Compo. Math., 73 (1990), 125 − 144.

CURVATURE AND HOLOMORPHIC MAPPINGS OF COMPLETE KÄHLER MANIFOLDS[1]

Peter Li and Shing-Tung Yau

Department of Mathematics
Univeristy of Arizona
Tucson
AZ 85721
and
Department of Mathematics
Harvard University
Cambridge
MA 02138

October 14, 1987 (Revised–April 17, 1989)

Table of Contents

§0 Introduction

A theorem of A. Huber in [H] asserts that if M is a complete noncompact 2-dimensional Riemannian manifold with the negative part of its Gaussian curvature being integrable, then M is parabolic in the sense that it does not admit any nonconstant bounded harmonic functions. Our goal of this project is to find an appropriate generalization or anolog of Huber's theorem in higher dimension. Realizing that Huber's theorem failed completely for real manifolds in dimension greater than 2 (see [L-T 1]), we turned our attention to the complex category. In particular, we will prove that if a complete Kähler manifold satisfies some integral curvature conditions then it does not admit any nonconstant positive pluriharmonic functions. This can be viewed as a generalization of Huber's theorem because in dimension 2, all real manifolds are Kähler manifolds and harmonic functions are pluriharmonic functions.

This paper is organized in the manner that §1 and §2 are discussions on Riemannian manifolds in general. The Kähler assumption will not be imposed on the manifold until

§3. In §1, we consider geomtric consequences when the Ricci curvature of M is assumed to satisfy an integrablity condition. In particular, we will show that the volume growth of M can be controlled by the growth of an L^p integral of the negative part of the pointwise lower bound of the Ricci curvature. In the original version of the paper we proved this fact when $p \geq n-1$, where n is the real dimension of M. However recently we found out that Gallot has studied a similar assumption on the Ricci curvature in [G] with a substantial overlap with our argument. In fact, his argument is more refined in the way that one only needs to assume that $p > \frac{n}{2}$. The proof presented in §1 will incorporate his argument and ours for the purpose of applying to our situation.

In §2, we study a certain class of differential inequalities which often arise in geometry. In particular, integrability conditions of nonnegative functions satisfying one of these differential inequalities will be derived. In the last section, we will apply the theory developed in §1 and §2 to holomorphic mappings from a complete Kähler manifold to a Hermitian manifold. We conclude by observing that all the computations, in fact, are valid for pluriharmonic mappings which is defined in [L].

The first author would like to thank S. Y. Cheng and A. Treibergs for many helpful discussions and their interest in this work. We would also like to express our gratitude to L. F. Tam for proof-reading our original manuscript and providing a simplier and slightly more general argument for Corollary 3.2 by way of Corollary 2.2.

§1 Curvature and Volume Growth

A higher dimensional anolog of the integral of the negative part of the Gaussian curvature on surfaces is the integral of some power of the negative part of the pointwise lower bound of the Ricci curvature. The following lemma obtained by a modification of Gallot's Theorem in [G] enables us to control the volume growth of the complete manifold in terms of the growth of such an integral.

Theorem 1.1. *Let M be a complete noncompact Riemannian manifold without boundary of dimension n. Let us denote $R(x)$ to be the pointwise lower bound of the Ricci curvature,*

$$Ric_{ij}(x) \geq R(x)\, g_{ij},$$

and $R_-(x) = \max\{0, -R(x)\}$ to be the negative part of $R(x)$. If the geodesic ball of radius r centered at $y \in M$ is denoted by $B_y(r)$, its volume is denoted by $V_y(r)$, and the area of its boundary is denoted by $A_y(r)$, then for any $p \geq n-1$ there exists constants $C_1, C_2 > 0$ depending only on n such that for any $r > 0$,

$$A_y(r) \leq C_1\, r^{n-1} + C_2\, r^{2n-4}\, V_y^{\frac{p-n+1}{p}}(r) \int_0^r \left(\int_{B_y(t)} R_-^p \, dV \right)^{\frac{n-1}{p}} dt.$$

Also if $n > 2$ then for any $\frac{n}{2} < p < n-1$, there exists constants $C_3, C_4, C_5 > 0$ depending only on n such that

$$\begin{aligned} A_y(r) \leq & C_3\, A_y(r_0) + C_4\, (r - r_0)^{2p-1} \int_{\partial B_y(r_0)} H_+^{2p-1}\, dA \\ & + C_5\, (r-r_0)^{2p-2} \int_{r_0}^{r_2} \int_{B_y(r_1) \setminus B_y(r_0)} R_-^p \, dV\, dr_1 \end{aligned}$$

for any $r > r_0 > 0$*, where* $H_+ = \max\{0, H\}$ *is the positive part of the mean curvature function on* $\partial B_y(r_0)$*.*

