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The Strong Rigidity of Locally Symmetric Complex Manifolds of Rank One and Finite Volume*

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Rigidity question have attracted much interest in the past. In the compact case, we have the famous work of Calabi and Vesentini [3] and Mostow [17]. Whereas Calabi and Vesentini proved a local version, namely that compact quotients of bounded symmetric domains admit no nontrivial deformations in case the domain is irreducible and of complex dimension at least 2, Mostow proved a global rigidity result, at the expense, however, of working only within the class of quotients of symmetric domains. Mostow's work is based on quasiconformal mappings. A different analytic approach was recently undertaken by Siu [22]. If M is a compact Kähler manifold diffeomorphic (or, more generally, homotopically equivalent) to a quotient N of an irreducible bounded symmetric domain, he studied a harmonic homotopy equivalence the existence of which is assured by the theorem of Eells and Sampson, and demonstrated that this map has to be a biholomorphic diffeomorphism itself, thus in particular obtaining the global rigidity of N within the class of Kähler manifolds and thereby generalizing the results of Calabi-Vesentini as well as of Mostow for quotients of bounded symmetric domains. A corresponding result for irreducible compact quotients of products of upper half planes, a case not covered by Siu's arguments, was then obtained through the work of Jost and Yau [11, 12], and Mok [15], thereby completting Siu's approach. On the other hand, Mostow's results were extended to noncompact locally symmetric spaces of finite volume by Prasad [19] and Margulis [14].

In the present work, we start an extension of Siu's results to the noncompact case. We study locally symmetric varieties of rank one, thereby generalizing some of Prasad's results, as well as irreducible quotients of products of upper half planes.

We shall prove:

Theorem 1. *Let D be an irreducible bounded symmetric domain in $\mathbb{C}^n$, $n \geqq 2$, of rank 1, i.e. the unit ball in complex space of dimension at least 2. Let N be a quotient of finite volume by a discrete torsion free subgroup Γ of* $\mathrm{Aut}(D)$.

Suppose $\bar{M}$ is a compact Kähler manifold, S a subvariety of $\bar{M}$ with normal crossings (i.e. S has possibly self intersections that locally look like the intersection of

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coordinate hyperplanes in $\mathbb{C}^n$ *and is otherwise regular) and suppose there exists a proper homotopy equivalence between* $M := \bar{M} \setminus S$ *and* N.

Then M *is* $\pm$ *biholomorphically equivalent to* N.

Note. A special case was obtained in [24].

Remark. The assumptions of the theorem are in particular satisfied if M is a quasiprojective manifold, properly homotopically equivalent to N. Here, "quasi-projective" means that M is a Zariski dense open subset of a projective algebraic variety. By Hironaka's theorem [8], the possible singularities of this projective algebraic variety can be resolved, i.e. replaced by complex hyposurfaces with normal crossings without changing the variety away from the singularities. The same remark applies to our next result.

Theorem 2. *Let* $H := \{x + iy \in \mathbb{C} : y > 0\}$ *denote the upper halfplane. Let* $N = H^n/\Gamma$, *where* Γ *is a discrete irreducible torsionfree subgroup of* $\mathrm{Aut}(H^n)$, $n \geqq 2$.

Let M *again be properly homotopically equivalent to* N *and admit a Kähler compactification* $\bar{M}$ *as in Theorem 1.*

Then, there exists a diffeomorphism $f: M \to N$ *with the property that for its lifting* $F = (F_1, \ldots, F_n): \tilde{M} \to H^n$ *to universal coverings, each* F_i *is* $\pm$ *holomorphic.*

As Siu did, we shall make strong use of harmonic maps and their properties. As a general reference for harmonic maps, one can use [10]. Because of the noncommpactness of the manifolds considered, we have to overcome several new technical difficulties compared to the compact case.

We first have to construct a proper homotopy equivalence of finite energy. In order to achieve this, we have to use the existence of suitable compactifications of M and N, in particular, the existence of a smooth compactification $\bar{M}$ of M with $\bar{M} \setminus M$ consisting of a union of smooth divisors with normal crossings. This special structure will also be important for showing that a harmonic map of finite energy (into which we deform the original map of finite energy) has maximal rank. On the other hand, we do not really need the locally symmetric structure of N but only a suitable decay of its metric towards the ends or cusps.

In the rank one case, this follows from Lemma 8 of [23]. The result of this Lemma already holds if N is a complete Kähler manifold whose sectional curvature is bounded between two negative constants. In addition, we shall also need strong negativity of the curvature of N in the sense of [22]. In our arguments, we need the more special assumption that N is locally symmetric of rank one only in the case where N has only one end. In this situation, we appeal to a result of Selberg that the fundamental group of N is residually finite, allowing us to reduce this case to the case where the image has more than one end by passing to a suitable finite cover. Apparently, a generalization of Selberg's result to negatively curved manifolds of finite volume is not known.

