sigma: [0, l] \to M$ be a geodesic segment such that $\sigma(0) = x$. Then we can find $\epsilon > 0$ such that $N_\epsilon = \{p \mid \text{dist}(p, \sigma[0, l]) \le \epsilon\}$ is a compact neighborhood of $\sigma[0, l]$. Hence by convergence of $\{(g_i)_*(e_j)\}_{i=1}^\infty$ (and the continuous dependence of ordinary differential equations), we see the existence of an integer I such that for $i \ge I$, $g_i(\sigma[0, l])$ is a subset of $g_I(N_{\epsilon/2})$. Clearly this implies $\{g_i\}$ converges on the geodesic segment $\sigma[0, l]$. Similarly one can show that $\{g_i\}$ converges in a neighborhood of σ (and hence in the frame bundle of this neighborhood by repeating the above argument.)

Let $p \in M$ be an arbitrary point. Then we can join p to x by a path γ. There exists a positive number $\epsilon > 0$ and finite number of points $\{\gamma(0) = x, \gamma(t_1), \dots, \gamma(t_m) = p\}$ such that every closed geodesic ball of radius ϵ around $\gamma(t_i)$ is compact convex and that γ is covered by the union of the open balls. By applying the above argument to the first ball, the second ball, etc., we see that the sequence $\{g_i\}$ converges at p. Therefore $\{g_i\}$ converges on M to an isometry g and $y = g(x) \in G(x)$. This finishes the proof of the proposition.

Note that we actually prove that the action of G is proper.

From now on, we shall assume that G is a closed subgroup of the group of isometries of M. In particular, all the orbits of G are closed submanifolds of M.

THEOREM 1. *Let G be a semi-direct product of a compact group K and another closed connected subgroup N which does not contain any non-trivial compact subgroup. Then differentiably M is a direct product $M_1 \times M_2$ where M_1 is diffeomorphic to N. Furthermore, the group N acts transitively on M_1 by left multiplication and trivially on M_2. For $k \in K$ and $(x, y) \in M_1 \times M_2$, we have $k(x, y) = (k_1(x), k_2(x, y))$. (The action of K on M_1 is quite regular because it has to preserve a left invariant metric of N.)*

Proof. First we claim that the group N acts freely on M. In fact, this follows because the isotropic group of the group of isometries is compact and N does not contain any non-trivial compact subgroup.

By Proposition 1, all the orbits of N are closed. Let $x \in M$ be an arbitrary point. Then there exists a geodesic ball $D_\epsilon(x)$ of radius ϵ around x such that for $y \in D_\epsilon(x)$, there exists at most one shortest geodesic that realizes the distance from y to $G(x)$. The tangent space of M at x splits into the orthogonal sum of the tangent space of $G(x)$ and another space V. Let $B_\epsilon(x)$ be the disk of radius ϵ in V. Then we claim that for all $y \in \exp_x(B_\epsilon(x))$, we have $G(y) \cap \exp_x(B_\epsilon(x)) = \{y\}$. In fact, let σ be the geodesic in $\exp_x(B_\epsilon(x))$ that realize the distance between y and $G(x)$. Then $g(\sigma)$ also realizes the distance between $g(y)$ and $G(x)$. If $g(y) \in \exp_x(B_\epsilon(x))$, $g(\sigma)$ must be the unique geodesic joining $g(y)$ and x. In particular, $g(x) = x$. Since N acts freely on M, this implies g is the identity and the claim is proved. It is easy to see from the claim that the orbit space M/N is a manifold and the natural projection $M \to M/N$ is a principal fibration.

Since N is contractible, we see that the fibration $M \to M/N$ is trivial and $M = M_1 \times M_2$ where M_1 is diffeomorphic to N, N acts by left translation on M_1 and trivially on M_2. We can now finish the proof of the theorem by noting that N is normal in G and the action of K must preserve the fiber structure.

COROLLARY. *Let M be a differentiable manifold which is not the product of a Euclidean space with some other manifold. Then for any Riemannian metric on M, the connected component of the group of isometries of this metric is compact.*

There is one instance where a group of infinite order can preserve a Riemannian metric. Namely, if a group G acts freely and properly discontinuously on a manifold M, then G preserves a complete Riemannian metric on M. This follows because the quotient space M/G is a manifold and admits a complete Riemannian metric. The required metric on M is obtained by lifting this metric.

For the rest of this section, we shall observe that for a non-compact manifold M to admit an infinite group which acts freely and properly discontinuously, certain topological obstruction exists.

Let $H_c^*(M)$ be the de Rham cohomology of M with compact support and $H^*(M)$ be the de Rham cohomology without compact support. Then under the above assumption, we claim that the map $H_c^i(M)\to H^i(M)$ is a trival map when $\dim H_c^i(M)<\infty$.