Proof. In terms of normal polar coordinates centered at the point y, the volume element of M can be written as $dV = a(\theta, r)\, dr\, d\theta$. The first variational formula gives

$$\frac{\partial a}{\partial r}(\theta, r) = a(\theta, r)\, H(\theta, r), \tag{1.1}$$

where $H(\theta, r)$ denotes the mean curvature of the geodesic sphere of radius r at the point (θ, r). The second variational formula yields

$$\frac{\partial^2 a}{\partial r^2}(\theta, r) = H^2(\theta, r)\, a(\theta, r) - Ric(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})\, a(\theta, r) - h_{ij}^2(\theta, r)\, a(\theta, r), \tag{1.2}$$

with h_{ij} being the second fundamental form of the geodesic sphere. However the inequality

$$\begin{aligned} h_{ij}^2 &\geq \frac{\left(\sum_{i=1}^{n-1} h_{ii}\right)^2}{n-1} \\ &= \frac{H^2}{n-1}, \end{aligned}$$

and the definition of $R(\theta, r)$ implies that (1.2) can be estimated by

$$\begin{aligned} \frac{\partial^2 a}{\partial r^2} &\leq \frac{n-2}{n-1}\, H^2\, a - R\, a \\ &= \frac{n-2}{n-1}\, a^{-1} \left(\frac{\partial a}{\partial r}\right)^2 - R\, a. \end{aligned} \tag{1.3}$$

If we set

$$f(\theta, r) = a^{\frac{1}{n-1}}(\theta, r),$$

then by differentiating and applying (1.1) and (1.3), we have

$$\frac{\partial f}{\partial r} = \frac{1}{n-1}\, H\, f, \tag{1.4}$$

and

$$\frac{\partial^2 f}{\partial r^2} \leq \frac{-1}{n-1}\, R\, f. \tag{1.5}$$

Moreover, f satisfies the initial conditions

$$f(\theta, 0) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial r}(\theta, 0) = 1, \tag{1.6}$$

because $a \sim r^{n-1}$ and $H \sim (n-1)\, r^{-1}$, as $r \to 0$. Integrating inequality (1.5) from 0 to r_0 and using (1.6), we obtain the inequality

$$\frac{\partial f}{\partial r}(\theta, r_0) \le \frac{-1}{n-1} \int_0^{r_0} R(\theta, r)\, f(\theta, r)\, dr + 1. \tag{1.7}$$

Let us now first consider the case when $p = n-1$. Integrating (1.7) from 0 to r_1 and using (1.6), we have

$$f(\theta, r_1) \le \frac{1}{n-1} \int_0^{r_1} \int_0^{r_0} R_-(\theta, r)\, f(\theta, r)\, dr\, dr_0 + r_1.$$

Using the definition of f, and the inequality $(a+b)^{n-1} \le 2^{n-2}(a^{n-1} + b^{n-1})$, for $a, b \ge 0$, we obtain

$$a(\theta, r_1) \le 2^{n-2} r_1^{n-1} + \frac{2^{n-2}}{(n-1)^{n-1}} \left(\int_0^{r_1} \int_0^{r_0} R_-(\theta, r)\, f(\theta, r)\, dr\, dr_0 \right)^{n-1}. \tag{1.8}$$

We shall point out that this inequality is only valid for those values of r_1 such that the point (θ, r_1) is within the cut locus of of y. If we denote the sets

$$S_y(r) = \{\theta \in \mathbf{S}_y^{n-1} \,|\, (\theta, r) \text{ is within cut locus of } y\},$$

we see that they satisfy the monotonicity property, $S_y(r_2) \subseteq S_y(r_1)$, if $r_1 \le r_2$. Integrating inequality (1.8) over the set $S_y(r_1)$ yields

$$\begin{aligned}
&A_y(r_1) \\
&\le 2^{n-2}\, r_1^{n-1} \int_{S_y(r_1)} d\theta + \frac{2^{n-2}}{(n-1)^{n-1}} \int_{S_y(r_1)} \left(\int_0^{r_1} \int_0^{r_0} R_-(\theta, r)\, f(\theta, r)\, dr\, dr_0 \right)^{n-1} d\theta \\
&\le 2^{n-2}\, r_1^{n-1}\, \omega_{n-1} \\
&\quad + \frac{2^{n-2}}{(n-1)^{n-1}} \int_{S_y(r_1)} \left(\int_0^{r_1} \int_0^{r_0} R_-^{n-1}(\theta, r)\, f^{n-1}(\theta, r)\, dr\, dr_0 \right) \left(\int_0^{r_1} \int_0^{r_0} dr\, dr_0 \right)^{n-2} d\theta \\
&= 2^{n-2}\, r_1^{n-1}\, \omega_{n-1} + \frac{1}{(n-1)^{n-1}}\, r_1^{2n-4} \int_{S_y(r_1)} \int_0^{r_1} \int_0^{r_0} R_-^{n-1}(\theta, r)\, f^{n-1}(\theta, r)\, dr\, dr_0\, d\theta \\
&\le 2^{n-2}\, r_1^{n-1}\, \omega_{n-1} + \frac{1}{(n-1)^{n-1}}\, r_1^{2n-4} \int_0^{r_1} \int_0^{r_0} \int_{S_y(r)} R_-^{n-1}(\theta, r)\, f^{n-1}(\theta, r)\, d\theta\, dr\, dr_0 \\
&= 2^{n-2}\, r_1^{n-1}\, \omega_{n-1} + \frac{1}{(n-1)^{n-1}}\, r_1^{2n-4} \int_0^{r_1} \int_{B_y(r_0)} R_-^{n-1} dV\, dr_0,
\end{aligned} \tag{1.9}$$

where ω_{n-1} denotes the area of the Euclidean unit (n-1)-sphere. This proves the case when $p = n-1$ with $C_1 = 2^{n-2}\omega_{n-1}$ and $C_2 = \frac{1}{(n-1)^{n-1}}$.