In the case of Theorem 2, we note that by a result of Selberg (cf. [9, p. 277]), Γ is commensurable with the Hilbert modular group $\tilde{\Gamma}$ of some totally real field K with $[K : \mathbb{Q}] = n$ (hence in particular arithmetic). Therefore, for our purposes, it is sufficient to know the behavior of the metric of $H^n/\tilde{\Gamma}$ near the cusps.

We also note that, concerning Theorem 2, our contribution lies in the construction of a proper harmonic homotopy equivalence of finite energy and maximal rank. The remaining arguments needed to prove this theorem are due to Mok [16].

1. The Metrics on Domain and Image

a) The Domain M

By assumption, M admits a compactification as a smooth Kähler manifold $\bar{M}$, with the property that $\bar{M} \setminus M$ consists of a union of complex hypersurfaces with normal crossings. The Kähler metric of $\bar{M}$ restricts to a Kähler metric ω on M.

Terminology. We call these hypersurface compactifying divisors, or short, cd's. Also, we call the boundary of a neighborhood of a compactifying divisor a bn.

We let $s_i = s_i^l \otimes \ldots \otimes s_i^{m_i}$ be local sections of the normal bundles of the cd's. In particular, s_i vanishes on the corresponding cd. We then take

$$g := \sum_i \partial\bar{\partial}(\varphi(|s_i|) \log(|\log|s_i^1|^2| \cdot \ldots \cdot |\log|s_i^{m_i}|^2|) + c\omega,$$

where φ is a suitable cut-off functions so that $\varphi(|s_i|)$ is identical one near the corresponding cd and vanishes outside a neighborhood of the cd, in particular for, say, $|s_i| \geqq 3/4$, and where $c>0$ is chosen sufficiently large to make g positive definite. g then is a Kähler metric with the following properties.

(K1) g is complete
(K2) (M, g) has finite volume
(K3) If D is a disk normal to a cd, i.e. a local complex curve in M intersecting the cd. at $0 \in D$, then the restriction of g to D behaves asymptotically (i.e. when approaching 0) like the Poincaré metric on the punctured disk

$$\partial\bar{\partial}\log(|\log|z|^2|) = \frac{1}{|z|^2(\log|z|^2)^2}|dz|^2$$

(K4) A cd metrically looks like a complex hypersurface in the following sense: if $\mathscr{C}$ is the collection of disks normal to the cd, then $\mathscr{C}$ has finite nonzero $(2n-2)$ dimensional Hausdorff measure, where $n = \dim_{\mathbb{C}} M$. (This is due to the part $c\omega$ in the definition of g.)

b) The Metric of Complete Negatively Curved Kähler Manifolds of Finite Volume

Let N be a complete Kähler manifold of finite volume with curvature bounded between two negative constants. Given an end of N, we choose a geodesic ray γ going into that end, an arbitrary initial point $t=0$ on γ and a parametrization $\gamma(t)$ by arclength.

If γ' is another ray going into the same end, then we can choose $\gamma'(0)$ in such a way that for the parametrization $\gamma'(t)$ by geodesic distance

$$\lim_{t\to\infty} d(\gamma(t), \gamma'(t)) = 0.$$

We do this for every geodesic ray extending into the given cusp and put

$$\varrho := \exp(-\exp t). \tag{1.1}$$

We put $N_{\varrho_0} := \{x \in N: \varrho(x) \leqq \varrho_0\}$. As N is negatively curved and of finite topological type (this follows, e.g., from [23]), though each each point of N_{ϱ_0} there is precisely one geodesic ray going into the given cusp, provided $\varrho_0 > 0$ is sufficiently small.

Hence, the level sets $\varrho = \text{const} \leqq \varrho_0$ are smooth and diffeomorphic to each other, and the diffeomorphism is explicitly obtained by moving along the geodesic rays.

Hence, in N_{ϱ_0}, we can choose local coordinates in such a way that the coordinates on the level sets $\varrho = \text{const}$ are independent of ϱ, i.e. invariant under moving along geodesic rays.

It then follows from Lemma 8 of [23] that the corresponding metric tensor of the hypersurfaces $\varrho = \text{const}$ is bounded by $\frac{l}{|\log \varrho|}$. In other words, if this hypersurface is described by the coordinates $u_i (i = 1, \ldots, 2n-1, \; n = \dim_{\mathbb{C}} N)$ and the metric tensor is denoted by $g_{u_i u_j}$, then

$$|g_{u_i u_j}(u_1, \ldots, u_{2n-1}, \varrho)| \leqq \frac{c}{|\log \varrho|}. \tag{1.2}$$

Here c is a fixed constant, depending only on the geometry of N.