According to [7], every closed form with compact support is cohomologous to a unique L^2-integrable harmonic form in the space of differential forms. Let V^i be the space of all L^2-harmonic i-forms on M which are cohomologous to closed i-forms with compact support. Then by our assumption $\dim V^i<\infty$.

Let G be a subgroup of the group of isometries of M which acts properly discontinuously. Then G also acts on V^i in the natural manner. Furthermore, G preserves the natural inner product on V^i.

If G is infinite and $\dim V^i>0$, there is a non-zero form $\omega\in V^i$ and a sequence of distinct elements $\{g_1, g_2, \ldots\}$ in G such that $\|g_j^*\omega-\omega\|_2<2^{-j}$. We claim that this is impossible. In fact, as G acts properly discontinuously, there is an open set $U\subset M$ such that $U\cap g_i(U)=\phi$ for $g_i\neq$ identity, $g_j(U)\cap g_k(U)\neq\phi$ for $j\neq k$ and $\int_U \omega\wedge_*\omega\neq 0$. Integrating $\omega\wedge_*\omega$ over the set $\cup_j g_j(U)$, we obtain

$$\|\omega\|_2^2\geq\int_{\cup_j g_j(U)}\omega\wedge_*\omega=\sum_j\int_U (g_j^*\omega)\wedge_*(g_j^*\omega)$$

$$\geq\sum_j\left(\frac{1}{2}\int_U\omega\wedge_*\omega-\|\omega-g_j^*\omega\|^2\right)=\infty$$

which is a contradiction. (We used the inequality $(1/2)a^2\leq b^2+(a-b)^2$.)

Therefore, we have proved the following

THEOREM 2. *Let M be a non-compact manifold such that for some infinite group G, G acts freely and properly discontinuously. Let $H_c^*(M)$ be the de Rham cohomology with compact support. Then if $\dim H_c^i(M)<\infty$, the map $H_c^i(M)\to H^i(M)$ is a trivial map.*

COROLLARY 1. *Let G and M be as in Theorem 2. Let ω_1 be a closed i-form with compact support and $\omega_2,\ldots,\omega_k$ be closed forms such that for some $j\geq 2$, ω_j has compact support. Then when $\dim H_c^i(M)<\infty$, $\omega_1\cup\omega_2\cup\cdots\cup\omega_k=0$ in $H_c^i(M)$.*

Proof. By the theorem, $\omega_1=d\theta_1$ for some $(i-1)$-form θ_1 in M. Therefore, $\omega_1\wedge\omega_2\wedge\cdots\wedge\omega_k=d(\theta_1\wedge\omega_2\wedge\cdots\wedge\omega_k)$ where $\theta_1\wedge\omega_2\wedge\cdots\wedge\omega_k$ has compact support.

COROLLARY 2. *Under the assumptions of Theorem 2, if $\dim H_c^i(M)<\infty$ for all i, the product structure of the cohomology ring of M with compact support is trivial.*

§2. TOPOLOGICAL OBSTRUCTIONS FOR GROUP ACTIONS

In this section, we shall assume that the group G is connected. Let $\mathscr{C}_G(M)$ be the complex of smooth G-invariant differential forms on M. Then if G is compact, one can check that the homology of the complex $\mathscr{C}_G(M)$ is the same as the cohomology of the de Rham complex of M.

THEOREM 3. *Let G be a closed connected subgroup of the group of isometries of some Riemannian manifold M. Let $\omega_1,\ldots,\omega_k$ be closed 1-forms in $\mathscr{C}_G(M)$. Suppose for each i and each closed one parameter subgroup $g(t)$ of G, there exists a point $x\in M$ such that*

$\omega_i[(d/dt)g(t)(x)|_{t=0}]=0$. *Then* $\omega_1\wedge\omega_2\wedge\cdots\wedge\omega_k$ *is a zero k-form if k is greater than the codimension of the principal orbit.*

Proof. Let X be a vector tangent to the principal orbit at some point x. Then for some one parameter group $g(t)\subset G$, X is tangent to the path $g(t)(x)$. Let $\tilde{X}$ be the global vector field on M generated by the one parameter group $g(t)$.

Then since $g(t)$ preserves the forms ω_i, we see that $L_{\tilde{X}}(\omega_i)=0$ for all i. Here $L_{\tilde{X}}$ is the Lie derivative with respect to $\tilde{X}$. However, by the well-known formula $L_{\tilde{X}}=d\circ i_{\tilde{X}}+i_{\tilde{X}}\circ d$ where $i_{\tilde{X}}$ is the interior derivative, it follows that $d(i_{\tilde{X}}\omega_i)=0$ for all i. In our case, this means $d[\omega_i(\tilde{X})]=0$ and $\omega_i(\tilde{X})$ is a constant function for all i.