For the values $p = s(n-1) > n-1$, we simply apply Hölder inequality to the second term on the right hand side of (1.9) and use the fact that the volume of the geodesic ball of radius r is an increasing function of r, and conclude that

$$A_y(r_1) \le C_1\, r_1^{n-1} + C_2\, r_1^{2n-4} \int_0^{r_1} V_y^{\frac{s-1}{s}}(r_0) \left(\int_{B_y(r_0)} R_-^{s(n-1)}\, dV \right)^{\frac{1}{s}} dr_0$$

$$\le C_1\, r_1^{n-1} + C_2\, r_1^{2n-4}\, V_y^{\frac{s-1}{s}}(r_1) \int_0^{r_1} \left(\int_{B_y(r_0)} R_-^{s(n-1)}\, dV \right)^{\frac{1}{s}} dr_0$$

We will now consider the case when $\frac{n}{2} < p < n-1$. Note that n must be at least 3 for this situation to occur. Following an argument of Gallot in [G], by setting $\delta = \frac{2p-n}{2p-1}$ one can rewrite inequality (1.5) in the form

$$\frac{\partial}{\partial r}\left(f^{-\delta}\, \frac{\partial f}{\partial r} \right) + \delta\, f^{-(1+\delta)} \left(\frac{\partial f}{\partial r} \right)^2 \le \frac{R_-}{n-1}\, f^{1-\delta}. \tag{1.10}$$

We claim that there is a constant $C_6 > 0$ depending only on n such that

$$\left| f^{-\delta}\, \frac{\partial f}{\partial r} \right|^{2p-2} \frac{\partial}{\partial r}\left(f^{-\delta}\, \frac{\partial f}{\partial r} \right) \le C_6\, R_-^p\, f^{n-1}.$$

This inequality is obvious if the left hand side is nonpositive. Otherwise, applying the algebraic inequality $(a+b)^p \ge \frac{p^p}{(p-1)^{p-1}}\, a\, b^{p-1}$ to the left hand side of inequaltiy (1.10), using the assumption on p and the definition of δ, and integrating this inequality from r_0 to r_1 over those values where $f^{-\delta}\, \frac{\partial f}{\partial r}$ is nonnegative, we conclude that its nonnegative part $\rho(\theta, r) = \max\{0, f^{-\delta}\, \frac{\partial f}{\partial r}(\theta, r)\}$ satisfies the estimate

$$\rho^{2p-1}(\theta, r_1) \le \rho^{2p-1}(\theta, r_0) + C_6(2p-1) \int_{r_0}^{r_1} R_-^p(\theta, r)\, a(\theta, r)\, dr.$$

Using the definition of δ, f, and (1.1), we rewrite this inequality as

$$H_+^{2p-1}(\theta, r_1)\, a(\theta, r_1) \le H_+^{2p-1}(\theta, r_0)\, a(\theta, r_0) + C_6(2p-1)(n-1)^{2p-1} \int_{r_0}^{r_1} R_-^p(\theta, r)\, a(\theta, r)\, dr,$$

where $H_+ = \max\{0, H\}$. Integrating over the set $S_y(r_1)$ and using the monotonicity property, this implies that

$$\int_{\partial B_y(r_1)} H_+^{2p-1}\, dA \le \int_{\partial B_y(r_0)} H_+^{2p-1}\, dA + C_6(2p-1)(n-1)^{2p-1} \int_{B_y(r_1)\setminus B_y(r_0)} R_-^p\, dV. \tag{1.11}$$

Applying the Cauchy-Schwarz inequality to the left hand side yields

$$\int_{\partial B_y(r_1)} H_+^{2p-1}\, dA \geq A_y^{-2(p-1)}(r_1) \left(\int_{\partial B_y(r_1)} H_+\, dA \right)^{2p-1}.$$

On the other hand, we have

$$\begin{aligned} \frac{\partial A_y}{\partial r}(r_1) &= \int_{\partial B_y(r_1)} H\, dA \\ &\leq \int_{\partial B_y(r_1)} H_+\, dA \end{aligned}$$

Therefore (1.11) implies

$$A_y^{\frac{-2(p-1)}{2p-1}}(r_1)\, \frac{\partial A_y}{\partial r}(r_1) \leq \left(\int_{\partial B_y(r_0)} H_+^{2p-1}\, dA + C_7 \int_{B_y(r_1)\setminus B_y(r_0)} R_-^p\, dV \right)^{\frac{1}{2p-1}}, \tag{1.12}$$

where $C_7 = C_6(2p-1)(n-1)^{2p-1}$. Integrating with respect to r_1 from r_0 to r_2, we conclude that

$$\begin{aligned} &A_y^{\frac{1}{2p-1}}(r_2) - A_y^{\frac{1}{2p-1}}(r_0) \\ &\qquad \leq (2p-1) \int_{r_0}^{r_2} \left(\int_{\partial B_y(r_0)} H_+^{2p-1}\, dA + C_7 \int_{B_y(r_1)\setminus B_y(r_0)} R_-^p\, dV \right)^{\frac{1}{2p-1}} dr_1. \end{aligned}$$