In the case of locally symmetric varieties of rank one, we can also give a more explicit description of the metric as follows (cf. [1]):

The complex unit ball

$$\left\{ z \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 < 1 \right\}$$

can be represented as a Siegel domain of genus 2, namely

$$\left\{ (w, u_1, \ldots, u_{n-1}) \in \mathbb{C}^n : \operatorname{Im} w - \sum_{i=1}^{n-1} |u_i|^2 > 0 \right\},$$

via the transformation

$$z_1 = \frac{w-i}{w+i}, \qquad z_{j+1} = \frac{u_j \cdot \sqrt{2}}{w+i} \; (j = 1, \ldots, n-1).$$

In this representation, the metric becomes

$$\partial \bar{\partial} \left(-\log (\operatorname{Im} w - \sum_{i=1}^{n-1} (|u_i|^2) \right). \tag{1.3}$$

If we consider a quotient of the unit ball by a discrete arithmetic group of isometries which has finite volume then near the so called cusps, i.e. the points where the fundamental domain reaches the boundary of the unit ball, this quotient is obtained by just putting

$$z = e^{2\pi i a w}, \qquad a \in \mathbb{R},$$

and then making the appropriate identifications in the u_i-directions. If we represent z in polar coordinates (ϱ, φ), then in particular

$$\operatorname{Im} w = \frac{-1}{2\pi a} \log \varrho$$

$\varrho = 0$ corresponds to the cusp.

In the unit ball, we can look at the disk

$$\{|z_1|<1,\ z_2=\ldots=z_n=0\}.$$

Subject to the operation of $S(U(n-1)\times U(1))$, this disk sweeps out the whole unit ball. In the Siegel domain representation, this disk becomes the half plane

$$\{\operatorname{Im} w>0,\ u_1=\ldots=u_{n-1}=0\}.$$

Since, as mentioned, the whole space can be recovered from the images of this half plane under isometries, it suffices to evaluate the metric on this half plane. In the quotient, we therefore have to consider

$$\partial\bar{\partial}\left(-\log\left(\frac{-1}{2\pi a}\log\varrho-\sum|u_i|^2\right)\right)$$

only where

$$\sum_{i=1}^{n-1}|u_i|^2=0. \tag{1.4}$$

Now

$$\frac{\partial^2}{\partial z\partial\bar{z}}\left(-\log\left(-\frac{1}{2\pi a}\log\varrho\right)\right)=\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho\frac{\partial}{\partial\varrho}\left(-\log\left(-\frac{1}{2\pi a}\log\varrho\right)\right)\right)=\frac{1}{\varrho^2(\log\varrho)^2}$$

and

$$\frac{\partial^2}{\partial u_i\partial\bar{u}_i}\left(-\log\left(-\frac{1}{2\pi a}\log\varrho-\sum|u_i|^2\right)\right)=-\frac{4\pi a}{\log\varrho}$$

because of (1.4).

c) The Metric of Hilbert Modular Varieties

These are irreducible noncompact quotients of finite volume of H^n, where $H=\{x+iy\in\mathbb{C}: y>0\}$ is the upper half plane. Apart from quotient singularities which disappear by passing to a finite covering and hence can be neglected for our purpose, these varieties again have cusp singularities. The kernel function now is

$$-\log(\operatorname{Im} w_1\ldots\operatorname{Im} w_n)$$

for $(w_1\ldots,w_n)\in H^n$, and near the cusp, one passes to the quotient by putting

$$\frac{i}{2}w_j=\sum_{k=1}^{n}a_{jk}\log z_k$$

where the a_{jk} are positive integers.

Putting $z_k=\varrho_k e^{i\varphi_k}$ with $\varrho_k>0$, the kernel function becomes

$$-\sum_{j=1}^{n}\log\left(\sum_{k=1}^{n}\left(-2a_{jk}\log\varrho_k\right)\right).$$

In order to compute the metric, we have to take $\partial\bar{\partial}$ of the kernel function, hence evaluate

$$-\frac{1}{\varrho_i}\frac{\partial}{\partial\varrho_i}\left(\varrho_i\frac{\partial}{\partial\varrho_i}\sum_j\log\left(\sum_k(-2a_{jk}\log\varrho_k)\right)\right)=\sum_{j=1}^n\frac{4a_{ji}^2}{\varrho_i^2}\frac{1}{\left(\sum_{k=1}^n(-2a_{jk}\log\varrho_k)\right)^2}.$$

Thus, if e.g. ϱ_ℓ tends to zero, in the z_ℓ-direction, the metric again behaves like the Poincaré metric on the punctured disk, whereas in the orthogonal directions, we get a decay of order

$$\frac{1}{(\log\varrho_\ell)^2}.$$

For more details on the preceding construction, we refer to Hirzebruch [9; particularly p. 193f., and p. 204ff.] as well as Ash et al. [1].

