

A Note on ODEs from Mirror Symmetry \diamond

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Abstract: We give close formulas for the counting functions of rational curves on complete intersection Calabi-Yau manifolds in terms of special solutions of generalized hypergeometric differential systems. For the one modulus cases we derive a differential equation for the Mirror map, which can be viewed as a generalization of the Schwarzian equation. We also derive a nonlinear seventh order differential equation which directly governs the instanton corrected Yukawa coupling.

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1. Introduction

In a seminal paper [1] physicists solved a problem in enumerative geometry, namely to count¹ the “number” n_d of rational curves² of arbitrary degree d on the quintic threefold X in \mathbb{P}^4 . The answer was given in terms of the large volume expansion of the correlation function, called Yukawa coupling by physicists, between three states \mathcal{O}_J of the twisted $N = 2$ topological σ -model on Q , which has the formal expansion [1][2][3]

$$K_{JJJ} = \langle \mathcal{O}_J \mathcal{O}_J \mathcal{O}_J \rangle = \int_X J \wedge J \wedge J + \sum_d \frac{d^3 n_d q^d}{(1 - q^d)}. \quad (1.1)$$

Here $q = e^{2\pi i t}$ and the modulus t parametrizes the complexified Kähler class of X , i.e. $(\text{Im}(t))^3 \propto \text{volume of } X$ and $\text{Re}(t)$ parametrizes an antisymmetric tensor field, which is the component of a harmonic $(1, 1)$ -form on X [4]. The correlation function (1.1), is sometimes referred to as an intersection of the quantum (co)homology of X . In the large volume limit the contribution of the instantons is damped out and (1.1) approaches the classical self intersection number between the cycle dual to the Kähler form J .

It is a remarkable fact, that this counting function (1.1) is expressible in a closed form in terms of solutions of a generalized hypergeometric system. This has been used in [5][6][7] to predict the number of rational curves on various Calabi-Yau spaces.

In the section 2 of this exposition we review the physical reasoning, which explains that fact and give, as a generalization, closed formulas for the counting functions on non-singular complete intersection Calabi-Yau spaces in products of weighted projective spaces. An important step in these calculations is the definition of the mirror map. We discuss therefore in section 3 and 4 the differential equation which governs the mirror map. As we will see in section 2 the most important quantity is the prepotential, from which the the correlation function (1.1) and the Weil-Peterson metric for the complex moduli space, can be derived. We will obtain in section 5 a differential equation for the prepotential.

¹ Only if the moduli space of the map from \mathbb{P}^1 to X , with three points fixed, is zero dimensional n_d counts the number of isolated rational curves. In general n_d has to be understood as the integral of the top Chern class of a vectorbundle over the moduli space of that map.

² These are called instantons by physicists as they correspond to classical solutions of the σ -model equations of motion.

2. Counting of rational curves and generalized hypergeometric functions

It was argued by Witten [3] that the states in the ($N = 2$) topological σ -model on an (arbitrary Calabi-Yau) space X are in one to one correspondence with the elements in the cohomology groups of X , for example the state \mathcal{O}_J above correspond to the Kähler form in $H^{1,1}(X, \mathbf{Z})$. As there is a natural involution symmetry in the $N = 2$ topological σ -model on Calabi-Yau manifolds exchanging the states corresponding to the cohomology groups $H^{3-p,q}(X)$ and $H^{p,q}(X)$, physicists suspect that Calabi-Yau spaces occur quite generally in mirror pairs³ X and X^* , in which the rôle of these cohomology groups are exchanged. In particular $h^{3-p,q}(X) = h^{p,q}(X^*)$ holds and the Euler number of X is therefore the negative of the Euler number of X^* . By the same token it is expected that the correlation functions among one type of states on X can be calculated by the same methods as its counterparts among the corresponding states on X^* . In particular the correlation function (1.1) is related by this argument[1][3] to the correlation function

$$K_{J^*J^*J^*} = \langle \mathcal{O}_{J^*} \mathcal{O}_{J^*} \mathcal{O}_{J^*} \rangle = \int_{X^*} \Omega \wedge b_{J^*}^\alpha \wedge b_{J^*}^\beta \wedge b_{J^*}^\gamma \Omega_{\alpha\beta\gamma}, \quad (2.1)$$

where Ω is the unique no-where vanishing holomorphic threeform on X^* and $b_{J^*}^\alpha \in H^1(X, TX) \cong H^{2,1}(X)$. In fact the expansion (1.1) and hence the successful prediction of rational curves on the quintic X was obtained in [1] by calculating the correlation function (2.1) on the mirror X^* using classical methods of the theory of complex structure deformation. The integral on the right-hand side of (2.1), introduced in [9] depends only, via Ω , on the choice of the complex structure modulus on X^* but not on the choice of complexified Kähler moduli on X^* , while (1.1) depends only on the complexified Kähler modulus on X , but not on the complex structure moduli of X . Due to mirror symmetry we should be able to identify the complex structure modulus space of X^* with the complexified Kähler structure modulus space of X . Hence after calculating the dependence of (2.1) on the complex structure modulus of X^* using the Picard-Fuchs equation, the decisive steps are to find the point of expansion in this modulus space of X^* , which corresponds to the large volume limit of X , and to determine the map from the complex structure modulus of X^* in (2.1) to the Kähler structure modulus t of X in (1.1). This map (or its inverse) will be called the mirror map.

³ See [8][6] and references therein for geometrical constructions of mirror pairs, which support this expectation.

For the quintic hypersurface in \mathbb{P}^4 , the mirror X^* can be constructed concretely as the canonical desingularized quotient $X^* = X/\widehat{(\mathbb{Z}_5^3)}$, where \mathbb{Z}_5^3 acts by phase multiplication on the homogeneous coordinates $(x_1 : \dots : x_5)$ of the \mathbb{P}^4 and is generated by $g_i : (x_i, x_5) \mapsto (\exp(2\pi i/5)x_i, \exp(8\pi i/5)x_5)$ $i = 1, 2, 3$. Here (2.1) depends on the one-dimensional complex structure deformation of X^* , which can be studied by considering the deformations of the quintic X , but restricted to the unique \mathbb{Z}_5^3 -invariant element $x_1x_2x_3x_4x_5$ in its local ring.

The Picard-Fuchs ODE can therefore be derived by the Dwork-Griffith-Katz reduction method from the standard residuum expression of the period [10] for $N = 5$

$$\tilde{\omega}_i(s_0, \dots, s_N) = \int_{\gamma} \int_{\Gamma_i} \frac{\mu}{\sum_{i=1}^N s_i x_i^N - s_0 \prod_{i=1}^N x_i}, \quad (2.2)$$

where γ is a small cycle in \mathbb{P}^{N-1} , $\Gamma_i \in H_{N-2}(M)$ and the measure is

$$\mu = \sum_i (-1)^i x_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_N.$$

Instead of using this generic algorithm, let us consider of the symmetries of (2.2) directly. Obviously

$$\hat{\omega}_i(\lambda^N s_0, \dots, \lambda^N s_N) = \lambda^{-N} \hat{\omega}_i(s_0, \dots, s_N) \quad (2.3)$$

$$\hat{\omega}_i(s_0, \dots, \lambda_i^N, \dots, \lambda_i^{-N} s_N) = \hat{\omega}_i(s_0, \dots, s_N), \quad \text{for } i = 1, \dots, N-1, \quad (2.4)$$

with $\lambda, \lambda_i \in \mathbf{C}^*$. Writing (2.3), (2.4) in infinitesimal form we obtain the differential equations

$$\left\{ \sum_{i=0}^N s_i \frac{\partial}{\partial s_i} + 1 \right\} \hat{\omega}_i(s) = 0 \quad , \quad (2.5)$$

$$\left(s_i \frac{\partial}{\partial s_i} - s_N \frac{\partial}{\partial s_N} \right) \hat{\omega}_i(s) = 0 \quad \text{for } i = 1, \dots, N-1. \quad (2.6)$$