Hence

$$\begin{aligned} A_y(r_2) \leq &C_3\, A_y(r_0) + C_4\, (r_2 - r_0)^{2p-2} \int_{r_0}^{r_2} \int_{\partial B_y(r_0)} H_+^{2p-1}\, dA\, dr_1 \\ &+ C_5\, (r_2 - r_0)^{2p-2} \int_{B_y(r_1)\setminus B_y(r_0)} R_-^p\, dV\, dr_1, \end{aligned}$$

which proves the theorem.

By integrating these estimates and using the fact that $A_y(r) = \frac{\partial V_y}{\partial r}(r)$, we deduced the following:

Corollary 1.2. *Let M be a complete Riemannian manifold without boundary. With the notation of Theorem 1.1, if for any $p > \frac{n}{2}$ the growth of the L^p-norm of R_- satisfies*

$$\int_{B_y(r)} R_-^p\, dV = o(r^k)$$

as $r \to \infty$, then

$$V_y(r) = o(r^{2p+k})$$

as $r \to \infty$.

Another corollary of Theorem 1.1 is the following generalization of Huber's theorem in [H].

Corollary 1.3. *Let M be a complete open surface whose negative part of its Gaussian curvature defined by $K_-(x) = \max\{0, -K(x)\}$ satisfies*

$$\int_{B_y(r)} K_- \, dA \leq \alpha_1 \, \log(r+1)$$

for some constant $\alpha_1 > 0$ and for all $r > 0$. Then M is parabolic.

Proof. By Theorem 1.1 and the curvature assumption, the length of the boundary of the geodesic ball of radius r centered at y satisfies

$$\begin{aligned} L_y(r) &\leq C_1\, r + C_2\, \alpha_1 \int_0^r \log(t+1)\, dt \\ &\leq C_1\, r + C_2\, \alpha_1 \, \{(r+1)\, \log(r+1) - r - 2\} \\ &\leq 2C_2\, \alpha_1 \, (r+1)\, \log(r+1), \end{aligned}$$

for r sufficiently large. Now we invoke the criterion (see [L-T 1]) for the parabolicity of M by checking the condition

$$\int_1^\infty \frac{1}{L_y(r)}\, dr = \infty.$$

§2 Differential Inequalities and Integrabilities

In this section we will focus our attention to positive functions defined on a complete manifold which satisfy a certain class of differential inequalities.

Theorem 2.1. *Let M be a complete noncompact Riemannian manifold without boundary. Assume u is a nonnegative function on M satisfying the differential inequality*

$$\Delta u - \frac{|\nabla u|^2}{u} \geq k\, u^{q+1} + g\, u,$$

for some constants $q \geq 0$, and for some functions $k \geq 0$ and g on M. Let us assume that the negative part of g define by $g_- = \max\{0, -g\}$ is integrable, and also that there exists a positive constant p and a point $y \in M$, such that for all r sufficiently large, the function u satisfies

$$\int_{B_y(r)} u^p \, dV = o(r^2).$$

Then

$$0 \geq \int_M k\, u^q \, dV + \int_{M^+} g\, dV,$$

where $M^+ = \{x \in M \,|\, u(x) > 0\}$.

Proof. Without loss of generality we may assume that $\int_M g\, dV < \infty$. Otherwise, we can replace g by the function

$$g_r(x) = \begin{cases} \min\{g(x), r\}, & \text{if } x \in B_y(r) \\ -\, g_-(x), & \text{if } x \notin B_y(r), \end{cases}$$

and let $r \to \infty$ after the theorem is proved for g_r. In fact, a consequence of the theorem is that the integral of g will be finite if $u \in L^q(M)$.

Let $\epsilon > 0$ be an arbitrary contant and $\phi(x)$ to be the cut-off function depending only on the distance from x to the point y which is defined by

$$\phi(x) = \begin{cases} 1 & \text{on } B_y(r) \\ 0 & \text{on } M \setminus B_y(2r) \end{cases}$$

with the properties that $\phi \geq 0$ and $|\nabla\phi|^2 \leq 3\, r^2$. Multiplying the differential inequality on both sides by the factor $\frac{\phi^2\, u^{p-1}}{u^p+\epsilon}$ and integrating over M yields

$$\int_M \frac{\phi^2\, u^{p-1}\, \Delta u}{u^p + \epsilon} - \int_M \frac{\phi^2\, u^{p-1} |\nabla u|^2}{u\,(u^p + \epsilon} \geq \int_M \frac{\phi^2\, k\, u^{p+q}}{u^p+\epsilon} + \int_M \frac{\phi^2\, g\, u^p}{u^p+\epsilon}. \tag{2.1}$$