2. Construction of a Finite Energy Map

On the domain, we let (r,θ) be polar coordinates on a disk transversal to the cd so that $r=0$ lies on the cd. We denote the coordinates in the other directions by x. The volume form then behaves like

$$\frac{1}{r(\log r)^2}\,dr\,d\theta\,dx. \tag{2.1}$$

On the image, we take the same coordinates ϱ, $u(=(u_1,\dots,u_{2n-1})$ locally) as in 1b. (Similarly, in the case of 1c.) We then choose a fixed differentiable homotopy equivalence from the CR-hypersurface $r=\text{const}$ onto the CR-hypersurface $\varrho=\text{const}$, and we let r and $\varrho=\varrho(r)$ correspond via

$$\log\varrho(r)=-(\log r)^2. \tag{2.2}$$

We denote the map constructed in this way by h. We can control the components of the inverse metric tensor of the domain by 1a) in the following way[1]

$$\gamma^{rr}\sim r^2(\log r)^2,\qquad \gamma^{\theta\theta}\sim(\log r)^2\qquad\text{by (K3)} \tag{2.3}$$

and

$$\gamma^{xx}\sim\text{const}\qquad\text{by (K4)}. \tag{2.4}$$

Likewise, by 1b), for the metric tensor of the image

$$g_{\varrho\varrho}=\frac{1}{\varrho^2(\log\varrho)^2}\qquad\text{by (1.1)} \tag{2.5}$$

and

$$g_{u_iu_j}\leqq\frac{c_0}{|\log\varrho|}\qquad(c_0=\text{const})\qquad\text{by (1.2)} \tag{2.6}$$

Thus, the energy of h is controlled via

$$E(h)\leqq c_1\int\gamma^{rr}g_{\varrho\varrho}\left(\frac{\partial\varrho}{\partial r}\right)^2+\gamma^{\theta\theta}g_{u_iu_j}\frac{\partial u_i}{\partial\varphi}\left\{\frac{\partial u_j}{\partial\varphi}+\gamma^{xx}c_2\right\}\frac{1}{r(\log r)^2}\,dr\,d\varphi\,dx,$$

[1] Super- and subscripts denote the corresponding coordinate directions

where c_1 and c_2 again are finite constants.

From (2.2), we derive

$$\frac{1}{\varrho}\frac{\partial \varrho}{\partial r} = -\frac{2}{r}\log r, \tag{2.7}$$

and then from (2.3)–(2.7)

$$E(h) \leqq c_3 < \infty .$$

We apply the same construction near each cusp, and we then extend these maps to the bounded parts of domain and image to get a proper homotopy equivalence, again called h, of finite energy. In a similar way, we can construct a proper homotopy equivalence of finite energy if the image is a Hilbert modular variety with a metric as in 1c).

3. The Harmonic Map and Its Properties

a) Existence

Since the image has nonpositive curvature, we can use the argument of [20][2], to deform the map h of finite energy, constructed in the preceding section, into a harmonic map f of finite energy. f then is a smooth map. We want to verify that f is also a proper homotopy equivalence.

b) f is Pluriharmonic

We again make use of the domain metric of 1a). Near each cusp, we choose a family φ_ε of cut-off functions depending only on $|s_i|$ where s_i is again the section of the normal bundle of the corresponding cd. We let $\varphi_\varepsilon \equiv 1$ outside a neighborhood of the cd, $\varphi_\varepsilon \to 1$ as $\varepsilon \to 0$. $\varphi_\varepsilon(|s_i|) \equiv 0$ for $|s_i| \log |s_i| \leqq \varepsilon$, $0 \leqq \varphi_\varepsilon(|s_i|) \leqq 1$ everywhere, and finally, we let $t^2(\log t)^2 \varphi_\varepsilon''(t)$ be bounded independently of ε as $\varepsilon \to 0$. Thus,

$$|\Delta \varphi_\varepsilon| \sim \big| |s_i|^2 (\log |s_i|)^2 \varphi_\varepsilon'' \big| \leqq c, \tag{3.1}$$

where c is independent of ε.

We then use the well-known formula (cf. e.g. [10, 1.6]) for the energy density $e(f) = |df|^2$

$$\Delta e(f) = |\Delta\, df|^2 + \langle df \cdot \mathrm{Ric}^M(e_i), df \cdot e_i\rangle - \langle R^N(df \cdot e_i, df \cdot e_j)\, df \cdot e_i, df \cdot e_j\rangle, \tag{3.2}$$

where (e_i) is an orthonormal frame on M, Ric^M is the Ricci tensor of M, and R^N the curvature tensor of N. On the other hand, by (3.1) and since f has finite energy,

$$\int_M \varphi_\varepsilon \Delta e(f) = \int_M \Delta \varphi_\varepsilon \cdot e(f) \tag{3.3}$$

is bounded independently of ε, and hence

$$\int_M \Delta e(f) < \infty . \tag{3.4}$$

[2] The noncompactness of the image presents no obstacle, since Hamilton's theorem [7] still holds

Since our domain metric has bounded Ricci curvature (cf. [4, Proposition (3.5)]),

$$\int \langle df \cdot \mathrm{Ric}^M(e_i),\ df \cdot e_i \rangle \leqq cE(f). \tag{3.5}$$