The trivial relation $x_1^N \dots x_N^N - (x_1 \dots x_N)^N \equiv 0$ leads to a further differential equation

$$\left\{ \prod_{i=1}^N \frac{\partial}{\partial s_i} - \left(\frac{\partial}{\partial s_0} \right)^N \right\} \hat{\omega}_i(s) = 0. \quad (2.7)$$

This system of differential equation (2.5)-(2.7) is precisely the type of generalized hypergeometric system, which was investigated by Gel'fand, Kapranov and Zelevinskii in [11] with the characters defined by

$$\begin{aligned} \chi_1 = (1, \underbrace{0, \dots, 0}_{(N-1)\text{-times}}), \quad \chi_2 = (1, 1, 0, \dots, 0), \dots \\ \dots, \chi_N = (1, 0, \dots, 0, 1), \quad \chi_{N+1} = (1, \underbrace{-1, \dots, -1}_{(N-1)\text{-times}}) \end{aligned} \quad (2.8)$$

and the exponents: $\vec{\beta} = (-1, 0, \dots, 0)$. The generator of the lattice \mathbf{L} of relations is

$$l = (-N; \underbrace{1, \dots, 1}_{N\text{-times}}). \quad (2.9)$$

The eqns. (2.5),(2.6) are satisfied identically by the ansatz

$$\hat{\omega}_i(s) = \frac{1}{s_0} \omega \left(\frac{\prod_{i=1}^N s_i}{s_0^N} \right). \quad (2.10)$$

By using the new coordinate for the complex structure modulus

$$z = (-1)^{l_0} \prod_{i=0}^N s_i^{l_i} \quad (2.11)$$

the eqn. (2.7) can be brought in the following convenient form

$$\theta_z \left[\theta^{N-1} - Nz \prod_{i=1}^{N-1} (N\theta + i) \right] \omega_i = 0, \quad (2.12)$$

where $\theta = z \frac{d}{dz}$. The generalized hypergeometric system defined by (2.8) and $\vec{\beta}$ is proven to be holonomic [11] and a formal powerseries expansion and (Euler) integral representations were likewise given. For the quintic ($N = 5$) the system has 5 solutions, but it is semi-resonant, which implies that the monodromy on the full solutions space is reducible. On the other hand the monodromy for the 4 periods on Q^* is known to be irreducible. The unique subsystem of the solutions of (2.12) on which the monodromy acts irreducible is given by the 4 solutions to

$$\left[\theta^{N-1} - Nz \prod_{i=1}^{N-1} (N\theta + i) \right] \omega_i = 0, \quad (2.13)$$

which identifies the later equation with the Picard-Fuchs equation of the mirror X^* .

The complex structure moduli space of a Calabi-Yau threefold exhibits special geometry, as it was explained in [12] using crucially the results of [13]. This structure is characterized by the existence of a section \tilde{F} of a holomorphic line bundle over the complex moduli space, which is a prepotential for structure constant(s) (2.1) and the Kähler potential K of the Weil-Petersson metric. There exists a special coordinate choice, given by a ratio of periods $\tilde{t} = \tilde{\omega}_1(z)/\tilde{\omega}_0(z)$ in which these relations read

$$\begin{aligned} K_{J^*J^*J^*} &= \partial_{\tilde{t}}^3 \tilde{F} \\ K &= -\log \left((\tilde{t} - \bar{\tilde{t}})(\partial_{\tilde{t}} \tilde{F} + \bar{\partial}_{\tilde{t}} \bar{\tilde{F}}) + 2\tilde{F} - 2\bar{\tilde{F}} \right). \end{aligned} \quad (2.14)$$

These coordinates can equivalently be characterized by the property that the period vector is expressible in terms of the prepotential as

$$\Pi(z) = (\tilde{\omega}_0(z), \tilde{\omega}_1(z), \tilde{\omega}_2(z), \tilde{\omega}_3(z)) = \tilde{\omega}_0(1, \tilde{t}, \partial_{\tilde{t}} \tilde{F}, 2\tilde{F} - \tilde{t}\partial_{\tilde{t}} \tilde{F}) \quad (2.15)$$

and vice versa

$$\tilde{F}(\tilde{t}) = \frac{1}{2\tilde{\omega}_0^2}(\tilde{\omega}_3\tilde{\omega}_0 + \tilde{\omega}_1\tilde{\omega}_2). \quad (2.16)$$

It has been argued [1][12] that the moduli space of the complexified Kähler structure of the the $N = 2$ topological σ -model exhibits also special geometry with (1.1) as structure constant(s) and that t is the special coordinate (especially for the last point see also [14]). Because of the analog of (2.14) for the Kähler structure modulus the prepotential $F(t)$ is determined by K_{JJJ} up to a quadratic polynomial in t :

$$F(t) = \frac{\int_X J \wedge J \wedge J}{3!} t^3 + \frac{a}{2} t^2 + bt + c + F_{inst}(q). \quad (2.17)$$

To identify t with \tilde{t} and $F(t)$ with $\tilde{F}(\tilde{t})$, we must find in the complex structure moduli space of X^* the special point z_1 , which corresponds to the large volume limit $\text{Im}(t) \rightarrow \infty$ of X . This can be done in the following heuristic way. First note the invariance of (1.1) under the shift symmetry $t \rightarrow t + 1$. In fact more generally, shifting the parameter of the antisymmetric background $\text{Re}(t)$ by an integer is a symmetry of the σ -model in the large volume region[3][4]. We require that the transformation of the ‘‘period’’ $(1, t, \partial_t F, 2F - t\partial_t F)$ under that symmetry should correspond to a monodromy operation on $\Pi(z)$ under

counterclockwise analytic continuation around z_1 . That is, we search a point $z = z_1$ in the complex modulus space with the specific monodromy action:

$$\vec{\Pi}(z) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ (a + \frac{K}{2}) & K & 1 & 0 \\ (2b - \frac{K}{6}) & (a - \frac{K}{2}) & -1 & 1 \end{pmatrix} \vec{\Pi}(z), \quad \text{with } K = \int_X J \wedge J \wedge J \quad (2.18)$$

which is unipotent of order 4. The importance of this monodromy requirement was pointed out in [15]. It is easy to see that among the three regular singular points $z = 0, 1/5^5, \infty$ of the ODE (2.13) with $N = 5$, the point that admits such a monodromy is $z = 0$, where the indicial equation is four-fold degenerate. Around this point, there is one powerseries solution given by

$$\omega_0(z) = \omega_0(z, \rho)|_{\rho=0} = \sum_{n \geq 0} c(n, \rho) z^{n+\rho} \Big|_{\rho=0}, \quad (2.19)$$

where the coefficients $c(\rho, n)$ can be expressed in terms of gamma-functions from the l in (2.9) as

$$c(n, \rho) = \frac{\Gamma(l_0(n + \rho) + 1)}{\prod_{i=1}^N \Gamma(l_i(n + \rho) + 1)}. \quad (2.20)$$