Integrating the first term on the left hand side by parts gives

$$\begin{aligned}\int_M \frac{\phi^2\, u^{p-1}\, \Delta u}{u^p+\epsilon} &= -2\int_M \frac{\phi\, u^{p-1}\, \langle \nabla\phi, \nabla u\rangle}{u^p+\epsilon} - (p-1)\int_M \frac{\phi^2\, u^{p-2}\, |\nabla u|^2}{u^p+\epsilon} \\ &\quad + p\int_M \frac{\phi^2\, u^{2p-2}\, |\nabla u|^2}{(u^p+\epsilon)^2} \\ &= -2\int_M \frac{\phi\, u^{p-1}\, \langle \nabla\phi, \nabla u\rangle}{u^p+\epsilon} + \int_M \frac{\phi^2\, u^{p-1}\, |\nabla u|^2}{u\,(u^p+\epsilon)} - \epsilon\, p\int_M \frac{\phi^2\, u^{p-2}\, |\nabla u|^2}{(u^p+\epsilon)^2} \\ &\leq \frac{1}{\epsilon\, p}\int_M |\nabla\phi|^2\, u^p + \int_M \frac{\phi^2\, u^{p-1}\, |\nabla u|^2}{u\,(u^p+\epsilon)}.\end{aligned}$$

Substituting this into inequality (2.1) yields

$$\frac{1}{\epsilon\, p}\int_M |\nabla\phi|^2\, u^p \geq \int_M \frac{\phi^2\, k\, u^{p+q}}{u^p+\epsilon} + \int_M \frac{\phi^2\, g\, u^p}{u^p+\epsilon}.$$

Using the estimate on $|\nabla\phi|^2$ and the assumption on u, the left hand side can be estimated by

$$\frac{1}{\epsilon\, p}\int_M |\nabla\phi|^2\, u^p \leq \frac{3}{\epsilon\, p\, r^2}\int_{B_y(2r)} u^p,$$

which tends to 0 as $r \to \infty$. Hence we arrive at the inequality

$$0 \geq \int_M \frac{k\, u^{p+q}}{u^p+\epsilon} + \int_M \frac{g\, u^p}{u^p+\epsilon}.$$

Now letting $\epsilon \to 0$ and observing that the second integral converges by Lebesgue convergence theorem and the first integral converges by the monotone convergence theorem to the desired inequality.

We would like to point out that if $k \equiv 0$ and $u > 0$, then the theorem implies that $\int_M g\, dV \leq 0$. This is a slight generalization of a theorem of the second author in [Y], where he assumed in addition that g is bounded from below. On the other hand if $k > 0$ and $\int_{M^+} g\, dV \geq 0$, then we can conclude that u must be identically 0.

Corollary 2.2. *Let M, u, g, and k satisfy the assumption of Theorem 2.1. In addition, let us also assume that M admits a positive Green's function. Then u must be identically 0.*

Proof. Assume that there is a point $z \in M$ such that $u(z) > 0$. Let $G_z(x)$ be the positive Green's function on M with a pole at z. Pick a sequence $G_i(x)$ of nonnegative smooth superharmonic functions on m which has the properties that $G_i \to G_z$ weekly in L^2. This can be achieved by simply capping-off G_z near the pole. Now consider the function $w = u\, e^v$, where $v = -\alpha\, G_i$ for some constant $\alpha > 0$. Observe that

$$\begin{aligned} \int_{B_y(r)} w^p &= \int_{B_y(r)} u^p\, e^{pv} \\ &\leq \int_{B_y(r)} u^p \\ &= o(r^2), \end{aligned}$$

by the assumption on u.

On the other hand, w satisfies the differential inequality

$$\begin{aligned} \Delta w - \frac{|\nabla w|^2}{w} &\geq k\, u^p\, w + g\, w + w\, \Delta v \\ &\geq w\,(g + \Delta v). \end{aligned}$$

Applying Theorem 2.1 to the function w and using the fact that $\Delta v \geq 0$, we have

$$\begin{aligned} -\int_M g &\geq \int_{w>0} \Delta v \\ &\geq \int_{u>0} \phi\, \Delta v, \end{aligned}$$

for all compactly supported function ϕ such that $\phi(z) = 1$ and $\phi \leq 1$. Integrating by parts and letting $i \to \infty$, we conclude that

$$\begin{aligned} \int_{u>0} \phi\, \Delta v &= -\int_{u>0} \alpha\, G_i\, \Delta\phi \\ &\to -\int_{u>0} \alpha\, G_z\, \Delta\phi \\ &= \alpha\, \phi(z) \\ &= \alpha, \end{aligned}$$

which gives a contradiction since α is arbitrary.

The next theorem allows us to deduce integrability conditions on nonnegative functions which satisfy a similar class of differential inequalities.