Since the sectional curvature of N is nonpositive, we obtain from (3.2), (3.4), (3.5)

$$\int_M |\nabla df|^2 < \infty. \tag{3.6}$$

For a smooth compact subset K of M, we then obtain

$$\left|\int_K \Delta e(f)\right| = \left|\int_{\partial K} \frac{\partial}{\partial n} e(f)\right| \leqq \left|\int_{\partial K} e(f)\right| \cdot \left|\int_{\partial K} |\nabla df|^2\right|. \tag{3.7}$$

Letting K run through a suitable exhaustion of M, (3.6), (3.7), and $E(f) < \infty$, imply

$$\int \Delta e(f) = 0. \tag{3.8}$$

As in [22], we put ($f = (f^1, \ldots, f^n)$ in local coordinates)

$$\bar\partial f^\alpha = \frac{\partial f^\alpha}{\partial \bar z^i}\, d\bar z^i,$$

$$D\bar\partial f^\alpha = \partial\bar\partial f^\alpha + \Gamma^\alpha_{\beta\gamma}\partial f^\beta \wedge \bar\partial f^\gamma, \text{ etc.}$$

employing in addition, however, a summation convention. Of course, $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the image N, and we denote the metric tensor by $(g_{\alpha\beta})$.

If ω is the Kähler form of M, then the argument leading to (3.8) also gives

$$\int \partial\bar\partial (g_{\alpha\beta}\bar\partial f^\alpha \wedge \partial \bar f^\beta) \wedge \omega^{n-2} = 0. \tag{3.9}$$

On the other hand, we have Siu's Bochner type identity [22, Sect. 3] (in integrated form)

$$\int_M \partial\bar\partial (g_{\alpha\beta}\bar\partial f^\alpha \wedge \partial \bar f^\beta) \wedge \omega^{n-2} = \int_M \{R_{\alpha\beta\gamma\delta}\bar\partial f^\alpha \wedge \partial \bar f^\beta \wedge \partial f^\gamma \wedge \bar\partial \bar f^\delta \wedge \omega^{n-2}$$
$$- g_{\alpha\beta} D\bar\partial f^\alpha \wedge \bar D \partial \bar f^\beta \wedge \omega^{n-2}\}. \tag{3.10}$$

As in [22, Sect. 4], since f is harmonic and the metric of the unit ball is strongly negative, both terms in the integral on the right hand side are pointwise nonpositive, hence

$$D\bar\partial f^\alpha \equiv \bar D \partial \bar f^\beta \equiv 0 \tag{3.11}$$

and

$$R_{\alpha\beta\gamma\delta}\bar\partial f^\alpha \wedge \partial \bar f^\beta \wedge \partial f^\gamma \wedge \bar\partial \bar f^\delta \equiv 0. \tag{3.12}$$

Lemma 1. *f is pluriharmonic, i.e. the restriction of f to any local complex curve in M is harmonic.*

c) Global Behavior

A neighborhood of a cd can be locally fibered by local holomorphic curves of the type of the unit disk that are transversal to the cd and that have the property that their intersection with a bn is a homotopically nontrivial curve in the bn. This follows from the assumption that M and N are homotopically equivalent and from Lemma 3 below. On each subdisk, we let again its center O correspond to the intersection point with the cd. Because of (K4) and $E(f) < \infty$, the restriction of f to almost every such disk has finite energy (and is harmonic as established in b). We now need

Lemma 2. *Let M be a manifold with (smooth) boundary and of finite volume, and let N be complete with nonpositive sectional curvature. Let $g: \partial M \to N$ be a smooth map. Let $f: M \to N$ be a harmonic map of finite energy with $f|\partial M = g$. Then there is no other harmonic map with finite energy and the same boundary values homotopic to f.*

Proof. We can take over the arguments of [21, Sect. 2], if we note that the function $\varphi = (\varrho^2+1)^{1/2}$ defined on p. 369 of [21] satisfies $\nabla\varphi = 0$ on ∂M in our case, so that we can still perform the integration by part used in the proofs of Lemmas 1 and 2 of [21]. Q.E.D.

In particular, under the assumptions of Lemma 1, any such harmonic map is necessarily energy minimizing. This will be essential for our subsequent reasoning.