The other solutions can be obtained by the well-known Frobenius method (see e.g. [16]):

$$\omega_p = \frac{1}{p!} \left(\frac{1}{2\pi i} \frac{\partial}{\partial \rho} \right)^p \omega_0(z, \rho) \Big|_{\rho=0}, \quad \text{for } p = 1, \dots, N - 2. \quad (2.21)$$

Their monodromy is dictated by the terms linear, quadratic und cubic in $\log(z)$. By comparing the monodromy of these solutions with (2.18) we conclude that the mirror map is given by

$$t = \frac{\omega_1(z)}{\omega_0(z)}. \quad (2.22)$$

Also from the monodromy requirement and using the special geometry relations (2.14) we get, independently of a, b, c , a unique expansion of (1.1) completely expressed in terms of special solutions to the GKZ system :

$$K_{JJJ} = \frac{\int_X J \wedge J \wedge J}{2} \partial_t^2 \left(\frac{\omega_2(z(t))}{\omega_0(z(t))} \right), \quad (2.23)$$

where we denote (the inverse of) the mirror map (2.22) by $z(t)$.

Let us finish this section with the generalization of the result (2.23) to nonsingular complete intersection Calabi-Yau spaces in products of k weighted projective spaces and give

closed formulas for the large radius expansions of the triple intersection (1.1) $\langle \mathcal{O}_{J_i} \mathcal{O}_{J_j} \mathcal{O}_{J_k} \rangle$, where J_i is the Kähler class induced from the i th weighted projective space. From these expansions one can read off the numbers of rational curves of any multidegree spaces, with respect to the Kähler classes induced from the projective spaces. These results were obtained in [7].

We consider in the following complete intersections of l hypersurfaces in products of k projective spaces. Since most formulas allow for an incorporation of weights we will state them for the general case. Denote by $d_j^{(i)}$ the degree of the coordinates of $\mathbb{P}^{n_i}[\vec{w}^{(i)}]$ in the j -th polynomial p_j ($i = 1, \dots, k; j = 1, \dots, l$). The residuum expression for the periods [10], with k perturbations satisfies again a GKZ-system, where the lattice of relations \mathbf{L} is generated by k generators $l^{(s)}$ ($s = 1, \dots, k, j = 1, \dots, l$)

$$l^{(s)} = (-d_1^{(s)}, \dots, -d_l^{(s)}; \dots, w_1^{(s)}, \dots, w_{n_s+1}, 0, \dots) \equiv \left(\{l_{0,j}^{(s)}\}; \{l_i^{(s)}\} \right), \quad (2.24)$$

from which one obtains k linear differential operators ($\theta_s = z_s \frac{d}{dz_s}$)

$$\begin{aligned} \mathcal{L}_s = & \prod_{j=1}^{n_s+1} \left(w_j^{(s)} \theta_s \right) \left(w_j^{(s)} \theta_s - 1 \right) \cdots \left(w_j^{(s)} \theta_s - w_j^{(s)} + 1 \right) \\ & - \prod_{j=1}^l \left(\sum_{i=1}^k d_j^{(i)} \theta_i \right) \cdots \left(\sum_{i=1}^k d_j^{(i)} \theta_i - d_j^{(s)} + 1 \right) z_s. \end{aligned} \quad (2.25)$$

The point⁴ $z = 0$ is again a point of maximal unipotent monodromy, and the unique powerseries solution is given

$$\omega_0(z) = \sum_{n_s \geq 0} c(n, \rho) z^{n+\rho} \Bigg|_{\rho=0}, \quad \text{with } c(n, \rho) = \frac{\prod_j \Gamma \left(-\sum_{s=1}^k l_{0,j}^{(s)} (n_s + \rho_s) + 1 \right)!}{\prod_i \Gamma \left(\sum_{s=1}^k l_i^{(s)} (n_s + \rho_s) + 1 \right)!}. \quad (2.26)$$

Again the system is semi resonant and the monodromy of (2.25) is reducible. Therefore one has to specify the subset of solutions, which correspond to the $2(k+2)$ period integrals on X^* . This problem was solved in [7] by factorizing the differential operators and the following convenient basis for the period vector was found:

$$\Pi(z) = \begin{pmatrix} w_0(z) \\ D_i^{(1)} w_0(z, \rho)|_{\rho=0} \\ D_i^{(2)} w_0(z, \rho)|_{\rho=0} \\ D_i^{(3)} w_0(z, \rho)|_{\rho=0} \end{pmatrix}. \quad (2.27)$$

⁴ Here and in the following we denote by z, n and ρ the k -tuples $z_1 \dots z_k, n_1, \dots, n_k$ and ρ_1, \dots, ρ_k . We use obvious abbreviations such as $z^n := \prod_{s=1}^k z_s^{n_s}$ etc.

Here the $D_i^{(k)}$ are differentials with respect to the parameter ρ_i , which are defined in terms of the classical intersection numbers among the Kähler classes J_i induced from the i 'th ambient space in the product space $\otimes_i \mathbb{P}_i^n$ as follows ($\partial_{\rho_i} := \left(\frac{1}{2\pi i}\right) \left(\frac{\partial}{\partial \rho_i}\right)$):

$$D_i^{(1)} := \partial_{\rho_i}, \quad D_i^{(2)} := \frac{\int_X J_i \wedge J_j \wedge J_k}{2} \partial_{\rho_j} \partial_{\rho_k} \quad \text{and} \quad D^{(3)} := -\frac{\int_X J_i \wedge J_j \wedge J_k}{6} \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k}. \quad (2.28)$$

By a straightforward generalization of the monodromy requirement one finds the generalization of (2.22)

$$t_i = \frac{\omega_i(z)}{\omega_0(z)}. \quad (2.29)$$

The following explicit expansions for the correlation function (1.1), which generalize (2.23)

$$\langle \mathcal{O}_{J_i} \mathcal{O}_{J_j} \mathcal{O}_{J_k} \rangle = \partial_{t_i} \partial_{t_j} \frac{D_k^{(2)} w_0|_{\rho=0}}{w_0}(t) = \int_X J_i \wedge J_j \wedge J_k + \sum_{d_1, \dots, d_k} \frac{n_{d_1, \dots, d_k}^r d_i d_j d_k}{1 - \prod_{i=1}^k q_i^{d_i}} \prod_{i=1}^k q_i^{d_i} \quad (2.30)$$

can be read off from the period vector (2.27), after normalizing by $1/w_0(z)$ and transforming the period vector by the inverse of (2.29) to the t variables. The prepotential was also given in [7] as

$$F(t) = \frac{1}{2} \left(\frac{1}{w_0} \right)^2 \left\{ w_0 D^{(3)} w_0 + D_l^{(1)} w_0 D_l^{(2)} w_0 \right\} (t) \Big|_{\rho=0}. \quad (2.31)$$

These formulas apply immediatly to all nonsingular complete intersections in weighted projective spaces. Let us summarize the observations, made for these series in [7]:

- a .) The mirror map (2.29) as well as its inverse have integral expansion.
- b .) The numbers n_{d_1, \dots, d_n}^r in (2.30) are integers.
- c .) The constants of the quadratic polynomial in t_i of multimoduli prepotential are $a_{ij} = 0$, $b_i = \left(\frac{1}{2\pi i}\right)^2 \int_X c_2 J_i \zeta(2)$ and $c = \left(\frac{1}{2\pi i}\right)^3 \int_X c_3 \zeta(3)$.
- d .) In all cases the invariants n_{d_1, \dots, d_r}^r coincide, as far as they can be checked, with the invariants of rational curves calculated with classical methods of algebraic geometry. For example, consider the Calabi-Yau manifolds defined by

$$p_1 = \sum a_{ijk} y_i y_j y_k = 0, \quad p_2 = \sum b_{ijk} x_i x_j x_k = 0, \quad p_3 = \sum c_{ij} y_i z_j = 0 \quad (2.32)$$

as complete intersections in $\mathbb{P}^3 \times \mathbb{P}^3$, where y_i are the homogeneous coordinates of the first \mathbb{P}^3 and z_i of the second. Then one obtains from (2.30) the following invariants n_{d_1, d_2}^r for the rational curves of bidegree less than 6:

(0,1)	81	(1,1)	729	(1,3)	6885
(0,2)	81	(2,2)	33534	(1,4)	18954
(0,3)	18	(3,3)	5433399	(1,5)	45927
(0,4)	81	(1,2)	2187	(2,3)	300348
(0,5)	81	(2,4)	1708047		
(0,6)	18				

The invariants for bidegree less than three coincide with the ones calculated by classical methods in [17].

In the remaining sections we want to investigate both the mirror map $z(t)$ and the prepotential $F(t)$. An important question is: are there any natural differential equations which govern z and F ? The answer to this questions is affirmative as we shall see.

3. Differential Equation for the Mirror Map by Examples

We will discuss in three examples in dimensions 1,2 and 3 respectively, the differential equation which governs the mirror map. We will state some general properties of the equation. Our original motivation for studying this equation was to understand the observations made experimentally on the mirror map and the Yukawa couplings.

3.1. Periods of Elliptic Curves

As a warm-up, we will first consider the most elementary example of Mirror Symmetry – for complex curves [18] [19]. This will be a brief exposition of some well-known classical construction – but in the context of Mirror Symmetry.

Consider the following one-parameter family of cubic curves in \mathbb{P}^2 :

$$X_s : x_1^3 + x_2^3 + x_3^3 - s x_1 x_2 x_3 = 0. \quad (3.1)$$

We may transform X_s by a $PGL(3, \mathbf{C})$ transformation to an elliptic curve in the Weierstrass form:

$$y^2 = 4x^3 - g_2 x - g_3 \quad (3.2)$$

where

$$\begin{aligned} g_2 &= 3s(8 + s^3) \\ g_3 &= 8 + 20s^3 - s^6. \end{aligned} \tag{3.3}$$

We would like to consider the variation of the period of the holomorphic 1-form $\frac{dx}{y}$ along a homology cycle Γ :

$$\omega_\Gamma = \int_\Gamma \frac{dx}{y}. \tag{3.4}$$

It can be shown that as a function of s , ω_Γ satisfies the second order ODE:

$$\frac{d^2\omega_\Gamma}{ds^2} + a_1(s)\frac{d\omega_\Gamma}{ds} + a_0(s)\omega_\Gamma = 0 \tag{3.5}$$

where

$$\begin{aligned} a_1 &= -\frac{d}{ds} \log \left(\frac{3}{2\Delta} (2g_2 \frac{dg_3}{ds} - 3 \frac{dg_2}{ds} g_3) \right) \\ a_0 &= \frac{1}{12} a_1 \frac{d}{ds} \log \Delta + \frac{1}{12} \frac{d^2}{ds^2} \log \Delta - \frac{1}{16\Delta} (g_2 \frac{dg_2}{ds} - 12 \frac{dg_3}{ds}) \end{aligned} \tag{3.6}$$

where $\Delta = g_2^3 - 27g_3^2$ is the discriminant of the above elliptic curve. By a change of coordinate $s \rightarrow z = s^{-3}$, equation (3.5) transforms into the hypergeometric equation (2.13) for $N = 3$ with regular singularities at $z = 0, 1/3^3, \infty$:

$$(\theta^2 - 3z(3\theta + 2)(3\theta + 1)) \omega_\Gamma = 0. \tag{3.7}$$

Thus the period ω_Γ is a linear combination of two standard hypergeometric functions.

We now do the following change of coordinates $s \rightarrow J = \frac{g_2^3}{\Delta}$, and write ω_Γ as $\sqrt{\frac{g_2}{g_3}} \Omega_\Gamma$. Then our equation (3.5) becomes

$$\frac{d^2\Omega_\Gamma}{dJ^2} + \frac{1}{J} \frac{d\Omega_\Gamma}{dJ} + \frac{31J - 4}{144J^2(1 - J)^2} \Omega_\Gamma = 0. \tag{3.8}$$

This equation has the following universal property: it is derived without the use of the explicit form of g_2, g_3 above, despite the fact that we begun with a particular realization (as a cubic in \mathbb{P}^2) of an elliptic curve. This means that if we have started from any other model for an elliptic curve, we will have arrived at the same equation (3.8), ie. this is the universal form of the Picard-Fuchs equation for the periods of the elliptic curves. Note also that under the above transformation, the ratio t of two periods $\omega_\Gamma, \omega_{\Gamma'}$ (which are two hypergeometric functions) remains the same.

We can now ask for a differential equation which governs the function $t(J)$ (which is a Schwarzian triangular function). This is the well-known Schwarzian equation:

$$\{t, J\} = 2 \left(\frac{3}{16(1-J)^2} + \frac{2}{9J^2} + \frac{23}{144J(1-J)} \right). \quad (3.9)$$

Here $\{z, x\}$ denotes the Schwarzian derivative $\frac{z'''}{z'} - \frac{3}{2} \left(\frac{z''}{z'} \right)^2$. Note that in this equation, by inverting $t(J)$ we may regard $J(t)$ as the dependent variable. Recall that the inverse function for the period ratio is precisely the mirror map. Thus $J(t)$ is our mirror map in this case and (3.9) is our differential equation which governs it. With a suitable choice of the period ratio t , $J(t)$ admits, up to overall constant, an integral q -series ($q = \exp(2\pi it)$) expansion

$$J(q) = \frac{1}{1728} (q^{-1} + 744 + 196884q + 21493760q^2 + \dots). \quad (3.10)$$

We can also relate the J -function for different realizations of the elliptic curves in different ways to solutions of GKZ systems. For example there exist three realizations of the elliptic curves as hypersurfaces in weighted projective spaces $\mathbb{P}^2(1, 1, 1)$, $\mathbb{P}^2(1, 1, 2)$ and $\mathbb{P}^2(1, 2, 3)$.

	constraint	diff. operator	$1728J(z)$
P_8	$x_1^3 + x_2^3 + x_3^3 - z^{-1/3}x_1x_2x_3 = 0$	$\theta^2 - 3z(3\theta + 2)(3\theta + 1)$	$\frac{(1 + 216z)^3}{z(1 - 27z)^3}$
X_9	$x_1^4 + x_2^4 + x_3^2 - z^{-1/4}x_1x_2x_3 = 0$	$\theta^2 - 4z(4\theta + 3)(4\theta + 1)$	$\frac{(1 + 192z)^3}{z(1 - 64z)^2}$
J_{10}	$x_1^6 + x_2^3 + x_3^2 - z^{-1/6}x_1x_2x_3 = 0$	$\theta^2 - 12z(6\theta + 5)(6\theta + 1)$	$\frac{1}{z(1 - 432z)}$

Here the differential operators are specified by factorizing the obvious differential operators from the general expression (2.25). By the expression for the $J(z)$ -function, which were obtained by transforming the constraints into the Weierstrass form, they can be brought in the form (3.8). The mirror map is related to the solutions of the GKZ system, by the formulas (2.19)-(2.22) using the generators of the lattice l given by (2.24). Concretely this yields, by inversion of (2.22), the following expansion for the functions $z(q)$

$$P_8 : z(q) = q - 15q^2 + 171q^3 - 1679q^4 + 15054q^5 - 126981q^6 + \dots$$

$$X_9 : z(q) = q - 40q^2 + 1324q^3 - 39872q^4 + 1136334q^5 - 31239904q^6 + \dots$$

$$J_{10} : z(q) = q - 312q^2 + 87084q^3 - 23067968q^4 + 5930898126q^5 - 1495818530208q^6 + \dots \quad (3.11)$$

The remarkable fact is that these expansions are already integers. Inserting them into the expressions for the $J(z)$ functions yields of course the expansion (3.10).