Theorem 2.3. *Let M be a complete noncompact Riemannian manifold without boundary. Assume that u is a nonnegative function on M satisfying the differential inequality*

$$\Delta u - \frac{|\nabla u|^2}{u} \geq k_0\, u^{q+1} + g\, u,$$

for some constants $q, k_0 > 0$, and for some function g on M. Then for any constant $p > 1$ and any fixed point $y \in M$, there eixsts constants $C_8, C_9 > 0$ depending only on k_0 such that the function u must satisfy

$$\int_{B_y(r)} u^{p+q-1} \leq C_8\, r^{-\frac{2(p+q-1)}{q}}\, V_y(2r) + C_9 \int_{B_y(2r)} g_-^{\frac{p+q-1}{q}}$$

for any $r > 0$. In particular, if

$$\int_{B_y(r)} g_-^{\frac{p+q-1}{q}} = o(r^2)$$

and

$$V_y(r) = o(r^{\frac{2(p+2q-1)}{q}})$$

as $r \to \infty$, then

$$\int_{B_y(r)} u^p = o(r^2)$$

as $r \to \infty$.

Proof. Let ϕ be the cut-off function defined in the proof of Theorem 2.1. Multiplying both sides of the differential inequality by $\phi^2\, u^{p-2}$ and integrate by parts yields

(2.2)
$$\begin{aligned} k_0 \int_M \phi^2\, u^{p+q-1} - \int_M \phi^2\, g_-\, u^{p-1} &\leq \int_M \phi^2\, u^{p-2}\, \Delta u - \int_M \phi^2\, u^{p-3}\, |\nabla u|^2 \\ &= -2 \int_M \phi\, u^{p-2}\, \langle \nabla \phi, \nabla u \rangle - (p-1) \int_M \phi^2\, u^{p-3}\, |\nabla u|^2 \\ &\leq \frac{1}{p-1} \int_M |\nabla \phi|^2\, u^{p-1}. \end{aligned}$$

However the right hand side can be estimated by

$$\int_M |\nabla \phi|^2\, u^{p-1} \leq \left(\int_M \phi^2\, u^{p+q-1} \right)^{\frac{p-1}{p+q-1}} \left(\int_M \phi^{-\frac{2(p-1)}{q}}\, |\nabla \phi|^{\frac{2(p+q-1)}{q}} \right)^{\frac{q}{p+q-1}}. \tag{2.3}$$

It is also clear that we can choose ϕ to satisfy the inequality

$$\phi^{-\frac{2(p-1)}{q}}\, |\nabla \phi|^{\frac{2(p+q-1)}{q}} \leq C_{10}\, r^{-\frac{2(p+q-1)}{q}},$$

hence (2.3) becomes

$$(2.4)\qquad \int_M |\nabla\phi|^2\, u^{p-1} \le \left(C_{10}\, r^{-\frac{2(p+q-1)}{q}}\, V_y(2r)\right)^{\frac{q}{p+q-1}} \left(\int_M \phi^2\, u^{p+q-1}\right)^{\frac{p-1}{p+q-1}}.$$

The second term on the left hand side of inequality (2.2) can be estimated by

$$\begin{aligned}\int_M \phi^2\, g_-\, u^{p-1} &\le \left(\int_M \phi^2\, u^{p+q-1}\right)^{\frac{p-1}{p+q-1}} \left(\int_M \phi^2\, g_-^{\frac{p+q-1}{q}}\right)^{\frac{q}{p+q-1}}\\ &\le \left(\int_M \phi^2\, u^{p+q-1}\right)^{\frac{p-1}{p+q-1}} \left(\int_{B_y(2r)} g_-^{\frac{p+q-1}{q}}\right)^{\frac{q}{p+q-1}}.\end{aligned}$$

The theorem follows by combining this and inequalities (2.2) and (2.4).

§3 Holomorphic Mappings and Holomorphic Functions

We are now ready to study holomorphic mappings from a Kähler manifold whose Ricci tensor satisfies certain integrability conditions.

Theorem 3.1. *Let M be a complete noncompact Kähler manifold without boundary of complex dimension m. Let $R(x)$ denote the pointwise lower bound of the Ricci curvature of M and $R_-(x)$ its negative part as defined in Theorem 1.1. Assume that $R_-(x)$ satisfies*

$$\int_M R_-\, dV < \infty,$$

and

$$\int_{B_y(r)} R_-^p\, dV = o(r^{\beta\,(p-1)})$$

for some $p > m$, and some $\beta < \frac{2}{m-1}$. Let ψ be a nonconstant holomorphic mapping from M into a complex Hermitian manifold N which has holomorphic bisectional curvature bounded from above by $K(z)$ for all $z \in N$. Suppose that the curvature of the image of M under ψ satisfies $K(\psi(x)) \le -B$ for all $x \in M$ and for some constant $B > 0$. If we denote the trace of the pulled-back metric tensor of N on M by

$$u = \operatorname{tr}\psi^*(ds_N^2).$$

Then it must satisfy the inequality

$$-\int_M R\, dV \ge -\int_M K(\psi(x))\, u(x)\, dV.$$

In particular, if either

$$\int_M R\, dV \ge 0,$$

or M admits a positive Green's function, then ψ has to be identically constant.