Lemma 3. *Let X be a complete (noncompact) manifold of nonpositive curvature. Let U be a neighborhood of an end of X. Let γ be a curve in U which is homotopically nontrivial in U but which can be homotoped in U into arbitrarily short curves. Then γ is homotopically nontrivial in X.*

Proof. We take some δ-neighborhood V_δ of ∂U. There exists $\varepsilon > 0$ with

$$\text{length}(\tau) > \varepsilon \tag{3.13}$$

for any homotopically nontrivial curve τ in V_δ. Let us assume also

$$\delta > \varepsilon . \tag{3.14}$$

Let U' be the component of $X \setminus V_\delta$ with

$$U' \cap U \neq \emptyset .$$

By assumption, we can move γ into U' and achieve

$$\text{length}(\gamma) < \varepsilon. \tag{3.15}$$

Let $\gamma(t)$ denote the images of $\gamma = \gamma(0)$ under the heat flow ($t \geqq 0$). If the initial curve γ was parametrized proportional to arclength,

$$\text{length}(\gamma(t)) \leqq \text{length}(\gamma) \text{ for all } t \geqq 0 . \tag{3.16}$$

For background information on the heat flow, see [10, Chap. 3].

By (3.13)–(3.16), and since γ and hence $\gamma(t)$ is homotopically nontrivial in U, no $\gamma(t)$ can be contained in V_δ, and hence

$$\gamma(t) \subset U$$

for all $t \geqq 0$.

Since on the other hand, γ can be homotoped to arbitrarily short curves by moving towards the end, $\gamma(t)$ cannot converge to a closed geodesic (such a geodesic would have to realize the minimal length in its homotopy class) and hence has to move towards the end as well.

If γ would be homotopic to a point p, then

$$p(t) \equiv p$$

would be another homotopic solution of the heat flow, and by the stability lemma of Hartman (cf. [10, 3.4])

$$\operatorname{dist}(p(t), \gamma(t))$$

had to be nonincreasing contradicting that $\gamma(t)$ moves towards the end. Q.E.D.

A similar argument yields

Lemma 4. *Let X be a complete, nonpositively curved manifold with at least two ends. Let U be a neighborhood of one end, and let γ be a homotopically nontrivial curve in U that can be homotoped into arbitrarily small curves by moving towards the corresponding end. Then it is not possible to shrink γ by moving towards a different end as well.*

Let D be a disk, i.e. a local holomorphic curve, transversal to a cd and having the property that its intersection with a bn is homotopically nontrivial in this bn. By Lemma 1, $f|D$ is harmonic. Since the energy of $f|D$ and the fact that $f|D$ is harmonic do not depend on the metric of D but only on the conformal structure (cf. [10, 1.3]), we can use the standard flat metric on each such D. Again, we let $O \in D$ be the intersection of D with the cd.

Lemma 5. *If the energy of $f|D$ is finite, then $f|D$ is proper, i.e. for any compact $K \subset N$ there is a neighborhood U_K of $O \in D$ with*

$$f^{-1}(K) \cap D \cap U_K = \emptyset .$$

Proof. We write $u := f|D$. Let $D_\varrho := \{(r, \varphi) \in D : 0 \leqq r \leqq \varrho\}$, where we use standard polar coordinates on D.

Since $E(u) < \infty$, for the energy of u on D_ϱ we have

$$E_\varrho(u) = \int_{r=0}^{\varrho} \int_{\varphi=0}^{2\pi} \left(u_r^2 + \frac{1}{r^2} u_\varphi^2 \right) \mathrm{r} d\varphi \, dr \to 0 \text{ as } l \to 0. \tag{3.17}$$

For $0 \leqq r_1 \leqq r_2$, we can find $\varphi_0 \in [0, 2\pi]$ with

$$\begin{aligned} d(u(r_1, \varphi_0), u(r_2, \varphi_0)) &\leqq \int_{r=r_1}^{r_2} \int_{\varphi=0}^{2\pi} |u_r| \, dr \, d\varphi \\ &\leqq \left(\int_{r_1}^{r_2} \int_{\varphi} \frac{1}{r} dr \, d\varphi \right)^{1/2} \cdot \left(\int_{r_1}^{r_2} \int_{\varphi} |u_r|^2 \, r dr d\varphi \right)^{1/2} \\ &\leqq \left(\log \frac{r_2}{r_1} \right)^{1/2} \cdot E_{r_2}(u)^{1/2} . \end{aligned} \tag{3.18}$$

Likewise, given $0 \leqq \varrho_1 < \varrho_2$, we can find $r_0 \in [\varrho_1, \varrho_2]$ with the property that for all $\varphi_1, \varphi_2 \in [0, 2\pi]$

$$\begin{aligned} d(u(r_0, \varphi_1), u(r_0, \varphi_2)) &\leqq \int_{\varphi=0}^{2\pi} |u_\varphi| \, d\varphi \leqq 2\pi \left(\int_0^{2\pi} |u_\varphi|^2 \, d\varphi \right)^{1/2} \\ &\leqq 2\pi (E_{\varrho_2}(u))^{1/2} \cdot \left(\int_{\varrho_1}^{\varrho_2} \frac{1}{r} dr \right)^{-1/2} \\ &= 2\pi \left(\log \frac{\varrho_2}{\varrho_1} \right)^{-1/2} E_{\varrho_2}(u)^{1/2}. \end{aligned} \tag{3.19}$$

Let now an end of M be given. This end is mapped under the map h constructed in Sect. 2 to an end of N. Let U be a neighborhood of this end of N, and let D intersect the given end of M. As h is a homotopy equivalence, $h(\partial D)$ is (homotopic to) a curve satisfying the assumption of Lemma 3.