The above construction (ie. the periods, the Picard-Fuchs equation and the Schwarzian equation for the elliptic curves) is of course classical. We will now give a similar construction for K3 surfaces (using quartics in \mathbb{P}^3) and for the quintics in \mathbb{P}^4 . At the end, we will have a Schwarzian equation which governs the period ratio (hence the Mirror map) in each of the cases. To our knowledge, this equation is new. Actually, we also have a similar construction for any Calabi-Yau complete intersection in a toric variety. But for the purpose of exposition, we must restrict ourselves to the above simple examples. Details for the general cases will be given in our forthcoming papers[20].

3.2. Periods of K3 surfaces

We consider the following one-parameter family of quartic hypersurfaces in \mathbb{P}^3 :

$$X_s : W_s(x_1, x_2, x_3, x_4) = x_1^4 + x_2^4 + x_3^4 + x_4^4 - s x_1 x_2 x_3 x_4 = 0. \quad (3.12)$$

The period of a holomorphic 2-form along a homology 2-cycle Γ_i in X_s is given by (2.2) with $N = 4$. The Picard-Fuchs equation (2.13) for the K_3 case is a third order ODE of Fuchsian type and has singularities at $z = 0, 1/4^4, \infty$. Thus the period ω_{Γ_i} is a linear combination of three generalized hypergeometric functions. There is one solution which is regular at $z = 0$. The other two given by (2.21) have singular behavior $\log z$ and $(\log z)^2$ respectively.

What is the analogue of the universal equation (3.8) in the case of K3 surfaces, ie. the Picard-Fuchs equation which is independent of the model for the K3 surfaces? To answer this, we should first interpret (3.8) as follows. Given a topological type of complex n -folds X , there is a universal moduli space M of complex structures on X . In the case of Calabi-Yau (or elliptic curves), there is a flat Gauss-Manin connection ∇_M on M . The period vector Ω of the holomorphic n -form of X is then a section which satisfies

$$\nabla_M \Omega = 0 \quad (3.13)$$

on a vector bundle $H^n(X, \mathbf{C}) \rightarrow E \rightarrow M$.

In the case of the elliptic curves, we may view J as the coordinate on M . The universal Picard-Fuchs equation (3.8) should be thought of as the equation (3.13) in the

local coordinate. It is an interesting problem to derive the analogue of such an equation in the case of K3 surfaces.

However in the absence of such an equation, we can still ask for the analogue of the Schwarzian equation, ie. a differential equation for the Mirror map which in this case is the local inverse of the function $t(z) = \omega_1(z)/\omega_0(z)$. To write this equation, it is convenient to first transform the Picard-Fuchs equation (2.13) to the form $(\frac{d^3}{dz^3} + q_1(z)\frac{d}{dz} + q_0(z))f$. This is obtained from (2.13) by a suitable change of dependent variable $\omega_\Gamma \rightarrow f$. Then for the quartic model of K3 surfaces above, the Schwarzian equation is the following fifth order ODE:

$$\begin{aligned} \{z, t\}_3 = & (-24T_2^2 + 6T_4)q_1z'^2 - 18T_2q_1^2z'^4 - 4q_1^3z'^6 + 12T_3(\partial_z q_1)z'^3 \\ & + 3(\partial_z q_1)^2z'^6 - 12T_2(\partial_z^2 q_1)z'^4 - 6q_1(\partial_z^2 q_1)^2z'^6 - 54T_3q_0z'^3 \\ & - 27q_0^2z'^6 + 36T_2(\partial_z q_0)z'^4 + 18(\partial_z q_0)q_1z'^6 \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} \{z, t\}_3 & := -8T_2^3 - 15T_3^2 + 12T_2T_4 \\ T_i & := \nabla^{i-2}\{z, t\} \end{aligned} \quad (3.15)$$

and $\nabla := \left(\frac{d}{dt} - k\frac{z''}{z'}\right)$. Note that prime here means $\frac{d}{dt}$. For each k the object $T_k dt^k$ is a rank k tensor under linear fractional transformations $t \rightarrow \frac{at+b}{ct+d}$, with $a, b, c, d \in \mathbb{C}$. Then ∇ above is a covariant derivative on this tensor. The eqn (3.14) has a solution given by the mirror map:

$$z(q) = q - 104q^2 + 6444q^3 - 311744q^4 + 13018830q^5 - 493025760q^6 + \dots \quad (3.16)$$

As for the classical Schwarzian equation, the new equation (3.14) is of course $SL(2, \mathbb{C})$ invariant. This implies that if $z(t)$ solves the equation, so does $z((at+b)/(ct+d))$ where a, b, c, d are entries of a usual $SL(2, \mathbb{C})$ matrix. Beside the invariance under this linear fractional transformation the differential equation (3.14) exhibits also invariances under nonlinear transformations, which were used in [20] to fix the numerical coefficients in (3.14)(3.16) uniquely. Once again we have observed experimentally that the q -series expansion of the mirror map z which satisfies (3.14) is in fact integral. In the case of elliptic curves (using the Weierstrass model), we have seen that the mirror map is given by the J function which is well-known to have an integral expansion. It would be interesting to establish a similar statement for $z(q)$ in the case of K3.

3.3. Periods of Quintic Threefolds

The periods of the quintic hypersurface in \mathbb{P}^4 were studied in the last section. Special geometry introduces the prepotential F as the new object of interest. The Weil-Peterson metric on the complex structure moduli space of mirror of the quintic X^* is described by F . Moreover the mirror hypothesis asserts that there is a special coordinate transformation given by a ratio of periods $t = \omega_1(z)/\omega_0(z)$, in which $\partial_t^3 F(t)$ gives the generating function for the number of rational curves in a generic quintic. It is therefore important to understand both the mirror map z and the prepotential F . Thus a relevant question is: are there natural differential equations which govern z and F ?