Proof. A direct computation (see [Lu], [C-C-L], and [L]) verifies that u satisfies the Bochner type differential inequality

$$\Delta u - \frac{|\nabla u|^2}{u} \geq -2K\,u^2 + 2R\,u.$$

Holomorphicity of ψ and the assumption that u is nonconstant implies that the zero set of u must be of measure zero on M. We claim that there is a constant $p' > m$ such that

$$\int_{B_y(r)} R_-^{p'}\,dV = o(r^2).$$

Indeed, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}\int_{B_y(r)} R_-^{p'}\,dV &\leq \left(\int_{B_y(r)} R_-\,dV\right)^{\frac{p-p'}{p-1}} \left(\int_{B_y(r)} R_-^{p}\,dV\right)^{\frac{p'-1}{p-1}}\\ &= o(r^{\beta\,(p'-1)}).\end{aligned}$$

The assumption on β allows us to choose $p' = \frac{2}{\beta} + 1 > m$.

Applying Corollary 1.2 and Theorem 2.3 by setting $p = p'$ and $q = 1$, we conclude that

$$\int_{B_y(r)} u^{p'}\,dV = o(r^2).$$

The theorem now follows from Theorem 2.1 and Corollary 2.2.

We would like to point out that the assumption that M possesses a positive Green's function is necessary even in dimension 2. In fact, let us consider a complete surface M with constant -1 curvature which has finite volume. By Huber's theorem, M is conformally equivalent, hence holomorphically equivalent to a compact surface with finite punctures. One can conformally change the metric to a complete metric which is flat in a neighborhood of each puncture. This new metric satisfies the hypothesis of Theorem 3.1 except the existence of a positive Green's function. However, it is holomorphically equivalent to a surface with constant -1 curvature, which gives a counter-example.

In the case if a Kähler manifold admits a nonconstant bounded holomorphic function, by scaling the holomorphic function, one can interpret it as a holomorphic mapping to the unit ball in **C**. On the other hand, the unit ball is biholomorphic to the Poincaré disk with the complete metric with -1 curvature. By taking the composition map, we obtain a holomorphic mapping from the Kähler manifold into the hyperbolic space form. Hence applying Theorem 3.1 to this setting we have the following:

Corollary 3.2. *Let M be a complete noncompact Kähler manifold without boundary of complex dimension m. Let $R(x)$ denote the pointwise lower bound of the Ricci curvature of M and $R_-(x)$ its negative part as defined in Theorem 1.1. Assume that $R_-(x)$ satisfies*

$$\int_M R_-\,dV < \infty,$$

and

$$\int_{B_y(r)} R_-^p \, dV = o(r^{\beta\,(p-1)})$$

for some $p > m$*, and some* $\beta < \frac{2}{m-1}$*. Then* M *does not admit any nonconstant bounded holomorphic functions.*

The argument in the proof of Theorem 3.1 relies on the Bochner differential inequality for the energy of holomorphic mappings. In fact, a larger class of mappings from a Kähler manifold also enjoy this differential inequality which was defined as pluriharmonic mappings in [L]. We will refer the reader to [L] for the computation and the proof of the following theorem.

Theorem 3.3. *Let* M *be a complete noncompact Kähler manifold without boundary of complex dimension* m*. Let* $R(x)$ *denote the pointwise lower bound of the Ricci curvature of* M *and* $R_-(x)$ *its negative part as defined in Theorem 1.1. Assume that* $R_-(x)$ *satisfies*

$$\int_M R_- \, dV < \infty,$$

and

$$\int_{B_y(r)} R_-^p \, dV = o(r^{\beta\,(p-1)})$$

for some $p > m$*, and some* $\beta < \frac{2}{m-1}$*. Let* ψ *be a nonconstant pluriharmonic mapping from* M *into a Riemannian manifold* N *which has Hermitian bisectional curvature bounded from above by* $K(z)$ *for all* $z \in N$*. Suppose that the curvature of the image of* M *under* ψ *satisfies* $K(\psi(x)) \leq -B$ *for all* $x \in M$ *and for some constant* $B > 0$*. If we denote the trace of the pulled-back metric tensor of* N *on* M *by*

$$u = \operatorname{tr} \psi^*(ds_N^2).$$

Then it must satisfy the inequality

$$-\int_M R \, dV \geq -\int_M K(\psi(x))\, u(x)\, dV.$$

In particular, if either

$$\int_M R \, dV \geq 0,$$

or M *admits a positive Green's function, then* ψ *has to be identically constant.*

By using the fact that the upper half plane is biholomorphic to the hyperbolic space form, we conclude the following:

Corollary 3.4. *Let* M *satisfies the same assumption as in Theorem 3.3. Then* M *does not admit any nonconstant positive pluriharmonic functions.*