By the assumption $\omega_i[(d/dt)g(t)(x)|_{t=0}]=0$, we conclude that $\omega_i(\tilde{X})\equiv 0$, and therefore the restriction of ω_i on every orbit is zero for all i. Hence, the ω_is are forms "lying" in the complement of the tangent space of each principal orbit. When k is greater than the dimension of the complement of these tangent spaces, $\omega_1\wedge\omega_2\wedge\cdots\wedge\omega_k$ is a zero form. Since the union of the principal orbits is an open dense set of M, $\omega_1\wedge\omega_2\wedge\cdots\omega_k$ is really a zero form on M.

COROLLARY 1. *Suppose the k-dimensional torus T_k acts differentiably and effectively on a manifold M of dimension m. If the orbit of every one-dimensional subtorus of T^k is homologous to zero, then for every set of T^k-invariant closed one forms $\omega_1,\omega_2,\ldots\omega_{m-k+1}$ on M; we have* $\omega_1\wedge\cdots\wedge\omega_{m-k+1}=0$.

Proof. This follows because the dimension of the principal orbit of an effective toral action is the same as the dimension of the torus.

To see that Theorem 3 is not trivial, we note the following

COROLLARY 2. *Let $N\times T^k$ be the product of a compact manifold N and the k-dimensional torus. Let M be any compact manifold with dimension equal to* $\dim N+k$. *Then if the Euler number (or some pontryagin number) of the connected sum $M\ \#\ (N\times T^k)$ is non-zero, $M\ \#\ (N\times T^k)$ does not admit any compact differentiable group action whose principal orbit has dimension* $>\dim N$. *In particular, if* $\dim N=0$, *$M\ \#\ T^k$ does not admit any compact connected differentiable group action.*

Note that in this corollary, we use an observation of R. Bott[12] that on a compact manifold with some non-zero pontryagin number, every Killing vector field must have a zero.

What happens if the hypothesis $\omega_i[(d/dt)g(t)x|_{t=0}]=0$ in Theorem 3 is dropped? In this respect, we have the following

PROPOSITION 2. *Let G be a connected Lie subgroup of the group of isometries of a Riemannian manifold. Let $\bar{G}$ be the Lie subgroup of G generated by the Lie subalgebra $[\mathfrak{G},\mathfrak{G}]$ of $\mathfrak{G}$ where $\mathfrak{G}$ is the Lie algebra of $\mathfrak{G}$. Suppose there are closed G invariant 1-forms $\omega_1,\ldots,\omega_k$ such that $\omega_1\wedge\cdots\wedge\omega_k\not\equiv 0$. Then* $\dim\bar{G}\leq(1/2)(n-k)(n-k+1)$.

Proof. Let $\tilde{\mathfrak{G}}=\{\tilde{X}|, \tilde{X}$ is a vector field on M induced by some $X\in\mathfrak{G}\}$. Then as in Theorem 2, $\omega(\tilde{X})=$ constant for all closed 1-forms $\omega\in\mathscr{C}_G(M)$ and $\tilde{X}\in\tilde{\mathfrak{G}}$. In particular, for all $\tilde{X}$, $\tilde{Y}\in\tilde{\mathfrak{G}}$, $\omega([\tilde{X},\tilde{Y}])=2d\omega(\tilde{X},\tilde{Y})-\tilde{X}(\omega(\tilde{Y}))+\tilde{Y}(\omega(\tilde{X}))=0$.

Let U be an open set of M where $\omega_1\wedge\cdots\wedge\omega_k\neq 0$. For each $x\in U$, we define $V_x=\{X|\omega_i(X)=0\forall_i\}$. Then the spaces V_x define a foliation of U which is invariant under G.

By the previous considerations, it is clear that for all $x\in U$, the set $\bar{G}(x)\cap U$ is a subset of a leaf of the foliation which has codimension k.

Let H be the isotropic group of G at a point $x\in U$. Let h_t be a one-parameter family of isometries in $H\cup\bar{G}$ such that for each t, $(h_t)_*$ fixes every vector in V_x. Since $h_t^*\omega_i=\omega_i$ for all i, we conclude that $(h_t)_*$ must also fix every vector in the orthogonal complement of V_x. As the action of $\bar{G}$ is effective, h_t must be degenerate to the identity and the action of $\bar{G}$ on each leaf of U is effective. This completes the proof of our claim and hence of the proposition.