For the mirror map z , there is a natural generalization of the Schwarzian equations (3.9)(3.14). Specifically, we claim that the mirror map $z(t)$ defined above satisfies the following seventh order ODE (see next section):

$$\{z, t\}_4 = -256q_0^3 z'^{12} + 128q_0^2 q_2^2 z'^{12} + \dots \quad (3.17)$$

where

$$\begin{aligned} \{z, t\}_4 := & -64T_2^6 - 560T_2^3 T_3^2 - 1275T_3^4 + 448T_2^4 T_4 + 2040T_2 T_3^2 T_4 - 192T_2^2 T_4^4 - \\ & 504T_4^3 + 1120T_2^2 T_3 T_5 + 840T_3 T_4 T_5 - 280T_2 T_5^2 + 20T_6 \{z, t\}_3. \end{aligned} \quad (3.18)$$

As in the cases of K3 surfaces and elliptic curves, this Schwarzian equation is also manifestly $SL(2, \mathbb{C})$ invariant. In the case of the quintic hypersurface, the mirror map which satisfies this equation has the q -expansion:

$$z(q) = q - 770q^2 + 171525q^3 - 81623000q^4 - 35423171250q^5 - 54572818340154q^6 + \dots \quad (3.19)$$

For the prepotential F , we have also derived a similar (seventh order) but considerably more complicated polynomial differential equation. We will discuss this in the last section.

4. Construction of the Schwarzian equations

We now give an exposition for the construction of the differential equation which governs our mirror map $z(t)$ in each case.

Note that in each case we begin with an n^{th} order ODE of Fuchsian type:

$$Lf := \left(\frac{d^n}{dz^n} + \sum_{i=0}^{n-1} q_i(z) \frac{d^i}{dz^i} \right) f = 0 \quad (4.1)$$

(n being 2,3 and 4 respectively for the elliptic curves, K3 surfaces and Calabi-Yau 3-folds.) In particular, the $q_i(z)$ are rational functions of z . Let f_1, f_2 be two linearly independent solutions of this equation and consider the ratio $t := f_2(z)/f_1(z)$. Inverting this relation (at least locally), we obtain z as a function of t . Our goal is to derive an ODE, in a canonical way, for $z(t)$.

We first perform a change of coordinates $z \rightarrow t$ on (4.1) and obtain:

$$\sum_{i=0}^n b_i(t) \frac{d^i}{dt^i} f(z(t)) = 0 \quad (4.2)$$

where the $b_i(t)$ are rational expressions of the derivatives $z^{(k)}$ (including $z(t)$). For example we have $b_n(t) = a_n(z(t))z'(t)^{-n}$. It is convenient to simplify the equation by writing (gauge transformation) $f = Ag$, where $A = \exp(-\int \frac{b_{n-1}(t)}{nb_n(t)})$, and multiplying (4.2) by $\frac{1}{Ab_n}$ so that it becomes

$$\tilde{L}g := \left(\frac{d^n}{dt^n} + \sum_{i=0}^{n-2} c_i(t) \frac{d^i}{dt^i} \right) g(z(t)) = 0 \quad (4.3)$$

where c_i is now a rational expression of $z(t), z'(t), \dots, z^{(n-i+1)}$ for $i = 0, \dots, n-2$. Now $g_1 := f_1/A$ and $g_2 := f_2/A = tg_1$ are both solution to the equation (4.3). In particular we have

$$\begin{aligned} P := \tilde{L}g_1 &= \left(\frac{d^n}{dt^n} + \sum_{i=0}^{n-2} c_i(t) \frac{d^i}{dt^i} \right) g_1 = 0 \\ Q := \tilde{L}(tg_1) - t\tilde{L}g_1 &= \left(n \frac{d^{n-1}}{dt^{n-1}} + \sum_{i=0}^{n-3} (i+1)c_{i+1}(t) \frac{d^i}{dt^i} \right) g_1 = 0. \end{aligned} \quad (4.4)$$

Note that since c_i is a rational expression of $z(t), z'(t), \dots, z^{(n-i+1)}$, it follows that P involves $z(t), \dots, z^{(n+1)}$ while Q involves only $z(t), \dots, z^{(n)}$. Eqns (4.4) may be viewed as a coupled system of differential equations for $g_1(t), z(t)$. Our goal is to eliminate $g_1(t)$ so that we obtain an equation for just $z(t)$. One way to construct this is as follows. By (4.4), we have

$$\begin{aligned} \frac{d^i}{dt^i} P &= 0, \quad i = 0, 1, \dots, n-2, \\ \frac{d^j}{dt^j} Q &= 0, \quad j = 0, 1, \dots, n-1. \end{aligned} \quad (4.5)$$

We now view (4.5) as a homogeneous *linear* system of equations:

$$\sum_{l=0}^{2n-2} M_{kl}(z(t), \dots, z^{(2n-1)}(t)) \frac{d^l}{dt^l} g_1 = 0, \quad k = 0, \dots, 2n-2, \quad (4.6)$$

where each (M_{kl}) is the following $(2n - 1) \times (2n - 1)$ matrix:

$$\begin{pmatrix} c_0 & c_1 & \dots & c_{n-2} & 0 & 1 & 0 & \dots & 0 \\ c'_0 & c_0 + c'_1 & \dots & c_{n-3} + c'_{n-2} & \cdot & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_0^{(n-2)} & (n-2)c_0^{(n-3)} + c_1^{(n-2)} & \dots & \dots & \cdot & \cdot & \cdot & 0 & 1 \\ c_1 & 2c_2 & \dots & (n-2)c_{n-2} & n & 0 & 0 & \dots & 0 \\ c'_1 & c_1 + 2c'_2 & \dots & (n-3)c_{n-3} + (n-2)c'_{n-2} & 0 & n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1^{(n-1)} & (n-1)c_1^{(n-2)} + 2c_2^{(n-1)} & \dots & \dots & \cdot & \cdot & 0 & \dots & n \end{pmatrix} \quad (4.7)$$

More precisely if we define the 1^{st} and n^{th} (n fixed) row vectors to be $(M_{1l}) = (c_0, c_1, \dots, c_{n-2}, 0, 1, 0, \dots, 0)$ and $(M_{nl}) = (c_1, 2c_2, \dots, (n-2)c_{n-2}, 0, n, 0, \dots, 0)$ respectively, then the matrix (M_{kl}) is given by the recursion relation:

$$M_{k+1,l} = M_{k,l-1} + M'_{k,l}, \quad l = 1, \dots, 2n - 1; \quad k = 1, \dots, n - 2, n, \dots, 2n - 2. \quad (4.8)$$

Thus the (M_{kl}) depends rationally on $z(t), \dots, z^{(2n-1)}(t)$. Since g_1 is nonzero, it follows that

$$\det \left(M_{kl}(z(t), \dots, z^{(2n-1)}(t)) \right) = 0. \quad (4.9)$$

This is what we call the Schwarzian equation associated with (4.1). Note that by suitably clearing denominators, this becomes a $(2n - 1)^{st}$ order polynomial ODE for $z(t)$ with constant coefficients. It is clear that this equation depends only on the data $q_i(z)$ we began with. In the case in which all the q_i are identically zero, we call the determinant in (4.9) the n^{th} Schwarzian bracket $\{z(t), t\}_n$.

Despite having a general form of the Schwarzian equation, it is useful to see a few simple examples. As the first example, consider the case $n = 2$:

$$\frac{d^2}{dz^2} f + q_0(z)f = 0. \quad (4.10)$$

The eqns (4.4) become

$$\begin{aligned} g_1'' + c_0 g_1 &= 0 \\ 2g_1' &= 0 \end{aligned} \quad (4.11)$$

where

$$c_0(t) := q_0(z(t))z'^2 - \frac{1}{2}\{z, t\}_2. \quad (4.12)$$

The corresponding linear system (4.6) has:

$$(M_{kl}) = \begin{pmatrix} c_0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad (4.13)$$

Hence the associated Schwarzian equation (4.9) in this case is

$$\det(M_{kl}) = 4c_0 = 2 \left(2q_0 z'^2 - \{z, t\}_2 \right) = 0 \quad (4.14)$$

which is the well-known classical Schwarzian equation.