Using Lemmas 3 and 4 and (3.17)–(3.19) given $\varepsilon > 0$, we can find some sufficiently small $R_0 > 0$, some $R_1 \in \left[\frac{R_0}{2}, R_0\right]$, $R_2 \in \left[\frac{R_0}{8}, \frac{R_0}{4}\right]$, and $\varphi_0 \in [0, 2\pi]$ so that $u(\partial D_{R_1})$, $u(\partial D_{R_2})$, and $u([R_2, R_1] \times \{\varphi_0\})$ are all contained in U and of length at most $\varepsilon/4$.

We then look at the curve ℓ in D obtained in the following way: first, keep $r = R_1$ fixed and let φ run from φ_0 to $\varphi_0 + 2\pi$. Then, keep $\varphi = \varphi_0$ (identified with $\varphi_0 + 2\pi$) fixed and let r run from R_1 and R_2. Then keep $r = R_2$ fixed and let φ again run from φ_0 to $\varphi_0 + 2\pi$. Finally, keep again $\varphi = \varphi_0$ fixed and let r run from R_2 to R_1. ℓ then is a homotopically trivial curve in $D \setminus \{0\} \subset M$ with $u(\ell) \subset U$ and

$$\text{length}(u(\ell)) < \varepsilon. \tag{3.20}$$

We then look at the lifts

$$\tilde{u}: D \setminus \{0\} \to \tilde{N}$$

to universal covers. The lift $\tilde{\ell}$ of ℓ is a closed curve and bounds a region B. We then look at the harmonic extension

$$\begin{aligned} \bar{u}&: B \to \tilde{N} \\ \bar{u}|\tilde{\ell} &= \tilde{u}|\tilde{\ell}. \end{aligned}$$

(3.20) implies that there is a ball $B(p, \varepsilon) \subset \tilde{N}$ of radius ε with

$$\bar{u}(\tilde{\ell}) \subset B(p, \varepsilon).$$

Since $\tilde{N}$ is simply connected and nonpositively curved, it follows from the maximum principle [applied to $\tilde{d}^2(p, \bar{u}(\cdot))$, where $\tilde{d}(\cdot, \cdot)$ is the distance function on $\tilde{N}$] that

$$\bar{u}(B) \subset B(p, \varepsilon). \tag{3.21}$$

On the other hand, by uniqueness (Lemma 2),

$$\bar{u} = \tilde{u}|B|.$$

This, together with (3.21), shows that u maps the whole annulus $R_2 \leqq r \leqq R_1$ into an ε-neighborhood of U.

In this way, we can cover D by smaller and smaller annuli that are mapped under u further and further towards the end, and we deduce that u is proper as claimed. Q.E.D.

We now let $\mathscr{D}$ be the collection of all disks of a local fibration of a neighborhood of a cd. by transversal disks with the above homotopy property. We want to control the energy of f on all disks in $\mathscr{D}$. We start with a disk $D \in \mathscr{D}$ on which the energy of f is finite. (3.17) and the chain of inequalities in (3.19) imply that we can always find some sufficiently small $r_0 > 0$ for which

$$\int_{\varphi=0}^{2\pi} |u_\varphi(r_0, \varphi)|^2 \, d\varphi := \delta(r_0, D)$$

becomes arbitrarily small. (Again, we put $u := f|D$.) Since f is smooth, we can find a neighborhood $\mathscr{U}$ of D in $\mathscr{D}$ such that for all $D' \in \mathscr{U}$

$$\delta(r_0, D') \leqq 2\delta(r_0, D).$$

On D'_{r_0}, we then construct a comparison map $v = v_{D'}$ as follows: for $r = r_0$, we put $v(r_0, \varphi) = f|D'(r_0, \varphi)$, and for $r = \frac{r_0}{2}$, $v\left(\frac{r_0}{2}, \varphi\right) = u(r_0, \varphi)$. For $r_0/2 \leqq r \leqq r_0$, we let v be the harmonic extension of its boundary values, and for $r \leqq r_0/2$, we define

$$v(r, \varphi) = u(2r, \varphi).$$

We see that there is a fixed constant K, independent of $D' \in \mathscr{U}$, that bounds the energy of $v_{D'}$ on D'_{r_0}. We note that $v_{D'}$ has the same boundary values as $f|D'$ on $\partial D'_{r_0}$.