COROLLARY. *Let M be a n-dimensional compact manifold such that for some cohomology classes $\omega_1,\ldots,\omega_n\in H^1(M,Q)$, $\omega_1\cup\cdots\cup\omega_n\neq 0$. Then the only compact connected Lie group which acts effectively and differentiably on M is a torus. Furthermore, the action must be locally free.*

Proof. The first part is clear from the proposition. The second part follows from the fact that $\omega_1(\tilde{X}) = \text{constant}$.

As was pointed out by B. Lawson, this corollary applies, for example, to complex submanifolds of a torus. In fact, suppose the complex n-dimensional torus T^n is covered by C^n. The projection of the differentials of the coordinate functions dz_i are closed holomorphic 1-forms on T^n. If the complex dimension of the submanifold M is m, it is not hard to see that among the above differentials, the wedge product of m of them is not identically zero when restricted to M. (If locally M is given by $(z_1, \dots, z_m, f_1(z_1, \dots, z_m), f_2(z_1, \dots, z_m), \dots, f_{n-m}(z_1, \dots, z_m))$, $dz_1 \wedge \cdots \wedge dz_m$ is non-zero on M.) Taking the conjugate of these holomorphic 1-forms, the hypothesis of the corollary can then be verified.

It should be noted that, in case the group G preserves the complex structure of M, a somewhat stronger result can be obtained. Namely, M has a finite cover of the form $T^k \times N$ where G acts by translation on T^k and trivially on N. This is essentially an easy application of the Bochner method because the Ricci curvature of the induced Kähler metric on M is non-positive.

Remark. This corollary was pointed out to us by J. P. Bourguignon.

PROPOSITION 3. *Let T^k be a k-dimensional subtorus of the group of isometries of some Riemannian manifold M. Let V be the space of closed 1-forms in $\mathscr{C}_G(M)$. Then there exists a codimension k subspace W of V such that for all forms $\omega_1, \omega_2, \dots, \omega_l$ in W with $l > \dim M - k$, we have $\omega_1 \wedge \omega_2 \cdots \wedge \omega_l \equiv 0$.*

Proof. Let $X_1, \dots, X_k$ be a basis of the Lie algebra of T^k and $\tilde{X}_1, \dots, \tilde{X}_k$ be their induced vector fields on M. Then for all $\omega \in \mathscr{C}_G(M)$, $\omega(\tilde{X}_i) = \text{constant}$ for all i.

Therefore, we have a homomorphism φ: $V \to R^k$ given by $\varphi(\omega) \to (\omega(\tilde{X}_1), \dots, \omega(\tilde{X}_k))$. The proposition follows from the equality $W = \ker \varphi$.

Now let us try to sharpen the conclusion of Theorem 3 by assuming that G is abelian.

THEOREM 4. *Let G be a closed abelian subgroup of the group of isometries of some Riemannian manifold. Suppose for each closed G-invariant 1-form ω and each one-parameter subgroup $g(t)$ of G, there exists a point $x \in M$ such that $(\omega((d/dt)g(t)x|_{t=0}) = 0$. Then for any closed G-invariant forms $\Omega_1, \dots, \Omega_k$ with $k > \dim M - \dim G$, we have $\Omega_1 \wedge \cdots \wedge \Omega_k = 0$.*

Proof. We shall assume $\Omega_1, \dots, \Omega_{k-1}$ are 1-forms and Ω_k is a 2-form. The rest of the argument follows by induction.

Let X be any vector field on M which is generated by a one-parameter subgroup of G. Then $i_X(\Omega_1 \wedge \cdots \wedge \Omega_k)$ can be written as a linear combination of forms $\Omega_1 \wedge \cdots \wedge i_X(\Omega_j) \wedge \cdots \wedge \Omega_k$ and $\Omega_1 \wedge \cdots \wedge i_X(\Omega_k)$. By the argument of Theorem 3, $i_X(\Omega_j) = 0$ for $j < k$ and $\Omega_1 \wedge \cdots \wedge i_X(\Omega_k) = 0$. (Since G is abelian, $i_X(\Omega_k)$ is invariant.) Therefore, $i_X(\Omega_1 \wedge \cdots \wedge \Omega_k) = 0$. Since the dimension of the principal orbit is $\dim G$ and $k > \dim M - \dim G$, we conclude that $\Omega_1 \wedge \cdots \wedge \Omega_k$ is zero as before.

Note that since $\Omega_1 \wedge \cdots \wedge \Omega_k$ vanishes in the form level, not only the cup product of the corresponding cohomology class vanishes, but also the Massey product of them vanishes when it can be defined (cf [8]).

There is a topological version of Theorem 3 for circle actions. For this purpose, we use the notation and definitions of [2].