For $n = 3$, we begin with the data

$$\left(\frac{d^3}{dz^3} + q_1(z) \frac{d}{dz} + q_0(z) \right) f = 0. \quad (4.15)$$

The transformed equation (4.3) in this case becomes

$$\left(\frac{d^3}{dt^3} + c_1(t) \frac{d}{dt} + c_0(t) \right) g = 0 \quad (4.16)$$

where

$$\begin{aligned} c_1(t) &:= q_1(z(t)) z'(t)^2 + 2\{z(t), t\}_2 \\ c_0(t) &:= q_0(z(t)) z'(t)^3 + q_1(z(t)) z'(t) z''(t) + \frac{3 z''(t)^3}{z'(t)^3} \\ &\quad - \frac{4 z''(t) z^{(3)}(t)}{z'(t)^2} + \frac{z^{(4)}(t)}{z'(t)}. \end{aligned} \quad (4.17)$$

The corresponding linear system (4.6) has:

$$(M_{kl}) = \begin{pmatrix} c_0 & c_1 & 0 & 1 & 0 \\ c'_0 & c_0 + c'_1 & c_1 & 0 & 1 \\ c_1 & 0 & 3 & 0 & 0 \\ c'_1 & c_1 & 0 & 3 & 0 \\ c''_1 & 2c'_1 & c_1 & 0 & 3 \end{pmatrix}. \quad (4.18)$$

Computing the associated Schwarzian equation, we get

$$\det(M_{kl}) := 27c_0^2 + 4c_1^3 - 18c_1 c'_0 - 3c_1'^2 + 6c_1 c''_1 = 0. \quad (4.19)$$

Substituting (4.17) into (4.19), we get the explicit form (3.14).

Now let's consider the case $n = 4$ which begins with

$$\left(\frac{d^4}{dz^4} + q_2(z) \frac{d^2}{dz^2} + q_1(z) \frac{d}{dz} + q_0(z) \right) f = 0. \quad (4.20)$$

We assume that this is the Picard-Fuchs equation for the periods of a 3 dimensional Calabi-Yau hypersurface. Then as pointed out earlier, there is a basis of solutions which takes the form (2.15). This implies that $q_1(z) = \frac{dq_2}{dz}$. The analogue of transformed equation (4.3) now becomes

$$\left(\frac{d^4}{dt^4} + c_2(t) \frac{d^2}{dt^2} + c_2'(t) \frac{d}{dt} + c_0(t) \right) g = 0 \quad (4.21)$$

where

$$\begin{aligned} c_2(t) &:= q_2(z) z'^2 + 5\{z(t), t\}_2 \\ c_0(t) &:= q_0(z) z'^4 + \frac{3}{2} \frac{dq_2(z)}{dz} z'^2 z'' - \frac{3}{4} q_2(z) z''^2 - \frac{135 z''^4}{16 z'^4} \\ &\quad + \frac{3}{2} q_2(z) z' z^{(3)} + \frac{75 z''^2 z^{(3)}}{4 z'^3} - \frac{15 z^{(3)2}}{4 z'^2} - \frac{15 z'' z^{(4)}}{2 z'^2} + \frac{3 z^{(5)}}{2 z'} \end{aligned} \quad (4.22)$$

The associated linear system (4.6) in this case is 7×7 . Computing the associated Schwarzian equation (4.9), we get

$$\begin{aligned} \det(M_{kl}) &:= 16 c_2^4 c_0 - 128 c_2^2 c_0^2 + 256 c_0^3 + 4 c_2^3 c_2'^2 \\ &\quad + 240 c_2 c_0 c_2'^2 - 15 c_2'^4 - 144 c_2^2 c_2' c_0' - 448 c_0 c_2' c_0' \\ &\quad + 256 c_2 c_0'^2 - 8 c_2^4 c_2'' + 128 c_0^2 c_2'' - 48 c_2 c_2'^2 c_2'' \\ &\quad + 48 c_2' c_0' c_2'' + 12 c_2^2 c_2''^2 - 48 c_0 c_2''^2 + 32 c_2^3 c_0'' \\ &\quad - 128 c_2 c_0 c_0'' + 48 c_2'^2 c_0'' + 32 c_2^2 c_2' c_2^{(3)} + 64 c_0 c_2' c_2^{(3)} \\ &\quad - 96 c_2 c_0' c_2^{(3)} + 8 c_2 c_2^{(3)2} - 8 c_2^3 c_2^{(4)} + 32 c_2 c_0 c_2^{(4)} - 12 c_2'^2 c_2^{(4)} = 0. \end{aligned} \quad (4.23)$$

Substituting (4.22) into (4.23), we get the 7th order ODE (3.17).

5. ODE for the Instanton corrected Yukawa Coupling

We can give an analogous construction of an ODE for the Yukawa coupling (1.1). To be brief, we will instead explain an approach which will in the end results in a simple *characterization*.

Let's consider first a classical problem: given a pair of polynomials $r(y, z), s(y, z)$, let X be the intersection of their zero loci (in \mathbf{C}^2) and X_y, X_z be the projections of X onto the y, z -directions. When can we construct (by quadrature) nontrivial polynomials $p(y), q(z)$ whose zero loci contain X_y, X_z respectively? The answer is simple: Hilbert's Nullstellensatz gives the following characterization. Namely $p(y), q(z)$ are constructible iff X is finite. Moreover there exists canonical choices for $p(y), q(z)$, namely the ones with

the minimal degrees. Unfortunately, no similar general characterization is known in the case of *differential* polynomials (see however [21]). However in the cases we consider, we can formulate our problem of constructing an ODE for the Yukawa coupling in a similar spirit. We can also construct the analogues of the $p(y), q(z)$ above.

We begin with (4.21) and (4.22). By applying (4.21) to the four solutions of the form $u(t), u(t)t, u(t)F'(t), u(t)(2F(t) - tF'(t))$ (cf. eqn (2.15)), we get a system of four equations which is equivalent to:

$$\begin{aligned}
u^{(4)} + c_2 u'' + c_2' u' + c_0 u &= 0 \\
4u''' + 2c_2 u' + c_2' u &= 0 \\
F^{(3)}(6u'' + c_2 u) + 4F^{(4)}u' + F^{(5)}u &= 0 \\
2F^{(3)}u' + F^{(4)}u &= 0.
\end{aligned} \tag{5.1}$$

Solving the last equation gives $u = (F^{(3)})^{-1/2}$. The second equation is redundant because it is the derivative of the third one. Thus the above system reduces to just

$$\begin{aligned}
A_2(z; t) - B_2(y; t) &= 0 \\
A_4(z; t) - B_4(y; t) &= 0
\end{aligned} \tag{5.2}$$

where $A_2(z; t), A_4(z; t), B_2(y; t), B_4(y; t)$ are differential rational functions defined by

$$\begin{aligned}
A_2(z; t) &:= c_2(t) = q_2(z)z'^2 + 5\{z(t), t\}_2 \\
A_4(z; t) &:= c_0(t) = q_0(z)z'^4 + \frac{3}{2}\frac{dq_2(z)}{dz}z'^2z'' - \frac{3}{4}q_2(z)z''^2 - \frac{135z''^4}{16z'^4} \\
&\quad + \frac{3}{2}q_2(z)z'z^{(3)} + \frac{75z''^2z^{(3)}}{4z'^3} - \frac{15z^{(3)2}}{4z'^2} - \frac{15z''z^{(4)}}{2z'^2} + \frac{3z^{(5)}}{2z'} \\
B_2(y; t) &:= 2y'' - \frac{y'^2}{2} \\
B_4(y; t) &:= \frac{y^{(4)}}{2} + \frac{y''^2}{4} - \frac{y''y'^2}{2} + \frac{y'^4}{16} \\
y(t) &:= \text{Log } F'''(t).
\end{aligned} \tag{5.3}$$