Since harmonic maps with finite energy are unique by Lemma 2, the restriction $f|D'_{r_0}$ for $D' \in \mathscr{U}$ has to coincide with the energy minimizing map with the same boundary values. First, this follows for those D' where we already know that the energy of f is finite. Since these disks are dense in $\mathscr{U}$, and since we now have uniform bounds for the energy on these disks, and since f is smooth, the result follows for all disks in $\mathscr{U}$. On the other hand, $\mathscr{D}$ is compact (since the cd.'s are compact) and hence is covered by a finite number of such neighborhoods. Therefore, we can apply the argument of the proof of Lemma 5 uniformly to all $D \in \mathscr{D}$ and deduce

Lemma 6. *Let (γ_n) be a sequence of (in M) homotopically nontrivial curves shrinking to a point in a cd of $\bar{M}$. Then the sequence $f(\gamma_n)$ of image curves is not contained in any compact subset of N. Moreover, this establishes a well-defined correspondence between the ends of M and N.*

d) f has Maximal Rank at Some Point

α) We first treat the case where M (and N) have at least two ends.

From Lemma 6 we see that we can control the mapping of the ends. Namely, f maps a given end in the domain to the same end in the image as the original proper homotopy equivalence h of Sect. 2 does.

On the other hand, since we have more than one end, a bn. of an end defines a nontrivial homology class of real dimension $2n-1$, and the image under f again is a nontrivial homology class.

This together with Lemma 6 implies that the functional determinant of f cannot vanish identically.

β) The case where M and N have only one end can be reduced to the preceding one as follows:

After a suitable conjugation, we can assume that the given cusp in N corresponds to the point at infinity in the universal cover $\tilde{N}$ (i.e. to $\operatorname{Im} w = \infty$ resp. $\operatorname{Im} w_1 = \ldots \operatorname{Im} w_n = \infty$ in the representations of 1b) resp. 1c)). The isotropy group of the cusp then has a subgroup of finite index leaving the level sets of the Bergmann kernel function invariant. The quotient of $\tilde{N}$ by this subgroup and hence also by the isotropy group has infinite volume, because geodesic rays running into the cusp are permuted and the distance between these rays increases when moving away from the cusp because of the nonpositivity of the sectional curvature. Hence $\pi_1(N)$ must contain an element γ not fixing the cusp. Since $\pi_1(N)$ is residually finite by a result of Selberg (cf. [2, p. 39]), it contains a normal subgroup Γ of finite index not containing γ. Then $\tilde{N}/\Gamma$ has more than one end. Therefore, by the argument of α), the lift of f to some finite coverings and hence also f has maximal rank somewhere.

For more details on the isotropy group of a cusp, cf. Eberlein [5] and Hirzebruch [9] (for the Hilbert modular varieties).

4. Proof of the Theorems

Theorem 2 follows from the argument of Mok [16] in conjunction with the preceding construction of a harmonic homotopy equivalence which has maximal rank somewhere, as established in 3d).

For Theorem 1, it first follows from Siu's arguments [22], using (3.12) and that the harmonic map f has maximal rank at some point, that f is $\pm$ holomorphic. N can be compactified as an algebraic variety by adding a simple point to each cusp. Note that this does not depend on the locally symmetric structure of N, but only on the fact that the sectional curvature of N is bounded between two negative constants, cf. [23]. The Schwarz lemma of [25] then implies in conjunction with Lemma 6 that f is proper.

If $\bar{M}$ is Hironaka's nonsingular compactification of M, then f can be extended as a holomorphic map $\bar{f}$: $\bar{M} \to \bar{N}$, cf. e.g. [13, Corollary 3.7, p. 100]. Also, the extension $\bar{f}$ is a map of degree ± 1. This is seen as follows: First, since $\pm$ holomorphic of maximal rank, it cannot have degree 0. Let g: $N \to M$ be a homotopy equivalence so that $h = f \circ g$ is homotopic to the identity of N. Let v_N be the volume form of N. We use our preceding construction (modifying h into a finite energy map and then use harmonic replacement on an increasing set of domains) to convert h into a proper harmonic self-map h' of N. It is clear from the construction that $\deg h = \int h * v_N)(\operatorname{vol} N)^{-1}$ remains unchanged, since integer valued. Namely, one just choses a domain D_n with $\left|\int_{D_n} h * v_N\right| \geqq (\deg h - \frac{1}{2}) \operatorname{vol} N$, observes that harmonic replacement on D_n does not change this number and then concludes that the degree can no longer jump to a smaller integer as the domain increases.

As above h' is $\pm$ holomorphic of maximal rank. The Schwarz lemma [25] then implies

$$|\deg f \cdot \deg g| = |\deg h'| \leqq 1$$

so that f is of degree ± 1.

In order to show that f is bijective on M, we can now proceed as in [22, p. 110f.], namely show that the set V of those points in M, where f is not locally homeomorphic, is empty, since otherwise V would be a complex hypersurface in M, extending into $\bar{M}$, whereas $f(V)$ would have codimension at least 2. The preimage of a generic point in $f(V)$ is a nontrivial compact analytic subvariety of M (since f is proper). This contradicts the fact that f is a homotopy equivalence.

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