Let the circle group T act topologically and effectively on a compact manifold M. Let M^T be its fixed point set. Then according to Floyd[3], there is only a finite number of orbit types. Furthermore (see [2]), if $\dim M \le n$, then $H^i(M/T, M^T) = 0$ for $i \ge n$ and we have the following exact sequences.

$$\to H^i(M/T) \to H^i(M) \to H^{i-1}(M/T, M^T) \to \cdots \tag{1}$$

$$\to H^i(M/T, M^T) \to H^i(M) \to H^{i-1}(M/T, M^T) \oplus H^i(M^T) \to H^{i+1}(M/T, M^T) \to \cdots \tag{2}$$

All these hold for rational coefficients.

From the information that $H^i(M/T, M^T) = 0$ for $i \ge n$, one derives easily that

$$H^n(M/T) \approx H^n(M^T) = 0 \tag{3}$$

The last equality follows because (2) shows that the restriction map $H^n(M)\to H^n(M^T)$ is surjective. Let p be any point in $M\backslash M^T$. Then the restriction map factors through $H^n(M)\to H^n(M\backslash p)\to H^n(M^T)$. As $H^n(M\backslash p)=0$, the inequality is proved.

If the fixed point set M^T is non-empty, then it follows from (1) that the natural map $H^1(M/T)\xrightarrow{\pi^*} H^1(M)$ is surjective. Let $\omega_1,\omega_2,\dots,\omega_n$ be any cohomology classes in $H^1(M)$. Then there are classes $\tilde\omega_1,\tilde\omega_2,\dots,\tilde\omega_n$ in $H^1(M/T)$ such that $\pi^*\tilde\omega_i=\omega_i$ for $1\le i\le n$. Since cup product is natural, we see that $\omega_1\cup\omega_2\cup\cdots\cup\omega_n=\pi^*(\tilde\omega_1\cup\cdots\cup\tilde\omega_n)$. This class is zero because of (3). Hence we have the following

THEOREM 5. *Let the circle group T acts effectively on a compact manifold M with dimension n. Then if the fixed point set of T is not empty, $\omega_1\cup\omega_2\cup\cdots\cup\omega_n=0$ for all cohomology classes $\omega_1,\dots,\omega_n\in H^1(M,Q)$.*

COROLLARY. *Let M be a compact manifold with Euler number not equal to two. Then the connected sum of M with a torus does not admit any circle action.*

Remark. If we replace T by a k-dimensional torus in Theorem 5, then we can conclude that $\omega_1\cup\cdots\cup\omega_{n-k+1}=0$ for cohomology classes $\omega_1,\dots,\omega_{n-k+1}\in H^1(M,Q)$ when $H^{n-k+1}(M^T)=0$. This is done by induction on the number of factors of $T^k=S^1\times\cdots\times S^1$. (We apply the above argument for the action of T^{k-1} on M/S^1, etc.) It should be noted that for smooth actions, the set M^T has codimension at least $2k$ and so $H^{n-k+1}(M^T)=0$.

In order to push the argument one step farther, we consider the following long exact sequence (from (2))

$$\begin{aligned}&0\to H^0(M/T,M^T)\to H^0(M)\to H^0(M^T)\\&\to H^1(M/T,M^T)\to H^1(M)\to H^1(M^T)\oplus H^0(M/T,M^T)\\&\to\cdots\\&\to H^{j-1}(M/T,M^T)\to H^{j-1}(M)\to H^{j-1}(M^T)\oplus H^{j-2}(M/T,M^T)\\&\to H^j(M/T,M^T)\to H^j(M)\to A\to 0.\end{aligned}$$

Then (read "dimension" in front of each term),

$$\sum_{i=0}^{j}(-1)^iH^i(M/T,M^T)-\sum_{i=0}^{j}(-1)^iH^i(M)+\sum_{i=0}^{j-1}(-1)^iH^i(M^T)+\sum_{i=1}^{j-1}(-1)^iH^{i-1}(M/T,M^T)+(-1)^jA=0 \tag{4}$$

Hence,

$$\begin{aligned}H^j(M/T,M^T)&=H^{j-1}(M/T,M^T)+(-1)^j\left\{\sum_{j=0}^{j}(-1)^iH^i(M)+\sum_{i=0}^{j-1}(-1)^{i+1}H^i(M^T)\right\}-A\\&\le H^{j-1}(M/T,M^T)+(-1)^j\sum_{i=0}^{j}(-1)^i\{H^i(M)-H^i(M^T)\}+H^j(M^T)\end{aligned} \tag{5}$$