Theorem 3.5. *Let M be a complete noncompact Kähler manifold without boundary of complex dimension m. let $S(x)$ denote the scalar curvature on M and $S_-(x)$ its negative part defined by $S_-(x) = \max\{0, -S(x)\}$. Assume that $S_-(x)$ satisfies*

$$\int_M S_- \, dV < \infty,$$

and that there exists a constant $p > 1$ and a point $y \in M$ such that

$$\int_{B_y(r)} S_-^p \, dV = o(r^2).$$

Suppose that the volume of the geodesic balls of radius r centered at y satisfy

$$V_y(r) = o(r^{2p+2}),$$

as $r \to \infty$. Let ψ be a nonconstant holomorphic mapping from M into a complex Hermitian manifold N which has the same dimension and with Ricci curvature bounded from above by $R_N(z)$ for all $z \in N$. Suppose that the curvature of the image of M under ψ satisfies $R_N(\psi(x)) \leq -B$ for all $x \in M$ and for some constant $B > 0$. If we denote the fourth power of the Jacobian of the map ψ by

$$v = \left(\frac{\psi^*(dV_N)}{dV_M} \right)^4,$$

and the trace of the pulled-back metric tensor of N on M by

$$u = \operatorname{tr} \psi^*(ds_N^2).$$

Then either ψ is totally degenerate, ie. v is identically 0, or u must satisfy

$$\begin{aligned} -\int_M S \, dV &\geq -\int_M R_N(\psi(x))\, u(x) \, dV \\ &\geq B \int_M u \, dV. \end{aligned}$$

In particular, if either

$$\int_M S \, dV \geq 0,$$

or M admits a positive Green's function, then ψ has to be totally degenerate.

Proof. It was derived in [C] that the function v satisfies the differential inequality

$$\Delta v - \frac{|\nabla v|^2}{v} \geq -2R_N \, u \, v + 2S \, v.$$

Arithmetic-geometric means implies that

$$u \geq \frac{m}{2}\, v^{\frac{1}{4m}}.$$

Letting $w = v^{\frac{1}{4m}}$, we can rewrite the differential inequality as

$$\Delta w - \frac{|\nabla w|^2}{w} \geq -\frac{1}{2}\, R_N\, w^2 + \frac{1}{2m}\, S\, w.$$

Now we can apply Theorem 2.3 to conclude that

$$\begin{aligned} \int_{B_y(r)} v^{\frac{p}{m}}\, dV &= \int_{B_y(r)} w^p\, dV \\ &= o(r^2). \end{aligned}$$

The theorem now follows from Theorem 2.1.

Corollary 3.5. *Let M be a complete noncompact Kähler manifold without boundary of complex dimension m. Let $S(x)$ and $R(x)$ denote the scalar curvature and the pointwise lower bound of the Ricci curvature on M, and $S_-(x)$ and $R_-(x)$ their negative parts respectively. Assume that $S_-(x)$ satisfies*

$$\int_M S_-\, dV < \infty,$$

and that there exists a constant $p > m$ such that

$$\int_{B_y(r)} R_-^p\, dV = o(r^2).$$

Let ψ be a nonconstant holomorphic mapping from M into a complex Hermitian manifold N which has the same dimension and with Ricci curvature bounded from above by $R_N(z)$ for all $z \in N$. Suppose that the curvature of the image of M under ψ satisfies $R_N(\psi(x)) \leq -B$ for all $x \in M$ and for some constant $B > 0$. If we denote the fourth power of the Jacobian of the map ψ by

$$v = \left(\frac{\psi^*(dV_N)}{dV_M}\right)^4,$$

and the trace of the pulled-back metric tensor of N on M by

$$u = \operatorname{tr} \psi^*(ds_N^2).$$

Then either ψ is totally degenerate, ie. v is identically 0, or u must satisfy

$$\begin{aligned} -\int_M S\, dV &\geq -\int_M R_N(\psi(x))\, u(x)\, dV \\ &\geq B \int_M u\, dV. \end{aligned}$$

In particular, if either

$$\int_M S\, dV \geq 0,$$

or M admits a positive Green's function, then ψ has to be totally degenerate.

Proof. The assumption on R_- and Corollary 1.2 implies the desired volume growth condition to apply Theorem 3.5. Now we observe that $S(x) \geq m\, R(x)$, and the corollary follows.

References

[C] S[redacted]S. Chern, *On holomorphic mappings of Hermitian manifolds of the same dimension*, Proc. Symp. Pure Math. **11** (1968), 157–170.

[C-C-L] [redacted]hen, S. Y. Cheng, and K. H. Look, *On the Schwarz lemma for complete Kähler manifolds*, Scientia Sinica **22** (1979), 1238–1247.

[G] S[redacted]Gallot, *Isoperimetric inequalities based on integral norms of Ricci curvature*, preprint 1988.

[H] A[redacted]Huber, *On subharmonic functions and differential geometry in the large*, Commentarii Math. Helv. **32** (1957), 13–72.

[L] P[redacted]Li, *On the structure of complete Kähler manifolds with nonnegative curvature near infinity*, preprint 1988.

[L-T1] P. Li [redacted]nd L. F. Tam, *Positive harmonic functions on complete manifolds with non-negative curvature outside a compact set*, Annals Math. **125** (1987), 171–207.

[L-T2] P. Li [redacted]nd L. F. Tam, *Symmetric Green's functions on complete manifolds*, Amer. J. Math. **109** (1987), 1129–1154.

[Lu] Y. [redacted]. Lu, *Holomorphic mappings of complex manifolds*, J. Diff. Geom. **2** (1968), 299–312.

[Y] S[redacted]T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Math. J. **25** (1976), 659–670.