The system (5.2) depends only on the two rational functions $q_2(z), q_0(z)$ via A_2, A_4 , whereas B_2, B_4 are independent of such data. Observe that if we assign a weight 0 to z, y and weight 1 to $\frac{d}{dt}$, then the expressions A_2, B_2 (resp. A_4, B_4) are formally homogeneous of weights 2 (resp. 4). Thus we can speak of the weight of a quasi-homogeneous differential polynomial of these expressions. Finally, we note that the system of equations (5.2) should

be regarded as a differential analogue of the system of polynomial equations $r(y, z) = 0 = s(y, z)$ considered above. With this analogy in mind, we now state the first of the two main results of this section:

Given a pair of rational function $(q_0(z), q_2(z))$ (which determines the Picard-Fuchs equation), there exists a differential polynomial P with the following properties:

- (i) P quasi-homogeneous;*
- (ii) $P(A_2(z; t), A_4(z; t))$ is identically zero (see eqns. (5.3));*
- (iii) $P(B_2(y; t), B_4(y; t)) = 0$ is a nontrivial 7th order ODE in y which has a solution $y(t) = \text{Log } F'''(t)$ where $F(t)$ is the prepotential;*
- (iv) P is minimal, ie. every differential polynomial satisfying (i)–(iii) has weight no less than that of P .*
- (v) The polynomial ODE $P(B_2(y; t), B_4(y; t)) = 0$ is $SL(2, \mathbf{C})$ invariant.*

We have observed in all the known examples that P is in fact characterized by the above properties, ie. P satisfying (i)–(v) is unique up to constant multiple. We conjecture that this is the case in general. Note that P depends on the data $(q_0(z), q_2(z))$ precisely via property (ii). The polynomial ODE $P(B_2(y; t), B_4(y; t)) = 0$ should be regarded as the differential analogue of $p(y) = 0$ considered above. In this analogy, the group $SL(2, \mathbf{C})$ plays the role for the *differential* polynomial $P(B_2(y; t), B_4(y; t))$ as the Galois group does for the *ordinary* polynomial $p(y)$. The simplest example of the above ODE for $F(t)$ is given by the considering the Picard-Fuchs equation for the complete intersection of 4 quadrics in \mathbf{P}^7 , which is given by (2.25) after factorizing θ^4 as $[\theta^4 - 16(2\theta - 1)^4] \omega_\Gamma = 0$. To write the ODE for $F(t)$ down, we first define the notations:

$$\begin{aligned}
 \rho &= 100B_4 - 9B_2^2 - 30B_2'' \\
 \chi &= -32\rho^2 B_2 - 45\rho'^2 + 40\rho\rho'' \\
 \delta &= 5\chi\rho' - 2\chi'\rho.
 \end{aligned} \tag{5.4}$$

Then our ODE for the prepotential in this case is

$$\begin{aligned}
& 3783403212890625 \chi^{18} + 52967644980468750 \chi^{15} \delta^2 + 292835408677734375 \chi^{12} \delta^4 \\
& + 833559395864062500 \chi^9 \delta^6 + 1301823644717109375 \chi^6 \delta^8 + 1064406315612768750 \chi^3 \delta^{10} \\
& + 357449882108765625 \delta^{12} + 9097175898878906250 \chi^{16} \rho^5 + 75543680906950781250 \chi^{13} \delta^2 \rho^5 \\
& - 55168781762820937500 \chi^{10} \delta^4 \rho^5 - 1235933279927738437500 \chi^7 \delta^6 \rho^5 \\
& - 2628328829388247068750 \chi^4 \delta^8 \rho^5 - 1669442421173622243750 \chi \delta^{10} \rho^5 \\
& - 31639592222462973709375 \chi^{14} \rho^{10} - 1041303693581386404075000 \chi^{11} \delta^2 \rho^{10} \\
& + 1397061241390545045311250 \chi^8 \delta^4 \rho^{10} - 978071752628929206000 \chi^5 \delta^6 \rho^{10} \\
& + 3088961515882945520173125 \chi^2 \delta^8 \rho^{10} + 726375263921582813504122500 \chi^{12} \rho^{15} \\
& - 2401967567306257982918892000 \chi^9 \delta^2 \rho^{15} + 2931906039367569842399977800 \chi^6 \delta^4 \rho^{15} \\
& - 2592007729730548310729752000 \chi^3 \delta^6 \rho^{15} + 26477211431856325292132500 \delta^8 \rho^{15} \\
& + 731773527868504699561324929024 \chi^{10} \rho^{20} - 1384453886791545382987331665920 \chi^7 \delta^2 \rho^{20} \\
& + 1003786188392583028918031769600 \chi^4 \delta^4 \rho^{20} \\
& - 92650299984331138894225408000 \chi \delta^6 \rho^{20} \\
& + 264379950716374035480557566033920 \chi^8 \rho^{25} \\
& - 323653884996678359415539902709760 \chi^5 \delta^2 \rho^{25} \\
& + 105122101152057682020817226956800 \chi^2 \delta^4 \rho^{25} \\
& + 48853700167414249640038923438653440 \chi^6 \rho^{30} \\
& - 33153423760664989683513831467253760 \chi^3 \delta^2 \rho^{30} \\
& + 528120679253988321156369324441600 \delta^4 \rho^{30} \\
& + 4965538896010513223822010617996247040 \chi^4 \rho^{35} \\
& - 1238934080748073699029086124292177920 \chi \delta^2 \rho^{35} \\
& + 265021771162266355900761945816768184320 \chi^2 \rho^{40} \\
& + 5822406825670998196401392296588763725824 \rho^{45} = 0
\end{aligned} \tag{5.5}$$

Note that since δ, χ, ρ are of weights 15,10,4 respectively, P is a quasi-homogeneous differential polynomial of weight 180. Each of the 37 terms in this polynomial corresponds to a partition of 180 by 15,10,4.

It turns out that there is a dual characterization for the Schwarzian equation (4.23) we have constructed:

There exists a differential polynomial Q with the following properties:

(i) Q is quasi-homogeneous;

(ii) $Q(B_2(y; t), B_4(y; t))$ is identically zero (see eqns. (5.3));

(iii) $Q(A_2(z; t), A_4(z; t)) = 0$ is a nontrivial ODE which has a solution given by the mirror map $z(t)$;

(iv) Q is minimal of weight 12, ie. every differential polynomial satisfying (i)–(iii) has weight at least 12;

(v) The polynomial ODE $Q(A_2(z; t), A_4(z; t)) = 0$ is $SL(2, \mathbf{C})$ invariant;

(vi) Q is universal, ie. it is independent of the data $(q_2(z), q_0(z))$ and it is characterized by (i)–(v) up to constant multiple;

(vii) $Q(A_2(z; t), A_4(z; t)) = 0$ coincides with $\det(M_{kl}) = 0$ in (4.23).

We will defer the detailed proofs of the above results to our forth-coming papers.

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