In particular,

$$\begin{aligned}H^{2j}(M/T,M^T)&\le H^{2j-1}(M/T,M^T)+\sum_{i=0}^{2j}(-1)^i\{H^i(M)-H^i(M^T)\}+H^{2j}(M^T)\\&\le H^{2j-2}(M/T,M^T)+\sum_{i=0}^{2j-1}(-1)^i\{H^i(M)-H^i(M^T)\}+H^{2j-1}(M^T)\\&\quad+\sum_{i=0}^{2j}(-1)^i\{H^i(M)-H^i(M^T)\}+H^{2j}(M^T)\\&=H^{2j-2}(M/T,M^T)+H^{2j}(M)-H^{2j}(M^T)+H^{2j-1}(M^T)+H^{2j}(M^T)\\&=H^{2j-2}(M/T,M^T)+H^{2j}(M)+H^{2j-1}(M^T)\\&\le\cdots\\&\le H^0(M/T,M^T)+\sum_{i=1}^{j}H^{2i}(M)+\sum_{i=1}^{j-1}H^{2i-1}(M^T)\end{aligned} \tag{6}$$

Therefore,

$$\dim H^{2j}(M/T,M^T)\le\sum_{i=1}^{j}\dim H^{2i}(M)+\sum_{i=1}^{j}\dim H^{2i-1}(M^T)+\begin{cases}1 & \text{if } M^T=\phi\\ 0 & \text{if } M^T\ne\phi\end{cases} \tag{7}$$

Similarly, we also have

$$\dim H^{2j+1}(M/T, M^T) \leq \dim H^0(M/T, M^T) + \sum_{i=0}^{j} \dim H^{2i+1}(M) + \sum_{i=0}^{j} \dim H^{2i}(M^T) - \dim H^0(M)$$
$$= \sum_{i=0}^{j} \dim H^{2i+1}(M) + \sum_{i=0}^{j} \dim H^{2i}(M^T) + \begin{cases} 0 & \text{if } M^T = \phi \\ -1 & \text{if } M^T = \phi \end{cases} \tag{8}$$

In order to state the next theorem, we define a number as follows: For each sequence $(I, J) = (i_1, \ldots, i_p, j_i, \ldots, j_q)$, let $c_{I,J}$ be the smallest integer for which there exists a subspace V of $H^{2i_1}(M) \oplus \cdots \oplus H^{2i_p}(M) \oplus H^{2j_1-1}(M) \oplus \cdots \oplus H^{2j_q+1}(M)$ such that codim $V = c_{I,J}$ and $\omega_1 \cup \cdots \cup \omega_k[M] = 0$ for $\omega_1, \ldots, \omega_k \in V$.

We shall now find an upper bound for the integers $c_{I,J}$. Observe that from sequence (1), there is a subspace in $H^i(M)$ with codimension $\leqq \dim H^{i-1}(M/T, M^T)$ such that all the classes in this subspace come from $H^i(M/T)$. Hence, using (3), (7) and (8), we can conclude that

$$c_{I,J} \leq \sum_{t=1}^{p} \sum_{m=0}^{i_t-1} \dim H^{2m+1}(M) + \sum_{t=1}^{p} \max\left[0, \sum_{m=0}^{i_t-1} \dim H^{2m}(M^T) - 1\right]$$
$$+ \sum_{t=1}^{q} \sum_{m=0}^{j_t} \dim H^{2m}(M) + q + \sum_{t=1}^{q} \max\left[0, \sum_{m=1}^{j_t} \dim H^{2j_t-1}(M^T) - 1\right] \tag{9}$$

Hence, by (7), (8), and (9) we have the following

THEOREM 4′. *Let the circle group T act effectively on a compact manifold with dimension n. Then inequality* (9) *holds. Furthermore, we have*

(i) *If the fixed set M^T is non-empty, $\sum_{j=1}^{i} \dim H^{2j}(M) = 0$ and there are cohomology classes $\omega_1, \ldots, \omega_k$ such that $\dim \omega_j \leq 2i+1$ and $\omega_1 \cup \cdots \cup \omega_k[M] \neq 0$, then $\sum_{i=1}^{j} \dim H^{2l-1}(M^T) > 0$.*

(ii) *If $\sum_{i=1}^{j} \dim H^{2i-1}(M) = 0$ and there are cohomology classes $\omega_1, \ldots, \omega_k$ with $\dim \omega_i \leq 2j$ and $\omega_1 \cup \cdots \cup \omega_k[M] \neq 0$. Then the fixed point set M^T is non-empty and $\sum_{i=0}^{j-1} \dim H^{2i}(M^T) > 1$.*

COROLLARY. *Let the circle group T act effectively on a compact manifold with dimension n. Suppose $H^1(M) = 0$ and there are elements $\omega_1, \ldots, \omega_k \in H^2(M)$ such that $\omega_1 \cup \cdots \cup \omega_k[M] \neq 0$. Then the fixed point set of T is non-empty and disconnected.*

Note that all simply connected Kähler manifolds satisfy the hypothesis of the corollary. This is true, in particular, for hypersurfaces in CP^n with $n \geq 3$.

Since Wu–Yi Hsiang (Cohomology theory of topological transformation groups, to appear in Ergebnisse der Math. Series, Springer–Verlag) has proved that if $\pi_{2i}(M) \otimes Q = 0$ for all $i > 0$, then the fixed point set of T is connected, we have the following

COROLLARY. *Let M be a compact manifold such that $\pi_{2i}(M) \otimes Q = 0$ for all $i > 0$, $H^1(M) = 0$ and there are classes $\omega_1, \ldots, \omega_k \in H^2(M)$ such that $\omega_1 \cup \cdots \cup \omega_k[M] \neq 0$. Then M admits no effective circle group action.*

The referee points out that if the manifold M is simply connected, then the hypothesis of this corollary can never be verified because in this case, $H^2(M) \neq 0$ implies $\pi_2(M) \otimes Q \neq 0$.

Remark. If the group action of T is semi-free, one can replace rational coefficient by integer coefficient in most cases.

Finally, let us remark that Atiyah and Hirzebruch proved in [1] that a spin manifold with $\hat{A}$-genus $\neq 0$ does not admit differentiable circle action.

From this, one has the following:

PROPOSITION 5. *Let G be a compact Lie group acting differentiably on a compact spin manifold M with dimension n. Suppose one of the characteristic numbers of M is not zero. Then if the principle orbit of M has codimension k, $\Omega \cup \omega_1 \cup \cdots \cup \omega_k = 0$ where Ω is the $(n-k)$-dimensional $\hat{A}$-class and $\omega_1, \ldots, \omega_k$ are 1-dimensional cohomology classes.*

Proof. As before, give M an invariant Riemannian metric so that Ω can be represented by curvature forms and $\omega_1, \ldots, \omega_k$ are closed G-invariant 1-forms. Let N be any principle orbit of the action. Then it is well-known that the normal bundle of N is stably trivial. Therefore, $\Omega|N$ is the $(n-k)$-dimensional $\hat{A}$-class of N. Since Ω is invariant under G and $\int_N \Omega = 0$ according to

Atiyah–Hirzebruch, we conclude that $\Omega|N = 0$. The rest of the argument is then similar to Theorem 1.

Remark. By a little more argument, which will appear later, one can in fact prove the following. Let M be a n-dimensional compact spin manifold which cannot be fibered over a circle. Suppose M admits an effective circle action. Then $\Omega \cup \omega_1 \cup \cdots \cup \omega_k = 0$ where Ω is the $(n-k)$-dimensional $\hat{A}$-class and $\omega_1, \ldots, \omega_k$ are one-dimensional cohomology classes.

§3. A FIXED POINT THEOREM FOR COMPACT GROUP ACTION

A famous theorem of Cartan says that for a complete simply connected manifold with non-positive curvature, any compact subgroup of isometries has a fixed point, We remark here that a similar theorem holds for positively curved manifolds. We shall use the result of Gromoll and Meyer[4]. The recent result of R. Greene and H. Wu on smoothing strictly convex function will also be used.

PROPOSITION 6. *Let M be a Riemannian manifold which admits a proper strictly convex function. Let G be any compact group of isometries. Then M is G-equivariantly diffeomorphic to the Euclidean space.*

Proof. Let f be the proper strictly convex functions. Then as usual, one can average the function f in the following manner: Define $\bar{f}(x) = \int_G f(g(x))\,dg$ where dg is the Haar measure on G. Since g is an isometry, $f(g(x))$ is still strictly convex and so is $\bar{f}(x)$. (Recall that a function is called strictly convex if its restriction to every geodesic is strictly convex.)

On the other hand, it is a simple exercise to show that $\bar{f}$ is also proper. Therefore, the invariant function $\bar{f}$ has a unique minimum at some point p. Such a point is clearly a fixed point of G.

In order to prove the proposition, we remark that recently R. Greene and H. Wu[9] have been able to prove that strictly convex function can be approximated by smooth strictly convex function. Therefore, we can assume f is smooth.

Clearly, p is the unique critical point of $\bar{f}$. Without loss of generality, we may assume $\bar{f}(p) = 0$. Applying the Morse lemma, we can find a coordinate system near p so that $\bar{f}(x) = \sum_{i=1}^{n} x_i^2$.

With the aid of this coordinate system, we now prove that for some small positive number ϵ, $\bar{f}^{-1}([0, \epsilon))$ is G-equivariantly diffeomorphic to the Euclidean space.

For each $g \in G$, let $J(g)$ be the jacobian matrix of g at the origin. Thus J defines the isotropic representation of G into the group of automorphisms of the tangent space of M at p. Since for each g, g preserves the form $\sum_{i=1}^{n} x_i^2$, it is clear that $J(g)$ is an orthogonal transformation with respect to the above coordinate system. Considering $J(g)^{-1}g$ as transformations from a small ball into itself, we can average them over G with respect to the Haar measure. Hence $H = \int_G J(g)^{-1}g$ is a smooth mapping from a small ball B into R^n.

By using Taylor's expansion, one sees that the jacobian matrix of H is the identity matrix at the origin. Therefore, by choosing B small enough, we may assume that H is a diffeomorphism from B into $H(B)$. It is also easy to see that we may assume $H(B)$ is strictly convex. Now for every $g \in G$, we have $J(g)H = Hg$. Hence, H serves as a G-equivariant diffeomorphism from B into $H(B)$ where G acts orthogonally on $H(B)$ with the origin belonging to $H(B)$. Since $H(B)$ is strictly convex, a radial deformation then shows $H(B)$ is G-equivariantly diffeomorphic to R^n. This proves that for small ϵ, $\bar{f}^{-1}([0, \epsilon))$ is G-equivariently to R^n. (Note that the idea of defining the transformation H is not new and goes back to H. Cartan's work on linearizing holomorphic vector fields near a critical point.)

Therefore, it remains to prove that $\bar{f}^{-1}([0, \infty))$ is equivariantly diffeomorphic to $\bar{f}^{-1}([0, \epsilon))$. This can be seen as follows.

Let F_t be the one parameter family of diffeomorphisms generated by $\nabla\bar{f}/|\nabla\bar{f}|^2$ so that $F_0(x) = x$ for all x. Then $\bar{f}(F_t(x)) = \bar{f}(x) + t$ and the map $\varphi(x) = F_{\bar{f}(x)/(\epsilon - \bar{f}(x))}(x)$ defines a diffeomorphism between $\bar{f}^{-1}(0, \epsilon)$ and $\bar{f}^{-1}(0, \infty)$. Using the local representation $\bar{f}(x) = \sum_{i=1}^{n} x_i^2$, we see that $F_t(x) = (\sqrt{[(t/|x|^2) + 1]})x$ and $\varphi(x) = (\sqrt{[(1/\epsilon - |x|^2) + 1]})x$ near p. Therefore, φ is the

COROLLARY. *Let M be a complete non-compact positively curved manifold. Let G be any compact group of isometries of M. Then M is G-equivariantly diffeomorphic to the Euclidean space.*

Proof. From the paper of Gromoll and Meyer (see also [9]), we know that on such manifolds, there is always a proper strictly convex function.

Remarks. 1. In Proposition 6, if we replace strict convexity by convexity, compact group of isometries need not have a fixed point. However, the following is true: Given a complete Riemannian manifold such that every geodesic ball is convex, then any compact group of isometries has a fixed point (see [10]).

2. A suitable equivariant version of the Gromoll–Cheeger theorem can be formulated and should not be hard to prove.

3. The non-equivariant version of Proposition 6 is also proved by R. Green and H. Wu [9].

Finally, let us apply the same procedure to holomorphic group actions. Recall that a function defined on a complex manifold is called strictly plurisurharmonic if its restriction to every complex curve is strictly subharmonic. A submanifold N of a complex manifold M is called totally real if $JT_xN \cap T_xN = (0) \forall x \in N$ where J is the complex structure of M and $T(N)$ is the tangent bundle of N.

PROPOSITION 7. *Let G be a compact group acting holomorphically on a complex manifold M. Suppose M admits a proper smooth strictly plurisubharmonic function. Then an orbit of G is totally real.*

Proof. As in Proposition 6, we can assume that the plurisubharmonic function f is invariant under the group action.

Let Z be the set of M where f attains its minimum. Then according to Harvey and Wells [11], for each point $z_0 \in Z$, there is a totally real submanifold N of M such that in a neighborhood of z_0, $Z \subset N$.

Since G acts holomorphically and leaves Z invariant, it is clear that $G(z_0)$ is totally real.

COROLLARY. *Let M be a two-dimensional Stein manifold whose tangent bundle is topologically trivial. Then for any compact connected semi-simple Lie group G acting holomorphically on M, G has a fixed point.*

Proof. It is well-known that every Stein manifold admits a proper strictly plurisubharmonic function. Therefore, Proposition 7 is applicable and $G(y_0)$ is totally real for some $y_0 \in M$.

If $\dim G(y_0) \leq 1$, then as G is semi-simple, $G(y_0) = y_0$.

It remains to consider the case $\dim G(y_0) = 2$. Since the tangent bundle of $G(y_0)$ is equivalent to its normal bundle and since M has trivial tangent bundle, the orbit $G(y_0)$ must be a torus. The semi-simplicity of G then shows that G must fix y_0.

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